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noML09: Deflected Subgradient Methods for a Dual Formulation of Convex Quadratic Separable Min Cost Flow Boxed Problems

The Problem and The Approach

We consider a dual approach to solve the Min Cost Flow box constrained convex quadratic separable problem (P) defined by

$$\nu_*(P) := \inf_{x \in S_{(P)}} \frac{1}{2} x^\intercal Q x + q^\intercal x$$

$$\text{where} \qquad S_{(P)} := \{x \,:\, Ex = b \;\wedge\; l \leq x \leq u\}$$

where, here and in the following, E, of size (m, n), is the node-arc incidence matrix of a directed graph, $Q \succcurlyeq 0$ is a diagonal matrix; we also let $f : \mathbb{R}^n \to \mathbb{R}$ denote the quadratic polynomial to be minimized, $f(x) := \frac{1}{2} x^\intercal Q x + q^\intercal x$.
We choose one of the simplest dual reformulations and approach its iterative solution implementing some subgradient methods; hence we touch upon deflected methods, automatic parameter tuning, heuristics and one of the "well known" properties of subgradient methods in dual reformulations.

Notation ▶ With subspace we mean a subset of $\mathbb{R}^k$, for some $k \in \mathbb{N}$. Comparisons between vectors in $\mathbb{R}^n$ are to be intended coordinate-wise, in the canonical basis; in the same spirit, the box described by $\{x : l \leq x \leq u\}$ is simply written as $[l, u]$. The linear subspace described by $\{x : Ex = b\}$ is named $\mathcal{H}_{E,b}$. $\|\cdot\|_F$ is the Frobenius norm defined as $\|A\|_F^2 := \operatorname{Tr} A^\intercal A = \sum_i \sigma_i^2$, where σ_i are the singular values of A. Functions are polymorphic, however the meaning should be evident from the context. Functions with a clear meaning for scalar values, such as max and min, when applied to vectors entail a broadcast, e.g. let u, v be two vectors in $\mathbb{R}^k$, then $\max(u, v) \in \mathbb{R}^k$ and $\max(u, v)_i = \max(u_i, v_i)$. Similarly, for any scalar q, $\max(u, q) \in \mathbb{R}^k$ and $\max(u, q)_i = \max(u_i, q)$. We conveniently define $\inf_\emptyset \cdot := +\infty$; analogous definition holds for $\sup_\emptyset$ et similia. We will stick to the *subgradient* appellative also when functions are concave.

1. The Dual Reformulation

The objective function in (P) is quadratic convex, the constraints are affine; box constraints can be effectively kept implicitly in the domain of definition of the Lagrangian. Strong duality holds, with the caveat that the optimal duality gap is to be considered null when both the optimal primal and optimal dual values are $+\infty$. This is a standard results and can be seen considering that constraints are affine and, in the box, the objective function is quadratic convex bounded so there exists a non-vertical separating hyperplane whenever $S_{(P)} \neq \emptyset$.

(D1) Dual Flux Conservation Constraints ▶ We relax the flux conservation constraints:

$$\nu_*(P) = \inf_{x \in S_{(D1)}} \sup_{\mu} L(x,\mu) \quad \geq \quad \sup_{\mu} \inf_{x \in S_{(D1)}} L(x,\mu) =: \nu^*(D1)$$

$$\text{where} \qquad L(x,\mu) := \frac{1}{2}x^\intercal Q x + q^\intercal x + \mu^\intercal (Ex - b)$$

$$\text{and} \qquad S_{(D1)} := [l,u].$$

The Lagrangian dual is defined by

$$L(\mu) := \inf_{x \in S_{(D1)}} L(x,\mu);$$

concurrently, the parametric set describing the relaxed primal points corresponding to a given dual point is defined as

$$X(\mu) := \arg \inf_{x \in S_{(D1)}} L(x,\mu). \tag{1}$$

Since the feasible space of the dual problem is a box and the Lagrangian is separable as a function of x for a fixed μ, $X(\mu)$ can be described componentwise[1], as the Cartesian product $X(\mu) = \prod X(\mu)_j$. Consider then the j-th component. If $Q_{jj} > 0$, thanks to convexity, it is sufficient to solve for x_j in

$$\frac{\partial L(x,\mu)}{\partial x_j} = (Qx + q + E^\intercal \mu)_j = 0$$

and then project to the nearest side of the box:

$$X(\mu)_j = \{\max(l_j, \min(u_j, -Q_{jj}^{-1}(q + E^\intercal \mu)_j))\}.$$

Else, if $Q_{jj} = 0$,

$$X(\mu)_j = \arg \min_{x_j \in [l_j,u_j]} (q + E^\intercal \mu)_j x_j = \begin{cases} \{l_j\} & \text{if } (q + E^\intercal \mu)_j > 0 \\ \{u_j\} & \text{if } (q + E^\intercal \mu)_j < 0 \\ [l_j, u_j] & \text{otherwise.} \end{cases}$$

Any specific value of $x(\mu) \in X(\mu)$ that we choose corresponds to the choice of a subgradient in the subdifferential

$$\partial L(\mu) = \{Ex - b : x \in X(\mu)\}; \tag{2}$$

without further specification, $x(\mu)$ represents an $x \in X(\mu)$. Clearly $L(\mu)$ is "piecewise differentiable" and

[1] In the same canonical basis, in which Q is diagonal

$$\forall g \in \partial L(\mu), \quad \|g\| \leq \|E\| \left(\|u\| + \|l\|\right) + \|b\| . \tag{3}$$

2. The Feasible Polyhedron

The feasible space $S_{(P)}$ of the primal problem is the intersection of the box $[l, u]$ with the linear subspace $\mathcal{H}_{E,b}$ described by flux conservation constraints, $Ex = b$. In order to devise a method to recast information we gain from calculations in the dual space to the primal space, some form of projection from $S_{(D1)}$ onto $S_{(P)}$[2] will be required. In the following paragraphs of this section we start with a short detour on basic properties of the incidence matrix E, to arrive to develop the theory to justify our implementation choices.

About E: bounding the distance from $S_{(P)}$ ▶ The first well known property of the incidence matrix E of a graph G is that, if the graph G is connected and it's not a singleton, the left kernel of E is generated by the vector of all 1s. Select the submatrix $E(T) \subseteq E$ corresponding to a spanning tree T of G. Then remove any node r, that is a row of $E(T)$; such node is conveniently designated as root of the tree. In this manner we obtained an invertible matrix that we call $\hat{E}(T_r)$. It's straightforward to calculate $\hat{E}(T_r)^{-1}$ with a simple analysis of its left action on $\hat{E}(T_r)$: consider the row i of $\hat{E}(T_r)^{-1}$; the result of its left action on $\hat{E}(T_r)$ is a linear combination of the rows of $E(T)$, and it should add to e_i, the row vector with a 1 at the ith place. There are two rows of $E(T)$ which are nonzero at column i, pick the one farther from the node r, with the appropriate sign to make it 1 at column i. If we continue, picking with the same sign the whole subtree which doesn't include r and rooted in the first vertex we selected, we'll obtain the whole i-th row of $\hat{E}(T_r)^{-1}$. In order to derive a bound independent from the particular topology of E, let T' range over all the possible trees with m nodes. Since each arc is appearing as a ± 1 in the matrix $\hat{E}(T'_r)^{-1}$ as many times as the number of its parents arcs in the tree, it's straightforward to determine greedily the trees leading to the sparsest and the densest $\hat{E}(T'_r)^{-1}$: the sparse one goes wide, e.g. is a star tree rooted in the center ($\hat{E}(T'_r)$ is the identity), the dense one, $T^d_r(m)$, goes far, e.g. is a path rooted at one of the extremal nodes ($\hat{E}(T^d_r(m))^{-1}$ is the upper triangular matrix filled with ± 1).
Two easy corollaries follow:

- It's more advantageous to calculate the left action of the inverse recursively, since each subtree can be considered as the appropriately union of its own proper subtrees and the root. As a consequence, given the sparse matrix $\hat{E}(T_r)$, calculating its action has the same O-cost as calculating the action of its inverse.

[2] or on a space of analogous description

- $\hat{E}(T_r)^{-1}$ contains only 0, ± 1, hence, denoting with σ the singular values, $\sigma_{\max}(\hat{E}(T_r)^{-1}) \leq \left\|\hat{E}(T_r)^{-1}\right\|_F \leq \left\|\hat{E}\left(T_r^d(m)\right)^{-1}\right\|_F = \sqrt{\binom{m}{2}}$.

Now consider the linear subspace $\mathcal{H}_{E,b}$ containing our feasible set; suppose that $x \in \mathcal{H}_{E,b'}$, with $b' = b + \Delta b$: *how far is x from $\mathcal{H}_{E,b}$?*
We can bound the distance with a function of the *least nonzero* singular value $\sigma_{\min}(E)$ and Δb. To evaluate $\sigma_{\min}(E)$ we can restrict our attention to connected graphs - otherwise we take the minimum among the $\sigma_{\min}$ of each connected component.
We can derive a lower bound for $\sigma_{\min}(E)$ from the upper bound for $\sigma_{\max}\left(\hat{E}(T_r)^{-1}\right)$.
Let $\mathfrak{S}_k^j$ be the set of subsets of $\{1,\ldots,k\}$ of cardinality j; for any $s \in \mathfrak{S}_k^j$ we can associate the canonical projection for Euclidean spaces $\mathbb{P}_s$. Let $V \subset \mathbb{R}^n$ be a linear subspace and restrict its dimension to be $m-1$; it holds that

$$\begin{aligned}
\sigma_{\min}(E) = \sup_V \inf_{\substack{v\in V\\ \|v\|=1}} \|Ev\| &\geq \sup_{s\in\mathfrak{S}_n^{m-1}} \inf_{\substack{v\in\mathbb{P}_s\mathbb{R}^n\\ \|v\|=1}} \|Ev\| \geq \\
&\geq \sup_{\substack{s\in\mathfrak{S}_n^{m-1}\\ t\in\mathfrak{S}_m^{m-1}}} \inf_{\substack{v\in\mathbb{P}_s\mathbb{R}^n\\ \|v\|=1}} \|\mathbb{P}_t Ev\| \geq \\
&\geq \sup_{T_r} \inf_{\substack{u\in\mathbb{R}^{m-1}\\ \|u\|=1}} \left\|\hat{E}(T_r)u\right\| = \\
&= \sup_{T_r} \sigma_{\min}\left(\hat{E}(T_r)\right) = \\
&= \frac{1}{\inf_{T_r} \sigma_{\max}\left(\hat{E}(T_r)^{-1}\right)} \geq \frac{\sqrt{2}}{\sqrt{m(m-1)}}.
\end{aligned}$$

The following corollary is then immediate.

Corollary 0.1 Distance of $\mathcal{H}_{E,b+\Delta b}$ from $\mathcal{H}_{E,b}$. *The distance of $\mathcal{H}_{E,\Delta b}$ from the origin is bounded by* $\sqrt{\binom{m}{2}}\,\|\Delta b\|$.

Proof: With a singular value decomposition, we can write $E = \sum_{i=1}^{\mathrm{rk}(E)} \sigma_i v_i u_i^\intercal$, $\sigma_i > 0$; if we let $x = \arg\min_{Ex'=\Delta b} \|x'\|$, then $\forall y \in \ker E,\ x \cdot y = 0$, hence it holds that $x = \sum_i u_i u_i \cdot x$. Since from $Ex = \Delta b$ we obtain $\sigma_i x \cdot u_i = \Delta b \cdot v_i$, we conclude with

$$\|x\| = \left\|\sum_i u_i \frac{\Delta b \cdot v_i}{\sigma_i}\right\| \leq \frac{\|\Delta b\|}{\sigma_{\min}} \leq \sqrt{\binom{m}{2}}\,\|\Delta b\|\,. \qquad \square$$

If the starting point was sufficiently inside the box $[l, u]$, e.g. at a distance greater than $\sqrt{\binom{m}{2}}\,\|\Delta b\|$ from the boundary, the aforementioned upper bound is valid also for the distance from the feasible set $S_{(P)}$.
Otherwise, from the amount of flow that a max-flow algorithm would move to satisfy the required flux conservation constraints, the distance from the feasible space is upper bounded by $n\,\|\Delta b\|_1 \leq n\sqrt{m}\,\|\Delta b\|$ [3]. Thus we have shown the following:

Proposition 0.2 Distance from $S_{(P)}$: bound. *Given an $x \in \mathcal{H}_{E,b'} \cap [l, u]$, the following upper bound for the distance of x from the feasible space $S_{(P)} \neq \emptyset$ holds:*

$$d\left(x, S_{(P)}\right) \leq \begin{cases} \sqrt{\binom{m}{2}}\,\|\Delta b\| & \text{if } d(x, \partial[l, u]) \geq \sqrt{\binom{m}{2}}\,\|\Delta b\| \\ n\,\|\Delta b\|_1 \leq n\sqrt{m}\,\|\Delta b\| & \text{otherwise.} \end{cases}$$

Spanning trees and cycles: $\ker E$ ▷ For the sake of completeness in the combinatorial interpretation of the geometry, we show how the choice of a tree T and a root r, to be removed, naturally leads to a choice of the basis for the circulation circuits of the graph. Let $\hat{E}(G \setminus T_r)$ be the incidence matrix of the graph G without the edges in T, where also the node r, and the respective row, has been removed. Consider the left action of $\hat{E}(T_r)^{-1}$ on a column of $\hat{E}(G \setminus T_r)$ corresponding to an arc $v_i \to v_j$ in $G \setminus T$, denoted e_{ij}. Each row of $\hat{E}(T_r)^{-1}$ corresponding to an arc in the unique path in T from the root r to a given node v_k, is nonzero at the column corresponding to the node v_k and vice versa, thus, taking care of the opposite signs, we deduce that the left action of $\hat{E}(T_r)^{-1}$ on the column of $\hat{E}(G \setminus T_r)$ corresponding to e_{ij} is the union of the two paths leaving from v_i and from v_j up to their lowest common ancestor in the rooted tree T. The two paths are taken with opposite signs and their union with the reversed arc e_{ij} is a circuit, which gives the combinatorial interpretation of the base of circuits described by the kernel basis $\begin{bmatrix} -\hat{E}(T_r)^{-1}\hat{E}(G \setminus T_r) \\ I \end{bmatrix}$.

Projecting on $S_{(P)}$ ▶ Proposition 0.2 gives upper bounds for the distance between an x living in the dual space $S_{(D1)}$ [4] and the primal space $S_{(P)}$. Given these theoretical results, how to actually project a given $x \in [l, u] \cap \mathcal{H}_{E,b+\Delta b}$ to the feasible polyhedron $[l, u] \cap \mathcal{H}_{E,b}$? We won't strictly need the Euclidean projection, but the common property which we require for any projection $\Pi : [l, u] \to S_{(P)}$ is

[3] From Holder inequality, $\|x\|_p = \left(\sum_{i=1}^{n} x_i^p \cdot 1\right)^{\frac{1}{p}} \leq n^{\frac{1}{p} - \frac{1}{q}}\,\|x\|_q$.
[4] i.e. $x \in [l, u] \cap \mathcal{H}_{E,b+\Delta b}$ for some Δb.

$$\lim_{x \to x^* \in S_{(P)}} \Pi x = x^*. \tag{Proj}$$

Exploiting such projections, we can obtain upper bounds to the optimal value $\nu_*(P)$ even if the iterates carried out in the dual space did not reach the optimum yet. The idea is to consider lower order approximations of the original problem, i.e. the 0th order approximation and the 1st order ones which are briefly explained in following paragraphs. For such models, performant combinatorial algorithms exist that open the way for a precise but cheaper projection onto the feasible space $S_{(P)}$.
Hereafter, property (Proj) is satisfied, as the variation of flow per arc remains bounded by $\|\Delta b\|_1$.
Order 0*: max flow* ▷ At order 0, there is no variable cost per edge; we can apply any max flow algorithm suited to a non-integer flux. Algorithms privileging shorter augmenting paths may grossly lead to a smaller variation in the objective value.
Order 1*: shortest paths* $\rightsquigarrow$ *linear min cost flow* ▷ We can assign a cost to each arc considering the gradient of the objective function or of the Lagrangian, then apply a linear min-cost max-flow algorithm. A relevant observation is that, for a given $x(\mu)$ satisfying equation (1), in exact arithmetic there is no negative cycle in the residual network. In fact, let $x(\mu) \in [l, u] \cap \mathcal{H}_{E,b'}$ and consider the case of linear costs described by $\nabla_x f(x(\mu))$; thus $x(\mu)$ is an optimal point of the problem (P'), that is the same as (P) except for the flux conservation constraints. Suppose there is a direction $d \in \ker E$ such that $d^\intercal \nabla f(x(\mu)) < 0$; since $x(\mu)$ is optimal for (P'), it must hold that $\forall \epsilon > 0,\ x + \epsilon d \notin [l, u]$. Analogous reasoning applies when the cost is described by $\nabla_x L(x(\mu), \mu)$.

The box who missed $\boldsymbol{\mathcal{H}_{E,b}}$: nearest point in the box ▶ Suppose we are given a box $[l', u']$ and a linear subspace $\mathcal{H}_{E,b}$ whose reciprocal intersection is empty; consider the problem to determine the point in the box that is nearest to the linear subspace, with respect to the Euclidean norm. From a geometric point of view, we have iterative approaches from the corresponding quadratic box constrained minimization problem (`MQBProblem` in the package):

$$\arg \min_{x \in [l', u']} \|Ex - b\|^2.$$

Any of the projected conjugate gradient iterations, though agnostic of the peculiar form of E, is suited to the problem.

Effective upper bounds to $\boldsymbol{\nu_*(P)}$ ▶ When $Q \succ 0$ even a simple max-flow projection is efficient, as shown in [11]; instead, when Q is singular, it's generally impossible to define a continuous $x : \mu \mapsto x(\mu) \in X(\mu)$. In particular, as $\mu \to \mu^*$, where $\mu^* \in \arg\sup L(\mu)$, there's no guarantee that

$x(\mu) \to x^*$, where $x^* \in \arg\inf_{x \in S_{(P)}} f(x)$ is a feasible primal optimal point of (P). Still, from equation (2), it does hold that

$$x^* \in \arg\min_{x \in X(\mu^*)} \|Ex - b\| = X(\mu^*) \cap \mathcal{H}_{E,b},$$

therefore, in exact arithmetic, we are able to calculate a primal optimal x^*, as an example with the projected conjugate gradient algorithm, where the box constraints are described by $X(\mu^*)$.

In a more realistic scenario[5], we have a μ_{lb} and a lower bound to the optimal value $L(\mu_{lb}) \leq \nu^*(D1)$; from this we can calculate the corresponding x_{lb}. The simple geometrical insight is that, if we are near to the optimal μ^* and L is not differentiable in μ^*, then a min-norm ϵ-subgradient could correspond to an x' nearer to the optimal primal point than x_{lb}.

In practice, consider the box $X_\epsilon(\mu) \supseteq X(\mu) = X_0(\mu)$, equal to $X(\mu)$ except for the fact the any component j corresponding to $Q_{jj} = 0$ is defined by

$$X_\epsilon(\mu)_j := \begin{cases} \{l_j\} & \text{if } (q + E^\intercal \mu)_j > \epsilon \\ \{u_j\} & \text{if } (q + E^\intercal \mu)_j < -\epsilon \\ [l_j, u_j] & \text{otherwise.} \end{cases}$$

Does a $\Delta\mu$ exist such that $X(\mu + \Delta\mu) = X_\epsilon(\mu)$? Let v be the vector in $\mathbb{R}^n$ defined by

$$v_j := \begin{cases} -(q + E^\intercal \mu)_j & \text{if } Q_{jj} = 0 \text{ and } |(q + E^\intercal \mu)_j| \leq \epsilon \\ 0 & \text{otherwise;} \end{cases}$$

then clearly it should hold $E^\intercal \Delta\mu = v$, which is solvable in $\Delta\mu$ if and only if the sum of the changes described by v in any cycle of the graph is null, that is $q^\intercal v = 0$ for any $q \in \ker E$. If any solution exists, it can be found inside a ball of radius $\sqrt{\binom{m}{2}}\, \|v\| \leq \epsilon \sqrt{\binom{m}{2} \dim \ker Q}$ centered in μ.

Though there's no guarantee that $X_\epsilon(\mu)$ is going to describe the subdifferential of the Lagrangian dual in a nearby point, the key observation is that redistributing flow through the network with capacity constraints restricted to $X_\epsilon(\mu)$, the Lagrangian variation is bounded by $\epsilon \left\| u_{X_\epsilon(\mu)} - l_{X_\epsilon(\mu)} \right\|_1$; the smaller the variation in the Lagrangian, the more we are moving toward the saddle point. Therefore the strategy is similar to the first order min-cost heuristic, where the costs are given by the gradient of the Lagrangian, but redistribution of flow along arcs without a quadratic part is prioritized. The nearer the starting point to the optimal solution, the more the strategy is expected to be effective.

[5] A full-fledged scenario would include floating point errors, e.g. a lower bound is such only up to a given approximation, to be evaluated

3. Subgradient Iterations

Classical subgradient iterations are the baseline upon which to build deflected iterations and against which to compare their performance. We deal with concave maximization problems, so we describe subgradient methods as applied to concave functions; for coherence, we will stick to the notation introduced for the problem $(D1)$ and denote the concave function under examination with L, the variable with μ, with $g(\mu) \in \partial L(\mu)$ any subgradient in the subdifferential $\partial L(\mu)$.

Convergence: general results ▶ Subgradient methods are very simple iterations usually written in one of the forms:

$$\mu^{k+1} = \mu^k + \alpha_{k+1} g(\mu^k) \qquad \text{(StepSize)},$$
$$\mu^{k+1} = \mu^k + \gamma_{k+1} \frac{g(\mu^k)}{\|g(\mu^k)\|} \qquad \text{(StepLength)},$$

where $\alpha_k, \gamma_k > 0$. Since subgradient methods are not strictly descent methods, we have to keep track of the best result found so far, i.e. the one with greatest function value, as of

$$\mu^k_{best} = \arg\max(L(\mu^1), \cdots, L(\mu^k)),$$
$$L^k_{best} = L(\mu^k_{best}) = \max(L(\mu^1), \cdots, L(\mu^k)).$$

Let L^* be the optimal value; from the definition, it's simple to show [7] that

$$L^* - L^k_{best} \leq \frac{\left\|\mu^1 - \mu^*\right\|^2 + \sum_{i=1}^k \alpha_i^2 \left\|g(\mu^i)\right\|^2}{2\sum_{i=1}^k \alpha_i}.$$

If a bound for the distance of the optimal point, $\left\|\mu^1 - \mu^*\right\| \leq R$, and a bound for the subgradient, $\forall \mu \; \|g(\mu)\| \leq G$, are available, i.e. equation (3), we obtain

$$L^* - L^k_{best} \leq \frac{R^2 + G^2 \sum_{i=1}^k \alpha_i^2}{2\sum_{i=1}^k \alpha_i}. \tag{4}$$

Note that the right-hand side of (4) is a convex function of the α_i, so that it has minimum when $\forall i, \; \alpha_i = \alpha$, that is

$$L^* - L^k_{best} \leq \frac{R^2}{2k\alpha} + \frac{G^2\alpha}{2};$$

fixed an ϵ, we find for $\alpha = \frac{\epsilon}{G^2}$ and $k = \frac{G^2 R^2}{\epsilon^2}$ that $L^* - L^k_{best} \leq \epsilon$; fixed the number of iterations k, we have for $\alpha = \frac{R}{G\sqrt{k}}$ that

$$L^* - L_{best}^k \leq \frac{RG}{\sqrt{k}}.$$

The upper bound is then confirming the intuitive idea that the step size should decrease together with the distance from the optimal value.
If we know the optimal value L^*, or an estimate for it, we can make our choice be adaptive with respect to the number of iterations, by maximizing the objective value step by step:

$$\gamma_{k+1} = L^* - L(\mu^k) \implies \sum_{i=1}^{k}(L^* - L(\mu^i))^2 \leq R^2G^2, \qquad \text{(PolyakStep)}$$

that is,

$$L^* - L_{best}^k \leq \frac{RG}{\sqrt{k}}.$$

In fact, a theorem by Nesterov shows that, for general concave functions with bounded subgradients, this is optimal up to a constant factor.

Theorem 0.3 Nesterov [4] Thm 3.2.5. *Suppose μ^{k+1} is computed by an* arbitrary *method as*

$$\mu^{k+1} = \mu^1 + span\{g(\mu^1), \cdots, g(\mu^k)\}$$

where the $g(\mu^i) \in \partial L(\mu^i)$ are arbitrary. Then there is a nonsmooth concave function with $\|g\| \leq G$ uniformly so that the above method obeys

$$L^* - L(\mu^k) \geq \frac{\|\mu^1 - \mu^*\| \, G}{2\sqrt{k}}.$$

The Polyak iterations reach the optimal worst case for generic concave functions. However, our objective function is not a generic concave function: it is piecewise quadratic/linear. Can we do better? Indeed, if we take a look at how the Polyak approach works on any of our instances, we get robust results, but still incredibly slow. In figure 3 it is evident how, even knowing the optimal value, 1 million of iterations are needed to reach a relative error of $1e-5$: for the same instance, one tenth of iterations of a restarted deflected subgradient method were enough to reach even three additional digits of precision. Similarly, the deflection methods built upon Polyak step sizes did not bring remarkable improvements on the convergence speed. It was while playing with the automatic tuner that a different strategy emerged.

4. Restarted Subgradient Methods

Our investigations on dynamic parameter tuning in deflected subgradient methods led to the experimental evidence that a strategy based on restarts and exponential decay of the step size yields a robust, efficient and simple enough method. Interestingly, in the case of standard subgradient iterations, with fixed step size, there is a nice analysis of the reasons for such effectiveness. A restarted subgradient method (RSG) runs in multiple stages warm-started by the solution from previous stages. Within each stage, the subgradient iteration of choice is performed for a fixed number of times with a constant step size. This step size is reduced geometrically from stage to stage, as described in Algorithm 1. Theorem 0.4 links the restarted subgradient complexity to the distance between the ϵ-level set and the optimal set; because of such property, specific iteration complexity are deduced based on the local growth property of the objective function in Corollary 0.5.
Let $\mathcal{L}_\epsilon$ denote the ϵ-level set of L, i.e. $\mathcal{L}_\epsilon := \{\mu : L(\mu) = L^* + \epsilon\}$, and let $\rho_\epsilon := \min_{\mu \in \mathcal{L}_\epsilon} \|\partial L(\mu)\|$. We denote with μ^* the nearest optimal point to μ.

Theorem 0.4 RSG [12] Thm 3. *Under the assumptions*

- $\forall \mu \; \|g(\mu)\| \leq G$,
- *the subgradient iterations will stay inside a region* Ω *such that* $\forall \mu \in \Omega$ *we have* $L^* - L(\mu) \leq \epsilon_0$,
- *the optimal set is non-empty convex compact*[6],

the total number of iterations for Algorithm 1 to find a 2ϵ*-optimal solution is at most* $O\left(t \left\lceil \log_r \left(\frac{\epsilon_0}{\epsilon}\right)\right\rceil\right)$, *where* $t \geq \frac{r^2 G^2}{\rho_\epsilon^2}$.
Denoting with $B_\epsilon := \max_{\mu \in \mathcal{L}_\epsilon} \|\mu - \mu^*\|$, *we have that* $\rho_\epsilon \geq \frac{\epsilon}{B_\epsilon}$, *so that the iteration complexity for obtaining a* 2ϵ*-optimal solution is* $O\left(\frac{r^2 G^2 B_\epsilon^2}{\epsilon^2} \left\lceil \log_r \left(\frac{\epsilon_0}{\epsilon}\right)\right\rceil\right)$.

In practice, an estimate of all the parameters in theorem 0.4 is not needed, i.e. an exponential search is enough to determine the number of iterations per stage required for a desired ϵ.
A nice corollary follows from theorem 0.4.

Corollary 0.5 RSG with Local Error Bounds [12] Crl 7. *If, for some constants* $\theta \in (0, 1]$ *and* $c > 0$,

[6] In our case, the optimal set is affine, in particular not compact. However, each direction generating the optimal set is orthogonal to the whole $\partial L(\mu)$ everywhere. Hence the algorithm is not affected by the degeneracy of the optimal set.

Algorithm 1: Restarted Subgradient

Input: the number of stages K and the number of iterations t per-stage, μ_0 and $r > 1$

Set $\alpha_1 = \frac{\epsilon_0}{rG^2}$ `// α : step size`

for $k = 1, \ldots, K$ **do**

  Call subgradient subroutine SG to obtain $\mu_k = SG(\mu_{k-1}, \alpha_k, t)$

  Set $\alpha_{k+1} = \frac{\alpha_k}{r}$

end

Output: μ_K

$$\|\mu - \mu^*\| \leq c(L^* - L(\mu))^\theta, \quad \forall \mu : L^* - L(\mu) \leq \epsilon,$$

then the iteration complexity for the RSG algorithm of Theorem 0.4 for obtaining a 2ϵ-optimal solution is $O\left(\frac{r^2 G^2 c^2}{\epsilon^{2(1-\theta)}} \log_r\left(\frac{\epsilon_0}{\epsilon}\right)\right)$, provided $t = \frac{r^2 G^2 c^2}{\epsilon^{2(1-\theta)}}$ and $K = \left\lceil \log_r\left(\frac{\epsilon_0}{\epsilon}\right)\right\rceil$.

Given that the local growth of $\mu - \mu^*$ as a power of $L(\mu) - L^*$ has an exponent $\theta \in \left[\frac{1}{2}, 1\right]$, we would expect for the number of iterations t per stage to satisfy approximately $t \propto \epsilon^{-\vartheta}$, where $\vartheta \in [0, 1]$ depends on $\dim \ker Q$.

5. Deflected Subgradient Iterations

Restarted subgradient methods turned out to be both simple to tune and effective approaches. They allowed to exploit the particular geometry of the problem at hand. In order to further accelerate subgradient methods we try to contrast the common zig-zagging behaviour by playing with the direction the algorithm chooses at each iteration.
In the literature such methods are known as deflected subgradient methods. They incorporate some form of memory (and of premonition, in the case of NesterovMomentum) of the previous iterations to synthesise a hopefully more promising direction for the step. We describe a couple of alternatives which approach the problem from different points of view; the description of each iteration is in the following table.

Name	Params	Iteration
Adagrad	$\alpha \in [0,1]$	$g^{k+1} \in \partial L(\mu^k)$ $s^{k+1} \leftarrow s^k + g^{k+1}.* g^{k+1}$ $d^{k+1} \leftarrow g^{k+1}./.\sqrt{s^{k+1}}$ $\mu^{k+1} \leftarrow \mu^k + \alpha d^{k+1}$
RMSProp	$\alpha \in [0,1]$ $\gamma \in [0,1]$	$g^{k+1} \in \partial L(\mu^k)$ $s^{k+1} \leftarrow \gamma s^k + (1-\gamma) g^{k+1}.* g^{k+1}$ $d^{k+1} \leftarrow g^{k+1}./.\sqrt{s^{k+1}}$ $\mu^{k+1} \leftarrow \mu^k + \alpha d^{k+1}$
Nesterov Momentum	$\alpha \in [0,1]$ $\beta \in [0,1]$	$g^{k+1} \in \partial L(\mu^k + \beta v^k)$ $v^{k+1} \leftarrow \beta v^k + \alpha g^{k+1}$ $\mu^{k+1} \leftarrow \mu^k + v^{k+1}$

Both Adagrad and RMSProp iterations reduce the movement along coordinates which are moving more than the others, see [13] for the theoretical justification of Adagrad, while RMSProp has been devised to obviate to the dampening of Adagrad. Incorporating them in a restarting scheme with decay of the step size parameter led to nice results. Nevertheless, the most performant method, generally requiring half of the iterations to reach the same precision, appeared to be NesterovMomentum with high momentum. Hence we present our short tentative convergence study of such iteration, and *leave as an unsolved question the theoretical convergence of the method*. We hint [14] as another possible approach we did not explore.

NesterovMomentum: tentative analysis ▶ At iteration k, the step direction is the sum of a momentum term, βv^k, and the contribution αg^{k+1} from the look-ahead subgradient, $g^{k+1} \in \partial L\left(\mu^k + \beta v^k\right)$. Unrolling the iteration equations, we derive

$$
\begin{aligned}
v^k &= \alpha \sum_{i=1}^{k} \beta^{k-i} g^i, \\
\mu^k &= \mu^0 + \sum_{i=1}^{k} v^i = \mu^0 + \alpha \sum_{i=1}^{k} g^i \frac{1-\beta^{k+1-i}}{1-\beta}, \\
g^k &\in \partial L\left(\mu^0 + \alpha \sum_{i=1}^{k-1} g^i \frac{1-\beta^{k+1-i}}{1-\beta}\right).
\end{aligned}
$$

Let's analyse if a reduction to the convergence results for ϵ-subgradients is viable. By the information transport property of ϵ-subgradients,

$$
\begin{aligned}
g \in \partial_\epsilon L(\mu) &\implies g \in \partial_{\epsilon'} L(\mu'),\\
\epsilon' = \epsilon + g \cdot (\mu' - \mu) &- L(\mu') + L(\mu),
\end{aligned}
$$

for the look-ahead subgradient we can write

$$
\begin{aligned}
g^{k+1} &\in \partial_{\rho^{k+1}} L\left(\mu^k\right),\\
\rho^{k+1} = L(\mu^k + \beta v^k) - L(\mu^k) &- \beta g^{k+1} \cdot v^k,
\end{aligned}
$$

while, for the momentum term, we obtain the formula

$$
\begin{aligned}
\frac{1-\beta}{\alpha(1-\beta^k)} v^k \in \partial_{\sigma^k} L\left(\mu^{k-1}\right) &\implies \frac{1-\beta}{\alpha(1-\beta^k)} v^k \in \partial_{\sigma^{k'}} L\left(\mu^k\right),\\
\sigma^{k'} = \sigma^k + L\left(\mu^{k-1}\right) &- L\left(\mu^k\right) + \frac{1-\beta}{\alpha(1-\beta^k)} v^k \cdot v^k.
\end{aligned}
$$

Finally, from $\frac{v^{k+1}}{\alpha(1+\cdots+\beta^k)} = \frac{1}{\alpha(1+\cdots+\beta^k)} \left(\alpha g^{k+1} + \beta\alpha(1+\cdots+\beta^{k-1}) \frac{v^k}{\alpha(1+\cdots+\beta^{k-1})}\right)$, we deduce

$$
\begin{aligned}
\frac{v^{k+1}}{\alpha(1+\cdots+\beta^k)} &\in \partial_{\sigma^{k+1}} L\left(\mu^k\right),\\
&\sigma^{k+1} \le \max(\rho^{k+1}, \sigma^{k'}) \le \rho^{k+1} + \sigma^{k'} \implies\\
&\implies \sigma^{k+1} \le \rho^{k+1} + L(\mu^{k-1}) - L(\mu^k) + \frac{1-\beta}{\alpha(1-\beta^k)} v^k \cdot v^k + \sigma^k \implies\\
&\implies \sigma^{k+1} \le L(\mu^0) - L(\mu^k) + \sum_{i=1}^{k} \left(\rho^{i+1} + \frac{1-\beta}{\alpha(1-\beta^i)} v^i \cdot v^i\right).
\end{aligned}
$$

It is not clear at this point how/if it is possible to reduce the proof of convergence of RestartedNesterovMomentum to the proof of convergence of Restarted(Approximate)Subgradients.

Arithmetic cost per iteration ▶ In the next table we describe the arithmetic cost of the main iterations we discussed; this allows to estimate, from the number of needed iterations, the performance of the algorithm. In particular, as shown in the data we selected, as presented in the next section, the RestartedNesterovMomentum is generally reaching at least the same accuracy as RestartedSubgradient in at most one half of the iterations (which is the reason why, for NesterovMomentum, we represent each iteration as two iterations in our plots). From the next table we can then observe how such deflected method is more performant than the simple RestartedSubgradient.

Iteration	$+$	$*$	max / min	$/$	$\sqrt{\cdot}$
Subgradient	$5n + 2m$	$n + m$	$2n$	0	0
NesterovMomentum	$5n + 4m$	$n + 3m$	$2n$	0	0
Adagrad	$5n + 3m$	$n + 2m$	$2n$	m	m
RMSProp	$5n + 3m$	$n + 4m$	$2n$	m	m

6. Experiments

Implementation ▷ The software is distributed as a Julia package. It is designed to allow for an easy exploration of the dynamics of (deflected) subgradient iterations, heuristics, meta-algorithms on this particular problem; because of this, the resulting product is not a 1-click solver and the focus is on flexibility rather than performance. A simple parallelized C++ implementation of the Restarted Nesterov Momentum is accompanying the Julia package; in our setup the parallel primitives of the C++ STL are leveraging Intel TBB libraries, with no gpu support at the time of writing. This allowed to test the algorithm on larger instances.
Testing material ▷ Separable convex quadratic min cost flow instance generators are freely available on the web, together with some already generated test-sets[7]. In addition to the available material, the simple random test generator used in our experiments is described in Algorithm 2; note that it generates a possibly disconnected multigraph.
Errors ▷ Calculations are carried out in the relaxed space of $(D1)$; points are exactly inside the box, but aren't usually exactly in $\mathcal{H}_{E,b}$, even after projection. To estimate the error on the objective value upper bound, for a point $x \in \mathcal{H}_{E,b+\Delta b}$, we use the estimate for $d(x, S_{(P)})$ from proposition 0.2 together with $|f(x + \Delta x) - f(x)| \approx |\partial f(x)\Delta x| \leq \|Qx + q\| \, \|\Delta x\|$.

Discarded strategies ▶ Most of the methods implemented in the package did not reveal to be effective for the problem at hand. In particular, we stress upon the following ones:

- *Projected conjugate subgradient* ▷ Since the Lagrangian dual is piecewise-quadratic, it's possible to define an exact line search; moreover, in exact arithmetic, once the right polyhedral region is found, with the conjugate gradient it could be possible to find the optimal point. The resulting conjugate gradient algorithm works fine when $Q \succ 0$, details can be found in the literature [11].
- *Ergodic sequences* ▷ There are general methods to derive primal iterates from the dual ones [3] [5] [6]; in particular we had implemented and tested the iterations described as "harmonic sequences" in [5]. However,

[7]As an example, the data available at the MCF web page by the Operations Research Group of University of Pisa.

Algorithm 2: Sketch of a Test Generator for QMCFBProblem

```
/* m : number of nodes                                  */
/* n : number of arcs                                   */
/* 𝔰 : ker Q                                            */
/* 𝔞 : ≈ active box constraints                         */
function get_test(m, n, 𝔰, 𝔞):
    E = incidence matrix m × n with random arcs
    x = random flow
    (l, u) = random capacity interval around x
    b = E*x            // so x is an internal feasible point

    Q[𝔰̄, 𝔰̄] = random diagonal ≻ 0
    Q[𝔰, 𝔰] = 0
    (𝔩, 𝔲) = random partition of 𝔞
    q[𝔩] = random > -Q[𝔩, 𝔩] * l[𝔩]
    q[𝔲] = random < -Q[𝔲, 𝔲] * u[𝔲]
    q[𝔞̄] = random ∈ (-Q[𝔞̄, 𝔞̄] * u[𝔞̄], -Q[𝔞̄, 𝔞̄] * l[𝔞̄])

    return (Q, q, E, b, l, u)
```

the techniques described in §2 appeared to be superior as for quality (stricter upper bounds to the optimal value) and flexibility (completely independent from the subgradient iteration of choice).

- *Meta algorithms* ▷ Deflected subgradient methods are empirically sensitive not only to parameters, but, by construction, also to the history of the iterations. The iterations can be embedded in a meta-algorithm which drives the dynamics of the parameters and the restarts (where the memory of the iteration is cleared). We tested the idea of fine-tuning with a zero order parameter search, the Nelder-Mead simplex algorithm. For some deflected methods, investigations led to the very simple, but performant, restart and parameter dynamics described in the foregoing sections, hence supplanting the idea of actually using the meta-algorithms to drive the subgradient method parameters. For some subgradient methods, the hyperparameter investigation did not fully confirm a simple strategy, except for the step-size parameter, which usually follows a grossly exponential decay (Figure 13).
- *Stopping criteria* ▷ Projection on the feasible polyhedron §2, as implemented at the time of writing, is computationally too expensive to be used as a stopping criterion. At the same time, the standard upper bounds for subgradient iterations are too loose to be useful. A possi-

ble strategy to attain efficient error-dependent projections could be to implement a scaling linear min-cost flow algorithm, like the Goldberg-Tarjan one, where the computational complexity is dependent on the desired precision, to be set accordingly to the bounds in Proposition 0.2.

Selected data ▶ About notation: we characterize each problem instance generated by Algorithm 2 with the following attributes:

- `nodes`: number of nodes
- `arcs`: number of arcs
- `singular`: (fraction of) zero elements on the diagonal of Q
- `active`: expected (fraction of) active constraints at the optimal point.

Instances generated with Pargen+Netgen+Qfcgen have a standardized name of the form `netgen`-n-ρ-k-cf-cq-$scale$-$singular$, where:

- ρ: then we have for the number of nodes $m = \left\lfloor \frac{1+\sqrt{1+\frac{32n}{\rho}}}{2} \right\rfloor$
- k: instance number
- $cf \in$ `[a, b]`: indicates if the fixed costs are generated to be high (`a`) or low (`b`) with respect to the linear costs generated by netgen
- $cq \in$ `[a, b]`: indicates if the quadratic costs are generated to be high (`a`) or low (`b`) with respect to the fixed costs
- $scale \in$ `[s, ns]`: if `s`, capacity are scaled by 0.7
- $singular$: number of diagonal elements of Q set to 0

We do not consider fixed costs in the present work, thus cf and cq together regulate the ratio between Q and q, in particular `a-a` corresponds to the highest ratio $\frac{Q}{q}$, `b-b` to the lowest one.

Deflected Subgradient methods ▷ A striking property of standard, non restarted, deflected subgradient methods, is their sensitivity to the parameters. This appears to be even more evident when the dual problem is C^1, hence we corroborate the statement with two anecdotal examples where Q is nonsingular. The precision attained for the solution is here measured with $\nabla_\mu L(\mu)$, which is more strictly related to the dynamics described by the iterations than the distance from the optimal point:

- Figure 1, represents the dynamics of Nesterov Momentum iterations while varying the parameters. Each line represents the dynamics for a different parameter setup. The best configuration we localized appeared to be a stationary point in parameter space.

- Figure 2, represents a similar dynamics for the RMSProp iterations. Here we represent another property of the iterations: by driving the parameters a bit further the stabler stationary position, we stumbled very near to the optimal point very soon.

After an initial investigation, with the help of the automatic parameter tuner, over all the implemented deflected subgradient methods (which can be found in `subgradient.jl`), we realized that the most performing deflected method was consistently the NesterovMomentum iteration, with sufficiently high momentum parameter, i.e. $\beta \geq 0.95$. In terms of call to the subgradient oracle, the method costs 1 call per iteration; however, the net effect of calling the oracle after moving forward (because of momentum) is that generally not more than *half of the iterations* are needed to obtain results comparable to the ones of other methods. Thus *we always depict NesterovMomentum iterations by considering each iteration as two iterations.*
Toward restarted subgradients $\triangleright$ We have seen that encapsulating subgradient iterations in a exponentially decaying stepsize scheme brings a new shape-dependent flavour to subgradient methods; in practice, the RestartedNesterovMomentum was the best subgradient method for the problems at hand.
Figures 9 10 11 12 13 plot the (locally) optimal parameters, as calculated from the automatic parameter tuner, for each step of the restarted algorithm, for the subgradient iterations of NesterovMomentum, Adagrad, RMSProp, Adam. Such plots are to be read as a hint to the exponential decay of the step-size parameter α but also to some ranges for the remaining parameters, keeping in mind that the automatic parameter tuner is looking for *locally optimal* parameters. To render the general trend more efficiently, in each figure three lines corresponding to a different choice of the number of iterations per stage are plotted.
Figure 4 confirms a part of Corollary 0.5, that is:

- It is necessary to increment the number of iterations per stage to get nearer to the optimum - in practice, later stages require more iterations. This is qualitatively compatible with the formula $t = \frac{r^2 G^2 c^2}{\epsilon^{2(1-\theta)}}$.
- The attainable precision is exponential in the number of stages, where the base is given by the geometric reduction factor of the step-size parameter. The plot confirms the worst-case formula $K = \left\lceil \log_r \left(\frac{\epsilon_0}{\epsilon} \right) \right\rceil$.

We can analyse in more detail the worst-case, geometry dependent, relation between the number of iterations per stage t and the desired error ϵ; Figures 5 6 7 render such relationship for instances with different local growth, at the same time also comparing the solution quality of the RestartedSubgradient (RSG) with the solution quality of the RestartedNesterovMomentum (RNM).

Sticking to the most performant method, RNM, we can analyse in more details what is the optimal number of stages/iterations per stage. We carry out the analysis on the test-set `netgen1000`, a test-set of `QMCFBProblems` with 1000 arcs, composed of 5 instances per parameter setup, generated with Netgen+Pargen+Qfcgen[8]. In Table 5 the optimal number of stage is plotted against the desired relative error, while in Table 6 the same is shown for the optimal number of iterations per stage. This is accomplished by locating with an exponential search the optimal setup of stages/iterations per stage. Hereafter the most evident properties emerging from the plots:

- The optimal number of stages is increasing with the singularity of the instance. In the extremal case of a differentiable Lagrangian Dual, a single stage is almost enough, because the gradient norm is decreasing to 0.
- The greater is Q with respect to the other parameters (the farther the optimal point of the unconstrained primal problem is from the feasible space) the higher is the number of stages; instead the number of iterations per stage does not appear to be sensible to such parameter.

For a better analysis of the summentioned property a dedicated instance generator is needed.
In Table 1 we compare the performance of RestartedNesterovMomentum versus the one of RestartedSubgradient, on the test-set `netgen1000`. The performance is measured in terms of the number of iterations required to attain a *relative primal-dual gap*, because this is the real precision measure that would be available in practice if we were to exploit the methods we are proposing to solve instances where other solvers fail.

Table 1. Restarted subgradient on Netgen test-set: iterations to $\epsilon_{rel}^{gap} < 10^{-6}$

Instance	RSG	RNM
netgen-1000-1-*-a-a-ns-0000	94968±54653	2371±161
netgen-1000-1-*-a-a-ns-0330	737469±36873	15883±2558
netgen-1000-1-*-a-a-ns-0660	888294±44415	148834±96942
netgen-1000-1-*-a-a-ns-1000	46820±40570	9370±7923
netgen-1000-1-*-a-b-ns-0000	691±566	361±222
netgen-1000-1-*-a-b-ns-0330	517818±378464	22395±12257
netgen-1000-1-*-a-b-ns-0660	1146590±229685	291165±118518
netgen-1000-1-*-a-b-ns-1000	61930±59037	5901±4375
netgen-1000-1-*-b-a-ns-0000	35039±26218	1739±293

[8] The three pass random generator available at the MCF web page

netgen-1000-1-*-b-a-ns-0330	340674±243933	16518±10369
netgen-1000-1-*-b-a-ns-0660	622770±256768	114400±79570
netgen-1000-1-*-b-a-ns-1000	30042±24654	3847±2308
netgen-1000-1-*-b-b-ns-0000	186±65	160±25
netgen-1000-1-*-b-b-ns-0330	144560±29086	34417±27232
netgen-1000-1-*-b-b-ns-0660	531289±351062	83620±58017
netgen-1000-1-*-b-b-ns-1000	40898±35502	5169±3752
netgen-1000-2-*-a-a-ns-0000	142372±28263	2148±124
netgen-1000-2-*-a-a-ns-0330	656398±65714	12754±2552
netgen-1000-2-*-a-a-ns-0660	691425±36036	65314±58140
netgen-1000-2-*-a-a-ns-1000	29730±24066	2970±1483
netgen-1000-2-*-a-b-ns-0000	1679±1065	407±133
netgen-1000-2-*-a-b-ns-0330	155124±101844	13052±5401
netgen-1000-2-*-a-b-ns-0660	215410±96123	27387±17658
netgen-1000-2-*-a-b-ns-1000	17612±16225	5573±4174
netgen-1000-2-*-b-a-ns-0000	13925±7237	1694±339
netgen-1000-2-*-b-a-ns-0330	163872±49147	8847±2190
netgen-1000-2-*-b-a-ns-0660	454868±241871	11281±4102
netgen-1000-2-*-b-a-ns-1000	9888±1903	2290±786
netgen-1000-2-*-b-b-ns-0000	142±95	153±20
netgen-1000-2-*-b-b-ns-0330	277066±129465	13838±3594
netgen-1000-2-*-b-b-ns-0660	395451±223366	110432±94674
netgen-1000-2-*-b-b-ns-1000	24463±18565	3345±1808
netgen-1000-3-*-a-a-ns-0000	97873±24794	2493±447
netgen-1000-3-*-a-a-ns-0330	589860±29493	11278±2061
netgen-1000-3-*-a-a-ns-0660	786634±65731	18240±4916
netgen-1000-3-*-a-a-ns-1000	27341±19887	7725±6252
netgen-1000-3-*-a-b-ns-0000	1421±1282	254±89
netgen-1000-3-*-a-b-ns-0330	80969±39950	4366±770
netgen-1000-3-*-a-b-ns-0660	208962±159807	17215±6392
netgen-1000-3-*-a-b-ns-1000	26595±20625	4115±2286
netgen-1000-3-*-b-a-ns-0000	22159±18457	1488±182
netgen-1000-3-*-b-a-ns-0330	116120±69950	9574±6240
netgen-1000-3-*-b-a-ns-0660	246111±164086	14245±8469
netgen-1000-3-*-b-a-ns-1000	17092±10995	3001±1591
netgen-1000-3-*-b-b-ns-0000	168±123	151±44
netgen-1000-3-*-b-b-ns-0330	70481±33609	19925±15124
netgen-1000-3-*-b-b-ns-0660	222825±153010	38057±28566
netgen-1000-3-*-b-b-ns-1000	66447±60029	17621±15941

TABLE 1

Test-set generated with Pargen+Netgen+Qfcgen, 5 instances per setup. Iterations to $\epsilon_{rel}^{gap} < 10^{-6}$.

Comparison with an external solver: Gurobi ▷ We tested the performance of the Julia implementation of RestartedNesterovMomentum (RNM) against the commercial solver Gurobi, *specialised for quadratic programs.* Experiments were performed with Julia 1.4.1 running on a Lenovo T430 / 2351AA6 with i5-3320M CPU @ 2.60GHz and 4GiB ram. Setting the desired relative precision of RNM to $\epsilon_{rel} < 10^{-8}$, on the test-set of Table 1, RNM resulted 10 to 50000 times slower than Gurobi, while for $\epsilon_{rel} < 10^{-5}$, Table 2, it was 7 to 500 times slower on average. Note that the number of stages was fixed to 30, so we could expect that the actual time cost, for $\epsilon_{rel} < 10^{-5}$, could be around half the one we report.
It would be interesting to analyse how RNM scales with respect to Gurobi; a glimpse of the scaling behaviour can be deduced comparing Table 2 to Table 3.

Table 2. Restarted subgradient on Netgen test-set, performance: RNM to $\epsilon_{rel} < 10^{-5}$ Vs Gurobi

Instance	RNM (ms)	Gurobi (ms)
netgen-1000-1-*-a-a-ns-0000	3546^{+1230}_{-1362}	31^{+2}_{-2}
netgen-1000-1-*-a-a-ns-0330	3747^{+789}_{-1795}	89^{+161}_{-59}
netgen-1000-1-*-a-a-ns-0660	5381^{+3053}_{-1291}	35^{+6}_{-5}
netgen-1000-1-*-a-a-ns-1000	914^{+1091}_{-631}	30^{+9}_{-8}
netgen-1000-1-*-a-b-ns-0000	302^{+174}_{-62}	35^{+4}_{-4}
netgen-1000-1-*-a-b-ns-0330	1253^{+1116}_{-742}	37^{+5}_{-3}
netgen-1000-1-*-a-b-ns-0660	6693^{+3316}_{-2919}	33^{+3}_{-2}
netgen-1000-1-*-a-b-ns-1000	1539^{+1752}_{-1290}	32^{+31}_{-9}
netgen-1000-1-*-b-a-ns-0000	1741^{+526}_{-756}	40^{+9}_{-7}
netgen-1000-1-*-b-a-ns-0330	1641^{+704}_{-705}	37^{+3}_{-4}
netgen-1000-1-*-b-a-ns-0660	14099^{+23643}_{-12234}	36^{+9}_{-6}
netgen-1000-1-*-b-a-ns-1000	997^{+1617}_{-755}	26^{+1}_{-1}
netgen-1000-1-*-b-b-ns-0000	284^{+98}_{-48}	33^{+9}_{-5}
netgen-1000-1-*-b-b-ns-0330	2049^{+1494}_{-1077}	43^{+36}_{-15}
netgen-1000-1-*-b-b-ns-0660	2476^{+1878}_{-1479}	35^{+6}_{-4}
netgen-1000-1-*-b-b-ns-1000	911^{+1546}_{-657}	26^{+8}_{-4}
netgen-1000-2-*-a-a-ns-0000	4073^{+3385}_{-2158}	43^{+32}_{-15}
netgen-1000-2-*-a-a-ns-0330	3677^{+2925}_{-1849}	34^{+5}_{-4}
netgen-1000-2-*-a-a-ns-0660	6224^{+4085}_{-4261}	34^{+3}_{-2}

netgen-1000-2-*-a-a-ns-1000	570^{+488}_{-277}	26^{+5}_{-2}
netgen-1000-2-*-a-b-ns-0000	316^{+63}_{-46}	32^{+6}_{-6}
netgen-1000-2-*-a-b-ns-0330	900^{+491}_{-369}	33^{+4}_{-4}
netgen-1000-2-*-a-b-ns-0660	3628^{+5544}_{-2557}	58^{+51}_{-28}
netgen-1000-2-*-a-b-ns-1000	929^{+1978}_{-689}	30^{+11}_{-5}
netgen-1000-2-*-b-a-ns-0000	1889^{+2438}_{-936}	33^{+5}_{-3}
netgen-1000-2-*-b-a-ns-0330	1580^{+481}_{-537}	35^{+4}_{-4}
netgen-1000-2-*-b-a-ns-0660	2218^{+1496}_{-1159}	45^{+29}_{-17}
netgen-1000-2-*-b-a-ns-1000	511^{+423}_{-251}	28^{+4}_{-3}
netgen-1000-2-*-b-b-ns-0000	353^{+112}_{-85}	42^{+31}_{-15}
netgen-1000-2-*-b-b-ns-0330	1461^{+686}_{-789}	33^{+3}_{-5}
netgen-1000-2-*-b-b-ns-0660	6345^{+10950}_{-5133}	43^{+28}_{-12}
netgen-1000-2-*-b-b-ns-1000	1160^{+1441}_{-834}	25^{+3}_{-2}
netgen-1000-3-*-a-a-ns-0000	3978^{+3715}_{-1953}	30^{+2}_{-2}
netgen-1000-3-*-a-a-ns-0330	4196^{+2499}_{-2361}	34^{+2}_{-3}
netgen-1000-3-*-a-a-ns-0660	4698^{+536}_{-1015}	37^{+2}_{-2}
netgen-1000-3-*-a-a-ns-1000	1348^{+2757}_{-1053}	30^{+9}_{-6}
netgen-1000-3-*-a-b-ns-0000	289^{+43}_{-52}	35^{+5}_{-3}
netgen-1000-3-*-a-b-ns-0330	572^{+160}_{-105}	31^{+4}_{-2}
netgen-1000-3-*-a-b-ns-0660	2129^{+828}_{-1184}	35^{+4}_{-2}
netgen-1000-3-*-a-b-ns-1000	924^{+998}_{-604}	38^{+26}_{-15}
netgen-1000-3-*-b-a-ns-0000	1524^{+713}_{-573}	31^{+5}_{-3}
netgen-1000-3-*-b-a-ns-0330	1216^{+872}_{-297}	36^{+5}_{-6}
netgen-1000-3-*-b-a-ns-0660	2509^{+6158}_{-1955}	35^{+3}_{-5}
netgen-1000-3-*-b-a-ns-1000	650^{+413}_{-162}	38^{+46}_{-15}
netgen-1000-3-*-b-b-ns-0000	448^{+158}_{-148}	32^{+7}_{-4}
netgen-1000-3-*-b-b-ns-0330	2342^{+7034}_{-2094}	37^{+4}_{-10}
netgen-1000-3-*-b-b-ns-0660	2872^{+5494}_{-2372}	38^{+13}_{-10}
netgen-1000-3-*-b-b-ns-1000	2704^{+6210}_{-2429}	36^{+31}_{-12}

Table 2

Performance of RNM, as implemented in the package, run up to a relative error $\epsilon_{rel} < 10^{-5}$, against Gurobi quadratic solver.

Table 3. Restarted subgradient on a Netgen instance, performance: RNM to $\epsilon_{rel} < 10^{-5}$ Vs Gurobi

Instance	RNM (ms)	Gurobi (ms)
netgen-50000-1-1-a-a-ns-00000	194116	2826
netgen-50000-1-1-a-a-ns-16500	2776811	2867

netgen-50000-1-1-a-a-ns-33000	2843126	4901
netgen-50000-1-1-a-a-ns-50000	2799640	4609

TABLE 3

Performance of RNM to $\epsilon_{rel} < 10^{-5}$ against Gurobi quadratic solver

To further analyse the scaling behaviour of RestartedNesterovMomentum, we have realized a simple implementation of the algorithm exploiting the parallel directives of C++ STL, run on a workstation assembled with used components from industry, Lenovo P700 / 1030 with E5-2580v3 CPU @ 2.50GHz and 128GiB ram, and built with GCC 9.3.0 on Ubuntu 20.04. We generated another test-set with Netgen+Pargen+Qfcgen composed of instances with number of arcs in $2^{10:17}$; for each number of arcs there are twelve instances described by the combinations of singularity percentage ($\left[0, \frac{1}{3}, \frac{2}{3}, 1\right]$) and sparsity of the graph (`[a, b, c]`). Results of the experiment are shown in Figure 14; the overall, not surprising, result is that for general instances the method can be considered a valid alternative only up to the first couple of digits, while for mildly singular, big (arcs $> 10^5$) instances it scales better than Gurobi and could be the best option among the two. Since the memory footprint of RNM method is minimal, it could be that for even bigger instances RNM could be the clear winner over Gurobi. We leave this as homework to the interested reader.

Heuristics ▷ The theoretical arguments of §2, about the quality of the many different possible projections on the feasible space $S_{(P)}$, have been fully confirmed experimentally. In Figure 8 is easy to check that:

- min-cost max-flow heuristics produce results of higher quality than max-flow ones
- the nearer the starting point to the optimal set, represented in the plot with a higher number of iterations per stage, the more the pre-projection with min-norm subgradient, as per equation 2, is relevant.

We did not analyse the performance in terms of computational efficacy, since all the algorithms in question are well studied and we chose to implement the simplest algorithms (Edmond-Karp for max-flow, Edmond-Karp with shortest path for the min-cost max-flow), which are not the most performant. It could be interesting to evaluate the computational efficiency of a cost/capacity-scaling min-cost max-flow heuristic also as a stopping criterion, e.g. at each restart of the algorithm. For a complete experimental evaluation of the computational efficiency, a performance oriented implementation of the algorithm of choice is necessary.

Epilogue

We analysed and specialized dual subgradient methods to convex quadratic separable min-cost flow boxed problems. The overall result is that, because of the geometry of the Lagrangian dual, restarted methods are effective in producing high quality solutions, as suggested by parameter tuning investigations. Experimentally we recorded the best performance with the RestartedNesterovMomentum with high momentum, for which a proof of convergence is missing.

From each point in the dual space we can calculate an approximate, almost feasible, point in the primal space via the projections on the feasible space $S_{(P)}$; an exact bounding interval for the optimal objective value is finally calculated through the error analysis illustrated in §2.

Of great help to the fast-paced investigation was the flexibility of both the Julia package we developed and the Julia language itself.

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Figures

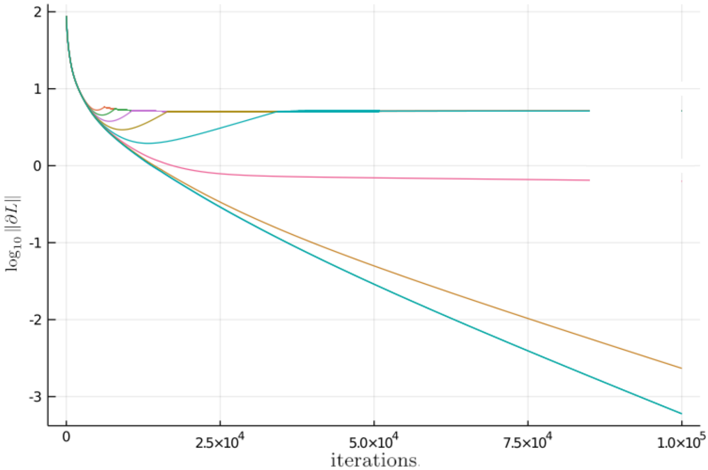


Figure 1. Instance: `singular`=0. NesterovMomentum iterations. Since L is differentiable, $\|\partial L\|$ is a plausible measure of the attained accuracy. The optimal setup appears to be a stationary point in the parameter space (the lowest line in the graph). Here the parameters of Nesterov Momentum iteration are encompassing a relative variation of only 10^{-8}, but the algorithm dynamic is completely changing.

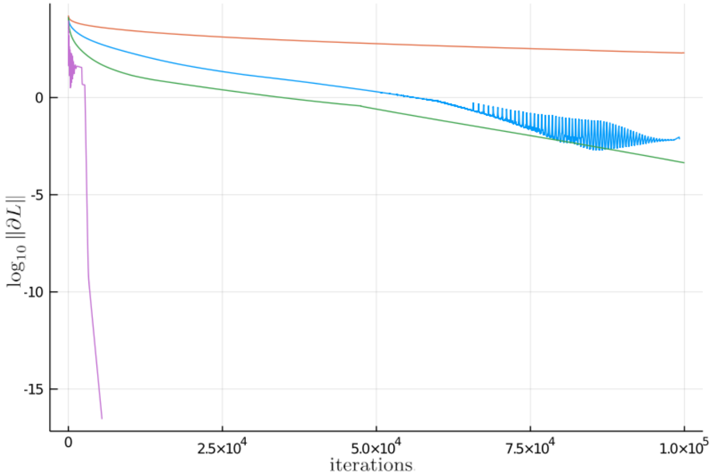


Figure 2. Instance: `singular`=0. RMSProp iterations. Since L is differentiable, $\|\partial L\|$ is a plausible measure of the attained accuracy. Here we drove the parameters a bit further the stable setup.

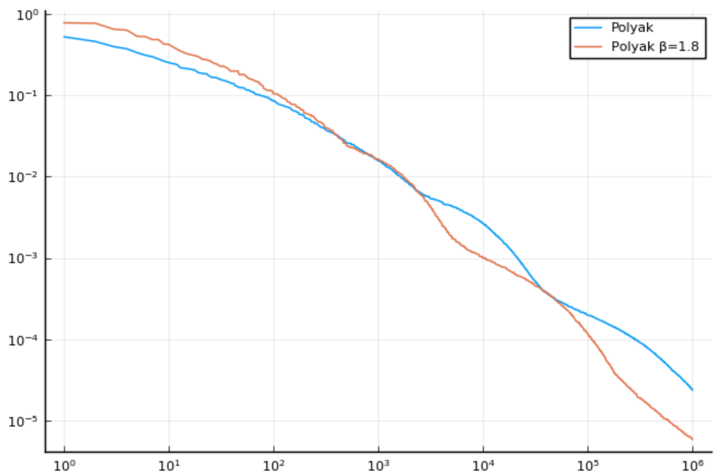


Figure 3. Polyak step size, relative error. With restarted methods, the same instance was solved in one-tenth of the iterations to one-thousandth of the relative error.

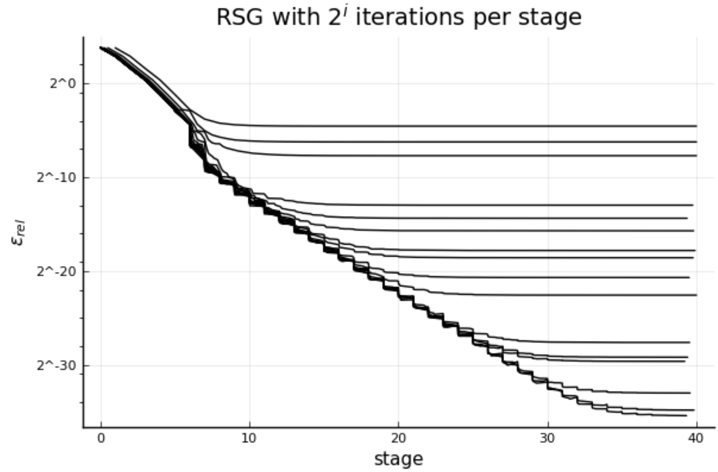


Figure 4. Instance: `singular`=0.55, `active`=0.87. Each line represents the relative error for the RestartedSubgradient with fixed number of iterations per stage, $t = 2^i$, $i \in \{0, 1, \cdots, 15\}$, over the 40 stages. This support the idea that t should increase as ϵ_{rel} decreases and that the number of (useful) stages is approximately $\log_r\left(\frac{\epsilon_0}{\epsilon}\right)$.

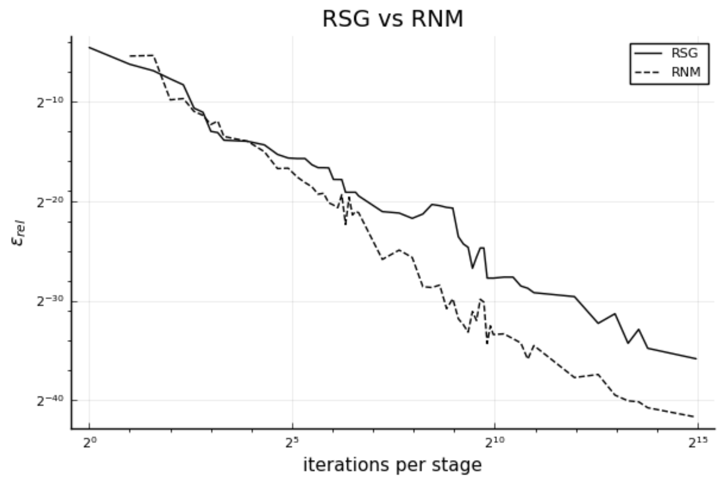


Figure 5. Instance: `singular`=0.55, `active`=0.87. Comparing RestartedSubgradient with RestartedNesterovMomentum. Note that for the RestartedSubgradient it approximately holds $t \propto \epsilon^{-\frac{1}{2}}$, where t is the number of iterations per stage. This is coherent with the argument on the local growth of L following Corollary 0.5.

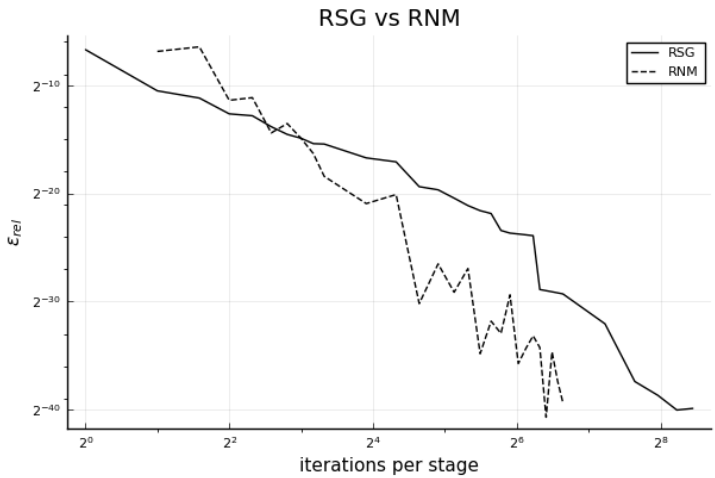


Figure 6. Instance: `singular`=0, `active`=0.87. Comparing RestartedSubgradient with RestartedNesterovMomentum. Note that for the RestartedSubgradient it approximately holds $t \propto \epsilon^{-\frac{1}{4}}$, where t is the number of iterations per stage. For a locally quadratic function we would expect a worst case $t \propto \frac{1}{\epsilon}$, as of Corollary 0.5.

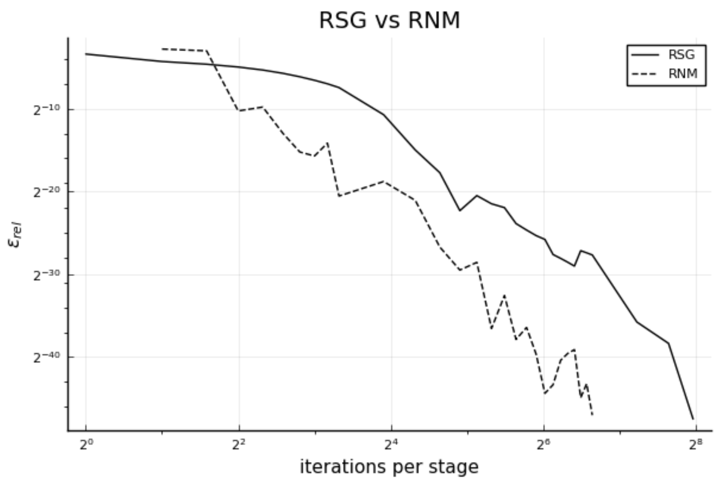


Figure 7. Instance: `singular`=1.0, `active`=0.87. Comparing RestartedSubgradient with RestartedNesterovMomentum. Note that the problem is linear; for the RestartedSubgradient there isn't really a dependence of the form $t \propto \epsilon^{-\vartheta}$. In fact, for a locally linear function, we would expect linear convergence, i.e. independence of t from the required ϵ, as of Corollary 0.5.

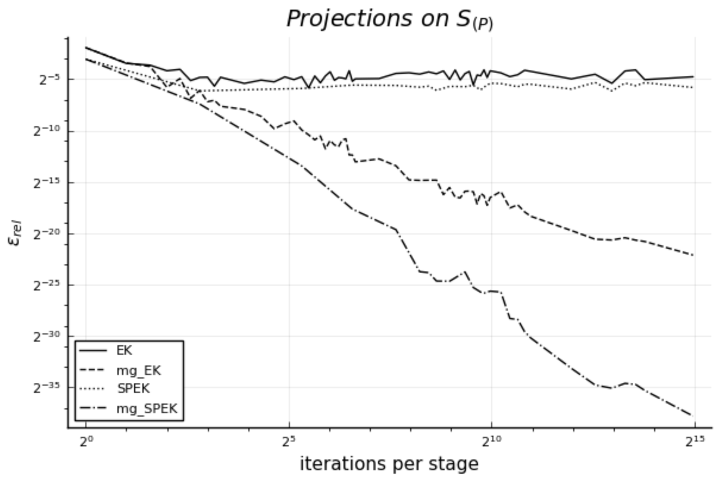


Figure 8. Instance: `singular`=0.5, `active`=0.87. Comparing efficacy of projections on the feasible space $S_{(P)}$. The argument in §2 is confirmed by the fact that, the more the precision of the result, the more the min-grad pre-projection is effective. `EK`: EdmondKarp (max-flow), `SPEK`: ShortestPath EdmondKarp (min-cost max-flow), `mg`: pre-projection with min-norm approximate subgradient.

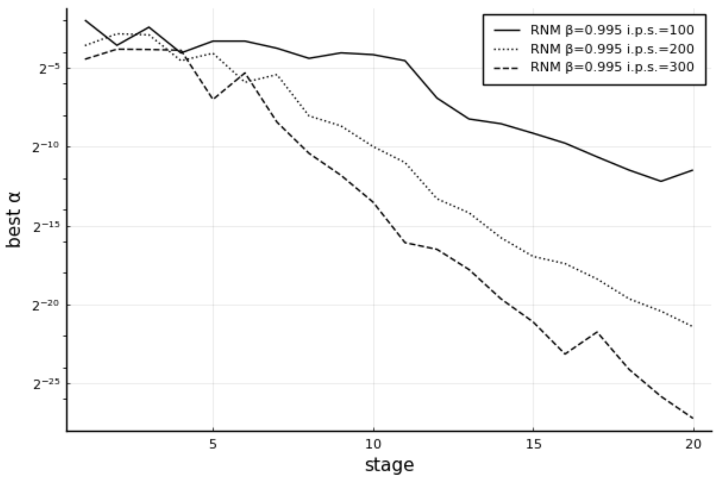


Figure 9. Instance: `netgen-1000-1-1-b-a-ns-0660`, `i.p.s`=iterations per stage. Automatic parameter tuner with restart on Nesterov Momentum with fixed β. The exponential decay of α from a stage to the next one is evident from this chart.

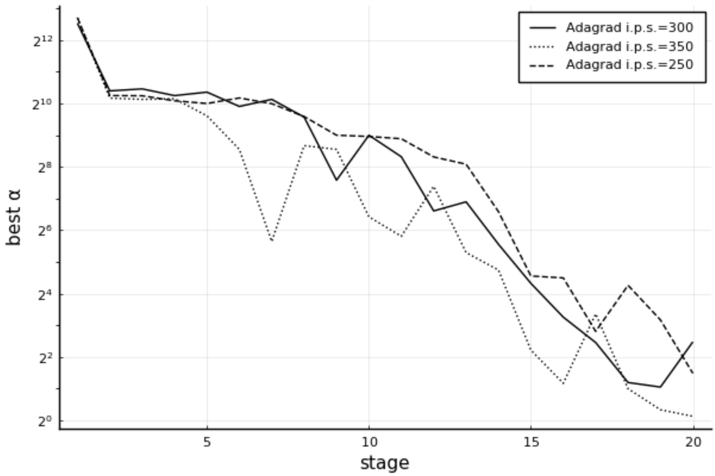


Figure 10. Instance: `netgen-1000-1-1-b-a-ns-0660`, `i.p.s`=iterations per stage. Automatic parameter tuner with restart on Adagrad. Also for this subgradient method the exponential decay of the step size parameter α, from a stage to the next one is evident from this chart. Note that the dampening of Adagrad leads to $\alpha \notin [0, 1]$ for all the best configurations.

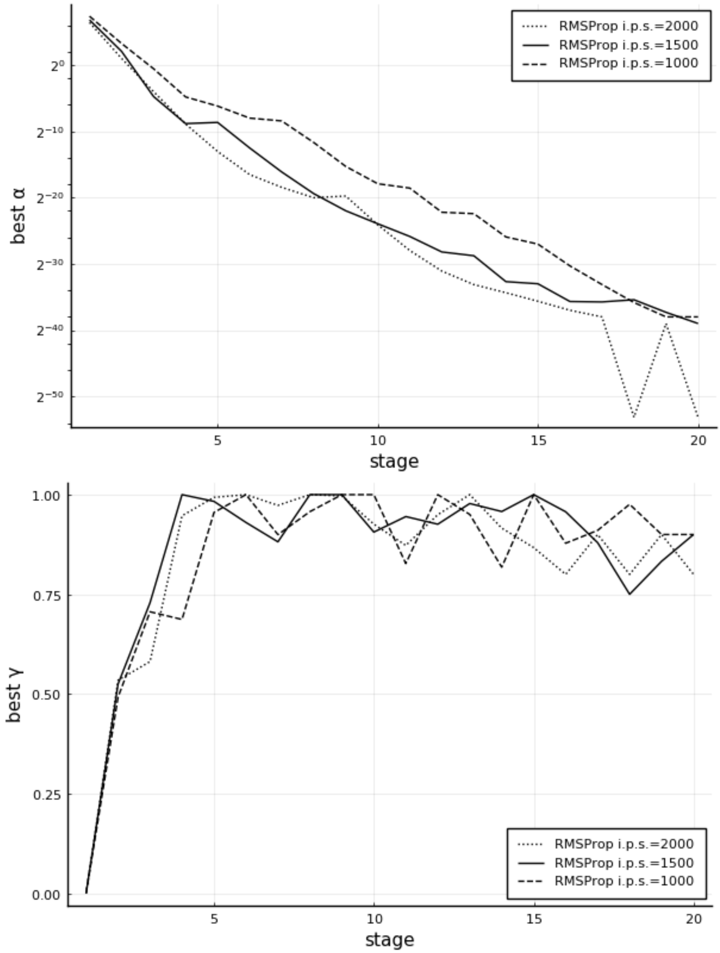


Figure 11. Instance: `netgen-1000-1-1-b-a-ns-0660`, `i.p.s`=iterations per stage. Automatic parameter tuner with restart on RMSProp. Typical behaviour: the best α is exponentially decaying and after the first steps the best γ is greater than 0.75.

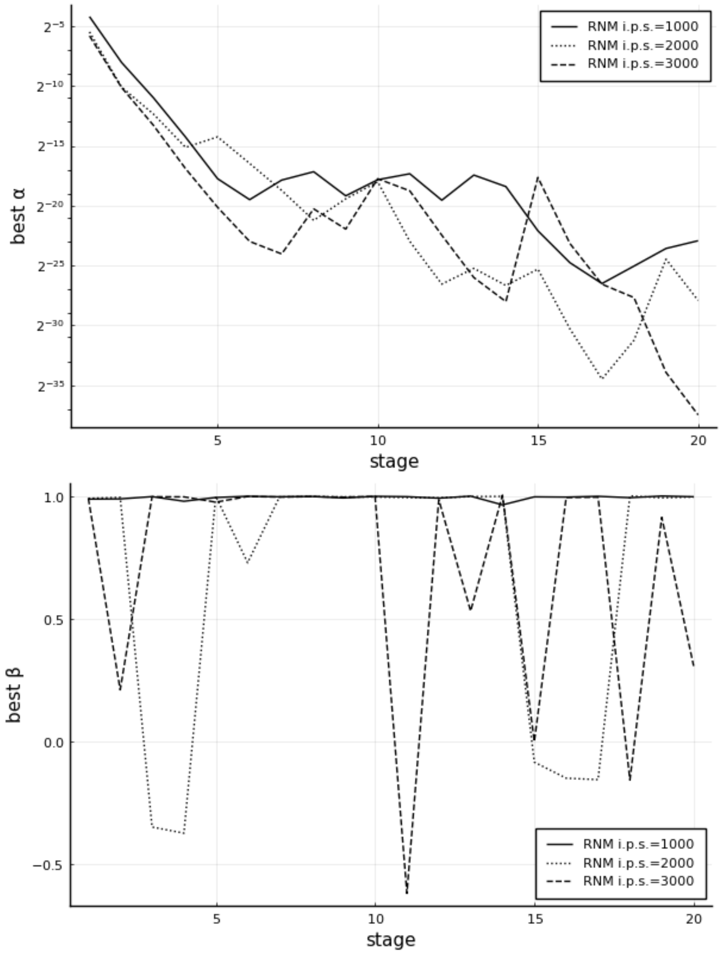


Figure 12. Instance: `netgen-1000-1-1-b-a-ns-0660`, `i.p.s`=iterations per stage. Automatic parameter tuner with restart on Nesterov Momentum. Typical behaviour quite confirmed by the plot: the best α is exponentially decaying and often $\beta > 0.95$ is a good choice.

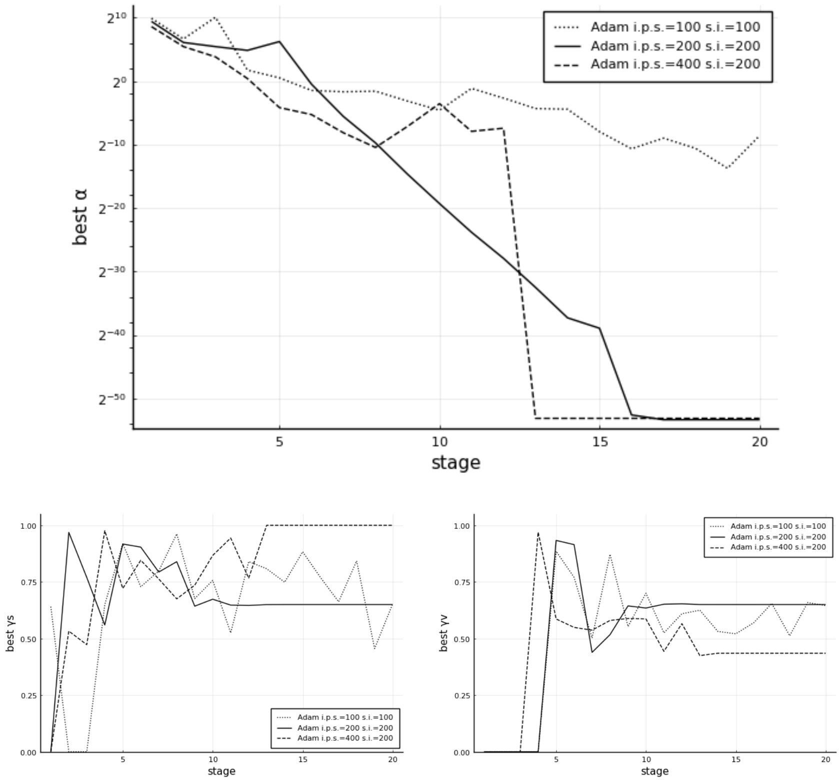


Figure 13. Instance: `netgen-1000-1-1-b-a-ns-0660`, `i.p.s`=iterations per stage. Automatic parameter tuner with restart on Adam. The best α is decaying.

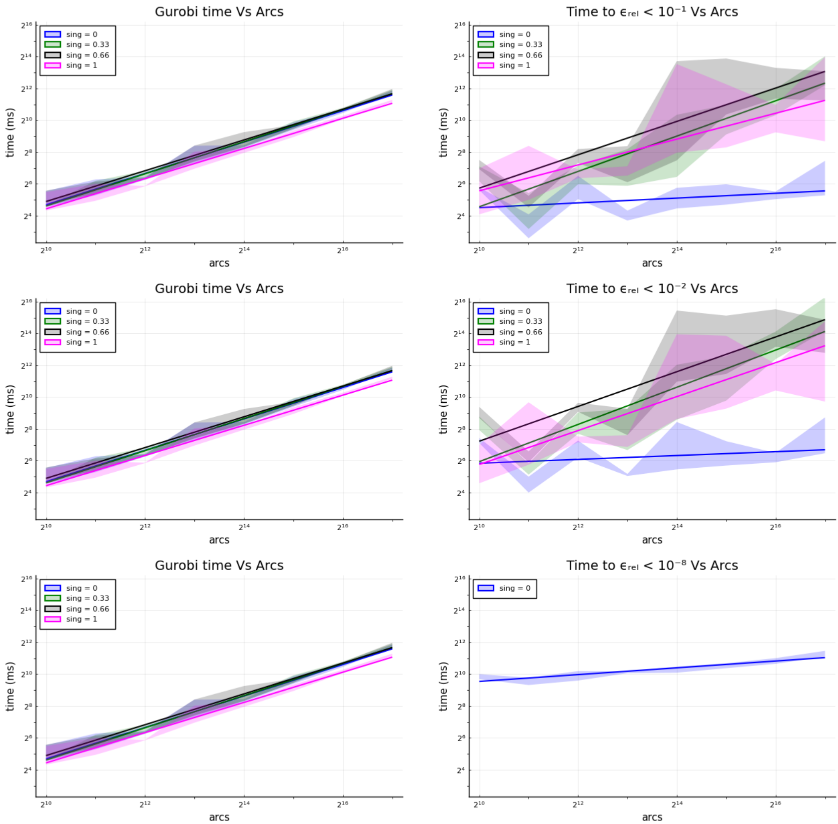
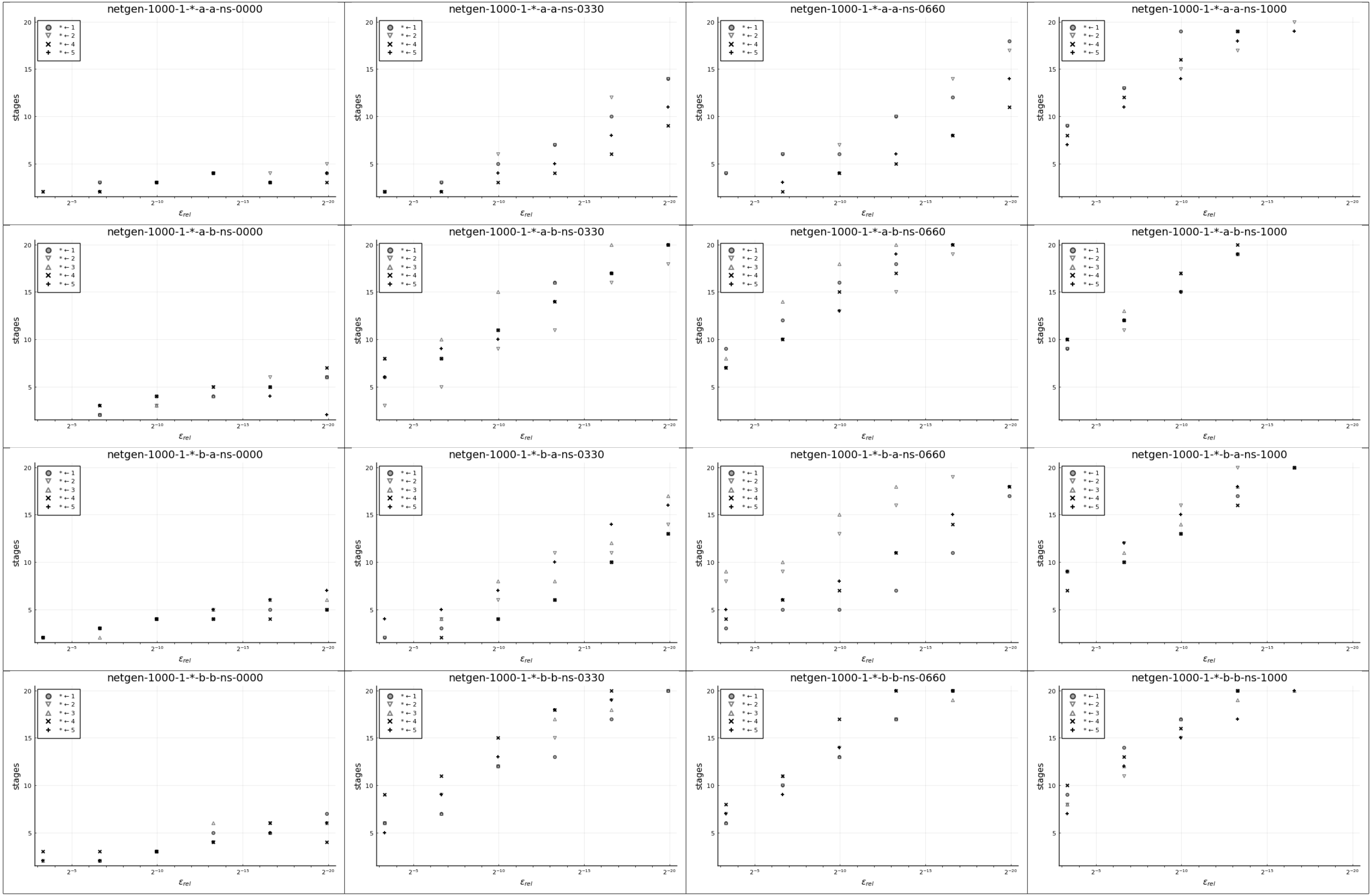
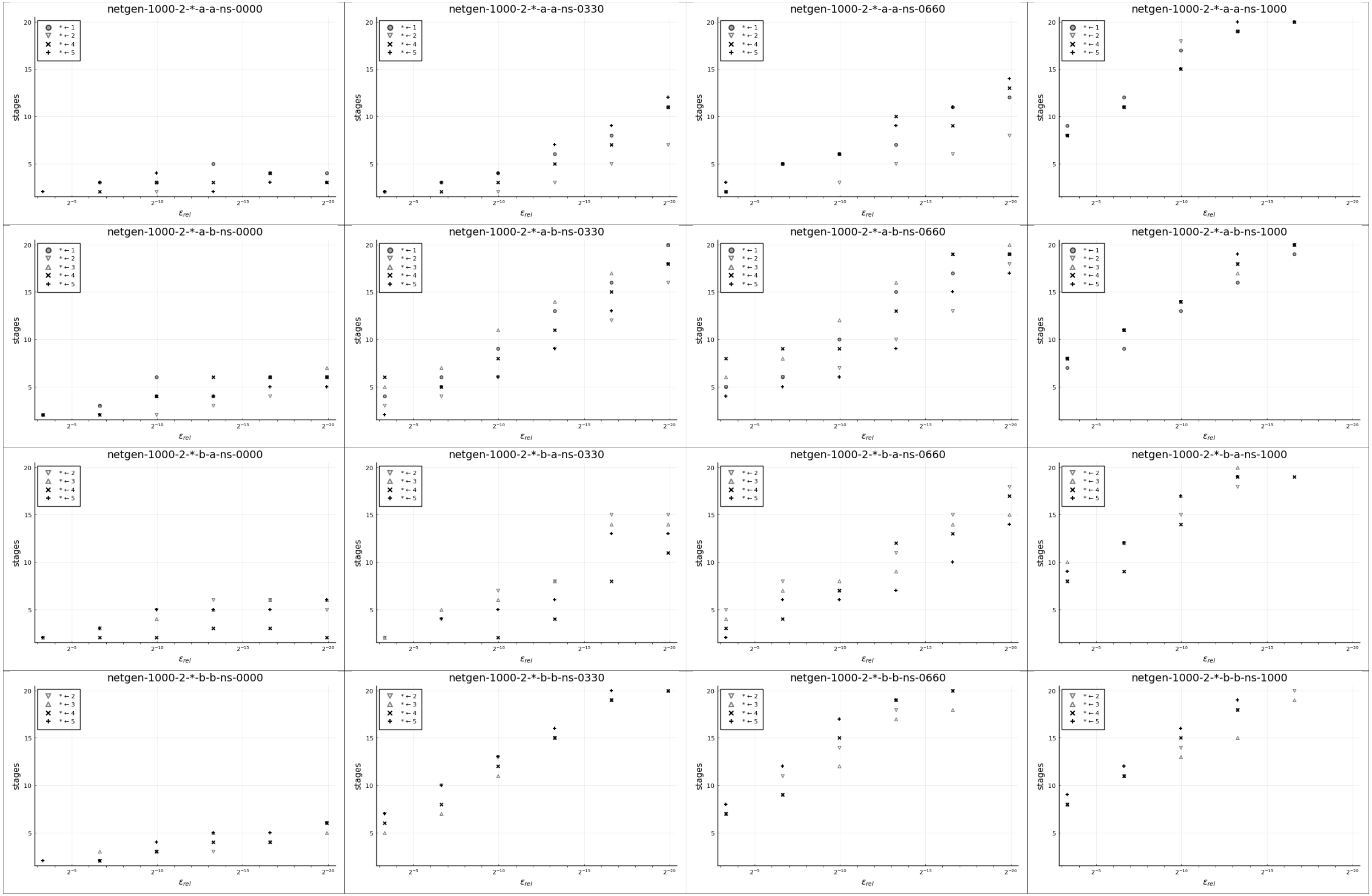


Figure 14. Scaling of RNM vs Gurobi. For singular instances, Gurobi is comparable to RNM run to precision $\epsilon_{rel} < 10^{-2}$. For nonsingular instances RNM scales way better than Gurobi. RNM has also a minimal memory footprint, so we propose its usage for mildly singular instances with more than 10^5 arcs.

Table 5. Restarted Nesterov Momentum : number of stages against ϵ_{rel}





netgen-1000-2-*-a-a-ns-0000
netgen-1000-2-*-a-a-ns-0330
netgen-1000-2-*-a-a-ns-0660
netgen-1000-2-*-a-a-ns-1000
netgen-1000-2-*-a-b-ns-0000
netgen-1000-2-*-a-b-ns-0330
netgen-1000-2-*-a-b-ns-0660
netgen-1000-2-*-a-b-ns-1000
netgen-1000-2-*-b-a-ns-0000
netgen-1000-2-*-b-a-ns-0330
netgen-1000-2-*-b-a-ns-0660
netgen-1000-2-*-b-a-ns-1000
netgen-1000-2-*-b-b-ns-0000
netgen-1000-2-*-b-b-ns-0330
netgen-1000-2-*-b-b-ns-0660
netgen-1000-2-*-b-b-ns-1000
stages
ε_{rel}
$* \leftarrow 1$
$* \leftarrow 2$
$* \leftarrow 3$
$* \leftarrow 4$
$* \leftarrow 5$

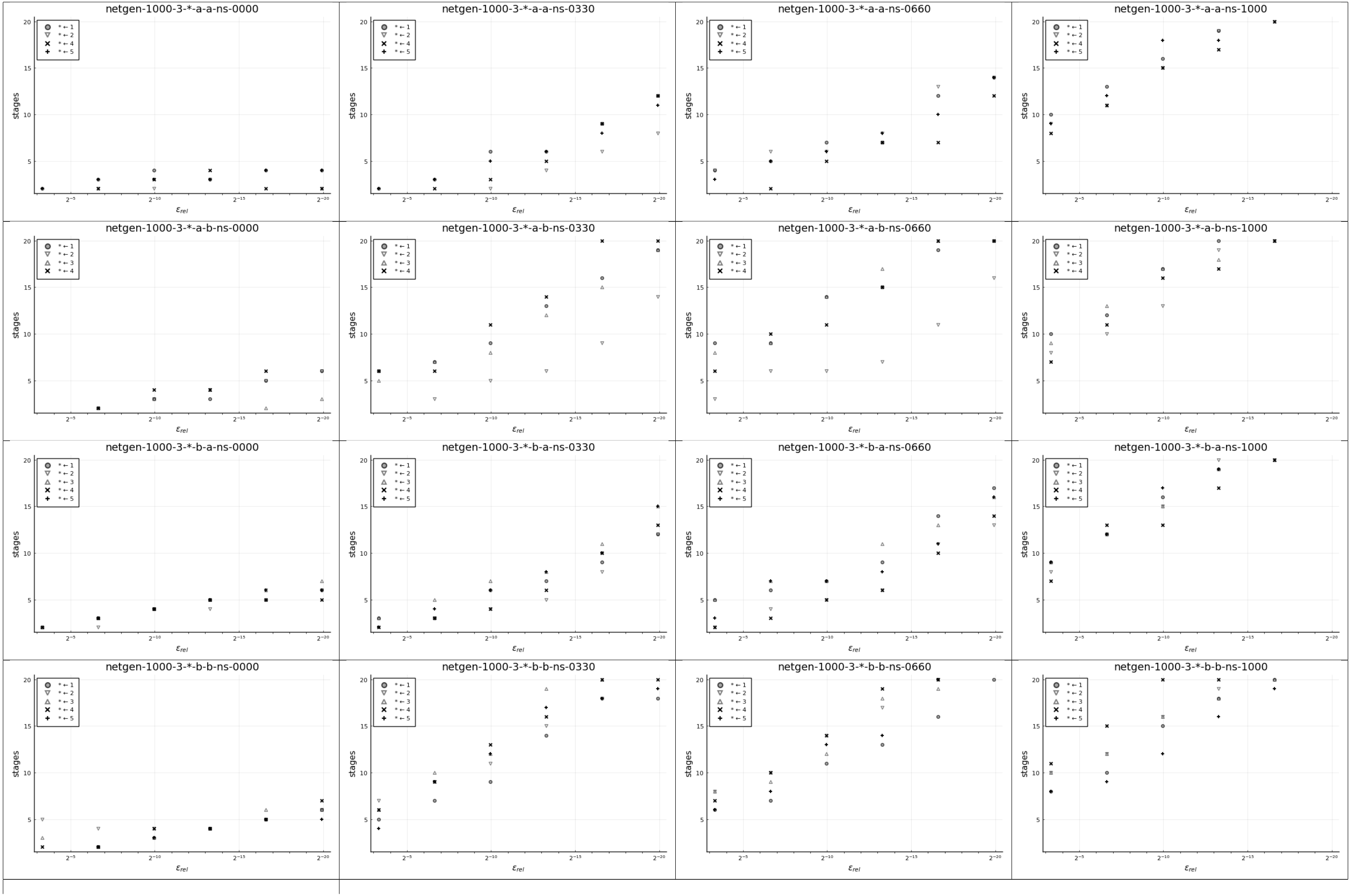
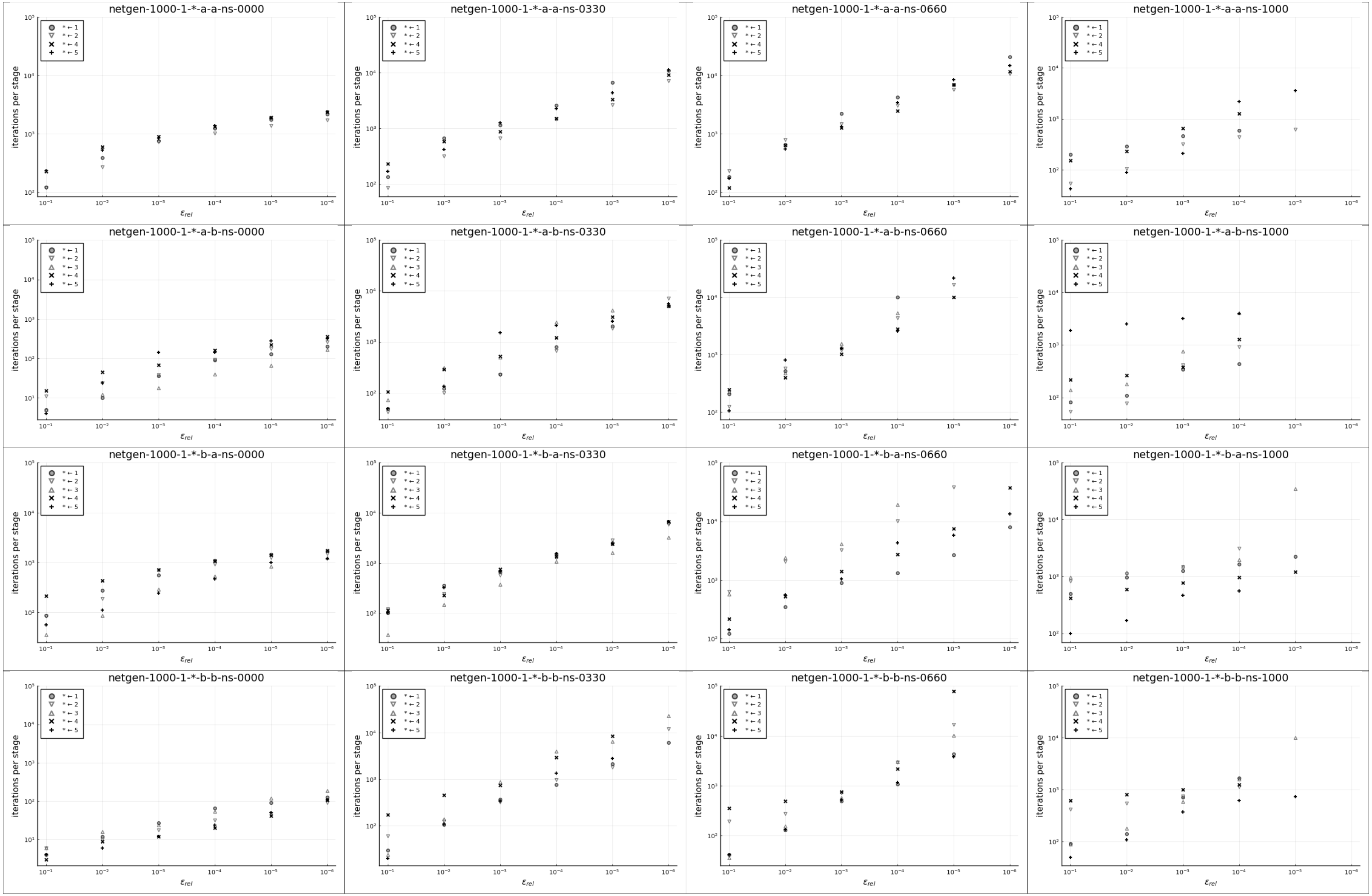
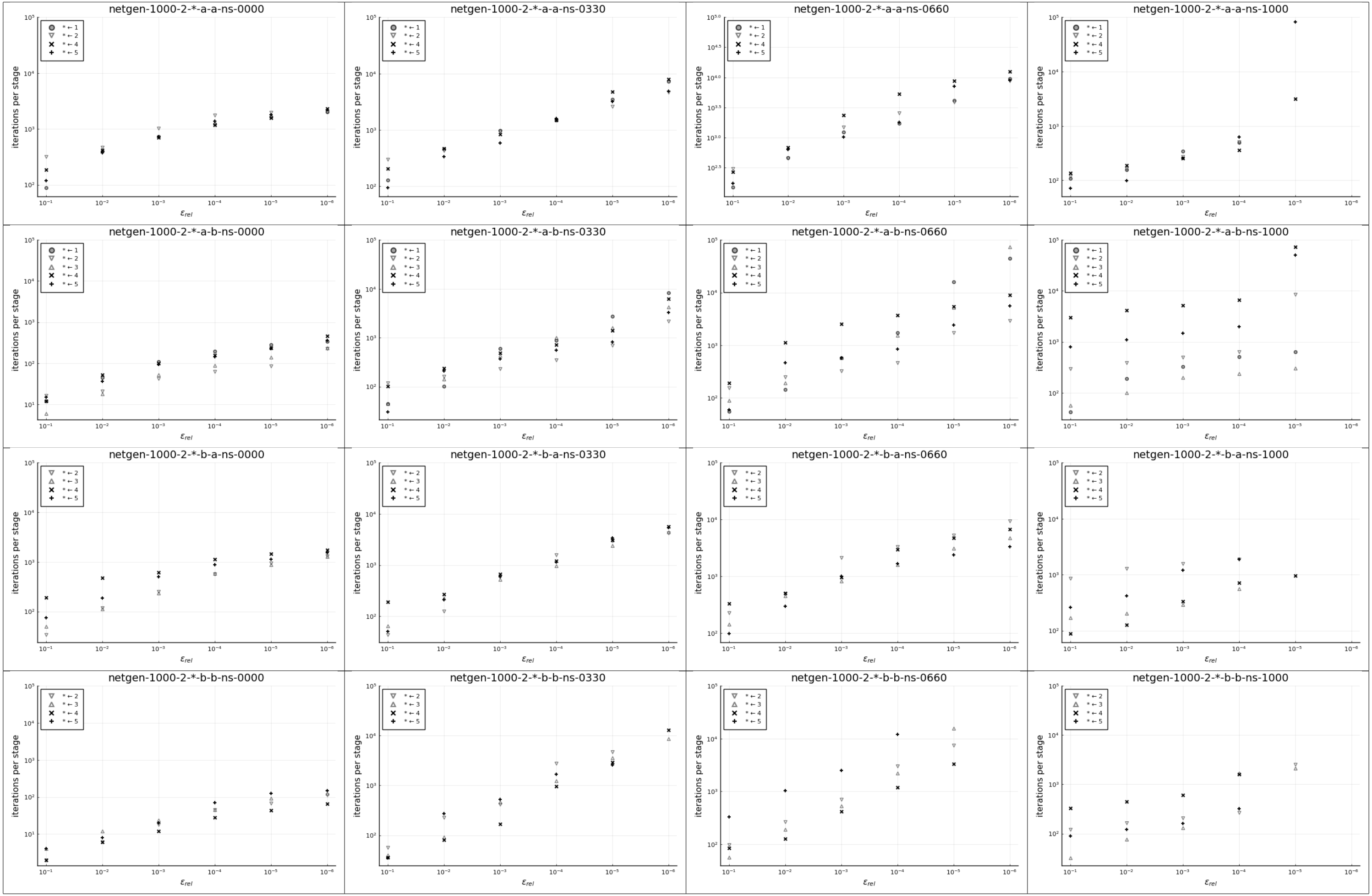
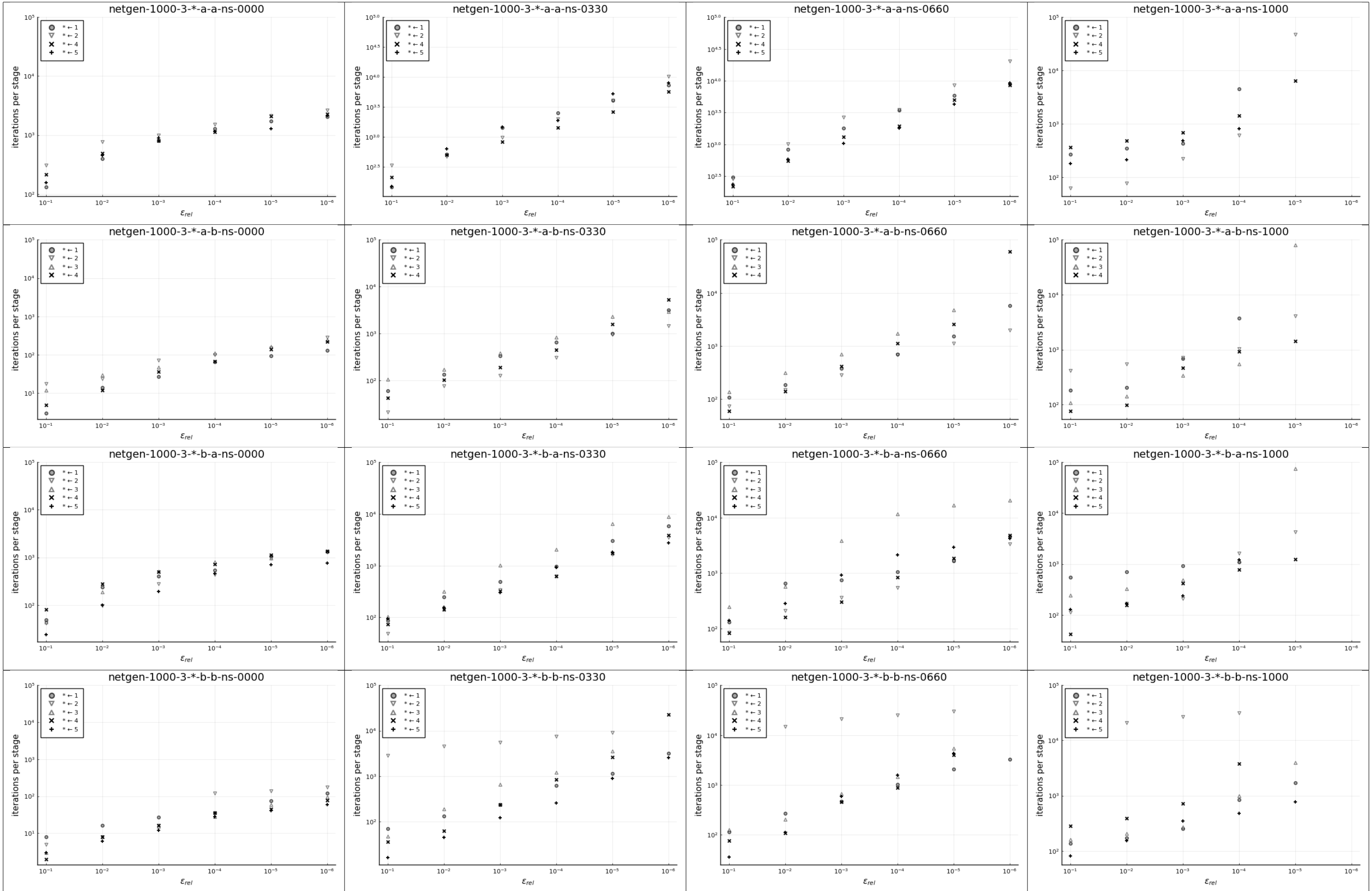

netgen-1000-3-*-a-a-ns-0000
netgen-1000-3-*-a-a-ns-0330
netgen-1000-3-*-a-a-ns-0660
netgen-1000-3-*-a-a-ns-1000
netgen-1000-3-*-a-b-ns-0000
netgen-1000-3-*-a-b-ns-0330
netgen-1000-3-*-a-b-ns-0660
netgen-1000-3-*-a-b-ns-1000
netgen-1000-3-*-b-a-ns-0000
netgen-1000-3-*-b-a-ns-0330
netgen-1000-3-*-b-a-ns-0660
netgen-1000-3-*-b-a-ns-1000
netgen-1000-3-*-b-b-ns-0000
netgen-1000-3-*-b-b-ns-0330
netgen-1000-3-*-b-b-ns-0660
netgen-1000-3-*-b-b-ns-1000
stages
ϵ_{rel}

Table 6. Restarted Nesterov Momentum : number of iterations per stage against ϵ_{rel}




netgen-1000-2-*-a-a-ns-0000
netgen-1000-2-*-a-a-ns-0330
netgen-1000-2-*-a-a-ns-0660
netgen-1000-2-*-a-a-ns-1000
netgen-1000-2-*-a-b-ns-0000
netgen-1000-2-*-a-b-ns-0330
netgen-1000-2-*-a-b-ns-0660
netgen-1000-2-*-a-b-ns-1000
netgen-1000-2-*-b-a-ns-0000
netgen-1000-2-*-b-a-ns-0330
netgen-1000-2-*-b-a-ns-0660
netgen-1000-2-*-b-a-ns-1000
netgen-1000-2-*-b-b-ns-0000
netgen-1000-2-*-b-b-ns-0330
netgen-1000-2-*-b-b-ns-0660
netgen-1000-2-*-b-b-ns-1000
iterations per stage
ε_{rel}


netgen-1000-3-*-a-a-ns-0000
netgen-1000-3-*-a-a-ns-0330
netgen-1000-3-*-a-a-ns-0660
netgen-1000-3-*-a-a-ns-1000
netgen-1000-3-*-a-b-ns-0000
netgen-1000-3-*-a-b-ns-0330
netgen-1000-3-*-a-b-ns-0660
netgen-1000-3-*-a-b-ns-1000
netgen-1000-3-*-b-a-ns-0000
netgen-1000-3-*-b-a-ns-0330
netgen-1000-3-*-b-a-ns-0660
netgen-1000-3-*-b-a-ns-1000
netgen-1000-3-*-b-b-ns-0000
netgen-1000-3-*-b-b-ns-0330
netgen-1000-3-*-b-b-ns-0660
netgen-1000-3-*-b-b-ns-1000
iterations per stage
ε_{rel}
* ← 1
* ← 2
* ← 3
* ← 4
* ← 5

Instance	RSG						RNM					
	mg EK		mg SPEK		Max iter	Iter	mg EK		mg SPEK		Max iter	Iter
	ϵ_{rel}	$\Delta\epsilon_{rel}$	ϵ_{rel}	$\Delta\epsilon_{rel}$			ϵ_{rel}	$\Delta\epsilon_{rel}$	ϵ_{rel}	$\Delta\epsilon_{rel}$		
1000-1-1-a-a-ns-0000	4e-06	2e-12	6e-07	2e-12	7e+04	1e+05	3e-07	4e-12	3e-10	5e-12	1e+03	2e+03
1000-1-1-a-a-ns-0330							2e-05	4e-12	4e-07	6e-12	1e+03	1e+04
1000-1-1-a-a-ns-0660												
1000-1-1-a-a-ns-1000	2e-11	6e-12	2e-11	1e-11	5e+02	1e+04	3e-11	6e-12	3e-11	1e-11	6e+01	2e+03
1000-1-1-a-b-ns-0000	7e-07	2e-12	7e-09	2e-12	1e+02	1e+02	5e-07	1e-12	1e-09	2e-12	6e+01	2e+02
1000-1-1-a-b-ns-0330	1e-05	2e-12	6e-07	3e-12	2e+04	3e+05	8e-06	1e-12	3e-07	3e-12	1e+03	2e+04
1000-1-1-a-b-ns-0660												
1000-1-1-a-b-ns-1000	7e-12	2e-12	7e-12	2e-12	1e+02	3e+03	1e-11	2e-12	1e-11	2e-12	6e+01	2e+03
1000-1-1-b-a-ns-0000	2e-06	2e-12	7e-07	2e-12	2e+04	4e+04	3e-06	3e-12	1e-09	2e-12	5e+02	2e+03
1000-1-1-b-a-ns-0330	3e-06	5e-12	2e-08	5e-12	3e+04	3e+05	4e-05	5e-12	2e-07	5e-12	5e+02	6e+03
1000-1-1-b-a-ns-0660	3e-06	2e-12	6e-08	4e-12	3e+04	4e+05	3e-06	6e-12	2e-08	6e-12	4e+03	5e+04
1000-1-1-b-a-ns-1000	8e-12	6e-12	8e-12	4e-12	1e+03	2e+04	7e-12	6e-12	7e-12	4e-12	3e+02	6e+03
1000-1-1-b-b-ns-0000	2e-06	4e-12	5e-09	6e-12	1e+02	1e+02	9e-08	2e-12	1e-11	2e-12	6e+01	1e+02
1000-1-1-b-b-ns-0330	4e-05	3e-12	7e-07	6e-12	4e+03	7e+04	5e-06	2e-12	7e-09	4e-12	2e+03	4e+04
1000-1-1-b-b-ns-0660	5e-05	1e-12	7e-07	2e-12	2e+04	3e+05	2e-05	1e-12	4e-07	2e-12	2e+03	4e+04
1000-1-1-b-b-ns-1000	3e-11	6e-12	3e-11	3e-12	2e+03	5e+04	2e-11	6e-12	2e-11	3e-12	5e+02	1e+04
1000-1-2-a-a-ns-0000	8e-08	2e-12	7e-10	9e-13	3e+04	4e+04	5e-06	2e-12	7e-08	2e-12	5e+02	2e+03
1000-1-2-a-a-ns-0330	4e-06	3e-12	4e-07	3e-12	7e+04	7e+05	7e-06	2e-12	7e-08	3e-12	1e+03	1e+04
1000-1-2-a-a-ns-0660	7e-06	5e-12	1e-07	6e-12	7e+04	9e+05	3e-06	5e-12	1e-08	5e-12	2e+04	2e+05
1000-1-2-a-a-ns-1000	5e-12	1e-11	4e-12	2e-12	3e+02	6e+03	1e-11	1e-11	1e-11	2e-12	6e+01	1e+03
1000-1-2-a-b-ns-0000	3e-06	8e-13	1e-06	1e-12	3e+02	2e+03	3e-05	2e-12	6e-07	1e-12	6e+01	4e+02
1000-1-2-a-b-ns-0330	3e-06	1e-12	9e-07	2e-12	8e+03	1e+05	3e-06	2e-12	5e-07	2e-12	1e+03	2e+04
1000-1-2-a-b-ns-0660												
1000-1-2-a-b-ns-1000	7e-12	8e-12	7e-12	3e-12	3e+02	6e+03	1e-11	8e-12	1e-11	3e-12	1e+02	3e+03
1000-1-2-b-a-ns-0000	6e-06	3e-12	5e-07	2e-12	2e+04	4e+04	3e-07	4e-12	4e-10	5e-12	5e+02	2e+03
1000-1-2-b-a-ns-0330	1e-05	3e-12	9e-08	3e-12	3e+04	4e+05	9e-06	3e-12	1e-07	3e-12	2e+03	3e+04
1000-1-2-b-a-ns-0660												
1000-1-2-b-a-ns-1000	6e-03	3e-12	2e-07	6e-12	4e+03	9e+04	2e-11	4e-12	2e-11	3e-12	3e+02	6e+03
1000-1-2-b-b-ns-0000	2e-06	1e-12	2e-08	9e-13	6e+01	1e+02	4e-09	2e-12	1e-14	2e-12	6e+01	1e+02
1000-1-2-b-b-ns-0330	5e-05	2e-12	6e-07	1e-12	8e+03	2e+05	6e-05	2e-12	1e-06	2e-12	3e+02	7e+03
1000-1-2-b-b-ns-0660	2e-05	4e-12	1e-07	2e-12	3e+04	7e+05	4e-05	2e-12	4e-07	2e-12	4e+03	9e+04
1000-1-2-b-b-ns-1000	2e-07	7e-13	1e-08	2e-12	5e+02	1e+04	1e-11	2e-12	1e-11	3e-12	1e+02	3e+03
1000-1-3-a-a-ns-0000												
1000-1-3-a-a-ns-0330												
1000-1-3-a-a-ns-0660												
1000-1-3-a-a-ns-1000												
1000-1-3-a-b-ns-0000	4e-08	8e-13	8e-11	1e-12	1e+02	1e+02	2e-08	2e-12	1e-11	2e-12	6e+01	1e+02

1000-1-3-a-b-ns-0330	4e-06	3e-12	4e-08	4e-12	7e+04	1e+06	1e-05	1e-12	2e-07	4e-12	2e+03	5e+04
1000-1-3-a-b-ns-0660	7e-06	5e-12	1e-07	5e-12	7e+04	1e+06	1e-05	3e-12	7e-07	3e-12	2e+04	4e+05
1000-1-3-a-b-ns-1000	2e-11	3e-12	2e-11	4e-12	4e+03	1e+05	3e-11	3e-12	3e-11	4e-12	3e+02	6e+03
1000-1-3-b-a-ns-0000	1e-06	2e-12	2e-08	3e-12	8e+03	9e+03	3e-05	2e-12	3e-07	2e-12	3e+02	1e+03
1000-1-3-b-a-ns-0330	2e-06	2e-12	5e-07	2e-12	8e+03	1e+05	1e-06	1e-12	1e-07	2e-12	2e+03	3e+04
1000-1-3-b-a-ns-0660												
1000-1-3-b-a-ns-1000	4e-12	2e-12	4e-12	2e-12	1e+03	2e+04	1e-11	2e-12	1e-11	2e-12	1e+02	3e+03
1000-1-3-b-b-ns-0000	3e-05	2e-12	6e-07	4e-12	1e+02	4e+02	2e-06	3e-12	7e-10	5e-12	6e+01	2e+02
1000-1-3-b-b-ns-0330	5e-05	4e-12	7e-07	3e-12	8e+03	1e+05	8e-05	1e-12	5e-07	3e-12	4e+03	8e+04
1000-1-3-b-b-ns-0660	1e-05	1e-12	2e-07	3e-12	7e+04	1e+06	4e-05	1e-12	3e-07	3e-12	4e+03	9e+04
1000-1-3-b-b-ns-1000	4e-12	6e-12	4e-12	5e-12	2e+03	4e+04	1e-11	6e-12	1e-11	5e-12	3e+02	6e+03
1000-1-4-a-a-ns-0000							3e-06	1e-12	1e-08	2e-12	1e+03	3e+03
1000-1-4-a-a-ns-0330							5e-07	3e-12	5e-09	3e-12	2e+03	2e+04
1000-1-4-a-a-ns-0660							1e-06	8e-12	4e-09	8e-12	2e+03	2e+04
1000-1-4-a-a-ns-1000	9e-11	8e-12	9e-11	8e-12	4e+03	9e+04	1e-11	8e-12	1e-11	8e-12	1e+03	2e+04
1000-1-4-a-b-ns-0000	9e-07	2e-12	4e-08	2e-12	5e+02	1e+03	5e-05	2e-12	1e-06	3e-12	6e+01	7e+02
1000-1-4-a-b-ns-0330	2e-05	3e-12	2e-07	3e-12	3e+04	6e+05	1e-05	1e-12	2e-07	2e-12	1e+03	2e+04
1000-1-4-a-b-ns-0660	3e-05	4e-12	8e-07	6e-12	3e+04	7e+05	2e-05	3e-12	1e-07	3e-12	8e+03	2e+05
1000-1-4-a-b-ns-1000	1e-03	1e-12	1e-06	3e-13	5e+02	1e+04	1e-11	2e-12	1e-11	5e-15	5e+02	1e+04
1000-1-4-b-a-ns-0000	4e-08	3e-12	3e-11	2e-12	7e+04	7e+04	2e-07	4e-12	1e-08	5e-12	5e+02	2e+03
1000-1-4-b-a-ns-0330	2e-06	2e-12	1e-08	2e-12	7e+04	7e+05	5e-06	2e-12	5e-08	3e-12	1e+03	1e+04
1000-1-4-b-a-ns-0660	7e-07	3e-12	1e-07	4e-12	7e+04	9e+05	9e-06	6e-12	5e-08	4e-12	2e+04	3e+05
1000-1-4-b-a-ns-1000	4e-12	1e-12	4e-12	8e-13	3e+02	5e+03	6e-12	1e-12	5e-12	8e-13	1e+02	3e+03
1000-1-4-b-b-ns-0000	9e-07	1e-12	3e-09	1e-12	6e+01	1e+02	2e-09	2e-12	3e-15	2e-12	6e+01	2e+02
1000-1-4-b-b-ns-0330	4e-05	1e-12	3e-07	2e-12	8e+03	2e+05	4e-05	2e-12	6e-07	4e-12	1e+03	2e+04
1000-1-4-b-b-ns-0660	3e-05	2e-12	4e-07	2e-12	8e+03	2e+05	4e-05	4e-12	7e-07	3e-12	1e+03	3e+04
1000-1-4-b-b-ns-1000	3e-11	3e-12	3e-11	1e-12	4e+03	1e+05	3e-03	6e-12	6e-07	5e-12	1e+02	3e+03
1000-1-5-a-a-ns-0000							3e-05	3e-12	4e-09	3e-12	1e+03	3e+03
1000-1-5-a-a-ns-0330							6e-06	4e-12	2e-08	4e-12	2e+03	2e+04
1000-1-5-a-a-ns-0660							9e-07	1e-12	5e-08	3e-12	2e+04	2e+05
1000-1-5-a-a-ns-1000	5e-10	5e-12	5e-10	3e-12	4e+03	8e+04	3e-12	5e-12	3e-12	3e-12	5e+02	1e+04
1000-1-5-a-b-ns-0000	8e-06	3e-12	5e-07	5e-12	1e+02	4e+02	1e-05	1e-12	1e-07	2e-12	6e+01	3e+02
1000-1-5-a-b-ns-0330	3e-05	5e-12	3e-07	5e-12	2e+04	3e+05	2e-05	2e-12	2e-07	4e-12	5e+02	1e+04
1000-1-5-a-b-ns-0660	4e-06	4e-12	3e-08	4e-12	7e+04	1e+06						
1000-1-5-a-b-ns-1000	1e-11	3e-12	1e-11	5e-12	8e+03	2e+05	3e-11	3e-12	3e-11	2e-12	3e+02	6e+03
1000-1-5-b-a-ns-0000	4e-06	3e-12	4e-07	3e-12	4e+03	9e+03	3e-06	1e-12	1e-07	2e-12	3e+02	1e+03
1000-1-5-b-a-ns-0330	4e-05	4e-12	2e-07	4e-12	2e+04	2e+05	2e-05	3e-12	4e-07	5e-12	5e+02	8e+03
1000-1-5-b-a-ns-0660	3e-06	4e-12	8e-07	6e-12	3e+04	6e+05	3e-06	4e-12	2e-07	5e-12	2e+03	3e+04
1000-1-5-b-a-ns-1000	9e-12	7e-12	9e-12	6e-12	3e+02	6e+03	2e-11	7e-12	2e-11	6e-12	6e+01	1e+03
1000-1-5-b-b-ns-0000	3e-05	2e-12	1e-07	3e-12	6e+01	2e+02	3e-08	3e-12	5e-14	4e-12	6e+01	1e+02
1000-1-5-b-b-ns-0330	1e-04	1e-12	8e-07	4e-12	8e+03	2e+05	7e-05	2e-12	3e-07	3e-12	1e+03	2e+04

1000-1-5-b-b-ns-0660	4e-05	1e-12	8e-08	5e-12	2e+04	3e+05	4e-06	3e-12	3e-09	5e-12	8e+03	2e+05
1000-1-5-b-b-ns-1000	2e-06	8e-13	3e-12	1e-13	3e+02	5e+03	9e-12	2e-12	9e-12	4e-12	6e+01	1e+03
1000-2-1-a-a-ns-0000	5e-06	4e-12	3e-07	5e-12	7e+04	1e+05	2e-06	5e-12	1e-07	4e-12	5e+02	2e+03
1000-2-1-a-a-ns-0330	3e-07	5e-12	9e-07	4e-12	7e+04	7e+05	7e-07	4e-12	2e-09	4e-12	1e+03	1e+04
1000-2-1-a-a-ns-0660	2e-06	4e-12	2e-07	5e-12	7e+04	7e+05	4e-07	6e-12	8e-09	7e-12	2e+03	2e+04
1000-2-1-a-a-ns-1000	1e-11	5e-12	1e-11	4e-12	5e+02	1e+04	3e-11	5e-12	3e-11	4e-12	6e+01	2e+03
1000-2-1-a-b-ns-0000	2e-07	2e-12	7e-10	2e-12	5e+02	5e+02	1e-06	3e-12	3e-10	4e-12	1e+02	4e+02
1000-2-1-a-b-ns-0330	2e-05	3e-12	6e-07	4e-12	2e+04	3e+05	1e-05	4e-12	4e-07	2e-12	1e+03	2e+04
1000-2-1-a-b-ns-0660	2e-05	3e-12	2e-07	4e-12	2e+04	3e+05	4e-06	4e-13	7e-09	3e-12	4e+03	7e+04
1000-2-1-a-b-ns-1000	2e-12	3e-12	2e-12	3e-12	3e+02	5e+03	4e-12	3e-12	3e-12	3e-12	6e+01	1e+03
1000-2-1-b-a-ns-0000												
1000-2-1-b-a-ns-0330												
1000-2-1-b-a-ns-0660												
1000-2-1-b-a-ns-1000												
1000-2-1-b-b-ns-0000												
1000-2-1-b-b-ns-0330												
1000-2-1-b-b-ns-0660												
1000-2-1-b-b-ns-1000												
1000-2-2-a-a-ns-0000							1e-09	1e-12	3e-12	1e-12	2e+03	2e+03
1000-2-2-a-a-ns-0330							3e-08	2e-12	1e-08	2e-12	2e+03	1e+04
1000-2-2-a-a-ns-0660							7e-07	2e-12	8e-07	1e-12	2e+03	2e+04
1000-2-2-a-a-ns-1000	7e-12	9e-13	7e-12	6e-13	3e+02	6e+03	2e-11	9e-13	2e-11	6e-13	6e+01	1e+03
1000-2-2-a-b-ns-0000	6e-07	9e-13	9e-08	1e-12	1e+03	3e+03	2e-06	6e-13	2e-09	7e-13	6e+01	3e+02
1000-2-2-a-b-ns-0330	1e-06	1e-12	8e-08	1e-12	4e+03	5e+04	5e-06	1e-12	6e-07	1e-12	3e+02	4e+03
1000-2-2-a-b-ns-0660	4e-06	1e-12	3e-07	2e-12	8e+03	1e+05	2e-05	4e-13	2e-07	9e-13	1e+03	2e+04
1000-2-2-a-b-ns-1000	6e-12	8e-12	6e-12	5e-12	5e+02	1e+04	1e-11	8e-12	1e-11	5e-12	6e+01	1e+03
1000-2-2-b-a-ns-0000	5e-08	2e-12	2e-10	2e-12	8e+03	7e+03	1e-05	1e-12	4e-08	1e-12	3e+02	1e+03
1000-2-2-b-a-ns-0330	1e-05	2e-12	5e-07	4e-12	2e+04	2e+05	1e-05	3e-12	2e-07	4e-12	5e+02	7e+03
1000-2-2-b-a-ns-0660	4e-05	5e-12	1e-06	6e-12	2e+04	4e+05	5e-06	9e-13	4e-07	2e-12	5e+02	9e+03
1000-2-2-b-a-ns-1000	9e-12	6e-12	9e-12	4e-12	5e+02	1e+04	2e-11	2e-12	2e-11	6e-12	1e+02	3e+03
1000-2-2-b-b-ns-0000	4e-05	1e-12	8e-07	4e-12	6e+01	2e+02	4e-07	3e-12	2e-10	2e-12	6e+01	1e+02
1000-2-2-b-b-ns-0330	4e-05	2e-12	3e-07	5e-12	3e+04	6e+05	6e-05	2e-12	6e-07	5e-12	1e+03	2e+04
1000-2-2-b-b-ns-0660	1e-05	6e-12	2e-08	4e-12	3e+04	7e+05	5e-05	2e-12	9e-07	4e-12	2e+03	5e+04
1000-2-2-b-b-ns-1000	1e-10	8e-12	1e-10	7e-12	2e+03	5e+04	5e-03	8e-13	4e-07	0e+00	1e+02	3e+03
1000-2-3-a-a-ns-0000												
1000-2-3-a-a-ns-0330												
1000-2-3-a-a-ns-0660												
1000-2-3-a-a-ns-1000												
1000-2-3-a-b-ns-0000	1e-06	3e-12	5e-09	2e-12	3e+02	2e+02	7e-06	2e-12	1e-07	1e-12	6e+01	3e+02
1000-2-3-a-b-ns-0330	1e-05	5e-12	2e-07	5e-12	2e+04	3e+05	5e-06	6e-12	4e-08	8e-12	1e+03	2e+04
1000-2-3-a-b-ns-0660	2e-05	4e-12	6e-07	5e-12	2e+04	3e+05	9e-06	6e-12	5e-08	5e-12	1e+03	2e+04

1000-2-3-a-b-ns-1000	4e-12	4e-12	3e-12	1e-11	6e+01	1e+03	5e-12	4e-12	5e-12	1e-11	6e+01	1e+03
1000-2-3-b-a-ns-0000	4e-06	1e-12	1e-07	1e-12	4e+03	7e+03	1e-05	3e-12	5e-07	4e-12	3e+02	2e+03
1000-2-3-b-a-ns-0330	6e-05	2e-12	3e-07	3e-12	8e+03	1e+05	1e-06	4e-12	6e-09	5e-12	5e+02	7e+03
1000-2-3-b-a-ns-0660	9e-06	1e-12	2e-07	3e-12	2e+04	2e+05	5e-06	3e-12	5e-08	3e-12	5e+02	7e+03
1000-2-3-b-a-ns-1000	1e-10	8e-12	1e-10	0e+00	3e+02	6e+03	3e-11	8e-12	3e-11	0e+00	6e+01	2e+03
1000-2-3-b-b-ns-0000	2e-06	3e-12	7e-09	3e-12	1e+02	1e+02	8e-08	2e-12	2e-11	3e-12	6e+01	1e+02
1000-2-3-b-b-ns-0330	6e-05	9e-13	5e-07	3e-12	8e+03	1e+05	8e-05	2e-12	9e-07	3e-12	5e+02	1e+04
1000-2-3-b-b-ns-0660	4e-05	2e-12	7e-07	1e-12	2e+04	3e+05	5e-05	2e-12	7e-07	5e-12	8e+03	2e+05
1000-2-3-b-b-ns-1000	4e-03	5e-12	2e-08	2e-11	2e+03	4e+04	4e-03	5e-12	6e-07	2e-11	1e+02	3e+03
1000-2-4-a-a-ns-0000	3e-05	9e-13	2e-07	2e-12	7e+04	2e+05	9e-07	2e-12	8e-11	2e-12	1e+03	2e+03
1000-2-4-a-a-ns-0330							3e-07	2e-12	5e-10	2e-12	2e+03	2e+04
1000-2-4-a-a-ns-0660							5e-06	2e-12	6e-07	2e-12	2e+04	2e+05
1000-2-4-a-a-ns-1000	5e-12	2e-12	5e-12	3e-12	5e+02	1e+04	9e-12	2e-12	9e-12	3e-12	1e+02	3e+03
1000-2-4-a-b-ns-0000	3e-06	3e-12	2e-07	4e-12	1e+03	2e+03	2e-06	3e-12	2e-08	5e-12	1e+02	5e+02
1000-2-4-a-b-ns-0330	6e-06	1e-12	4e-07	4e-12	4e+03	7e+04	3e-06	7e-12	3e-07	5e-12	5e+02	9e+03
1000-2-4-a-b-ns-0660	8e-07	2e-12	1e-07	2e-12	2e+04	3e+05	4e-06	2e-12	1e-07	3e-12	5e+02	1e+04
1000-2-4-a-b-ns-1000	4e-12	2e-11	4e-12	2e-11	1e+03	2e+04	1e-11	2e-11	1e-11	2e-11	5e+02	1e+04
1000-2-4-b-a-ns-0000							2e-06	2e-12	5e-07	3e-12	5e+02	2e+03
1000-2-4-b-a-ns-0330							7e-05	4e-12	2e-08	5e-12	1e+03	9e+03
1000-2-4-b-a-ns-0660	3e-07	4e-12	2e-07	7e-12	7e+04	9e+05	7e-06	3e-12	6e-07	7e-12	1e+03	2e+04
1000-2-4-b-a-ns-1000	2e-12	3e-12	2e-12	4e-13	5e+02	1e+04	4e-12	2e-12	4e-12	4e-13	6e+01	1e+03
1000-2-4-b-b-ns-0000	1e-07	2e-12	4e-11	2e-12	6e+01	5e+01	2e-09	2e-12	8e-15	1e-12	6e+01	1e+02
1000-2-4-b-b-ns-0330	9e-06	4e-12	3e-07	2e-12	8e+03	2e+05	6e-06	3e-12	9e-08	2e-12	5e+02	1e+04
1000-2-4-b-b-ns-0660	4e-05	1e-12	6e-07	2e-12	8e+03	2e+05	3e-05	4e-12	8e-07	2e-12	5e+02	1e+04
1000-2-4-b-b-ns-1000	9e-03	8e-12	2e-07	5e-12	3e+02	6e+03	1e-11	9e-12	1e-11	6e-12	3e+02	6e+03
1000-2-5-a-a-ns-0000	1e-06	2e-12	1e-07	2e-12	7e+04	1e+05	6e-08	2e-12	5e-12	2e-12	1e+03	2e+03
1000-2-5-a-a-ns-0330	4e-06	3e-12	2e-08	2e-12	7e+04	6e+05	1e-06	5e-12	7e-09	7e-12	1e+03	1e+04
1000-2-5-a-a-ns-0660	2e-06	2e-12	4e-08	2e-12	7e+04	7e+05	3e-05	8e-12	4e-07	9e-12	5e+02	7e+03
1000-2-5-a-a-ns-1000	6e-12	2e-12	6e-12	4e-12	4e+03	9e+04	1e-11	2e-12	1e-11	4e-12	3e+02	6e+03
1000-2-5-a-b-ns-0000	5e-07	4e-13	2e-07	6e-13	1e+03	3e+03	6e-04	9e-13	9e-07	1e-12	6e+01	5e+02
1000-2-5-a-b-ns-0330	1e-06	6e-13	2e-07	8e-13	8e+03	1e+05	7e-07	4e-13	9e-08	6e-13	1e+03	2e+04
1000-2-5-a-b-ns-0660	5e-05	8e-13	1e-07	2e-12	8e+03	1e+05	4e-05	4e-13	7e-07	6e-13	1e+03	2e+04
1000-2-5-a-b-ns-1000	4e-03	4e-12	8e-07	1e-12	2e+03	5e+04	1e-11	1e-12	1e-11	2e-12	5e+02	1e+04
1000-2-5-b-a-ns-0000	1e-06	3e-12	1e-07	3e-12	2e+04	3e+04	7e-07	2e-12	1e-10	3e-12	5e+02	2e+03
1000-2-5-b-a-ns-0330	2e-06	2e-12	3e-07	2e-12	2e+04	2e+05	1e-06	4e-12	5e-09	2e-12	1e+03	1e+04
1000-2-5-b-a-ns-0660	8e-07	8e-13	1e-07	9e-13	3e+04	3e+05	2e-06	4e-12	2e-08	5e-12	1e+03	1e+04
1000-2-5-b-a-ns-1000	4e-12	7e-13	4e-12	2e-12	5e+02	1e+04	2e-11	7e-13	2e-11	2e-12	1e+02	3e+03
1000-2-5-b-b-ns-0000	2e-06	2e-12	1e-08	3e-12	6e+01	2e+02	9e-08	3e-12	3e-11	2e-12	6e+01	2e+02
1000-2-5-b-b-ns-0330	9e-05	2e-12	7e-07	5e-12	8e+03	2e+05	4e-05	6e-12	2e-07	5e-12	5e+02	1e+04
1000-2-5-b-b-ns-0660	2e-04	1e-12	1e-06	9e-13	2e+04	4e+05	8e-05	5e-12	5e-07	1e-11	8e+03	2e+05
1000-2-5-b-b-ns-1000	1e-11	4e-12	1e-11	4e-12	3e+02	6e+03	2e-11	4e-12	2e-11	4e-12	6e+01	2e+03

1000-3-1-a-a-ns-0000	1e-05	3e-12	4e-07	5e-12	3e+04	7e+04	4e-06	5e-12	1e-07	7e-12	5e+02	2e+03
1000-3-1-a-a-ns-0330	8e-07	3e-12	4e-08	6e-12	7e+04	6e+05	8e-06	3e-12	4e-08	3e-12	1e+03	1e+04
1000-3-1-a-a-ns-0660	3e-06	7e-12	4e-07	9e-12	7e+04	7e+05	1e-06	1e-11	2e-07	2e-11	1e+03	1e+04
1000-3-1-a-a-ns-1000	2e-09	7e-12	2e-09	2e-12	2e+03	5e+04	3e-11	4e-12	3e-11	2e-12	1e+03	2e+04
1000-3-1-a-b-ns-0000	3e-07	2e-12	3e-09	2e-12	1e+02	1e+02	8e-08	1e-12	1e-10	3e-12	6e+01	1e+02
1000-3-1-a-b-ns-0330	2e-05	3e-12	3e-07	3e-12	4e+03	7e+04	7e-06	4e-12	3e-07	2e-12	3e+02	5e+03
1000-3-1-a-b-ns-0660	3e-06	3e-12	7e-07	4e-12	4e+03	8e+04	4e-06	3e-12	3e-07	4e-12	5e+02	1e+04
1000-3-1-a-b-ns-1000	9e-12	3e-12	9e-12	5e-12	2e+03	5e+04	4e-11	4e-12	4e-11	8e-12	3e+02	6e+03
1000-3-1-b-a-ns-0000	7e-06	5e-12	1e-06	6e-12	8e+03	5e+04	4e-09	5e-12	1e-11	3e-12	5e+02	2e+03
1000-3-1-b-a-ns-0330	2e-06	4e-12	5e-08	4e-12	2e+04	2e+05	1e-05	3e-12	7e-07	5e-12	5e+02	7e+03
1000-3-1-b-a-ns-0660	1e-06	6e-13	2e-07	1e-12	3e+04	4e+05	5e-08	1e-12	2e-08	1e-12	1e+03	1e+04
1000-3-1-b-a-ns-1000	2e-11	9e-12	2e-11	0e+00	3e+02	6e+03	4e-11	9e-12	4e-11	0e+00	6e+01	2e+03
1000-3-1-b-b-ns-0000	1e-06	6e-13	2e-07	9e-13	1e+02	4e+02	4e-08	2e-12	7e-12	2e-12	6e+01	1e+02
1000-3-1-b-b-ns-0330	1e-05	1e-12	9e-07	1e-12	2e+03	4e+04	1e-05	9e-13	6e-07	8e-13	3e+02	5e+03
1000-3-1-b-b-ns-0660	1e-05	7e-13	4e-07	2e-12	4e+03	7e+04	2e-06	6e-13	4e-08	2e-12	5e+02	9e+03
1000-3-1-b-b-ns-1000	3e-12	8e-12	3e-12	3e-12	1e+03	2e+04	1e-02	4e-12	9e-07	1e-12	6e+01	2e+03
1000-3-2-a-a-ns-0000							2e-07	2e-12	8e-13	2e-12	2e+03	3e+03
1000-3-2-a-a-ns-0330							1e-07	2e-12	4e-10	1e-12	2e+03	1e+04
1000-3-2-a-a-ns-0660	6e-06	8e-12	7e-07	8e-12	7e+04	9e+05	7e-07	8e-12	1e-08	8e-12	2e+03	3e+04
1000-3-2-a-a-ns-1000	4e-09	5e-12	4e-09	5e-13	2e+03	5e+04	1e-11	4e-12	1e-11	2e-12	1e+02	3e+03
1000-3-2-a-b-ns-0000	1e-08	2e-12	5e-10	2e-12	2e+03	3e+03	1e-05	1e-12	2e-07	2e-12	6e+01	3e+02
1000-3-2-a-b-ns-0330	3e-07	9e-13	1e-08	4e-13	4e+03	4e+04	1e-06	5e-13	3e-07	6e-13	3e+02	3e+03
1000-3-2-a-b-ns-0660	6e-07	8e-13	2e-07	8e-13	4e+03	5e+04	1e-05	5e-13	9e-08	1e-12	1e+03	1e+04
1000-3-2-a-b-ns-1000	2e-12	2e-12	2e-12	1e-12	2e+03	4e+04	2e-09	2e-12	2e-09	1e-12	3e+02	6e+03
1000-3-2-b-a-ns-0000	2e-08	1e-12	1e-09	9e-13	8e+03	8e+03	5e-07	1e-12	4e-07	2e-12	3e+02	1e+03
1000-3-2-b-a-ns-0330	1e-06	1e-12	4e-07	2e-12	8e+03	8e+04	4e-06	2e-12	4e-08	1e-12	5e+02	5e+03
1000-3-2-b-a-ns-0660	3e-06	6e-13	5e-07	2e-12	8e+03	8e+04	2e-06	2e-12	4e-08	6e-13	5e+02	6e+03
1000-3-2-b-a-ns-1000	6e-09	2e-12	6e-09	1e-12	5e+02	1e+04	9e-12	1e-11	9e-12	8e-13	3e+02	6e+03
1000-3-2-b-b-ns-0000	4e-05	2e-12	2e-07	3e-12	6e+01	2e+02	3e-07	1e-12	1e-11	3e-12	6e+01	2e+02
1000-3-2-b-b-ns-0330	3e-05	3e-12	3e-07	3e-12	4e+03	8e+04	3e-06	2e-12	3e-09	2e-12	1e+03	2e+04
1000-3-2-b-b-ns-0660	1e-05	4e-12	3e-07	5e-12	8e+03	2e+05	3e-06	3e-12	4e-09	4e-12	4e+03	9e+04
1000-3-2-b-b-ns-1000	1e-11	4e-12	1e-11	1e-11	8e+03	2e+05	5e-11	4e-12	5e-11	1e-11	2e+03	5e+04
1000-3-3-a-a-ns-0000												
1000-3-3-a-a-ns-0330												
1000-3-3-a-a-ns-0660												
1000-3-3-a-a-ns-1000												
1000-3-3-a-b-ns-0000	1e-06	8e-13	7e-07	2e-12	5e+02	2e+03	1e-06	2e-12	5e-08	3e-12	6e+01	3e+02
1000-3-3-a-b-ns-0330	2e-05	2e-12	8e-07	2e-12	4e+03	7e+04	3e-05	2e-12	5e-07	1e-12	3e+02	5e+03
1000-3-3-a-b-ns-0660	2e-06	1e-12	3e-07	2e-12	3e+04	6e+05	3e-06	5e-12	8e-07	6e-12	1e+03	2e+04
1000-3-3-a-b-ns-1000	6e-12	1e-12	6e-12	7e-13	3e+02	6e+03	2e-11	1e-12	2e-11	7e-13	6e+01	2e+03
1000-3-3-b-a-ns-0000	2e-06	2e-12	3e-08	5e-12	2e+04	3e+04	3e-05	1e-12	6e-07	2e-12	3e+02	1e+03

1000-3-3-b-a-ns-0330	3e-06	4e-12	2e-07	3e-12	2e+04	2e+05	2e-08	6e-12	3e-08	5e-12	2e+03	3e+04
1000-3-3-b-a-ns-0660	1e-05	3e-12	2e-07	3e-12	3e+04	5e+05	3e-06	2e-12	1e-08	5e-12	2e+03	3e+04
1000-3-3-b-a-ns-1000	7e-12	6e-12	7e-12	4e-13	2e+03	5e+04	1e-11	6e-12	1e-11	4e-13	1e+02	3e+03
1000-3-3-b-b-ns-0000	4e-09	3e-12	1e-13	4e-12	6e+01	7e+01	9e-10	2e-12	4e-14	2e-12	6e+01	1e+02
1000-3-3-b-b-ns-0330	2e-05	2e-12	2e-07	3e-12	8e+03	2e+05	1e-05	3e-12	3e-07	3e-12	2e+03	4e+04
1000-3-3-b-b-ns-0660	8e-06	7e-13	5e-07	4e-13	3e+04	6e+05	1e-05	9e-13	3e-07	4e-13	2e+03	4e+04
1000-3-3-b-b-ns-1000	3e-03	1e-11	9e-07	3e-12	5e+02	1e+04	9e-12	5e-12	9e-12	6e-12	3e+02	6e+03
1000-3-4-a-a-ns-0000							2e-05	2e-12	2e-07	3e-12	1e+03	3e+03
1000-3-4-a-a-ns-0330							3e-07	3e-12	9e-08	4e-12	1e+03	9e+03
1000-3-4-a-a-ns-0660							2e-08	2e-12	2e-10	2e-12	2e+03	2e+04
1000-3-4-a-a-ns-1000	5e-12	1e-11	5e-12	1e-11	5e+02	1e+04	6e-09	1e-11	6e-09	1e-11	6e+01	1e+03
1000-3-4-a-b-ns-0000	9e-07	2e-12	3e-08	2e-12	1e+02	2e+02	5e-08	2e-12	5e-10	3e-12	6e+01	3e+02
1000-3-4-a-b-ns-0330	1e-05	1e-12	1e-07	1e-12	8e+03	1e+05	2e-05	3e-12	7e-07	3e-12	3e+02	5e+03
1000-3-4-a-b-ns-0660	5e-05	2e-12	8e-07	2e-12	4e+03	8e+04	3e-05	3e-12	4e-07	3e-12	1e+03	2e+04
1000-3-4-a-b-ns-1000	5e-11	3e-12	3e-11	5e-12	5e+02	1e+04	7e-12	3e-12	6e-12	2e-12	1e+02	3e+03
1000-3-4-b-a-ns-0000	1e-09	3e-12	6e-10	6e-12	2e+04	2e+04	1e-04	3e-12	9e-07	2e-12	3e+02	2e+03
1000-3-4-b-a-ns-0330	4e-06	2e-12	6e-07	2e-12	8e+03	9e+04	6e-06	1e-12	7e-08	3e-12	5e+02	5e+03
1000-3-4-b-a-ns-0660	1e-06	3e-12	3e-08	3e-12	2e+04	2e+05	2e-07	3e-12	6e-09	4e-12	1e+03	1e+04
1000-3-4-b-a-ns-1000	4e-07	5e-12	2e-12	5e-12	5e+02	1e+04	2e-11	5e-12	2e-11	8e-12	6e+01	1e+03
1000-3-4-b-b-ns-0000	3e-07	1e-12	1e-09	2e-12	6e+01	1e+02	3e-09	2e-12	0e+00	2e-12	6e+01	1e+02
1000-3-4-b-b-ns-0330	4e-05	8e-13	7e-07	3e-12	2e+03	4e+04	3e-06	2e-12	2e-07	4e-12	1e+03	2e+04
1000-3-4-b-b-ns-0660	6e-06	2e-12	3e-07	1e-12	4e+03	8e+04	4e-06	2e-12	2e-07	3e-12	1e+03	2e+04
1000-3-4-b-b-ns-1000	2e-11	9e-12	2e-11	7e-13	4e+03	1e+05	5e-11	9e-12	5e-11	7e-13	1e+03	3e+04
1000-3-5-a-a-ns-0000	2e-06	2e-12	2e-07	3e-12	7e+04	1e+05	8e-08	2e-12	4e-12	2e-12	1e+03	2e+03
1000-3-5-a-a-ns-0330							4e-06	2e-12	2e-08	1e-12	1e+03	1e+04
1000-3-5-a-a-ns-0660							3e-05	7e-13	6e-07	2e-12	1e+03	1e+04
1000-3-5-a-a-ns-1000	2e-07	3e-12	2e-10	3e-12	3e+02	6e+03	2e-11	1e-11	2e-11	1e-11	6e+01	2e+03
1000-3-5-a-b-ns-0000												
1000-3-5-a-b-ns-0330												
1000-3-5-a-b-ns-0660												
1000-3-5-a-b-ns-1000												
1000-3-5-b-a-ns-0000	9e-09	2e-12	1e-11	2e-12	4e+03	4e+03	2e-06	2e-12	2e-09	2e-12	3e+02	1e+03
1000-3-5-b-a-ns-0330	9e-06	4e-12	4e-07	3e-12	4e+03	5e+04	3e-06	2e-12	5e-07	2e-12	3e+02	3e+03
1000-3-5-b-a-ns-0660	6e-07	2e-12	9e-07	2e-12	8e+03	1e+05	2e-07	3e-12	6e-07	2e-12	5e+02	8e+03
1000-3-5-b-a-ns-1000	8e-03	7e-12	3e-07	6e-12	5e+02	1e+04	2e-11	2e-12	2e-11	5e-12	1e+02	3e+03
1000-3-5-b-b-ns-0000	1e-08	2e-12	1e-13	3e-12	6e+01	5e+01	6e-09	5e-12	1e-15	8e-12	6e+01	1e+02
1000-3-5-b-b-ns-0330	1e-04	3e-12	5e-07	3e-12	2e+03	4e+04	1e-05	2e-12	1e-08	2e-12	5e+02	9e+03
1000-3-5-b-b-ns-0660	8e-05	3e-12	3e-07	3e-12	8e+03	2e+05	7e-05	4e-12	3e-07	4e-12	1e+03	2e+04
1000-3-5-b-b-ns-1000	1e-02	8e-12	1e-06	5e-12	3e+02	6e+03	3e-12	9e-13	2e-12	1e-11	1e+02	3e+03