Sample PDF for Final Project

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Second-Order Equations

In this section we extend the method of Laplace transforms to second-order, constantcoefficient, forced linear equations, that is, equations of the form

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = f(t)$$

where p and q are constants. To begin, we recall that the functions

$$\sin\omega t \quad \cos\omega t \quad e^{at}\sin\omega t \quad e^{at}\cos\omega t$$

appear often. So first we compute the Laplace transformations for those functions

Laplace transform of sine and cosine

We can use the definition

$$\mathcal{L}[\sin\omega t] = \int_0^\infty \sin\omega t e^{-st}\,dt$$

or use complex exponentials. Lets take advantage of the fact that we already know that $y(t) = \sin\omega t$ is the solution to the IVP

$$\frac{d^2y}{dt^2} + \omega^2 y = 0, \qquad y_0 = 0, \qquad y'(0) = \omega$$

So just compute the Laplace transform of the solution to the IVP

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + \omega^2\mathcal{L}[y] = \mathcal{L}[0]$$

Since $\mathcal{L}[0] = 0$

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + \omega^2\mathcal{L}[y] = 0$$

Therefore

$$\mathcal{L}[y] = \mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$$

Similarly for cos we get that

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}$$

Which illustrates how we find the laplace transforms of sin and cos functions

Shifting the Origin on the s-axis

Next for the equations $e^{at}\sin\omega t$ and $e^{at}\cos\omega t \implies$ the Laplace transform of a product is complicated BUT if one of the factors is exponential, the factor combines with e^{-st} in the definition of the transform

Suppose we are given $f(t)$ and we know that $\mathcal{L}[f(t)]$ os $F(s)$. To compute the transform of $e^{at}f(t)$ recall the definition

$$\mathcal{L}[e^{at}f(t)] = \int_0^\infty e^{at}f(t)e^{-st}\,dt$$

$$= \int_0^\infty f(t)e^{-(s-a)t)}\,dt$$

$$= F(s-a)$$

So our last rule is: If $\mathcal{L}[f] = F(s)$ then,

$$\mathcal{L}[e^{at}f(t)] = F(s-a)$$

$\implies$ multiplying $f(t)$ by e^{at} corresponds to replacing the argument s fof its Laplace Transform by $s-a$

Example

Since we know $\mathcal{L}[\cos 2t] = \frac{2}{s^2+4} \implies$ to compute the transform of $e^{-3t}\cos 2t$, replace s by $s+3$ on the right hand side

$$\mathcal{L}[e^{-3t}\cos 2t] = \frac{s+3}{(s+3)^2} = \frac{s+3}{s^2+6s+13}$$

We can also use this rule to compute $\mathcal{L}^{-1}$, for example to compute

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+5}\right]$$

we find the roots of the denominator to be $-1 \pm 2i \implies$ since the roots are complex, complete the square (if they werent complex we could factor $\rightarrow$ partial fractions to compute $\mathcal{L}^{-1}$)

$$s^2+2s+5 = (s+1)^2+4$$

Then we note that

$$\frac{1}{(s+1)^2+4} = F(s+1)$$

where

$$F(s) = \frac{1}{s^2+4}$$

We know that

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{2}{s^2+4}\right] = \frac{1}{2}\sin 2t$$

So using the rule we have

$$\mathcal{L}^{-1} = \left[\frac{1}{(s+1)^2+4}\right] = \frac{1}{2}e^{-t}\sin 2t$$

A Forced Harmonic Oscillator

An Oscillator with Discontinuous Forcing

Sinusoidal Forcing

Resonant Forcing