Extension of the FKG Inequality

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***Abstract*—The 1971 Fortuin-Kasteleyn-Ginibre (FKG) inequality for two monotone functions on a distributive lattice is well studied and has shown to have many applications in statistical mechanics and other fields of mathematics. In 2008 Sahi conjectured an extended version of this inequality, called E_n for all n > 2 monotone functions on a distributive lattice. Here we consider a special version, namely F_n and examine its properties on the space on n monotone Boolean functions. We show that F_n does not satisfy the quasi-concave property by formulating a counter-example. We prove that F_3, over one-dimensional monotone Boolean functions, does not possess non-zero minima and we examine a method to prove a more general version.**

***Index Terms*—Higher Correlation, FKG inequality, monotone functions, quasi-concavity**

I. Introduction

We consider a finite distributive lattice L equipped with a probability measure function satisfying the following property,

$$\mu(x \vee y)\mu(x \wedge y) \geq \mu(x)\mu(y)$$

Henceforth, we will call this kind of lattice equipped with the probability measure function as a FKG poset. By the FKG Inequality, for any two positive monotone functions f,g on L,

$$E(fg) - E(f)E(g) \geq 0$$

In other words, the FKG Inequality states that any two positive monotone functions on a FKG poset are positively correlated.

For example [3], let $G = (V, E)$ be a random graph on V obtained by picking every edge, independently, with probability p. Let P denote the property that the graph is Planar and H denote the property that the graph is Hamiltonian. P is a monotonically decreasing property since every graph G' on the same vertices which is a sub-graph of G is also planar. H is a monotonically increasing property since every graph G' on the same vertices which contains G as a sub-graph is also Hamiltonian. The set of edges can be viewed as a Boolean lattice and taking μ to be the product measure, we can apply the FKG Inequality to get:

$$Pr(P \wedge H) \leq Pr(P)Pr(H) \Leftrightarrow Pr(P|H) \leq Pr(P)$$

Intuitively, knowing that G is Hamiltonian suggests that it has many edges and hence seems to indicate that G is less likely to be planar.

The multi-linear functional defined in [1], E_n, can be viewed as an extension for the FKG Inequality. E_n is defined as below:

1) Decompose a permutation σ in the symmetric group S_n as a product of disjoint cycles:

$$\sigma = (i_1, ..., i_p)(j_1, ..., j_q)...$$

2) For σ as above, let C_σ denote the number of cycles in σ and define:

$$E_\sigma(f_1, ..., f_n) = E(f_{i_1}...f_{i_p})E(f_{j_1}...f_{j_q})...$$

3) Combining the above two expressions, we get E_n:

$$E_n(f_1, ..., f_n) = \sum_{\sigma \in S_n} (-1)^{C_\sigma - 1} E_\sigma(f_1, ..., f_n)$$

In [2], Sahi made the following conjecture on the sequence of multi-linear functions E_n:

Conjecture: If $f_1, ..., f_n$ are positive, monotone functions on an FKG poset, then

$$E_n(f_1, ..., f_n) \geq 0$$

F_n is defined almost identically to E_n except instead of point-wise multiplication of functions we take point-wise minima of functions. We also restrict the range of the monotone Boolean functions to the closed interval [0,1]. For example the expression for F_3 is:

$$F_3(f, g, h) = 2E(f * g * h) + E(f)E(g)E(h) - E(f)E(g * h)$$

$$-E(g)E(f * h) - E(h)E(f * g)$$

where $f * g$ is the point-wise minima taken across all points of the Boolean Lattice. Showing that $F_n \geq 0$ is a stronger version of the general $E_n \geq 0$. When we consider characteristic functions i.e., $\{0, 1\}$ valued functions, F_n reduces to E_n. Characteristic functions over sets are some of the simplest examples of monotonically increasing functions and showing that E_n is non-negative over them would be an important result.

The rest of the paper is divided into three sections. We explain the notion of quasi-concavity in F_n and provide a counter-example to show why it fails. Then we explore an idea to search for certain kinds of minima of F_n and describe how the idea lead to proving non-negativity of F_3 for a small class of functions. Finally, we conclude with remarks of the proof and F_n in general and also mention potential applications of this correlation inequality.

II. Quasi-concavity

We start by introducing monotone Boolean functions and understanding their geometric representation. With this perspective in mind, we define quasi-concavity and describe it in the context of F_3. We conclude with a counter-example that shows that the property fails to hold for F_3.

A. Monotone Boolean Functions

A Boolean function takes Boolean variables as input; the dimension of the function is given by the number of Boolean variables it is a function of. A one dimensional Boolean function can be treated as a point in $[0,1]^2$. Its value at 0 gives one coordinate and its value at 1 gives the other coordinate.A two dimensional Boolean function can be treated as a point in $[0,1]^4$. Its values at $\{00\},\{01\},\{10\},\{11\}$ give the four coordinates. Since these functions are also monotonically increasing they must also satisfy the condition that:

$$x \leq y \implies f(x) \leq f(y)$$

B. Geometric Perspective

The set of one-dimensional monotone Boolean functions MBF_1 has a simple geometric visualization.

$$MBF_1 = \{(x,y) \in [0,1]^2 | x \leq y\}$$

The set of two-dimensional monotone Boolean functions MBF_2 has a slightly more complex geometric visualization.

$$MBF_2 = \{(x_1,x_2,x_3,x_4) \in [0,1]^4 | x_1 \leq \min(x_2,x_3), \\ x_4 \geq \max(x_2,x_3)\}$$

F_n can be understood as a function over k-space to $[0,1]$, where $k = n * 2^d$ and d is the dimension of each MBF.

C. Quasi-concavity

Definition: Let S be a convex set. A function $f : S \to \mathbb{R}$ is quasi-concave if for each $a,b \in S$, $f(a+tb) \geq \min(f(a), f(b))$ $\forall t \in [0,1]$.

Figure 1 shows the graph of four piece-wise linear functions. All the functions except the one in the bottom-right are quasi-concave. This function attains a value strictly lesser than the minimum of the values attained at the end points of the interval.

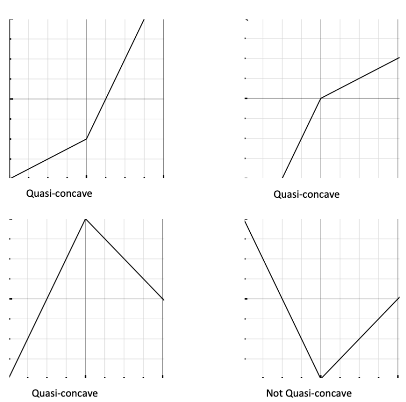

Fig. 1. Graphs of piece-wise linear functions

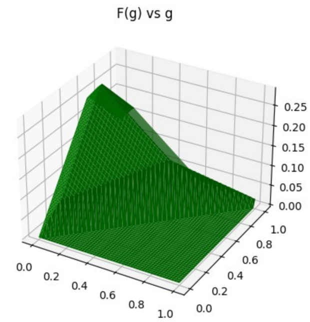

Fig. 2. F3 as a function of g (f0=[0.6,0.6], f1=[0.1,1])

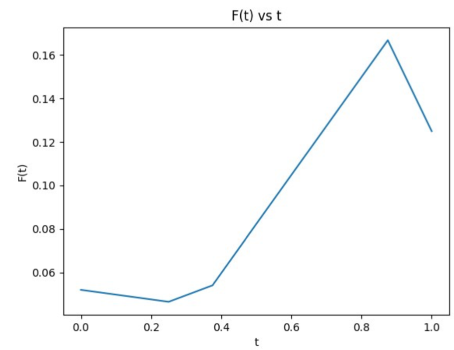

Fig. 3. F3 as a function of t (f0=[0.6,0.6], f1=[0.1,1], g0=[0,1], g1=[0.9,0.9])

The analog of F_n in a 2-dimensional plane is a piece-wise linear function. In reality it is a piece-wise planar graph; F_n is almost linear except at the points at which the min function changes its nature.

If a function is quasi-concave on its domain, then it is easy to see that the function can only take its minima at the extreme points of the convex set (points which do not lie along any line contained in the convex set). Thus, verifying non-negativity of F_n across its domain would reduce to checking non-negativity at its extreme points. An easy way to check quasi-concavity is to fix $n-1$ functions in F_n and vary the n^{th} function. In this case, the convex set S is the set of all d-dimensional MBFs.

D. Counter-example

Figure 2 is a plot of F_3 over monotone Boolean functions in one dimension. The first two functions of F_3 are fixed; $f_0 = (0.6, 0.6)$ and $f_1 = (0.1, 1)$ and F_3 varies over the last function. Since one dimensional MBFs can be treated as points in a subset of $[0,1]^2$, the domain of F_3 is this subset and it's range is $[0,1]$. The figure depicts the piece-wise planar nature of F_3 stated in the previous subsection.

To show that quasi-concavity fails, three points a, b and c belonging to a convex set are needed which satisfy the following condition: One point, say a, must be a convex combination of the other two, b and c , and the value of the function at a must be strictly lesser than the minimum of the function at b and c.

Having sampled f_0 and f_1, fix points $g_0 = [0,1]$ and $g_1 = [0.9, 0.9]$. Figure 3 is the graph of $F_3(f_0, f_1, tg_0 + (1-t)g_1)$ over $t \in [0,1]$. Clearly, the quasi-concave condition is violated
.

III. Searching for Minima

F_3 fails the quasi-concavity test which means that it may have minima in its non-extreme points. These putative minima may be of two kinds:

1) Zero minima: Minima at which F_3 is zero
2) Non-zero minima: Minima at which F_3 is positive

There are numerous examples of zero minima; the simplest one is two of the functions being identically zero while the third one can be any function belonging to the convex set. Then F_3 is zero and the point is a non extreme point.

To test if a point is actually a minima or not, we came up with the idea of a singular coordinate shift. The three d-dimensional functions that F_3 takes can be interpreted as k-dimensional points in a subset of $[0,1]^k$ where $k = 3 * 2^d$. We pick a coordinate and perturb it by some $\varepsilon > 0$. However, this ε cannot be arbitrarily large. The adjusted coordinate must still satisfy the monotone Boolean property of the function it is a value of. Moreover, we also require that the adjusted value does not exceed or be less than the corresponding coordinate values of the other functions.

These restrictions ensure that a linear change in the coordinate corresponds to a linear change in the value of F_3. By this logic, if a coordinate can be increased or decreased, then F_3 can also be increased and decreased. Points like these cannot be a local minima.

We use the idea introduced above to show that F_3 over one-dimensional MBFs has no non-zero minima.

Theorem III.1. *F_3 over one-dimensional monotone Boolean functions has no non-zero minima.*

Proof. We prove this by contradiction. We assume that F_2 does not have any non-zero minima. Suppose F_3 does have a non-zero minima, say $(f_1, f_2, g_1, g_2, h_1, h_2)$. This point represents three one-dimensional MBFs and so it must satisfy certain conditions:

- $0 \leq f_i, g_i, h_i \leq 1$
- $f_1 \leq f_2, g_1 \leq g_2, h_1 \leq h_2)$

F_3 is also symmetric, i.e., functions f, g and h can be permuted without changing the values of F_3, so, without loss of generality, we can assume that $f_1 \leq g_1 \leq h_1$. This leaves us with six cases based on the ordering of $\{f_2, g_2, h_2\}$.

- *Case 1:* $f_2 \leq g_2 \leq h_2$
  - Step 1: $(f_1, f_2, g_1, g_2, h_1, h_2)$ Consider the perturbation $(f_1, f_2, g_1, g_2, h_1, h_2 + \varepsilon)$. The coefficient of ε in the expression for F_3 is
  $$(E(f)E(g) - E(f))/2 = (E(g) - 1)E(f)/2 \leq 0$$
  so to preserve the minimality of the point, ε should be zero. This happens only if $h_2 = 1$.
  - Step 2: $(f_1, f_2, g_1, g_2, h_1, 1)$ Consider the perturbation $(f_1, f_2, g_1, g_2, h_1 + \varepsilon, 1)$. The coefficient of ε in the expression for F_3 is
  $$(E(f)E(g) - E(f))/2 = (E(g) - 1)E(f)/2 \leq 0$$
  so to preserve the minimality of the point, ε should be zero. This happens only if $h_1 = 1$.

  $F_3(f_1, f_2, g_1, g_2, 1, 1) = F_2(f_1, f_2, g_1, g_2)$, which does not have a non-zero minima. Hence, we arrive at a contradiction.
- *Case 2:* $g_2 \leq f_2 \leq h_2$
  - Step 1: $(f_1, f_2, g_1, g_2, h_1, h_2)$ Consider the perturbation $(f_1, f_2, g_1, g_2, h_1, h_2 + \varepsilon)$. The coefficient of ε in the expression for F_3 is
  $$(E(f)E(g) - E(f * g))/2 \leq 0$$
  (by the FKG Inequality) so to preserve the minimality of the point, ε should be zero. This happens only if $h_2 = 1$.
  - Step 2: $(f_1, f_2, g_1, g_2, h_1, 1)$ Consider the perturbation $(f_1, f_2, g_1, g_2, h_1 + \varepsilon, 1)$. The coefficient of ε in the expression for F_3 is
  $$(E(f)E(g) - E(f * g))/2 \leq 0$$
  (by the FKG Inequality) so to preserve the minimality of the point, ε should be zero. This happens only if $h_1 = 1$.

  $F_3(f_1, f_2, g_1, g_2, 1, 1) = F_2(f_1, f_2, g_1, g_2)$, which does not have a non-zero minima. Hence, we arrive at a contradiction.
- *Case 3:* $f_2 \leq h_2 \leq g_2$
  - Step 1: $(f_1, f_2, g_1, g_2, h_1, h_2)$ Consider the perturbation $(f_1, f_2, g_1, g_2 + \varepsilon, h_1, h_2)$. The coefficient of ε in the expression for F_3 is
  $$(E(f)E(h) - E(f))/2 = E(f)(E(h) - 1)/2 \leq 0$$
  so to preserve the minimality of the point, ε should be zero. This happens only if $g_2 = 1$.
  - Step 2: $(f_1, f_2, g_1, 1, h_1, h_2)$ Consider the perturbation $(f_1, f_2, g_1, 1, h_1, h_2 + \varepsilon)$. The coefficient of ε in the expression for F_3 is
  $$(E(f)E(g) - 2E(f))/2 = E(f)(E(g) - 2)/2 \leq 0$$
  so to preserve the minimality of the point, ε should be zero. This happens only if $h_2 = 1$.
  - Step 3: $(f_1, f_2, g_1, 1, h_1, 1)$ Consider the perturbation $(f_1, f_2, g_1, 1, h_1 + \varepsilon, 1)$. The coefficient of ε in the expression for F_3 is
  $$(E(f)E(g) - E(f))/2 = E(f)(E(g) - 1)/2 \leq 0$$
  so to preserve the minimality of the point, ε should be zero. This happens only if $h_1 = 1$.

  $F_3(f_1, f_2, g_1, 1, 1, 1) = F_2(f_1, f_2, g_1, 1)$, which does not have a non-zero minima. Hence, we arrive at a contradiction.
- *Case 4:* $h_2 \leq f_2 \leq g_2$
  - Step 1: $(f_1, f_2, g_1, g_2, h_1, h_2)$ Consider the perturbation $(f_1, f_2, g_1, g_2 + \varepsilon, h_1, h_2)$. The coefficient of ε in the expression for F_3 is
  $$(E(f)E(h) - E(f * h))/2 \leq 0$$

(by the FKG Inequality) so to preserve the minimality of the point, ε should be zero. This happens only if $g_2 = 1$.

- Step 2: $(f_1, f_2, g_1, 1, h_1, h_2)$ Consider the perturbation $(f_1, f_2 + \varepsilon, g_1, 1, h_1, h_2$. The coefficient of ε in the expression for F_3 is

$$(E(g)E(h) - E(g * h) - E(h))/2 \leq 0$$

so to preserve the minimality of the point, ε should be zero. This happens only if $f_2 = 1$.

- Step 3: $(f_1, 1, g_1, 1, h_1, h_2)$ Consider the perturbation $(f_1, 1, g_1, 1, h_1 + \varepsilon, h_2)$. The coefficient of ε in the expression for F_3 is

$$(E(f)E(g) - E(f))/2 = E(f)(E(g) - 1)/2 \leq 0$$

so to preserve the minimality of the point, ε should be zero. This happens only if $h_1 = h_2$.

- Step 4: $(f_1, 1, g_1, 1, h_2, h_2)$ Consider the perturbation $(f_1, 1, g_1 + \varepsilon, 1, h_2, h_2)$. The coefficient of ε in the expression for F_3 is

$$(E(f)E(h) - E(f) - E(f * h))/2 \leq 0$$

so to preserve the minimality of the point, ε should be zero. This happens only if $g_1 = h_2$.

- Step 5: $(f_1, 1, h_2, 1, h_2, h_2)$ Consider the perturbation $(f_1 + \varepsilon, 1, h_2, 1, h_2, h_2)$. The coefficient of ε in the expression for F_3 is

$$(2 + E(g)E(h) - E(g) - E(h) - E(g * h))/2 =$$

$$((1 - E(g))(1 - E(h)) + (1 - E(g * h)))/2 \geq 0$$

so to preserve the minimality of the point, ε should be zero. This happens only if $f_1 = 0$.

Finally, we have the point $(0, 1, h_2, 1, h_2, h_2)$ which means that F_3 is a function of h_2. Any decrease of h_2 allows us to diminish the value of F_3. This would contradict the minimality of the point and thus the only possibility is $h_2 = 0$. This gives the point $(0, 1, 0, 1, 0, 0)$ which has F_3 value 0. This contradicts the non-zero property of the point.

- *Case 5:* $g_2 \leq h_2 \leq f_2$
  - Step 1: $(f_1, f_2, g_1, g_2, h_1, h_2)$ Consider the perturbation $(f_1, f_2 + \varepsilon, g_1, g_2, h_1, h_2)$. The coefficient of ε in the expression for F_3 is

$$(E(g)E(h) - E(g))/2 = E(g)(E(h) - 1)/2 \leq 0$$

so to preserve the minimality of the point, ε should be zero. This happens only if $f_2 = 1$.

  - Step 2: $(f_1, 1, g_1, g_2, h_1, h_2)$ Consider the perturbation $(f_1, 1, g_1, g_2, h_1, h_2 + \varepsilon)$. The coefficient of ε in the expression for F_3 is

$$(E(f)E(g) - E(g) - E(f * g))/2 \leq 0$$

so to preserve the minimality of the point, ε should be zero. This happens only if $h_2 = 1$.

  - Step 3: $(f_1, 1, g_1, g_2, h_1, 1)$ Consider the perturbation $(f_1, 1, g_1, g_2, h_1 + \varepsilon, 1)$. The coefficient of ε in the expression for F_3 is

$$(E(f)E(g) - E(f * g))/2 \leq 0$$

so to preserve the minimality of the point, ε should be zero. This happens only if $h_1 = 1$.

$F_3(f_1, 1, g_1, g_2, 1, 1) = F_2(f_1, 1, g_1, g_2)$, which does not have a non-zero minima. Hence, we arrive at a contradiction.

- *Case 6:* $h_2 \leq g_2 \leq f_2$
  - Step 1: $(f_1, f_2, g_1, g_2, h_1, h_2)$ Consider the perturbation $(f_1, f_2 + \varepsilon, g_1, g_2, h_1, h_2)$. The coefficient of ε in the expression for F_3 is

$$(E(g)E(h) - E(g * h))/2 \leq 0$$

(by the FKG Inequality) so to preserve the minimality of the point, ε should be zero. This happens only if $f_2 = 1$.

  - Step 2: $(f_1, 1, g_1, g_2, h_1, h_2)$ Consider the perturbation $(f_1, f_2, g_1, g_2 + \varepsilon, h_1, h_2$. The coefficient of ε in the expression for F_3 is

$$(E(f)E(h) - E(f * h) - E(h)/2 \leq 0$$

so to preserve the minimality of the point, ε should be zero. This happens only if $g_2 = 1$.

  - Step 3: $(f_1, 1, g_1, 1, h_1, h_2)$ Consider the perturbation $(f_1, 1, g_1, 1, h_1 + \varepsilon, h_2)$. The coefficient of ε in the expression for F_3 is

$$(E(f)E(g) - E(f * g))/2 \leq 0$$

so to preserve the minimality of the point, ε should be zero. This happens only if $h_1 = h_2$.

  - Step 4: $(f_1, 1, g_1, 1, h_2, h_2)$ Consider the perturbation $(f_1, 1, g_1 + \varepsilon, 1, h_2, h_2)$. The coefficient of ε in the expression for F_3 is

$$(E(f)E(h) - E(f) - E(f * h))/2 \leq 0$$

so to preserve the minimality of the point, ε should be zero. This happens only if $g_1 = h_2$.

  - Step 5: $(f_1, 1, h_2, 1, h_2, h_2)$ Consider the perturbation $(f_1 + \varepsilon, 1, h_2, 1, h_2, h_2)$. The coefficient of ε in the expression for F_3 is

$$(2 + E(g)E(h) - E(g) - E(h) - E(g * h))/2 =$$

$$((1 - E(g))(1 - E(h)) + (1 - E(g * h)))/2 \geq 0$$

so to preserve the minimality of the point, ε should be zero. This happens only if $f_1 = 0$.

Finally, we have the point $(0, 1, h_2, 1, h_2, h_2)$ which means that F_3 is a function of h_2. Any decrease of h_2 allows us to diminish the value of F_3. This would contradict the minimality of the point and thus the only possibility is $h_2 = 0$. This gives the point $(0, 1, 0, 1, 0, 0)$ which has F_3 value 0. This contradicts the non-zero property of the point.

This completes the proof. ■

IV. CONCLUSION

Studying the nature of the minima of F_n is an important step in attempting to show that F_n is non-negative. An important step in this direction is verifying if quasi-concavity holds for F_n. We were able to show that the quasi-concave property fails to hold for F_3. A more rigorous analysis and understanding of the types of minima was required once we came up for a counterexample for quasi-concavity. Minima of F_3 can be of two types: zero minima and non-zero minima. We proved that F_3 over one-dimensional monotone Boolean functions has no non-zero minima.

Handling the large number of cases involved in proving that non-zero minima do not exist for F_3 over MBFs of any dimension is computationally complex. A simpler method might be to first shift individual entries (coordinate values that occur only in one function) and then attempt to shift shared entries (coordinate values that occur in more than one function). This is a more generalized method and it might lead to a more concise proof than trying to tackle the problem case by case.

A correlation inequality in 3 monotone functions would have applications in probability theory, combinatorics, stochastic processes and statistical mechanics. It would be exciting to see a stronger version of E_3 being applied in areas like uniform random spanning tree measures, symmetric exclusion processes, random cluster models (with q < 1), balanced and Rayleigh matroids [3],[4],[5],[6].

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