IBDP HL Mathematics

Exploration

An investigation into the magnification of objects in oceans with respect to depths and thermoclines.

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Introduction

In 2012 I had taken my first Scuba Diving course, and since then I have dived numerous times at several sites throughout South East Asia. On nearly every single one of these dives I had experienced, time and again, going through an underwater phenomenon known as thermoclines. These are regions near the surface that experience relatively rapid temperature drops. Figures 1 and 2 below help to illustrate this.

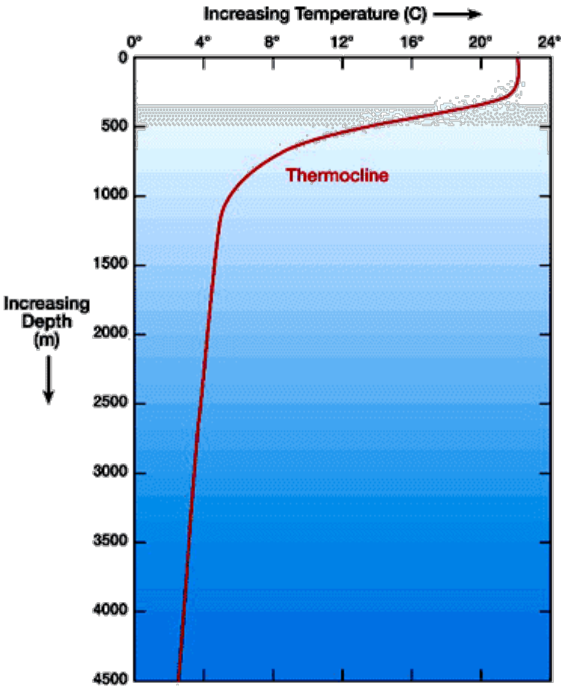


Figure 1: Rough graph of Temp. vs Depth
(source: http://marinebio.org/oceans/temperature/)

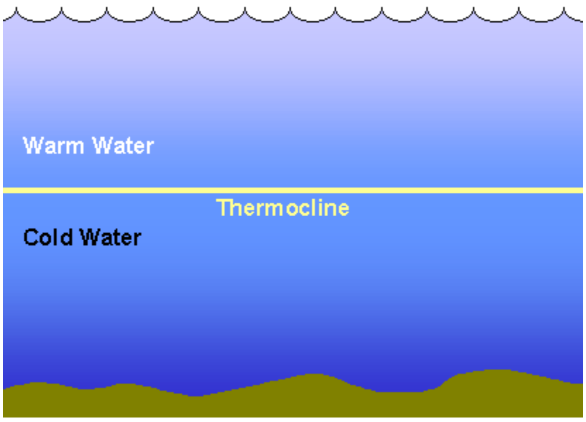


Figure 2: Crude Illustration of thermocline phenomenon
(source: http://www.scuba-tutor.com/diving-environment/dive-site-conditions/thermocline.php)

When approaching and passing through a thermocline, distinct visual differences can be observed. Firstly, a very faint but visual divide between the hot and cold sections of the thermocline can be seen, along which temperatures drop immensely - on one of my dives the temperature dropped from around 22°C all the way to 9°C. There is a visual divide, but due to it being very faint, is not easy to capture on camera, however figure 3 below can act as an example of it.

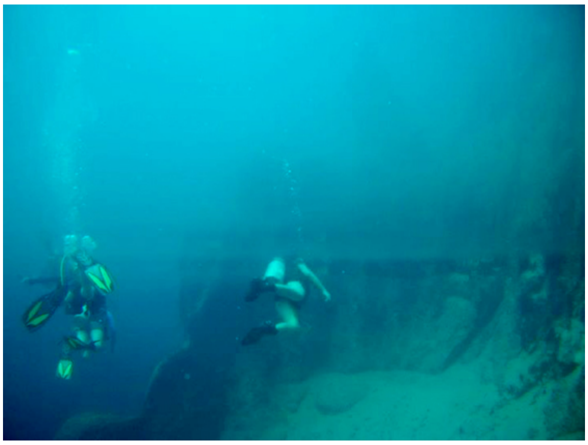


Figure 3: Thermocline divide
(source: http://zackyoung.ca/2014/08/thermocline/)

The second observation which can be made - the one upon which this investigation will be dealing with - is the refraction of objects placed across a thermocline. The effect is very much like if you placed your hand in a pool where it would appear to bend when observed from above the pool. This occurs due to the fact that your hand is being placed in two different fluid mediums of different refractive index.

However, despite occurring within one fluid medium, thermoclines still exhibit a change in refractive index. This is due to the change in temperature effecting the relative density of the fluid in different regions, causing light to refract across this change in temperature. This is similar to the effect where when driving on highways on hot days pools of water appear to be seen on the road ahead. This happens because there are hot layers of air near the hot road and cooler (denser) layers of air higher up. A ray of light is gradually refracted more and more towards the horizontal. Eventually it meets a hot layer near the ground at an angle greater than the critical angle, and total internal reflection takes place (as illustrated in figure 4 below).

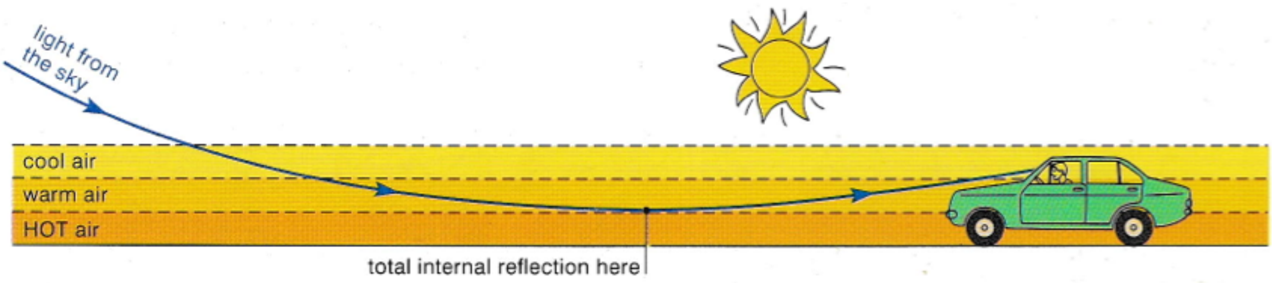


Figure 4: Highway hot-day mirage illustration.
(source: "Physics for You" (textbook) by Keith Johnson. 2011 edition. Page 189)

Although total internal reflection does not occur in thermoclines, refraction does occur due to the variations in density due to variations in temperature.

Diving through these thermoclines always intrigued me (in addition to making me wish I had some sort of thick underwater down-jacket), especially since it always seemed like I was making a strange wobble motion every time I went through them. Hence, my goal for this investigation is to mathematically model this visual difference experienced while going through these thermoclines, and produce a real world equation that maps this visual effect as a function of depth. More specifically, this visual effect will be interpreted as the 'magnification' effect of diving under water, not only through thermocline depths but even further as well.

Overview

Let us first establish a few important variables we will have to deal with:
ρ = Density
n = refractive index
t = temperature
d = depth
M = magnification

In order to utilise a holistic Magnification as a function of depth EQUATION, we first need to find a function that maps refractive index to depth - $n(d)$. In order to do so we will have to derive a composite function from three available EQUATION's as described below. (Full EQUATION's will be expressed later).

1. $\rho(n)$ $\rightarrow$ Density as a function of Refractive Index.
2. $n(\rho)$ $\rightarrow$ Refractive Index as a function of Density. *inverse of step 1*
3. $\rho(t)$ $\rightarrow$ Density as a function of Temperature.
4. $t(d)$ $\rightarrow$ Temperature as a function of Depth.

We will ultimately find a composite function of refractive index, n, in terms of depth , d. i.e. $n\big(\rho\big(t(d)\big)\big)$.

Once $n(d)$ is derived, we substitute it into another derived formula for magnification, M, of an object given certain fixed parameters, which will be described later. The resulting equation will then be able to fully map the magnification of a certain object with depth. This will also be discussed in further detail later.

Derivation for *n*(*d*)
Refractive index as a function of depth

(Note: Specific constants used in the explanations of individual earlier EQUATION's will be discussed in the Final Derivation section.)

The function $\rho(n)$ describes density in terms of refractive index. Note that the actual formula is in terms of λ (specific wavelength of a light ray) and n. However we shall keep λ as a constant of our choosing which shall be discussed later on. The formula for $\rho(n)$ is as follows. (Laurent Weiss, 2012)

$$\rho_r(\lambda,n)=(a_2\lambda^2+a_1\lambda+a_0)e^{(b_2\lambda^2+b_1\lambda+b_0)n} \rightarrow \text{EQUATION } \mathbf{1}$$

where,

a_2 = 1.45781289 × 10^{-9}
a_1 = - 2.35046033 × 10^{-6}
a_0 = 5.32669029 × 10^{-3}
b_2 = - 4.85379246 × 10^{-7}
b_1 = 7.59564314 × 10^{-4}
b_0 = 3.77629295

It should be noted that the LHS of EQUATION 1 represents relative density→ $\rho_r=\dfrac{\rho}{\rho_o}$, where ρ is final density (a variable), and ρ_o is the initial density (a constant determined relative to situation).

Analysing EQUATION 1 in terms of λ we can see that the exponent of ***e*** (the ***b*** coefficient quadratic) exhibits a downward convex parabola. The domain of wavelengths in the visible light section of the electromagnetic spectrum is between 390nm and 700nm. In terms of the ***b*** coefficient quadratic, this domain only represents the positive gradient section of the quadratic. What this means is that the higher the wavelength of light, the more 'compressed' the curve of EQUATION 1 becomes. As a result, longer wavelengths will return higher density values for a given input of ***n***. In terms of the ***a*** coefficient quadratic, an upward convex parabola is exhibited. However because its coefficients are relatively far smaller than the coefficients of the ***b*** quadratic, the range of values returned for the visible light domain 390nm < λ < 700nm are negligible. Hence, varying λ between its domain compresses EQUATION 1's curve far more than it shifts it along the *x* axis. It would be more useful to analyse how these quadratics effect returned values once we have found its inverse, as described with EQUATION 2 on the next page.

We need to however find n in terms of ρ, and hence need to get the inverse of EQUATION 1. If we take a and b to temporarily represent $a_2\lambda^2+a_1\lambda+a_0$ and $b_2\lambda^2+b_1\lambda+b_0$ respectively, we get the following expression which we can then solve for n.

$$\frac{\rho}{\rho_0} = (a)e^{(b)n} \quad \rightarrow \quad n = \frac{1}{(b)}\ln\left(\frac{\rho}{\rho_0(a)}\right)$$

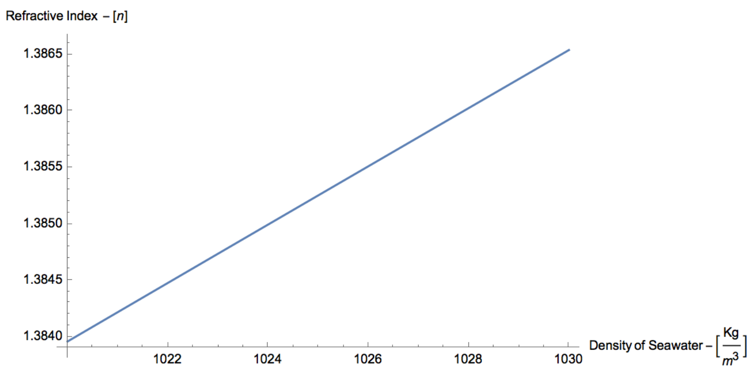
Substituting our values for a and b into the above EQUATION for n, we get the following expression for $n(\rho)$:

$$n(\rho) = \frac{\ln\left(\frac{\rho}{\rho_0(a_2\lambda^2 + a_1\lambda + a_0)}\right)}{(b_2\lambda^2 + b_1\lambda + b_0)} \rightarrow \text{EQUATION } \mathbf{2}$$

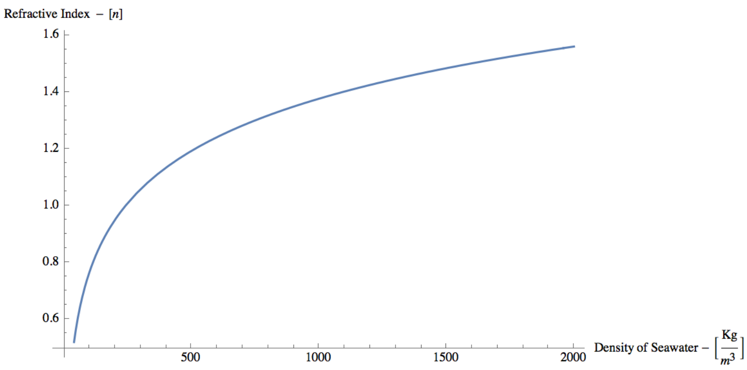
Domain $\rightarrow$ *1020 Kg/m³ < ρ < 1030 Kg/m³*

We shall let the constant ρ_o be 1029 Kg/m³ as it is a worldwide average. For the wavelength constant - λ - in EQUATION 3 we shall take it to be 650nm = 6.5 x 10^{-7} metres. This value comes from using my hand, and therefore my red glove, as an example for the object to be observed. λ is wavelength of red light. λ can obviously be changed in the future, but given that the order of magnitude of wavelengths in the visible light range is very small, changing the values of λ will likely only make minor alterations to the final outcome of values in this investigation. The fact that λ will always be very small, and that the λ constant is present only in denominator sections of EQUATION 2, shows that EQUATION 2 will likely always return large values.

For the range of EQUATION 2, we get a curve that very closely approximates a Linear curve, as can be seen in curve 1 below. The range for this linear curve (Curve 1) is given due to the fact that the density of seawater in non extreme cases are likely to only vary between 1020 and 1030 Kg/m³. When plotting EQUATION 2 for a larger range however we get a far more different looking curve as shown in Curve 2 below.



Curve 1: Refractive Index vs Density of Seawater for given range
Graph of EQUATION 2 with given domain
(created with Wolfram Mathematica software)



Curve 2: Refractive Index vs Density of Seawater for a large
Graph of EQUATION 2 with larger domain
(created with Wolfram Mathematica software)

Taking advantage of the linear characteristic of Curve 1 would ideally simplify the overall derivation. Hence, by using the co-ordinates of the points where the the upper and lower bounds of the average density range (1020Kg/m³ < *d* < 1030Kg/m³) cross the Curve produced by EQUATION 2, we form a linear equation expressing $n(\rho)$ as shown by EQUATION 2A below.

$$n(\rho) = 0.000258 \cdot \rho + 1.1208 \rightarrow \text{EQUATION 2A}$$

Domain $\rightarrow$ *1020 < ρ < 1030*

The significance of the "+1.1208" component in EQUATION 2A should be noted in that it as $\lim_{\rho \to 0}$, n approaches 1.1208. The reason $\rho = 0$ is referred to as a limit is because in the physical world it would be impossible for water to achieve zero density, let alone even approaching it. We are however only concerned with going as low as 1020Kg/m^3 in density as it is our lower bound, in which case the "+1.1208" component is still significant for the given range as the coefficient of ρ in EQUATION 2A is extremely small, which would return very small values for very large inputs.

Below we have EQUATION **3**, (which ultimately will derive EQUATION **5**) expressing $\rho(t)$, pressure as a function of temperature. We can use this to form a composite function of $n(\rho(t))$. (Engineering Toolbox)

$$\rho(t) = \frac{\rho_o}{\left(1+\beta(t_o - t)\right)} = \frac{1029}{\left(1+\beta(t_o - t)\right)} \rightarrow \text{EQUATION } \mathbf{3}$$

where,

ρ_o = initial density (kg/m^3) = 1029 kg/m^3 as a worldwide average.
t = final temperature (°C)
t_o = initial temperature (°C) = varies depending on location (even when in close vicinity)
β = expansivity coefficient of a fluid, in this case water.**

***Note: EQUATION 3 requires a positive difference between t_o and t, hence having a modulus effect. However for the purposes of this investigation it shall be reduced to t_o — t as temperature decreases with depth regardless under normal circumstances.*

In EQUATION 3 we can see that as t increases and gets very large, t_o and the additional '1' will become less and less significant in the function. β in contrast will multiply the affect of an increase of t, as a result (given the constant numerator), as temperature increases and gets very large, the density of the water should approach zero. This is intuitively acceptable as with an increase in temperature the water particles would hold more kinetic energy on average and would hence be pushing away every other particle more and more, which would cause a given no. of particles to collectively take up more volume in space. As a result for a given volume fewer particle's would be present hence density (mass/volume) would decrease.

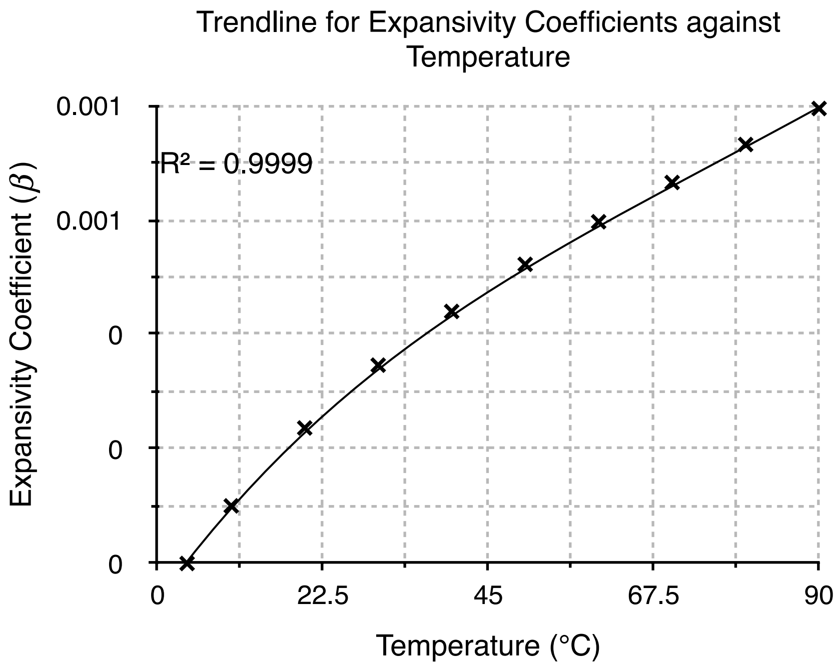
t_o is treated as a constant and it will vary immensely depending on location. However out of intuition, and given the tropical climate model used in this investigation (as described in page 6), t_o is unlikely to vary outside 18°C and 27°C. That being said, given the range of t being between 0°C and 90°C (as per data available - Table 1 on the following page), and the fact that t will always have to be less than t_o (as depth increases temperature decreases), t_o will almost always be proportionally closer to t as t changes between 0°C and 90°C. Hence the difference between t_o and t will likely be relatively small in comparison to the value of t. Being in the denominator section of EQUATION 3 this would decrease the magnitude of the entire denominator section, resulting in higher values of ρ on average. The effect is that of a control on the exaggeration of returned ρ values for every given input t value.

It should be noted that the β variable, or the expansivity coefficient of water, in EQUATION 3 also varies with temperature. Values for β are publicly available from 0°C to 90°C in increments of 10 (Elert). Because of data only being available to this range for β, EQUATION 3 will only be restricted to said range. This range can be seen to be overcompensating as divers are extremely unlikely to dive in waters greater that 35°C, should such a thing even exist. It is in good nature however to leave ample room for possible unexpected deviation, however unlikely. The range of

0°C to 90°C for β will therefore be fully considered in the derivations in this investigation. Moving on, these values of β vs. temperature were plotted on a scatter graph and cubic regression was found, with an R^2 value of 0.99991042. This data for β is shown in Table 1 and Graph 1 below.

Temperature (℃)	Expansivity Coefficient (β)
4	0
10	0.000088
20	0.000207
30	0.000303
40	0.000385
50	0.000457
60	0.000522
70	0.000582
80	0.00064
90	0.000695

Table 1:Values for Expansivity Coefficients for given temperatures



Graph 1: Cubic trend line for Temperature plotted against Expansivity coefficient
(created with 'Numbers' software)

The cubic regression plotted in Graph 1 above (which is also the formula for β against temperature) is be given as follows.

$$\beta(t) = at^3 + bt^2 + ct + q \rightarrow \textbf{EQUATION 4}$$

Domain $\rightarrow$ 4°C < t < 90°C

where,

$a = 5.0378616230461 \times 10^{-10} = 0.00000000050378616230461$
$b = -\ 1.1717518030341 \times 10^{-7} = -\ 0.00000011717518030341$
$c = 1.4733031500992 \times 10^{-5} = 0.000014733031500992$
$q = -\ 4.827213258101 \times 10^{-5} = -\ 0.00004827213258101$

Due to the fact that $\beta(t)$ follows a cubic trend, as t increases and gets very large, β would too. This is coherent with the intuitive explanation of the trend of EQUATION 3 for when t gets very large, as explained on page 4 previously. However, our range is only limited to 90°C. From 4°C to 90°C, as can be seen in graph 1, β continually increases but at a decreasing rate. The effect this has on EQUATION 3 is that rate at which ρ falls will decrease as t increases. This is, as described before, because of the fact that β is the coefficient of t.

Substituting EQUATION 3 into 4 we get the following EQUATION 5.

$$\rho(t) = \frac{1029}{\left(1+\left(at^3+bt^2+ct+q\right)\left(t_o-t\right)\right)} \rightarrow \textbf{EQUATION 5}$$

$$Domain \rightarrow 4℃ < t < 90℃$$

To get $n(\rho(t))$, refractive index as a function of temperature, we substitute EQUATION 5 into 2A, giving us EQUATION 6 below.

$$n\left(\rho\left(t\right)\right) = 0.000258 \cdot \left(\frac{1029}{\left(1+\left(at^3+bt^2+ct+q\right)\left(t_o-t\right)\right)}\right) + 1.1208$$

simplifying further we get,

$$n(t) = \left(\frac{0.265482}{\left(1+\left(at^3+bt^2+ct+q\right)\left(t_o-t\right)\right)}\right) + 1.1208 \rightarrow \textbf{EQUATION 6}$$

$$Domain \rightarrow 4℃ < t < 90℃$$

From the above simplification we can see that the numerator of EQUATION 5 - 1029 - has been very significantly reduced to 0.265482. The significance of this is that every value of *t* inputed into EQUATION 6 will give significantly smaller values as opposed to if it were to be inputed into EQUATION 5. This is a valid transformation as the output variable of EQUATION 6 is *n* - the refractive index - as opposed to ρ for EQUATION 5; refractive index values generally vary within the range of 1 to 1.5, which far less than density figures which in this investigation vary between 1020 and 1030. As a result our input *t* values are adjusted appropriately to match the expected *n* values that would then be processed to calculate magnification as described later in this investigation.

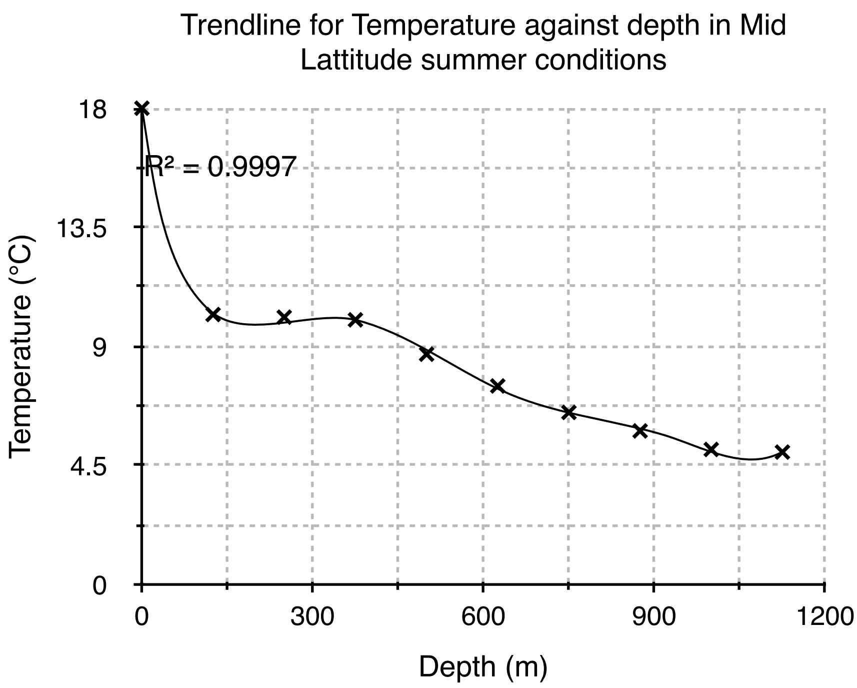
As $\lim_{t \to 0}$, the denominator component of EQUATION 6 will approach 1 , leaving the rest of the EQUATION to simply approach the value that the remaining of EQUATION 6 (0.265482 + 1.1208) will return which is 1.38628. This implies that as $\lim_{t \to 0} \rightarrow \lim_{n \to 1.38628}$.

EQUATION 6 can now be furthered by transforming it into a function for *n* in terms of *d*. To do this a function of *t(d)* will have to be found.

Data for temperature vs depth - *t(d)* - varies immensely in oceans world wide. I have therefore chosen EQUATION 7 below to represent a typical/expected scenario for middle latitude locations in the summer, as it is where I live (Singapore) and where most of my dive locations have been in, as well as the season recreational divers usually dive in. Data was found for *d* vs *t* and plotted on scatter graph - represented by Table 2 and Graph 2 in the following page respectively - on which a 6th degree polynomial regression was found, with gave the highest R^2 value of all other regressions that excel could compute, which was 0.980019974434246. (UCLA Marine Science Centre, 5/22/03).

Depth (m)	Temp. Mid Latitude Summer (℃)
0	18
125	10.2
250	10.1
375	10
500	8.7
625	7.5
750	6.5
875	5.8
1000	5.1
1125	5

Table 2: Values for Temperatures for given Depths in Middle Latitude Areas during the summer



Graph 2: 6th degree polynomial trend line for depth plotted against temperature.
(created with 'Numbers' software)

The 6th degree polynomial regression plotted in Graph 2 (which is also the formula for depth against temperature) is be given by EQUATION 7 as follows.

$$t(d) = Ad^6 + Bd^5 + Cd^4 + Dd^3 + Ed^2 + Fd + G \rightarrow \text{EQUATION 7}$$

$$\text{Domain} \rightarrow 0m < d < 1125m$$
$$\text{Range} \rightarrow 5℃ < t < 18℃$$

where,

$A = 1.9694322197862 \times 10^{-17}$
$B = -1.30006517447945 \times 10^{-13}$
$C = 3.30824274190628 \times 10^{-10}$
$D = -4.0519082794487 \times 10^{-7}$
$E = 2.4457172000597 \times 10^{-4}$
$F = -0.0725148751101585$
$G = 17.5278278773512$

It can be seen that the variables of higher exponents have smaller coefficients. A change in the product of the coefficients and their variables of higher magnitudes would therefore be less significant than a change in the product of coefficients and variables of lower magnitudes (the variable being *d* of course). As a result values would therefore not tend to deviate as dramatically from the constant *G*. This would make the distances between the vertexes of all present minima and maxima very small, as can be seen in the ‘ slightly bumpy downward slope’ of graph 2. It is also interesting to note that except for *A* and *B* the difference in orders of magnitude of the coefficients are always 3. This could possibly also indicate a close to constant decrease in the order of magnitude of the distances between the maxima and minima present in the curve. This cannot be accurately observed in Graph 2 as these Graph window and the curve variation for small increments in domain are too small to observe.

Also, despite the fact that the coefficients of *A* and *B* are very small in magnitude compared to *C, D, E, F, & G,* these sections of EQUATION 7 are still considered in the derivation as without them, the shape of the Temperature to Depth curve would drastically change after certain point in depth. Without the $Ad^6 + Bd^5$ section of the equation, temperature entries returned for a given amount of Depth would be every similar up till the said point in depth as part of this curve would be identical to the original. However after this point it will dramatically differ as the shape of the polynomial itself would change. Hence, we keep EQUATION 7 as it has been found.

It should be noted that the domain of EQUATION 7 does not produce a range of *t* that is greater than the domain EQUATION 6, so the EQUATION's are sound in terms of their current domains and ranges.

Since substituting EQUATION 7 into EQUATION 6 will produce a very large and complex function to display, the next step in the derivation for *n(d)* - EQUATION 8 - will be expressed as a composite function. Values can be substituted into EQUATION 8 with respect to its composites later.

$$n(t(d)) = \left(\frac{0.265482}{\left(1 + a\cdot(t(d))^3 + b\cdot(t(d))^2 + c\cdot(t(d)) + q\right)\cdot\left(t_o - t(d)\right)} \right) + 1.1208$$

→ EQUATION **8**

$$Domain \rightarrow 0m < d < 1125m$$

We know have derived an EQUATION for *n(d)* - EQUATION 8 above. We can now proceed to the derivation of our magnification model formula.

Derivation for Magnification formula

(Reefnet, n.d.)

Let us consider the following two diagrams and refer to them for the rest of this derivation.

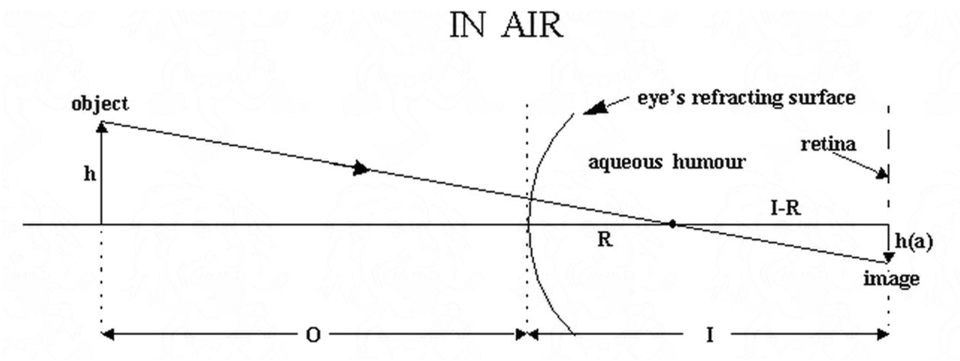


Diagram 1: Image and object refraction in air with respect to human eye as lens.
(Source: http://scubageek.com/articles/wwwbigr.html)

(Note: The observable object is represented in Diagram 1 by **h***, and the image by* **h(a)***)*
(Note: The variable **h(a)** *is not a function in* **h***)*
From diagram 1 we can express **h(a)** in terms of **h** using similar triangles, as in EQUATION 9:

$$h(a) = h\frac{(I - R)}{(O + R)} \rightarrow \text{EQUATION } \mathbf{9}$$

We can now introduce diagram 2 below.

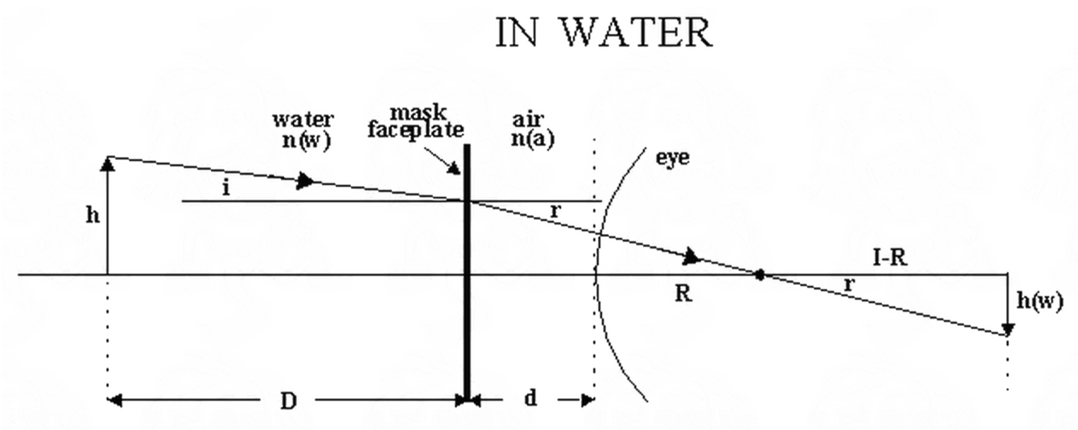


Diagram 2: Image and object refraction in combination of water and air - divided by mask - with respect to human eye as lens.
(Source: http://scubageek.com/articles/wwwbigr.html)

(Note: The observable object is represented in Diagram 2 by **h***, and the image by* **h(w)***)*
(Note: The variable **h(w)** *is not a function in* **h***)*

From Diagram 2 can can express the following EQUATION's 10 and 11 (in degrees):

$$h(w) = (I - R)\tan(r) \rightarrow \text{EQUATION } \mathbf{10}$$

$$h = D\tan(i) + (d + R)\tan(r) \rightarrow \text{EQUATION } \mathbf{11}$$

Substituting EQUATION's 10 and 11 into 9, and considering EQUATION 12 below (where **M** = magnification), we get the following EQUATION 13:

$$M = \frac{h(w)}{h(a)} \rightarrow \textbf{EQUATION 12}$$

$$M = \frac{O+R}{D\dfrac{\tan(i)}{\tan(r)}+(d+R)} \rightarrow \textbf{EQUATION 13}$$

Snell's law, as described by the following EQUATION 14, describes the illustration of the refraction of the ray of light in diagram 2 at the faceplate of the mask.

$$\frac{\sin(i)}{\sin(r)} = \frac{n(a)}{n(w)} \rightarrow \textbf{EQUATION 14}$$

For small angles, such as the paraxial rays shown at angles **i** and **r** in diagram 2, sines and tangents are approximately equivalent. We can therefore interpret EQUATION 14 as the following EQUATION 15.

$$\frac{\tan(i)}{\tan(r)} = \frac{n(a)}{n(w)} \rightarrow \textbf{EQUATION 15}$$

Noting that O = D + d , substituting EQUATION's 15 into 13, we get the following EQUATION 16 which is equivalent to the magnification of the object in water - **h(w)** - with respect to the height of the image in air - **h(a)**.

$$M = \frac{D+d+R}{D\dfrac{n(a)}{n(w)}+(d+R)} \rightarrow \textbf{EQUATION 16}$$

where,

D = Distance from object to mask faceplate (metres)
d = Distance from tip of refractive lens, eyeball, to max faceplate (metres) = 0.025 metres
R = Radius of refractive lens, i.e. eyeball (metres) = 0.01225 metres
n(a) = Refractive index of air = 1.000293
n(w) = Refractive index of water. However this varies according to EQUATION 8, hence **n(w)** = *n(d)*
(Note: The variable ***n*(a)** is *not a function in a. It is merely a constant.)*

Let us assume R (radius of human eyeball) to be 0.01225 metres on average. (Britannica Inc)

Specific to the dive mask I own, 'd' would approximately be 2.5 cm. We shall therefore take d = 0.025m as our constant for d in EQUATION 16.

(*Note: the* 'd' *constant in EQUATIONS 11, 13 & 16 is different from the 'd' constant in EQUATIONS 7 & 8.)*

Final Derivation - combining Magnification and *n(d)*

We can now get a final EQUATION that models the magnification of an object with depth for a given depth to temperature data set as given by EQUATION 7. We therefore simply substitute EQUATION 8 into 16 to get the following EQUATION 17.

$$M(t(d)) = \frac{D + d + R}{D\frac{n(a)}{n(d)} + (d + R)}$$

$\rightarrow$ EQUATION 17 - *(more simply, i.e. not substituting in entire function contents)*

$$M(t(d)) = \frac{D + d + R}{D\dfrac{n(a)}{\left(\dfrac{0.265482}{\left(1 + a\cdot(t(d))^3 + b\cdot(t(d))^2 + c\cdot(t(d)) + q\right)\cdot(t_o - t(d))}\right) + 1.1208} + (d + R)}$$

$$\textit{Domain} \rightarrow \textit{0m} < d < \textit{1125m}$$

$\rightarrow$ EQUATION **17** - *(in detail, i.e substituting in entire function contents)*

We can substitute all the remaining constants (as described previously in this exploration) into EQUATION 17. Thus far unfilled in constants are listed below.

n(a) = 1.00029 (*refractive index of air at standard pressure and temperature*)
t_o = 18°C (*as described in Table 2 and Graph 2)*
D = 0.5 metres (*assumed distance for observed object. Can change but will be kept constant for the purposes of this investigation*)
d = 0.025 meters
R = 0.01225 metres

Because the resultant function is very large, a composite of *t*(*d*) - EQUATION 7 - and M(*t*) will be used. As a result, we get function 1 below directly describing the relationship between depth and magnification for the example scenario as given in Table 2 and Graph 2 on page 8.

$$t(d) = \left(1.96943\times10^{-17}\right)\cdot d^6 - \left(1.30007\times10^{-13}\right)\cdot d^5 + \left(3.30824\times10^{-10}\right)\cdot d^4 - \left(4.05191\times10^{-7}\right)\cdot d^3 + (0.000244572)\cdot d^2 - (0.0725148751101585)\cdot d + (17.4553)$$

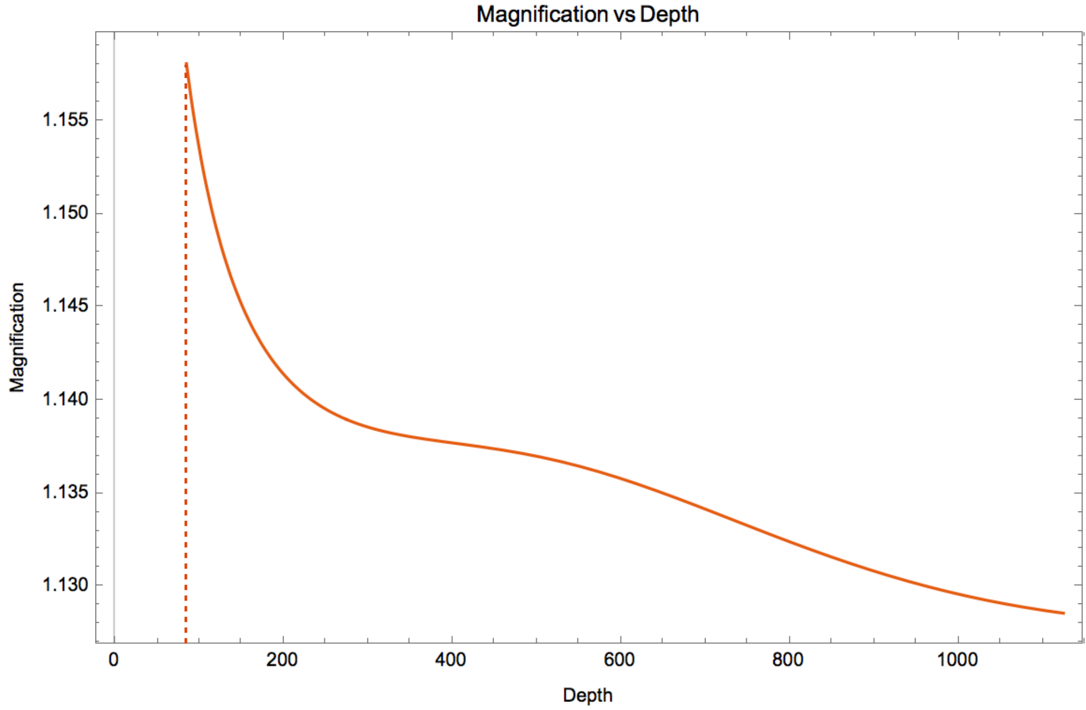
&

$$M(t) = \cfrac{0.57325}{0.03725 + \cfrac{0.500145}{1.1208 + \cfrac{0.265482}{(18 - t)\cdot\left(\left(5.03786\cdot10^{-10}\right)\cdot t^3 - \left(1.17175\cdot10^{-7}\right)\cdot t^2 + (0.000014733)\cdot t + 0.999952\right)}}}$$

$$\textit{Domain} \rightarrow \textit{0m} < d < \textit{1125m}$$

$\rightarrow$ **FUNCTION 1** $\rightarrow$ $M(t(d))$

Plotting the above Function 1, we get a Graph 3 below illustrating the relationship between the magnification of an object for the given range of depth.



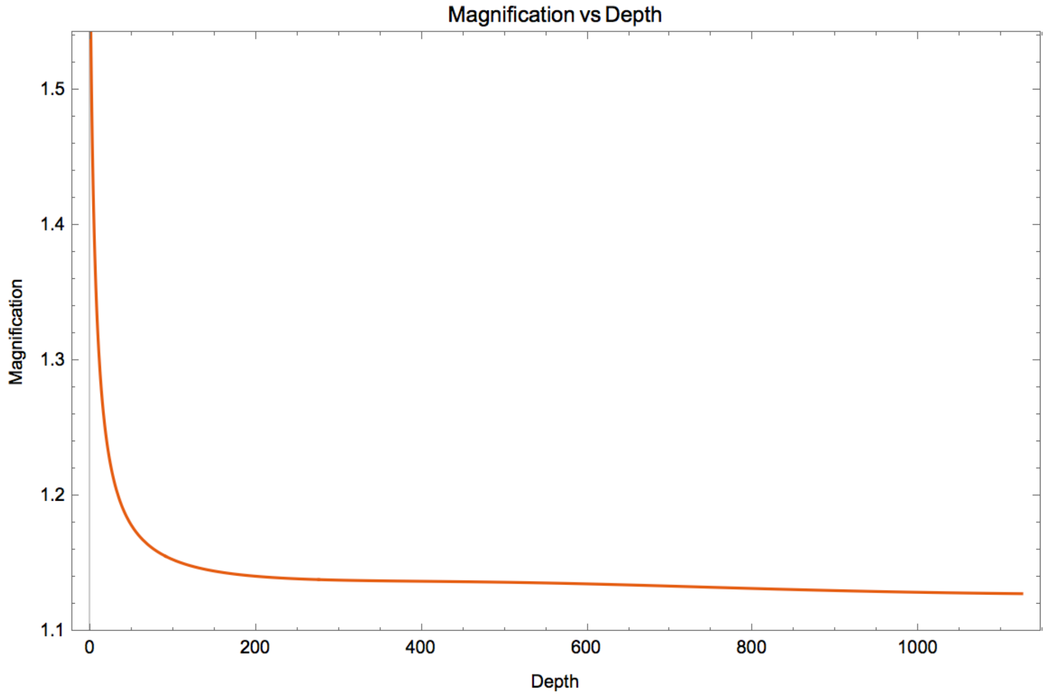
Graph 3A: Magnification vs depth as according to Function 1. *Limited Range*
(created with Wolfram Mathematica software)

Graph 3A exhibits a downward trend overall. However the graph doesn't exhibit the full range given its domain; the curve only shows magnification values for points between approximately 90 to 1125 metres. According to our table of figures in Table 2 we know the lower limit of depth - 0m - gives us 18°C. Looking at the M(t) part of Function 1, we can see that as $\lim_{t\to 18} \rightarrow \lim_{M\to 0}$. This is due to the fact that the function (18—t) exists in the bottom most denominator component of Function 1 as high lighted in yellow in the graphic below reproducing the M(t) section of Function 1:

$$M(t)=\cfrac{0.57325}{0.03725+\cfrac{0.500145}{1.1208+\cfrac{0.265482}{(18-t)\cdot\left(\left(5.03786\cdot 10^{-10}\right)\cdot t^{3}-\left(1.17175\cdot 10^{-7}\right)\cdot t^{2}+(0.000014733)\cdot t+0.999952\right)}}}$$

The structure of the function of M(t) is basically a fraction who's denominator is the sum of a constant and another second order fraction who's denominator is also the sum of a constant and a final third order fraction. As depth, *d,* approaches 0 ($\lim_{d\to 0}$) temperature, *t,* approaches 18 ($\lim_{t\to 18}$) as depicted by EQUATION 7 on page 8. As $\lim_{t\to 18}$, the denominator of the third order fraction of M(t) approaches 0. As this happens, the denominator of the second order fraction becomes very large, which then makes the denominator of the first order fraction very small, resulting in the function M(t) returning large values for M. As a result of this, as $\lim_{t\to 18}$, we get a very large values for Magnification M relative to the rest of the values for M in the rest of the flatter section of the graph. This therefore gives an almost asymptotic impression as depth nears zero as can be seen in Graph 3B on the following page which plots Function 1 on page 12, except for a greater range of Y axis values.

In the red highlighted sections we can see that the coefficients of t^3 & t^2 are relatively very small. However they are still considered in this investigation for the same reason as $Ad^6 + Bd^5$ were kept in EQUATION 7 on page 8. The reason being that if $(5.03786 \cdot 10^{-10}) \cdot t^3 - (1.17175 \cdot 10^{-7}) \cdot t$ were removed and the function M(t) hence simplified, the polynomial structure would completely change. However for some domain of t inputs, the returned M values would be very similar as the simplified curve would be closely related to the original curve within this domain. But, once outside this domain of t, the nature of the simplified curve will change and differ from the original curve, which would result in dramatic differences between the returned M values from the original M(t) function and the simplified one. This is undesirable as it would lead to severe inaccuracy once outside said domain of t.



Graph 3A: Magnification vs depth as according to Function 1.
Full Range of Depth
(created with Wolfram Mathematica software)

The previous paragraph mentions 'an almost asymptotic impression' exhibited by Graph 3B. It should be clarified that it is only referred to as 'asymptotic' purely and simply because of its immediate appearance. In actuality however, the curve crosses the Y axis at 1.54265. This is still relatively much larger than the average values for magnification in the rest of the curve.

This drastic drop in magnification from 0 to 90 metres - as described earlier with the 'almost asymptotic impression' - can be intuitively acceptable due to the immediate change of the path of light rays as it transfers from one medium - air - to another - sea water. Magnification is a relative measure, where in the actual magnification is based on the initial size of an object, which in this case is measured at 0 meters at 18°C where it can be considered the meeting point of air and the water surface. Near this boundary of change of medium (i.e. near the surface of sea water) the light rays are bound to be affected far more than if it were travelling deeper into the single medium due to the sudden physical changes when crossing the boundary. This therefore explains the fact that magnifications are significantly much higher between 0 and 90 metres as opposed to deeper depths as illustrated in Graph 3B.

The drastic change of the rate of decrease in magnification with depth at around the 90 meter region can also be explained with the observation of Graph 2 on page 8 for it also exhibits a high rate of decrease in temperature between the range of 0 to around 100 metres. Since Graph 2 reflects the *t(d)* curve which is the initial influence on Function 1, Graph 3B and 3A exhibit a similar characteristic.

The actual reason for this dramatic drop is dependent on the physical circumstances on the scenario. The point of rapid change in rate of decrease of Magnification with respect to depth (at around the 100 meter region) could be due to a very large thermocline. Divers aren't ever likely to dive below 80 meters anyhow, especially recreational divers, but to have a wide range such as that used in this exploration is potentially useful especially in the case of industrial applications. This will be discussed further in the following evaluation section.

Evaluation

The model as derived in this investigation uses an extremely wide domain for depth - 0 to 1125 metres. This is however only a result of the wide range of values present in the data set used to portray an assumed scenario (Table 2 and Graph 2). This 'data set' is expressed as *t(d)* in Function 1 in the form of a closely approximating regression. Ideally, one would be able to put any reasonable function that reflects their data in the place of *t(d)*, which they could have collected through their own means. The point of the use of Table 2's data set was therefore as an example to 'test' the remainder of the function and whether the output result was reasonable, which it was, as described in the previous section. Seeing as how the data set used in Table 2 is an approximation to tropical mid latitude like locations, it can still be valid indicator of magnification vs depth in this region. *t(d)* can vary immensely according to circumstance, and likewise the positions of thermoclines can vary between very shallow and very deep depths. If personal data were to be collected it would be highly likely that dramatic changes in magnification characteristics due to the presence of thermoclines (much like those exhibited and explained with Function 1 and Graphs 3A and 3B) would be present at shallower depths, in which case dramatic changes in the M(*d*) function would occur at shallower depths too, up to 5 meters even.

One major limitation of this derived model as it stands is the fact that the *t(d)* function on only uses positive values of °C, that too only above 4°C. Diving in subzero waters aren't uncommon, but accounting for it in this model would mean introducing negative numbers into the M(*t*) function. This would cause the function to return absurd values as the expansivity regression as shown in Table 1 and Graph 1 only account for temperature values above 0°C. It should however also be noted that sub zero waters only exist in an unfrozen state when their salinity is higher than average, in which case their density would vary higher than average. Accounting this into this model would mean having to simply change ρ_o in EQUATION 3 appropriately. In this perspective the model does have some degree of accountability for arctic conditions.

If the scenario obeys this models prescribed domain for temperature, then it can be a reasonably versatile tool. For example, if programmed into diving computers or instruments with temperature sensors and depth sensors with fast enough sample rates, information on actual and apparent heights of observed objects could be fed back in realtime to any user of said dive computers or instruments. Going even further, the model could even be implemented into the instruments of ROV's and other industrial equipment that would have to potential to travel to far greater depths than a regular diver. The information fed back could thus be very useful in terms of data collection, factor monitoring, and advanced telemetry for situational awareness.

By deriving and examining this model I was able to dig far deeper into and understand a lot more of an observation of which I had until now only taken for granted and gotten used to - the apparent change of sizes of objects underwater. Hopefully, by using this model in the near future, I may be able to properly appreciate the saying - "some things aren't as small as they seem!".

Bibliography

WOLFRAM RESEARCH. *Wolfram Mathematica Version 10.* Mathematical processing software used in the development of this Investigation.

Formula for Calculating density of fluids when changing pressure and temperature [Online]. Engineering Toolbox. Available: http://www.engineeringtoolbox.com/fluid-density-temperature-pressure-d_309.html [Accessed 9/Feb/2015].

Cubical expansion when changing temperature - EQUATION 3 [Online]. Engineering Toolbox. Available: http://www.engineeringtoolbox.com/fluid-density-temperature-pressure-d_309.html [Accessed 9/Feb/2015].

"The Ocean and Temperature - MarineBio.org". MarineBio Conservation Society. Web. Available: http://marinebio.org/oceans/temperature/ [Accessed 11:13 AM 2/17/2015.]

UCLA Marine Science Centre, 5/22/03. *Worksheet - Investigating the Vertical Thermal Structure of the Ocean: Temperature vs Depth at Three Different Latitudes.* [Online]. UCLA Marine Science Centre. Available: http://www.msc.ucla.edu/oceanglobe/pdf/thermo_plot_lab.pdf [Accessed 9/Feb/2015].

BERGMAN, J. 16/Feb/2011. Temperature of Ocean Water. Available: http://www.windows2universe.org/earth/Water/temp.html [Accessed 9/Feb/2015].

BERGMAN, J. 31/Aug/2001. Density of Ocean Water. Available: http://www.windows2universe.org/earth/water/density.html [Accessed 9/Feb/2015].

ELERT, G. *Thermal Expansion* [Online]. Available: http://physics.info/expansion/ [Accessed 9/Feb/2015].

I. THORMÄHLEN, J. S., U. GRIGULL. 1985. *Refractive Index of Water and Its dependenceon Wavelenght, Temperature, and Density* Lehrstuhl A für Thermodynamik, Technische Universität, München, Arcisstr. 21, D-8000 München 2.

LAURENT WEISS, Abdel Tazibt, Albert Tidu, Michel Aillerie. Water density and polarizability deduced from the refractive index determined by interferometric measurements up to 250 MPa. Journal of Chemical Physics, American Institute of Physics (AIP), 2012, 136, pp.124201. <10.1063/1.3698481>. <hal-00699507> Available: https://hal.archives-ouvertes.fr/hal-00699507/document [Accessed 16/Feb/2015].

NAVE, R. *Simple Magnifier* [Online]. Available: http://hyperphysics.phy-astr.gsu.edu/hbase/geoopt/simmag.html [Accessed 16/Feb/2015].

NAVE, R. *Seawater* [Online]. Available: http://hyperphysics.phy-astr.gsu.edu/hbase/chemical/seawater.html [Accessed 16/Feb/2015].

REEFNET. Do Things Seem 33% Bigger Underwater? Available: http://scubageek.com/articles/wwwbigr.html. [Accessed 16/Feb/2015].

ROSS, H. E. & NAWAZ, S. 2003. *Why do objects appear enlarged under water?*

STEWART, R. H. 16/Sep/2006. *Chapter 6 - Temperature, Salinity, and Density* [Online]. Available: http://oceanworld.tamu.edu/resources/ocng_textbook/chapter06/chapter06_09.htm [Accessed 9/Feb/2015].

BRITANNICA INC. Encyclopedia Britannica Macropedia: Sensory Reception. USA, 1997.