Discrete Analogue of the McAdams Poker Model

Searching for Optimal Strategies and Nash Equilibria

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Introduction and Motivation

We base our work on the card game developed by Duke Economist **David McAdams**, aptly named 'the **World's Simplest Poker**', is a zero-sum game that features a two-player, symmetric blinds system. It is structured as follows:

1. **Dealing**: Both players receive a single card with a value that can be modeled as a random variable between $[0, 1]$.
2. **Deciding**: After seeing their card, both players are presented with the option to either *bet* or *fold*.
3. **Outcome**: Should both players choose the same action, the game proceeds to a *showdown* (in which the player with the card of higher value wins the round). Otherwise, the betting player wins by *forfeit*.

We apply the model in the discrete setting, replacing the real-valued cards with a random element from the discrete set $\{1, 2, 3, ...n\}$. Through this lens, we ask the following:

- What is the optimal playing strategy?
- When should a player bluff?
- Does there exist a Nash Equilibrium?

Mechanics

The Poker model adheres to the following mechanics, in relation to Player A and Player B:

Action	Result	Profits
Bet, Bet	$c_A > c_B$	$(2, -2)$
Bet, Bet	$c_A < c_B$	$(-2, 2)$
Fold, Fold	$c_A > c_B$	$(1, -1)$
Fold, Fold	$c_A < c_B$	$(-1, 1)$
Bet, Fold	Player A Wins	$(1, -1)$
Fold, Bet	Player B Wins	$(-1, 1)$

Deterministic Strategies

As per the dynamics of the game, we can define a the following two deterministic strategies that players can adopt:

- **Pure Cutoff Strategies:** A player bets if and only if their card $c \geq k$, where k is some arbitrary cutoff in the set $\{1, 2, 3...n\}$, and folds otherwise.
- **Pure Betting Strategies:** A player bets if and only if their card $c \in \mathcal{B}$, where $\mathcal{B}$ is their discrete betting set, and folds otherwise.

Expected Value Visualization

Building Intuition for Optimal Strategies

We started by deriving a closed-form solution for the expected payout of a player given cutoff strategies. We found the relationship

$$\mathbb{E}_A(k_A, k_B) = \frac{3k_Ak_B - k_B^2 - 2k_A^2 + 2k_A - 2k_B + nk_A - nk_B}{n(n-1)},$$

depicted in Figure 1 for the case $n = 50$. By varying k_A and fixing k_B, we can draw the following conclusions from the graph:

- **Opponent plays safe**: When your opponent bets conservatively ($k_B \approx n$), your best move would be to play riskier ($k_A = 0$).
- **Opponent plays risky**: When your opponent bets riskily ($k_B \approx 0$), your best move would be to be slightly more conservative ($k_A \approx \frac{1}{4}n$).

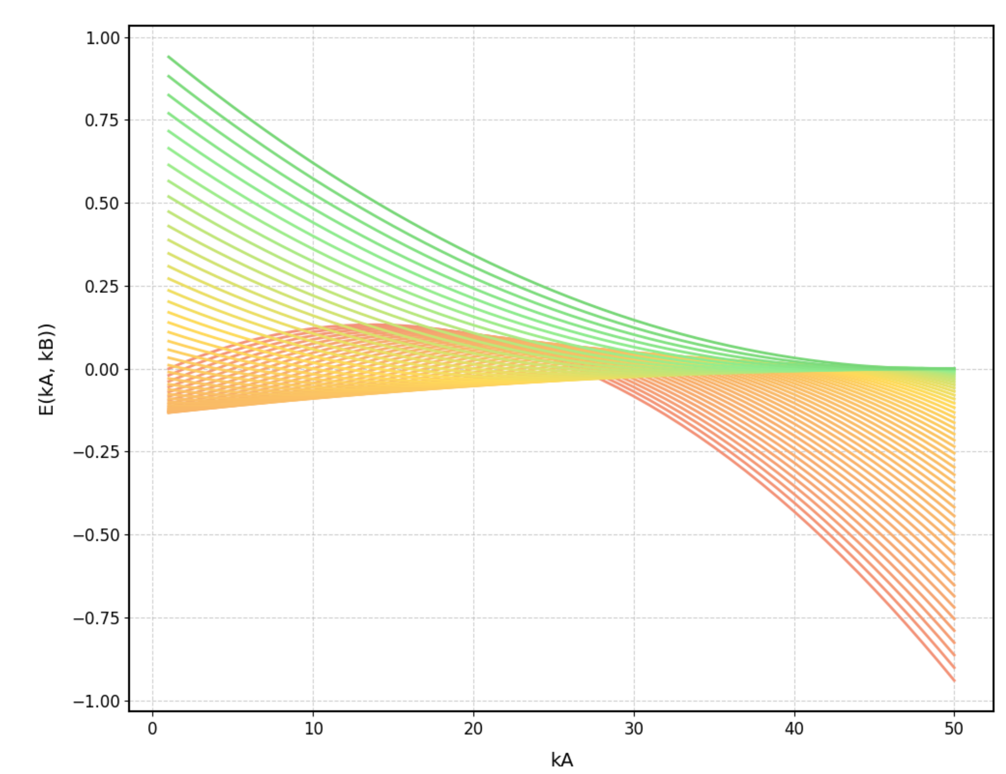

Figure 1: $\mathbb{E}_A(k_A, k_B)$ with constant k_B and $1 \leq k_A \leq n$

Similarly, we can expand our analysis to three dimensions, where we allow both k_A and k_B to vary freely over the range $[1, n]$. Figure 2 depicts the lattice for $\mathbb{E}_A$. Notice that, by the nature of the lattice, there exists **no global minimum** in $\mathbb{E}_A$.

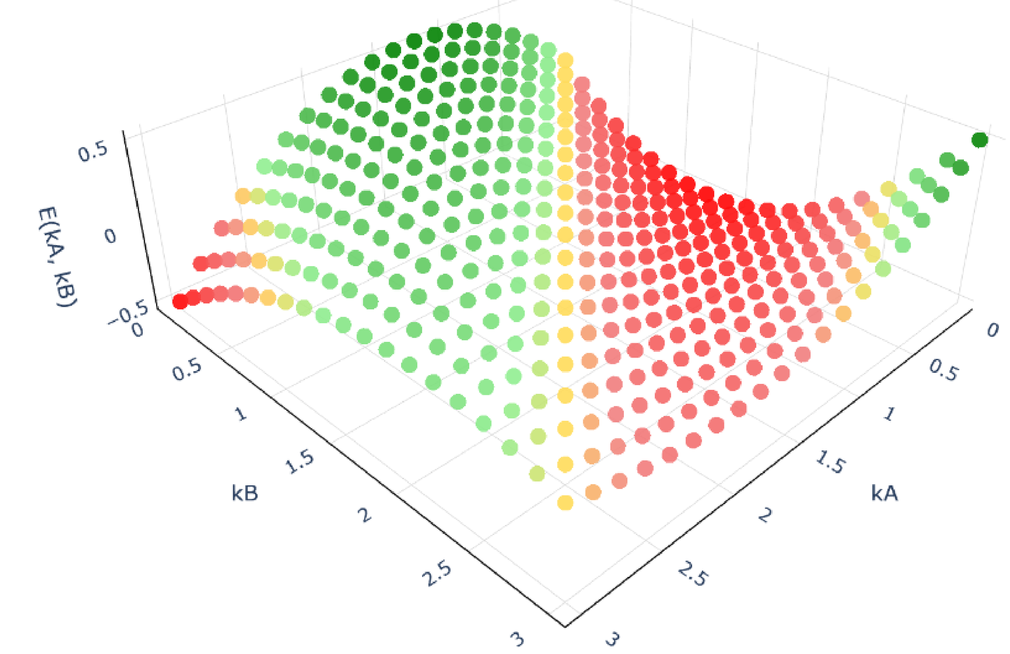

Figure 2: $\mathbb{E}_A(k_A, k_B)$ when $1 \leq k_A, k_B \leq n$

Remark. The exact optimal k_A when the opponent plays risky is given by:

$$k_A = \frac{2 + n + 3k_B}{4},$$

where we can then perform case-by-case analysis for $2+n+3k_B \pmod 4$. However, it should be noted that

- **Cutoff Strategies are non-dominant**: As can be seen in Table 1, $\forall k_x, \exists \mathcal{B}_x$ such that $\mathbb{E}_A(\mathcal{B}_x) \geq \mathbb{E}_A(k_x)$.

As such, we can now direct our efforts to betting set, and as we will later introduce, general strategies, in our search for dominant strategies and Nash Equilibria.

Intransitivity & Iterated Strategies

By analyzing payouts for all possible pairs $\mathcal{B}_A, \mathcal{B}_B$ of betting sets (depicted in Table 1 for $n = 3$), Player A can find their best possible response $\mathcal{B}_A$ given $\mathcal{B}_B$.

1. **Usage**: Suppose Player B chooses $\mathcal{B}_B = \{2, 3\}$. Looking at the corresponding row, we see that $\mathcal{B}_A = \{3\}$ maximizes Player A's payout.
2. **Player B Counter-strategy**: As Player B aims to minimize Player A's profit, they would change to $\mathcal{B}_B = \{1, 3\}$.
3. **Player A Counter-counter-strategy**: Player A can then change to $\mathcal{B}_A = \{1, 2, 3\}$ in order to maximize their payout.
4. **Intransitivity**: Player B then chooses $\mathcal{B}_B = \{2, 3\}$, after which player A chooses $\mathcal{B}_A = \{3\}$, returning us to our starting point. This shows that the game is intransitive, as there exists a strategy cycle.

$\mathcal{B}_B \backslash \mathcal{B}_A$	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
$\emptyset$	0	4	2	0	6	4	2	6
$\{1\}$	-4	0	1	-1	5	3	3	8
$\{2\}$	-2	-1	0	1	1	2	3	4
$\{3\}$	0	1	-1	0	1	-1	-1	0
$\{1, 2\}$	-6	-5	-1	0	0	1	5	6
$\{1, 3\}$	-4	-3	-2	-1	-1	0	1	2
$\{2, 3\}$	-2	-4	-3	1	-5	-1	0	-2
$\{1, 2, 3\}$	-6	-8	-4	0	-6	-2	2	0

Table 1: $\mathbb{E}_A$ for betting sets $\mathcal{B}_A, \mathcal{B}_B$ when $n = 3$.

Nash Equilibria & Dominance

As part of our game theoretic approach to the game, we search for the following:

- A **Nash Equlibrium**, where neither player has an incentive to change strategies in response to their opponent's current strategy.
- A **Dominant Strategy**, where one player always achieves their maximum payout by choosing a particular strategy.

We do this by laying the following groundwork:

- **Defining General Strategies**: A player's most general strategy can be expressed as a vector of probabilities,

$$\vec{p} = (p_1, p_2, p_3, \ldots, p_n),$$

where Player A bets on card c_i with probability p_i.

- **Applying Nash's Theorem**: By *Nash's Theorem*, any game with a finite number of players and a finite number of pure strategies has at least one Nash Equilibrium.

Consider the case $n = 3$. We proved that, among the aforementioned General Strategies, the unique Nash Equilibrium occurs when both players adopt the strategy

$$(p_1, p_2, p_3) = (1/3, 1/3, 1).$$

Additionally, we proved that there is **no dominant strategy** for the discrete McAdams Model.

Future Directions

The results of our work show promise, and we hope to move forward in the following sectors:

- **Deterministic Nash Equilibria**: We know, through computation, that there exists no deterministic Nash Equilibria for $n \leq 10$. Does there exist any deterministic Nash Equilibria for $n > 10$?
- **General Nash Equilibria**: Is there a pattern we can extrapolate to find General Strategy Nash Equilibria for any arbitrary n?

Links

https://github.com/aryan-cs/poker-like-games/tree/discrete-poker

https://aryan-cs.github.io/poker-like-games