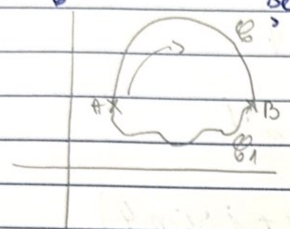
CURS4-MSI

$f \in \mathcal{H}(D)$, $\exists F$ cu prop: $F'(z) = f(z)$

Dem

$\int_{\mathcal{C}} f(z)\,dz = 0 \leftarrow$ am dem. deja $\Rightarrow z = z(t),\ t\in(a,b)$

at $\int_{\mathcal{C}} f(z)\,dz$ nu depinde decât de originea $\mathcal{C}$
și de extremitate



$$\int_{\mathcal{C}} f(z)\,dz = -\int_{\underline{\mathcal{C}}} f(\bar{z})\,dz$$

$\underline{\mathcal{C}}$ $z = z(t)$ $t\in(a,b)$ $z(a)$ ar $\mathcal{C}$ $z(b)$ extr

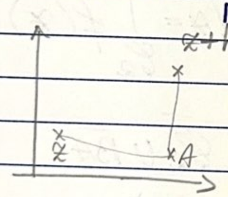
$\bar{\mathcal{C}}$ $z = z(a+b-t)$, $t\in(a,b)$

$$\int_{\mathcal{C}} f(z)\,dz = -\int_a f(z(a+b-\overset{()'=-1}{t}))\cdot dt \quad \overset{a+b-t=t}{=}$$

$$= +\int_b^a f(z(t))\,\overset{z'(t)}{dt} = -\int_a^b f(z(t))\cdot z'(t)\,dt$$

$\mathcal{C}$ a->b
$\bar{\mathcal{C}}$ b->a
Gamma

$$\Gamma = \mathcal{C}\cup\bar{\mathcal{C}}_1 \Rightarrow 0 = \int_\Gamma f(z)\,dz = \int_{\mathcal{C}} f(z)\,dz - \int_{\mathcal{C}_1} f(z)\,dz.$$



$$\Rightarrow F(z) = \int_{\mathcal{C}} f(\cancel{z})\,d\cancel{z}w$$

$$F(z+h) - F(z) = \int_{Az+h} f(w)\,dw - \int_{Az} f(w)\,dw$$

$$\Rightarrow \int_{z\,z+h} f(w)\,dw \quad \left(\begin{array}{l} w(t) = (1-t)\cdot z + t\cdot(z+h) \\ t\in(0,1)\end{array}\right) =$$

$$= h\int_0^1 f[(1-t)\cdot z + t(z+h)]\,dt$$

$$\frac{F(z+h) - F(z)}{h} = \int_0^1 f[(1-t)\cdot z + t(z+h)]\,dt$$

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = \int_0^1 \lim_{h\to 0} f[(1-t)z + t(z+h)]\,dt$$

H - olomorfă

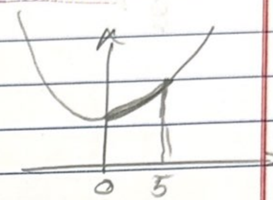
$$-\int_0^1 f(z)\,dt = f(z)\int_0^1 dt = f(z)$$

$$F'(z) = f(z)$$

$5t+j(t^2+4)$

$$\int_{\mathcal{C}} e^z\,dz \;;\quad \mathcal{C}: z = 5t+j(t^2+4),\ t\in(0,1)$$

$$x = 5t,\ y = t^2+4 \qquad y = \frac{x^2}{25}+4$$



$$\int_{\mathcal{C}} e^z\,dz = e^{z(1)} - e^{z(0)} = e^{5+5j} - e^{4j}$$

$$= e^5\cdot(\cos 5 + j\sin 5) - (\cos 4 + j\sin 4)$$

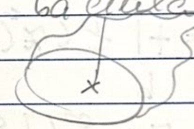
$$\int_{\mathcal{C}} \frac{1}{(z+1)^2}\,dz = -\frac{1}{z(1)+1} + \frac{1}{z(0)+1} = -\frac{1}{6+5j} + \frac{1}{1+4j}$$

$\boxed{T}$ (Cauchy) , închisă & parcursă în sens direct

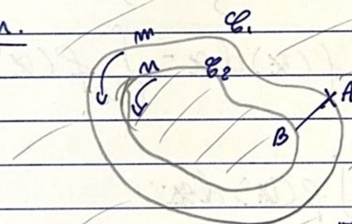
$f\in \mathcal{H}(D)$, $(\mathcal{C})\subset D$, $a\in Int(\mathcal{C})$:

$$f(a) = \frac{1}{2\pi j}\int_{\mathcal{C}} \frac{f(z)}{z-a}\,dz.$$

Căciula



dem.



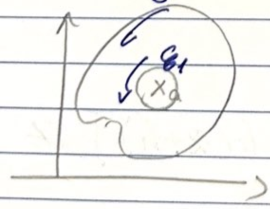
$$\int_{\mathcal{C}_1} f(z)\,dz = \int_{\mathcal{C}_2} f(z)\,dz.$$

$$\Gamma = \overset{\frown}{AmA} \cup \overline{AB} \cup \overset{(BmB)}{\mathcal{C}_2} \cup \overline{BA}$$

$$\Rightarrow \int_{\Gamma} f(z)\,dz = 0 \;;\quad 0 = \int_{\mathcal{C}_1} f(z)\,dz + \cancel{\int_{\overline{AB}} f(z)\,dz} - \int_{\mathcal{C}_2} f(z)\,dz \cancel{- \int_{\overline{AB}} f(z)\,dz}$$

$$\Rightarrow \int_{\mathcal{C}_1} f(z)\,dz = \int_{\mathcal{C}_2} f(z)\,dz$$

$$g(z) = \frac{f(z)}{z-a}$$



$$\Rightarrow \int_{\mathcal{C}} \frac{f(z)}{z-a}\,dz = \int_{|z-a|=r} \frac{f(z)}{z-a}\,dz = \frac{1}{2\pi j}\cdot f$$

$|z-a| = r \rightarrow$ cerc

$$\int_{|z-a|=r} \frac{1}{z-a}\,dz = 2\pi j \Rightarrow 1 = \frac{1}{2\pi j}\cdot \int_{|z-a|=r} \frac{dz}{|z-a|}$$

$$f(a)=\frac{1}{2\pi j}\int_{|z-a|=r}\frac{f(a)}{z-a}\,dz$$

$$\frac{1}{2\pi j}\int_{|z-a|=r}\frac{f(z)}{z-a}\,dz-\frac{1}{2\pi j}\cdot f(a)=\frac{1}{2\pi j}\int_{|z-a|r}\frac{f(z)-f(a)}{z-a}\,dz.=$$

$$=\frac{1}{2\pi j}\int_0^{2\pi}\frac{f(a+re^{jt})-f(a)}{re^{jt}}\cdot rje^{jt}\,dt$$

$\hookrightarrow 0$ când $r\to 0$

$$\Rightarrow f(a)=\frac{1}{2\pi j}\int_{\mathcal{C}}\frac{f(z)}{z-a}\,dz.$$

$\Rightarrow$ o f olomorfă are deriv de $\forall$ ordin

$$\left(\frac{1}{z+a}\right)^{(n)}_a \Rightarrow\left(\frac{1}{\alpha z+\beta}\right)^{(n)}=\frac{(-1)^n\alpha^n n!}{(\alpha z+\beta)^{n+1}}$$ ← mem

$$\left((\alpha z+\beta)^{-1}\right)^{(n)}$$

$$\Rightarrow\left(\frac{1}{z+a}\right)^n_a=\frac{(-1)^n(-1)^n n!}{(z-a)^{n+1}}$$

I (Cauchy pt. derivate)

Dc f e olomorfă în dom. mărg de $\mathcal{C}$ at f are derivate de $\forall$ ordin în int curb $\mathcal{C}$ și are loc formula.

$$\boxed{f(z)=\frac{n!}{2\pi j}\int_{\mathcal{C}}\frac{f(w)}{(w-z)^{n+1}}\,dw}$$

$f(x)=x^2,\ x\in[-1,0)$
$\quad\ \ x^3,\ x\in[0,\infty)$

$$f'(x)\begin{cases}2x,& x\in[-1,0)\\ 3x^2,& x\in[0,\infty\end{cases}$$

$\nexists f''(z) \Rightarrow 2\neq 6x$

$$f(x)=\begin{cases}e^{-\frac{1}{x^2}} & x\neq 0\\ 0 & x=0\end{cases}\qquad f^{(n)}(0)=0\ \forall n\in\mathbb{N}$$

$$f(x)=\sum_{n=0}^{\infty}x^n\frac{f^{(n)}(0)}{n!}=0$$

$\hookrightarrow$ nu

$$f(a) = \frac{1}{2\pi j} \int_{|z-a|=r} \frac{f(a)}{z-a}\, dz$$

$$\frac{1}{2\pi j} \int_{|z-a|=r} \frac{f(z)}{z-a}\, dz - \frac{1}{2\pi j}\cdot f(a) = \frac{1}{2\pi j} \int_{|z-a|=r} \frac{f(z)-f(a)}{z-a}\, dz = \qquad z-a = re^{jt}$$

$$= \frac{1}{2\pi j} \int_0^{2\pi} \frac{f(a+re^{jt})-f(a)}{re^{jt}} \cdot rje^{jt}\, dt$$

$\hookrightarrow 0$ când $r \to 0$

$$\Rightarrow f(a) = \frac{1}{2\pi j} \int_{\mathcal{C}} \frac{f(z)}{z-a}\, dz.$$

$\Rightarrow$ o f olomorfă are deriv de $\forall$ ordin

$$\left(\frac{1}{z+a}\right)^{(n)}_a \Rightarrow \left(\frac{1}{\alpha z+\beta}\right)^{(n)} = \frac{(-1)^n \alpha^n n!}{(\alpha z+\beta)^{n+1}}$$ ← memorează !

$$\left((\alpha z+\beta)^{-1}\right)^{(n)}$$

$$\Rightarrow \left(\frac{1}{z+a}\right)^{n}_a = \frac{(-1)^n (-1)^n n!}{(z-a)^{n+1}}$$

I (Cauchy pt. derivate)

De f e olomorfă în dom. mărg at $\qquad$ deci $\mathcal{C}$
f are derivate de $\forall$ ordin în int curb $\mathcal{C}$ și are loc formula.

$$\boxed{f(z) = \frac{n!}{2\pi j} \int_{\mathcal{C}} \frac{f(w)}{(w-z)^{n+1}}\, dw}$$

Γ $f(x) = \begin{cases} x^2, & x \in [-1, 0) \\ x^3, & x \in [0, \infty) \end{cases}$

$f'(x) = \begin{cases} 2x, & x \in [-1, 0) \\ 3x^2, & x \in [0, \infty) \end{cases}$

$\nexists f''(x) \Rightarrow 2 \neq 6x$

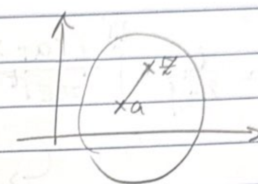
$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad f^{(n)}(0) = 0\ \forall n \in \mathbb{N}$

$$f(x) = \sum_{n=0}^{\infty} x^n \frac{f^{(n)}(0)}{n!} = 0$$

$\hookrightarrow$

$\boxed{T}$ Dacă f e olom în int cerc $|z-a|=R$, $R>0$
f e analitică, adică:

- $f(z) = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} f^{(n)}(a)$



dem

$$f(z) = \frac{1}{2\pi j} \int_{|z-a|=R} \frac{f(w)}{w-z}\, dw \qquad |z-a|<R$$

$$\frac{1}{w-z} = \frac{1}{w-a-(z-a)} = \frac{1}{(w-a)\left(1-\frac{z-a}{w-a}\right)}$$

$$\left|\frac{z-a}{w-a}\right| < 1 \quad \forall\, w \text{ pe cerc} \quad ; \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n,\ z<1$$

$$\int_{|z-a|=R} f(w) \Big| \quad \frac{1}{w-z} = \frac{1}{w-a} \sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^n} = \sum_{n=0}^{\infty} (z-a)^n \cdot \frac{1}{(w-a)^{n+1}}$$

$$\frac{1}{2\pi j} \int_{|w-a|=R} \frac{f(w)}{w-z} = \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} \underbrace{\frac{n!}{2\pi j} \int_{|w-a|=r} \frac{f(w)}{(w-a)^{n+1}}}_{}$$

-> deriv de ordin n în punct

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{n!} \cdot f^{(n)}(a)$$

ex. • Serie Taylor: $z_0 = 0$, $f(z) = \cos z \,\mathrm{ch}\, z$.

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

$$f(z) = \cos z \cdot \cos(2jz) = \frac{1}{2}\left(\cos z(1+2j) + \cos z(1-2j)\right)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n \cdot z^{2n} \left((1+2j)^{2n} + (1-2j)^{2n} \right)}{(2n)!} \qquad (\sqrt{5})^{2n} \cos(2n \cdot \mathrm{arctg}\, 2)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n} \cdot 2\,\mathrm{Re}(1+2j)^{2n}}{(2n)!}$$

$$f^{(201)}(0) = 0$$

$$f^{(200)}(0) \Rightarrow \frac{f^{(200)}(0)}{(200)!} = \frac{\mathrm{Re}(1+2j)^{200}}{(200)!}$$

T (Liouville) $f \in \mathcal{H}(\mathbb{C})$, $\exists M>0$, a.i. $|f(z)| \le M \ \forall z \in \mathbb{C} \Rightarrow$ f e constant

Dem. $f'(z) = \frac{1}{2\pi j} \int_{|w-z|=R} \frac{f(w)}{(w-z)^2} dw = \int_0^{2\pi} \frac{f(Re^{jt}+z)}{R^2 e^{2jt}} \cdot Rje^{jt} dt$

$z \in \mathbb{C}$ fixat $\nearrow$ $|w-z| = R$

$w - z = Re^{jt}$

$$|f'(z)| = \frac{1}{2\pi} \frac{1}{R} \cdot \left| \int_0^{2\pi} f(re^{jt}+z)\, dt \right|$$

$$\le \frac{1}{2\pi R} \int_0^{2\pi} |f(Re^{jt}+z)| dt \le \frac{1}{2\pi R} \int_0^{2\pi} M\, dt$$

$$= \frac{1}{2\pi R} 2\pi M = \frac{M}{R}. \quad \forall R>0$$

$$\Rightarrow |f'(z)| \le \frac{M}{R}$$

Π $R \to \infty \Rightarrow |f'(z)| \le 0 \Rightarrow |f'(z)| = 0 \Rightarrow f'(z) = 0$

$\Rightarrow f(z) =$ constantă