MSI - SEMINAR 2.

3a) Să se det f ~~ixom~~ olomorfe $f = u + iv$ știind că

$u(x,y) = x^2 + axy - y^2$

$f(j) = 1 + 5j$

$\Rightarrow u, v$ - armonice

$\Rightarrow \Delta u = \Delta v = 0$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x + a \qquad \frac{\partial u}{\partial y} = ax - 2y$$

$$\frac{\partial^2 u}{\partial x} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2 \quad \Big\} \Rightarrow \Delta u = 0 \ \forall a$$

Dacă f e olomorfă am dem. că tre satisfăcute cond lui Ri. Cauchy și Rieman

$$C\text{-}R: \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

$$\Rightarrow \frac{\partial v}{\partial y} = 2x + ay.$$

$$\frac{\partial v}{\partial x} = 2y - ax$$

$$\Rightarrow dv = (2y - ax) \cdot dx + (2x + ay)dy.$$

$\frac{a}{2}x^2$

$\frac{a}{2}y^2$

$2xy$

$$v(x,y) = 2xy - \frac{a}{2}x^2 + \mathscr{C}(y)$$

$$\frac{dv'}{dy} = 2x - 0 + \mathscr{C}_y'(y) = 2x + ay.$$

$$\Rightarrow \mathscr{C}_y'(y) = ay \Rightarrow \mathscr{C}(y) = \frac{a^2}{2}y.$$

$$\Rightarrow v = 2xy - \frac{ax^2}{2} + \frac{ay^2}{2} = \frac{a}{2}(y^2 - x^2) + 2xy + c.$$

$$f(z) = x^2 + axy - y^2 + j\left(\frac{a^2}{2}(y^2 - x^2) + 2xy\right) + cj$$

$$f(z) = \left(\frac{z+\bar{z}}{2}\right)^2 + a \cdot \frac{z^2 - \bar{z}^2}{4j} - \left(\frac{z - \bar{z}}{2j}\right)^2 +$$

$$j\left(\frac{z^2 - \bar{z}^2}{2j} + \frac{a}{2}\left(\frac{(z-\bar{z})^2}{-4} - \frac{(z+\bar{z})^2}{4}\right)\right) + jc$$

$\bar{z} = x - jy.$

$\left.\begin{array}{l}\frac{\partial u}{\partial x} = 1 \\ \frac{\partial v}{\partial y} = -1\end{array}\right\} \Rightarrow$ nu e olomorfă, nu tre să avem $\bar{z}$

$$f(z) = \frac{z^2}{4} - a\frac{z^2}{4}j + \frac{z^2}{4} + \frac{z^2}{2} + \frac{a}{2}\frac{z^2}{4} - \frac{a}{2}\frac{z^2}{4} + jc$$

$$f(z) = \frac{z^2}{4}\left(1 - aj + 1 + 2 - a\right) + jc.$$

$$f(j) = \frac{-1}{4}\left(4 - a - aj\right) + jc = 1 + 5j$$

$\Rightarrow \frac{a-4}{4} = 1. \Rightarrow a = 8$

$\frac{a+4c}{4} = 5 \quad 4c = 20 - 8 \Rightarrow c = 3.$

2) Să se det toate f. olomorfe pe $\mathbb{C}$ pt care $|f(z)| = 1.$

$f(z) = u(x,y) + j\,v(x,y)$

$|f(z)| = 1 = \sqrt{u^2 + v^2} \Rightarrow u^2 + v^2 = 1. \; /()'_x$

Cond C.R.

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \end{cases}$$

$\Rightarrow 2\left(u(x,y)\frac{\partial u}{\partial x}\right) + 2v(xy)\frac{\partial v}{\partial y} = 0 \quad /2$

$()'_y \Rightarrow 2u(x,y)\frac{\partial u}{\partial y} + 2v(x,y)\frac{\partial v}{\partial x} = 0 \quad /2$

$$u(x,y)\frac{\partial u}{\partial x} - v(x,y)\frac{\partial u}{\partial y} = 0$$

$$u(x,y)\frac{\partial u}{\partial y} + v(x,y)\frac{\partial u}{\partial x} = 0$$

$$\begin{vmatrix} u(x,y) & -v(x,y) \\ v(x,y) & u(x,y) \end{vmatrix} = u^2(x,y) + v^2(x,y)$$

I $Det = 0 \Rightarrow u = v = 0 \Rightarrow f(z) = 0 \Rightarrow |f(z)| = 0$ fals

II $Det \neq 0 \quad u^2 + v^2 > 0$

$\Rightarrow$ sist lin omogen $\Rightarrow$ sist are sol unică τ aia 0

$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$

$\Rightarrow du = (0) \Rightarrow u(x,y) = c_1$

$\Rightarrow dv = 0 \Rightarrow v(x,y) = c_2.$

$\Rightarrow f(z) = c_1 + jc_2$ (const complexă

$\left.\begin{array}{l} k \in \mathbb{C} \\ \text{Ip.} \end{array}\right\} \Rightarrow |k| = 1.$

$\Rightarrow f(z) = e^{j\theta}$

3) Să se det toate f. olomorfe

$f(z) = u(x,y) + i\,v(x,y)$ $\Rightarrow$ u, v = armonice

$v(x,y) = \varphi\left(\frac{x}{y}\right)$

$\varphi \in C^\infty \mathbb{R}$

$\Delta u = 0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

$\frac{\partial u}{\partial x} = \varphi'\left(\frac{x}{y}\right)\frac{1}{y}$

$\frac{\partial^2 u}{\partial x^2} = \varphi''\left(\frac{x}{y}\right)\frac{1}{y^2}$

$\frac{\partial u}{\partial y} = \varphi'\left(\frac{x}{y}\right)\cdot\frac{-x}{y^2}$

$\frac{\partial u}{\partial y} = \varphi''\frac{x^2}{y^4} + \varphi'\left(\frac{x}{y}\right)\cdot\frac{2x}{y^{3}}$

$\Delta u = \frac{1}{y^2} \cdot \varphi''\left(\frac{x}{y}\right) + \varphi''\left(\frac{x}{y}\right)\frac{x^2}{y^4} + \varphi'\left(\frac{x}{y}\right) \cdot \frac{2x}{y^3}$

$\Delta u = \varphi''\left(\frac{x}{y}\right)\left(\frac{1}{y^2} + \frac{x^2}{y^4}\right) + \varphi'\left(\frac{x}{y}\right) \cdot \frac{2x}{y^3} = 0 / \cdot y^2$

↳ ec. diferențiale

$\Delta u = \varphi''\left(\frac{x}{y}\right)\left(1 + \left(\frac{x}{y}\right)^2\right) + \varphi'\left(\frac{x}{y}\right) \cdot \frac{2x}{y} = 0$

Not $\frac{x}{y} = t \Rightarrow u = \varphi''(t)\,(1+t)^2 + 2\varphi'\left(\frac{x}{y}\right)t = 0$

$(1+t^2) = 2t \Rightarrow t^2 - 2t + 1 = 0 \Rightarrow (t-1) = 0$

$\Rightarrow t = 1.$

$\left((1+t) \cdot \varphi'(t)\right)' = 0 \Rightarrow (1+t^2) \cdot \varphi'(t) = C_1.$

$\Rightarrow \varphi'(t) = \frac{c_1}{1+t^2} = \varphi(t) = c_1 \operatorname{arctg} t \Rightarrow$

$u(x,y) = \Rightarrow c = \frac{c_1}{c_2} \cdot \operatorname{arctg} \frac{y}{x} + c_2.$

$\partial v = \frac{\partial v}{\partial x} \cdot dx + \frac{\partial v}{\partial y} \cdot dy \overset{R.C.}{=}$

$= -\frac{x}{y^2} - \cancel{\frac{c_1}{c_2} \operatorname{arctg}}\; c_1\left(\frac{1}{x^2+y^2}\right) \cdot dx \;\; \frac{\partial v}{\partial y}\, \partial y$

$\Rightarrow \cancel{+c\frac{t}{y}} \cdot \varphi \;\; \frac{1}{y} \cdot C\left(\frac{1}{1+\frac{x^2}{y^2}}\right)$

$\Rightarrow dv = -\frac{x\, c_1}{x^2 y^2} + x^2 + y^2 \cdot dx + \frac{y\, c_1}{x^2 \cdot y^2} \cdot dy.$

$v(x, y) = \int \frac{+c_1}{x^2 + y^2} + x^2 + y^2\, dx$

$v(x,y) = +C_1 \, \frac{1}{2} \ln(x^2 + y^2) + \mathscr{C}(y)$

$v'_y(y) = \frac{+c_1 y}{x^2 y^2} + \mathscr{C}'(y)$

$v'_y(y) = y^2\, dy + \cancel{dy} \;\; \frac{y^2}{x^2+y^2} - \cancel{\frac{y^2}{x^2 y^2}} = 0$

$\Rightarrow C_3$

$\left.\begin{array}{l} u(x,y) \\ v(x,y,y) \end{array}\right\} \Rightarrow f(z) = c_1 \operatorname{arctg} \frac{x}{y} + c_2 + j \cdot \frac{c_1}{c_2} \cdot \ln (x^2 + y^2) + c_3$

4) a) Dem $|\cos z| \le ch|z|$

b) $\frac{2e^{-xy}}{1+e^{-xy}} \le |\,tg\, z - j| \le \frac{2e^{-xy}}{1e^{-xy}}$

$$\boxed{ch\, x = \frac{e^{x}-e^{-x}}{2}}$$

$$|\cos z| = \left|\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}\right| \le \sum_{n=0}^{\infty} \left|(-1)^n \frac{z^2}{(2n)!}\right|$$

$$= \sum_{n=0}^{\infty} \frac{|z|^{2n}}{(2n)!} = ch\,|z|$$

$$ch\, x = \frac{e^{x}+e^{-x}}{2} = \frac{1}{2}\left(\sum_{n=0}^{\infty} \frac{x^n}{n!} + \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}\right)$$

$$ch = \frac{\cancel{1}}{\cancel{2}} \sum_{n=0}^{\infty} \cancel{2} \frac{z^{2n}}{(2n)!}$$

$$\boxed{\cos j\cdot z = ch\, z} \qquad \cos z = \frac{e^{jz}+e^{-jz}}{2}$$

Dem.

$$\cos jz = \frac{e^{j\cdot jz}+e^{-jz}}{2} = \frac{e^{z}+e^{-z}}{2} = ch\, z$$

b) $\frac{2e^{-xy}}{1+e^{-xy}} \le |\,tg\, z - j\,| \le \frac{2e^{-xy}}{1-e^{-xy}}$ $\quad z = x+jy.$

$$|\,tg\, z - j| = \left|\frac{\sin z}{\cos z} - j\right| = \left|\frac{j(e^{jz}-e^{-jz})}{e^{jz}+e^{-jz}} - j\right|$$

$$= \left|\frac{\sin z - j\cos z}{\cos z}\right| = \frac{|\sin z - j\cos z|}{|\cos z|}$$

$$= j\,\frac{|-j\sin z - \cos z|}{|\cos z|} =$$

$$= \frac{|\cos z + j\sin z|}{|\cos z|} = \frac{|e^{j\cdot z}|}{|\cos z|} = \frac{|e^{-y}|}{|\cos z|}$$

$z = x+jy \Rightarrow jz = jx - y.$

$|\cos z| = |\cos(x + jy)| = |\cos x \cos jy \cancel{+} \sin x \cdot \sin jy$

$= |\cos x \cdot ch\, y - \sin x \cdot j\, sh\, y)$

$\sin(jy) = \dfrac{e^{\cancel{-j}y} - e^{jy}}{2j} = j\dfrac{(e^{y} - e^{-y})}{2} = j\, sh\, y$

$sh\, \cancel{x} = \dfrac{e^{x} - e^{-x}}{2}$

$= \sqrt{\cos^2 x\, ch^2 y + \sin^2 x \cdot sh^2 y}$ => restu