Aerodynamics II summary: 2017-2018 edition

Based on *Fundamentals of Aerodynamics, Fifth Edition* by John D. Anderson, Jr.

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Contents

Preface

Please note that this summary contains a lot more derivations than my summaries typically do (which makes it a fair bit longer than I would have wanted). However, I've included them because they regularly literally just ask you to derive an equation (especially for derivations in the first two chapters). Hence, it's advisable to actually read through the derivations I've included in the summary for they might ask one of those, and otherwise you at least get a better feeling for how to derive stuff, which is quite a handy skill to have[1]. The "end-result" of the derivations (which are still way more important than derivations themselves) are denoted in red boxes, so those red boxes are the especially important formulas.

Furthermore, the lecturer waits a bit until he publishes some old exams (iirc until just before the Christmas break), and once he does, he doesn't publish official solutions for most of them (at least not for all of them). Fortunately, I made extensive solutions for all the exams for which there are no official solutions so that should help a fair bit and I'll upload them as soon as the lecturer publishes the old exams.

Finally, some study-advice:

- First of all, print the formula sheet as soon as you start studying. Honestly, I never print stuff (not even my own summaries), but this formula sheet is honestly worth so much as there are also some graphs on there that you need like 50% of the time. It saves you so much time compared to having to look up the graphs in the book (or in the pdf on your laptop).
- Secondly, studying for this exam simply takes a lot of time. In my view, the best way to study for the exam is to just take a few days of your Christmas break to carefully read through the entire summary, and not bother with the old exams yet. Really read the summary carefully, and try to make sure you everything, and do all of the examples I included (*all* of them). This will give you a nice solid understanding of the course material, and after the Christmas break you can start doing old exams (more or less after the third Structural Analysis partial exam/final presentation for project). The first old exam you try will be really hard unfortunately (and will probably take a full day to do), but once you've done two or three old exams you'll start getting the hang of it and it'll go a bit faster. Nonetheless, even when you master the course material, a single exam will take you at least 2.5 hours, even if you're a fast worker normally. They're just really, really long (although fortunately, you guys already have my solution manuals for them, so you won't have to spend fucking hours trying to come up with the correct solution).
- Thirdly, regarding how hard each chapter is:
  - Chapter 7 isn't that hard, it's mostly just a recap of various things you've already seen before (it touches upon quite some stuff though, so it may seem to go a bit fast). Most important is that you understand the notion of total conditions, i.e. section 7.5, as we'll use that a lot.
  - Chapter 8 contains a number of derivations that can be asked on the exam. However, it is most important that you make sure you understand the notion of critical conditions (and the fact that there are formulas on the formula sheet that you can use). If you get hopeless from the amount of mathematics involved, there are a bunch of examples towards the end of the chapter which will clarify a lot, so don't get freaked out by the mathematics[2].
  - Chapter 9 is like chapter 8, except that now the shock waves are at angle. This alters the calculations slightly, but the idea behind it largely remains the same (again, do the examples I included, that should clarify a lot). It's a lot of information to process, and chapters 8 and 9 will probably take you the longest of all chapters (probably takes a couple of days to read through all of it).
  - Chapter 10 is like one concept you really have to understand well. However, although it is just one concept, it's deceptively difficult in the sense that you'll really often have, 'oh I understand, no wait I don't, wait I do, no fuck how was it', etc. etc. Again, do the examples I included to get some clarification.
  - Chapters 11 and 12 are pretty easy, as long as you ignore the mathematical derivations (which you

[1] Although I should add: don't make it the main focus when you're studying chapters 7 and 8 (9 - 12 don't really contain many derivations you need to know). Being able to derive stuff is like what makes the difference between an 9 and a 10 (i.e. they are the hardest questions on the exams imo).

[2] In fact, you'll barely ever use the formulas in the chapter; instead you'll use tables that list the results of the calculations.

are allowed to ignore). I didn't include any examples on it, but once you start doing old exams, you'll note that it's pretty much the same everything.

All in all, don't get demotivated by chapter 9 (and to a lesser extent chapter 10); chapters 11 and 12 are a breeze compared to those so you have something to look forward to.

- Fourthly, take part in the high-speed wind tunnel experiment. There are so many good things about it. First, it's a good motivation to start studying early (because you'll need to have studied chapters 7-10 before doing the practical), and then it only takes one afternoon+evening to prepare for the practical[3]. It's really worth it for a full bonus point on the exam, imo. It's also quite fun to do, and you can choose your own group of three people so you can just do it with friends. Just make sure you have one person in there who can program in either Python or Matlab and you're set.

[3] Do prepare! Preferably, you'll have the entire report finished before you actually do the practical itself, so that you only have to import the data generated during your experiment and then you're finished. Two hours to write the report on the spot is really little and if you don't prepare you simply won't get any bonus points at all.

7 Compressible Flow: Some Preliminary Aspects

7.2 A brief review of thermodynamics

7.2.1 Equation of state

FORMULAS: EQUATION OF STATE

$$p = \rho R T \tag{7.1}$$
$$p v = R T \tag{7.2}$$

where p is the pressure, ρ the density, R the specific gas constant, which for air equals 287 J/kg/K, T the temperature and v the specific volume; $v = 1/\rho$.

Do note that v is also used as y-component of the velocity, but usually this shouldn't cause confusion.

7.2.2 Internal energy and enthalpy

FORMULA: ENTHALPY

$$h = e + pv \tag{7.3}$$

where h is the specific enthalpy and e the specific internal energy.

You can see the enthalpy as the energy necessary to give a bunch of molecules the internal energy it has, plus the energy necessary to create room for the molecules (as pv seems awfully similar to work). Furthermore, we had the following relations:

FORMULAS: SPECIFIC HEATS

For a perfect gas:

$$de = c_v dT \tag{7.4}$$
$$dh = c_p dT \tag{7.5}$$

where c_v and c_p are the specific heats at constant volume and constant pressure, respectively. For a **calorically perfect gas**, these are constant and

$$e = c_v T \tag{7.6}$$
$$h = c_p T \tag{7.7}$$

In this book, we only deal with calorically perfect gases.

Furthermore, we have the following nice relationships:

FORMULAS

$$\gamma \equiv \frac{c_p}{c_v} \tag{7.8}$$
$$c_p - c_v = R \tag{7.9}$$

Where $c_p \approx 1008$ J/kg/K, $c_v \approx 720$ J/kg/K, $\gamma = 1.4$ and $R = 287$ J/kg/K for dry air. Note that it is helpful to know these numbers by heart. From these two equations, two further relationships can be derived. First, expressing c_p as a function of γ and R:

$$\begin{aligned} c_p - c_v &= R \\ c_p - \frac{c_p}{\gamma} &= R \\ c_p\left(1 - \frac{1}{\gamma}\right) &= R \\ c_p &= \frac{R}{1 - \frac{1}{\gamma}} \\ c_p &= \frac{\gamma R}{\gamma - 1} \end{aligned}$$

Furthermore, as $c_v = \frac{c_p}{\gamma}$,

$$c_v = \frac{R}{\gamma - 1}$$

FORMULAS

$$c_p = \frac{\gamma R}{\gamma - 1} \tag{7.10}$$

$$c_v = \frac{R}{\gamma - 1} \tag{7.11}$$

7.2.3 *First law of thermodynamics*

FORMULA: FIRST LAW OF THERMODYNAMICS

$$\delta q + \delta w = de \tag{7.12}$$

Where δq is a small *amount* of heat, δw a small *amount* of work and de a small *change* in internal energy.

There are a lot of ways in which heat can be added and work done on the system. We focus on three types of processes:

THEOREM: IMPORTANT THERMODYNAMIC PROCESSES

1. **Adiabatic process**: no heat is added or taken away from the system.
2. **Reversible process**: no dissipative phenomena occur, i.e. no viscosity, thermal conductivity and mass diffusion.
3. **Isentropic process**: both adiabatic and reversible.

For a reversible process, you can show that $\delta w = -p\,dv$, and hence the first law becomes

FORMULA

$$\delta q - p\,dv = de \tag{7.13}$$

7.2.4 *Entropy and the second law of thermodynamics*

You probably know that we are supposed to remember what entropy was, but if you don't remember it any more: it was a state variable (i.e. you can measure it at one specific point in time) which described the "randomness" of the molecular motion. Entropy can never be destroyed, but only created. We can compare this with a flywheel spinning at high rotational speeds in a cold box filled with air: naturally, due to friction (dissipative effects), the flywheel will stop spinning and the surrounding air will heat up. A higher temperature means that the molecules vibrate more, which is an unorganized motion. We compare this with the initial state, where the molecules in the flywheel were definitely moving, but it was all in a very organized matter; they all rotated in the same way, after all. We thus see that it is natural to go from an organized state to the unorganized state. The opposite way never happens: if you put a flywheel in a hot box of air, the flywheel won't suddenly start accelerating. So, entropy can only ever increase in an isolated system. Do note that you can take entropy out of a system of course, for example by taking heat away from the system; however, this entropy is then not destroyed, it's simply stored somewhere else. Now, we have that

FORMULAS: SECOND LAW OF THERMODYNAMICS

$$ds \geq \frac{\delta q}{T} \geq 0 \tag{7.14}$$

For reversible processes, $ds = \frac{\delta q}{T}$, for irreversible processes, a little extra entropy is added (namely the entropy that is 'irreversible'), hence then $ds > \frac{\delta q}{T}$. So, assuming an reversible process, equation (7.13) becomes

FORMULA

$$T\,ds = de + p\,dv \tag{7.15}$$

However, we also know that $h = e + pv$ and hence $dh = de + p\,dv + v\,dp$ (simply the chain rule), and hence $de + pdv = dh - vdp$:

FORMULA

$$T\,ds = dh - v\,dp \tag{7.16}$$

Now, we know that $de = c_v dT$ and $dh = c_p dT$. Hence, we have

$$\begin{aligned} T\,ds &= c_v dT + p\,dv \\ T\,ds &= c_p dT - v\,dp \end{aligned}$$

Which, by dividing by T, becomes

$$\begin{aligned} ds &= c_v \frac{dT}{T} + \frac{p\,dv}{T} \\ ds &= c_p \frac{dT}{T} - \frac{v\,dp}{T} \end{aligned}$$

However, with $pv = RT$, or $v/T = R/p$ into the last term of the second equation, we get

$$\begin{aligned} ds &= c_P \frac{dT}{T} - R\frac{dp}{p} \\ s_2 - s_1 &= \int_{T_1}^{T_2} c_p \frac{dT}{T} - \int_{p_1}^{p_2} R\frac{dp}{p} \\ s_2 - s_1 &= c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} \end{aligned}$$

The exact same procedure can be done for the upper equation, which results in

$$s_2 - s_1 = c_v \ln\frac{T_2}{T_1} + R\ln\frac{v_2}{v_1}$$

FORMULAS

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1} \tag{7.17}$$

$$s_2 - s_1 = c_v \ln\frac{T_2}{T_1} + R\ln\frac{v_2}{v_1} \tag{7.18}$$

7.2.5 Isentropic relations

For isentropic processes, $s_2 - s_1 = 0$ as the process is adiabatic, hence $\delta q = 0$. Hence, we can rewrite equations (7.17) and (7.18) a fair bit: first focussing on equation (7.17):

$$\begin{aligned} 0 &= c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} \\ \ln \frac{p_2}{p_1} &= \frac{c_p}{R} \ln \frac{T_2}{T_1} \\ \frac{p_2}{p_1} &= \left(\frac{T_2}{T_1}\right)^{c_p/R} \end{aligned}$$

However, as $c_p = \frac{\gamma R}{\gamma - 1}$, we have that $\frac{c_p}{R} = \frac{\gamma}{\gamma-1}$:

$$\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}}$$

Similarly, we can rewrite equation (7.18):

$$\begin{aligned} 0 &= c_v \ln \frac{T_2}{T_1} + R \ln \frac{v_2}{v_1} \\ \frac{v_2}{v_1} &= \left(\frac{T_2}{T_1}\right)^{-\frac{c_v}{R}} \end{aligned}$$

However, $v = \frac{1}{\rho}$ and $\frac{c_v}{R} = \frac{1}{\gamma-1}$, hence

$$\begin{aligned} \frac{\rho_1}{\rho_2} &= \left(\frac{T_2}{T_1}\right)^{-\frac{1}{\gamma-1}} \\ \frac{\rho_2}{\rho_1} &= \left(\frac{T_2}{T_1}\right)^{\frac{1}{\gamma-1}} \end{aligned}$$

Or:

FORMULA: ISENTROPIC RELATIONS

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} \tag{7.19}$$

7.3 Definition of compressibility

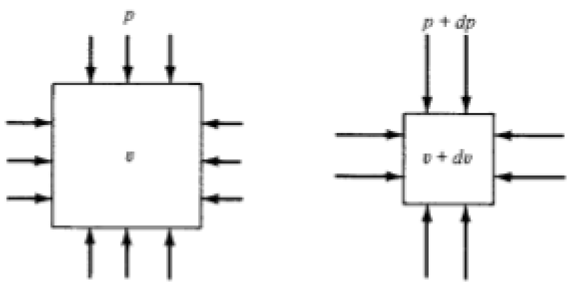


Figure 7.1: Definition of compressibility.

Look at figure 7.1. Naturally, if we increase the pressure by a small amount dp, then the specific volume will change by a small amount dv (dv will be negative if dp is positive, obviously). So, it makes sense to have our compressibility τ have something to do with $\frac{dv}{dp}$, i.e. the change in specific volume per unit change in pressure. Now, we need to normalize this by dividing by the specific volume itself[1]. Furthermore, we know that $\frac{dv}{dp}$ will always be negative, but people in aerodynamics are generally quite positive people and hence add a minus sign to always have positive compressibility:

FORMULA: COMPRESSIBILITY

$$\tau = -\frac{1}{v}\frac{dv}{dp} \tag{7.20}$$

Now, you can compress a gas in two ways: by keeping the temperature constant, and by not keeping the temperature constant. If you want to keep the temperature constant, you need to remove heat from the gas as the gas will heat up if the gas is compressed, i.e. the entropy will change. On the other hand, if you let the temperature just be what it wants to be, you don't remove any heat and the entropy will remain constant as long as you compress reversibly. Hence, we have two compressibilities:

FORMULA: COMPRESSIBILITY

The **isothermal compressibility** τ_T is defined as

$$\tau_T = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_T \tag{7.21}$$

where T indicates that the temperature is held constant.
The **isentropic compressibility** τ_s is defined as

$$\tau_s = -\frac{1}{v}\left(\frac{\partial v}{\partial p}\right)_s \tag{7.22}$$

where s indicates that the entropy is held constant.

These are thermodynamic properties which can be looked up somewhere. Now, sometimes we like to use density rather than specific volume because density is more straightforward. You can then derive that

FORMULA: COMPRESSIBILITY

$$\tau = \frac{1}{\rho}\frac{d\rho}{dp} \tag{7.23}$$

$$d\rho = \rho\tau dp \tag{7.24}$$

Finally,

FORMULA: MACH NUMBER

$$M \equiv \frac{V}{a} \tag{7.25}$$

7.4 Governing equations for inviscid, incompressible flow

For an inviscid, compressible flow we have:

[1] Because if some gas would have an enormous specific volume, then naturally the absolute change in specific volume per unit change in pressure would be much larger, but we're interested in the relative change, of course.

FORMULA: CONTINUITY, MOMENTUM AND ENERGY EQUATION

$$\underbrace{\frac{\partial}{\partial t}\oiiint_{\mathcal{V}} \rho d\mathcal{V}}_{\text{Time rate of decrease of mass inside CV}} + \underbrace{\oiint_{S} \rho \mathbf{V} d\mathbf{S}}_{\text{Net mass flow out of CV}} = 0 \tag{7.26}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0 \tag{7.27}$$

FORMULA: MOMENTUM EQUATION

$$\underbrace{\frac{\partial}{\partial t}\oiiint_{\mathcal{V}} \rho \mathbf{V} d\mathcal{V}}_{\text{Change of momentum of mass inside CV}} + \underbrace{\oiint_{S} (\rho \mathbf{V} \cdot d\mathbf{S})\mathbf{V}}_{\text{Momentum flow out of CV}} = \underbrace{-\oiint_{S} p d\mathbf{S}}_{\text{Pressure force over CV surface}} + \underbrace{\oiiint_{\mathcal{V}} \rho \mathbf{f} d\mathcal{V}}_{\text{Sum of body forces on CV}} \tag{7.28}$$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho f_x \tag{7.29}$$

$$\rho \frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho f_y \tag{7.30}$$

$$\rho \frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho f_z \tag{7.31}$$

FORMULA: ENERGY EQUATION

$$\begin{aligned}&\underbrace{\oiiint_{\mathcal{V}} \rho \dot{q} d\mathcal{V}}_{\text{Rate of heat added to CV}} - \underbrace{\oiint_{S} p d\mathbf{S} \cdot \mathbf{V}}_{\text{Rate of work on CV due to pressure}} + \underbrace{\oiiint_{V} \rho \mathbf{f} d\mathcal{V} \cdot \mathbf{V}}_{\text{Rate of work on CV due to body forces}} \\ &= \underbrace{\frac{\partial}{\partial t}\oiiint_{\mathcal{V}} \rho \left(e + \frac{V^2}{2}\right) d\mathcal{V}}_{\text{Rate of change of total energy inside CV}} + \underbrace{\oiint_{S} (\rho \mathbf{V} \cdot d\mathbf{S}) \left(e + \frac{V^2}{2}\right)}_{\text{Net rate of flow of total enery into CV}}\end{aligned} \tag{7.32}$$

$$\rho \frac{D\left(e + V^2/2\right)}{Dt} = \rho \dot{q} - \nabla \cdot p\mathbf{V} + \rho (\mathbf{f} \cdot \mathbf{V}) \tag{7.33}$$

Absolutely beautiful equations you probably won't end up using anyway. However, they become much more useful if we assume steady, adiabatic flow with no body forces. We can then analyse a streamtube depicted in figure 7.2. This is very similar in concept to a streamline, but now it's more like a collection of streamlines which together form a tube, of which the boundaries are streamlines. Hence, there is no mass flow across these boundaries, which significantly simplifies the integrals.

First, from the continuity equation, we now have

FORMULA: CONTINUITY EQUATION FOR STREAMTUBE

$$\rho V A = \dot{m} = \text{constant} \tag{7.34}$$

Which isn't totally brand new information. The momentum equation reduces to

FORMULA: MOMENTUM EQUATION FOR STREAMTUBE

$$A\left(p + \rho V^2\right) = \text{constant} \tag{7.35}$$

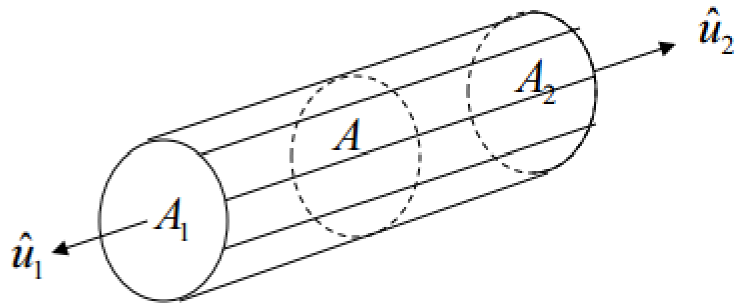


Figure 7.2: A streamtube.

Why? We now obviously have

$$\oiint_S (\rho \mathbf{V} \cdot d\mathbf{S})\, \mathbf{V} = -\oiint_S p\, d\mathbf{S}$$

Which can be reduced to (as there is no flow crossing the cylindrical boundaries (only the base areas):

$$\begin{aligned} \int_{A_1} (\rho \mathbf{V} \cdot d\mathbf{S})\, \mathbf{V} + \int_{A_2} (\rho \mathbf{V} \cdot d\mathbf{S})\, \mathbf{V} &= -\int_{A_1} p\, d\mathbf{S} - \int_{A_2} p\, d\mathbf{S} \\ -\rho_1 A_1 V_1^2 + \rho_2 A_2 V_2^2 &= p_1 A_1 - p_2 A_2 \end{aligned}$$

Confused by how the signs work for the integrals? Remember that $d\mathbf{S}$ is pointing *outwards*; hence, for A_1, it points in the negative direction, hence that gets another minus sign. Note that this means that

FORMULA: MOMENTUM EQUATION

$$A\left(p + \rho V^2\right) = \text{constant} \tag{7.36}$$

Which, if the cross-sectional area A is constant, means that

FORMULA: MOMENTUM EQUATION

$$p + \rho V^2 = \text{constant} \tag{7.37}$$

Please note that this looks *very* similar to Bernoulli's equation, but just please don't be the stupid kid who's going to apply Bernoulli's equation on the exam: not only did that equation contain a $\frac{1}{2}$ in front of the second term, it was derived for incompressible flow, and most importantly, we derived this for a streamtube with constant cross-sectional area; Bernoulli applied to a streamline which is vastly different than this.

Finally, the energy equation reduces beautifully too:

$$\begin{aligned} -\oiint_S p\, d\mathbf{S} \cdot \mathbf{V} &= \oiint_S (\rho \mathbf{V} \cdot d\mathbf{S}) \left(e + \frac{V^2}{2}\right) \\ \oiint_S \rho \mathbf{V} \left(e + \frac{V^2}{2} + \frac{p}{\rho}\right) d\mathbf{S} &= 0 \end{aligned}$$

Now, as $h = e + \frac{p}{\rho}$, and H, the **total enthalpy**, equals $H = h + \frac{V^2}{2}$, we get

$$-\rho_1 A_1 V_1 H_1 + \rho_2 A_2 V_2 H_2 = 0$$

Again, $d\mathbf{S}$ points outwards to the surface, so for A_1, it's negative, for A_2 it's positive. Now, as $\rho A V = \dot{m}$, we get that $\dot{m}_1 H_1 = \dot{m}_2 H_2$, but by the continuity equation, $\dot{m}_1 = \dot{m}_2$, hence $H_1 = H_2$, hence the total enthalpy is constant, and

FORMULA: ENERGY EQUATION

$$e + \frac{V^2}{2} + \frac{p}{\rho} = \text{constant} \tag{7.38}$$

Now, we can use these reduced versions of the governing equations together with the equation of state ($p = \rho RT$) and internal energy ($e = c_v T$) to end up at 5 equations with 5 unknowns:

THEOREM: THE FIVE GOVERNING EQUATIONS FOR STEADY, INVISCID, ADIABATIC, COMPRESSIBLE FLOW WITH NO BODY FORCES

$$\begin{aligned} \rho V A = \dot{m} &= \text{constant} && (7.39) \\ A\left(p + \rho V^2\right) &= \text{constant} && (7.40) \\ e + \frac{V^2}{2} + \frac{p}{\rho} &= \text{constant} && (7.41) \\ p &= \rho R T && (7.42) \\ e &= c_v T && (7.43) \end{aligned}$$

7.5 Definition of total (stagnation) conditions

Static conditions are the pressure, temperature and density you feel when you ride along with the gas at the local flow velocity. Total/stagnation conditions are the pressure, temperature and density you feel when the flow is brought to a halt.

Now, suppose we bring the flow is brought to a halt *adiabatically* (that is, there is no heat transfer). We saw before that the total enthalpy then is constant throughout a streamtube. Hence, if we'd have a streamtube which contained a stagnation point, we'd have that

$$H = h + \frac{V^2}{2} = h + \frac{0^2}{2} = h_0$$

And everywhere along the streamtube, we'd have $h + \frac{V^2}{2} = h_0$. Moreover, if all the streamlines of the flow originate from a common uniform freestream (usually happens), then h_0 is the same for every streamline, and hence we have that for such steady, adiabatic flow that

FORMULA

$$h + \frac{V^2}{2} = h_0 = \text{constant} \tag{7.44}$$

Furthermore, as $h_0 = c_p T_0$, we also have that

FORMULA

$$T_0 = \text{constant} \tag{7.45}$$

and T_0 is called the total temperature. Please note that above two equations only hold for *adiabatic* flows.

Now, suppose we go even further and bring the flow to a halt adiabatically *and* reversibly (so isentropically). Then, in addition to the temperature, both pressure and density are constant:

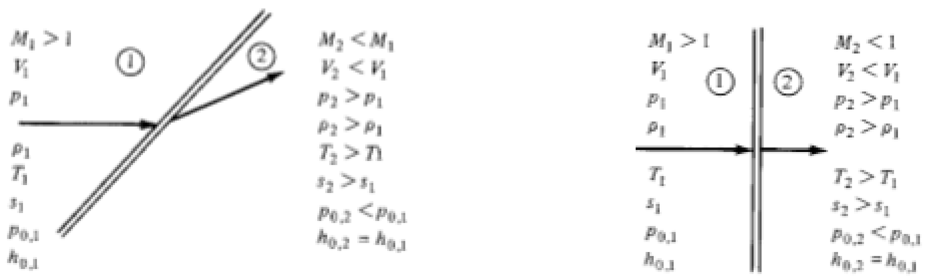
FORMULAS

$$\begin{aligned} p_0 &= \text{constant} && (7.46) \\ \rho_0 &= \text{constant} && (7.47) \end{aligned}$$

To conclude this section: please note that the total quantities are defined *everywhere* in the flow (at each single point in the flow). However, it's only if we know that it's an adiabatic flow that we can say that the total temperature is constant at every point in the flow, and only if we know that it's an isentropic flow that we can say that the total density and pressure are constant as well.

7.6 *Some aspects of supersonic flow: shock waves*

You may have heard of the concept of shock waves before. Why did they exactly occur? Suppose you're a wing flying at relatively low speed (subsonic). Remember that sound waves were, in fact, nothing more than pieces of information you transmit, at the speed of sound. So, air molecules near the wing will be nice molecules and transmit information via sound waves to the molecules upstream that a wing is coming up and that they have to move out of the way to make sure the wing doesn't hit them, and that they need to adjust their pressure and density etc. as Bernoulli told them to, based on the airfoil shape. However, when the wing goes faster than the speed of sound, these air molecules may very well try to transmit this information, but the wing moves faster. Hence, the air molecules don't have any time to adjust their density etc., but they need to do it anyway to make sure their continuity equation is still correct etc. So, this occurs in a shock, which becomes a shock wave. We see these waves depicted in figure 7.3a and 7.3b.



(a) Oblique shock wave.

(b) Normal shock wave.

Figure 7.3: Qualitative pictures of flow through oblique and normal shock waves.

Oblique shock waves make an angle with the flow, normal shock waves are normal to the flow. Note that for an oblique shockwave, the flow is supersonic before the wave. However, after the shock wave, the flow is compressed: this increases the density, pressure and temperature, but reduces the Mach number[2] and velocity. In addition, the entropy is now higher, whereas the total pressure is lower. The enthalpy is still constant, however, as we typically have an adiabatic flow (we don't place our windtunnels in microwaves or refrigerators, most of the time).

Oblique shock waves may be so strong that the flow becomes subsonic after; for normal shock waves, this is *always* the case. Please remember the relations shown in figure 7.3; you'll come across them rather frequently. Most importantly, remember that the flow becomes compressed after a shock wave. There are also waves where the flow suddenly expands; these are fittingly called expansion waves, but we won't be discussing them until the end of the course.

7.7 *Kinetic Gas Theory*

Please note that this section ends with relatively straightforward formulas. However, as I've said before, derivations seem to be rather important in this course, so try to follow the steps as best as you can. The goal of this section is to show how ou can calculate c_v and c_p from the number of atoms in a molecule and R.

First, we introduce the perfect gas model:

- Volume of molecules can be neglected with respect to total volume;
- Molecules act as rigid spheres (elastic collisions);
- No forces between molecules: they only interact during the collision.

We have, by Newton,

$$m \cdot \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}$$

[2]As the flow is compressed, the density is higher. To have conservation of mass, i.e. $\dot{m}$ is constant, the velocity must thus be lower.

where $\mathbf{r}$ is the position vector from the origin. Multiplication by $\mathbf{r}$ gives:

$$m\mathbf{r}\cdot\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}\cdot\mathbf{r}$$

Which can be shown to equal

$$\begin{aligned}
m\frac{d}{dt}\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right) - m\left(\frac{d\mathbf{r}}{dt}\right)^2 &= \mathbf{F}\cdot\mathbf{r}\\
m\frac{d}{dt}(\mathbf{r}\cdot\mathbf{c}) - m\mathbf{c}^2 = \mathbf{F}\cdot\mathbf{r}
\end{aligned}$$

Where $\mathbf{c} = \frac{d\mathbf{r}}{dt}$ (the velocity vector). This equation is for a single molecule, so let's take the average:

$$\overline{m\frac{d}{dt}(\mathbf{r}\cdot\mathbf{c})} - m\overline{\mathbf{c}^2} = \overline{\mathbf{F}\cdot\mathbf{r}}$$

Furthermore, note that

$$\overline{\frac{d}{dt}(\mathbf{r}\cdot\mathbf{c})} = \frac{d}{dt}\overline{(\mathbf{r}\cdot\mathbf{c})} = 0$$

And hence

$$m\overline{\mathbf{c}^2} = -\overline{\mathbf{F}\cdot\mathbf{r}}$$

Now, let's assume that we have a spherical vessel with radius r_0 with N molecules inside. The only force acting on it is the pressure force, which acts in opposite direction to the surface's normal vector. We now have

$$\begin{aligned}
Nm\overline{\mathbf{c}^2} &= -N\cdot\overline{\mathbf{F}\cdot\mathbf{r}} = -p\cdot 4\pi r_0^2\cdot -r_0\\
Nm\overline{\mathbf{c}^2} &= P4\pi r_0^3 = 3P\mathcal{V}
\end{aligned}$$

With $\rho = \frac{Nm}{\mathcal{V}}$, we can write

$$\frac{p}{\rho} = \frac{1}{3}\overline{\mathbf{c}^2} = RT$$

Furthermore, $e_T = \frac{1}{2}\overline{\mathbf{c}^2}$, and hence

$$RT = \frac{2}{3}e_T$$

Or

$$e_T = \frac{3}{2}RT$$

Comparing with $e = c_V T$, we see that $c_V = \frac{3}{2}R$. Furthermore,

$$\begin{aligned}
h &= e + \frac{p}{\rho} = e + RT\\
c_P T &= c_V T + RT\\
c_P &= c_V + R = \frac{3}{2}R + R = \frac{5}{2}R
\end{aligned}$$

And hence

$$\gamma = \frac{c_P}{c_V} = \frac{\frac{5}{2}R}{\frac{3}{2}R} = \frac{5}{3}$$

This was all for a monatomic gas. Let's now focus on a diatomic gas. For a diatomic gas, we have five degrees of freedom: three translational motions, and two rotational motions (the roll motion has negligible moment of inertia and hence stores a negligible amount of energy). For a monatomic gas, we only had three degrees of freedom, namely only the translational motions, for which we saw, rather coincidently, that

$$e = \frac{3}{2}RT$$

Indeed, for a diatomic gas with five degrees of freedom, it becomes

$$e = \frac{5}{2}RT$$

Indeed, in general we have for a gas of n degrees of freedom that

$$\begin{aligned} c_V &= \frac{n}{2}R \\ c_P &= \frac{n+2}{2}R \\ \gamma &= \frac{n+2}{n} \end{aligned}$$

So for a diatomic gas, $\gamma = 1.4$. Now, note that if the gas gets very hot, the molecules will start to vibrate, adding more degrees of freedom, hence decreasing γ as the temperature of the gas goes up. However, between 3 K and 600 K, $\gamma = 1.4$ is pretty constant for dry air.

8 Normal Shock Waves and Related Topics

8.2 The basic normal shock equations

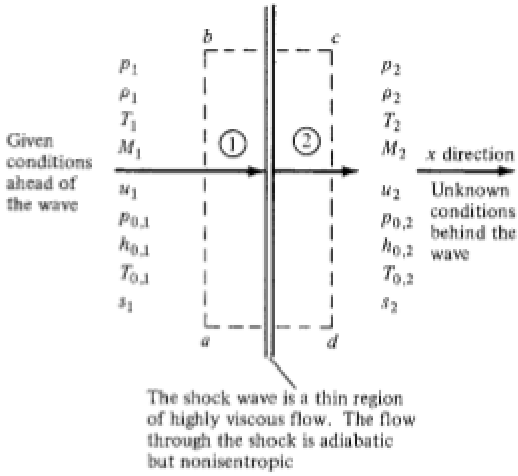


Figure 8.1: Sketch of a normal shock wave.

Yeah I don't exactly get the point of this section to be honest, but anyway. Let's look at figure 8.1, with the control volume formed by the dashed lines. We observe:

- Steady flow: $\frac{\partial}{\partial t} = 0$;
- Adiabatic flow: $\dot{q} = 0$. We're not heating the control volume after all;
- Inviscid flow: there are no viscous effects on the sides of the control volume (there certainly are huge viscous effects within the shock wave itself, but these do not act on the control volume itself);
- No body forces; $\mathbf{f} = \mathbf{0}$.

Unless I'm grossly missing something, derivations are then exactly the same as in chapter 7.4 to arrive at

THEOREM: THE FIVE GOVERNING EQUATIONS FOR STEADY, INVISCID, ADIABATIC, COMPRESSIBLE FLOW WITH NO BODY FORCES

$$\rho_1 u_1 = \rho_2 u_2 \tag{8.1}$$
$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{8.2}$$
$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{8.3}$$
$$p_2 = \rho_2 R T_2 \tag{8.4}$$
$$h_2 = c_p T_2 \tag{8.5}$$

Comparing with the equations derived in chapter 7.4,

$$\begin{aligned} \rho V A = \dot{m} &= \text{constant} \\ A\left(p + \rho V^2\right) &= \text{constant} \\ e + \frac{V^2}{2} + \frac{p}{\rho} &= \text{constant} \\ p &= \rho R T \\ e &= c_v T \end{aligned}$$

we see only small differences:

- As the area is the same on both sides, A is left out everywhere;
- We use u rather than V;
- In the third equation, $e + \frac{p}{\rho}$ is replaced by h, as $h = e + pv = e + \frac{p}{\rho}$;
- We use $h_2 = c_p T_2$ rather than $e = c_v T$.

But I honestly don't get why they're talking about all of this again for four pages.

8.3 *Speed of sound*

Sound propagation is associated to the molecular speed (you don't have to derive this, honestly):

$$\bar{c} = \sqrt{\frac{8RT}{\pi}}$$

So, we expect that the speed of sound also depends on temperature.

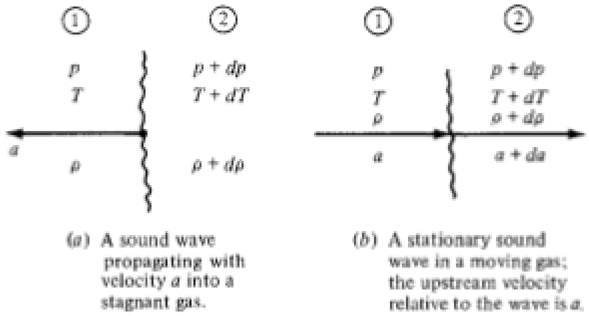


Figure 8.2: Sketch of a normal shock wave.

Now, let's consider figure 8.2. It shows two things that we can do here: we can stand still and look at a sound wave travelling by; in the undisturbed region (region 1), the air has properties p, T and ρ. When the shock wave has passed, the air will have properties $p + dp$, $T + dT$ and $\rho + d\rho$. Now, let's move along with the sound wave, travelling towards the left, so that the relative wind seems to come from the left. When you look towards the left, you see the undisturbed flow, with properties p, T, ρ and a (a being the velocity/speed of sound). When you look towards the right, you see the mess your sound wave has left behind, because now it has properties $p + dp$, $T + dT$, $\rho + d\rho$ and $a + da$ (yes, the velocity may also have changed, we don't know yet whether it has stayed the same). Hence, if we built a control volume around the sound wave, we can apply the continuity equation rather simply:

$$\begin{aligned} \rho a &= (\rho + d\rho)(a + da) \\ \rho a &= \rho a + \rho da + a\,d\rho + d\rho da \end{aligned}$$

Now, as $d\rho$ is small, and da is small, $d\rho$ and da will be negligibly small, and with ρa disappearing on both sides, we have that

$$a = -\rho \frac{da}{d\rho}$$

Now, we can use the momentum equation:

$$p + \rho a^2 = (p + dp) + (\rho + d\rho)(a + da)^2$$

We can again ignore the products of differentials, and we end up at

$$0 = dp + 2\rho a da + a^2 d\rho$$

Or

$$da = \frac{dp + a^2 d\rho}{-2a\rho}$$

But, from the continuity equation, $da = -a\frac{d\rho}{\rho}$ and hence

$$\begin{aligned} -a\frac{d\rho}{\rho} &= \frac{dp + a^2 d\rho}{-2a\rho} \\ a &= \frac{dp/d\rho + a^2}{2a} \\ 2a^2 &= \frac{dp}{d\rho} + a^2 \\ a^2 &= \frac{dp}{d\rho} \end{aligned}$$

The flow through a sound wave is isentropic, hence, we can write

FORMULA

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \tag{8.6}$$

However, this is a rather ugly formula because no one wants to differentiate, so let's look further. First, we have an isentropic flow, hence

$$\frac{p_1}{p_2} = \left(\frac{\rho_1}{\rho_2}\right)^\gamma$$

Or,

$$\frac{p}{\rho^\gamma} = \text{constant} = c$$

Or $p = c\rho^\gamma$. Hence,

$$\left(\frac{\partial p}{\partial \rho}\right)_s = c\gamma\rho^{\gamma-1} = \left(\frac{p}{\rho^\gamma}\right)\gamma\rho^{\gamma-1}$$

Note that $\rho^{\gamma-1} = \frac{\rho^\gamma}{\rho}$, and hence we get that:

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \left(\frac{p}{\rho^\gamma}\right)\gamma\frac{\rho^\gamma}{\rho} = \frac{\gamma p}{\rho}$$

FORMULA

$$a = \sqrt{\frac{\gamma p}{\rho}} \tag{8.7}$$

But with $\frac{p}{\rho} = RT$ (gas law), we get

FORMULA

$$a = \sqrt{\gamma RT} \tag{8.8}$$

Furthermore, recall that $\tau = \frac{1}{\rho}\frac{d\rho}{dp}$, or

$$\tau_s = \frac{1}{\rho\left(\frac{\partial p}{\partial \rho}\right)_s}$$

Now, with $\left(\frac{\partial p}{\partial \rho}\right)_s = a^2$, we have that

$$\tau_s = \frac{1}{\rho a^2}$$

FORMULA

$$a = \sqrt{\frac{1}{\rho\tau_s}} \tag{8.9}$$

Finally, for another physical meaning of the speed of sound, consider the ratio between the kinetic energy and internal energy, i.e. the ratio between organized motion and random motion. Per unit mass, these equal $\frac{V^2}{2}$ and e, respectively. Hence, we have that:

$$\frac{\frac{V^2}{2}}{e} = \frac{\frac{V^2}{2}}{c_v T} = \frac{\frac{V^2}{2}}{\frac{RT}{\gamma-1}} = \frac{\frac{\gamma}{2}V^2}{\frac{a^2}{(\gamma-1)}} = \frac{\gamma(\gamma-1)}{2}M^2$$

8.4 *Special forms of the energy equation*

This section introduces a few concepts and equations that probably seem rather abstract at first and for which you don't really know what you could practically use them for (at least, that was my experience when writing this summary). So, at the very least, try to understand what each new symbol means (a^* etc.) and what is meant by critical conditions.

We already discussed the concepts of total enthalpy, total temperature, total pressure and total density. We can also introduce the concept of total speed of sound: what we'll see is that if the flow has a higher velocity, the speed of sound will actually go down. How can that? Let's start at

$$h + \frac{u^2}{2} = h_0$$

We know that $h = c_P T$, and with $c_P = \frac{\gamma R}{\gamma - 1}$ and $\gamma RT = a^2$, this means that

$$\begin{aligned} \frac{\gamma RT}{\gamma-1} + \frac{u^2}{2} &= \frac{\gamma RT_0}{\gamma-1} \\ \frac{a^2}{\gamma-1} + \frac{u^2}{2} &= \frac{a_0^2}{\gamma-1} \end{aligned}$$

where a_0 is the total speed of sound. As this is the speed of sound when the flow is adiabatically brought to a standstill, it's also referred to as the stagnation speed of sound (as it's basically the speed of sound in the stagnation point). Furthermore, note what happens to the speed of sound when the flow velocity increases: it clearly must decrease. So, just like temperature, density, and pressure, the local speed of sound changes over the flow (the total speed of sound, however, is constant).

Something interesting happens when we plot the two ratios $\frac{u}{a_0}$ and $\frac{a}{a_0}$, i.e. the ratio between the flow velocity and the stagnation speed of sound and the ratio between the local speed of sound and the stagnation speed of sound. Note that $\frac{u}{a_0}$ is absolutely not the Mach number; the Mach number is $\frac{u}{a}$. The shape of the graph is quite logical: a_0 is a constant, so it basically only shows the relation between u and a. We already established that if u increased, a would decrease and vice versa, so if we look at the equation

$$\frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{a_0^2}{\gamma-1}$$

where a and u are the variables, we basically see an ellipse.

There are three points of interest in this graph: first of all, we have the point where the ellipse crosses the x-axis. In this point, the flow velocity equals 0 and thus the local speed of sound equals the total speed of sound, meaning that $\frac{a}{a_0} = 1$. This point contains the **stagnation conditions**.

Secondly, we have the point where the ellipse crosses the y-axis, i.e. where the local speed of sound equals 0. In this case, the ratio $\frac{u}{a_0}$ can be calculated as follows:

$$\begin{aligned} \frac{u^2}{2} &= \frac{a_0^2}{\gamma-1} \\ \frac{u^2}{a_0^2} &= \frac{2}{\gamma-1} \\ \frac{u}{a_0} &= \sqrt{\frac{2}{\gamma-1}} \end{aligned}$$

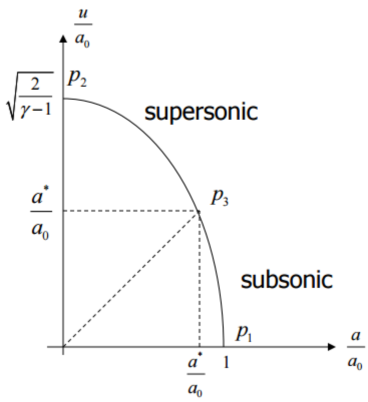


Figure 8.3: A nice graph.

We also see this value denoted in the graph. Note that the **limit flow velocity** (as a tends to 0) equals $u = a_0\sqrt{\frac{2}{\gamma-1}}$.

Now, the third point of interested requires a bit more elaborate explanation. This point is the point where $u = a$, i.e. where $M = 1$, the point where conditions are sonic. This is the point where p_3 is written. Everything to the right of this point is subsonic: we clearly see that we go down in u, but that a goes up, meaning that $M < 1$. Everything to the left of this point is supersonic: u goes up, but a goes down, meaning that $M > 1$. Let's call the speed of sound at which $u = a$ a^*, such that $u = a = a^*$. We can then write

$$\frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{a^{*2}}{\gamma-1} + \frac{a^{*2}}{2}$$

or

FORMULA

$$\frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{\gamma+1}{2(\gamma-1)}a^{*2} \tag{8.10}$$

So, if you know a and u at one point in the flow, you can compute a^*. Again, note that a^* (just like T_0, a_0 (and also ρ_0 and p_0 if the flow is isentropic)) is a characteristic value of the entire flow and does not change from point to point. Furthermore, a^*, T^*, p^*, etc. are called the critical quantities (so critical speed of sound, critical temperature, etc.), which are achieved under **critical conditions** (so $M = 1$).

Alternatively, we can relate a^* to a_0:

$$\begin{aligned}
\frac{a^{*2}}{\gamma-1} + \frac{a^{*2}}{2} &= \frac{a_0^2}{\gamma-1} \\
\frac{2a^{*2} + (\gamma-1)a^{*2}}{2(\gamma-1)} &= \frac{a_0^2}{\gamma-1} \\
a^{*2}(\gamma+1) &= 2a_0^2 \\
a^* &= a_0\sqrt{\frac{2}{\gamma+1}}
\end{aligned}$$

FORMULA

$$a^* = a_0\sqrt{\frac{2}{\gamma+1}} = \sqrt{\gamma R T^*} \tag{8.11}$$

So, if you T_0 (the temperature at the stagnation point, e.g. the temperature you measure in the reservoir of a wind tunnel, or the temperature at the stagnation point of your airfoil/wing), you can compute a_0 and thus a^*. Alternatively, if you know T^*, you can compute a^* directly.

Now, onto some more useful stuff. It should be quite clear that it's arguably relatively easy to measure the temperature, density and pressure in the stagnation point of a flow, so that you know T_0, ρ_0 and p_0. The two most straightforward stagnation points that we'll see in practice are of course the stagnation points on an airfoil/wing and the reservoir of a wind tunnel. Based on these quantities, we can actually do some calculations, if we know the values of temperature, density and pressure at a different point[1]. Remember that we have

$$c_pT_1 + \frac{u_1^2}{2} = c_pT_2 + \frac{u_2^2}{2} = c_pT_0 = \text{constant}$$

We can rewrite this to

$$\begin{aligned} c_PT_0 &= c_PT + \frac{u^2}{2} \\ \frac{T_0}{T} &= 1 + \frac{u^2}{2c_PT} = 2 + \frac{u^2}{2\frac{\gamma RT}{\gamma-1}} = 1 + \frac{u^2}{\frac{2a^2}{\gamma-1}} = 1 + \frac{\gamma-1}{2}\left(\frac{u}{a}\right)^2 \end{aligned}$$

FORMULA

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2 \tag{8.12}$$

Note that this equation is only valid when one of the temperatures is the temperature in stagnation conditions; it is not valid for any T_1 and T_2. The other equations follow from the isentropic relations:

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^\gamma = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}}$$

FORMULAS

$$\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}} \tag{8.13}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}} \tag{8.14}$$

These equations are very important; they should be branded on your mind (according to the book). These equations are so important, that values of $\frac{T_0}{T}$, $\frac{p_0}{p}$ and $\frac{\rho_0}{\rho}$ obtained from these equations are tabulated as functions of M on the formula sheet (appendix A). Now, at sonic conditions, i.e. when $M = 1$, we have that the critical properties are (for $\gamma = 1.4$):

FORMULAS

$$\frac{T^*}{T_0} = \frac{2}{\gamma+1} = 0.833 \tag{8.15}$$

$$\frac{p^*}{p_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}} = 0.528 \tag{8.16}$$

$$\frac{\rho^*}{\rho_0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{\gamma-1}} = 0.634 \tag{8.17}$$

These numbers will appear in subsequent discussions.

Finally, to conclude this section: it's sometimes convenient to introduce a "characteristic" Mach number M^* defined as

CHARACTERISTIC MACH NUMBER

$$M^* \equiv \frac{u}{a^*} \tag{8.18}$$

[1] It goes without saying that of course you can also do these type of question in reverse direction, but it's just to make you understand what's the use of this.

Note that this quantity, contrary to a^*, T^* etc. is *not* constant throughout the value, but changes from point to point as u changes from point to point. Now, let's find the relation between the actual Mach number $M = \frac{u}{a}$ and the characteristic Mach number, by looking at equation (8.10)

$$\begin{aligned}
\frac{a^2}{\gamma-1} + \frac{u^2}{2} &= \frac{\gamma+1}{2(\gamma-1)}a^{*2} \\
\frac{(a/u)^2}{\gamma-1} + \frac{1}{2} &= \frac{\gamma+1}{2(\gamma-1)}\left(\frac{a^*}{u}\right)^2 \\
\frac{(1/M)^2}{\gamma-1} &= \frac{\gamma+1}{2(\gamma-1)}\left(\frac{1}{M^*}\right)^2 - \frac{1}{2} \\
\left(\frac{1}{M}\right)^2 &= \frac{\gamma+1}{2}\left(\frac{1}{M^*}\right)^2 - \frac{\gamma-1}{2} = \frac{\frac{\gamma+1}{M^{*2}} - (\gamma-1)}{2}
\end{aligned}$$

FORMULA

$$M^2 = \frac{2}{\frac{\gamma+1}{M^{*2}} - (\gamma-1)} \tag{8.19}$$

Alternatively, you can solve above derivation for M^*:

$$\begin{aligned}
\frac{(1/M)^2}{\gamma-1} &= \frac{\gamma+1}{2(\gamma-1)}\left(\frac{1}{M^*}\right)^2 - \frac{1}{2} \\
\frac{(1/M)^2}{\gamma-1} + \frac{1}{2} &= \frac{\gamma+1}{2(\gamma-1)}\left(\frac{1}{M^*}\right)^2 \\
\frac{2(1/M)^2 + (\gamma-1)}{2(\gamma-1)} &= \frac{\gamma+1}{2(\gamma-1)}\left(\frac{1}{M^*}\right)^2 \\
\left(\frac{1}{M^*}\right)^2 &= \frac{2(1/M)^2 + (\gamma-1)}{\gamma+1} = \frac{2+(\gamma-1)M^2}{(\gamma+1)M^2}
\end{aligned}$$

FORMULA

$$M^{*2} = \frac{(\gamma+1)M^2}{2+(\gamma-1)M^2} \tag{8.20}$$

From this, it is apparent that M^* qualitatively largely acts in the same fashion as M: if $M = 1$, then $M^* = 1$ as well; if $M > 1$, then $M^* > 1$; if $M < 1$, then $M^* < 1$. The only difference occurs if $M \to \infty$; then $M^* \to \sqrt{\frac{\gamma+1}{\gamma-1}}$. This equation will be used frequently in the next section.

To conclude this section: a large number of equations and a few new symbols have been introduced. Try to understand the basic concept behind them; in subsequent sections and chapters, we'll use them, so remember that you can find some useful information in this section.

Example 1

Consider a point in an airflow where the local Mach number, static pressure, and static temperature are 3.5, 0.3 atm and 180 K, respectively. Calculate the local values of p_0, T_0, T^*, a^* and M^*.

From the formula sheet, we have that for $M = 3.5$, $p_0/p = 76.27$ and $T_0/T = 3.45$.[a]. Thus,

$$\begin{aligned}
p_0 &= \left(\frac{p_0}{p}\right)p = 76.27 \cdot 0.3 = 22.9\,\text{atm} \\
T_0 &= \left(\frac{T_0}{T}\right)T = 3.45 \cdot 180 = 621\,\text{K}
\end{aligned}$$

Note that these are not local but hold for the entire flow (as long as the flow is adiabatic and isentropic (through T_0 would still hold for the entire flow even if it wasn't isentropic)). T^* is the temperature when

$M = 1$. We already know that at critical conditions, $\frac{T^*}{T_0} = 0.833$, and thus

$$T^* = \left(\frac{T^*}{T_0}\right) T_0 = 0.833 \cdot 621 = 517.5\,\mathrm{K}$$
$$a^* = \sqrt{\gamma R T^*} = \sqrt{1.4 \cdot 287 \cdot 517.5} = 456\,\mathrm{m/s}$$

Now, we only need to know M^*. For this, we need to know u (or V), for which we need to know the local speed of sound (so a and not a^*), and then multiply this by the local Mach number of 3.5:

$$a = \sqrt{\gamma R T} = \sqrt{1.4 \cdot 287 \cdot 180} = 268.9\,\mathrm{m/s}$$
$$V = M a = 3.5 \cdot 268.9 = 941\,\mathrm{m/s}$$
$$M^* = \frac{V}{a^*} = \frac{941}{456} = 2.06$$

Note that we could have also computed M^* directly:

$$M^{*2} = \frac{(\gamma+1)\,M^2}{2\,(\gamma-1)\,M^2} = \frac{2.4 \cdot 3.5^2}{2 + 0.4 \cdot 3.5^2} = 4.26$$
$$M^* = 2.06$$

Key is that you look at your formula sheet and find equations that you can use.

[a] How to look this up? In the left column, look for 0.3500+01 (the +01 indicate you have to multiply by 10^1). Then you find in the second column that $\frac{p_0}{p} = 0.7627 + 02$, i.e. $0.7627 \cdot 10^2 = 76.27$. Similarly, in the fourth column, we see $\frac{T_0}{T} = 0.345 + 01$, i.e. $0.345 \cdot 10^1 = 3.45$.

8.5 *When is a flow compressible?*

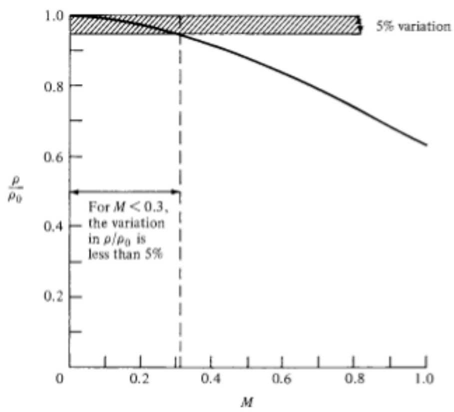


Figure 8.4: Isentropic variation of density with Mach number.

Using

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{\frac{1}{\gamma-1}}$$

figure 8.4 plots the ratio $\frac{\rho}{\rho_0}$ as a fnction of M. It is clear that for $M < 0.3$, the variation in pressure is smaller than 5%. Therefore, as a rule of thumb, you can assume incompressible flow if the local Mach number in a flow *never* exceeds 0.3. So, not only the freestream Mach number must be smaller than 0.3; if there's any place along the airfoil where it's larger than 0.3, then you shouldn't be using incompressible flow anymore as well.

8.6 Calculation of normal shock-wave properties

Remember from the beginning of this chapter:

THEOREM: THE FIVE GOVERNING EQUATIONS FOR STEADY, INVISCID, ADIABATIC, COMPRESSIBLE FLOW WITH NO BODY FORCES

$$\rho_1 u_1 = \rho_2 u_2 \tag{8.21}$$
$$p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \tag{8.22}$$
$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{8.23}$$
$$p_2 = \rho_2 R T_2 \tag{8.24}$$
$$h_2 = c_p T_2 \tag{8.25}$$

In this section, we're finally going to find out what we can do with all of these beautiful relations. We're also going to see that a^* and M^* were slightly more useful than it may have seemed at first. However, again we have that a lot of equations will appear; only the formulas in red boxes are useful end results and these are pretty much all given on the formula sheet. However, do try to understand where they come from.

Now, we can divide equation (8.22) by (8.21):

$$\begin{aligned}\frac{p_1}{\rho_1 u_1} + u_1 &= \frac{p_2}{\rho_2 u_2} + u_2\\ \frac{p_1}{\rho_1 u_1} - \frac{p_2}{\rho_2 u_2} &= u_2 - u_1\end{aligned}$$

We also remember that $a = \sqrt{\frac{\gamma p}{\rho}}$, i.e. $\frac{p}{\rho} = \frac{a^2}{\gamma}$:

$$\frac{a_1^2}{\gamma u_1} - \frac{a_2^2}{\gamma u_2} = u_2 - u_1$$

The energy equation (8.23), which could be rewritten as equation (8.10) as

$$\frac{a^2}{\gamma - 1} + \frac{u^2}{2} = \frac{\gamma+1}{2(\gamma-1)} a^{*2}$$

and thus as

$$\begin{aligned}a_1^2 &= \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} u_1^2\\ a_2^2 &= \frac{\gamma+1}{2} a^{*2} - \frac{\gamma-1}{2} u_2^2\end{aligned}$$

Note that a^* is a constant value for the entire flow, and thus we don't speak of a_1^* and a_2^* as they're simply the same. We can now substitute this into the previously found equation to get

$$\begin{aligned}\frac{\gamma+1}{2}\frac{a^{*2}}{\gamma u_1} - \frac{\gamma-1}{2\gamma} u_1 - \frac{\gamma+1}{2}\frac{a^{*2}}{\gamma u_2} + \frac{\gamma-1}{2\gamma} u_2 &= u_2 - u_1\\ \frac{\gamma+1}{2\gamma u_1 u_2}\left(u_2 - u_1\right) a^{*2} + \frac{\gamma-1}{2\gamma}\left(u_2 - u_1\right) &= u_2 - u_1\\ \frac{\gamma+1}{2\gamma u_1 u_2} a^{*2} + \frac{\gamma-1}{2\gamma} &= 1\\ \frac{\gamma+1}{2\gamma u_1 u_2} a^{*2} &= 1 - \frac{\gamma-1}{2\gamma} = \frac{2\gamma - (\gamma-1)}{2\gamma} = \frac{\gamma+1}{2\gamma}\end{aligned}$$

This final expression can be rewritten to

$$a^{*2} = u_1 u_2 \tag{8.26}$$

and is called the **Prandtl relation** and is a useful intermediate relation for normal shock waves. We can deduce the first important equation from this:

$$\begin{aligned} 1 &= \frac{u_1}{a^*}\frac{u_2}{a^*} \\ 1 &= M_1^* M_2^* \\ M_2^* &= \frac{1}{M_1^*} \end{aligned}$$

Now, using equation (8.20) and substituting that into this:

$$\begin{aligned} \frac{(\gamma+1)\,M_2^2}{2+(\gamma-1)\,M_2^2} &= \left[\frac{(\gamma+1)\,M_1^2}{2+(\gamma-1)\,M_1^2}\right]^{-1} \\ \frac{\gamma+1}{\frac{2}{M_2^2}+(\gamma-1)} &= \left[\frac{(\gamma+1)\,M_1^2}{2+(\gamma-1)\,M_1^2}\right]^{-1} \\ (\gamma+1)\,\frac{(\gamma+1)\,M_1^2}{2+(\gamma-1)\,M_1^2} &= \frac{2}{M_2^2}+(\gamma-1) \\ \frac{(\gamma+1)^2\,M_1^2}{2+(\gamma-1)\,M_1^2}-(\gamma-1) &= \frac{2}{M_2^2} \\ \frac{(\gamma+1)^2\,M_1^2-(\gamma-1)\left(2+(\gamma-1)\,M_1^2\right)}{2+(\gamma-1)\,M_1^2} &= \frac{2}{M_2^2} \\ \frac{\gamma^2M_1^2+2\gamma M_1^2+M_1^2-2\gamma+2-\gamma^2M_1^2+2\gamma M_1^2-M_1^2}{2+(\gamma-1)\,M_1^2} &= \frac{2}{M_2^2} \\ \frac{4\gamma M_1^2-2\,(\gamma-1)}{2+(\gamma-1)\,M_1^2} &= \frac{2}{M_2^2} \\ \frac{1}{M_2^2} &= \frac{2\gamma M_1^2-(\gamma-1)}{2+(\gamma-1)\,M_1^2}=\frac{\gamma M_1^2-(\gamma-1)\,/2}{1+(\gamma-1)\,/2M_1^2} \end{aligned}$$

FORMULA

$$M_2^2=\frac{1+[(\gamma-1)]\,M_1^2}{\gamma M_1^2-(\gamma-1)\,/2} \tag{8.27}$$

Yes, the derivation required some work, but look at the end result: if we know the Mach number in front of the normal shock wave (NSW), we can immediately compute the Mach number behind the shockwave, without requiring any information on stuff like temperature, pressure etc. Furthermore, note that if $M_1 = 1$, then $M_2 = 1$ as well. This is the cse of an infinitely weak normal shock waved, defined as a **Mach wave**. If $M_1 > 1$, then $M_2 < 1$: as M_1 increases above 1, the normal shock wave becomes stronger, and M_2 becomes progressively less than 1. In the limit however, as $M_1 \to \infty$, $M_2 \to \sqrt{\frac{\gamma-1}{2\gamma}} \approx 0.378$ (for air).

Now, let's find relation for the thermodynamic properties ρ_2/ρ_1, p_2/p_1 and T_2/T_1 across NSW. Rearranging equation (8.21) (continuity equation) and using (8.26) (Prandtl relation), we have

$$\frac{\rho_2}{\rho_1}=\frac{u_1}{u_2}=\frac{u_1^2}{u_2u_1}=\frac{u_1^2}{a^{*2}}=M_1^{*2}$$

Using equation (8.20), we obtain

FORMULA

$$\frac{\rho_2}{\rho_1}=\frac{u_2}{u_1}=\frac{(\gamma+1)\,M_1^2}{2+(\gamma-1)\,M_1^2} \tag{8.28}$$

Note how we only need the Mach number in front of the NSW and the density in front of it to compute the density after it.

Now, to obtain the pressure ratio, we use the momentum equation, equation (8.22) and combine it with the continuity equation, equation (8.21):

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \rho_1 u_1^2 - \rho_1 u_1 u_2 = \rho_1 u_1 \left(u_1 - u_2\right) = \rho_1 u_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

Now, dividing by p_1, and remembering that $a_1^2 = \frac{\gamma p_1}{\rho_1}$:

$$\frac{p_2 - p_1}{p_1} = \frac{\gamma \rho_1 u_1^2}{\gamma p_1}\left(1 - \frac{u_2}{u_1}\right) = \frac{\gamma u_1^2}{a_1^2}\left(1 - \frac{u_2}{u_1}\right) = \gamma M_1^2 \left(1 - \frac{u_2}{u_1}\right)$$

Now, for $\frac{u_2}{u_1}$, we can simply take the inverse of equation (8.28):

$$\begin{aligned}
\frac{p_2 - p_1}{p_1} = \frac{p_2}{p_1} - 1 &= \gamma M_1^2 \left[1 - \frac{2 + (\gamma - 1) M_1^2}{(\gamma + 1) M_1^2}\right] = \gamma M_1^2 - \frac{2\gamma + \gamma(\gamma - 1) M_1^2}{\gamma + 1} \\
&= \frac{\gamma^2 M_1^2 + \gamma M_1^2 - 2\gamma - \gamma M_1^2 + \gamma M_1^2}{\gamma + 1} = \frac{2\gamma M_1^2 - 2\gamma}{\gamma + 1} = \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)
\end{aligned}$$

FORMULA

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right) \tag{8.29}$$

Again, we only need to know p_1 and M_1 to compute p_2.

For the temperature ratio, recall the equation of state $p = \rho R T$, leading to

$$\frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)\left(\frac{\rho_1}{\rho_2}\right)$$

Now, we already found formulas for the pressure ratio and density ratio, so we can just substitute them, and with $h = c_p T$, this means that

FORMULA

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)\right]\frac{2 + (\gamma - 1) M_1^2}{(\gamma + 1) M_1^2} \tag{8.30}$$

These ratios are very important: they are a function of M_1 only. Therefore, every NSW for which the Mach number in front of the wave is the same has exactly the same ratios; even if T_1, p_1 and ρ_1 are different.

Furthermore, note that for $M_1 = 1$, $\frac{p_2}{p_1} = \frac{\rho_2}{\rho_1} = \frac{T_2}{T_1} = 1$; that is, we have the case of a normal shock wave of vanishing strength - a Mach wave. However, if $M_1 > 1$, $\frac{p_2}{p_1}$, $\frac{\rho_2}{\rho_1}$ and $\frac{T_2}{T_1}$ progressively increase above 1. In case of $M_1 \to \infty$, we have for air ($\gamma = 1.4$):

$$\begin{aligned}
\lim_{M_1 \to \infty} M_2 &= \sqrt{\frac{\gamma - 1}{2\gamma}} = 0.378 \\
\lim_{M_1 \to \infty} \frac{\rho_2}{\rho_1} &= \frac{\gamma + 1}{\gamma - 1} = 6 \\
\lim_{M_1 \to \infty} \frac{p_2}{p_1} &= \infty \\
\lim_{M_1 \to \infty} \frac{T_2}{T_1} &= \infty
\end{aligned}$$

Note that the temperature and pressure become unbounded but that the density reaches a rather moderate finite limit.

Now, note something strange on all of these four nice equations in the red boxes: mathematically, they should work for $M_1 < 1$ as well: however, subsonic flow cannot contain normal shock waves, so this does not make perfect sense. We invoke the principle of entropy to show that indeed these formulas do not hold for $M_1 < 1$: we can write

$$\begin{aligned} s_2 - s_1 &= c_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1} = c_p \ln \left\{ \left[1 + \frac{2\gamma}{\gamma+1} \left(M_1^2 - 1 \right) \right] \frac{2 + (\gamma - 1) M_1^2}{(\gamma+1) M_1^2} \right\} \\ &\quad - R \ln \left[1 + \frac{2\gamma}{\gamma+1} \left(M_1^2 - 1 \right) \right] \end{aligned}$$

Now, from the second law of Thermodynamics, we must have $s_2 - s_1 \geq 0$. You can verify yourself that for $M_1 =$, $s_2 - s_1 = 0$ (so it's valid there), and if $M_1 > 1$, then $s_2 - s_1 > 0$ so it's valid then as well. However, for $M_1 < 0$, $s_2 - s_1 < 0$, and this is not allowed by the second law; consequently, only cases involving $M_1 \geq 1$ are valid; that is, normal shock waves can occur only in supersonic flow.

Now, you must be wondering, why does the entropy increase across the shock wave? Remember that shock waves are very thin (10^{-5} cm), across which large gradients in velocity and temperature occur. That means that the mechanisms of friction and thermal conduction are strong. These are dissipative, irreversible mechanisms that always increase the entropy.

Now, what happens to the total temperature T_0 and total pressure p_0 over a shock wave? Remember that the total temperature and pressure where achieved by bringing the fluid to rest isentropically. Look at figure 8.5: the fluid element in region 1, ahead of the shock, has properties M_1, p_1, T_1 and s_1. When it's brought to rest isentropically, it has properties p_{1a}, $T_{0,1}$, but as it's isentropically brought to rest, the entropy still equals s_1. The same can be said for the flow in region 2. Now, how does $T_{0,2}$ compare with $T_{0,1}$ and $p_{0,2}$ with $p_{0,1}$?

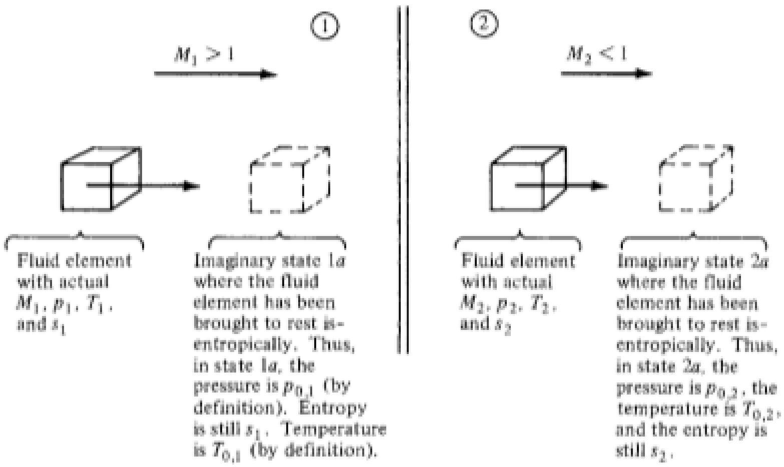


Figure 8.5: Total conditions ahead of and behind a normal shock wave.

We've already seen what happens with T_0: we derived the energy equation, which meant that

$$\begin{aligned} h_1 + \frac{u_1^2}{2} &= h_2 + \frac{u_2^2}{2} \\ c_p T_1 + \frac{u_1^2}{2} &= c_p T_2 + \frac{u_2^2}{2} \\ c_p T_{0,1} &= c_p T_{0,2} \end{aligned}$$

That is, $T_{0,1} = T_{0,2}$ and the total temperature is constant across a stationary NSW. This was expected: flow across a shock wave is adiabatic, and we already demonstrated in section 7.5 that the total temperature in a steady, adiabatic, inviscid flow of a calorically perfect gas is constant.

Now, the total pressure *does* change:

$$s_2 - s_1 = c_p \ln \frac{T_{0,2}}{T_{0,1}} - R \ln \frac{p_{0,2}}{p_{0,1}}$$

However, as $T_{0,2} = T_{0,1}$, this reduces to

$$\begin{aligned} s_2 - s_1 &= -R \ln \frac{p_{0,2}}{p_{0,1}} \\ \frac{p_{0,2}}{p_{0,1}} &= e^{-(s_2 - s_1)/R} \end{aligned}$$

We know that $s_2 - s_1 > 0$ for a normal shock wave (when $M_1 = 1$, i.e. $s_2 - s_1 = 0$, it was called a Mach wave, after all) and thus we know that the total pressure decreases across a shock wave. Moreover, since $s_2 - s_1$ is a function of M_1 only (we derived that), we know that the total pressure ratio $\frac{p_{0,2}}{p_{0,1}}$ is a function of M_1 only as well.

This ratio $\frac{p_{0,2}}{p_{0,1}}$, in combination with the other four important equations we derived, are plotted in figure 8.6. Furthermore, the outcomes are so important that they are tabulated as a function of M_1 in appendix B on the formula sheet for $\gamma = 1.4$ (so for air).

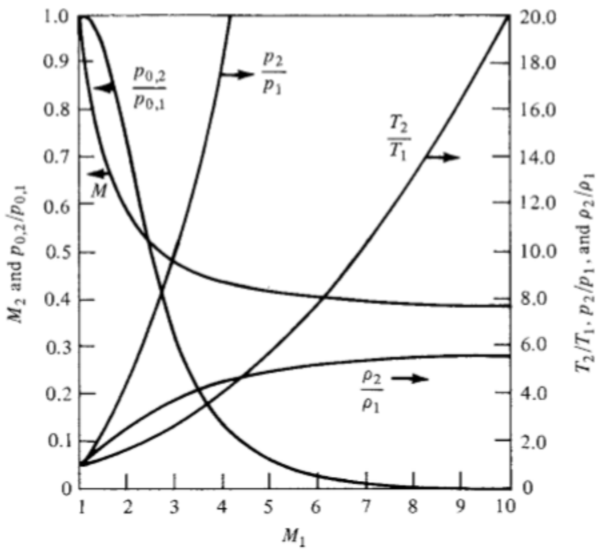


Figure 8.6: The variation of properties across a normal shock wave as a function of upstream Mach number: $\gamma = 1.4$.

Example 2

Consider a normal shock wave in air where the upstream flow properties are $u_1 = 680\,\mathrm{m/s}$, $T_1 = 288\,\mathrm{K}$ and $p_1 = 1\,\mathrm{atm}$. Calculate the velocity, temperature and pressure downstream of the shock.

First, we must compute the Mach number, by computing the speed of sound: $a = \sqrt{\gamma RT} = \sqrt{1.4 \cdot 287 \cdot 288} = 340\,\mathrm{m/s}$, so that $M = \frac{680}{340} = 2$. We then find on the formula sheet, for the entry in the left most column of 0.200+01, that $\frac{p_2}{p_1} = 4.5$, $\frac{T_2}{T_1} = 1.687$ and $M_2 = 0.5774$, so that

$$\begin{aligned} p_2 &= \frac{p_2}{p_1} p_1 = 4.5 \cdot 1 = 4.5\,\text{atm} \\ T_2 &= \frac{T_2}{T_1} T_1 = 1.687 \cdot 288 = 486\,\text{K} \\ a_2 &= \sqrt{\gamma R T_2} = \sqrt{1.4 \cdot 287 \cdot 486} = 442\,\text{m/s} \\ u_2 &= M_2 a_2 = 0.5774 \cdot 442 = 255\,\text{m/s} \end{aligned}$$

Example 3

Consider a shock wave in a supersonic airstream where the pressure upstream of the shock is 1 atm. Calculate the loss of total pressure across the shock wave when the upstream Mach number is (a) $M_1 = 2$ and (b) $M_1 = 4$. Compare these two results and comment on their implication.

To find the total pressure *before* the NSW (i.e. $p_{0,1}$), we must use appendix A, column $\frac{p_0}{p}$, which we find to equal 7.824 for $M = 2$. Thus the total pressure in front of the NSW equals

$$p_{0,1} = 7.824 \cdot 1 = 7.824\,\text{atm}$$

To find $p_{0,2}$, there are two methods. First, appendix gives the ratio $\frac{p_{0,2}}{p_1}$, allowing for immediate computation:

$$p_{0,2} = \frac{p_{0,2}}{p_1} p_1 = 5.64 \cdot 1 = 5.64\,\text{atm}$$

Alternatively, one can use the ratio $\frac{p_{0,2}}{p_{0,1}}$ listed in appendix B to find:

$$p_{0,2} = \frac{p_{0,2}}{p_{0,1}} p_{0,1} = 0.7209 \cdot 7.824 = 5.64\,\text{atm}$$

so that the loss of total pressure is 2.184 atm.

We can do the exact same computation for (b), yielding us $p_{0,2} = 21.07$ atm and $p_{0,1} = 151.8$ atm, meaning the total pressure loss equals 130.7 atm.
Now, what is important to understand is that for any flow, total pressure is a precious commodity. Any loss of total pressure reduces the flow's ability to do useful work. So, you don't want those very large pressure drops. Therefore, if you are going to suffer normal shock waves in a flow, ceteris paribus, you want the normal shock to occur at the lowest possible upstream Mach number. We already saw that at $M = 2$, the pressure loss was only 2.184 atm, but that at $M = 4$, this increased to a whopping 130.7 atm.

Example 4

A ramjet engine is an air-breathing propulsion device with essentially no rotating machinery (no rotating compressor blades, turbine, etc.). The basic generic parts of a conventional ramjet are sketched in figure 8.7. The flow, moving from left to right, enters the inlet, where it is compressed and slowed down. The compressed air then enters the combustor at very low subsonic speed, where it is mixed and burned with a fuel. The hot gas then expands through a nozzle. The net result is the production of thrust toward the left in figure 8.7. In this figure the ramjet is shown in a supersonic freestream with a detached shock wave ahead of the inlet. The portion of the shock just to the left of point is a normal shock. (A detached normal shock wave in front of the inlet of a ramjet in a supersonic flow is not the ideal operating condition; rather, it is desirable that the flow pass through one or more *oblique* shock waves before entering the inlet. Oblique shock waves are discussed in chapter 9.) After passing through the shock wave, the flow from point 1 to point 2, located at the entrance to the combustor, is isentropic (so no normal shock wave there). (a) The ramjet is flying at Mach 2 at a standard altitude of 10 km, where the air pressure and temperature are 2.65×10^4 N/m^2 and 223.3 K, respectively. Calculate the air temperature and pressure at point 2 when the Mach number at that point is 0.2. (b) Repeat these computations when $M_\infty = 10$, but the Mach number at point 2 remains equal to 0.2.

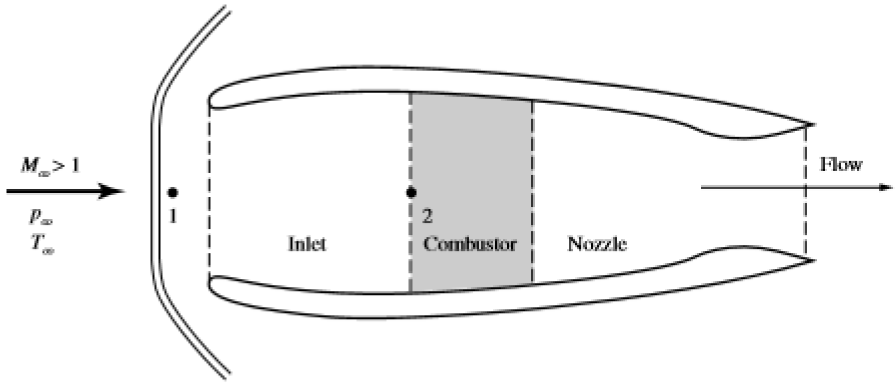


Figure 8.7: Schematic of a conventional subsonic-combustion ramjet engine.

This one is already slightly less straightforward than the previous two and require you to actually think a bit. In the beginning, we only know the pressure, temperature and Mach number in front of the shock wave, and the Mach number at point 2. Between the freestream and point 1, there's a NSW, so we know we can use appendix B for that. However, between point 1 and 2, there's no NSW present; we can only use the isentropic relations for that. We know that for the isentropic relations, we must know the stagnation conditions; i.e. the total pressure and total temperature at point 1. To find them, we must first know the total pressure and total temperature before the NSW; for this, we can use appendix A. There, we see that for $M = 0.2000 + 01$, $\frac{p_0}{p} = 0.7824 + 01$ and $\frac{T_0}{T} = 0.1800 + 01$, so that

$$\begin{aligned} p_{0,\infty} &= \left(\frac{p_{0,\infty}}{p_\infty}\right) p_\infty = 7.824 \cdot 2.65 \cdot 10^4 = 2.07 \times 10^5\,\mathrm{N/m^2} \\ T_{0,\infty} &= \left(\frac{T_{0,\infty}}{T_\infty}\right) T_\infty = 1.8 \cdot 223.3 = 401.9\,\mathrm{K} \end{aligned}$$

We already know that the total temperature does not change over a NSW, so $T_{0,1} = T_{0,2} = 401.9$ K. Now, again using appendix A, we find that for $M = 0.2000 + 00$, $\frac{T_0}{T} = 0.1008 + 01$, i.e. that

$$T_2 = \left(\frac{T_{0,2}}{T_2}\right)^{-1} T_{0,2} = 1.008^{-1} \cdot 401.9 = 399\,\mathrm{K}$$

Now, the pressure is slightly more work. We know that the total pressure over a NSW changes; the ratio $\frac{p_{0_2}}{p_{0_1}}$ for $M = 0.2000 + 01$ equals $0.7209 + 00$, so we have

$$p_{0_1} = \left(\frac{p_{0_1}}{p_{0_\infty}}\right) p_{0_\infty} = 0.7209 \cdot 2.07 \cdot 10^5 = 1.49 \times 10^5\,\mathrm{N/m^2}$$

Now, we know that for an isentropic process, $p_{0_1} = p_{0_2}$, and thus $p_{0_2} = 1.49 \times 10^5$ N/m^2 as well. We can now use appendix A to convert this to static pressure for $M = 0.2000 + 00$: we then have $\frac{p_0}{p} = 0.1028 + 01$, i.e.

$$p_2 = \left(\frac{p_0}{p_2}\right)^{-1} p_0 = 1.028^{-1} \cdot 1.49 \cdot 10^5 = 1.45 \times 10^5\,\mathrm{N/m^2}$$

Note that this only deviates very little from the total pressure: this was expected from aero I, where we assumed Bernoulli's law to be valid ($p + \frac{1}{2}\rho V^2 = p_0$, so that for relatively small values of V, p did not deviate much from p_0. Furthermore, note that $p_2 = 1.45 \times 10^5\,\mathrm{N/m^2} = 1.42\,\mathrm{atm}$. Combined with the temperature of 399 K, these are quite reasonable results and you can easily build a structure that withstands that.

For (b), we can repeat the exact same procedure, leading us to $T_2 = 4653\,\mathrm{K}$ and $p_2 = 3.34 \times 10^6\,\mathrm{N/m^2} = 32.7\,\mathrm{atm}$. It goes without saying that this is rather extreme: even if you find materials that are able to resist such heat, you still have to build a structure able to cope with such enormous amounts of pressure. Furthermore, the fuel decomposes rather than burns at such high temperatures. All in all, in a conventional ramjet, where the flow is slowed down to a low subsonic Mach number before entering the combustor, will not work at high, hypersonic Mach numbers. The solution to this problem is not to slow the flow inside the engine to low subsonic speeds, but rather to slow it only to a lower but still supersonic speed. In this manner, the temperature and pressure increase inside the engine will be smaller and can be made tolerable. In such a ramjet, the entire flowpath through the engine remains at supersonic speed, including the combustor. This necessitates the injection and mixing of the fuel in a supersonic stream - a challenging technical problem. This type of ramjet, where the flow is supersonic throughout, is called a supersonic combustion ramjet - SCRAMjet.

Please also note from this example: if you already know properties of a flow at a certain point (be it supersonic or subsonic), you can calculate other properties at the *same* point by use of appendix A. If you want to go to properties at a different point: if the two points are separated by a NSW, you must use appendix B. If the two points are not separated by a NSW, you may not use appendix B (it's simply wrong), but you must use the fact that it's isentropic (most likely, at least), so that p_0 and T_0 are the same for both points. Note that T_0 is also the same, even if there's a NSW in between.

Then, finally, two examples regarding using the appendices:

Example 5

The velocity and temperature behind a normal shock wave are $329\,\mathrm{m/s}$ and $1500\,\mathrm{K}$, respectively. Calculate the velocity in front of the shock wave.

First, computing the Mach number behind the shock wave:

$$\begin{aligned} a_2 = \sqrt{\gamma R T_2} &= \sqrt{1.4 \cdot 287 \cdot 1500} = 776.3\,\mathrm{m/s} \\ M_2 &= \frac{u_2}{a_2} = \frac{329}{776.3} = 0.4238 \end{aligned}$$

Now, you must "reverse" look this up in appendix B. However, there's no entry for 0.4238 unfortunately; 0.4245 (associated with $M = 0.4450+01$) and 0.4236 (associated with $M = 0.4500+01$) are the ones closest. We can, however, simply interpolate:

$$M_1 = 4.45 + \frac{0.4245 - 0.4238}{0.4245 - 0.4236}(4.5 - 4.45) = 4.4898$$

Now, we must also know the speed of sound in order to compute the flow velocity in front of the wave, for which we must know the temperature and for which we once again must interpolate:

$$\frac{T_2}{T_1} = 4.788 + \frac{0.4525 - 0.4238}{0.4525 - 0.4236}(4.875 - 4.788) = 4.856$$

And thus we can deduce

$$\begin{aligned} T_1 &= \frac{T_2}{T_2/T_1} = \frac{1500}{4.856} = 308.9\,\mathrm{K} \\ a_1 &= \sqrt{\gamma R T_1} = \sqrt{1.4 \cdot 287 \cdot 308.9} = 352.3\,\mathrm{m/s} \\ u_1 &= M_1 a_1 = 4.489 \cdot 352.3 = 1581\,\mathrm{m/s} \end{aligned}$$

Example 6

Repeat the previous example, but use the "nearest entry" in tables rather than interpolating.

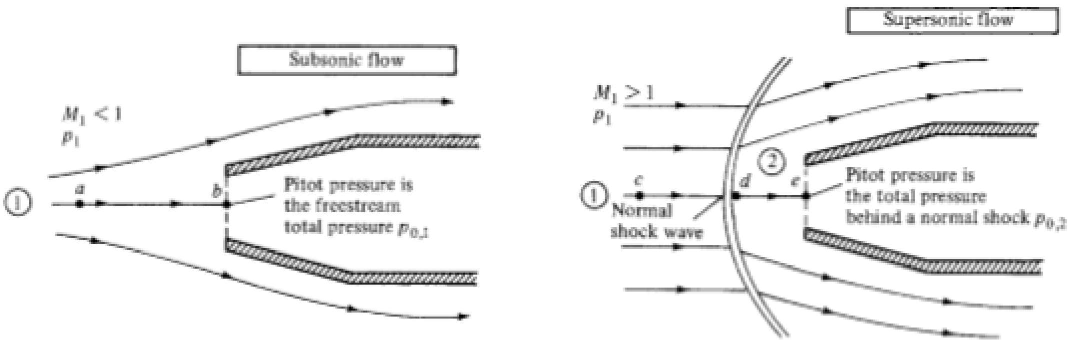
Now, the nearest entry is $M_2 = 0.4236$ so $M_1 = 4.5$ and $T_2/T_1 = 4.875$, and thus

$$\begin{aligned} T_1 &= \frac{T_2}{T_2/T_1} = \frac{1500}{4.875} = 307.7\,\mathrm{K} \\ a_1 &= \sqrt{\gamma R T_1} = \sqrt{1.4 \cdot 287 \cdot 307.7} = 351.6\,\mathrm{m/s} \\ u_1 &= M_1 a_1 = 4.5 \cdot 351.6 = 1582\,\mathrm{m/s} \end{aligned}$$

Now, note how close this result is to the 1581 m/s found by using interpolation (0.06%, that's pretty insignificant). Anderson also noted this and can't be bothered by the hassle of interpolating, so we'll be using the method of closest entry when using the tables as well.

8.7 *Measurement of velocity in a compressible flow*

We use Pitot tubes for this, and you should by know, having done at least 1.5 years of aerospace engineering, have a basic idea of how Pitot tubes work.



(a) Oblique shock wave.

(b) Normal shock wave.

Figure 8.8: Qualitative pictures of flow through oblique and normal shock waves.

8.7.1 *Subsonic compressible flow*

Consider a Pitot tube in a subsonic, compressible flow, as sketched in figure 8.8a. The mouth of the Pitot tube, near b, is a stagnation region, so it measures the total pressure, $p_{0,1}$. If the freestream static pressure p_1 is known (which is easy to measure), then the Mach number in region 1 can be obtained from equation (8.13):

$$\begin{aligned} \frac{p_{0,1}}{p_1} &= \left(1 + \frac{\gamma - 1}{2} M_1^2\right)^{\gamma/(\gamma-1)} \\ \left(\frac{p_{0,1}}{p_1}\right)^{(\gamma-1)/\gamma} &= 1 + \frac{\gamma - 1}{2} M_1^2 \\ \frac{\gamma - 1}{2 M_1^2} &= \left(\frac{p_{0,1}}{p_1}\right)^{(\gamma-1)/\gamma} - 1 \end{aligned}$$

FORMULA

$$M_1^2 = \frac{2}{\gamma - 1}\left[\left(\frac{p_{0,1}}{p_1}\right)^{(\gamma-1)/\gamma} - 1\right] \tag{8.31}$$

The flow velocity can then be determined by recalling that $u_1 = M_1 a_1$ so that

$$u_1^2 = \frac{2a_1^2}{\gamma - 1}\left[\left(\frac{p_{0,1}}{p_1}\right)^{(\gamma-1)/\gamma} - 1\right]$$

So, next to $p_{0,1}$ and p_1, we must also know the freestream speed of sound (or the freestream temperature so that we can calculate it) in order to compute u_1. Note that for *incompressible* flow, which relied solely on Bernoulli's law, the latter was not necessary to know.

8.7.2 Supersonic flow

As the freestream is supersonic and the Pitot tube presents an obstruction for a supersonic flow, a strong bow shock wave forms in front of the Pitot tube, as shown in figure 8.8b. This shock wave is pretty much normal to the flow close to the streamline cde, however, so we can treat it as a normal shock wave. Now, note what happens: a fluid element moving along streamline cde will be first decelerated nonisentropically between c and d, and then isentropically between d and e. We know, however, that for the nonisentropic deceleration the total pressure does not stay the same (it does for the isentropic deceleration). As a result, the pressure at point e is *not* the total pressure of the freestream, but rather the total pressure behind a normal shock wave, $p_{0,2}$. This is the Pitot pressure read at the end of the tube. Note that $p_{0,2} < p_{0,1}$, due to the entropy increase across the shock. However, knowing $p_{0,2}$ and p_1 is still sufficient. We can write:

$$\frac{p_{0,2}}{p_1} = \frac{p_{0,2}}{p_2}\frac{p_2}{p_1}$$

We know from equation (8.13) that

$$\frac{p_{0,2}}{p_2} = \left(1 + \frac{\gamma - 1}{2}M_2^2\right)^{\gamma/(\gamma-1)}$$

where, from equation (8.27),

$$M_2^2 = \frac{1 + \left[(\gamma - 1)/2\right] M_1^2}{\gamma M_1^2 - (\gamma - 1)/2}$$

also, from equation (8.29), we have

$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)$$

Combining all of this leads to

FORMULA: RAYLEIGH PITOT TUBE FORMULA

$$\frac{p_{0,2}}{p_1} = \left(\frac{(\gamma + 1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma - 1)}\right)^{\gamma/(\gamma-1)} \frac{1 - \gamma + 2\gamma M_1^2}{\gamma + 1} \tag{8.32}$$

This equation is called the **Rayleigh Pitot tube formula**. It relates the Pitot pressure $p_{0,2}$ and the freestream static pressure p_1 to the freestream Mach number M_1. It allows for the calculation of M_1 from a known $p_{0,2}/p_1$. For convenience, the ratio $\frac{p_{0,2}}{p_1}$ is tabulated versus M_1 in appendix B.

Now, you must be kicking yourself, thinking, this last one is the most complicated one, surely that one is the best one to use? The answer is no. The Rayleigh Pitot tube formula only holds when the flow is supersonic, i.e. if $M_1 > 1$. If this is not the case, you are *not* allowed to use it (it's just downright wrong). After all, when deriving it, we assumed there'd be a NSW. For $M < 1$, you can look at figure 8.9 for how accurate they are: for $M < 0.3$, they are nearly one-to-one; beyond that, the isentropic one definitely deserves preference.

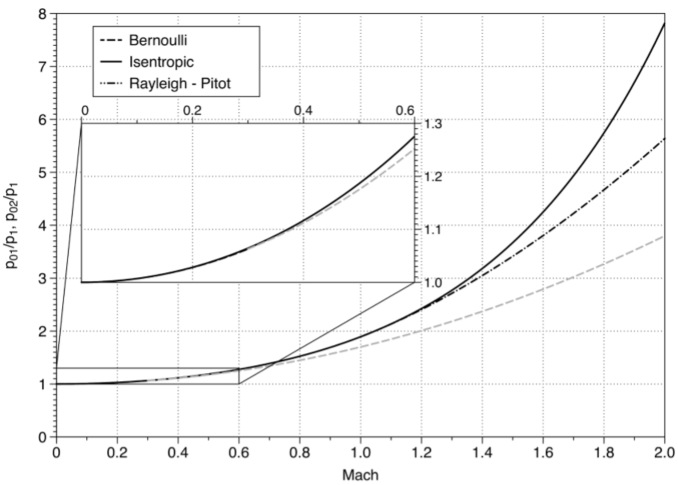


Figure 8.9: Pressure ratios as a function of Mach number.

9 Oblique Shock and Expansion Waves

9.1 Introduction

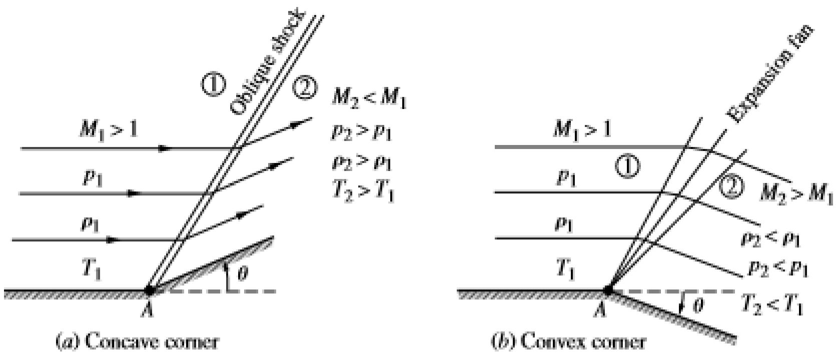


Figure 9.1: Supersonic flow over a corner.

We discussed normal shock waves in the previous chapter: these were shock waves normal to the upstream flow. However, we often see oblique shock waves; shock waves that are not normal to the flow. Where pressure is increased discontinuously across an oblique shock wave, there are also oblique expansion waves, where pressure is decreased continuously. We see this difference in figure 9.1. In the left part of figure 9.1, the bulk of the gas is above the wall, and the streamlines are turned upward, into the main bulk of the flow; the flow is turned into itself. This will cause an oblique shockwave. The originally horizontal streamlines are all deflected by an angle θ. Across the wave, the Mach number discontinuously decreases, and the pressure, density and temperature discontinuously increase. However, if we take a look at the right part of figure 9.1, where the wall is turned downward through the angle θ, then we see that, as the flow at the wall must be tangent to the wall, the streamlines are turned downward, away from the main bulk of the flow; the flow is turned away from itself. An expansion wave will occur. This expansion wave is in the form of a fan centered at the corner. The originally horizontal streamlines are deflected smoothly and continuously through the expansion fan such that the streamlines behind the wave are parallel to each other and inclined downwards at the deflection angle θ. Across the expansion wave, the Mach number increases, and the pressure, temperature and density decrease. Thus, an expansion wave is the direct antithesis of a shock wave.

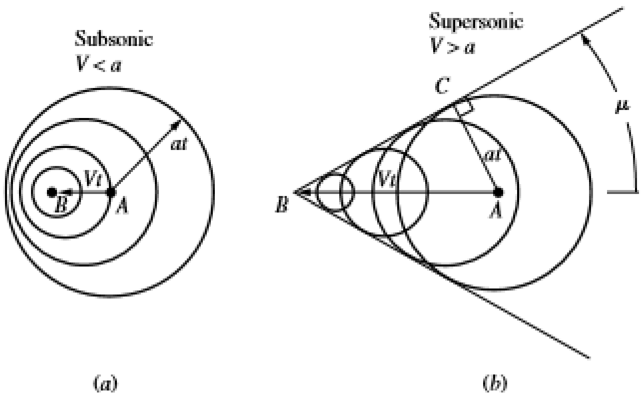


Figure 9.2: Another way of visualizing the propagation of disturbances in (a) subsonic and (b) supersonic flow.

Now, why are most waves oblique rather than normal to the upstream flow? For this, consider a moving "beeper", as shown in figure 9.2. If we move at subsonic speeds as in the left part of figure 9.2, then the sound waves will not interfere with each other. However, if we move at supersonic speeds as in the right part of figure 9.2, then there'll be nice wave fronts appearing, forming a disturbance envelope line given by the straight line BC, which is tangent to the family of circles. This line of disturbances is defined as a Mach wave. In addition, the angle ABC that the **Mach wave** makes with respect to the direction of motion of the beeper is defined as the **Mach angle** μ, which we can readily find to be:

$$\sin\mu = \frac{at}{Vt} = \frac{a}{V} = \frac{1}{M}$$

FORMULA

$$\mu = \arcsin\frac{1}{M} \tag{9.1}$$

Now, the Mach wave, that is, the envelope of disturbances in the supersonic flow, is clearly oblique to the direction of motion. If the disturbances are stronger than a simple sound wave however, then the wave front becomes stronger than a Mach wave, creating an oblique shock wave at an angle β to the freestream, where $\beta > \mu$. In fact, a Mach wave is simply an infinitely weak oblique shock.

9.2 Oblique shock relations

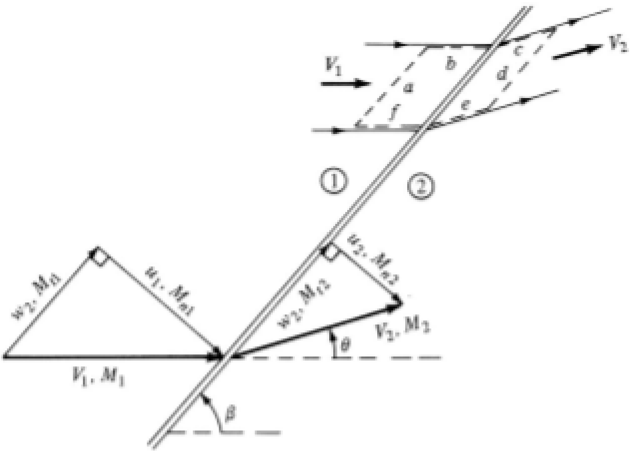


Figure 9.3: Oblique shock geometry.

Now, onto the fun stuff. Just like for NSW, we can calculate relations between conditions before the OSW and after the OSW. Consider the OSW sketched in figure 9.3. The angle between the shock wave and the upstream flow direction is defined as the **wave angle** and is denoted by β. The upstream flow (region 1) is horizontal, with a velocity V_1 and Mach number M_1. The downstream flow (region 2) is inclined upward through the deflection angle θ and has velocity V_2 and Mach number M_2. The upstream velocity V_1 is split into components tangential and normal to the shock wave w_1 and u_1, respectively, with the associated tangential and normal Mach numbers $M_{t,1}$ and $M_{n,1}$, respectively. Similarly, the downstream velocity is split into tangential and normal components w_2 and u_2, respectively, with the associated Mach numbers $M_{t,2}$ and $M_{n,2}$.

Consider the control volume shown by the dashed lines in the upper part of figure 9.3. a and d are both tangential to the shock wave, b, c, e and f are all parallel to the streamlines. Remember that the continuity equation meant that

$$\oiint \rho\mathbf{V}\cdot d\mathbf{S} = 0$$

Note that faces b, c, e and f are all parallel to the flow, so they do not produce a mass flow and do not contribute to the surface integral. You may be inclined to now take[1]

$$-\rho_1 V_1 A_1 + \rho_2 V_2 A_2 = 0$$

where $A_1 = A_2$ = area of faces a and d. However, you're then forgetting that only the velocity *normal* to the surface counts (due to the dot product), so we need to take u_1 and u_2 (this is clearly visible from the lower part of figure 9.3). Thus, the continuity equation for an OSW reduces to (as $A_1 = A_2$)

FORMULA

$$\rho_1 u_1 = \rho_2 u_2 \tag{9.2}$$

Now, onto the momentum equation, which was

$$\oiint_S (\rho \mathbf{V} \cdot d\mathbf{S}) \mathbf{V} = -\oiint_S p\, d\mathbf{S}$$

Now, as this is a vector equation, be can actually decompose it into a tangential and normal (to the shock wave) component. First, considering the tangential component:

$$\oiint_S (\rho \mathbf{V} \cdot d\mathbf{S})\, w = -\oiint_S (p\, dS)_{\text{tangential}}$$

Now, note what happens to $\oiint_S (p\, dS)_{\text{tangential}}$: $d\mathbf{S}$ is perpendicular to the control surface, but a and d themselves are tangential to the shock wave. This means that for a and d, $d\mathbf{S}$ points in perpendicular direction to the wave and therefore the contribution of $(p\, dS)_{\text{tangential}}$ is zero over a and d. Furthermore, p_1 and p_2 are both constant; i.e. the pressure acting on b is equal to the pressure acting on f, and the same holds for c and e. As b and f both have the same length (it's a parallelogram), the contributions $(p\, dS)_{\text{tangential}}$ cancel out, and the same can be said for the contributions of c and e (as the pressure is equal in magnitude but acts in opposite direction). This means that $(\rho \mathbf{V} \cdot d\mathbf{S})\, w$ must be zero; we already saw that $\oint (\rho \mathbf{V} \cdot d\mathbf{S})$ only took into account faces a and d and thus we get:

$$-\left(\rho_1 u_1 A_1\right) w_1 + \left(\rho_2 u_2 A_2\right) w_2 = 0$$

Dividing by $\rho_1 u_1 = \rho_2 u_2$ gives

FORMULA

$$w_1 = w_2 \tag{9.3}$$

which is an important result: the tangential component of the flow velocity is constant across an oblique shock. Now, onto the component normal to the shockwave of the momentum equation. Once again, the equal and opposite pressure forces on b and f and on c and e cancel, but the pressure integral evaluated over faces a and d yields the net sum $-p_1 A_1 + p_2 A_2$ (again, pressure points inward on the surface for face a, hence the minus sign). Thus, we have

$$-\left(\rho_1 u_1 A_1\right) u_1 + \left(\rho_2 u_2 A_2\right) u_2 = -\left(-p_1 A_1 + p_2 A_2\right)$$

With $A_1 = A_2$, this leads to

FORMULA

$$p_1 + \rho_1 u_1^2 = p_2 u_2^2 \tag{9.4}$$

Finally, considering the energy equation, i.e.

$$\oiint_S \rho \left(e + \frac{V^2}{2}\right) \mathbf{V} \cdot d\mathbf{S} = -\oiint_S p \mathbf{V} \cdot d\mathbf{S}$$

[1] The minus sign appears in front of the first term because the velocity points inwards (it enters after all), but the surface normal points outward, by definition.

Again, noting that the flow is tangent to faces b, c, f and e, and hence $\mathbf{V}\cdot d\mathbf{S} = 0$ on these faces, and thus:

$$-\rho_1\left(e_1 + \frac{V_1^2}{2}\right)u_1A_1 + \rho_2\left(e_2 + \frac{V_2^2}{2}\right)u_2A_2 = -\left(-p_1u_1A_1 + p_2u_2A_2\right)$$

But with $A_1 = A_2$, this can be reduced to

$$\begin{aligned}
-\rho_1u_1\left(e_1 + \frac{p_1}{\rho_1} + \frac{V_1^2}{2}\right) &= \rho_2u_2\left(e_2 + \frac{p_2}{\rho_2} + \frac{V_2^2}{2}\right) = 0 \\
\rho_1u_1\left(h_1 + \frac{V_1^2}{2}\right) &= \rho_2u_2\left(h_2 + \frac{V_2^2}{2}\right)
\end{aligned}$$

With $\rho_1u_1 = \rho_2u_2$ (continuity equation), this means that

$$h_1 + \frac{V_1^2}{2} = h_2 + \frac{V_2^2}{2} \tag{9.5}$$

Now, note that $V_1^2 = u_1^2 + w_1^2$ and $V_2^2 = u_2^2 + w_2^2$, and that $w_1 = w_2$ and thus

FORMULA

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{9.6}$$

Now, look again at those beautiful formulas in the red boxes in this section: these are exactly the same formulas as they were for NSWs. Hence, precisely the same algebra as applied to the normal shock equations in section 8.6 will lead to identical expressions for changes across an oblique shock in terms of the normal component of the upstream Mach number $M_{n,1}$ (so not in terms of the upstream Mach number directly). We can quickly get the normal component from the upstream Mach number M_1, simply by

$$M_{n,1} = M_1 \sin\beta$$

Then we have, using equation (8.27), (8.28) and (8.29),

$$\begin{aligned}
M_{n,2}^2 &= \frac{1 + \left[(\gamma-1)/2\right]M_{n,1}^2}{\gamma M_{n,1}^2 - (\gamma-1)/2} \\
\frac{\rho_2}{\rho_1} &= \frac{(\gamma+1)M_{n,1}^2}{2 + (\gamma-1)M_{n,1}^2} \\
\frac{p_2}{p_1} &= 1 + \frac{2\gamma}{\gamma+1}\left(M_{n,1}^2 - 1\right)
\end{aligned}$$

And the temperature ratio $\frac{T_2}{T_1}$ can be found from $\frac{p_2}{p_1}\frac{\rho_1}{\rho_2}$ (equation of state), leading to

$$\frac{T_2}{T_1} = \frac{h_2}{h_1} = \left[1 + \frac{2\gamma}{\gamma+1}\left(M_{n,1}^2 - 1\right)\right]\frac{2 + (\gamma-1)M_{n,1}^2}{(\gamma+1)M_{n,1}^2}$$

Note that we are only allowed to plug in the *normal* components of the Mach number. So, you first need to convert M_1 to $M_{n,1}$, compute $M_{n,2}$ and convert this back to M_2 by applying simple geometry in the form of

$$M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)}$$

Now, small problem: though θ can be easily determined from geometry (it's just the angle the wall makes with the horizontal, remember), β is not so easily obtained. Nevertheless, we can obtain it if we know M_1 and θ[2].

[2]Alternatively, which is how the book looks at it, you can say that θ is a function of M_1 and β, but it's practically the same.

From simple geometry of figure 9.3, we have

$$\begin{aligned}\tan\beta &= \frac{u_1}{w_1}\\ \tan(\beta-\theta) &= \frac{u_2}{w_2}\end{aligned}$$

Now, remember that $w_1 = w_2$, so that dividing the two gives, with $u_1\rho_1 = u_2\rho_2$ so that $\frac{u_1}{u_2} = \frac{\rho_2}{\rho_1}$

$$\frac{\tan(\beta-\theta)}{\tan\beta} = \frac{u_2}{u_1} = \frac{\rho_1}{\rho_2} = \frac{2+(\gamma-1)M_1^2\sin^2\beta}{(\gamma+1)M_1^2\sin^2\beta}$$

which can be rewritten (do not recommend doing this):

FORMULA: θ-β-M RELATION

$$\tan\theta = 2\cot\beta\frac{M_1^2\sin^2\beta-1}{M_1^2(\gamma+\cos 2\beta)+2} \tag{9.7}$$

where $\cot = \frac{1}{\tan}$. This $\theta-\beta-M$ relation specifies θ as a unique function of M_1 and β.

We can plot the outcomes of this equation in figure 9.4 (note that it stretches over two pages, before you start wondering how freaking weird the left figure seems). How to read this graph? First, look up which line you have to take, determined by the Mach number. Note that each graph is two-valued (i.e. for each x-value there exist two y-values). Then, look up your deflection angle θ on the x-axis, and from that, look up β, which will present you two possible values for β. Which one to pick will be explained shortly.

Let's discuss one-by-one, all of the important properties of this graph:

1. We see that for any given upstream Mach number M_1, there is a maximum deflection angle θ_{max}. For example, for $M = 2.6$, we see that θ can only be as large as 31°, otherwise the graph has no value for β for you. If the physical geometry is such that $\theta > \theta_{\text{max}}$, then nature will create a detached shock, as shown in figure 9.5. If M_1 is larger, than the maximum allowable θ also becomes larger, though it reaches a limit at 45.5°.
2. For any given θ less than θ_{max}, there are two straight oblique shock solutions for a given M_1 (e.g., if $M_1 = 2.0$ and $\theta = 15°$, then either $\beta = 45.3°$ or 79.8°). The smaller value of β is called the weak shock solution, and the larger value is the strong shock solution. These two cases are illustrated in figure 9.6. Why do we call them weak and strong? For the larger angle, $M_{1,n}$ is larger, and we know from our beautiful formulas that the pressure ratio $\frac{p_2}{p_1}$ will then be larger, thus that the flow will be compressed more than if the angle is smaller. That's why we call the shock wave for the larger angle a strong shock wave as opposed to the weak one. In reality, the weak shock solution usually prevails (don't bother too much with why). This leads to the following important remark: the collection of points connecting all the values of θ_{max} has been drawn in figure 9.4, which divides the weak (below this line) and strong (above this line) shock solutions. Note that slightly below this curve there's another curve, which is the dividing line above which $M_2 < 1$ and below which $M_2 > 1$. What do we learn from this? For the weak shock solution, M_2 will be larger than 1 as well (so it'll be supersonic after the shock wave as well). Only at values of θ very close to θ_{max} will $M_2 < 1$, but then it'll still be close to 1. So: the Mach number downstream of a straight, attached oblique shock is almost always supersonic.
3. If $\theta = 0$, then either $\beta = 90°$ (as all the graphs come together at the left-top corner), which corresponds to a normal shock wave, or a different angle, μ, which corresponds to the Mach wave illustrated in figure 9.2. In both cases, the flow streamlines naturally experience no deflection across the wave.
4. (In the following two bullet points, we will assume the weak shock solution exclusively.) Consider an experiment where the angle θ is fixed. If we increase M_1, then β will decrease, as shown in figure 9.7. Although the decrease in β means that $M_{n,1}$ would become smaller, the increase in M_1 itself more than compensates for this, meaning that $M_{n,1}$ actually becomes higher. This means that the shock wave becomes stronger as well. The other way around, if M_1 is decreased, then β will increase and the shock wave will become weaker. If M_1 is decreased sufficiently, then the shock wave will become detached, as θ_{max} will become smaller than θ.
5. Now, the other way around: we keep M constant, but increase θ. Then β will also increase. As M_1 stays constant, $M_{n,1}$ will increase, as shown in figure 9.8, meaning that the shock wave will become stronger.

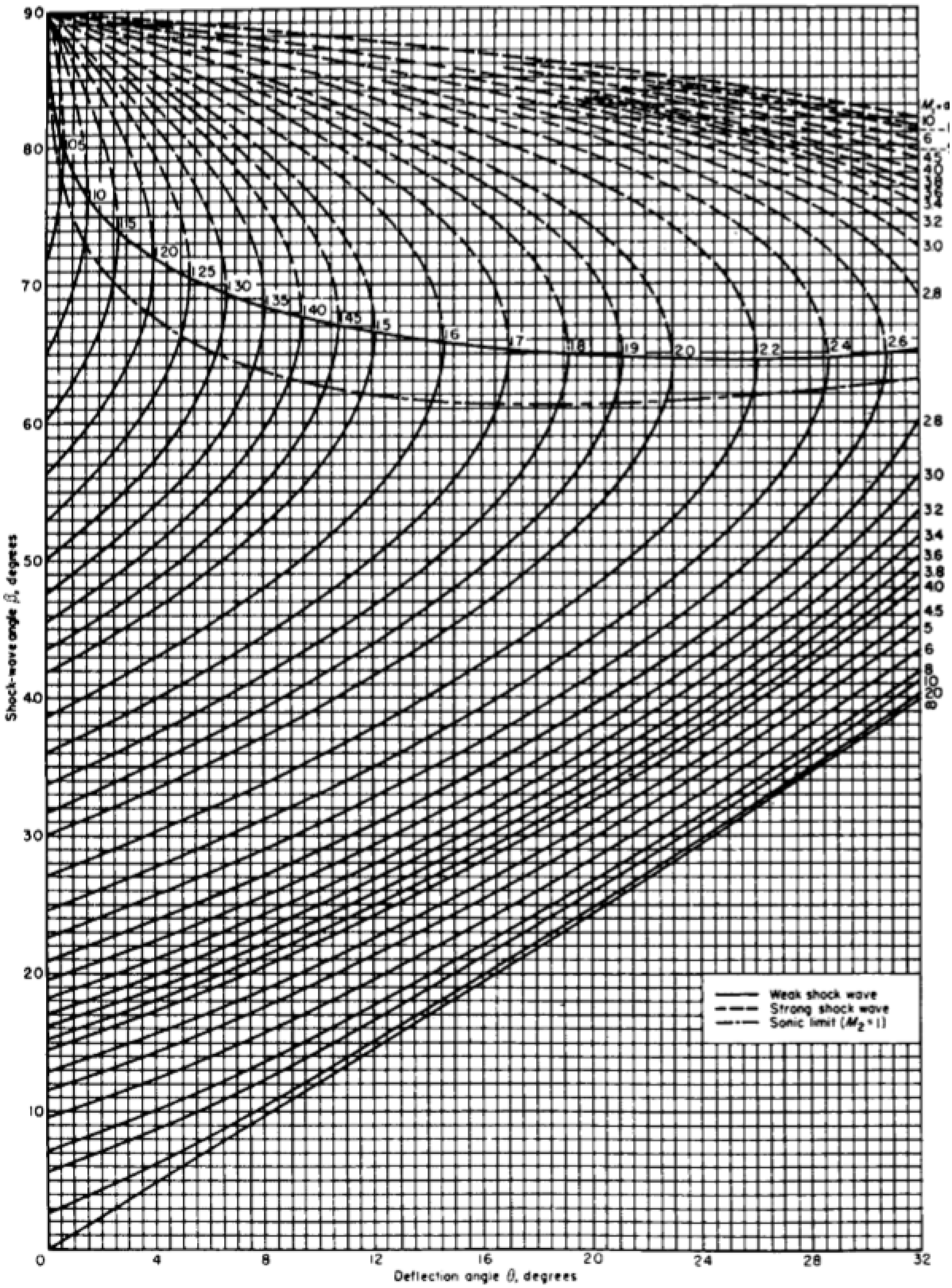


Figure 9.4: Oblique shock properties: $\gamma = 1.4$. The $\theta - \beta - M$ diagram.

So, in short:

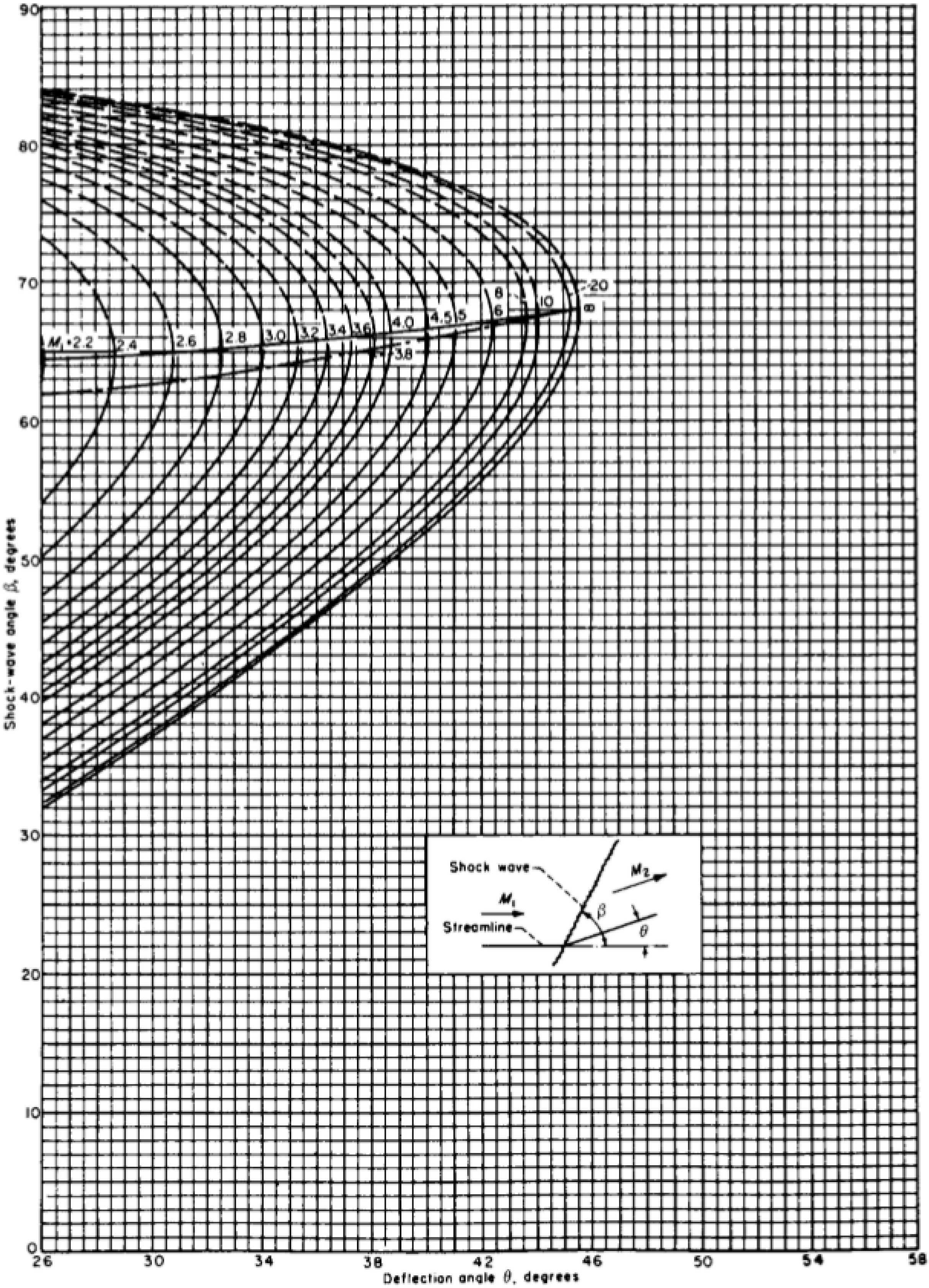


Figure 9.4 (cont.): Oblique shock wave properties: $\gamma = 1.4$.

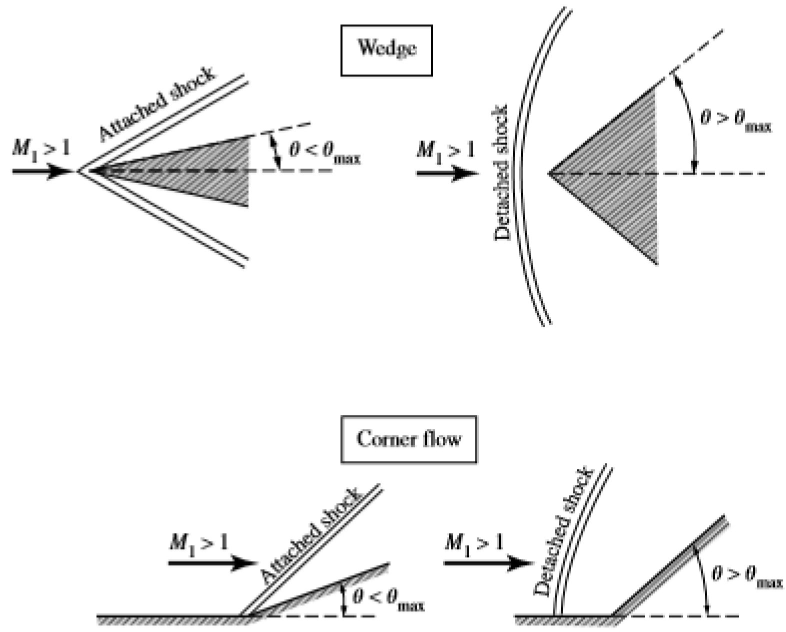


Figure 9.5: Attached and detached shocks.

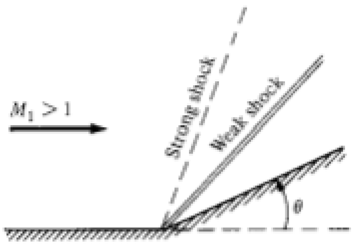


Figure 9.6: The weak and strong cases.

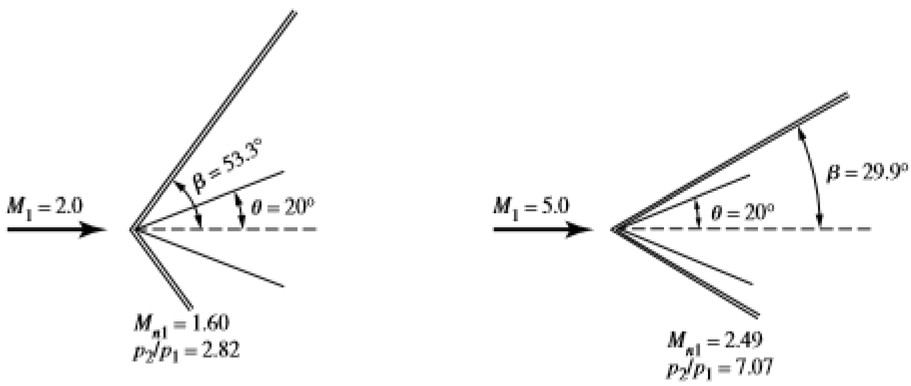


Figure 9.7: Effects of increasing the upstream Mach number.

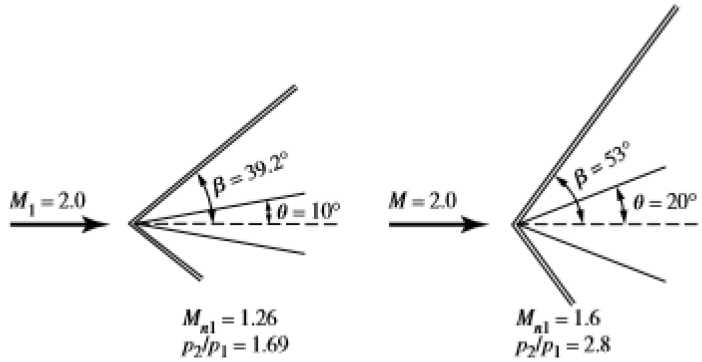


Figure 9.8: Effect of increasing the deflection angle.

IMPORTANT PROPERTIES OF OSW

1. For each M_1, there's a maximum deflection angle θ_{max}.
2. For each M_1 and θ, there exist two solutions for β; nature will typically pick the lowest value.
3. For $\theta = 0°$, two solutions for β exist, namely 90° or an angle μ, which correspond to a NSW and a Mach wave, respectively.
4. If M is increased, β is decreased and the shock wave becomes stronger.
5. If θ is increased, β is increased and the shock wave becomes stronger.

Example 1

Consider an OSW with a wave angle of 30°. The upstream flow Mach number is 2.4. Calculate the deflection angle of the flow, the pressure and temperature ratios across the shock wave, and the Mach number behind the wave.

From figure 9.4, we see that for $M_1 = 2.4$ and $\beta = 30°$, $\theta = 6.5°$. Now, to compute the pressure etc. behind the shock wave, we need to use the NSW relations from appendix B, but for that, we need to know the Mach number *normal* to the shock wave. With an angle $\beta = 30°$, this equals

$$M_{n,1} = M_1 \sin\beta = 2.4 \sin 30° = 1.2$$

From appendix B, we then have

$$\begin{aligned} \frac{p_2}{p_1} &= 1.513 \\ \frac{T_2}{T_1} &= 1.128 \\ M_{n,2} &= 0.8422 \end{aligned}$$

and now we need to convert back to M_2, using

$$M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)} = \frac{0.8422}{\sin(30 - 6.5)} = 2.11$$

Note that it is indeed annoying to convert between the total Mach number and Mach number normal to the OSW, but you've gotta do it unfortunately.

Example 2

Consider a supersonic flow with $M = 2$, $p = 1\,\text{atm}$ and $T = 288\,\text{K}$. This flow is deflected at a compression corner through 20°. Calculate M, p, T, p_0 and T_0 behind the oblique shock wave.

From figure 9.4, for $M_1 = 2$ and $\theta = 20°$, $\beta = 53.4°$. Thus, $M_{n,1} = 2\sin 53.4° = 1.606$. From appendix B, for $M_{n,1} = 1.60$ (rounded to the nearest entry), we then have

$$\begin{aligned} M_{n,2} &= 0.6684 \\ p_2 &= \frac{p_2}{p_1} p_1 = 2.82 \cdot 1 = 2.82\,\text{atm} \\ T_2 &= \frac{T_2}{T_1} T_1 = 1.388 \cdot 288 = 399.7\,\text{K} \end{aligned}$$

Furthermore, $M_2 = \frac{M_{n,2}}{\sin(\beta-\theta)} = \frac{0.6684}{\sin(53.4-20)} = 1.21$. Now, for the total conditions, you are *not* allowed to use $\frac{p_{0,2}}{p_1}$ in appendix B: this was derived for normal shocks only, and in its derivation, M_2, the actual flow Mach number, was used, rather than the normal component. For NSW, these two are equal, hence it may only be used for NSW. So, we need to look up $p_{0,2}/p_{0,1}$ (appendix B) and $p_{0,1}/p_1$ (appendix A) instead:

$$p_{0,2} = \frac{p_{0,2}}{p_{0,1}} \frac{p_{0,1}}{p_1} p_1 = 0.8952 \cdot 7.824 \cdot 1 = 7.00\,\text{atm}$$

For the total temperature, we have that it is constant across the shock, and thus

$$T_{0,2} = T_{0,1} = \frac{T_{0,1}}{T_1} T_1 = 1.8 \cdot 288 = 518.4\,\text{K}$$

Example 3

Consider an oblique shock wave with $\beta = 35°$ and a pressure ratio $p_2/p_1 = 3$. Calculate the upstream Mach number.

From appendix B, we have for $p_2/p_1 = 3$ that $M_{n,1} = 1.64$ (nearest entry) and thus

$$M_1 = \frac{M_{n,1}}{\sin\beta} = \frac{1.66}{\sin 35°} = 2.86$$

Example 4

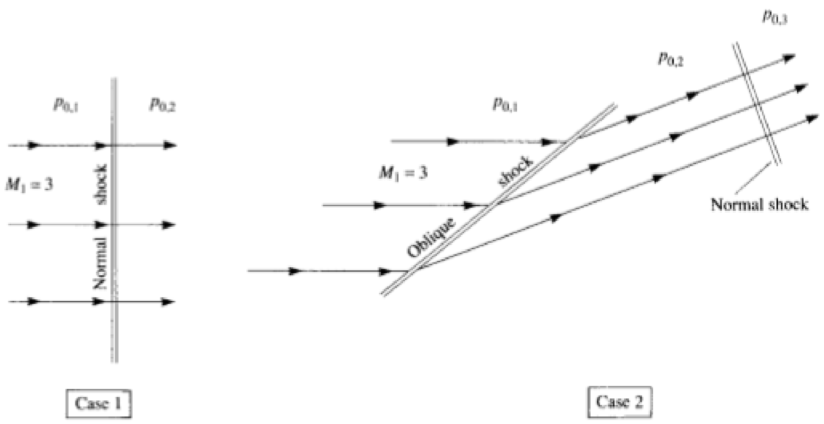


Figure 9.9: Example 4.

Consider a Mach 3 flow. It is desired to slow this flow to a subsonic speed. Consider two separate ways of achieving this: (1) the Mach 3 flow is slowed down by passing directly through a normal shock wave; (2) the Mach 3 flow first passes through a normal shock. These two cases are sketched in figure 9.9. Calculate the ratio of the final total pressure values for the two cases.

For (1), we can simply take directly from appendix B that

$$\left(\frac{p_{0,2}}{p_{0,1}}\right)_{\text{case 1}} = 0.3283$$

Computations are a little more work for case 2, as we have two pressure ratios to take into account. For the OSW, we have $M_{n,1} = M_1 \sin\beta = 3\sin 40° = 1.93$. We then have from appendix B that $\frac{p_{0,2}}{p_{0,1}} = 0.7535$ and $M_{n,2} = 0.588$. We now need to know the total Mach number after the OSW, which equals

$$M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)} = \frac{0.588}{\sin(40-22)} = 1.90$$

From appendix B, we then have $\frac{p_{0,3}}{p_{0,2}} = 0.7674$. Thus, we have

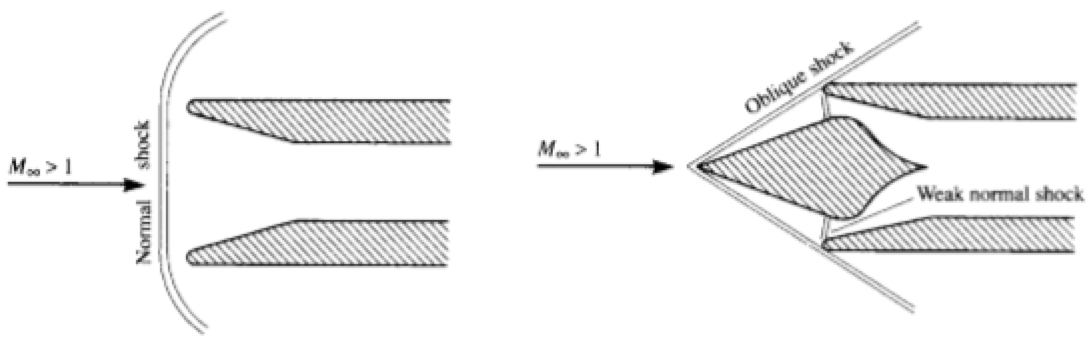
$$\left(\frac{p_{0,3}}{p_{0,2}}\right)_{\text{case 2}} = \left(\frac{p_{0,2}}{p_{0,1}}\right)\left(\frac{p_{0,3}}{p_{0,2}}\right) = 0.7535 \cdot 0.7674 = 0.578$$

We thus see that

$$\left(\frac{p_{0,3}}{p_{0,1}}\right)_{\text{case 2}} / \left(\frac{p_{0,2}}{p_{0,1}}\right)_{\text{case 1}} = \frac{0.578}{0.3283} = 1.76$$

So, in the second case, the final total pressure is 76% higher than in the first case. As said before, the total pressure of a flow is an indicator of how much useful work can be done by the gas. So, it's actually more efficient to have one OSW and one NSW to slow the flow down than to have only one NSW. The physical reason for this is straightforward: the loss in total pressure across a NSW becomes particularly severe as the upstream Mach number increases; the $\frac{p_{0,2}}{p_{0,1}}$ column in appendix B confirms this. If the Mach number of a flow can be reduced before passing through a NSW, the loss in total pressure is much less because the normal shock is weaker. This is the function of the oblique shock in case 2, namely to reduce the Mach number of the flow before passing through the normal shock. Although there is a total pressure loss across the OSW as well, it is much less than across a NSW for the same Mach number. The net effect of the OSW reducing the flow Mach number before passing through the NSW more than makes up for the total pressure loss across the OSW, with the beneficial result that the multiple shock system in case 2 produces a smaller loss in total pressure than a single NSW at the same freestream Mach number.

This example also explains why supersonic aircraft have oblique shock inlets, as shown in figure 9.10.



(a) Normal shock inlet.

(b) Oblique shock inlet.

Figure 9.10: Illustration of (a) normal shock inlet and (b) oblique shock inlet.

9.3 *Supersonic flow over wedges and cones*

Everything we've done so far is for 2D flows: wedges have been the primary example of a solid object placed inside a flow (similar to a cylinder for the 2D subsonic flow discussed in aero I). However, naturally, we also have 3D flows, and the equivalent of that in 3D is a cone. If you still remember, the 3D aspect had a relieving effect on the flow around a sphere, as the flow had more space to escape to so the flow velocity did not have to increase that much. The same holds for supersonic flow: to satisfy the continuity equation, the flow does not have to compress as much compared to the 2D case. This means that the shockwave will be weaker as well, and the wave angle will be smaller as well, as shown in figure 9.11. Note that as the shockwave is weaker, the pressure on the cone p_c will be less than the wedge surface pressure p_2 as well (and that M_c will be larger than M_2). And that's all we'll discuss about cones for this course.

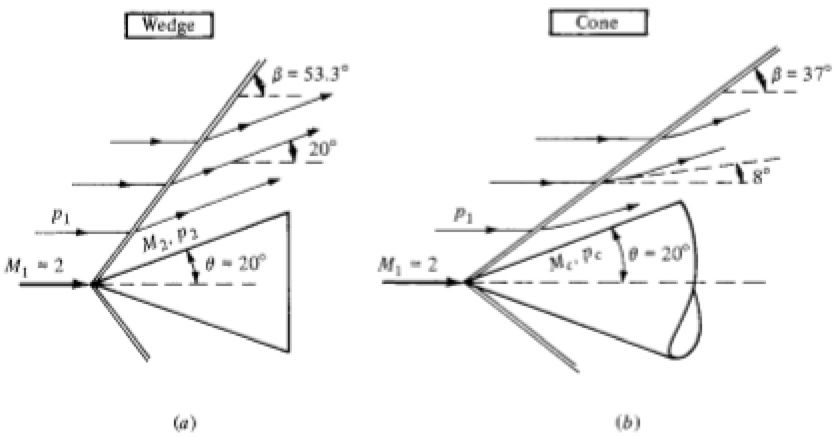


Figure 9.11: Relation between wedge and cone flow; illustration of the three-dimensional relieving effect.

Example 5

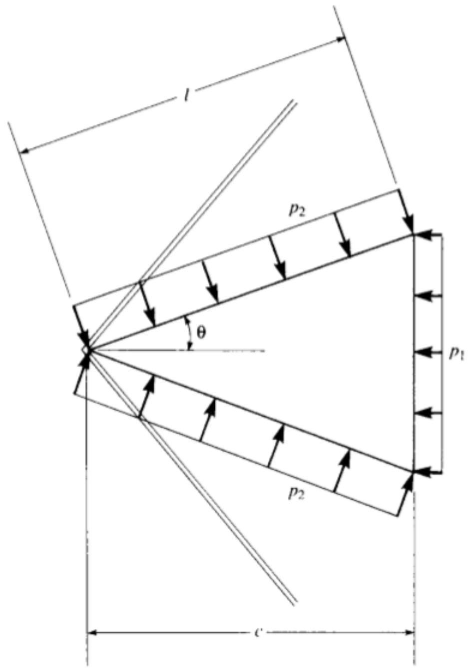


Figure 9.12: Example 5.

Consider a wedge with a 15° half angle in a Mach 5 flow, as sketched in figure 9.12. Calculate the drag coefficient for this wedge. Assume that the pressure over the base is equal to the freestream static pressure, as shown in figure 9.12.

The drag coefficient is given by $c_d = \frac{D'}{q_1 S} = \frac{D'}{q_1 c \cdot 1} = \frac{D'}{q_1 c}$. Now, onto D': the contribution of p_2 is given in positive x-direction by

$$2p_2 \cdot l \sin\theta$$

as the component of the pressure in positive x-direction equals $p_2 \sin\theta$ and the pressure acts on two times the length l (as there is both an upside and downside). For p_1, the vertical length equals $2l\sin\theta$, and thus the force per unit span in positive x-direction equals

$$-2p_1 \cdot l \sin\theta$$

And thus the drag per unit span becomes

$$D' = 2p_2 l \sin\theta - 2p_1 l \sin\theta = (2l\sin\theta)\left(p_2 - p_1\right)$$

With the substitution of $l = \frac{c}{\cos\theta}$, this can be written as

$$\begin{aligned} D' &= (2c\tan\theta)\left(p_2 - p_1\right) \\ c_d &= 2\tan\theta\left(\frac{p_2 - p_1}{q_1}\right) \end{aligned}$$

Now, finding a fancy expression for q_1: we have (remember $a_1^2 = \frac{\gamma p_1}{\rho_1}$)

$$q_1 \equiv \frac{1}{2}\rho_1 V_1^2 = \frac{1}{2}\rho_1 \frac{\gamma p_1}{\gamma p_1} V_1^2 = \frac{\gamma p_1}{2a_1^2} = \frac{\gamma}{2} p_1 M_1^2$$

so that we can write

$$c_d = 2\tan\theta \cdot \frac{p_2 - p_1}{(\gamma/2)\, p_1 M_1^2} = \frac{4\tan\theta}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right)$$

Now, as $M_1 = 5$ and $\theta = 15°$, we have from figure 9.4 that $\beta = 24.2°$ and thus $M_{n,1} = 5 \cdot \sin 24.2° = 2.05$, and from appendix B we have for $M = 2.05$ that $\frac{p_2}{p_1} = 4.736$, leading to

$$c_d = \frac{4\tan\theta}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right) = \frac{4\tan 15°}{1.4 \cdot 5^2}(4.736 - 1) = 0.114$$

Note that during aero I, for 2D inviscid flow, we saw that the drag coefficient was 0 (D'Alabmbert's paradox). However, now, even though we also have 2D inviscid flow, the drag coefficient is finite. Indeed, the paradox does not hold for freestream Mach numbers such that shock waves appear, as shocks are dissipative, drag-producing mechanisms. For this reason, the drag in this case is called **wave drag**, denoted by $c_{d,w}$ (the wave drag coefficient, that is).

Now, one final remark: note how the drag coefficient is a function of M only; we didn't need any information on the value of the freestream pressure etc.

9.4 *Shock interactions and reflections*

In reality, shock waves will of course not extend towards infinity, but will encounter other stuff as well. This section will deal with numerous interesting cases: these obviously do not cover all of the possible cases, but cover the most important ones:

- First of all, consider the OSW generated by a concave corner, as shown in figure 9.13. The deflection angle

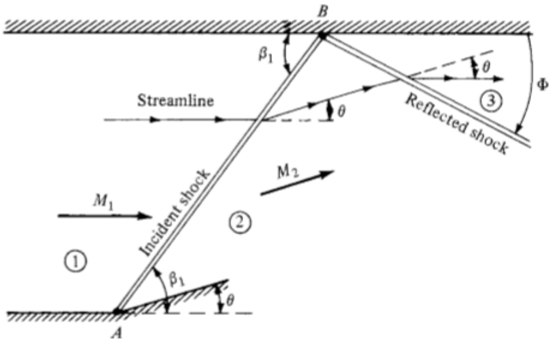


Figure 9.13: Regular reflection of a shock wave from a solid boundary.

θ causes a wave angle β_1. The shock wave generated at A, called the **incident shock wave**, impinges on the upper wall at B. Now, what happens at B? Basically, the flow now see a concave corner at B, with angle equal to θ. Hence, a new OSW is created, with angle Ψ. This second shock is called the **reflected shock wave**. As the Mach number decreased over incident shock wave, the wave angle of the reflected shock will be larger. Computations regarding this are relatively straightforward, though it can be annoying: the properties of the reflected shock are uniquely defined by M_2 and θ, and M_2 is in turn uniquely defined by M_1 and θ, so that the properties in region 3 behind the reflected shock as well as then angle ψ are easily determined from the given conditions of M_1 and θ as follows:

1. Calculate the properties in region 2 from the given M_1 and θ. This gives us M_2.
2. Calculate the properties in region 3 from the value of M_2 calculated above and the angle θ.

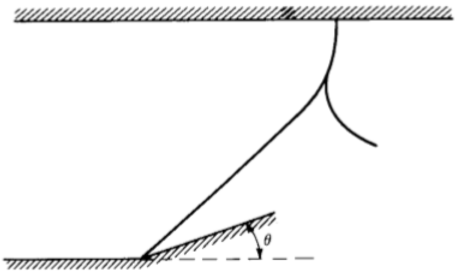


Figure 9.14: Mach reflection.

- Above situation can be complicated however: I already said that the flow after the incident shock wave experiences a lower Mach number, but it sees the same deflection angle θ. Now, it may be the case, that the initial Mach number is so low, that although an incident shock wave is formed, the Mach number at point B is too low to form a new shock wave (remember that $\theta_{\max}$ decreases as M_1 is decreased). Nature handles this by forming the wave pattern shown in figure 9.14. The exact properties of this pattern are beyond the scope of this course. This wave pattern is called a **Mach reflection**.

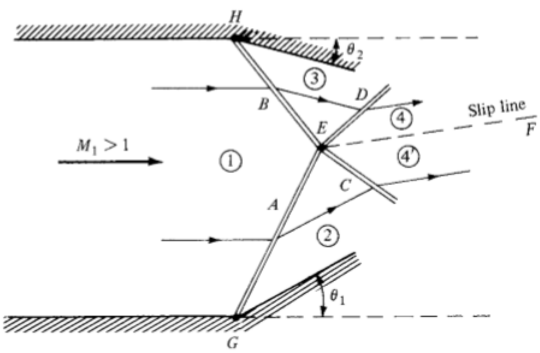


Figure 9.15: Intersection of right- and left-running shock waves.

- Another type of shock interaction is shown in figure 9.15, which looks complicated af but is still pretty straightforward. One shock wave (shock wave A) propagates from point G; this one is called a **left-running wave** if you stand on top of the wave, and look in the direction in which the flow moves, then you see the shock wave running in front of you to the left. Similarly, shock wave B is called a **right-running wave**. In point E, the shock waves come together; A is refracted and continues as D, and B is refracted and continues as C. Why does this happen? The flow through A is deflected upwards, whereas the flow through B is deflected downwards, meaning they'd be colliding, which cannot happen. Hence, both must be again deflected, causing two new shockwaves, to make sure they run parallel to each other and do not run into each other. The region behind D is denoted by region 4, the region behind C is denoted by region 4'. These regions are separated by a **slip line** EF. Across the slip line, the pressures are constant (so $p_4 = p_{4'}$) and the direction (but not necessarily the magnitude) of the velocity is the same as well, namely, parallel to the slip line. Everything else is different however, most notably the entropy ($s_4 \neq s_{4'}$). The conditions which must hold across the slip line, along with the known M_1 and θ, uniquely determine the shock-wave interaction shown in figure 9.15. For an example regarding how to compute the angles that the shock waves and slip lines make, see example 7.

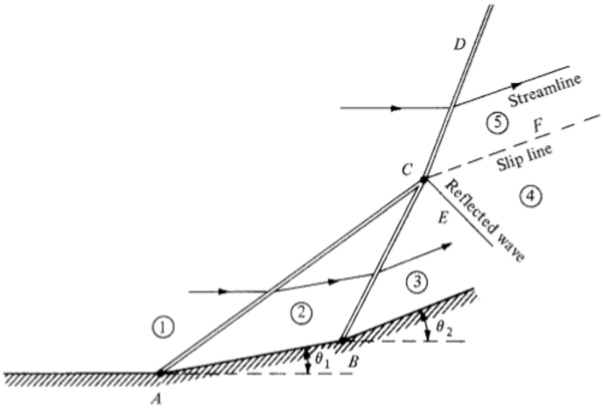


Figure 9.16: Intersection of two left-running shock waves.

- Finally, figure 9.16 illustrates the intersection of two left-running shocks generated at corners A and B. This intersection occurs at point C, where the two merge and propagate as the stronger shock CD, usually along with a weak reflected wave CE. This reflective wave is necessary to adjust the flow so that the velocities in regions 4 and 5 are in the same direction. Again, a slip line CF trails downstream of the intersection point. For an example about this, see example 8.

Example 6

Consider an oblique shock wave generated by a compression corner with a 10° deflection angle. The Mach number of the flow ahead of the corner is 3.6; the flow pressure and temperature are standard sea level conditions. The OSW subsequently impinges on a straight wall opposite the compression corner. The geometry for this flow is given in figure 9.13. Calculate the angle of the reflected shock wave ψ relative to the straight wall. Also, obtain the pressure, temperature and Mach number behind the reflected wave.

From figure 9.4, we have that $\beta = 24°$. Thus, $M_{n,1} = 3.6 \sin 24° = 1.464$. Then, from appendix B, we easily obtain

$$\begin{aligned}
M_{n,2} &= 0.7157 \\
\frac{p_2}{p_1} &= 2.32 \\
\frac{T_2}{T_1} &= 1.294 \\
M_2 &= \frac{M_{n,2}}{\sin(\beta - \theta)} = \frac{0.7157}{\sin(24 - 10)} = 2.96
\end{aligned}$$

Now, for the reflected shock, we thus have $M_2 = 2.96$ and $\theta = 10°$, and thus $\beta_2 = 27.3°$. However, this is *not* the angle the reflected shock makes w.r.t. the upper wall; it is the angle between the reflected shock and the direction of the flow in region 2. The shock angle relative to the wall is, by simple geometry in figure 9.13, equal to $\phi = \beta_2 - \theta = 27.3 - 10 = 17.3°$. However, we still have that the Mach number relative to the reflected shock is $M_2 \sin\beta_2 = 2.96 \sin 27.3° = 1.358$. Then, from appendix B, we find

$$\begin{aligned}\frac{p_3}{p_2} &= 1.991\\ \frac{T_3}{T_2} &= 1.229\\ M_{n,3} &= 0.7572\\ M_3 &= \frac{M_{n,3}}{\sin\left(\beta_2-\theta\right)} = \frac{0.7572}{\sin(27.3-10)} = 2.55\end{aligned}$$

For standard sea level conditions, $p_1 = 1\,\text{atm}$ and $T = 288\,\text{K}$, and thus we have

$$\begin{aligned}p_3 &= \frac{p_3}{p_2}\frac{p_2}{p_1}p_1 = 1.991 \cdot 2.32 \cdot 1 = 4.619\,\text{atm}\\ T_3 &= \frac{T_3}{T_2}\frac{T_2}{T_1}T_1 = 1.229 \cdot 1.294 \cdot 288 = 458\,\text{K}\end{aligned}$$

Note that the reflected shock wave is indeed weaker than the incident shock, visible from the fact that $\frac{p_3}{p_2} < \frac{p_2}{p_1}$.

Example 7

The inlet of a supersonic aircraft generates two shock waves that interact at point S, as shown in figure 9.17. As a result of the interaction two waves emanate (dotted lines) from point S. The freestream Mach number is 3 and the wedge deflection angles are $\theta_1 = 5°$ and $\theta_2 = 10°$.

(i) Determine both angles β_1 and β_2 of the shock waves produced by the wedges as well as the Mach number and pressure downstream of each wave.

(ii) Say whether the waves resulting from the interaction are of compression or expansion type. Justify your answer.

(iii) Sketch the flow field (streamlines, shock waves, possible expansion waves, slip-lines) before and after the interaction.

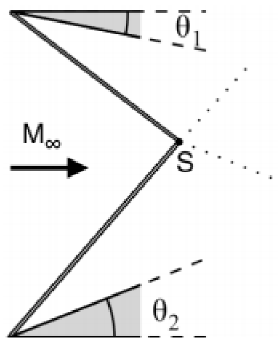


Figure 9.17: Example 7.

For (i), first, for shock wave 1, originating due to θ_1. For $M = 3$ and $\theta_1 = 5°$, we find from the $\theta - \beta - M$ relation that $\beta_1 = 23°$. This means that $M_{n,\infty} = 3 \cdot \sin 23° = 1.17$, and from appendix B

(using $M = 0.1180 + 01$)

$$\begin{aligned} M_{n,1} &= 0.8549 \\ M_1 &= \frac{0.8549}{\sin(23-5)} = 2.767 \\ \frac{p_1}{p_\infty} &= 1.458 \end{aligned}$$

Similarly, for $\theta_2 = 10°$, we have $\beta = 27°$. This means that $M_{n,\infty} = 3 \cdot \sin 27° = 1.362$, and thus, using $M = 0.1360 + 01$,

$$\begin{aligned} M_{n,2} &= 0.7572 \\ M_2 &= \frac{0.7572}{\sin(27-10)} = 2.590 \\ \frac{p_2}{p_\infty} &= 2.055 \end{aligned}$$

For (ii), β_1 deflects the flow downward whereas β_2 deflects the flow upward. Therefore, shock waves (so compression type) are necessary to turn the flow into itself, to prevent the flows from running into each other.

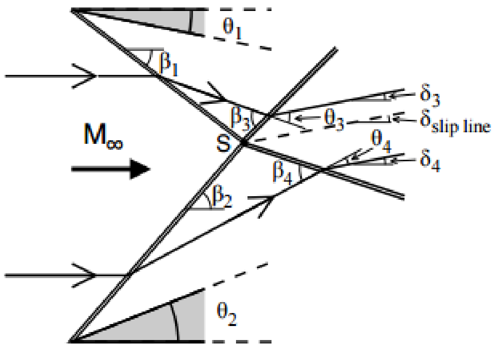


Figure 9.18: Example 7iii.

For (iii), see also figure 9.18. We must determine $\delta_{\text{slip line}}$. How do we do that? It's quite a pain in the ass, unfortunately, requiring iteration. We know from the slip line that the pressure of the flow that has passed through the lower two shock waves must be equal to the pressure of the flow that has passed through the upper two shock waves. So, we must determine δ such that p_3 and p_4 are more or less equal (with 3 being the region between the upper shock wave and the slip line and 4 the region between the lower shock wave and the slip line). Now, note that the deflection angle of the upper part of the flow will be $\theta_1 + \delta_{\text{slip line}}$, whereas for the lower part, it'll be $\theta_2 - \delta_{\text{slip line}}$.
For starters, let's take $\delta_{\text{slip line}} = 3°$. We then have that the deflection angle for the upper flow will be 8°; for the lower part, it'll be 7°. Then, for $M_1 = 2.767$ and $\theta_3 = 8°$, we have from the $\theta - \beta - M$ relation that $\beta_3 = 28°$, and $M_{n,1} = M_1 \sin\beta = 2.767 \sin 28° = 1.299$. Using appendix B, we thus have

$$\begin{aligned} \frac{p_3}{p_1} &= 1.805 \\ \frac{p_3}{p_\infty} &= 1.805 \cdot 1.458 = 2.63 \end{aligned}$$

For the lower flow, we find for $M_2 = 2.590$ and $\theta = 7°$ that $\beta_4 = 28°$, and thus $M_{n,2} = 2.590 \sin 28° =$

1.216, which means we can use appendix B to find

$$\frac{p_4}{p_2} = 1.570$$
$$\frac{p_4}{p_\infty} = 1.570 \cdot 2.055 = 3.226$$

This means that the lower part is significantly more compressed than the upper part of the flow. Thus, for an iteration, we must make $\delta_{\text{slip line}}$ larger, say, 5°. Then $\theta_3 = 10°$ and $\theta_4 = 5°$. We then do the exact same steps. This results in $\beta_3 = 29°$ and $\beta_4 = 26.5°$, leading to $M_{n,1} = 1.341$ and $M_{n,2} = 1.156$, and thus in

$$\frac{p_3}{p_1} = 1.928$$
$$\frac{p_3}{p_\infty} = 2.81$$
$$\frac{p_4}{p_2} = 1.403$$
$$\frac{p_4}{p_\infty} = 2.88$$

Not entirely sure on how accurate it is, but I think your first guess should be the angle $\theta_2 - \theta_1$ as long as they both remain small. That should get you pretty close the first time you do it.

Example 8

Consider a supersonic Mach 4 flow over a double compression ramp. The first ramp angle is $\delta_1 = 5°$ and the second is $\delta_2 = 10°$ with respect to the free stream. At both compression corners a shock wave (OSW_{01} and OSW_{12}) is generated and these shocks interact in point I. From the interaction a shock (OSW_{03}) and a second wave (indicated by the dotted line) results.

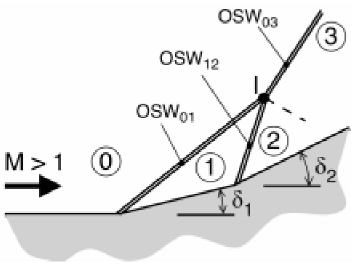


Figure 9.19: Example 8.

(i) Sketch the flow field including shock waves, expansion waves, streamlines and sliplines.
(ii) Determine the angles of the shocks OSW_{01}, OSW_{12} and OSW_{03}.
(iii) Determine the pressure in regions 2 and 3 and comment on the nature (compression or expansion) of the wave indicated by the dotted line.

For (i), pretty much everything is already drawn in figure 9.19. Only some streamlines and a slip line is missing. How to draw this can be seen in figure 9.16.

For (ii), let's do them one-by-one. First, OSW_{01}: with $M = 4$ and $\theta = 5°$, this means that $\beta_1 = 18°$, and thus $M_{n,0} = 4 \cdot \sin 18° = 1.236$. From appendix B, this means that $M_{n,1} = 0.8183$, meaning that $M_1 = \frac{0.8183}{\sin(18-5)} = 3.638$.

For OSW_{12}, we have $M = 3.638$ and $\theta = 10 - 5 = 5°$. Thus, $\beta_2 = 19.5°$ (and thus $M_{n,1} = 3.638 \sin 19.5° = 1.214$, which, according to appendix B, means that $M_{n,2} = 0.8300$, and thus $M_2 = \frac{0.8300}{\sin(19.5-5)} = 3.315$).

For OSW_{03}, we have that the deflection angle equals 10°: quite clearly from the figure, region 3 is already in the zone where $\delta_2 = 10°$. Thus, with $M = 4$ and $\theta = 10°$, this leads to $\beta_3 = 22°$, (meaning that $M_{n,0} = 4\sin 22° = 1.498$, which means that $M_{n,3} = 0.7011$, and thus $M_3 = \frac{0.7011}{\sin(22-10)} = 3.3721$).

First, let's compute the pressure in region 2 by computing the pressure in region 1. We have from appendix B that for $M_{n,0} = 1.236$ that $\frac{p_1}{p_0} = 1.627$, and for $M_{n,1} = 1.214$ we get $\frac{p_2}{p_1} = 1.513$, so that $\frac{p_2}{p_0} = 1.627 \cdot 1.513 = 2.462$. For region 3, we can directly compute that $\frac{p_3}{p_0} = 2.458$. So, the pressure is slightly higher region 2 than in region 3. Thus, through the wave denoted by the dotted line, the flow must be expanded, so it's an expansion wave.

9.5 *Detached shock wave in front of a blunt body*

I don't think this section is actually something you're supposed to know, but it is helpful in understanding how exactly shock waves happen.

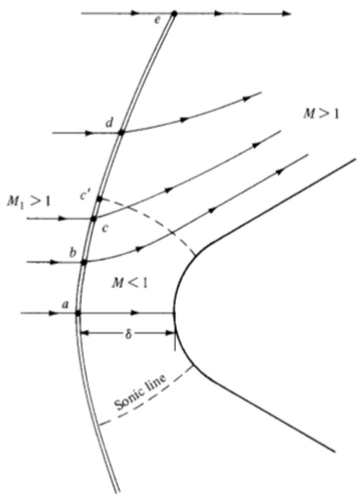


Figure 9.20: Flow over a supersonic body.

Quite often, we see bow shocks, as shown in figure 9.20. The wave stands a distance δ in front of the nose of the blunt body, where δ is called the **shock detachment distance**. At point a, the shock wave is normal to the upstream flow; hence, point a corresponds to a NSW. Away from point a, the shock wave gradually becomes curved and weaker, eventually evolving into a Mach wave at large distances from the body (illustrated by point e in figure 9.20).

A curved bow shock wave is one of the instances in nature when you can observe all possible shock solutions at once for a given freestream Mach number. We can see what happens in figure 9.21. First of all, as established already, at point a, there's a NSW, and the solution in figure 9.21 is denoted by a: the deflection angle θ does not correspond to the deflection angle of a physical wall (there isn't a wall there is there), but to the deflection angle of the flow itself (which is 0° as it'll stay horizontal). Far away from the body, at e, the deflection angle will also be 0°, corresponding to a Mach wave. Note that the NSW corresponds to the strong shock and e to the weak shock.

Now, what happens in between a and e? Note that at a, the NSW ensures that the flow is subsonic afterwards. If we go from a to a point b, then we see in figure 9.21 that although the wave angle is still almost 90°, the deflection angle has increased significantly, meaning that the flow is diverted away from the horizontal (which is necessary as there's a blunt body coming up). If we move to point c, we reach the maximum deflection angle,

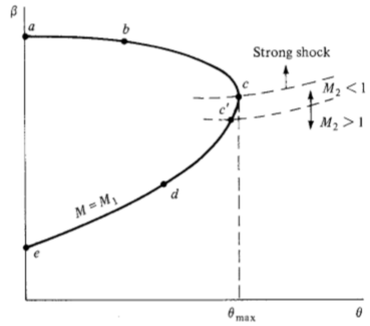


Figure 9.21: $\theta - \beta - M$ diagram for the sketch shown in figure 9.20.

but the flow is still subsonic. Only at point c' becomes the flow sonic; hence, the flow field behind the curved bow shock and the blunt body is a mixed region of both subsonic and supersonic flow, divided by the **sonic line**. At d, the deflection angle decreases again, and the bow shock becomes more horizontal as β decreases, until finally at e, a Mach wave forms and the deflection angle is 0°.

The main takeaway is that the deflection angle is not "defined" as the angle the blunt body makes, but rather the angle the flow makes after passing through a shock. So, don't think that if there's no wall, there can't be a deflection of the flow, etc.

9.6 Prandtl-Meyer expansion waves

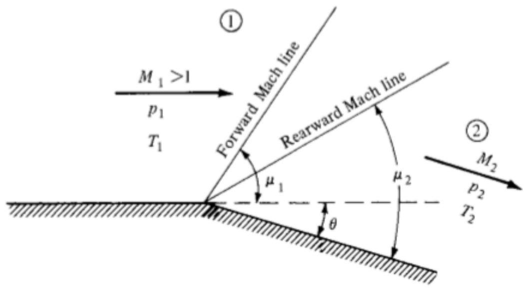


Figure 9.22: Prandtl-Meyer expansion.

Or PMEWs. This is the final difficult section of this chapter. We see the expansion fan in figure 9.22 as a continuous region of expansion that can be visualized as an infinite number of Mach waves, each making an angle μ with the local flow direction; bounded by a forward and rearward Mach line. Since the expansion through the wave takes place across a continuous succession of Mach waves, and since $ds = 0$ for each Mach wave, the expansion is isentropic. An expansion wave emanating from a sharp convex corner as sketched in figure 9.22 is called a **centered expansion wave**.

Now, what are we gonna do in this section? Given the upstream flow (region 1) and θ, calculate the downstream flow (region 2). Let us proceed.

Consider a very weak wave produced by an infinitesimally small flow deflection $d\theta$ as sketched in figure 9.23. We consider the limit as $d\theta \to 0$, so that it essentially a Mach wave at the angle μ to the upstream flow. The velocity ahead of the wave is V. This will be increased by the infinitesimal amount dV. Any change in velocity across a wave takes place *normal* to the wave; the tangential component is unchanged across the wave (remember section 9.2 where we proved this). In figure 9.23, the horizontal line segment AB with length V is drawn behind the wave; line segment AC is drawn to represent $V + dV$ and BC is normal to the wave because it represents the line along which the change in velocity occurs. Examining the geometry in figure 9.23, and using the law of

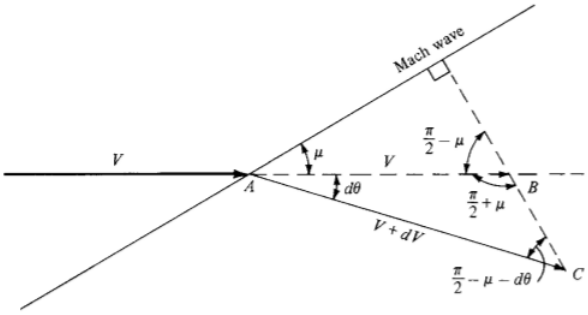


Figure 9.23: Geometrical construction for the infinitesimal changes across an infinitesimally weak wave (in the limit, a Mach wave).

sines for triangle ABC, we see that

$$\begin{aligned}\frac{V+dV}{\sin\left(\frac{\pi}{2}+\mu\right)} &= \frac{V}{\sin\left(\frac{\pi}{2}-\mu-d\theta\right)}\\ \frac{V+dV}{V} &= \frac{\sin\left(\frac{\pi}{2}+\mu\right)}{\sin\left(\frac{\pi}{2}-\mu-d\theta\right)}\end{aligned}$$

But we also have $\sin\left(\frac{\pi}{2}+\mu\right)=\sin\left(\frac{\pi}{2}-\mu\right)=\cos\mu$, and thus

$$\sin\left(\frac{\pi}{2}-\mu-d\theta\right)=\cos\left(\mu+d\theta\right)=\cos\mu\cos d\theta-\sin\mu\sin d\theta$$

leading to

$$1+\frac{dV}{V}=\frac{\cos\mu}{\cos\mu\cos d\theta-\sin\mu\sin d\theta}$$

and for small $d\theta$, $\cos d\theta\approx 1$ an $\sin d\theta\approx d\theta$:

$$1+\frac{dV}{V}=\frac{\cos\mu}{\cos\mu-d\theta\sin\mu}=\frac{1}{1-d\theta\tan\mu}$$

Now, onto your favourite subject in mathematics, series expansion:

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots$$

and ignoring terms of second order and higher, this leads to

$$1+\frac{dV}{V}=1+d\theta\tan\mu$$

and thus

$$d\theta=\frac{dV/V}{\tan\mu}$$

Now, only the term $\tan\mu$ isn't that nice. However, we know that $\sin\mu=\frac{1}{M}$, which, if we'd draw the associated triangle, leads to figure 9.24. From this, we also see that $\tan\mu=\frac{1}{\sqrt{M^2-1}}$, meaning that we can write

$$\boxed{d\theta=\sqrt{M^2-1}\frac{dV}{V} \qquad (9.8)}$$

FORMULA

If you don't exactly see what $\frac{dV}{V}$ means: it's basically the change in velocity normalized (by dividing by V, you get a relative change rather than an absolute change). This equation relates the infinitesimal change in velocity

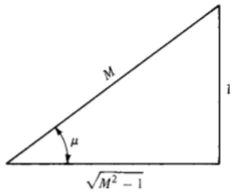


Figure 9.24: Right triangle associated with the Mach angle.

dV to the infinitesimal deflection $d\theta$ across a wave of vanishing strength. In the precise limit of a Mach wave, of course dV and hence $d\theta$ are zero. Now, quite naturally, we can integrate all of this:

$$\int_0^\theta = \theta = \int_{M_1}^{M_2} \sqrt{M^2-1}\frac{dV}{V}$$

Now, the integration on the right hand side is a kind of a bitch, as we must express $\frac{dV}{V}$ in terms of M. Just follow along with what I'm doing because I don't think you would have come up with this yourself. Remember that $M = \frac{V}{a}$ and thus $V = Ma$, or $\ln V = \ln M + \ln a$. Differentiating this yields

$$\frac{dV}{V} = \frac{dM}{M} + \frac{da}{a}$$

However, we then use the equation of state to write

$$\left(\frac{a_0}{a}\right)^2 = \frac{\gamma RT_0}{\gamma RT} = \frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2$$

Leading to

$$\begin{aligned} a &= a_0\left(1+\frac{\gamma-1}{2}M^2\right)^{-1/2} \\ da &= -\frac{a_0}{2}(\gamma-1)M\left(1+\frac{\gamma-1}{2}M^2\right)^{-3/2}dM = -a_0\left(\frac{\gamma-1}{2}\right)M\left(1+\frac{\gamma-1}{2}M^2\right)^{-3/2}dM \\ \frac{da}{a} &= -\left(\frac{\gamma-1}{2}\right)M\left(1+\frac{\gamma-1}{2}M^2\right)^{-1}dM \end{aligned}$$

Finally, we have our expression for $\frac{dV}{V}$:

$$\begin{aligned} \frac{dV}{V} &= \frac{dM}{M} - \left(\frac{\gamma-1}{2}\right)M\left(1+\frac{\gamma-1}{2}M^2\right)^{-1}dM = \frac{dM}{M}\left(1-\frac{\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}M^2}\right) = \frac{dM}{M}\left(\frac{1+\frac{\gamma-1}{2}M^2-\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}M^2}\right) \\ &= \frac{1}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M} \end{aligned}$$

Now, we can integrate quite nicely here:

$$\theta = \int_{M_1}^{M_2} \frac{\sqrt{M^2-1}}{1+\left[(\gamma-1)/2\right]M^2}\frac{dM}{M}$$

which is absolutely awful to integrate. Instead, we'll introduce the **Prandtl-Meyer function**, denoted by υ as

$$\upsilon(M) \equiv \frac{\sqrt{M^2-1}}{1+\left[(\gamma-1)/2\right]M^2}\frac{dM}{M}$$

Note that this is a *indefinite integral*. Carrying out the integration, this becomes

FORMULA

$$\upsilon(M) = \sqrt{\frac{\gamma+1}{\gamma-1}} \tan^{-1} \sqrt{\frac{\gamma-1}{\gamma+1}\left(M^2-1\right)} - \tan^{-1}\sqrt{M^2-1} \tag{9.9}$$

Now, before you start complaining where's the constant of integration, it can be left out here, as you will use the definite integral at all times, as we now have

FORMULA

$$\theta = \upsilon\left(M_2\right) - \upsilon\left(M_1\right) \tag{9.10}$$

Note that this also means that $\upsilon(1) = 0$ which is convenient. How can we know now the properties in region 2 from the known properties in region 1 and the known deflection angle θ?

1. For the given M_1, obtain $\upsilon\left(M_1\right)$ from appendix C (these outcomes are tabulated there).
2. Calculate $\upsilon\left(M_2\right)$ from $\theta = \upsilon\left(M_2\right) - \upsilon\left(M_1\right)$ which should not pose any problem at all, really.
3. Obtain M_2 from appendix C corresponding to the value of $\upsilon\left(M_2\right)$ from the previous step.
4. The expansion wave is isentropic, hence p_0 and T_0 are constant. This means that $\frac{p_{0,2}}{p_{0,1}}$ and $\frac{T_{0,2}}{T_{0,1}}$ are both 1. Furthermore, you can either use appendix A now to compute the pressure and temperature, or use the equations

FORMULAS

$$\frac{T_2}{T_1} = \frac{T_2/T_{0,2}}{T_1/T_{0,1}} = \frac{1+\left[(\gamma-1)/2\right] M_1^2)}{1+\left[(\gamma-1)/2\right] M_2^2} \tag{9.11}$$

$$\frac{p_2}{p_1} = \frac{p_2/p_{0,2}}{p_1/p_{0,1}} = \left(\frac{1+\left[(\gamma-1)/2\right] M_1^2)}{1+\left[(\gamma-1)/2\right] M_2^2}\right)^{\gamma/(\gamma-1)} \tag{9.12}$$

I'll be using the appendix but you're free to use these (more accurate) formulas.

Example 9

A supersonic flow with $M_1 = 1.5$, $p_1 = 1\,\text{atm}$ and $T_1 = 288\,\text{K}$ is expanded around a sharp corner through a deflection angle of 15°. Calculate M_2, p_2, T_2, $p_{0,2}$, $T_{0,2}$ and the angles that the forward and rearward Mach lines make w.r.t. the upstream flow direction.

From appendix C, for $M_1 = 1.5$, we have $\upsilon_1 = 11.91°$, and thus with $\theta = \upsilon_2 - \upsilon_1$, we have

$$\upsilon_2 = \upsilon_1 + \theta = 11.91 + 15 = 26.91°$$

which is approximately $M_2 = 2.000$ (look up $\upsilon = 0.2638+02$, the closest entry), for which $\mu = 30°$, so the angle with the upstream flow direction for the rearward Mach line is simply $\mu - \theta = 30 - 15 = 15°$. For the forward Mach line, it is simply

$$\mu = \arcsin\frac{1}{M} = \arcsin\frac{1}{1.5} = 41.81°$$

Now, onto the pressure. Using appendix A, you can straightforwardly look up that $\frac{p_{0,2}}{p_2}$ for $M_2 = 2.000$ equals 7.824, and $\frac{p_{0,1}}{p_1}$ for $M_1 = 1.5$ equals 3.671, and since $p_{0,2} = p_{0,1}$ we can readily deduce the value of p_2:

$$p_2 = \frac{p_2}{p_{0,2}}\frac{p_{0,2}}{p_{0,1}}\frac{p_{0,1}}{p_1}p_1 = \frac{1}{7.824}\cdot 1 \cdot 3.671 \cdot 1 = 0.469\,\text{atm}$$

We can do exactly the same for T_2:

$$T_2 = \frac{T_2}{T_{0,2}}\frac{T_{0,2}}{T_{0,1}}\frac{T_{0,1}}{T_1}T_1 = \frac{1}{1.8}\cdot 1 \cdot 1.45 \cdot 288 = 232\,\text{K}$$

And the total pressure and total temperature are even easier to compute:

$$p_{0,2} = p_{0,1} = \frac{p_{0,1}}{p_1} p_1 = 3.671 \cdot 1 = 3.671\,\text{atm}$$
$$T_{0,2} = T_{0,1} = \frac{T_{0,1}}{T_1} T_1 = 1.45 \cdot 288 = 417.6\,\text{K}$$

Again, it's mostly just a matter of looking stuff up, which arguably is not very fun, but it's not difficult. It goes without saying however, that looking up $\frac{p_2}{p_1}$ and $\frac{T_2}{T_1}$ from appendix B makes you look like a complete fool.

9.7 Shock-expansion theory: applications to supersonic airfoils

We've already seen this happen before, in example 5 of this chapter. Indeed, it's arguably easier than for subsonic flows. Why? Because for subsonic flows, the properties of the flow changed continuously, meaning you had to integrate pressures and stuff. However, for supersonic flows, the pressure only changes at certain places.

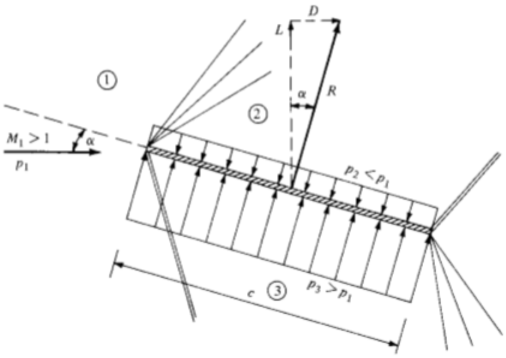


Figure 9.25: Flat plate at an angle of attack in a supersonic flow.

Let's see how we'd compute the lift on a thin airfoil shown in figure 9.25. Rather elementary, we haev per unit span that

$$R' = (p_3 - p_2)\, c$$
$$L' = (p_3 - p_2)\, c \cos\alpha$$
$$D' = (p_3 - p_2)\, c \sin\alpha$$

p_3 can easily be determined from OSW stuff, and p_2 can easily be determined from PMEW stuff. Using this stuff is called the **shock-expansion theory**. A more complicated example is shown in figure 9.26, although it's still really straightforward. Calculate p_2 from OSW, and p_3 from PMEW, and you're done if you simply use (due to line symmetry around the x-axis):

$$D' = 2\left(p_2 l \sin\epsilon - p_3 l \sin\epsilon\right) = 2\left(p_2 - p_3\right)\frac{t}{2} = \left(p_2 - p_3\right) t$$

9.10 Viscous flow: shock-wave/boundary layer interaction

Not something you really need to know for the exam, but it's interesting to note why our computations won't be 100% correct, due to the presence of a boundary layer: suppose we have a wall, with a boundary layer on top of it, as shown in figure 9.27. A shock wave impinges on the wall somewhere in the middle. We know that this shock wave causes a huge adverse pressure gradient (as the pressure is suddenly significantly increased over a

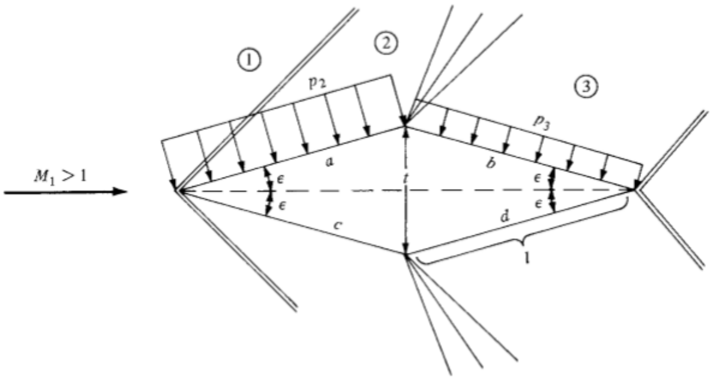


Figure 9.26: Diamond-wedge airfoil at zero angle of attack in a supersonic flow.

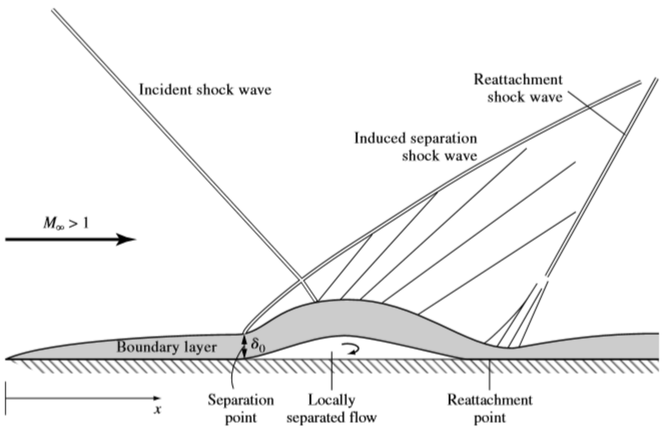


Figure 9.27: Schematic of the shock-wave/boundary-layer interaction.

shock wave). Hence, the boundary layer separates, turning the flow into itself at the separation point, causing another shock wave, called the induced separation shock wave. After a while, the boundary layer reattaches, and at the reattachment point, the flow is once again turned into itself, causing a reattachment shock wave. In between, the flow is turned away from itself, causing expansion waves. Note that this occurs both for laminar and turbulent boundary layers.

10 Flow through Nozzles, Diffusers and Wind Tunnels

10.2 Governing equations for quasi-one-dimensional flow

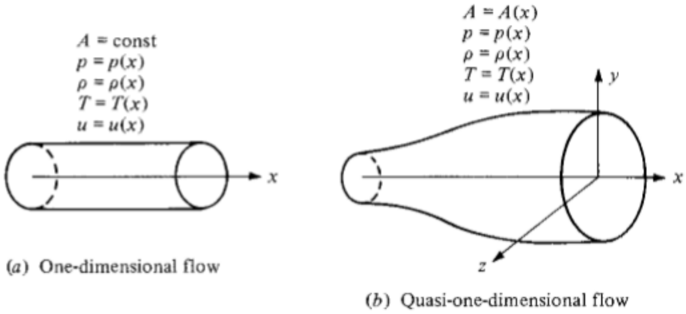


Figure 10.1: One-dimensional and quasi-one-dimensional flows.

Quasi-one-dimensional flow is a flow that is *assumed* to vary with x only (i.e. only in the direction of the flow, but it does not vary along any cross section at a given x station). The quasi part is added to distinguish it from real one-dimensional flow, which has constant cross-sectional area (see figure 10.1).

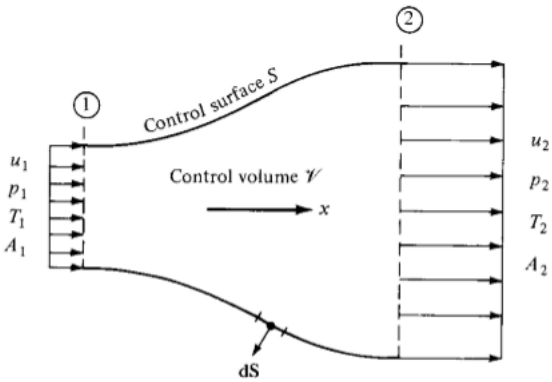


Figure 10.2: Finite control volume for quasi-one-dimensional flow.

Quite logically, we have that the continuity equation equals (using figure 10.2 as control volume):

FORMULA

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2 \tag{10.1}$$

The momentum equation is slightly more annoying:

$$\oiint_S (\rho \mathbf{V} \cdot d\mathbf{S})\, \mathbf{V} = -\oiint_S p\, d\mathbf{S}$$

Although it reduces to

$$\oiint_S (\rho \mathbf{V} \cdot d\mathbf{S})\, u = -\oiint_S (p\, dS_x)$$

Now, the integral on the left simply becomes $-\rho_1 u_1^2 A_1 + \rho_2 u_2^2 A_2$ (remember that $\mathbf{V} \cdot d\mathbf{S}$ is negative for A_1, as u_1 points in the opposite direction of dS). However, the right integral is more of a pain in the ass. We have four surfaces: the two flat surfaces A_1 and A_2, and the two curved surfaces connecting them. Over A_1 and A_2, the integral equals $-p_1A_1 + p_2A_2$ (A_1 gets a negative sign because dS points to the left for A_1). For the curved surfaces, it's slightly less obvious. We must integrate the pressure "between" the two areas A_1 and A_2, by evaluating the integral

$$-\int_{A_1}^{A_2} -p\,dA = \int_{A_1}^{A_2} p\,dA$$

where dA is the x-component of the area dS (so the area dS projected on an axis perpendicular to the x-axis). The negative sign appears in front of $p\,dA$ due to the fact that dS clearly points in negative direction along the entire upper and lower surface in figure 10.2. This means that the momentum equation can be written as

$$-\rho_1u_1^2A_1 + \rho_2u_2^2A_2 = -\left(-p_1A_1 + p_2A_2\right) + \int_{A_1}^{A_2} p\,dA$$

or

FORMULA

$$p_1A_1 + \rho_1u_1^2A_1 + \int_{A_1}^{A_2} p\,dA = p_2A_2 + \rho_2u_2^2A_2 \tag{10.2}$$

Now, the energy equation:

$$\oiint_S \rho\left(e + \frac{V^2}{2}\right)\mathbf{V}\cdot d\mathbf{S} = -\oiint_S p\mathbf{V}\cdot d\mathbf{S}$$

Applied to the control volume in figure 10.2, this yields

$$\begin{aligned}
\rho_1\left(e_1 + \frac{u_1^2}{2}\right)\left(-u_1A_1\right) + \rho_2\left(e_2 + \frac{u_2^2}{2}\right)\left(u_2A_2\right) &= -\left(-p_1u_1A_1 + p_2u_2A_2\right)\\
p_1u_1A_1 + \rho_1A_1\left(e_1 + \frac{u_1^2}{2}\right) &= p_2u_2A_1 + \rho_2u_2A_2 + \rho_2u_2A_2\left(e_2 + \frac{u_2^2}{2}\right)
\end{aligned}$$

Now, divide this by the continuity equation $\rho_1u_1A_1 = \rho_2u_2A_2$ to get

$$\frac{p_1}{\rho_1} + e_1 + \frac{u_1^2}{2} = \frac{p_2}{\rho_2} + e_2 + \frac{u_2^2}{2}$$

and with $h = e + pv = e + \frac{p}{\rho}$, this becomes

FORMULA

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} \tag{10.3}$$

Note that this means that h_0 = constant. We now need to two more equations, namely

FORMULAS

$$\begin{aligned}
p_2 &= \rho_2RT_2 \qquad (10.4)\\
h_2 &= c_pT_2 \qquad (10.5)
\end{aligned}$$

Now, for the three previous equations, let's find the differential expressions for them, because we're gonna use them later on. We have for the continuity equation:

$$\rho uA = \text{constant}$$

FORMULA

$$d(\rho u A) = 0 \tag{10.6}$$

For the momentum equation, it's slightly more complicated. We can write $p_2 = p + dp$, $\rho_2 = \rho + d\rho$, etc., so that we have (note that as we assume dA to be small, we can assume that p is constant over dA and thus that the integral reduces to its integrand):

$$pA + \rho u^2 A + p\,dA = (p + dp)(A + dA) + (\rho + d\rho)(u + du)^2 (A + dA)$$

Now, all products of differentials can be ignored as they'll be very small. Thus, it reduces to

$$A\,dp + Au^2 d\rho + \rho u^2 dA + 2\rho u A du = 0$$

But, we can write the continuity equation as

$$(d\rho)\,uA + \rho\,(du)\,A + \rho u\,(dA) = 0$$

(simply the chain rule), so that upon rewriting and multiplying by u, we get

$$\rho u^2 dA + \rho u A\,du + Au^2 d\rho = 0$$

Now, upon comparing this with the earlier found result from the momentum equation and subtracting this equation from that one, we realize that we can write

$$Adp + \rho u A du = 0$$

and thus

FORMULA: EULER EQUATION

$$dp = -\rho u\,du \tag{10.7}$$

This equation is called the **Euler equation**.

For the energy equation, we have

$$h + \frac{u^2}{2} = \text{constant}$$

FORMULA

$$dh + u\,du = 0 \tag{10.8}$$

Now, you may have been wondering, why are we doing all of this again if we did in chapter 2 of aero I as well? The reason for this is quite fundamental: in chapter 2, we derived the equations for truly three-dimensional flows. We were able to decompose all of the equations into x, y and z components, but when we did this, we maintained the aspect of truly one-dimensional flow: that means, the cross-sectional area does not change. This means that if we for example had used the result from chapter 2 for the continuity equation, we'd have ended up at

$$d(\rho u) = 0$$

In other words, we have an A in this chapter which wasn't there in chapter 2. That is because we derived these equations for *quasi*-one-dimensional flow, which means that the cross-sectional area *does* change. It's not really important for anything as long as you just use the equations we derive in this chapter, but it's an important assumption to keep in mind.

Now, let's go back to the differential form of the continuity equation, i.e. $d(\rho u A)$. We can write this as

$$uA\,d\rho + \rho A\,du + \rho u\,dA = 0$$

as shown before. Dividing by $\rho u A$ thus yields

$$\frac{d\rho}{\rho} + \frac{du}{u} + \frac{dA}{A} = 0$$

Now, let's replace $\frac{d\rho}{\rho}$ by something. From the differential form of the momentum equation, $dp = -\rho u\, du$, we have

$$\frac{dp}{\rho} = \frac{dp}{d\rho}\frac{d\rho}{\rho} = -u\, du$$

Furthermore, let's assume that we don't have any shock waves in the flow, so that the flow is isentropic, so that any change in density $d\rho$ w.r.t. a change in pressure dp takes place isentropically, i.e.

$$\frac{dp}{d\rho} \equiv \left(\frac{\partial p}{\partial \rho}\right)_s = a^2$$

So that

$$\begin{aligned} \frac{dp}{d\rho}\frac{d\rho}{\rho} = a^2\frac{d\rho}{\rho} &= -u\, du \\ \frac{d\rho}{\rho} &= -\frac{u\, du}{a^2} = -\frac{u^2}{a^2}\frac{du}{u} = -M^2\frac{du}{u} \end{aligned}$$

and thus we can rewrite the continuity equation as

$$-M^2\frac{du}{u} + \frac{du}{u} + \frac{dA}{A} = 0$$

leading to the familiar

FORMULA: AREA-VELOCITY RELATION

$$\frac{dA}{A} = \left(M^2 - 1\right)\frac{du}{u} \tag{10.9}$$

This formula is called the **area-velocity relation**. From this formula, we can deduce the following three important pieces of information:

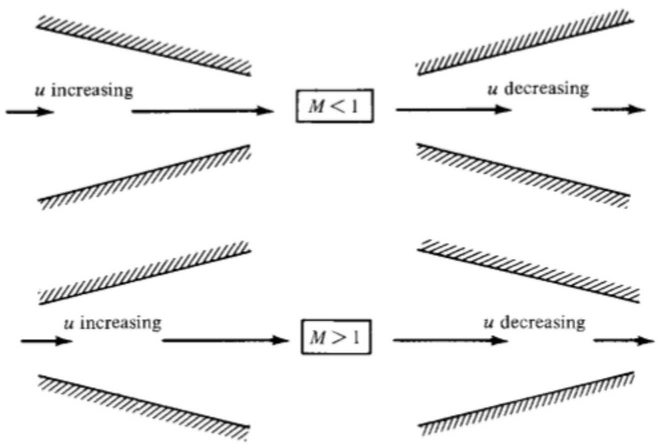
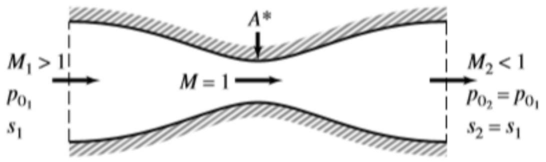


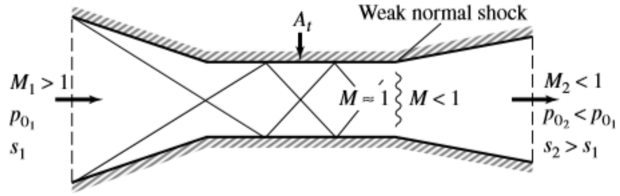
Figure 10.3: Compressible flow in converging and diverging ducts.

1. For $0 \leq M < 1$ (subsonic flow), du will be positive for negative values of dA, i.e., if the area decreases, the flow velocity will increase and vice versa. This is illustrated at the top of figure 10.3.
2. For $M > 1$ (supersonic flow), du will be positive for positive values of dA, i.e., if the area increases, the flow velocity will increase and vice versa. This is illustrated at the bottom of figure 10.3.
3. For $M = 1$, $dA = 0$ even though du can take any value. This corresponds to a minimum area, called the **throat**.

So, to create a supersonic wind tunnel by accelerating the gas isentropically, we need to have a convergent-divergent duct, as shown in figure 10.4a. The minimum area of the duct is called the throat. Whenever the flow accelerates from subsonic to supersonic speeds, the flow must pass through a throat, where $M = 1$ at the throat. The converse is also true: if we wish to slow a supersonic flow down to subsonic speeds, we need to use a convergent-divergent duct as well.



(a) Supersonic nozzle.



(b) Supersonic diffuser.

Figure 10.4: Illustration and comparison of a supersonic nozzle and a supersonic diffuser.

10.3 Nozzle flows

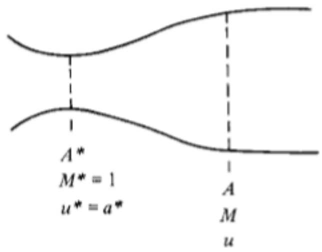


Figure 10.5: Geometry for the derivation of the area-Mach number relation.

Now, going into a little more detail. Assume that sonic flow exists at the throat, where the area is A^*. The Mach number and velocity at the throat are denoted by M^* and u^* respectively. Since the flow is sonic at the throat, $M^* = 1$ and $u^* = a^*$.[1] At any other section of this duct, the area, Mach number and velocity are denoted by A, M and u respectively. The continuity equation thus dictates that

$$\rho^* u^* A^* = \rho u A$$

and with $u^* = a^*$, this can be rewritten to

$$\frac{A}{A^*} = \frac{\rho^*}{\rho}\frac{a^*}{a} = \frac{\rho^*}{\rho_0}\frac{\rho_0}{\rho}\frac{a^*}{u}$$

But we already derived in chapter 8 that

$$\begin{aligned}
\frac{\rho^*}{\rho_0} &= \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} \\
\frac{\rho_0}{\rho} &= \left(1+\frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)}
\end{aligned}$$

$$\left(\frac{u}{a^*}\right)^2 = M^{*2} = \frac{\left[(\gamma+1)/2\right]M^2}{1+\left[(\gamma-1)/2\right]M^2}$$

Squaring the original equation so that we can make the substitutions leads to:

$$\begin{aligned}
\left(\frac{A}{A^*}\right)^2 &= \left(\frac{\rho^*}{\rho_0}\right)^2\left(\frac{\rho_0}{\rho}\right)^2\left(\frac{a^*}{u}\right)^2 \\
&= \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}\left(1+\frac{\gamma-1}{2}M^2\right)^{2/(\gamma-1)}\frac{1+\left[(\gamma-1)/2\right]M^2}{\left[(\gamma+1)/2\right]M^2} \\
&= \left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{2/(\gamma-1)}\cdot\frac{2+(\gamma-1)M^2}{\gamma+1}\cdot\frac{1}{M^2} = \frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{2/(\gamma-1)+1}
\end{aligned}$$

[1] If you haven't noticed before, but just like in chapter 8, we use asterisks to denote the sonic conditions.

The exponent can be more nicely written as

$$\frac{2}{\gamma-1}+1=\frac{2}{\gamma-1}+\frac{\gamma-1}{\gamma-1}=\frac{\gamma+1}{\gamma-1}$$

leading to

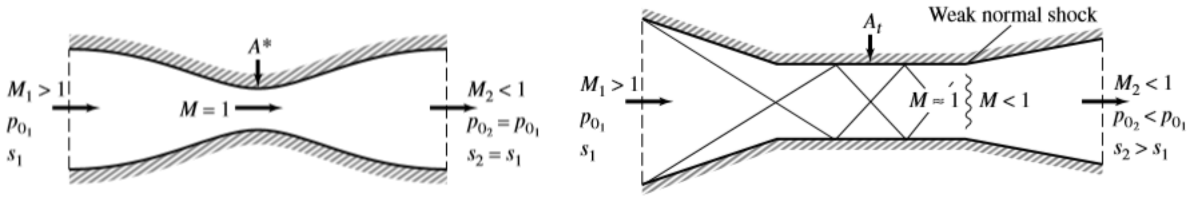
FORMULA: AREA-MACH NUMBER RELATION

$$\left(\frac{A}{A^*}\right)^2=\frac{1}{M^2}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}M^2\right)\right]^{(\gamma+1)/(\gamma-1)} \tag{10.10}$$

This equation is very important and is called the **area-Mach number relation**. This tells us that M is a function of $\frac{A}{A^*}$. Note that $A \geq A^*$, $A < A^*$ is not physically possible for an isentropic flow. Furthermore, there are always two solutions for M for a given $\frac{A}{A^*}$, a subsonic value and a supersonic value. The results for $\frac{A}{A^*}$ as a function of M are tabulated in appendix A. Naturally, you can also use this appendix as a reverse look-up. We see in appendix A indeed that while $M < 1$, if we increase M, $\frac{A}{A^*}$ decreases; for $M = 1$, $\frac{A}{A^*} = 1$; while $M > 1$, if we increase M, $\frac{A}{A^*}$ increases.

Now, the question remains, how exactly can we now calculate the properties anywhere along the wind tunnel, if we know the area A as a function of x (so how the cross-sectional area changes along the duct)? For this, it is important to consider how exactly a wind tunnel works. If we build a convergent-divergent duct, does a flow immediately start flowing? Of course not. We can create a flow by having the exit have a lower pressure than the inlet (quite obvious, I hope). Now, the inlet is usually fed by an enormous reservoir, where it's at a standstill (so $M \approx 0$ there), so that $p_0 = p \approx 1\,\text{atm}$, and $T_0 = T = 288\,\text{K}$. We then create a pressure differential by lowering the pressure in the exit of the throat ourselves; this will pressure the air to flow from the reservoir to the exit.

Now, we realize something important from appendix A. If we want to have a supersonic flow at the exit, which has a fixed ratio $\frac{A_e}{A^*}$ (where A_e is obviously the cross-sectional area of the exit), then there exists only only *one* solution for M and, more importantly right now, only *one* solution for $\frac{p}{p_0}$. If we have this *exact* pressure $p_{e,6}$ (you'll see why we use a subscript 6 later on), the flow properties will be as shown in figure 10.6. If we don't have this exact pressure, the flow will be different. We'll analyse shortly what'll occur then.



(a) Geometry and Mach number variation for $p_e = p_{e,6}$. (b) Pressure and temperature variation for $p_e = p_{e,6}$.

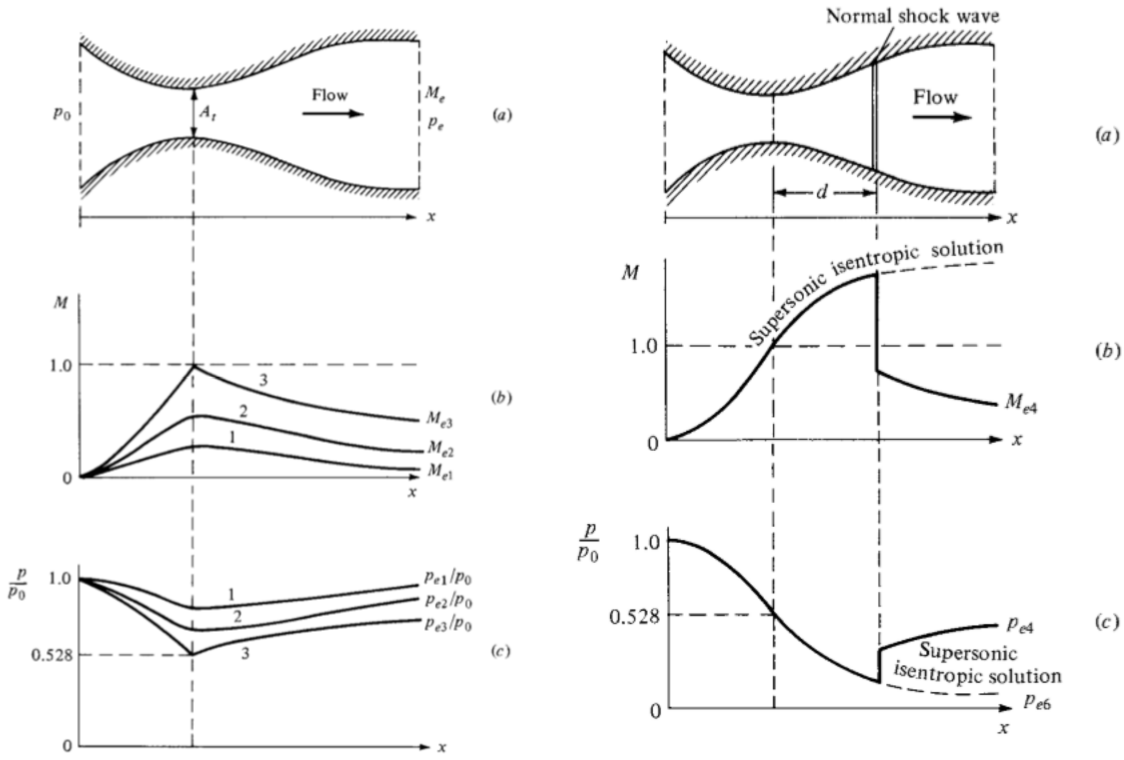
Figure 10.6: Isentropic supersonic nozzle flow.

For now, let's assume we're able to get exactly that $p_{e,6}$ at the exit. How can we then compute the flwo properties at any point along the duct (if we know how A varies with x)? It's then relatively straightforward:

1. The local Mach number as a function of x is obtained from the tabulated values in appendix A, by looking up $\frac{A}{A^*}$ in the right most column. If you are in the convergent portion of the nozzle, take the subsonic Mach numbers; if you are in the divergent part, take the supersonic Mach numbers.
2. For the variation of temperature, pressure and density, also use appendix A.

Note that all properties are basically only dependent on $\frac{A}{A^*}$. Now, like I promised you, we'll now discuss what happens if we are *not* able to get that pressure $p_{e,6}$ at the exit.

First of all, let's get back to basics: as I said, we create a flow in a convergent-divergent nozzle by decreasing the exit pressure. If we don't do anything ourselves, then $p_e = p_0$, and now flow exists. Now, suppose we start decreasing p_e:



(a) Isentropic subsonic nozzle flow.

(b) Supersonic nozzle flow with a normal shock inside the nozzle.

Figure 10.7: Isentropic supersonic nozzle flow.

1. Let's decrease it very slightly, to a value $p_{e,1} = 0.999p_0$. In this case, only a gentle flow will be created, and a very low-speed subsonic flow will be created. The local Mach number will increase slightly through the convergent portion, reaching a maximum value at the throat, as shown by curve 1 in figure 10.7ab. Downstream of the throat, the local Mach number will decrease in the divergent section, reaching a very small but finite value $M_{e,1}$ at the exit. Correspondingly, the pressure in the convergent section gradually decreases from p_0 at the inlet to a minimum value at the throat, and then will gradually increase to $p_{e,1}$ at the exit. This variation is shown as curve 1 in figure 10.7ac. Please note: as the flow is not sonic at the throat, A_t (the throat area) is not equal to A^*, as A^* is the *sonic* throat area. Rather, A^* takes on the character of a reference area; it's the area the flow in figure 10.7a would have had it been sonic. Quite obviously, this means that $A^* < A_t$.
2. Let's decrease it slightly more. Then the wind will blow slightly faster, but only slightly, as shown by curve 2 in figure 10.7a.
3. We can decrease it up until a value $p_{e,3}$. At this value, the flow is accelerated just enough to reach sonic speeds at the throat.

As an intermezzo, if you haven't quite understood yet what's wrong with appendix A: after all, appendix A only gives one subsonic Mach number and one $\frac{p}{p_0}$ for each $\frac{A}{A^*}$, then how can we just change p_e freely (which changes $\frac{p}{p_0}$ at the exit), doesn't that mean we've invalidated appendix A or something[2]? The reason is that A^* is the sonic throat area, not the actual throat area A_t. So, if you change p_e, you change A^* and you would have to calculate A^* manually, and *then* you can use appendix A, using $\frac{A}{A^*}$. This is why if you use the ratio $\frac{A}{A_t}$, none of the stuff you look up in appendix A is correct. There's another remarkable property that follows from this: there are infinitely many isentropic flow solutions possible throughout a duct, by varying p_e. However, there is only *one* supersonic flow solution, namely the one achieved by having $p_e = p_{e,6}$. Any pressure other than that (that still ensures supersonic flow) will not be isentropic any more, as I'll explain hereafter:

4. If we decrease the pressure beyond $p_{e,3}$ to a pressure $p_{e,4}$, then the flow becomes supersonic after the

[2]Appendix A is said to be valid for isentropic flow, and subsonic flow is absolutely isentropic (as long as you assume it to be inviscid).

throat, following the path followed by the supersonic isentropic solution achieved by having $p_e = p_{e,6}$. However, after a while, a NSW appears to make sure the pressure at the exit coincides with $p_{e,4}$. A NSW will *always* make a flow subsonic, and for subsonic flows, we know that the Mach number decreases with increase in cross-sectional area, thus the Mach number goes down after the NSW. The flow is isentropic before and after the NSW, but not over it, meaning you cannot use appendix A to describe flow after the NSW, as p_0 and A^* will change over the NSW. I can really recommend doing example 3c with regards to this, because it'll be much clearer there how you have to compute stuff.

There's another important property that you may have noticed: the variation of the Mach number and pressure before the throat is exactly equal to the variation of those achieved by having $p_e = p_{e,3}$, even though that flow was still completely subsonic. How can that? We know that the flow can only become supersonic at the throat, so it has to be fully subsonic before. Hence, even if we decrease p_e below $p_{e,3}$, the Mach number at the throat will remain $M = 1$. In a sense, the flow at the throat, as well as upstream of the throat, becomes "frozen". Once the flow flow becomes sonic at the throat, distrubances cannot work their way upstream. Note that this means that the mass flow, $\dot{m} = \rho^* u^* A^*$ will remain constant (and it will remain constant throughout the entire flow, as the mass flow has to be the same everywhere). This situation, when the flow goes sonic at the throat, and the mass flow remains constant now matter how low p_e is reduced, is called **choked flow**. This is depicted in figure 10.8.

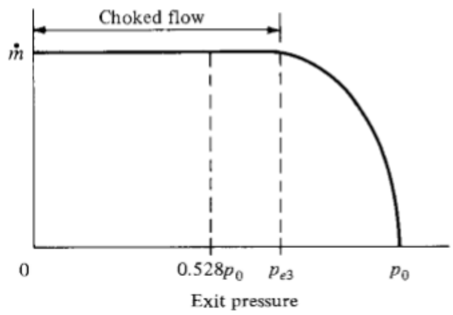


Figure 10.8: Variation of mass flow with exit pressure; illustration of choked flow.

5. As we reduce p_e below $p_{e,4}$, the NSW moves downstream, closer to the nozzle exit. At a certain value of exit pressure, $p_e = p_{e,5}$, the normal shock stands precisely at the exit. This is sketched in figure 10.9a, 10.9b and 10.9c. Just in front of the NSW, the properties of the flow will be $M_{e,6}$ and $p_{e,6}$, but the NSW makes them $M_{e,5}$ and $p_{e,5}$ so that the pressure is in coincident with the pressure you have set yourself. Note that $p_{e,5}$ is still quite a bit larger than $p_{e,6}$.

Now, before we go further with decreasing p_e, the book has an important point to make: you may have wondered what's behind the nozzle. These surroundings could be another reservoir, or the atmosphere or anything, really. Let's say it's another reservoir. Then it's much more likely that we control the pressure *inside* the back reservoir instead of the pressure at the exit of the nozzle (I mean, how would you control the pressure at exactly the exit? In flow direction, it's an infinitesimal section, it's really difficult to control the pressure there directly). Indeed, we therefore use the back pressure p_B rather than the exit pressure p_e. Before you start killing yourself thinking we now need to rediscuss all of the previously made points, don't worry, we don't. Remember that inviscid, subsonic flow is *always* isentropic, and that NSWs always make flow subsonic. So, for every $p_e \leq p_{e,5}$, the flow is subsonic and thus isentropic at the exit. The fact that the flow is isentropic means that the flow properties do not change discontinuously, and so we know, since p_B is measured *just* behind the exit, $p_e = p_B$. So, for all of the previously discussed cases, nothing changes. We can now go back to our discussion:

6. Imagine we reduce p_B below $p_{e,5}$, but still above $p_{e,6}$, i.e. $p_{e,6} < p_B < p_{e,5}$. What happens then? We saw that for $p_{e,4}$, the pressure followed the isentropic solution until a NSW occured; in $p_{e,5}$, the pressure followed the isentropic solution right until the exit, and that's where the NSW occured. If we decrease the pressure even further, the pressure follows the isentropic solution until even after the exit, meaning that the pressure of the flow as it enters the reservoir will still equation $p_{e,6}$. However, the pressure of the reservoir itself will be p_B (we controlled that one ourselves, after all), which is higher than $p_{e,6}$. Thus, $p_{e,6}$ needs to be compressed to make sure it matched p_B, which is done by virtue of an OSW, as shown in figure 10.9d. The flow is said to be **overexpanded**, which needs to be compressed.

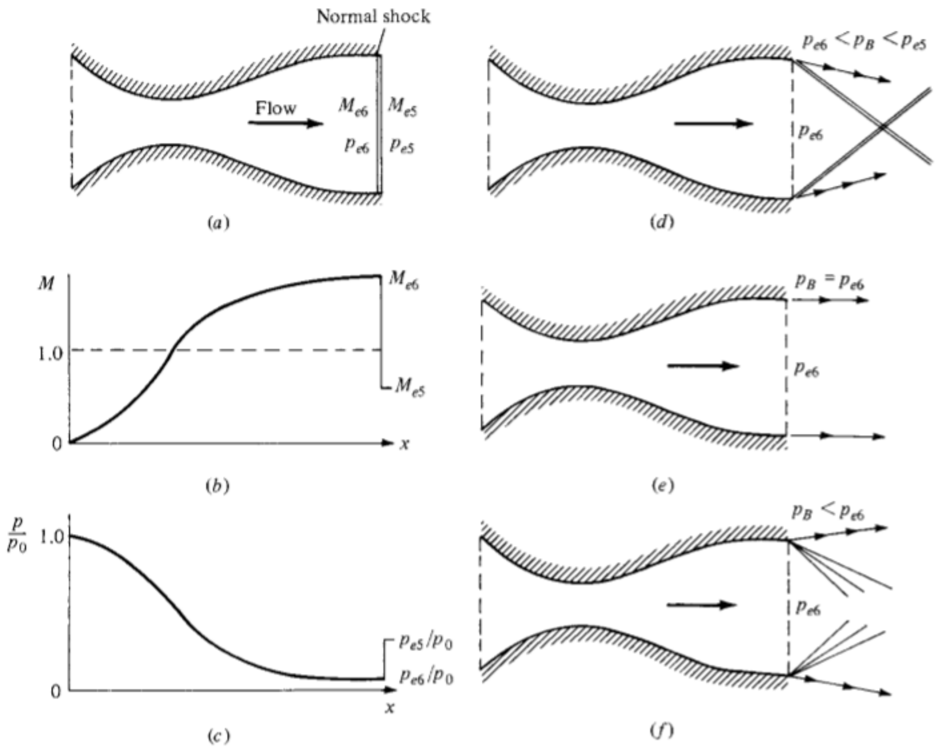


Figure 10.9: Supersonic nozzle flows with waves at the nozzle exit: (a), (b), and (c) pertain to a normal shock at the exit, (d) overexpanded nozzle, (e) isentropic expansion to the back pressure equal to the exit pressure, (f) underexpanded nozzle.

7. Now, suppose we reduce p_B to equal $p_{e,6}$. In this case, the flow does not need to be compressed nor expanded, so it's all good.
8. Finally, suppose we reduce p_B to be even less than $p_{e,6}$. In this case, the pressure of the flow as it enters the reservoir will be higher than the pressure inside the reservoir itself, meaning it needs to chill and expand, which is done by PMEWs as shown in figure 10.9f. The flow is said to be **underexpanded**, which is expanded to make it have $p_{e,6} = p_B$.

Example 1

Consider the isentropic supersonic flow through a convergent-divergent nozzle with an exit-to-throat ratio of 10.25. The reservoir pressure and temperature are 5 atm and 600 K, respectively. Calculate M, p and T at the nozzle exit.

Note that the flow is isentropic and supersonic, so this is the "ideal" case, with no shocks whatsoever. We have from appendix A, from the supersonic portion of it, that for $\frac{A_e}{A^*} = 10.25$:

$$\begin{aligned}
M_e &= 3.95 \\
\frac{p_e}{p_0} &= \frac{1}{142} \\
p_e &= \frac{p_e}{p_0} p_0 = \frac{1}{142} \cdot 5 = 0.035\,\text{atm} \\
\frac{T_e}{T_0} &= \frac{1}{4.12} \\
T_e &= \frac{T_e}{T_0} T_0 = \frac{1}{4.12} \cdot 600 = 145.6\,\text{K}
\end{aligned}$$

Example 2

Consider the isentropic flow through a convergent-divergent nozzle with an exit-to-throat area ratio of 2. The reservoir pressure and temperature are 1 atm and 288 K, respectively. Calculate the Mach number, pressure and temperature at both the throat and the exit for the cases where

(a) The flow is supersonic at the exit;
(b) The flow is subsonic throughout the entire nozzle except at the throat, where $M = 1$;
(c) The pressure at the exit is 0.973 atm.

For (a), we have that the flow at the throat is sonic, and thus (using appendix A)

$$\begin{aligned} M_t &= 1.0 \\ p_t &= p^* = \frac{p^*}{p_0} p_0 = 0.528 \cdot 1 = 0.528\,\mathrm{atm} \\ T_t &= T^* = \frac{T^*}{T_0} T_0 = 0.833 \cdot 288 = 240\,\mathrm{K} \end{aligned}$$

Then, onto the exit. There, the flow is supersonic, and thus (using the supersonic portion of appendix A), we have for $\frac{A_e}{A^*} = 2$ that

$$\begin{aligned} M_e &= 2.2 \\ p_e &= \frac{p_e}{p_0} p_0 = \frac{1}{10.69} \cdot 1 = 0.0935\,\mathrm{atm} \\ T_e &= \frac{T_e}{T_0} T_0 = \frac{1}{1.968} \cdot 288 = 146\,\mathrm{K} \end{aligned}$$

For (b), at the throat, the flow is apparently still sonic. This means that the conditions at the throat are still exactly the same as before. However, for the exit, it's now obviously different. We need to use the subsonic portion of appendix A to find for $\frac{A_e}{A^*} = 2$ that

$$\begin{aligned} M_e &= 0.3 \\ p_e &= \frac{p_e}{p_0} p_0 = \frac{1}{1.064} \cdot 1 = 0.94\,\mathrm{atm} \\ T_e &= \frac{T_e}{T_0} T_0 = \frac{1}{1.018} \cdot 288 = 282.9\,\mathrm{K} \end{aligned}$$

For (c), we saw in (b) that the maximum exit pressure to have sonic conditions at the throat equalled 0.94 atm; our current exit pressure is higher than that so we'll have subsonic flow everywhere. We thus do not now A^* yet. However, we can easily compute that

$$\frac{p_0}{p_e} = \frac{1}{0.973} = 1.028$$

With the help of the almighty appendix A, we find for $\frac{p_0}{p_e} = 1.028$ that

$$\begin{aligned} M_e &= 0.2 \\ \frac{A_e}{A^*} &= 2.964 \end{aligned}$$

Furthermore, with $\frac{A_e}{A_t} = 2$, or $\frac{A_t}{A_e} = \frac{1}{2}$, we find

$$\frac{A_t}{A^*} = \frac{A_t}{A_e}\frac{A_e}{A^*} = \frac{1}{2} \cdot 2.964 = 1.482$$

And again, we look in appendix A to find for $\frac{A_t}{A^*} = 1.482$ that

$$M_t = 0.44$$

When you've done these calculations a few times, it becomes really easy.

Example 3

Consider a convergent-divergent nozzle with an exit-to-throat area ratio of 1.53. The reservoir pressure is 1 atm. Assuming isentropic flow, except for the possibility of a normal shock wave inside the nozzle, calculate the exit Mach number when the exit pressure is
(a) 0.94 atm
(b) 0.886 atm
(c) 0.75 atm
(d) 0.154 atm

Before we do anything, let's just realize what this question is probably asking from us: we saw that there were a number of different important exit pressure values; probably, the four different pressures each correspond to a different important value. The first, possibly first two, will probably be about subsonic flow, the other two about supersonic flow.
For (a), we can compute $\frac{p_0}{p_e} = \frac{1}{0.94} = 1.064$. From appendix A, this leads to $M_e = 0.3$ and $\frac{A_e}{A^*} = 2.035$. If the flow were to be sonic at the throat, i.e. $M_t = 1$, then it'd need to have $\frac{A_t}{A^*} = 1$, obviously. So, let's compute $\frac{A_t}{A^*}$:

$$\frac{A_t}{A^*} = \frac{A_t}{A_e}\frac{A_e}{A^*} = \frac{1}{1.53} \cdot 2.035 = 1.33$$

The value of 1.53 was literally given in the question, and the 2.035 was looked up. This means that $A_t > A^*$, meaning the flow is not sonic at the throat. This means no shock wave exists, and thus $M_e = 0.3$ is valid, as this means that the flow is completely isentropic and thus appendix A is correct. The following subquestions deal with what happens if A_t is not larger than A^*.

For (b), we can compute $\frac{p_0}{p_e} = \frac{1}{0.886} = 1.129$, from which we find that $M_e = 0.42$ and $\frac{A_e}{A^*} = 1.529$. Again, the ratio $\frac{A_t}{A^*}$ equals

$$\frac{A_t}{A^*} = \frac{A_t}{A_e}\frac{A_e}{A^*} = \frac{1}{1.53} \cdot 1.529 \approx 1.00$$

And thus we have $A_t = A^*$. We thus have precisely sonic flow at the throat, and it is subsonic everywhere else. Hence, we have a fully isentropic flow and $M_e = 0.42$.

For (c), stuff does get annoying. We have $\frac{p_0}{p_e} = \frac{1}{0.75} = 1.333$, for which

$$\frac{A_e}{A^*} = 1.127$$

and thus

$$\frac{A_t}{A^*} = \frac{A_t}{A_e} \cdot \frac{A_e}{A^*} = \frac{1}{1.53} \cdot 1.127 = 0.7366$$

and thus we'd have $A_t < A^*$, which is not possible. Thus, there must be a shock wave inside the nozzle, somewhere, which causes both p_0 and A^* to change across the shock. Finding M_e involves finding the location of the NSW, which is rather tedious but we need to do it anyway. We first *assume* a NSW exists somewhere in the nozzle, at a location where $\frac{A_2}{A_t} = 1.204$ (see also figure 10.10).

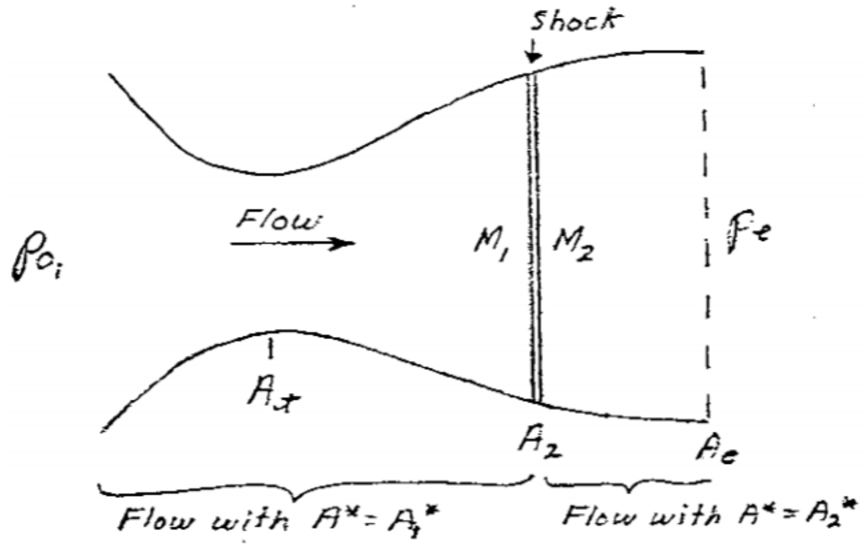


Figure 10.10: Schematic for 3c.

We define the following parameters:

- A_1^* is the sonic throat area for the flow ahead of the shock, simply equal to the "real" area A_t.
- A_2^* is the sonic throat area for the flow behind the shock. What does this mean? We've basically seen two variables deciding the flow in the wind tunnel: the throat area, and $\frac{p_e}{p_0}$. Decreasing the throat area obviously causes the flow to speed up more, and decreasing p_e causes a higher pressure difference which also accelerates the flow. By carefully setting $\frac{A_e}{A_t}$ and $\frac{p_e}{p_0}$, we can achieve flow indicated by the third bullet point of the enumeration in this section: the flow is completely subsonic everywhere, except for the throat where it is exactly sonic. This was associated with $p_e = p_{e,3}$. In this question, however, p_e is smaller than $p_{e,3}$ (we saw that $p_{e,3}$ equalled 0.866 atm, whereas p_e now equals 0.75 atm), meaning the wind tunnel has basically become "too strong". We can, however, counter this by increasing the area of throat (while keeping the exit area the same), which "weakens" the wind tunnel. If we increase it, we see that $p_{e,3}$, the exit pressure necessary to have sonic flow at the throat and subsonic flow everywhere else, decreases; eventually, we'll see that $p_{e,3}$ has decreased to 0.75 atm. This associated sonic throat area is defined as A_2^*. Note that $A_2^* > A_1^*$.
- $p_{0,1}$ is the total pressure ahead of the shock.
- $p_{0,2}$ is the total pressure behind the shock. Note that $p_{0,2} < p_{0,1}$.

Now, what do we want to do with this? If we didn't have a NSW there, we know that we ended up at a really low pressure, $p_{e,6}$. Thus, we want to place our NSW at such a location that it increases the pressure so that at the exit, it exactly matches p_e. We can compute the pressure p_e by

$$p_e = \frac{p_e}{p_{0,2}} \frac{p_{0,2}}{p_{0,1}} p_{0,1}$$

$\frac{p_e}{p_{0,2}}$ can be determined by determining, $\frac{A_e}{A_2^*}$, and $\frac{p_{0,2}}{p_{0,1}}$ can be determined from the Mach number in front of the Mach wave.

As I said, let's first place it at a location where $\frac{A_2}{A_t} = 1.204$. We then know from appendix A that just in front of the NSW, $M_1 = 1.54$. From appendix B, we then know that $\frac{p_{0,2}}{p_{0,1}} = 0.9166$. So, we only need to find one pressure ratio. To compute $\frac{A_e}{A_2^*}$, we first need to know $\frac{A_2}{A_2^*}$. For $M_1 = 1.54$, we find from appendix B that $M_2 = 0.6874$. From appendix A, we then have that $\frac{A_2}{A_2^*} = 1.1018$. We thus have

$$\frac{A_e}{A_2^*} = \frac{A_e}{A_t} \frac{A_t}{A_2} \frac{A_2}{A_2^*} = 1.53 \cdot \frac{1}{1.204} \cdot 1.1018 = 1.4$$

for which we have from appendix A $M_e = 0.47$ and $\frac{p_{0,2}}{p_e} = 1.163$ (taking the subsonic part as we know that the flow in region 2 is subsonic). Thus,

$$p_e = \frac{p_e}{p_{0,2}}\frac{p_{0,2}}{p_{0,1}}p_{0,1} = \frac{1}{1.163} \cdot 0.9166 \cdot 1 = 0.788\,\text{atm}$$

which is higher than 0.75 atm. To lower p_e, we needed to move our NSW downstream, for example assume now that $\frac{A_2}{A_t} = 1.301$. Doing the exact same computations as before:

$$\begin{aligned} M_1 &= 1.66 \\ \frac{p_{0,2}}{p_{0,1}} &= 0.872 \\ M_2 &= 0.6512 \\ \frac{A_2}{A_2^*} &= 1.1356 \end{aligned}$$

$$\begin{aligned} \frac{A_e}{A_2^*} &= 1.53 \cdot \frac{1}{1.301} \cdot 1.1356 = 1.335 \\ M_e &= 0.50 \\ \frac{p_{0,2}}{p_e} &= 1.1862 \\ p_e &= \frac{1}{1.1862} \cdot 0.872 \cdot 1 = 0.735\,\text{atm} \end{aligned}$$

which is slightly lower than 0.75 atm. By interpolation, let's use

$$\frac{A_2}{A_t} = 1.301 - (1.301 - 1.204) \cdot \frac{0.75 - 0.735}{0.788 - 0.735} = 1.274$$

Doing the exact same computations once more

$$\begin{aligned} M_1 &= 1.63 \\ \frac{p_{0,2}}{p_{0,1}} &= 0.8838 \\ M_2 &= 0.6596 \\ \frac{A_2}{A_2^*} &= 1.1265 \\ \frac{A_e}{A_2^*} &= 1.53 \cdot \frac{1}{1.274} \cdot 1.1265 = 1.353 \\ M_e &= 0.49 \\ \frac{p_{0,2}}{p_e} &= 1.178 \\ p_e &= \frac{1}{1.178} \cdot 0.8838 \cdot 1 = 0.75\,\text{atm} \end{aligned}$$

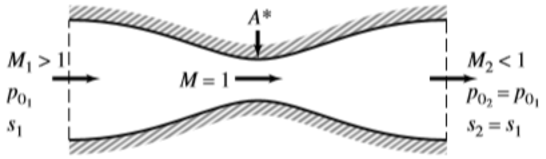
and thus $M_e = 0.49$. Finally we're done with this question.

For (d), we find $\frac{p_0}{p_e} = 6.49$. From appendix A we find $\frac{A_e}{A^*} = 1.53$, which is precisely the given area ratio of the nozzle. Hence, for this case, we have a completely isentropic expansion, where $M_e = 1.88$.

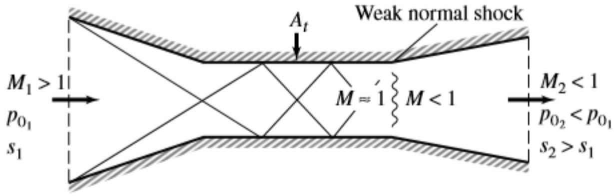
The most important thing for these type of exercises is that you are able to come up with the relations between several ratios (e.g. $p_e = \frac{p_e}{p_{0,2}}\frac{p_{0,2}}{p_{0,1}}p_{0,1}$ etc.).

10.4 Diffusers

A diffuser is a duct designed to slow down an incoming gas flow to lower velocity at the exit of the diffuser. The incoming flow can be subsonic or supersonic, but the shape of the diffuser is vastly different (remember that for supersonic flow, dA had to be negative to slow the flow down; for subsonic flow, dA had to be positive). Now, how do we design a proper diffuser? For that, you need to remember that total pressure was an important property of a flow, indicating how much work it can do when it's brought to a stand still. It's therefore logical that you want to have as little total pressure loss as possible. Consequently, an ideal diffuser would be characterized by an isentropic compression to lower velocities, as sketched in figure 10.11a.



(a) Ideal (isentropic) supersonic diffuser.



(b) Actual supersonic diffuser.

Figure 10.11: The ideal (isentropic) diffuser compared with the actual situation.

However, this ideal diffuser is impossible to achieve: the flow would need to be compressed isentropically, for which the shape of the diffuser would need to be absolutely perfectly designed and built. Note that the flow is turned into itself, unavoidably causing OSWs, which destroys the isentropic nature of the flow. Moreover, in real life, the flow is viscous; there will be an entropy increase within the boundary layers on the walls of the diffuser, so an ideal, isentropic supersonic diffuser is only a wish, not something we can actually build.

An actual supersonic diffuser is shown in figure 10.11b: there's a convergent part (usually straight walls), a constant area part and a divergent part. Due to the interaction of the shock waves with the viscous flow near the wall, the shock pattern eventually weakens and becomes quite diffuse, sometimes ending in a weak NSW at the end of the constant throat area. Finally, the subsonic flow downstream of the constant-area throat is further slowed by moving through a divergent section. At the exit, clearly $s_2 > s_1$, and thus $p_{0,2} < p_{0,1}$. Designing a diffuser means making this total pressure loss as small as possible; i.e. to make $\frac{p_{0,2}}{p_{0,1}}$ as close to unity as possible, although we usually fall far short of that goal. Furthermore, note that due to the entropy increase over the shock waves and in the boundary layers, the real diffuser throat area A_t is larger than A^*; that is, in figure 10.11, $A_t > A^*$.

10.5 Supersonic wind tunnels

Imagine you want to create a Mach 2.5 uniform flow in a laboratory for experiments etc. How do you do it? From appendix A, we easily see that we must have $\frac{p_0}{p_e} = 17.09$ (we're not using a diffuser now). So, we could have the reservoir pressure be 17.09 atm, or the exit pressure be 0.0585 atm, and we'd be done. However, both of these things are rather costly: it's difficult to pressurize an entire reservoir to 17.09 atm, and it's difficult to lower the pressure to such low pressures. Difficult things cost money and we don't like that, so we'd prefer a smaller ratio $\frac{p_e}{p_0}$ to have the pressure difference be smaller. One way to do this is shown in figure 10.12: we simply try to create a NSW at the end of the constant area section, and place our test model in front of this NSW: the test model still experiences $M = 2.5$, but due to the NSW, the pressure is signficantly increased after it, meaning that the pressure differential isn't as large anymore ($\frac{p_0}{p_e} = 2.4$).

However, this "normal shock diffuser" bears some issues:

- A NSW is the strongest possible shock, hence creating the largest total pressure loss. If we could replace the NSW with a weaker shock, the total pressure loss would be less, and the required ratio p_0/p_e would be even less as well.
- It is extremely difficult to hold a NSW stationary at the duct exit; flow unsteadiness and instabilities would cause the shock to move somewhere else and to fluctuate constantly in position. Thus, we could never be

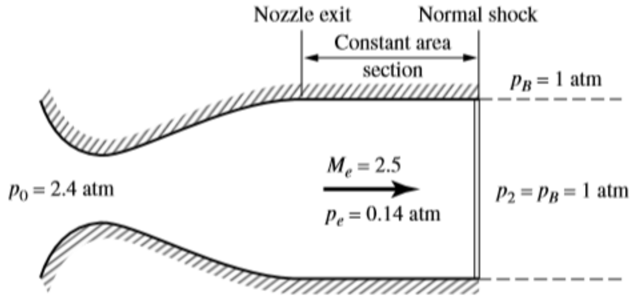


Figure 10.12: Nozzle exhausting into a constant-area duct, where a normal shock stands at the exit of the duct.

certain about the quality of the flow in the constant-area duct.

- As soon as the test model is introduced into the constant-area section, the oblique waves from the model would propagate downstream, causing the flow to become two-or-three dimensional. The normal shock sketched in figure 10.12 could not exist in such a flow.

So, let's find a different solution. Let's try the wind tunnel sketched in figure 10.13, which also includes a diffuser. We have a convergent-divergent nozzle feeding a uniform supersonic flow into the constant-area duct, which is called the **test section**. This flow is subsequently slowed to a low subsonic speed by means of a diffuser. This arrangement - a convergent-divergent nozzle, a test section and a convergent-divergent diffuser - is a **supersonic wind tunnel**.

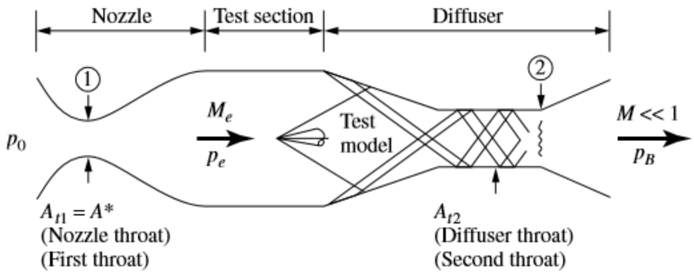


Figure 10.13: Sketch of a supersonic wind tunnel.

Now, how does the total pressure loss in figure 10.13 compare to the total pressure loss in figure 10.12? There's no clear-cut answer to this, but generally speaking, having multiple OSWs and then a weak NSW (as in figure 10.13) results in a smaller total pressure loss than having one strong NSW. We also saw this for example with our scramjet in example 4 of chapter 9. This is not *always* true, because in real life, the shock waves interact with the boundary layers on the walls, causing local thickening and even possible separation of the boundary layers, creating an additional total pressure loss. Moreover, the simple aspect of skin friction generates a total pressure loss as well. Hence, actual oblique shock diffusers may have efficiencies greater or less than a hypothetical normal shock diffuser. Nevertheless, virtually all supersonic wind tunnels use oblique shock diffusers.

Note that the supersonic wind tunnel of figure 10.13 has two throats: the nozzle throat with area $A_{t,1}$, called the **first throat** and the diffuser throat with area $A_{t,2}$, called the **second throat**. The question now is, does $A_{t,2}$ differ from $A_{t,1}$, and if so, how does it differ?

Note that the mass flow $\dot{m} = \rho u A$ must be constant throughout both throats. Furthermore, for the first throat, we have that $\rho = \rho_1^*$, $u = a_1^*$ (as the flow is sonic) and $A = A_{t,1}$. For the second throat, we have $\dot{m}_2 = \rho_2 u_2 A_{t,2}$:

$$\rho_1^* a_1^* A_{t,1} = \rho_2 u_2 A_{t,2}$$

As the thermodynamic state of the gas is irreversibly changed in going through the shock waves, clearly ρ_2 and possible u_2 are different from ρ_1^* and a_1^*. Hence, $A_{t,2} \neq A_{t,1}$. So, how much do they differ?

Assume the flow is sonic at both stations 1 and 2. We then can rewrite the equation for the mass flow as

$$\frac{A_{t,2}}{A_{t,1}} = \frac{\rho_1^* a_1^*}{\rho_2^* a_2^*}$$

Remember from section 8.4 that a^* is constant for an adiabatic flow, and that flow across shock waves is adiabatic (it's just not isentropic), and thus $a_1^* = a_2^*$. Thus, we have, using the equation of state, where T^* is constant as well for an adiabatic flow (it's even constant for an non-isentropic flow, after all)

$$\frac{A_{t,2}}{A_{t,1}} = \frac{\rho_1^*}{\rho_2^*} = \frac{p_1^*/RT_1^*}{p_2^* RT_2^*} = \frac{p_1^*}{p_2^*}$$

And from equation (8.16), we have

$$p^* = p_0 \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}$$

and thus we can write

FORMULA

$$\frac{A_{t,2}}{A_{t,1}} = \frac{p_{0,1}}{p_{0,2}} \tag{10.11}$$

We always have across a shock wave that $p_{0,2} < p_{0,1}$, so we always have $A_{t,2} > A_{t,1}$. Only in the case of an ideal isentropic diffuser, where p_0 is constant, would $A_{t,2} = A_{t,1}$.

Note that this is only useful to size the second throat relative to the first throat if we know the total pressure ratio across the tunnel. In the absence of such information, for the preliminary design of supersonic wind tunnels, the total pressure ratio across a normal shock is assumed.

Now, suppose you did a bad job and $A_{t,2}$ is less than the value given by equation (10.11). What happens? In this case, the diffuser will "choke"; the diffuser cannot pass the mass flow coming from the isentropic, supersonic expansion through the nozzle. In this case, nature adjusts the flow through the wind tunnel by creating shock waves in the nozzle, which in turn reduce the Mach number in the test section, producing weaker shocks in the diffuser with an attendant overall reduction in the total pressure loss; that is, nature adjusts the total pressure loss such that $p_{0,1}/p_{0,2} = p_{0,1}/p_B$ satisfies equation (10.11). Sometimes, this adjustment is so severe that a NSW stands inside the nozzle, which makes the flow through the test section and diffuser totally subsonic, which is not kinda what we want from a supersonic wind tunnel. The supersonic wind tunnel is now said to be **unstarted**. The only way to rectify this is to make $A_{t,2}/A_{t,1}$ large enough so that the diffuser can pass the mass flow from the isentropic expansion in the nozzle, so that equation (10.11) is satisfied along with a shock-free isentropic nozzle expansion.

Example 4

For the preliminary design of a Mach 2 supersonic wind tunnel, for which $A_{t,1} = 0.024\,\mathrm{m}^2$, calculate the minimum required diffuser throat area to ensure proper working of the wind tunnel.

Now, the big question is, how large should $\frac{p_{0,1}}{p_{0,2}}$ be; after all, this is highly dependent on how the shock pattern in figure 10.13 works. Therefore, for these type of questions, you take the worst case scenario: a single, strong NSW. We have from appendix B that for a NSW with $M_1 = 2$ that $p_{0,2}/p_{0,1} = 0.7209$, and thus

$$\frac{A_{t,2}}{A_{t,1}} = \frac{p_{0,1}}{p_{0,2}} = \frac{1}{0.7209} = 1.387$$

and thus the area of the diffuser throat must equal at least $0.333\,\mathrm{m}^2$

10.6 *Viscous flow: shock-wave/boundary-layer interaction inside nozzles*

We saw in section 10.3 that for $p_e = p_{e,4}$, there was a NSW standing somewhere in the nozzle. However, in reality, there will be interaction with the boundary layer. One of the possible flow fields resulting from this interaction is shown in figure 10.14. The adverse pressure gradient across the shock causes the boundary layer to separate from the nozzle wall. A lambda-type shock pattern occurs at the two feet of the shock near the wall, and the core of the nozzle flow, now separated from the wall, flows downstream at almost constant area. This is further depicted in figure 10.15: especially in part (a), we see the boundary layer separation trailing downstream.

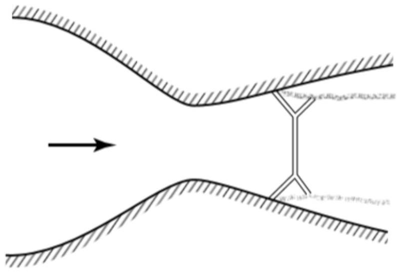


Figure 10.14: Sketch of an overexpanded nozzle flow with flow separation.

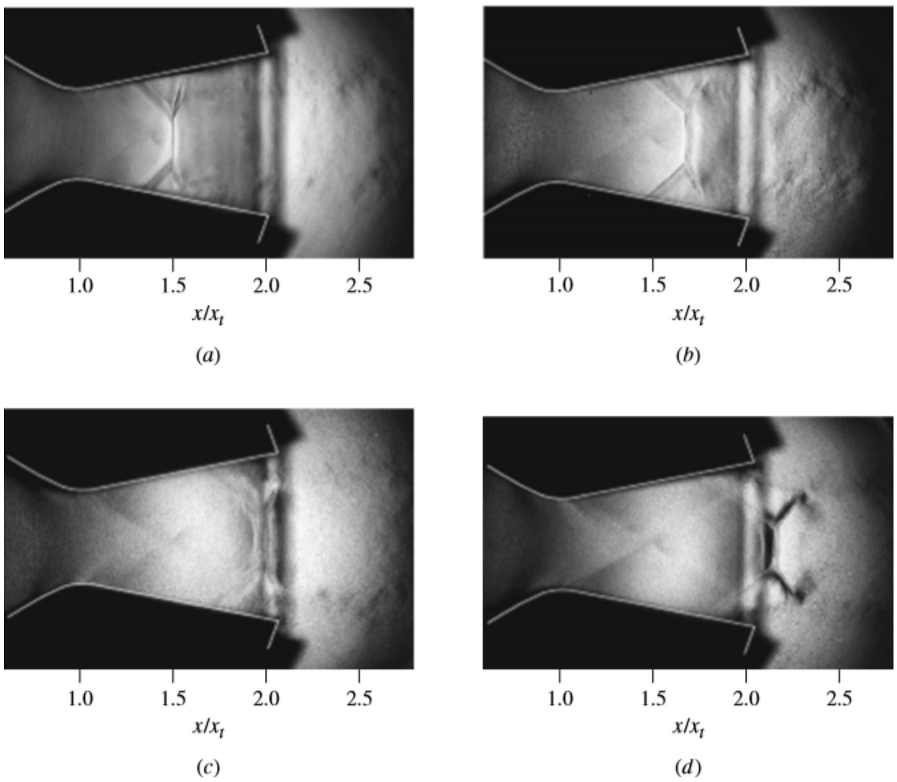


Figure 10.15: Schlieren photographs of the shock-wave/boundary-layer interaction inside an overexpanded nozzle flow. Exit-to-reservoir pressure ratio is (a) 0.5, (b) 0.417, (c) 0.333, (d) 0.294.

11 Subsonic Compressible Flow over Airfoils: Linear Theory

This chapter contains a lot of mathematical derivations, so just try to follow them; you'll see that in the end, we'll simplify them a bit. Furthermore, I'm pretty sure that you don't need to know the derivations of sections 11.2 to 11.5, as they focus more on mathematics than real physics (yes that's usually the case with derivations, but you'll see what I mean). The exam questions about this chapter (and chapter 12 because it's more or less the same topic) are the easiest questions on the exam, imo, so don't worry too much if you don't follow the derivations (and remember that you get a nice big formula sheet).

11.2 The velocity potential equation

In chapter 2 of Aero I, we looked into the velocity potential for incompressible flow. However, we can just as well define a velocity potential for inviscid, *compressible*, subsonic flow (which is always irrotational). The velocity potential is defined such that (in vector form and cartesian velocity components):

$$\mathbf{V} = \nabla\phi \tag{11.1}$$

$$u = \frac{\partial\phi}{\partial x} \tag{11.2}$$

$$v = \frac{\partial\phi}{\partial y} \tag{11.3}$$

Now, let us obtain an equation for ϕ which combines the continuity, momentum and energy equations. From the continuity equation, we have

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = \rho\frac{\partial u}{\partial x} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + \rho\frac{\partial v}{\partial y} = 0$$

Using the derivatives for u and v, this can be written as

$$\rho\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\rho}{\partial y} + \rho\frac{\partial^2\phi}{\partial y^2} = \rho\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\rho}{\partial y} = 0$$

Now, we can use the momentum equation to find an expression for the partial derivatives of ρ. Remember Euler's equation:

$$dp = -\rho V\,dV = -\frac{\rho}{2}d\left(V^2\right) = -\frac{\rho}{2}d\left(u^2+v^2\right) = -\frac{\rho}{2}d\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]$$

Now, we have by definition, as the considered flow is isentropic:

$$\frac{dp}{d\rho} = \left(\frac{\partial p}{\partial\rho}\right)_s = a^2$$

or $dp = a^2 d\rho$, meaning we can write

$$\begin{aligned}
a^2 d\rho &= -\frac{\rho}{2} d\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] \\
d\rho &= -\frac{\rho}{2a^2} d\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] \\
\frac{\partial\rho}{\partial x} &= -\frac{\rho}{2a^2}\frac{\partial}{\partial x}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] = -\frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}\right) \\
\frac{\partial\rho}{\partial y} &= -\frac{\rho}{2a^2}\frac{\partial}{\partial y}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] = -\frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2}\right)
\end{aligned}$$

If you don't quite see the final two steps: the x-derivative of $(\partial\phi/\partial x)^2$ equals (chain rule)

$$2\frac{\partial\phi}{\partial x}\cdot\frac{\partial}{\partial x}\frac{\partial\phi}{\partial x} = 2\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2}$$

similar to why the derivative of u^2 equals $2u \cdot u'$

Now, we can substitute these two expressions for the partial derivatives of the density into the equation:

$$\begin{aligned}
\rho\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial\phi}{\partial x}\frac{\partial\rho}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial\rho}{\partial y} &= \rho\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{\partial\phi}{\partial x}\cdot -\frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y}\right) \\
&\quad + \frac{\partial\phi}{\partial y}\cdot -\frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2}\right) \\
&= \left[\rho - \frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2} + \left[\rho - \frac{\rho}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2} - \frac{2\rho}{a^2}\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial xy} = 0
\end{aligned}$$

Divide everything by ρ to get

FORMULA: VELOCITY POTENTIAL EQUATION

$$\left[1 - \frac{1}{a^2}\left(\frac{\partial\phi}{\partial x}\right)^2\right]\frac{\partial^2\phi}{\partial x^2} + \left[1 - \frac{1}{a^2}\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial^2\phi}{\partial y^2} - \frac{2}{a^2}\left(\frac{\partial\phi}{\partial x}\right)\left(\frac{\partial\phi}{\partial y}\right)\frac{\partial^2\phi}{\partial xy} = 0 \qquad (11.4)$$

We can also write a in terms of ϕ:

$$a^2 = a_0^2 - \frac{\gamma-1}{2}V^2 = a_0^2 - \frac{\gamma-1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right] \qquad (11.5)$$

where a_0 is a known property of the flow. Now, we see that we actually have one big differential equation, written in terms of only. Based on two boundary conditions, we are theoretically able to find a solution for ϕ, which allows us to calculate the velocity of the flow everywhere. Unfortunately, there exists no analytical solution for ϕ just yet, so we just assume that on exam questions, we are given a function for ϕ. How can we, based on ϕ, calculate a number of things?

1. We can calculate u and v from equations (11.2) and (11.3).
2. We can calculate a from equation (11.5).
3. We can calculate M from $M = V/a = \sqrt{u^2+v^2}/a$.
4. We can calculate T, p and ρ from the isentropic flow relations equations (or appendix A). The values T_0, p_0 and ρ_0 are obtained from freestream conditions.

So far, we have a nonlinear partial differential equation, which we don't like very much, so we want to linearise it.

11.3 The linearised velocity potential equation

We can see the velocity components u and v as the freestream flow velocity components plus some small increments, called **perturbations**, i.e.

$$u = V_\infty + \hat{u} \qquad v = \hat{v}$$

as shown in figure 11.1. $\hat{u}$ and $\hat{v}$ are called the **perturbation velocities**. These obviously are not necessarily small.

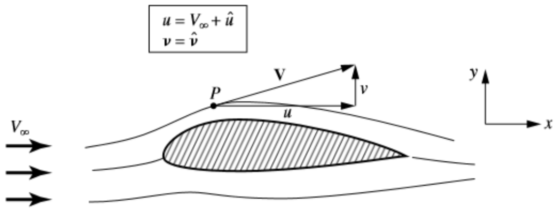


Figure 11.1: Uniform flow and perturbed flow.

Now, as $\mathbf{V} = \nabla\phi$, we can write

$$\begin{aligned}
\phi &= V_\infty x + \hat{\phi} \\
\frac{\partial\hat{\phi}}{\partial x} &= \hat{u} \\
\frac{\partial\hat{\phi}}{\partial y} &= \hat{v} \\
\frac{\partial\phi}{\partial x} &= V_\infty + \frac{\partial\hat{\phi}}{\partial x} \qquad \frac{\partial\phi}{\partial y} = \frac{\partial\hat{\phi}}{\partial y} \\
\frac{\partial^2\phi}{\partial x^2} &= \frac{\partial^2\hat{\phi}}{\partial x^2} \qquad \frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\hat{\phi}}{\partial y^2} \qquad \frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\hat{\phi}}{\partial x\partial y}
\end{aligned}$$

Substituting this in equation (11.4) and multiplying everything by a^2 gives

$$\left[a^2 - \left(V_\infty + \frac{\partial\hat{\phi}}{\partial x}\right)^2\right]\frac{\partial^2\hat{\phi}}{\partial x^2} + \left[a^2 - \left(\frac{\partial\hat{\phi}}{\partial y}\right)^2\right]\frac{\partial^2\hat{\phi}}{\partial y^2} - 2\left(V_\infty + \frac{\partial\hat{\phi}}{\partial x}\right)\left(\frac{\partial\hat{\phi}}{\partial y}\right)\frac{\partial^2\hat{\phi}}{\partial x\partial y} = 0$$

and this formula is called the **perturbation velocity potential equation**. However, it's still nonlinear, so let's finally linearise it. For this, let's write it as

$$\left[a^2 - \left(V_\infty + \hat{u}\right)^2\right]\frac{\partial\hat{u}}{\partial x} + \left(a^2 - \hat{v}^2\right)\frac{\partial\hat{u}}{\partial y} - 2\left(V_\infty + \hat{u}\right)\hat{v}\frac{\partial\hat{u}}{\partial y} = 0$$

Now, using the energy equation, we have

$$\frac{a_\infty^2}{\gamma-1} + \frac{V_\infty^2}{2} = \frac{a^2}{\gamma-1} + \frac{\left(V_\infty + \hat{u}\right)^2 + \hat{v}^2}{2}$$

Substituting this and rewriting everything (I honestly don't think you need to know how to do this, it's just so much work):

$$\begin{aligned}
\left(1 - M_\infty^2\right)\frac{\partial\hat{u}}{\partial x} + \frac{\partial\hat{v}}{\partial y} &= M_\infty^2\left[(\gamma+1)\frac{\hat{u}}{V_\infty} + \frac{\gamma+1}{2}\frac{\hat{u}^2}{V_\infty^2} + \frac{\gamma-1}{2}\frac{\hat{v}^2}{V_\infty^2}\right]\frac{\partial\hat{u}}{\partial x} \\
&+ M_\infty^2\left[(\gamma+1)\frac{\hat{u}}{V_\infty} + \frac{\gamma+1}{2}\frac{\hat{v}^2}{V_\infty^2} + \frac{\gamma-1}{2}\frac{\hat{u}^2}{V_\infty^2}\right]\frac{\partial\hat{v}}{\partial y} + M_\infty^2\left[\frac{\hat{v}}{V_\infty}\left(1 + \frac{\hat{u}}{V_\infty}\right)\left(\frac{\partial\hat{u}}{\partial y} + \frac{\partial\hat{v}}{\partial x}\right)\right]
\end{aligned}$$

Now, let us consider small pertuabtions. For slender bodies at a small angle of attack, we have

$$\frac{\hat{u}}{V_\infty}, \frac{\hat{v}}{V_\infty} << 1 \qquad \frac{\hat{u}^2}{V_\infty^2}, \frac{\hat{v}^2}{V_\infty^2} <<< 1$$

So, then we have, by comparing the terms that appear in front of each partial derivative, that:

- For $0 \leq M_\infty \leq 0.8$ or $M_\infty \geq 1.2$ that the magnitude of

$$M_\infty^2 \left[(\gamma + 1)\frac{\hat{u}}{V_\infty} + \frac{\gamma + 1}{2}\frac{\hat{u}^2}{V_\infty^2} + \frac{\gamma - 1}{2}\frac{\hat{v}^2}{V_\infty^2}\right]\frac{\partial \hat{u}}{\partial x}$$

  will be small in comparison to

$$\left(1 - M_\infty^2\right)\frac{\partial \hat{u}}{\partial x}$$

  and therefore we should ignore the former term.
- For $M_\infty < 5$, we have

$$M_\infty^2 \left[(\gamma - 1)\frac{\hat{u}}{V_\infty} + \cdots\right]\frac{\partial \hat{v}}{\partial y}$$

  is small in comparison to $\partial\hat{v}/\partial y$, and thus we can ignore the first term. Also for $M_\infty < 5$, the term

$$M_\infty^2 \left[\frac{\hat{v}}{V_\infty}\left(1 + \frac{\hat{u}}{V_\infty}\right)\left(\frac{\partial \hat{u}}{\partial y} + \frac{\partial \hat{v}}{\partial x}\right)\right] \approx 0$$

Thus, the equation beautifully reduces to

FORMULA

$$\left(1 - M_\infty^2\right)\frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} = \left(1 - M_\infty^2\right)\frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{11.6}$$

If you would have looked at this the first time, it may have looked strange as to why the x derivatives do get that fancy $1 - M_\infty^2$ but the y-derivatives didn't; now you know that this has all to do with the freestream velocity V_∞ which is only active in x-direction. Please note the assumptions made for this equation to be valid (not exact, but reasonably valid):

- Small perturbations: thin bodies at small angles of attack;
- Subsonic and supersonic Mach numbers, but not transonic, sonic and hypersonic.

Additionally, it assumes inviscid, steady flow. Remember these assumptions as they sometimes ask literally for them.

Now, onto the pressure coefficient, because that's quite useful. We have

$$\begin{aligned} C_p &\equiv \frac{p - p_\infty}{q_\infty} \\ q_\infty &= \frac{1}{2}\rho_\infty V_\infty^2 = \frac{1}{2}\frac{\gamma p_\infty}{\gamma p_\infty}\rho_\infty V_\infty^2 = \frac{\gamma}{2}p_\infty\left(\frac{\rho_\infty}{\gamma p_\infty}\right)V_\infty^2 = \frac{\gamma}{2}p_\infty\frac{V_\infty^2}{a_\infty^2} = \frac{\gamma}{2}p_\infty M_\infty^2 \\ C_p &= \frac{2}{\gamma M_\infty^2}\frac{p - p_\infty}{p_\infty} \end{aligned}$$

FORMULA

$$C_p = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right) \tag{11.7}$$

Now, we again want to linearise it. To do this, remember, with $c_p = \gamma R / (\gamma - 1)$, that

$$\begin{aligned}
T + \frac{V^2}{2c_p} &= T_0 + \frac{V_\infty^2}{2c_p} \\
T - T_\infty &= \frac{V_\infty^2 - V^2}{2\gamma R / (\gamma - 1)} \\
\frac{T}{T_\infty} - 1 &= \frac{\gamma - 1}{2} \frac{V_\infty^2 - V^2}{\gamma R T_\infty} = \frac{\gamma - 1}{2} \frac{V_\infty^2 - V^2}{a_\infty^2}
\end{aligned}$$

Using

$$V^2 = \left(V_\infty + \hat{u}\right)^2 + \hat{v}^2 = V_\infty^2 + 2\hat{u}V_\infty + \hat{u}^2 + \hat{v}^2$$

this can be written as

$$\begin{aligned}
\frac{T}{T_\infty} &= 1 - \frac{\gamma - 1}{2a_\infty^2}\left(2\hat{u}V_\infty + \hat{u}^2 + \hat{v}^2\right) \\
\frac{p}{p_\infty} &= \left(\frac{T}{T_\infty}\right)^{\gamma/(\gamma-1)} = \left[1 - \frac{\gamma - 1}{2a_\infty^2}\left(2\hat{u}V_\infty + \hat{u}^2 + \hat{v}^2\right)\right]^{\gamma/(\gamma-1)} \\
&= \left[1 - \frac{\gamma - 1}{2} M_\infty^2 \left(\frac{2\hat{u}}{V_\infty} + \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2}\right)\right]^{\gamma/(\gamma-1)}
\end{aligned}$$

Now, we didn't linearise anything yet, which we will do now. Let's again assume

$$\frac{\hat{u}}{V_\infty} << 1 \qquad \frac{\hat{u}^2}{V_\infty^2}, \frac{\hat{v}^2}{V_\infty^2} <<< 1$$

In this case, we can write it as

$$\frac{p}{p_\infty} = (1 - \epsilon)^{\gamma/(\gamma-1)}$$

where ϵ is small. We can use binomial expansion, and neglecting higher-order terms (because ϵ^2 will be negligibly small):

$$\begin{aligned}
\frac{p}{p_\infty} &= 1 - \frac{\gamma}{\gamma - 1}\epsilon + \cdots = 1 - \frac{\gamma}{\gamma - 1} \cdot \frac{\gamma - 1}{2} M_\infty^2 \left(\frac{2\hat{u}}{V_\infty} + \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2}\right) + \cdots \\
&= 1 - \frac{\gamma}{2} M_\infty^2 \left(\frac{2\hat{u}}{V_\infty} + \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2}\right) + \cdots
\end{aligned}$$

Substituting this into the expression for the pressure coefficient, we have

$$\begin{aligned}
C_p &= \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right) = \frac{2}{\gamma M_\infty^2}\left[1 - \frac{\gamma}{2} M_\infty^2 \left(\frac{2\hat{u}}{V_\infty} + \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2}\right) + \cdots - 1\right] \\
&= -\frac{2\hat{u}}{V_\infty} - \frac{\hat{u}^2 + \hat{v}^2}{V_\infty^2}
\end{aligned}$$

but with $\hat{u}^2 / V_\infty^2$ and $\hat{v}^2 / V_\infty^2 <<< 1$, this becomes

FORMULA

$$C_p = -\frac{2\hat{u}}{V_\infty} \tag{11.8}$$

To round up this section, let us find our two boundary conditions for ϕ. We have at infinitely clearly that ϕ = constant, i.e. $\hat{u} = \hat{v} = 0$. Furthermore, at the surface, if θ is the angle between the tangent to the surface and the freestream flow, we must have

$$\tan\theta = \frac{v}{u} = \frac{\hat{v}}{V_\infty + \hat{u}}$$

which is the flow-tangency condition at the body surface. As $\hat{u} << V_\infty$, we can write

$$\hat{v} = V_\infty \tan\theta$$

$$\boxed{\frac{\partial\hat{\phi}}{\partial y} = V_\infty \tan\theta \qquad (11.9)}$$

FORMULA

Very nice indeed. Again, note that all of the equations derived are *approximations*, which have the two important limitations I bulleted on the previous page.

11.4 *Prandtl-Glauert compressibility correction*

This section is basically only one big derivation. Unfortunately, I'm not really able to summarize it because all of it is just plain mathematics. However, I'm pretty sure you don't need to be able to derive it yourself.

How wonderful would it be if we could correct our incompressible flow results so that we can get compressible results without actually doing experiments on high flow velocities. Prandtl and Glauert thought so at least, hence they derived a correction for it. Consider the subsonic, compressible, inviscid flow over the airfoil depicted in figure 11.2, for which the shape is given by $y = f(x)$.

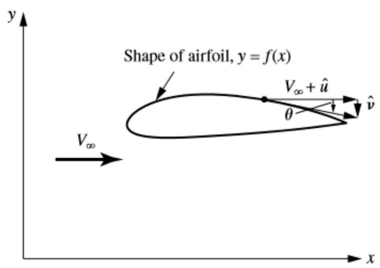


Figure 11.2: Airfoil in physical space.

Now, let $\beta^2 \equiv 1 - M_\infty^2$, so that we can write

$$\beta^2 \frac{\partial^2\hat{\phi}}{\partial x} + \frac{\partial^2\hat{\phi}}{\partial y^2} = 0$$

Now, let us transform the variables x and y into a new space, ξ and η, such that

$$\xi = x \qquad (11.10)$$
$$\eta = \beta y \qquad (11.11)$$
$$\hat{\phi}(\xi,\eta) = \beta\hat{\phi}(x,y) \qquad (11.12)$$

From the chain rule for partial differentiation, we have

$$\frac{\partial\hat{\phi}}{\partial x} = \frac{\partial\hat{\phi}}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\hat{\phi}}{\partial\eta}\frac{\partial\eta}{\partial x} \quad \text{and} \quad \frac{\partial\hat{\phi}}{\partial y} = \frac{\partial\hat{\phi}}{\partial\xi}\frac{\partial\xi}{\partial y} + \frac{\partial\hat{\phi}}{\partial\eta}\frac{\partial\eta}{\partial y}$$

From equations (11.10) and (11.11) we clearly have

$$\frac{\partial\xi}{\partial x} = 1 \quad \frac{\partial\xi}{\partial y} = 0 \quad \frac{\partial\eta}{\partial x} = 0 \quad \frac{\partial\eta}{\partial y} = \beta$$

We can plug this in the chain rules, and we can also use equation (11.12) to find

$$\begin{aligned}\frac{\partial\hat{\phi}}{\partial x} &= \frac{\partial\hat{\phi}}{\partial\xi} = \frac{1}{\beta}\frac{\partial\bar{\phi}}{\partial\xi}\\ \frac{\partial\hat{\phi}}{\partial y} &= \beta\frac{\partial\hat{\phi}}{\partial\eta} = \frac{\partial\bar{\phi}}{\partial\eta}\end{aligned}$$

Note the difference between the hats and bars. We can differentiate both equations with respect to x and y, using the chain rule, to find

$$\frac{\partial^2\hat{\phi}}{\partial x^2} = \frac{1}{\beta}\frac{\partial^2\bar{\phi}}{\partial\xi^2} \quad \text{and} \quad \frac{\partial^2\hat{\phi}}{\partial y^2} = \beta\frac{\partial^2\bar{\phi}}{\partial\eta^2}$$

Thus, we can write the initial equation as

$$\begin{aligned}\beta^2\frac{1}{\beta}\frac{\partial^2\bar{\phi}}{\partial\xi^2} + \beta\frac{\partial^2\bar{\phi}}{\partial\eta^2} &= 0\\ \frac{\partial^2\bar{\phi}}{\partial\xi^2} + \frac{\partial^2\bar{\phi}}{\partial\eta^2} &= 0\end{aligned}$$

What's the use of this? Note that this is the Laplace equation, actually, which we said only holds for *incompressible flow*. So, by the use of the transformation described, we actually transform to an in *incompressible flow* in (ξ,η) space. This is a very important, as we'll see shortly. First, a small intermezzo. If the original airfoil shape is given by $f(x)$, then we have that the tangency condition is, with $df/dx = \tan\theta$:

$$V_\infty\frac{df}{dx} = \frac{\partial\hat{\phi}}{\partial y} = \frac{1}{\beta}\frac{\partial\bar{\phi}}{\partial y} = \frac{\partial\bar{y}}{\partial\eta}$$

Let the shape of the airfoil in the transformed space by given by $\eta = q(\xi)$, then we have

$$V_\infty\frac{dq}{d\xi} = \frac{\partial\bar{\phi}}{\partial\eta}$$

Comparing the right hand side of both equations, we clearly have

$$\frac{df}{dx} = \frac{dq}{d\xi}$$

So, the shape of the airfoil in the transformed space is the as in the physical space. Hence, the above transformation relates the compressible flow over an airfoil in (x,y) space to the incompressible flow in (ξ,η) space over the same airfoil (so we're not just doing some random transformations, don't worry). Anyway, we can write the pressure coefficient as

$$C_p = \frac{-2\hat{u}}{V_\infty} = -\frac{2}{V_\infty}\frac{\partial\hat{\phi}}{\partial x} = -\frac{2}{V_\infty}\frac{1}{\beta}\frac{\partial\bar{\phi}}{\partial x} = -\frac{2}{V_\infty}\frac{1}{\beta}\frac{\partial\bar{\phi}}{\partial\xi} = -\frac{2}{V_\infty}\frac{\bar{u}}{\beta} = \frac{1}{\beta}\left(-\frac{2\bar{u}}{V_\infty}\right)$$

What's the significance of this? $\bar{u}$ is the perturbation velocity for the incompressible flow, and thus, $(-2\bar{u}/V_\infty)$ is simply the (linearised) pressure coefficient for the *incompressible flow*. Denote this pressure coefficient by $C_{p,0}$, and using $\beta \equiv \sqrt{1-M_\infty^2}$, we get

FORMULA: PRANDTL-GLAUERT RULE

$$C_p = \frac{C_{p,0}}{\sqrt{1-M_\infty^2}} \tag{11.13}$$

Similarly, we have

FORMULAS

$$c_l = \frac{c_{l,0}}{\sqrt{1-M_\infty^2}} \tag{11.14}$$

$$c_m = \frac{c_{m,0}}{\sqrt{1-M_\infty^2}} \tag{11.15}$$

You can do the same correction for the drag coefficient of course, but it's rather useless to do so, because remember d'Alamberts paradox: for inviscid, incompressible flow over a closed, two-dimensional body, the drag is theorically zero, so $c_{d,0}$ would be zero as well. Only when M becomes sufficiently high to produce locally supersonic flow, then shock waves are created and a positive wave drag is produced. However, you shouldn't use the Prandtl-Glauert rule beyond $M_\infty > 0.7$ anyway.

11.5 Improved compressibility corrections

If you feel rebellious, you can also use the following two equations:

$$C_p = \frac{C_{p,0}}{\sqrt{1-M_\infty^2} + \left[M_\infty^2 / \left(1+\sqrt{1-M_\infty^2}\right)\right] C_{p,0}/2}$$

$$C_p = \frac{C_{p,0}}{\sqrt{1-M_\infty^2} + \left(M_\infty^2 \left\{1+\left[(\gamma-1)/2\right] M_\infty^2\right\} / 2\sqrt{1-M_\infty^2}\right) C_{p,0}}$$

The upper one is actually widely adopted by the aeronautical industry since World War II.

11.6 Critical Mach number

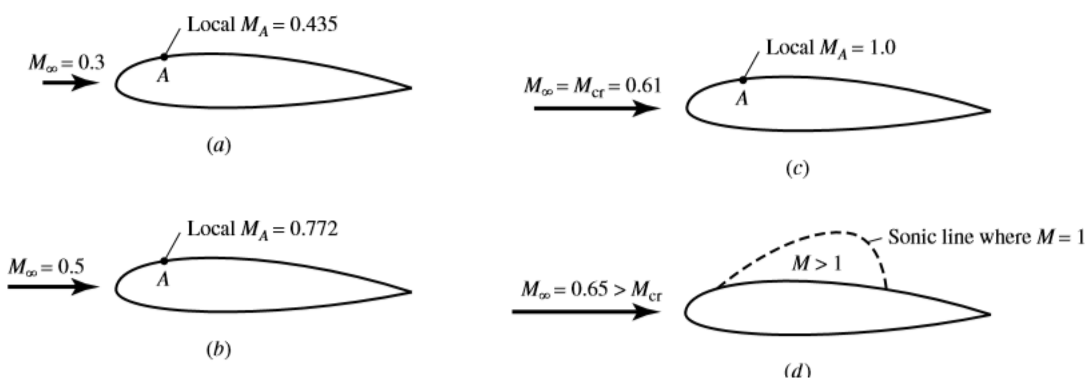


Figure 11.3: Definition of critical Mach number. Point A is the location of minimum pressure on the top surface of the airfoil.

The critical Mach number is the freestream Mach number at which anywhere along the airfoil, a Mach number of $M = 1$ is achieved (not everywhere, but anywhere). Thus, for the airfoil shown in figure 11.3, M_{cr} clearly equals 0.61. However, most of the time in life, you won't be given such nice drawings showing the maximum Mach number at the airfoil, but you have to compute it yourself. Instead, you have to calculate it yourself. How do we do this? There are basically two components deciding what the critical Mach number is. First, we have the following. For isentropic flow, we have

$$\frac{p_A}{p_\infty} = \frac{p_A/p_0}{p_\infty/p_0} = \left(\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_A^2}\right)^{\frac{\gamma}{\gamma-1}}$$

$$C_{p,A} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_A}{p_\infty}-1\right) = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_A^2}\right)^{\frac{\gamma}{\gamma-1}} - 1\right]$$

We can now compute the **critical pressure coefficient**, denoted by $C_{p,cr}$. This is the value of the local C_p when the local Mach number is unity. At this point, we have $M_A = 1$, and thus

$$C_{p,cr} = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma}{\gamma-1}} - 1\right] \tag{11.16}$$

Note that we have a critical pressure coefficient for every value for M_∞: even if our flow is very low speed (so you wouldn't expect the flow to reach $M = 1$ anywhere along the airfoil), there is still a critical pressure coefficient; this is the pressure coefficient such that *if* there had been a point on the airfoil where locally $M = 1$, the pressure coefficient locally equals the critical pressure coefficient.

Remember that the pressure coefficient basically indicated how much the flow is sped up, relative to the freestream velocity: very negative pressure coefficients indicated that the flow is locally much faster than the freestream velocity. If our flow has a very low freestream velocity (say $M_{cr} = 0.01$), the pressure coefficient required to make the flow sonic at a point must be *incredibly high*: not only does it have to be sped up in an absolute sense a lot (namely 0.99), but also relatively (it needs to become 100 times as high). This means that at low freestream Mach numbers, the critical pressure coefficient is very negative.

At higher freestream Mach numbers (say $M_\infty = 0.8$), the flow does not have to accelerate as much: it only needs to accelerate 0.2, which is also just 25% higher. Thus, the critical pressure coefficient will be far less negative.

The point I'm trying to make is that clearly, the critical pressure coefficient of an airfoil depends on the freestream Mach number. However, there is only one unique critical Mach number for a given airfoil. How do we determine this?

1. From either experimental or theoretical data, we compute the low-speed incompressible value of the pressure coefficient $C_{p,0}$ at the minimum pressure point on the given airfoil.
2. Using any of the compressibility corrections, we can calculate the maximum pressure coefficient at any freestream Mach number (as long as $M_\infty < 1$). We can plot this together with the previously found formula for critical pressure coefficient as done in figure 11.4.
3. Somewhere, there is a point where the pressure coefficient corresponds to locally sonic flow. The intersection of curves B and C represents the point corresponding to sonic flow at the minimum pressure location on the airfoil. The value of M_{cr} at this intersection is the critical Mach number.

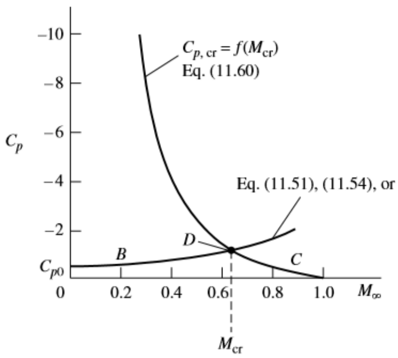


Figure 11.4: Estimation of critical Mach number.

The critical pressure coefficient at the critical Mach number is simply given by

FORMULA

$$C_{p,cr} = \frac{2}{\gamma M_{cr}^2}\left[\left(\frac{1+\frac{\gamma-1}{2}M_{cr}^2}{1+\frac{\gamma-1}{2}}\right)^{\frac{\gamma}{\gamma-1}} - 1\right] \tag{11.17}$$

Now, consider one thin airfoil and one thick airfoil. Recall that

$$C_p = -\frac{2\hat{u}}{V_\infty}$$

The flow over a thin airfoil is only slightly perturbed from the freestram. Hence, the expansion over the top surface is mild, and $C_{p,0}$ at the minimum pressure point is a negative number of only small absolute magnitude, as shown in figure 11.5. On the other hand, a thick airfoil causes a much larger perturbation, and thus $C_{p,0}$ at the minimum pressure point is a negative number of a rather large absolute magnitude, also shown in figure 11.5. This leads to the thick airfoil having a smaller critical Mach number than the thin airfoil, and it is for this reasons that most high-speed aircraft, thin airfoils are used.

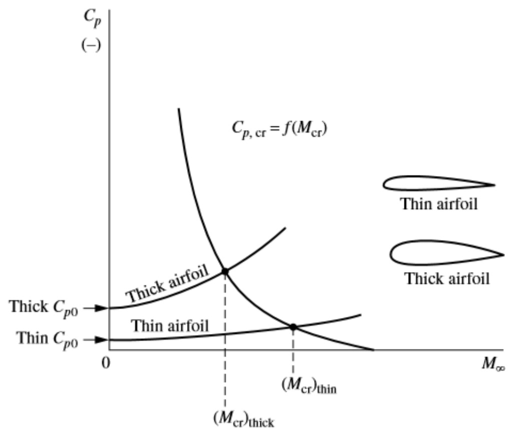


Figure 11.5: Effect of airfoil thickness on critical Mach number.

11.6.1 *A comment on the location of minimum pressure (maximum velocity)*

You may have found it weird that an airfoil may have its location of maximum thickness at around 0.30c, but that its location of minimum pressure (which is also the location of maximum velocity) is located much more in front of that. After all, the continuity equation needed to be satisfied, and the most critical point for that is the location of maximum thickness, so you'd expect the velocity to be maximum there. However, nature places the maximum velocity at a point which satisfies the physics of the whole flow field (also away from the surface), not just what is happening in a local region of the flow.

11.7 *Drag-divergence Mach number: the sound barrier*

Look at figure 11.6.

Let's just go step by step what happens if we start measuring the drag coefficient c_d of an airfoil at a low subsonic speed.

1. If we gradually increase M_∞, c_d remains relatively constant all the way to the critical Mach number.
2. If we carefully increase beyond M_{cr}, say, to point *d* in figure 11.6, then there is a finite region of supersonic flow on the airfoil, as shown in figure *d* of figure 11.3.
3. However, as we increase to nudge M_∞ higher, we encounter a point where the drag coefficient suddenly starts to increase significantly. This is given as point *e* in figure 11.6. The value of M_∞ at which this sudden increase in drag starts is defined as the **drag-divergence Mach number**. This large increase in drag is associated with an extensive region of supersonic flow over the airfoil, terminating in a shock wave as sketched in the insert in figure 11.6. Do note: the drag-divergence Mach number is not a clear-cut value; a "significant increase" is defined arbitrarily so just use a value at which it starts to increase significantly (my point is, it's not a well-defined value as M_{cr}).
4. Going to point f in figure 11.6, the insert shows that as M_∞ approaches unity, the flow on both the top and bottom surfaces can be supersonic, terminated by shock waves. These shock waves generally cause severe flow separation downstream of the shocks, with an attendant large increase in drag.
5. At or around Mach 1, c_d peaks and then actually decreases as we enter the supersonic regime, as the

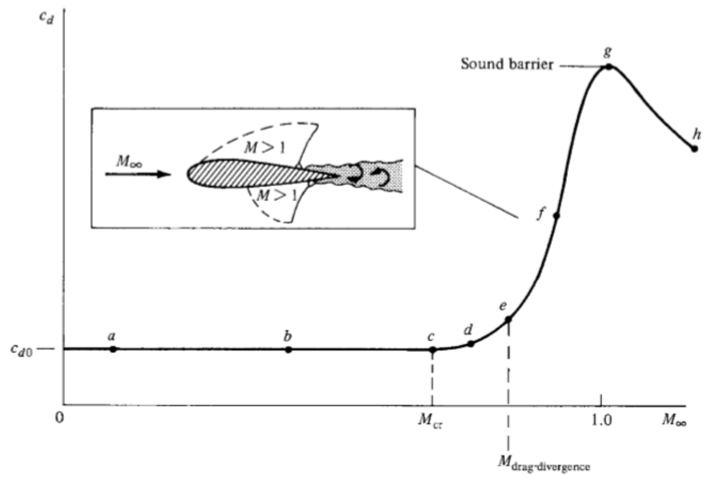
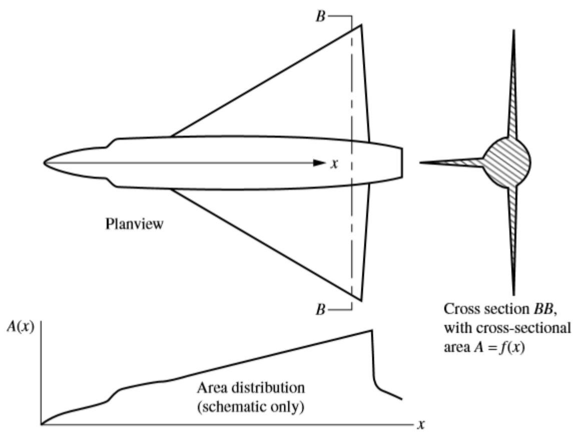


Figure 11.6: Sketch of the variation of profile drag coefficient with freestream Mach number, illustrating the critical and drag-divergence Mach numbers and showing the large drag rise near Mach 1.

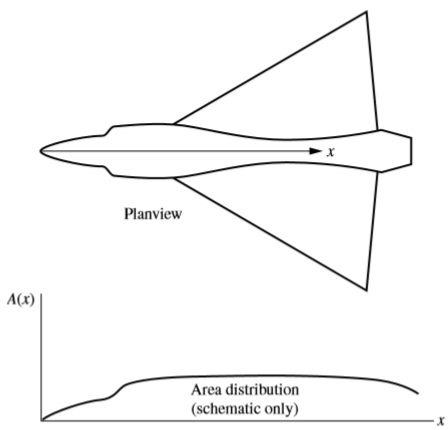
shock waves become weaker[1]. Point g is called the sound barrier.

11.8 *The area rule*

Originally, in WWII, fighter aircraft had the shape shown in figure 11.7a: on the bottom, the *cross-sectional* area is plotted as a function of x. With the appearance of the wing, the cross-sectional area increases constantly; at the end, when the wing terminates, the cross-sectional area suddenly decreases a lot. Now, it was already known for a century by ballisticians (people specialized in artilley and that kind of stuff) that a body had to be smooth with no discontinuities. So, one guy tried this for an aircraft as well, and an aircraft using this area distribution is shown in figure 11.7b. The aircraft shown in figure 11.7b countered the problem of the wings increasing the cross-sectional area by decreasing the cross-sectional area of the fuselage continuously with increasing x. This keeps the total cross-sectional area of the aircraft more or less constant.



(a) A schematic of a non-area ruled aircraft.



(b) A schematic of an area ruled aircraft.

Figure 11.7: Area rule.

This design philosophy is called the **area rule**. Applying the area rule can decrease the c_d at $M = 1$ by up to a factor 2.

[1] Why? If the deflection angle θ remains the same, and M is increased, β becomes smaller, meaning the waves became weaker and weaker.

11.9 The supercritical airfoil

It has been touched upon before, but one way of delaying the drag divergence number is by simply increasing M_{cr}, which could be done by using a thinner airfoil, as shown in figure 11.5. However, there are limits on how thin a practical airfoil can be; most importantly, an airfoil requires thickness for structural strength, and there must be room for storage of fuel.

So, how can we increase $M_{\text{drag-divergence}}$ even further? Rather than increasing M_{cr}, we can strive to increase the Mach number increment between M_{cr} and $M_{\text{drag-divergence}}$, that is, referring to figure 11.6, let us increase the distance between points e and c. This philosophy has been pursued since 1965, leading to the design of a new family of airfoils called **supercritical airfoils**. The shape of a supercritical airfoil is compared with an NACA 64-series airfoil in figure 11.8. The supercritical airfoil has a relatively flat top, thus encouraging a region of supersonic flow with lower lcoal values of M than the NACA 64 series. In turn, the terminating shock is weaker, thus creating less drag. Similar trends can be seen by comparing the C_p distributions for the NACA 64 series. Similar trends can be seen for the C_p distributions.

In fact, the figures for the NACA 64-series airfoil pertain to a lower freestream Mach number, $M_\infty = 0.69$, than the figures for the supercritical airfoil, $M_\infty = 0.79$. Clearly, the supercritical airfoil shows more desirable flow-field characteristics. This results in a higher value of $M_{\text{drag-divergence}}$, as shown in figure 11.9.

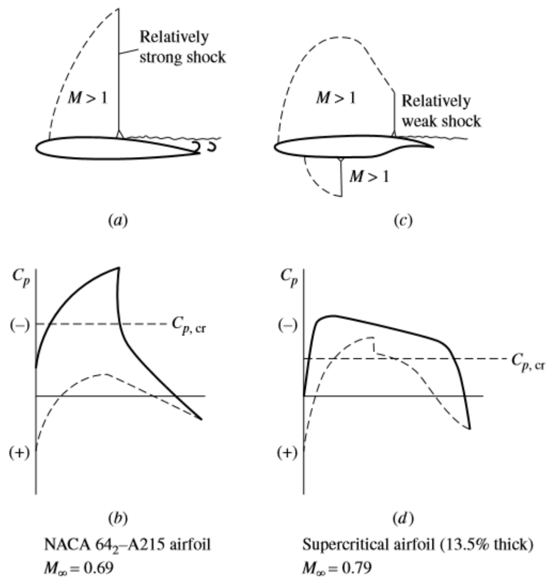


Figure 11.8: Standard NACA 64-series airfoil compared with a supercritical airfoil at cruise lift conditions.

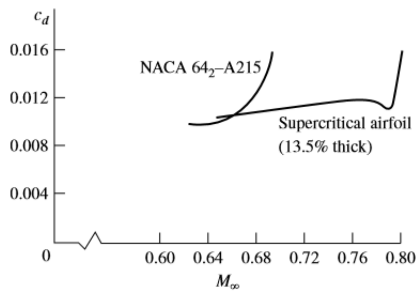


Figure 11.9: The drag-divergence properties of a standard NACA 64-series airfoil and a supercritical airfoil.

Because the top of the supercritical airfoil is relatively flat, the forward 60% of the airfoil has negative camber, lowering the lift. To compensate, the lift is increased by having extreme positive camber on the rearward 30% of the airfoil, hence the funny cusplike shape of the bottom surface near the trailing edge.

12 Linearized supersonic flow

If you paid attention, you would have noticed that chapter 11 does not provide us an easy way to calculate the variation of the pressure coefficient for a supersonic flow: the formula

$$C_p = -\frac{2\hat{u}}{V_\infty}$$

requires knowledge of $\hat{u}$, which is not straightforward to collect. Alternatively, although you could deduce the pressure coefficient variation over an airfoil for incompressible flow using stuff from Aero I, the compressibility correction

$$C_p = \frac{C_{p,0}}{\sqrt{1 - M_\infty^2}}$$

only worked for subsonic Mach numbers. This chapter aims to find a formula to straightforwardly calculate the pressure coefficient anywhere on a supersonic airfoil (as long as it's not too large and the angle of attack is small), so that we can from that deduce the lift and drag, using knowledge of Intro I, basically.

12.2 Derivation of the linearized supersonic pressure coefficient formula

Let us write

$$\left(1 - M_\infty^2\right)\frac{\partial^2\hat{\phi}}{\partial x^2} + \frac{\partial\hat{\phi}}{\partial y^2} = 0$$

as (notice that we multiply everything by -1)

$$\lambda^2\frac{\partial^2\hat{\phi}}{\partial x^2} - \frac{\partial^2\hat{\phi}}{\partial y^2} = 0$$

with $\lambda = \sqrt{M_\infty^2 - 1}$. A solution to this equation is the functional relation

$$\hat{\phi} = f(x - \lambda y)$$

that is, any function which rather than the argument x uses the argument $x - \lambda y$ satisfies this equation. That is a rather daring statement to make, so let's see why. The partial derivative of this with respect to x can be written as

$$\frac{\partial\hat{\phi}}{\partial x} = f'(x - \lambda y)\frac{\partial(x - \lambda y)}{\partial x} = f'(x - \lambda y)\cdot 1 = f'$$

If you're a bit confused: $f'(x - \lambda y)$ denotes the differentiation of f with respect to its argument, $x - \lambda y$. We must multiply this with the partial derivative of the argument over x, which equals 1 rather obviously. We write f' instead of $f'(x - \lambda y)$ just for brevity. Again differentiating f' gives

$$\frac{\partial^2\hat{\phi}}{\partial x^2} = f''$$

and similarly:

$$\begin{aligned}
\frac{\partial\hat{\phi}}{\partial y} &= f'(x - \lambda y)\frac{\partial(x - \lambda y)}{\partial y} = -\lambda\cdot f' \\
\frac{\partial^2\hat{\phi}}{\partial y^2} &= \lambda^2 f''
\end{aligned}$$

Plugging this into the original equation gives

$$\lambda^2 \frac{\partial^2 \hat{\phi}}{\partial x^2} - \frac{\partial^2 \hat{\phi}}{\partial y^2} = \lambda^2 f'' - \lambda^2 f'' = 0$$

So we've proven that *any* function of $x - \lambda y$ is a solution. Therefore, we know that $\hat{\phi}$ is constant along lines of $x - \lambda y =$ constant (after all, the argument is same if you plug it into the function, so it should generate the same outcome). The slope of these lines can be deduced as follows:

$$\begin{aligned} x - \lambda y &= \text{constant} \\ x &= \text{constant} + \lambda y \\ dx &= 0 + \lambda dy \\ \frac{dy}{dx} &= \frac{1}{\lambda} = \frac{1}{\sqrt{M_\infty^2 - 1}} \end{aligned}$$

Now, recall figure 12.1, where we clearly also see that

$$\frac{1}{\sqrt{M_\infty^2 - 1}} = \tan \mu$$

And thus a line along which $\hat{\phi}$ is constant, and thus a line with slope $dy/dx = 1/\sqrt{M_\infty^2 - 1}$ is also a Mach line. This is depicted in figure 12.2.

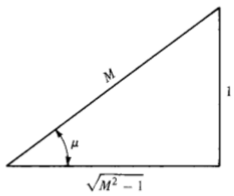


Figure 12.1: Triangle associated with the Mach angle.

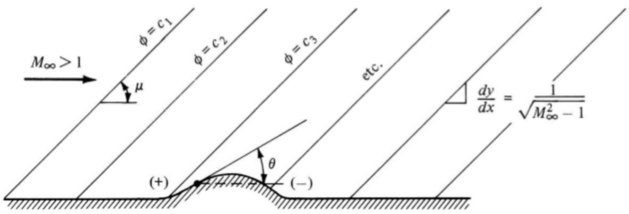


Figure 12.2: Linearized sueprsonic flow.

All disturbances created at the wall propagate unchanged away from the wall along Mach waves. All Mach waves have the same slope, only dependent on M_∞. This is one visible result of the linearization of the flow: the results assume small perturbations, that is, the hump in figure 12.2 is small, and thus θ is small. We know by know that in reality, oblique shock waves and expansion waves will be created around the hump, but linearized supersonic flow theory does not take these into account.

Note that all the Mach waves point downstream above the wall, so once again we see that disturbances at the wall only propagate downstream.

Now, onto finding an expression for the pressure coefficient: we rather simply get:

$$\begin{aligned} \hat{u} &= \frac{\partial \hat{\phi}}{\partial x} = f' \\ \hat{v} &= \frac{\partial \hat{\phi}}{\partial y} = -\lambda f' \\ \hat{u} &= -\frac{\hat{v}}{\lambda} \end{aligned}$$

Furthermore, we also have derived equation (11.9) before, which stated that

$$\hat{v} = \frac{\partial \hat{\phi}}{\partial y} = V_\infty \tan\theta \approx V_\infty \theta$$

Thus, we have

$$\hat{u} = -\frac{V_\infty \theta}{\lambda}$$

Now, remember $C_p = -2\hat{u}/V_\infty$ and $\lambda \equiv \sqrt{M_\infty^2 - 1}$, then we have

FORMULA

$$C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}} \tag{12.1}$$

Note that if you are given a formula for the curve of an airfoil/wall, then you can also use $\theta = dy/dx$. Note how easy and powerful formula this is: it allows you to directly calculate the pressure coefficient based on the deflection angle. It holds for any slender two-dimensional body where θ is small. To conclude this section, compare this beautiful formula with the Prandtl-Glauert correction of the previous chapter: there, we saw that $C_p \propto (1 - M_\infty^2)^{-1/2}$, whereas this formula says that $C_p \propto (M_\infty^2 - 1)^{-1/2}$. This is remarkable and depicted in figure 12.3. However, keep in mind that both of them do not hold for the transonic range.

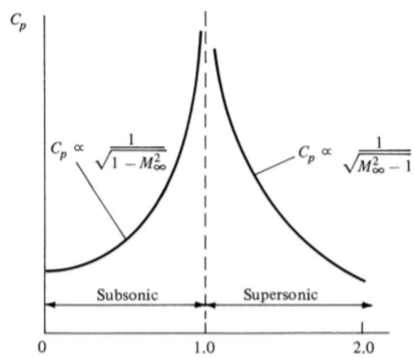


Figure 12.3: Variation of the linearized pressure coefficient with Mach number (schematic).

12.3 *Application to supersonic airfoils*

Now, let's finally do some application. Look at figure 12.4. Using the method of previous section, you may have one slight issue: what is the sign convention for θ? Well, you can make it quite formal, but it's easiest if you just remember: if a surface is inclined *into* the freestream direction, then θ is positive. If a surface is inclined *away from* the freestream, then θ is negative. Looking at figure 12.4, we then clearly see that

$$C_{p,A} = \frac{2\theta_A}{\sqrt{M_\infty^2 - 1}} \quad \text{and} \quad C_{p,B} = \frac{2\theta_B}{\sqrt{M_\infty^2 - 1}}$$
$$C_{p,C} = \frac{-2\theta_C}{\sqrt{M_\infty^2 - 1}} \quad \text{and} \quad C_{p,D} = \frac{-2\theta_D}{\sqrt{M_\infty^2 - 1}}$$

Now, let's consider a flat plate as shown in figure 12.5. The top surface is inclined away from the flow, and thus

$$C_{p,u} = \frac{-2\alpha}{\sqrt{M_\infty^2 - 1}}$$

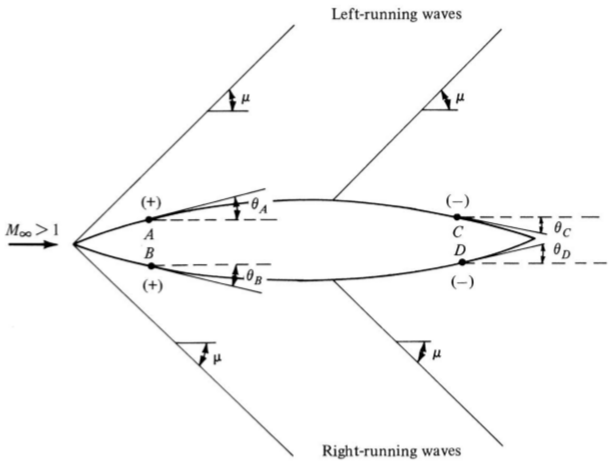


Figure 12.4: Linearized supersonic flow over an airfoil.

The lower surface is inclined into the flow, and thus

$$C_{p,l} = \frac{2\alpha}{\sqrt{M_\infty^2 - 1}}$$

Both are constant over their surfaces. The normal force coefficient for the flat plate can be easily obtained:

$$c_n = \frac{1}{c}\int_0^c \left(C_{p,l} - C_{p,u}\right) dx = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \frac{1}{c}\int_0^c dx = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

The axial force coefficient is given by

$$c_a = \frac{1}{c}\int_{\text{LE}}^{\text{TE}} \left(C_{p,u} - C_{p,l}\right) dy$$

but this quite obviously reduces to zero as a flat plate has zero thickness. We then have

$$\begin{aligned} c_l &= c_n \cos\alpha - c_a \sin\alpha \approx c_n - c_a\alpha = c_n \\ c_d &= c_n \sin\alpha - c_a \cos\alpha \approx c_n\alpha - c_a = c_n\alpha \end{aligned}$$

leading to

FORMULAS

$$c_l = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \tag{12.2}$$

$$c_d = \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}} \tag{12.3}$$

These are the lift and wave-drag coefficients for supersonic flow over a flat plate. Keep in mind that these are from linearized theory nad therefore only valid for small α.

For a thin airfoil of arbitrary shape, we still have

$$c_l = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}}$$

but for the drag-coefficient, it actually becomes

$$c_d = \frac{4}{\sqrt{M_\infty^2 - 1}} \left(\alpha^2 + g_c^2 + g_t^2\right)$$

where g_c and g_c are functions of the airfoil camber and thickness, respectively (you don't need to know how to compute them).

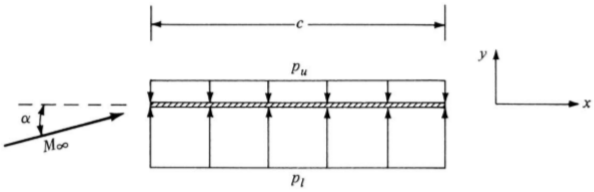


Figure 12.5: A flat plate at angle of attack in a supersonic flow.

12.3.1 Induced drag

You may be thinking, is there no such thing as wing tip effects for supersonic flight? Yes there are, but due to the fact that the speeds are supersonic, these are limited to the region inside the Mach cone with its vertex at the tip leading edge (just imagine the Mach waves to be 3D). Already at Mach 2, where $\mu = \arcsin \frac{1}{2} = 30°$, we realize that only a small portion of the wing is affected by tip effects (whereas the entire wing suffered wing tip effects for the subsonic case).

Index