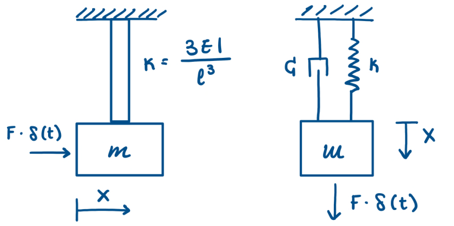
Problems vibrations tutorial

January 2019



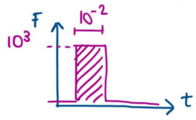
$m = 1000$ kg
$\zeta = 0'01$
$I = 0'5$ m^4
$\ell = 2$ m
$E_{al} = 7 \cdot 10^{10}$ N/m^2

$\Delta t = 10^{-2}$ s
$F_0 = 10^3$ N

What are we doing in this problem?

This problem asks for the system's response under an impulse loading. This response is called the impulse response $h(t)$.
The formula sheet for the exam includes all the impulse responses you'll need for the exam; but here we show you how to derive it.

- How the impulse function works



An impulse input quickly strikes the body and then it goes away, leaving it to vibrate freely. Think about it as what happens to a spring when you compress it: it only starts wobbling up and down only once you release it. The way it wobbles is determined by the force you apply (if you pull a lot, the spring will kick back very fast; if you pull only a little the spring will barely move). But once you release it, the spring vibrates freely, the force is no longer acting on it.

Thus, think of this case as a free vibration with very specific initial conditions. Don't try to look for a particular solution using Laplace or convolution; that is only used if the force acts continuously on the system as it moves (but for the impulse the force just goes away!)

1 Problem identification

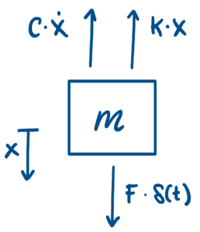
This system has a damper with a damping coefficient $\zeta = 0'01$ → it is underdamped

There is also a force acting on the system. It has an impulse modulus (area underneath the graph) of

$F = F_0 \cdot \Delta t = 10^3 \cdot 10^{-2} = 10\ N \cdot S$

the force has an impulse form (that's what the $\delta(t)$ means)

2 FBD, KD and EOM



$\sum F_x = m \cdot a_x$ →

$F_x(t) - Kx - c\dot{x} = m\ddot{x}$

$F_x(t) = m\ddot{x} + c\dot{x} + Kx$

$f(t) = \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x$

3 Homogeneous solution ($f(t) = 0$)

$f(t) = \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

→ Substitute $x = e^{rt}$
$\dot{x} = re^{rt}$
$\ddot{x} = r^2 e^{rt}$

$f(t) = r^2 e^{rt} + 2\zeta\omega_n \cdot r e^{rt} + \omega_n^2 e^{rt} = 0$

$f(t) = r^2 + 2\zeta\omega_n \cdot r + \omega_n^2 = 0$

$$r = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2}$$

$$r = \underbrace{-\zeta\omega_n}_{-\lambda} \pm \underbrace{\omega_n\sqrt{\zeta^2-1}}_{\omega_d \cdot i}$$ → Remember: $\zeta < 1$

→ Substitute back to $x = e^{rt}$

$x = e^{(-\lambda \pm \omega_d i)t}$

$f(t) = A_1 e^{-\lambda t} \cdot \cos(\omega_d t) + A_2 e^{-\lambda t} \cdot \sin(\omega_d t)$

$= A_0 e^{-\lambda t} \cdot \sin(\omega_d t + \phi)$

$$\boxed{x(t) = e^{-\zeta\omega_n t}(A_1 \cdot \cos(\omega_d t) + A_2 \cdot \sin(\omega_d t))}$$

(4) Boundary conditions

→ The homogeneous equation had unknown coefficients (A_1 and A_2). We can use the forcing term to find the initial conditions, and from there get the coefficients

$$x(t) = m\ddot{x} + c\dot{x} + kx = F\ \delta(t)$$

→ The total response is calculated by integrating the effect of the force over the time it acts

$$\int_{-\infty}^{\infty} m\ddot{x}(t)\,dt + \underbrace{\int_{-\infty}^{\infty} c\dot{x}(t)\,dt + \int_{-\infty}^{\infty} kx(t)\,dt}_{} = F\cdot\int_{-\infty}^{\infty}\delta(t)\,dt$$

Both these functions are continuous, so they are =0 over the span $t^- \to t^+$. While the force is acting, the body doesn't change positions or speed

$$m\,[\dot{x}(t^+) - \dot{x}(t^-)] = F$$

→ The body is at rest before the impulse starts

$\dot{x}(t^-) = 0$

→ $m\dot{x}(t^+) = F$ → $\dot{x}(t^+) = F/m$

- Now, keep in mind that the body is released (so it starts vibrating) at $t = t^+$. So, as we are analysing how the body vibrates, t^+ is our $t=0$.

$$\begin{cases} x(t) = e^{-\zeta w_n t}\,(A_1\cdot\cos(w_d t) + A_2\cdot\sin(w_d t)) \\ \dot{x}(t) = -\zeta w_n\, e^{-\zeta w_n t}\,(A_1\cdot\cos(w_d t) + A_2\cdot\sin(w_d t)) \\ \qquad + e^{-\zeta w_n t}\,[A_1 w_d\cdot\cos(w_d t) + A_2\cdot w_d\cdot\sin(w_d t)] \end{cases}$$

→ initial conditions: $x(0) = 0$

$\dot{x}(0) = P_0/m$

$$x(t) = e^{-\zeta w_n t}\,[A_1\cdot\cos(w_d t) + A_2\cdot\sin(w_d t)] = 0 \rightarrow \boxed{A_1 = 0}$$

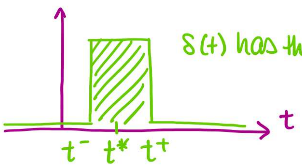
$$\dot{x}(t) = P_0/m \rightarrow -\zeta_n a_1 + a_2 w_d = P_0/m \rightarrow A_2 = P_0/m\cdot w_d$$

Final solution: $x(t) = P_0\cdot\dfrac{1}{m\cdot w_d}\,e^{-\zeta w_n t}\cdot\sin(w_d t)$

This is the response of the system to an impulse, which is also noted as $h(t)$

Impulse response $\delta(t)$

An impulse is a "push" on the body, starting at t^- and ending at t^+



$\delta(t)$ has the property that $\int_{-\infty}^{\infty}\delta(t)\,dt = 1$

The impulse function is defined as:

$$x(t)\begin{cases} 0 & t \le t^* \\ P_0\, h(t-t^*) & t > t^* \end{cases}$$

→ $h(t)$ is the system impulse response, which we just derived in the previous step

5 Fill in the numbers

→ Remember that $P_0 = F = F_0 \Delta t$

$$x(t) = F_0 \cdot \Delta t \cdot \frac{1}{m \cdot W_d} \cdot e^{-\zeta W_n t} \cdot \sin(W_d \cdot t)$$

$$K = \frac{3EI}{\ell^3} \qquad W_n = \sqrt{K/m} \qquad W_d = W_n \sqrt{1-\zeta^2}$$

$m = 1000$ Kg

$\ell = 4$m

$F = 1000$ N

$\zeta = 0'01$

$I = 0'5\ m^4$

$\Delta t = 10^{-2}$ s

$E = 7'1 \cdot 10^{10}\ N/m^2$

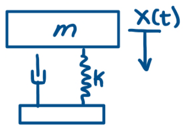
$K = 1'664 \cdot 10^{9}\ N/m$

$W_n = 1289$ rad/s

$W_d = 1288$ rad/s

$$x(t) = 1000 \cdot 10^{-2} \cdot \frac{1}{1000 \cdot 1288} \cdot e^{-0'01 \cdot 1289 \cdot t} \cdot \sin(1288\, t)$$

$$\boxed{x(t) = 7'763 \cdot 10^{-6} \cdot e^{-12'89 t} \cdot \sin(1288\, t)}$$



the mass is dropped from a height h. The impact on the ground can be modelled as an impulse. Calculate the response of m once it hits the ground

(Assume an underdamped system)

What are we doing in this problem?

This problem is the same as problem 3.8.

The initial conditions are given in a slightly different way

① Defining the boundary conditions

If you remember, $\dot{x}(t^+) = F/m$

We don't know F or m, but we can know $\dot{x}(t^+)$

→ Assume the body starts from rest ($v_0 = 0$) and from a height h

$$x = 1/2\, a t^2 \rightarrow h = 1/2\, g t^2 \rightarrow t^* = \sqrt{2h/g}$$

$$v_0 = \sqrt{2gh}$$

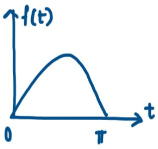
→ The spring impacts the ground at $t = 0$

→ Substitute in the response for underdamped impulse response ($h(t)$)

$$\boxed{\text{Final solution: } x(t) = \frac{\sqrt{2gh}}{\omega_d} \cdot e^{-\zeta \omega_n t} \cdot \sin(\omega_d t)}$$

Calculate the response of an underdamped system with an impulse input of the form

$f(t) = F_0 \cdot \sin(t)$



What are we doing in this problem?

Laplace is a neat little trick to make our lives easier when analysing a random input.
Note that laplace only works on harmonic loadings (unlike convolution, which works always)

How to go about laplace

In the exam you will be given a sheet with laplace transforms, both from the time domain to s-domain and back.

Both when you make the direct and inverse laplace, you have to manipulate the function until getting something that is on the formula sheet.

NOTE: Laplace transform is not affected by the multiplication of constants. Neither the direct transform nor the inverse transform:

$$\mathcal{L}(K \cdot x(t)) = K \cdot \mathcal{L}(x(t)) \qquad \mathcal{L}^{-1}(K \cdot x(t)) = K \cdot \mathcal{L}^{-1}(x(t))$$

① Equations of motion

It's said on the text that the system is underdamped
Just go to the formula sheet and find x(t) for an underdamped system.

$$m\ddot{x} + c\dot{x} + Kx = F(t)$$

$$\underbrace{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x}_{} = \underbrace{f_0(t)}_{f_0 = F_0/m}$$

→ Why are we using x(t) and not h(t)? They are two different things! x(t) is the equation of motion of the system; h(t) is a particular response to the system when forced by an impulse. For doing Laplace, we have to look at the equations of motion.

② Laplace transform of the left-hand side

$$\begin{aligned} x &= X(s) \\ \dot{x} &= sX(s) - x(0) \\ \ddot{x} &= s^2 -(s) - s\,x(0) - \dot{x}(0) \end{aligned}$$

→ Assumptions : starts from rest
$\dot{x}(0) = 0$

starts from $x=0$
$x(0) = 0$

→ $s^2 X(s) + 2\zeta w_n^2 s X(s) + w_n^2 X(s)$

This one is very straight forward. Just go to the formula sheet and the expression for the transform of $x(t)$, $\dot{x}(t)$ and $\ddot{x}(t)$ is explicitly given.

③ Laplace transform of the right-hand side (the forcing function)

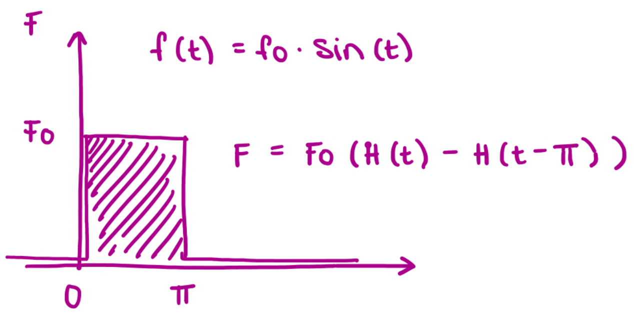
This one is a little bit trickier. The forcing function can be written as

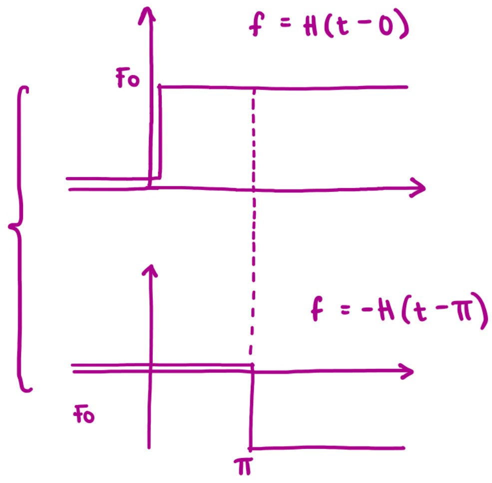
$$f(t) = \underbrace{f_0 \cdot \sin(t)}_{f(t)} \cdot \underbrace{\delta(t)}_{\delta(t)}$$

If you go to the formula sheet, you'll see that there is no formula to transform $f(t) \cdot \delta(t)$. There is a formula to transform $\delta(t)$ and $\delta(t-t_0)$, but no for $f(t) \cdot \delta(t)$.

However, there is a transform expression for $f(t-a)\,H(t-a)$, with $H(t-a)$ being the step (or Heavyside function). Impulse functions ($\delta(t)$) can be "transformed" into step functions:

Step (or heaviside) function $H(t-t^*)$





$$H(t-t^*) \begin{cases} 0 & -\infty < t < t^* \\ 1 & t^* \leq t \leq \infty \end{cases}$$

$$f_0(t) = f_0 \left[H(t) - H(t-\pi) \right] \cdot \sin(t)$$

Laplace transform

$$f_0(t) = f_0 \cdot \left[\sin(t) H(t) - \sin(t)\, H(t-\pi) \right]$$

$$\mathcal{L}\left(f(t-a)\, H(t-a)\right) = e^{-as}\, F(s)$$

$$f(x) = \sin(Kt) \rightarrow F(s) = \frac{K}{s^2 + K^2}$$

$$\mathcal{L}\left(\sin(t) H(t)\right) = e^{-0s} \cdot \frac{1}{s^2+1}$$

$$\mathcal{L}\left(\sin(t)\, H(t-\pi)\right) = \mathcal{L}\left(-\sin(t-\pi)\, H(t-\pi)\right) = -e^{-\pi s} \cdot \frac{1}{s^2+1}$$

④ Arrange the laplace terms and compute the inverse

$$s^2 X(s) + 2\zeta \omega_n^2 s X(s) + K X(s) = f_0 \frac{(1+e^{-\pi s})}{s^2+1}$$

$$X(s) = f_0 \left(1+e^{-\pi s}\right) \cdot \frac{1}{s^2+1} \cdot \frac{1}{s^2+2\zeta\omega_n^2 s+\omega_n^2}$$

→ There is nothing in the formula sheet that resembles this. But we can turn the product of fractions into a sum of fractions which look like something on the formula sheet

Partial fractions

$$\frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{1}{(s^2+1)(s^2+2\zeta\omega_n^2 s+\omega_n^2)}$$

$$(As+B)(s^2+2\zeta\omega_n s+\omega_n^2) + (Cs+D)(s^2+1) = 1$$

$$As^3 + 2\zeta\omega_n \cdot As^2 + A\cdot\omega_n^2\cdot s + Bs^2 + 2\zeta\omega_n\cdot Bs + \omega_n^2\cdot B + Cs^3 + Cs + Ds^2 + D = 1$$

$$\left.\begin{array}{lll} s^3: & A + C & = 0 \\ s^2: & 2\zeta\omega_n\cdot A + B + D & = 0 \\ s^1: & A\cdot\omega_n^2 + 2\zeta\omega_n B + C & = 0 \\ s^0: & \omega_n^2 B + D & = 1 \end{array}\right\} \quad A = \frac{-2\zeta\omega_n}{(1-\omega_n^2)^2+(2\zeta\omega_n)^2} \quad C = \frac{2\zeta\omega_n}{(1-\omega_n^2)^2+(2\zeta\omega_n)^2}$$

$$B = \frac{\omega_n^2-1}{(1-\omega_n^2)^2+(2\zeta\omega_n)^2} \quad D = \frac{(1-\omega_n^2)+(2\zeta\omega_n)^2}{(1-\omega_n^2)+(2\zeta\omega_n)^2}$$

$$x(s) = f_0(1+e^{-\pi s})\cdot\frac{1}{s^2+1}\cdot\frac{1}{s^2+2\zeta\omega_n^2 s+K}$$ → if we call $G(s) = \frac{1}{s^2+1}\cdot\frac{1}{s^2+2\zeta\omega_n^2 s+K}$

$$X(s) = f_0\left(\underbrace{G(s)}_{g(t)} + \underbrace{e^{-\pi s}\cdot G(s)}_{g(t-\pi)H(t-\pi)}\right)$$

→ inverse laplace: $g(t)$ $\quad$ $g(t-\pi)H(t-\pi)$

$$x(t) = f_0\left(g(t) + g(t-\pi)H(t-\pi)\right)$$

→ $g(t) = \mathcal{L}^{-1}(G(s))$; $\quad G(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2\zeta\omega_n s+\omega_n^2}$ → let's change this into something that it's on the formula sheet

$$\frac{As}{s^2+1} \to A\cdot\cos(t)$$

$$\frac{B}{s^2+1} \to B\cdot\sin(t)$$

$$\frac{Cs}{s^2+2\zeta\omega_n s+\omega_n^2} \to s^2+2\zeta\omega_n s+\omega_n^2 = s^2-2as+a^2+K^2$$

- $-2a = 2\zeta\omega_n \to a = \zeta\omega_n$
- $\omega_n^2 - (\zeta\omega_n)^2 = K^2 \to K = \omega_d$

→ $(s^2 - \zeta\omega_n) + \omega_d^2$

→ $C\cdot e^{-\zeta\omega_n t}\cdot\cos(\omega_d t)$

there's nothing on the formula sheet that looks like this. But the form $(s-a)^2+K^2$ fits this denominator very nicely

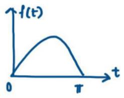
$$\frac{D}{s^2+2\zeta\omega_n s+\omega_n^2} \to \left(\frac{D-C\zeta\omega_n}{\omega_d}\right)e^{-\zeta\omega_n t}\cdot\sin(\omega_d t)$$

Final solution: $x(t) = f_0\left(g(t) + g(t-\pi)H(t-\pi)\right)$

$$g(t) = A\cdot\cos(t) + B\cdot\sin(t) + C\cdot e^{-\zeta\omega_n t}\cdot\cos(\omega_d t) + \left(\frac{D-C\zeta\omega_n}{\omega_d}\right)e^{-\zeta\omega_n t}\cdot\sin(\omega_d t)$$

Calculate the response of an underdamped system with an impulse input of the form

$f(t) = F_0 \cdot \sin(t)$



Convolution Method $x_p(t) = \int_{-\infty}^{t} F(\tau)\, h(t-\tau)\, d\tau = \int_{0}^{t} f(t-\tau)\, h(\tau)\, d\tau$

- $f(t)$ is the forcing term
- $h(t)$ is the impulse response of the system (which is already given on the formula sheet).
  - ↳ Mind that there are two "$h(t)$" functions on the formula sheet: one for an impulse and another for step input. The one you'll need to use is determined by the type of input. For this case, it is given that the input is an impulse

- Which convolution formula to use?
  - → You can only use $x_p = \int_0^t f(t-\tau)\, h(\tau)\, d\tau$ if $f(t) = 0$ for $t<0$
  - → You can use $x_p = \int_{-\infty}^{t} f(\tau)\, h(t-\tau)\, d\tau$ for any function
    - ↳ If the function is $f(t) = 0$ for $t<0$, the integral becomes

$$\int_{-\infty}^{0} f(\tau)\, h(t-\tau)\, d\tau + \int_0^t f(\tau)\, h(t-\tau)\, d\tau = \int_0^t f(\tau)\, h(t-\tau)\, d\tau$$

So that $x_p(t) = \int_0^t f(t-\tau)\, h(\tau)\, d\tau = \int_0^t f(\tau)\, h(t-\tau)\, d\tau$

Which means that if $f(t)=0$ for $t<0$, either integral can be used

- Solving the convolution: you'll most likely have to use integration by parts

$$\int_a^b u v' = [uv]_a^b - \int_a^b u' v$$

Best functions for u:
Logarithmic functions
Inverse trigonometric functions
Algebraic functions
Trigonometric functions
Exponential functions

- As you can see, there are two distinct moments in which the force acts: from $0 \to \pi$ (force is acting) and from $\pi \to \infty$ (the force is 0)
  Thus, the convolution response has two parts:
  - $t \leq \pi$ → $\int_0^{\pi} f(\tau)\, h(t-\tau)\, d\tau$
  - $t > \pi$ → for $t>\pi$, the force has already acted and thus we have to consider its effects on the response:

$$\int_0^{\pi} f(\tau)\, h(t-\tau)\, d\tau + \int_{\pi}^{t} f(\tau)\, h(t-\tau)\, d\tau$$

But after π, the force $f(t) = 0$

$$\int_0^{\pi} f(\tau)\, h(t-\tau)\, d\tau$$

① Identify the system and set up the EOM

→ Spring - damper system

$$m\ddot{x} + c\dot{x} + kx = F_0 \cdot \sin(t)$$

system impulse response: $h(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \cdot \sin \omega_d (t)$

$$\int_0^t F(\tau)\, h(t-\tau)\, d\tau = \int_0^t F(\tau) \cdot \frac{1}{m\omega_d} \underbrace{e^{-\zeta\omega_n (t-\tau)}}_{e^{-\zeta\omega_n t} \cdot e^{\zeta\omega_n \tau}} \cdot \sin(\omega_d (t-\tau))\, d\tau$$

$$x(t) = \frac{1}{m \cdot \omega_d} \cdot e^{-\zeta\omega_n t} \int_0^t F(\tau)\, e^{\zeta\omega_n \tau} \sin(\omega_d (t-\tau))\, d\tau$$

② Solve the convolution

For $t \leq \pi$ → $F(t) = F_0 \cdot \sin(t)$

$$x(t) = \frac{F_0}{m\omega_d} \cdot e^{-\zeta\omega_n t} \int_0^t \sin(\tau)\, e^{\zeta\omega_n \tau} \cdot \sin(\omega_d (t-\tau))\, d\tau$$

→ Useful identities to know:

$$\cos A \cdot \cos B = \tfrac{1}{2}(\cos(A+B) + \cos(A-B))$$
$$\sin A \cdot \sin B = \tfrac{1}{2}(\cos(A-B) - \cos(A+B))$$
$$\sin A \cdot \cos B = \tfrac{1}{2}(\sin(A+B) + \sin(A-B))$$

$$\boxed{\begin{aligned} \text{Solution: } x(t) = \frac{F_0}{m \cdot \omega_d} \cdot e^{-\zeta\omega_n t} \cdot \Bigg[& \frac{1}{2(1+2\omega_d+\omega_n^2)} \left\{ e^{\zeta\omega_n t}\left[(\omega_n - 1)\sin t - \zeta\omega_n \cdot \cos(t)\right] - (\omega_d - 1)\sin(\omega_d t) - \zeta\omega_n \cdot \cos(\omega_d t) \right\} \\ & + \frac{1}{2(1+2\omega_d+\omega_n^2)} \left\{ e^{\zeta\omega_n t}\left[(\omega_n - 1)\sin t - \zeta\omega_n \cdot \cos(t)\right] + (\omega_d - 1)\sin(\omega_d t) - \zeta\omega_n \cdot \cos(\omega_d t) \right\} \Bigg] \end{aligned}}$$

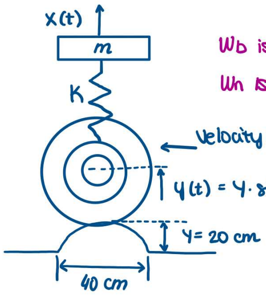
For $t > \pi$ → after $t = \pi$, the force stops but the behaviour of the system has been affected by the presence of the force:

$$x(t) = \int_0^t f(\tau) \cdot h(t-\tau)\, d\tau = \int_0^\pi f(\tau)\, h(t-\tau)\, d\tau + \underbrace{\int_\pi^t f(\tau)\, h(t-\tau)\, d\tau}_{f(t) = 0 \text{ after } t = \pi}$$

$$x(t) = \int_0^\pi f(\tau)\, h(t-\tau)\, d\tau$$

$$\boxed{\begin{aligned} \text{Final solution: } \frac{F_0}{m\omega_d} \cdot e^{-\zeta\omega_n t} \cdot & \left[\frac{1}{2(1+2\omega_d+\omega_n^2)} \left\{ e^{\zeta\omega_n t} \left[(\omega_d - 1)\sin[\omega_d(t-\pi)] - \zeta\omega_n \cdot \cos[\omega_d(t-\pi)]\right] - (\omega_d - 1) \cdot \sin \omega_d t - \zeta\omega_n \cdot \cos \omega_d t \right\} \right. \\ & \left. \frac{1}{2(1+2\omega_d+\omega_n^2)} \left\{ e^{\zeta\omega_n t} \left[(\omega_d + 1)\sin[\omega_d(t-\pi)] + \zeta\omega_n \cdot \cos[\omega_d(t-\pi)]\right] + (\omega_d - 1) \cdot \sin \omega_d t - \zeta\omega_n \cdot \cos \omega_d t \right\} \right] \end{aligned}}$$

Find an expression of the car's deflection versus the velocity of the car. Note that this is a moving base problem

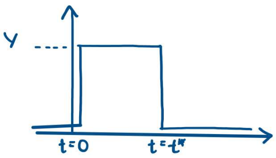


W_b is the angular velocity of the bump

W_n is the angular velocity of the car

→ Undamped base excitation

→ The bump acts as a step function on the car



Thus, the impact on the body will have the form

$$F_y = Y \cdot \sin(W_b t)\left[H(t) - H(t-t^*)\right]$$

① Free body diagram and EOM

$$-K(x(t) - y(t)) = m\ddot{x}$$
$$m\ddot{x} + K(x(t) - y(t)) = 0$$
$$\ddot{x} + W_n(x(t) - y(t)) = 0$$

→ Relative motion between car and wheel

$$z(t) = x(t) - y(t) \rightarrow x(t) = z(t) + y(t)$$
$$\ddot{x}(t) = \ddot{z}(t) + \ddot{y}(t)$$

$$m\ddot{z}(t) + m\ddot{y}(t) + Kz(t) = 0$$
$$m\ddot{y}(t) = -(m\ddot{z} + Kz(t)) \rightarrow \ddot{y}(t) = -(\ddot{z}(t) + W_n^2 z(t))$$
$$y(t) = Y \cdot \sin(W_b t) \rightarrow \dot{y}(t) = Y \cdot W_b \cdot \cos(W_b t)$$
$$\ddot{y}(t) = -Y \cdot W_b^2 \cdot \sin(W_b t)$$

$$Y \cdot W_b^2 \cdot \sin(W_b t)\left[H(t) - H(t-t^*)\right] = \ddot{z}(t) + W_n^2 z(t)$$

② Convolution method $z(t) = \int_0^t f(t-\tau)\, h(\tau)\, d\tau$

→ The response of the system (undamped) is

$$x(t) = \frac{1}{mW_n} \cdot \sin(W_n t)$$

$$z(t) = \int_0^t Y \cdot \sin(W_b(t-\tau)) \cdot \frac{1}{mW_n} \sin(\tau)\, d\tau$$

$$= \frac{Y}{m \cdot W_n} \int_0^t \sin(W_b(t-\tau)) \sin(\tau)\, d\tau \rightarrow \sin(at+c) \cdot \sin(bt) = \tfrac{1}{2}\cos((a-b)t+c) - \cos((a+b)t+c)$$

→ Performing integration by parts:

$$z(t) = \frac{YW_b^2}{W_n} \cdot \frac{1}{2} \cdot \left[\frac{\sin(W_b t - (W_n + W_b)\tau)}{-(W_n + W_b)} - \frac{\sin(W_b t + (W_n - W_b)\tau}{W_n - W_b}\right]$$

$$\boxed{z(t) = \frac{YW_b^2}{W_n} \cdot \frac{1}{W_n^2 - W_b^2}(W_n \cdot \sin(W_b t) - W_b \cdot \sin(W_n t)}$$

③ Response for high speed ($W_b \gg W_n$)

Multiply and divide by W_b to get some nice fractions that can go away

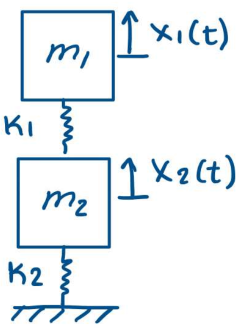
$$z(t) = \frac{YW_b^2}{W_n} \cdot \frac{W_b}{W_n^2 - W_b^2}\left(\underbrace{\frac{W_n}{W_b}}_{W_n \ll W_b \rightarrow 0} \cdot \sin(W_b t) - \sin(W_n t)\right)$$

$$z(t) = \frac{-Y \cdot W_b^2}{W_n} \cdot \frac{W_b}{W_n^2 - W_b^2} \cdot \sin(W_n t)$$

$$\rightarrow W_b = \frac{2\pi}{2\ell} v = \frac{\pi}{\ell} v$$

$$z(t) = \frac{-Y \cdot \left(\frac{\pi}{\ell}\right)^3 v^3}{W_n(W_n^2 - W_b^2)} \cdot \sin(W_n t) \rightarrow |z(v)| = \frac{Y\left(\frac{\pi}{\ell}\right)^3}{W_n(W_n^2 - W_b^2)} v^3$$

Determine the natural frequencies for this system:



$k_1 = 10^3$ N/m
$k_2 = 10^4$ N/m
$m_1 = 50$ kg
$m_2 = 2000$ kg

① Type of system

Multiple degrees of freedom, undamped, freely vibrating system

② Free body diagram and equations of motion

Block 1

m_1

$\downarrow k_1(x_1 - x_2)$

$$m_1\ddot{x}_1 = -k_1(x_1 - x_2)$$
$$m_1\ddot{x}_1 + k_1x_1 - k_1x_2 = 0$$

Block 2

$\uparrow k_1(x_1 - x_2)$

m_2

$\downarrow k_2 \cdot x_2$

$$m_2\ddot{x}_2 = -k_2x_2 + k_1(x_1 - x_2)$$
$$m_2\ddot{x}_2 - k_1x_1 + (k_2+k_1)x_2$$

Matrix form: $\bar{M}\ddot{\bar{x}} + \bar{K}\bar{x} = 0$

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}\ddot{\bar{x}} + \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_2+k_1 \end{pmatrix}\bar{x} = 0$$

$$\ddot{\bar{x}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} \qquad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

③ Natural frequencies

Natural frequencies: $|-\omega^2\bar{M} + K| = 0$

$$\begin{vmatrix} -\omega^2 m_1 + k_1 & -k_1 \\ -k_1 & -m_2\omega^2 + (k_2+k_1) \end{vmatrix} = 0$$

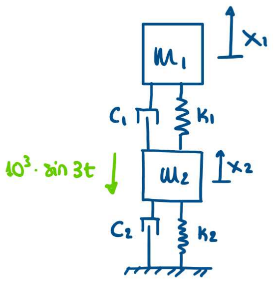
Could you have used modal analysis? Yes, you would have arrived to the same result, but with a lot more work. This is a nice shortcut for second-degree systems.

$$\begin{vmatrix} -2000\,\omega^2 + 1000 & -1000 \\ -1000 & -50\,\omega^2 + 11000 \end{vmatrix} = 100\,000\,\omega^4 - 2'205 \cdot 10^7\,\omega^2 + 10^7 = 0$$

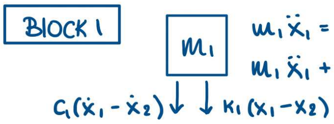
Final solution: $\omega_1^2 = 0'454 \rightarrow \omega_1 = 0'674$ rad/s
$\omega_2^2 = 220'046 \rightarrow \omega_2 = 14'8$ rad/s

The model of problem 4.14 is complemented with two dampers $c_1 = 0'01$ and $c_2 = 0'2$. Calculate the response to a harmonic input on the second mass of $f(t) = 10 \cdot \sin(3t)$

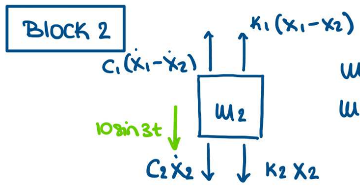
! These problems are very sensitive to rounding!
Use 5 o 6 decimals or the algebraic form of the number



① Free body diagram, EOM and matrix form



$$m_1\ddot{x}_1 = -k_1(x_1 - x_2) - c_1(\dot{x}_1 - \dot{x}_2)$$
$$m_1\ddot{x}_1 + k_1(x_1 - x_2) + c_1(\dot{x}_1 - \dot{x}_2) = 0$$



$$m_2\ddot{x}_2 - 10 \cdot \sin 3t = k_1(x_1 - x_2) + c_1(\dot{x}_1 - \dot{x}_2) - c_2\dot{x}_2 - k_2 x_2$$
$$m_2\ddot{x}_2 - k_1 x_1 + (k_1 + k_2)x_2 - c_1\dot{x}_1 + (c_2 + c_1)\dot{x}_2 = 10 \cdot \sin 3t$$

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \ddot{\bar{x}} + \begin{pmatrix} -c_1 & -c_1 \\ -c_1 & (c_1 + c_2) \end{pmatrix} \dot{\bar{x}} + \begin{pmatrix} k_1 & -k_1 \\ -k_1 & (k_1 + k_2) \end{pmatrix} \bar{x} = \begin{pmatrix} 0 \\ 10 \cdot \sin 3t \end{pmatrix}$$

② Solve the modal damping problem

- $\bar{M}\ddot{\bar{x}} + \bar{K}\bar{x} = 0$

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \ddot{\bar{x}} + \begin{pmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{pmatrix} \bar{x} = 0$$

- $M^{-1/2} = \begin{pmatrix} 1/\sqrt{m_1} & 0 \\ 0 & 1/\sqrt{m_2} \end{pmatrix}$

- Normalized stiffness matrix

$$\tilde{K} = \bar{M}^{-1/2}\,\bar{K}\,\bar{M}^{-1/2} = \begin{pmatrix} k_{11}/m_1 & k_{12}/\sqrt{m_1 m_2} \\ k_{21}/\sqrt{m_2 m_1} & k_{22}/m_2 \end{pmatrix}$$

$$\tilde{K} = \begin{pmatrix} k_1/m_1 & -k_1\sqrt{m_1 m_2} \\ -k_1\sqrt{m_1 m_2} & (k_1 + k_2)/m_2 \end{pmatrix} = \begin{pmatrix} 1/2 & -3'162 \\ -3'162 & 220 \end{pmatrix}$$

- Eigenvalues and eigenvectors of $\tilde{K}$

→ $\tilde{K}$ is perpendicular → second eigenvector will be perpendicular to the first
→ Eigenvectors must be normalized

- Eigenvalues: $\det(\tilde{K} - \gamma \bar{I}) = 0$

$$\begin{vmatrix} k_1/m_1 - \gamma & -k_1/\sqrt{m_2 m_1} \\ -k_1/\sqrt{m_1 m_2} & (k_1 + k_2)/m_2 - \gamma \end{vmatrix} = 0 \rightarrow \begin{vmatrix} 1/2 - \gamma & -3'162 \\ -3'162 & 220 - \gamma \end{vmatrix} = 0$$

$$110 - 1/2\gamma - 220\gamma + \gamma^2 - 3'162^2 = 0 \rightarrow \gamma^2 - 220'5\gamma + 100'19 = 0$$

$$\gamma_2 = 220'05 \rightarrow \omega_2 = 14'8 \text{ rad/s}$$
$$\gamma_1 = 0'454 \rightarrow \omega_1 = 0'674 \text{ rad/s}$$

- Eigenvectors

→ Eigenvector for $\gamma_2 = 220'05$

$$(\hat{K} - \gamma_1 I)\bar{v} = 0 \rightarrow \begin{pmatrix} 1/2 - 220'05 & -3'161 \\ -3'161 & 220 - 220'05 \end{pmatrix} \begin{pmatrix} v_1^2 \\ v_2^2 \end{pmatrix} = 0$$

→ Assume $v_1^1 = 1 \rightarrow -219'55 - 3'161 v_2^2 = 0$

$$v_2^2 = -69'45 \qquad v^2 = \begin{pmatrix} 1 \\ -69'45 \end{pmatrix}$$

→ Normalize → $\dfrac{1}{\sqrt{1^2 + 69'45^2}} \begin{pmatrix} 1 \\ -69'45 \end{pmatrix} = \begin{pmatrix} 0'1044 \\ -0'999 \end{pmatrix}$

→ Orthogonality: $v^1 = \begin{pmatrix} -0'999 \\ -0'104 \end{pmatrix}$

$$P = \begin{pmatrix} -0'999 & 0'014 \\ -0'014 & -0'999 \end{pmatrix}$$ → Let's change signs because it looks better

$$P = \begin{pmatrix} 0'999 & -0'014 \\ 0'014 & 0'999 \end{pmatrix} \rightarrow P^T = \begin{pmatrix} 0'999 & 0'014 \\ -0'014 & 0'999 \end{pmatrix}$$

3 Modal equations

- Modal force vector $\hat{F} = P^T \cdot M^{-1/2} \cdot \bar{F}(t) = \begin{pmatrix} 0'999 & 0'014 \\ -0'014 & 0'999 \end{pmatrix} \begin{pmatrix} 1/\sqrt{m_1} & 0 \\ 0 & 1/\sqrt{m_2} \end{pmatrix} \begin{pmatrix} 0 \\ 10 \cdot \sin(3t) \end{pmatrix}$

$$= \begin{pmatrix} 0'0223 & 1'97 \cdot 10^{-3} \\ 0 & 0'1414 \end{pmatrix} \begin{pmatrix} 0 \\ 10 \cdot \sin(3t) \end{pmatrix}$$

$$\hat{F}(t) = \begin{pmatrix} 0'02 \\ 1'414 \end{pmatrix} \sin(3t) = \begin{matrix} f_1 \\ f_2 \end{matrix}$$

- Modal equations:

$$\ddot{r}_1 + 2\zeta_1 w_{n,1} \dot{r}_1 + w_{n,1}^2 r_1 = f_1$$
$$\ddot{r}_2 + 2\zeta_2 w_{n,2} \dot{r}_2 + w_{n,2}^2 r_2 = f_2$$

$$\ddot{r}_1 + 0'01348 \dot{r}_1 + 0'454 r_1 = 0'02 \cdot \sin(3t)$$
$$\ddot{r}_2 + 5'92 \dot{r}_2 + 220 r_2 = 1'41 \cdot \sin(3t)$$

→ System starts from $\bar{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \dot{\bar{x}}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$r_0 = S^{-1} \bar{x}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\dot{r}_0 = S^{-1} \dot{\bar{x}}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Modal equation 1

$$\ddot{r} + 0'01348\,\dot{r} + 0'454\,r = 0'02 \cdot \sin 3t$$

$$s^2 F(s) + 0'01348\, s F(s) + 0'454\, F(s) = 0'06 \cdot \frac{1}{s^2+9}$$

$$F(s) = 0'06 \cdot \underbrace{\frac{1}{s^2+9} \cdot \frac{1}{s^2+0'01348s+0'454}}_{\text{Partial fractions}}$$

$$\frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+0'01348s+0'454} = \frac{1}{(s^2+9)(s^2+0'0135s+0'454)}$$

$$As^3 + A\,0'0135\,s^2 + A \cdot 0'454s + Bs^2 + 0'0135Bs + 0'454B + Cs^3 + Ds^2 + 9Cs + 9D = 1$$

$$\begin{aligned}
&\text{I)} \quad s^3: \quad A + C = 0 \\
&\text{II)} \quad s^2: \quad 0'0135A + B + D = 0 \\
&\text{III)} \quad s: \quad 0'454A + 0'0135B + 9C = 0 \\
&\text{IV)} \quad -: \quad 0'454B + 9D = 1
\end{aligned}$$

$$\begin{aligned}
&\text{I)} \quad A = -C \\
&\text{II)} \quad 0'0135A + B + D = 0 \\
&\text{III)} \quad 0'454A + 0'0135B - 9A = 0 \\
&\text{IV)} \quad 0'454B + 9D = 1
\end{aligned}
\quad \Bigg\} \quad
\begin{aligned}
&0'0135A + B + D = 0 \\
&8'546A - 0'0135B = 0 \\
&0'454B + 9D = 1
\end{aligned}$$

$A = -1'848 \cdot 10^{-4} \approx 0$

$B = -0'117$

$C = 1'848 \cdot 10^{-4} \approx 0$

$D = 0'117$

$$F(s) = 0'06 \cdot \left(\frac{-0'117}{s^2+9} + \frac{0'117}{s^2+0'0135+0'454} \right)$$

$$F(s) = 0'06 \cdot 0'117 \left(\underbrace{\frac{1}{s^2+0'0135s+0'454}} - \frac{1}{s^2+9} \right)$$

$s^2 + 0'0135s + 0'454 = s^2 - 2as + a^2 + k^2$

$a = -6'75 \cdot 10^{-3}$

$k = 0'6737$

$$F(s) = 7'02 \cdot 10^{-3} \left[\frac{0'6737}{(s+6'75\cdot 10^{-3})^2 + 0'6737^2} \cdot \frac{1}{0'6737} \right] - 7'02 \cdot 10^{-3} \cdot \left[\frac{1}{3} \cdot \frac{3}{s^2+3^2} \right]$$

$\downarrow$

Inverse Laplace

$$\boxed{r_1(t) = 0'0105 \cdot e^{-6'7\cdot 10^{-3} t} \cdot \sin(0'6737\,t) - 2'4 \cdot 10^{-3} \cdot \sin(3t)}$$

$\boxed{\text{Modal equation 2}}$

$$\ddot{r} + 5'92\ \dot{r} + 220\ r = 4'41\ \sin 3t$$

$$s^2 F(s) + 5'92 \cdot s\ F(s) + 220 \cdot F(s) = 4'23 \cdot \frac{1}{s^2+9}$$

$$F(s) = \underbrace{4'23 \cdot \frac{1}{s^2+9} \cdot \frac{1}{s^2 + \qquad s +}}_{\text{Partial fractions}}$$

$$\frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+5'92 \cdot s + 220} = \frac{1}{(s^2+9)(s^2+5'92\ s+220)}$$

$$As^3 + A\ 5'92\ s^2 + A \cdot 220\ s + Bs^2 + 5'92Bs + 220\ B + Cs^3 + Ds^2 + 9Cs + 9D = 1$$

I) s^3 : $A + C = 0$

II) s^2 : $5'92\ A + B + D = 0$

III) s : $220\ A + 5'92\ B + 9C = 0$

IV) $-$: $220\ B + 9D = 1$

$A \simeq 0$

$B = 4'7 \cdot 10^{-3}$

$C \simeq 0$

$D = -3'92 \cdot 10^{-3}$

$$F(s) = 4'23 \cdot \left(\frac{4'7 \cdot 10^{-3}}{s^2+9} - \underbrace{\frac{3'92 \cdot 10^{-3}}{s^2+5'95s+220}} \right)$$

$s^2 + 5'92s + 220 = s^2 - 2as + a^2 + k^2$

$a = -2'96$

$k = 14'53$

$$F(s) = 0'01981 \left[\frac{14'53}{(s+2'96)^2 + 14'53^2} \cdot \frac{1}{14'53} \right] - 0'0165 \left[\frac{1}{3} \cdot \frac{3}{s^2+3^2} \right]$$

$\downarrow$

Inverse Laplace

$$\boxed{r_2(t) = 0'0136 \cdot e^{-2'96t} \cdot \sin(14'53t) - 0'055 \cdot \sin(3t)}$$

④ Transform into physical coordinates

$$\bar{x}(t) = M^{-1/2}\ P\ \bar{r}(t)$$

Determine whether the following system will experience resonance

$$\ddot{\bar{x}} + \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \bar{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \sin(0'618\,t)$$

→ A system, with a certain natural frequency, is pulled by a force that will make the system vibrate with a certain frequency.
When both frequencies are the same, there is resonance

→ Undamped forced system
→ Resonance: natural frequencies are the same as response frequencies

$$\breve{K} = \bar{M}^{-1/2}\, \bar{K}\, \bar{M}^{-1/2} = \begin{pmatrix} K_{11}/m_1 & K_{12}/\sqrt{m_1 m_2} \\ K_{21}/\sqrt{m_2 m_1} & K_{22}/m_2 \end{pmatrix} \rightarrow \bar{M} = I$$

$$\breve{K} = K = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

1 Natural frequencies of the system

$\omega_n \Rightarrow \left| -\omega^2 \bar{M} + \bar{K} \right| = 0$

→ In this case, $\bar{M} = I$

$$\begin{vmatrix} -\omega^2 + 2 & -1 \\ -1 & -\omega^2 + 1 \end{vmatrix} = 0 \rightarrow (-\omega^2 + 2)(-\omega^2 + 1) - 1 = 0$$

$$\omega^4 - \omega^2 - 2\omega^2 + 2 - 1 = 0$$
$$\omega^4 - 3\omega^2 + 1 = 0$$

$$\gamma = \omega^2$$

$$\gamma = \frac{3 \pm \sqrt{9-4}}{2} \quad \begin{cases} \gamma_1 = 2'618 \rightarrow \omega_1 = 1'618 \text{ rad/s} \\ \gamma_2 = 0'38196 \rightarrow \omega_2 = 0'618 \text{ rad/s} \end{cases}$$

2 Eigenvectors and modal force matrix

- Eigenvectors for $\gamma_1 = 2'618$

$$(\breve{K} - \gamma_1 I)\, v_1 = 0 \rightarrow \begin{pmatrix} 2-2'618 & -1 \\ -1 & 1-2'618 \end{pmatrix} \begin{pmatrix} v_1^1 \\ v_2^1 \end{pmatrix} = 0$$

→ Assume $v_1^1 = 1$ → $v_2 = 0'618$

$$\bar{v}^1 = \begin{pmatrix} 1 \\ 0'618 \end{pmatrix} \rightarrow \text{Normalize}: \frac{1}{\sqrt{1^2 + 0'681^2}} \begin{pmatrix} 1 \\ 0'618 \end{pmatrix}$$

$$\bar{v}^1 = \begin{pmatrix} 0'851 \\ 0'526 \end{pmatrix} \rightarrow \text{Orthogonality}: \bar{v}^2 = \begin{pmatrix} -0'526 \\ 0'851 \end{pmatrix}$$

$$P = \begin{pmatrix} 0'851 & -0'526 \\ 0'526 & 0'851 \end{pmatrix}$$

- Modal force matrix

$$\breve{f} = P^T \cdot M^{-1/2} \cdot F(t) = \begin{pmatrix} 0'851 & -0'526 \\ 0'526 & 0'851 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(0'618\,t)$$

$$\breve{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0'851 \\ 0'526 \end{pmatrix} \sin(0'618\,t)$$

③ Modal equations

→ Forced undamped vibration → go to formula sheet!

$\boxed{\text{Modal equation 1}}$

$\ddot{r} + w_n^2 r = f_1 \rightarrow \ddot{r} + 1'61^2 \cdot r = 0'851 \cdot \sin(0'618\, t)$

Natural frequency ≠ response of the frequency

↳ No resonance

$\boxed{\text{Modal equation 2}}$

$\ddot{r} + w_n^2 r = f_2 \rightarrow \ddot{r} + 0'618^2 \cdot r = 0'526 \sin(0'618\, t)$

Natural frequency = response of the frequency

↳ Resonance