Aerodynamics II solution manual: 2018-2019 edition

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QUICK LINKS

Preface

Please note that just like you, I do not have the official solutions. However, I have tried to compare questions as much as possible with the 4 exams for which official solutions exist, so they should be relatively alright. If you nevertheless disagree with my solutions, please let me know.

As a small note, for the questions where you are given an airfoil shape and must determine the pressure coefficients, I have continuously done it by using the small angle approximation $\theta \approx dy/dx$, whereas in reality $\theta = \text{atan}\, dy/dx$. The instructors seem to prefer the second one, but I suppose it doesn't really matter, I mean we assume small angles basically everywhere and it is more important that you know how to do the rest of the computations.

Furthermore, I can recommend doing the exams in the order I have included the solutions. Last year, I assumed people would do the exams in the order I included them as I was making them at the time. Since the exams are rather repetitive (i.e. similar questions appear every now and then), I explained such questions in more detail in the solutions for the initial exams; later on they got more brief as I figured people'd have understood the question from the previous exams. I also think mine are a bit more extensive than the few official solutions that do exist, so I'd suggest starting with the exams I listed first as well.

Part I

Exam July 2012

1.1. Problem 1

1.1.1. Question i

Linear theory may be applied under the assumption:

- Perturbations are small (small thickness) and angle of attack is small;
- Flow is either subsonic or supersonic, but not transonic.

1.1.2. Question ii

We have $C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}$. For positive values of dy/dx, the flow is turned into itself on the upper surface, so for the upper surface, we have $\theta = dy/dx$, as θ should be positive when the flow is turned into itself. We have

$$y = -h\frac{x^2}{c^2} \qquad \rightarrow \qquad \frac{dy}{dx} = -h\frac{2x}{c^2}$$

and thus

$$C_{p,u} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}} = \frac{-4hx}{c^2\sqrt{M_\infty^2 - 1}}$$

For the lower surface, we have that $\theta = -\frac{dy}{dx}$ as the flow is turned away from itself for positive values of $\frac{dy}{dx}$, and thus

$$C_{p,l} = \frac{-2\theta}{\sqrt{M_\infty^2 - 1}} = \frac{4hx}{c^2\sqrt{M_\infty^2 - 1}}$$

The pressure plots are then really straightforward as shown in figure 1.1.

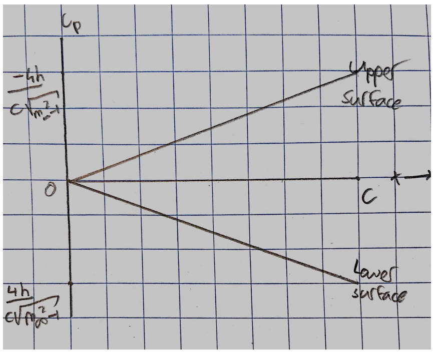

Figure 1.1: Pressure plots.

1.1.3. Question iii

Although the angle of attack is strictly speaking not equal to zero degrees (just look at what angle the chord line makes with the free stream flow), due to the way the coordinate system is drawn, the lift coefficient is simply equal to the force normal to the x-axis (which normally is defined as the normal coefficient):

$$c_l = \frac{1}{c}\int_0^c \left(c_{p,l} - c_{p,u}\right)dx = \frac{1}{c}\int \frac{4hx}{c^2\sqrt{M_\infty^2 - 1}} - \frac{-4hx}{c^2\sqrt{M_\infty^2 - 1}}dx = \frac{1}{c}\int_0^c \frac{8hx}{c^2\sqrt{M_\infty^2 - 1}}dx = \frac{1}{c}\left[\frac{4hx^2}{c^2\sqrt{M_\infty^2 - 1}}\right]_0^c = \frac{4h}{c\sqrt{M_\infty^2 - 1}}$$

Similarly, we simply have for the drag coefficient that it equals the "regular" axial coefficient, due to the way the coordinate system is drawn. We thus have, per definition:

$$\begin{aligned} c_d &= \frac{1}{c}\int_0^c \left(C_{p,u}\frac{dy_u}{dx} - C_{p,l}\frac{dy_l}{dx}\right)dx = \frac{1}{c}\int_0^c \left(\frac{-4hx}{c^2\sqrt{M_\infty^2-1}}\cdot\frac{-2hx}{c^2} - \frac{4hx}{c^2\sqrt{M_\infty^2-1}}\cdot\frac{-2hx}{c^2}\right)dx \\ &= \frac{h^2}{c^5\sqrt{M_\infty^2-1}}\int_0^c 16x^2 dx = \frac{h^2}{c^5\sqrt{M_\infty^2-1}}\left[\frac{16x^3}{3}\right]_0^c = \frac{16h^2}{3c^2\sqrt{M_\infty^2-1}} \end{aligned}$$

Had the coordinate system been drawn such that the x-axis coincided with the chordline, then you would have had to use

$$\begin{aligned} c_l &= c_n - c_a\alpha \\ c_d &= c_n\alpha + c_a \end{aligned}$$

1.2. Problem 2

The second law of thermodynamics states that

$$ds \geq \frac{\delta q}{T} + ds_{\text{irreversible}}$$

If we have the first law of thermodynamics stating that

$$\delta q + \delta w = de$$

and then assuming a reversible process such that $\delta q = T\,ds$, we can write

$$T\,ds + \delta w = de$$

We also have $\delta w = -p\,dv$, and thus

$$T\,ds - p\,dv = de \qquad \rightarrow \qquad T\,ds = de + p\,dv$$

Now, $h = e + pv$ and $dh = e + v\,dp + p\,dv$, and thus $de + p\,dv = dh - v\,dp$, and thus we can write

$$T\,ds = dh - v\,dp$$

We then divide everything by T to get

$$ds = \frac{dh}{T} - \frac{v\,dp}{T}$$

where $dh = c_p dT$ (with c_p assumed constant) and $pv = RT$ or $v/T = R/p$ and thus

$$ds = \frac{c_p dT}{T} - \frac{R\,dp}{p}$$

Integration leads to

$$s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

If we imaginarily bring both the flow in front of the NSW and after the NSW isentropically to a halt, and denote these (stagnation) conditions by a subscript a, i.e.

$$s_{2a} - s_{1a} = c_p \ln\frac{T_{2a}}{T_{1a}} - R\ln\frac{p_{2a}}{p_{1a}}$$

From the definition of stagnation conditions, we have $T_{1a} = T_{0,1}$, $T_{2a} = T_{0,2}$, $p_{2a} = p_{0,2}$, $p_{1a} = p_{0,1}$, $s_{1a} = s_1$ and $s_{2a} = s_2$, thus leading to

$$s_2 - s_1 = c_p \ln \frac{T_{0,2}}{T_{0,1}} - R \ln \frac{p_{0,2}}{p_{0,1}}$$

From comparison of equations 8.30 and 8.38, it is quickly seen that $T_{0,2} = T_{0,1}$, and for a NSW, $s_2 > s_1$, thus we have

$$-R \ln \frac{p_{0,2}}{p_{0,1}} > 0$$

And thus we must have $p_{0,2} < p_{0,1}$.

Please note: I do not know exactly how detailed this derivation should be on the exam: there is a official solution manual for one exam containing almost the same question (but there you had to start at the first law of thermodynamics), but they just say you have to read the book. The derivation I've shown is quite long, but I interpret starting at the second law of thermodynamics that you are not allowed to use any equation on the formula sheet that was derived using the second law of thermodynamics, so that's why I'm deriving equation 7.25 on the formula sheet again. On the other hand, I do use equations 8.30 and 8.38 without deriving them because they do not use the second law of thermodynamics in their derivation.

1.3. Problem 3

1.3.1. Question i

First, converting it to the minimum pressure at $M_\infty = 0$:

$$c_{p,0} = c_p \sqrt{1 - M_\infty^2} = -0.25 \sqrt{1 - 0.5^2} = -0.2165$$

Then to $M_\infty = 0.8$:

$$c_p = \frac{c_{p,0}}{\sqrt{1 - M_\infty^2}} = \frac{-0.2165}{\sqrt{1 - 0.8^2}} = -0.3608$$

1.3.2. Question ii

The critical Mach number is the lowest freestream Mach number at which at any point along the airfoil, $M = 1$ is reached.

1.3.3. Question iii

The pressure coefficient is given by

$$C_p \equiv \frac{p - p_\infty}{q_\infty} = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\frac{p - p_\infty}{\rho_\infty a_\infty^2 M_\infty^2} = \frac{2}{M_\infty^2}\frac{p - p_\infty}{\rho_\infty (\gamma R T_\infty)} = \frac{2}{\gamma M_\infty^2}\frac{p - p_\infty}{p_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

Now, computing the pressure ratio p_{cr}/p_∞:

$$\frac{p_{cr}}{p_\infty} = \frac{p_{cr}/p_0}{p_\infty/p_0} = \left(\frac{1 + \frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M_A^2}\right)^{\gamma/(\gamma-1)}$$

with $M_A = 1$, this becomes simply

$$\frac{p_{cr}}{p_\infty} = \left(\frac{1 + \frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2}}\right)^{\gamma/(\gamma-1)}$$

so that the critical pressure is given by

$$c_{p,cr} = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}}\right)^{\gamma/(\gamma-1)} - 1\right]$$

Then the critical Mach number can be found by setting

$$\frac{-0.2165}{\sqrt{1-M_\infty^2}} = \frac{2}{\gamma M_\infty^2}\left[\left(\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}}\right)^{\gamma/(\gamma-1)} - 1\right]$$

Plotting this equation on your graphical calculator gives you the solution $M_{cr} = 0.8202$.

1.4. Problem 4

1.4.1. Question i

In other words, there is a NSW at station 1, and then isentropic expansion between station 1 and 2. The total temperature is constant over a NSW and during isentropic expansion, so $T_{0,2} = T_{0,1} = T_{0,\infty}$. We have that $T_{0,\infty}/T_\infty = 4.200$ for the freestream conditions by using appendix A for $M = 4$, and thus $T_{0,2} = T_{0,\infty} = 4.200 \cdot 180 = 756\,\text{K}$. We then have for $M = 0.25$ that $T_{0,2}/T_2 = 1.013$ for $M = 0.25$ (again, appendix A, and then interpolating between $M = 0.24$ and $M = 0.26$), thus the static pressure at station 2 equals

$$T_2 = \frac{756}{1.013} = 746\,\text{K}$$

For the static pressure, we must first look up the total pressure after the NSW by looking up in column $p_{0,2}/p_1$ in appendix B. There, we see for $M = 4$ that $p_{0,1}/p_\infty = 21.07$, and thus

$$p_{0,1} = 21.07 \cdot 5000 = 105\,350\,\text{Pa} = 105.350\,\text{kPa}$$

As the total pressure is constant during isentropic expansion, we can immediately use appendix A, column p_0/p to find that for $M = 0.25$, $p_2/p_{0,2} = 1.0455$ (again by interpolating between $M = 0.24$ and $M = 0.26$) so that

$$p_2 = \frac{105.350}{1.0455} = 100.765\,\text{kPa}$$

As I said, the total pressure is constant during isentropic expansion, thus we have

$$p_{0,2} = p_{0,1} = 105.350\,\text{kPa}$$

1.4.2. Question ii

If there is a OSW generating a flow deflection of 30° at the inlet, then we find from the graph of the $\theta-\beta-M$ relation that $\beta = 45°$. Doing then all of the computations using methods I described in the

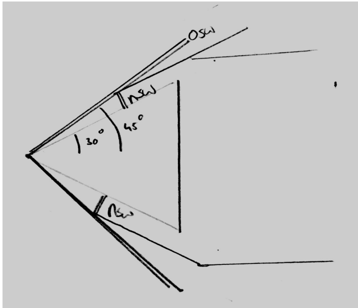

Figure 1.2: Wedge.

summary, and calling the region between the OSW and NSW region 4:

$$
\begin{aligned}
T_{0,2} &= T_{0,1} = T_{0,4} = T_{0,\infty} = 757\,\mathrm{K} && \text{Total temperature is constant everywhere} \\
T_2 &= \frac{T_2}{T_{0,2}} T_{0,2} = \frac{1}{1.013} \cdot 757 = 746\,\mathrm{K} && \text{Just use appendix A} \\
p_{0,\infty} &= \frac{p_{0,\infty}}{p_\infty} p_\infty = 151.8 \cdot 5 = 759\,\mathrm{kPa} && p_{0,\infty}/p_\infty \text{ for } M = 4 \text{ in appendix A} \\
M_{n,\infty} &= M_\infty \sin\beta = 4 \sin 45^\circ = 2.828 && \text{Mach number normal to OSW in freestream} \\
M_{n,4} &= 0.4847 && \text{Looked up in appendix B for } M_1 = 2.850 \\
M_4 &= \frac{M_{n,4}}{\sin(\beta - \theta)} = \frac{0.4847}{\sin(45-30)} = 1.8727 \\
p_{0,4} &= \frac{p_{0,4}}{p_{0,\infty}} p_{0,\infty} = 0.3733 \cdot 759 = 283\,\mathrm{kPa} && \text{Total pressure after OSW, looked up in appendix B for } M_1 = 2.850 \\
p_{0,1} &= \frac{p_{0,1}}{p_{0,4}} p_{0,4} = 0.7765 \cdot 283 = 220\,\mathrm{kPa} && \text{Total pressure after NSW, looked up in appendix B for } M_1 = 1.880 \\
p_{0,2} &= p_{0,1} = 220\,\mathrm{kPa} && \text{Isentropic flow} \\
p_2 &= \frac{p_2}{p_{0,2}} p_{0,2} = \frac{1}{1.0445} \cdot 220 = 210\,\mathrm{kPa} && \text{Static pressure at station 2, looked up from appendix A for } M = 0.25
\end{aligned}
$$

Please note, before you do something stupid like me, *do not use* appendix B to look up $\frac{p_{0,4}}{p_\infty}$ over the OSW (even though we did so for the NSW). This ratio only holds if your shock wave is normal to the flow.

1.4.3. Question iii

The total enthalpy is given by

$$H = c_p T + \frac{V^2}{2} = c_p T_0$$

Thus, if we have

$$
\begin{aligned}
H_3 &= 1.5 H_2 \\
c_p T_{0,3} &= 1.5 c_p T_{0,2} \\
T_{0,3} &= 1.5 T_{0,2}
\end{aligned}
$$

Now, as the total density remains constant, we get that

$$p_{0,3} = \rho_{0,3} R T_{0,3} = \rho_{0,2} R \cdot 1.5 T_{0,2} = 1.5 p_{0,2}$$

Why is all of this important? The pressure at the exit is equal to the freestream pressure, i.e. 5 kPa. Thus, the ratio between the total pressure at station 3 and the exit pressure greatly affects how our flow will be throughout the nozzle. There are three distinct possibilities for the Mach number at the exit, as shown in figure 1.3. Calculating the Mach number at the exit is different for each flow regime:

- For the fully subsonic one (i.e., $p_e > p_{e,3}$), the subsonic branch of appendix A needs to be used and an area correction needs to be applied.
- For the subsonic-supersonic-subsonic one, the location of the NSW needs to be determined, after which you can combine appendices A and B to compute M_e. This is the case for $p_{e,5} < p_e < p_{e,3}$.
- For the subsonic-supersonic one, the supersonic branch of appendix A can be used. This is the case when $p_e < p_{e,5}$. If the exit pressure is not equal to $p_{e,6}$, then the shock waves only appear *behind* the exit, so we don't have to deal with their shit.

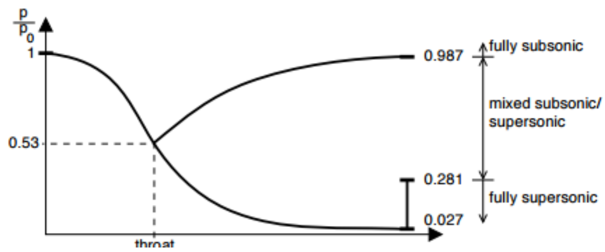

Figure 1.3: Important flow regimes. Numerical values do not correspond to this question.

Now, let us calculate the range of each regime. For the fully subsonic one, we see that for $A/A^* = 5$ in the subsonic part of appendix A, $M_e \approx 0.12$ and $p_0/p \approx 1.010$, i.e. $p/p_0 \approx 0.99$.

For the second range, the most extreme scenario is a NSW located at the exit itself. If this is the case, then the Mach number in front of the exit will be $M_e = 3.15$ (using nearest entry for the supersonic branch of appendix A for $A/A^* = 5$), which has $p_0/p = 45.93$. For a NSW with $M_1 = 3.15$, $p_2/p_1 = 11.41$, and thus

$$p_e/p_0 = p_2/p_1 \cdot p_1/p_0 = 11.41 \cdot \frac{1}{45.93} = 0.248$$

For the fully supersonic case, we again have simply $p_0/p = 45.93$, or $p/p_0 = 0.02177$. Furthermore, $M_e = 3.15$.

Now, using the results of questions i and ii, we have that for case i), $p_{0,3} = 1.5 p_{0,2} = 1.5 \cdot 105.350 = 158.025\,\text{kPa}$. This means that

$$\frac{p_e}{p_{0,3}} = \frac{5}{158.025} \approx 0.0316$$

This means that the flow will be in the region that shocks will only appear behind the exit, and thus $M_e = 3.15$. As the pressure is higher than the 0.02177 required for the isentropic supersonic solution, the flow is underexpanded and Prandtl-Meyer expansion waves appear behind the exit.

For case ii), $p_{0,3} = 1.5 p_{0,2} = 1.5 \cdot 220 = 330\,\text{kPa}$ and thus

$$\frac{p_e}{p_{0,3}} = \frac{5}{330} \approx 0.0152$$

This means that the flow will be in the region that shocks will only appear behind the exit, and thus again, $M_e = 3.15$. As the pressure is lower than the 0.02177 required for the isentropic supersonic solution, the flow is underexpanded and Prandtl-Meyer expansion waves appear behind the exit.

Note: if the pressure ratio was in the fully subsonic flow regime, then you'd have to apply an area correction, sthen the question would be comparable to example 3a of the summary. Had the pressure ratio been such that a NSW had appeared, then the question would be comparable to example 3c.

1.5. Problem 5

1.5.1. Question i

We have

$$c_pT + \frac{u^2}{2} = c_pT_0$$

with

$$c_p = \frac{\gamma R}{\gamma - 1}$$

which equals 1004 J/kg/K for air. We then have that the total temperature equals

$$1004 \cdot 154 + \frac{(2a)^2}{2} = 1004 \cdot 154 + \frac{\left(2 \cdot \sqrt{\gamma RT}\right)^2}{2} = 1004 \cdot 154 + 2 \cdot 1.4 \cdot 287 \cdot 154 = 1004 \cdot T_0$$

and thus $T_0 = 277.26\,\mathrm{K}$. For the limit velocity, $T = 0$, so

$$\frac{u_{\text{lim}}^2}{2} = c_pT_0 = 1004 \cdot 277.26$$

and thus $u_{\text{lim}} = 746\,\mathrm{m/s}$.

1.5.2. Question ii

We can repeat all of the calculations, with the only difference being that γ has now become (as Helium is monoatomic, and thus three degrees of freedom):

$$\gamma = \frac{n+2}{n} = \frac{3+2}{3} = 1.67$$

and R_{helium} equals

$$R_{\text{He}} = \frac{R_0}{\mu_{\text{He}}} = \frac{8314}{4} = 2078.5\,\mathrm{J/kg/K}$$

and thus

$$c_p = \frac{\gamma R}{\gamma - 1} = \frac{1.67 \cdot 2078.5}{1.67 - 1} = 5196.25\,\mathrm{J/kg/K}$$

Now, plugging all of this in leads to

$$T_0 = \frac{c_pT + \frac{u^2}{2}}{c_p} = \frac{5196.25 \cdot 154 + 2 \cdot 1.67 \cdot 2078.5 \cdot 154}{5196.25} = 359.74\,\mathrm{K}$$

$$u_{\text{lim}} = \sqrt{2c_pT_0} = 2\sqrt{2 \cdot 5196.25 \cdot 359.74} = 1934\,\mathrm{m/s}$$

Part II

August 2012

1.6. Problem 1

1.6.1. Question i

See figure 1.4.

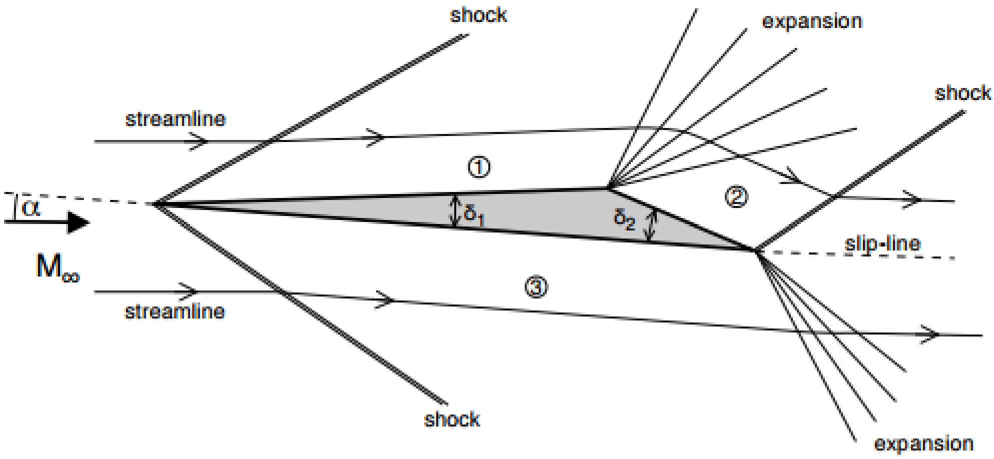

Figure 1.4: Sketch of flow.

1.6.2. Question ii

Look at figure 1.5

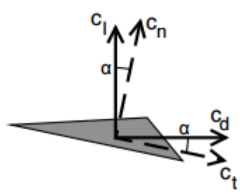

Figure 1.5: Force coefficients.

We clearly have

$$\begin{aligned} c_l &= c_n \cos\alpha - c_t \sin\alpha \approx c_n - c_t\alpha \\ c_d &= c_n \sin\alpha + c_t \cos\alpha \approx c_n\alpha + c_t \end{aligned}$$

First, computing c_n. Using the numbers denoted in figure 1.4 for the three regions, and looking for clarification at figure 1.6 we have that

$$\begin{aligned} c_n &= C_{p,3} - \left(0.8C_{p,1} + 0.2C_{p,2}\right) \\ c_a &= 0.08C_{p,1} - 0.08C_{p,2} \end{aligned}$$

We have in general,

$$C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}$$

We have to components determining θ: the angle of attack α, and the geometric angle a surface makes. Let's first consider region 1. There, increasing α makes the flow turned out of itself more, meaning that

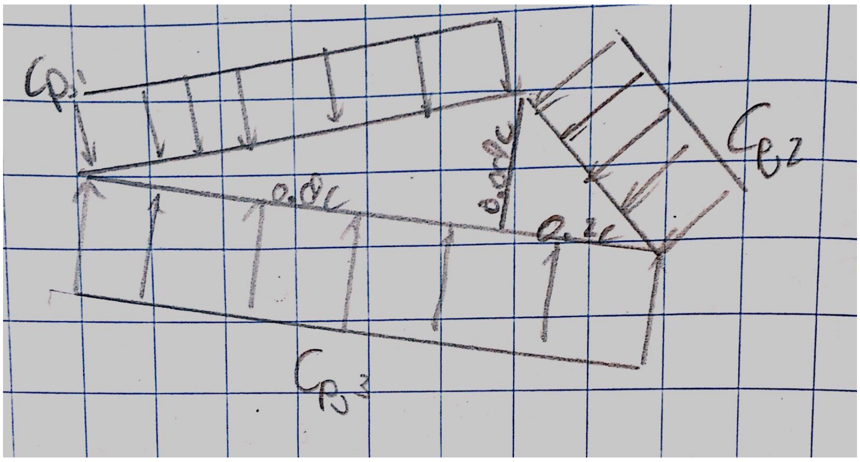

Figure 1.6: A sketch.

θ has a negative relation with α. On the other hand, the slope of the upper surface makes the flow turned into itself, so θ has a positive relation with the slope of the surface of region 1. The slope equals approximately $\frac{0.08c}{0.8c} = 0.1$, and thus

$$\theta = -\alpha + 0.1 = -\frac{3}{180}\pi + 0.1 = 0.04764\,\text{rad}$$

Leading to

$$C_{p,1} = \frac{2 \cdot 0.04764}{\sqrt{M_\infty^2 - 1}} = \frac{2 \cdot 0.04764}{\sqrt{2.4^2 - 1}} = 0.04367$$

For region 2, exactly the same relationship for θ holds. However, the slope now equals $-0.08c/0.2c = -0.4$, and thus

$$\theta = -\alpha + -0.4 = -\frac{3}{180}\pi + -0.4 = -0.452\,\text{rad}$$

Leading to

$$C_{p,2} = \frac{2 \cdot -0.452}{\sqrt{2.4^2 - 1}} = -0.4147$$

For region 3, the flow is turned into itself for increasing α, meaning θ has a positive relation with α. Furthermore, there is no geometric slope, so we have simply

$$C_{p,3} = \frac{2 \cdot \frac{3}{180}\pi}{\sqrt{2.4^2 - 1}} = 0.04800$$

We can now compute everything:

$$\begin{aligned}
c_n &= C_{p,3} - \left(0.8C_{p,1} + 0.2C_{p,2}\right) = 0.04800 - (0.8 \cdot 0.04367 + 0.2 \cdot -0.4147) = 0.0960 \\
c_a &= 0.08C_{p,1} - 0.08C_{p,2} = 0.08 \cdot 0.04367 - 0.2 \cdot -0.4147 = 0.03667 \\
c_l &= c_n - c_a\alpha = 0.0960 - 0.03667 \cdot \frac{3}{180}\pi = 0.09408 \\
c_d &= c_n\alpha + c_a = 0.0960 \cdot \frac{3}{180}\pi + 0.03667 = 0.04169
\end{aligned}$$

1.6.3. Question iii

Using methods of chapter 9, we find:

$$\begin{aligned}
\theta_1 &= -\alpha + 0.1 = -\frac{3}{180}\pi + 0.1 = 0.04764\,\text{rad} = 2.730^\circ \\
\beta &= 26.5^\circ \qquad \text{From } \theta-\beta-M \text{ relation for } M = 2.4 \text{ and } \theta = 2.730^\circ \\
M_{n,\infty} &= M_\infty \sin\beta = 2.4\cdot\sin 26.5^\circ = 1.0708 \\
M_{n,1} &= 0.9277 \qquad \text{Using the method of nearest entry for appendix B} \\
M_1 &= \frac{M_{n,1}}{\sin(\beta-\theta_1)} = \frac{0.9277}{\sin(26.5-2.730)} = 2.30 \\
\frac{p_1}{p_\infty} &= 1.194 \qquad \text{Using the method of nearest entry for appendix B} \\
\frac{p_{0,1}}{p_1} &= 12.50 \qquad \text{Using appendix A} \\
v(M_1) &= 34.28^\circ \\
v(M_2) &= \theta + v(M_1) = (0.1+0.4)\cdot\frac{180}{\pi} + 34.28 = 62.93^\circ \\
M_2 &= 3.8 \qquad \text{Using the method of nearest entry for appendix C} \\
\frac{p_{0,2}}{p_2} &= \frac{p_{0,1}}{p_2} = 115.9 \qquad \text{Using appendix A} \\
\frac{p_2}{p_\infty} &= \frac{p_2}{p_{0,1}}\frac{p_{0,1}}{p_1}\frac{p_1}{p_\infty} = \frac{1}{115.9}\cdot 12.50\cdot 1.194 = 0.1288
\end{aligned}$$

The pressure coefficient itself can then be found by

$$C_p = \frac{p_2-p_\infty}{q_\infty} = \frac{p_2-p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\frac{p_2-p_\infty}{\rho_\infty M_\infty^2 a_\infty^2} = \frac{2}{M_\infty^2}\frac{p_2-p_\infty}{\rho_\infty\gamma R T_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_2}{p_\infty}-1\right) = \frac{2}{1.4\cdot 2.4^2}(0.1288-1) = -0.2161$$

We clearly see that the magnitude of the pressure coefficient is much smaller than for the linearized theory. This can be explained due to the assumption of a thin airfoil for linearized theory; it can be argued that an airfoil of 8% thickness is not thin anymore.

1.7. Problem 2

1.7.1. Question i

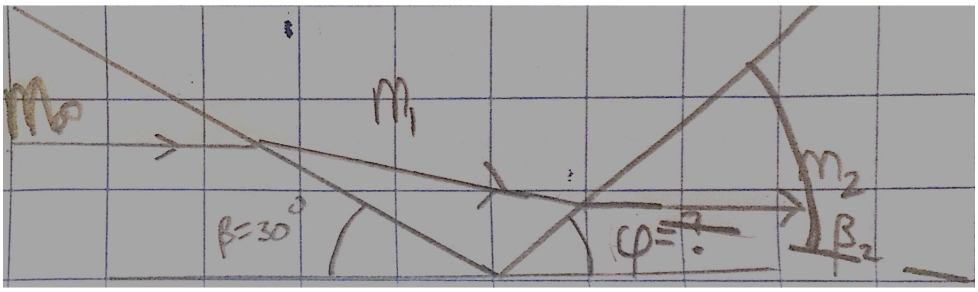

Figure 1.7: A sketch.

See figure 1.7. The first shock will deflect the flow by an angle θ. However, to prevent the flow from entering into the wall, a second OSW is created, deflecting the flow back so that the flow after the second is parallel to the freestream flow. Properties for the first OSW are easily found:

$$\begin{aligned}
\theta &= 12.5^\circ \\
M_{n,\infty} &= M_\infty \sin\beta = 3.0 \cdot \sin 30^\circ = 1.5 \\
M_{n,1} &= 0.7011 \\
M_1 &= \frac{M_{n,1}}{\sin(\beta-\theta)} = \frac{0.7011}{\sin(30-12.5)} = 2.33
\end{aligned}$$

Now, let us think. The flow needs to be deflected an angle of 12.5° and has a Mach number of 2.33. From the $\theta-\beta-M$ relation, we then have $\beta_2 = 36.5^\circ$. However, this is *not* the angle between the shock and the wall; this is the angle between the shock and the direction of the flow in region 1. The angle we're looking for, ϕ, actually equals $\phi = \beta_2 - \theta = 36.5 - 12.5 = 24^\circ$.

1.7.2. Question ii

If we increase β, two things happen:

- The shock becomes stronger, decreasing M_1, decreasing θ_{max} for the second wave.
- The deflection after the first wave becomes higher, making it more difficult for the second wave to be strong enough to deflect it sufficiently.

Let us thus increase β to, say, 50°. We then have that

$$\begin{aligned}
\theta &= 29^\circ \\
M_{n,\infty} &= M_\infty \sin\beta = 3.0 \sin 50^\circ = 2.298 \\
M_{n,1} &= 0.5344 \\
M_1 &= \frac{M_{n,1}}{\sin(\beta-\theta)} = \frac{0.5344}{\sin(50-29)} = 1.491
\end{aligned}$$

We clearly see that for $M_1 = 1.491$, $\theta_{\text{max}} = 12^\circ$ and thus the flow cannot be reflected. Let's decrease $\beta = 40^\circ$.

$$\begin{aligned}
\theta &= 20.5^\circ \\
M_{n,\infty} &= M_\infty \sin\beta = 3.0 \sin 40^\circ = 1.928 \\
M_{n,1} &= 0.5918 \\
M_1 &= \frac{M_{n,1}}{\sin(\beta-\theta)} = \frac{0.5918}{\sin(40-20.5)} = 1.773
\end{aligned}$$

For $M_1 = 1.773$, $\theta_{\text{max}} \approx 17^\circ$, so we're getting close. Let's decrease β to 38°. Then:

$$\begin{aligned}
\theta &= 18^\circ \\
M_{n,\infty} &= M_\infty \sin\beta = 3.0 \sin 38^\circ = 1.847 \\
M_{n,1} &= 0.6078 \\
M_1 &= \frac{M_{n,1}}{\sin(\beta-\theta)} = \frac{0.6078}{\sin(38-18)} = 1.777
\end{aligned}$$

For $M_1 = 1.777$, we see from the graph on the formula sheet that $\theta_{\text{max}} \approx 18^\circ$. Thus, we deduce that the maximum shock angle for the incoming shock for which a regular reflection is still possible is approximately 38°. Doing this question quickly requires just some basic estimation skills on where the correct answer will be, actually.

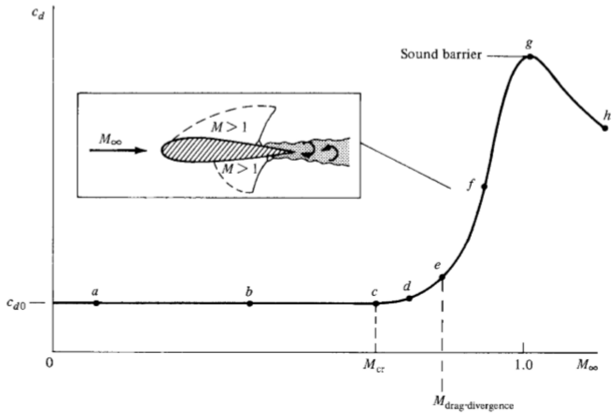

Figure 1.8: Drag divergence and stuff.

1.8. Question 3

- At low subsonic speeds, the drag coefficient is more or less constant.
- Once M_{cr} is reached, the drag starts to gradually increase. Beyond this point, there are sonic regions over the airfoil, which produce weak shock waves, creating a small amount of wave drag.
- Shortly thereafter, after M_{dd}, an extensive region of supersonic flow over the airfoil appears, increasing the wave drag significantly.
- It increases until about the Mach number reaches unity, when the sound barrier is reached and the drag coefficient is maximum.
- It decreases thereafter as the shock waves become weaker with increasing Mach number, as the shock wave angle decreases and thus the shock waves become weaker. Furthermore, directly after $M=1$, the flow around the entire airfoil is supersonic, except possibly just in front of the airfoil if a detached shock wave is formed. There will either be a detached shock wave in front of the airfoil, or an oblique shock wave at the leading edge of the airfoil. Both of these do not cause boundary layer separation (as there is simply no boundary layer at that point yet), contrary to the shock waves which formed at $M_\infty < 1$, thus also reducing the drag due to boundary layer separation due to shock waves.

1.9. Problem 4

1.9.1. Question i

The energy equation can be written for steady, adiabatic flow as

$$h_0 = h + \frac{V^2}{2} = \text{constant}$$

We can thus write

$$
\begin{aligned}
c_p T + \frac{V^2}{2} &= c_p T_0 \\
\frac{\gamma R}{\gamma - 1} T + \frac{M^2 a^2}{2} &= \frac{\gamma R}{\gamma - 1} T_0 \\
\frac{\gamma R}{\gamma - 1} T + \frac{\gamma R T}{2} M^2 &= \frac{\gamma R}{\gamma - 1} T_0 \\
\frac{T}{\gamma - 1} + \frac{T}{2} M^2 &= \frac{T_0}{\gamma - 1} \\
T\left(1 + \frac{\gamma - 1}{2} M^2\right) &= T_0 \\
\frac{T_0}{T} &= 1 + \frac{\gamma - 1}{2} M^2
\end{aligned}
$$

With the isentropic relation being

$$
\frac{p_2}{p_1} = \left(\frac{T_2}{T_1}\right)^{\gamma/(\gamma-1)}
$$

this leads to

$$
\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}
$$

1.9.2. Question ii

In front of the stagnation point, there *will* be a NSW, thus we must use the Rayleigh-Pitot formula for the total pressure (not doing this is just downright wrong, my apologies for the earlier mistake). Over a NSW, the total temperature is constant so we can calculate that straightforwardly.

$$
\begin{aligned}
\frac{p_{0,2}}{p_1} &= \left[\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma-1)}\right]^{\gamma/(\gamma-1)} \frac{1 - \gamma + 2\gamma M_1^2}{\gamma + 1} \\
p_0 &= p\left[\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma-1)}\right]^{\gamma/(\gamma-1)} \frac{1 - \gamma + 2\gamma M_1^2}{\gamma + 1} = 5 \cdot \left[\frac{(1.4+1)^2 \cdot 2.02^2}{4 \cdot 1.4 \cdot 2.02^2 - 2(1.4-1)}\right]^{1.4/(1.4-1)} \frac{1 - 1.4 + 2 \cdot 1.4 \cdot 2.02^2}{1.4 + 1} = 28.72\,\text{kPa} \\
\frac{T_0}{T} &= 1 + \frac{\gamma - 1}{2} M^2 \\
T_0 &= T\left(1 + \frac{\gamma - 1}{2} M^2\right) = 216 \cdot \left(1 + 0.2 \cdot 2.02^2\right) = 392.27\,\text{K}
\end{aligned}
$$

1.10. Problem 5

1.10.1. Question i

Computing the mass flow is always easiest at the throat, because there the flow velocity is equal to the local speed of sound at the throat. We have for the mass flow:

$$
\dot{m} = \rho u A = \rho a A
$$

We know A at the throat to be simply

$$A = \pi \frac{d^2}{4} = \pi \cdot \frac{0.94^2}{4} = 0.694\,\text{m}^2$$

Furthermore, we have for the speed of sound that

$$\begin{aligned} a &= \sqrt{\gamma R T} \\ T &= \frac{T}{T_0} T = \frac{1}{1.2} \cdot 4000 = 3333.33\,\text{K} \qquad \text{Using appendix A for } M = 1 \\ a &= \sqrt{1.4 \cdot 287 \cdot 3333.33} = 1157\,\text{m/s} \end{aligned}$$

Furthermore, the total density can be computed using

$$\begin{aligned} p_0 &= \rho_0 R T_0 \\ \rho_0 &= \frac{p_0}{R T_0} = \frac{70 \cdot 10^5}{287 \cdot 4000} = 6.098\,\text{kg/m}^3 \end{aligned}$$

such that the local density at the throat equals

$$\rho = \frac{\rho}{\rho_0} \rho_0 = \frac{1}{1.577} \cdot 6.098 = 3.867\,\text{kg/m}^3 \qquad \text{Using appendix A for } M = 1$$

and thus

$$\dot{m} = \rho u A = 3.867 \cdot 1157 \cdot 0.694 = 3104.68\,\text{kg/s}$$

1.10.2. Question ii

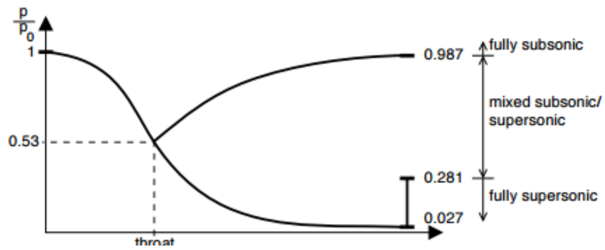

Figure 1.9: Important flow regimes. Numerical values do not correspond to this question. The top graph is associated with $p_e = p_{e,3}$, the middle one with $p_e = p_{e,5}$ and the bottom one with $p_e = p_{e,6}$.

We have three important flow regimes. If the exit pressure to total pressure ratio is higher than or equal to $p_{e,3}/p_0$, than the flow is fully subsonic. If $p_{e,5} \leq p_e < p_{e,3}$, then a normal shock wave is created between the throat and the exit. If $p_{e,6} < p_b < p_{e,5}$, then the flow within the entire nozzle is isentropic, but is overexpanded as the flow needs to be compressed to have the pressure of the flow equal the back pressure when it leaves the nozzle. If $p_b < p_{e,6}$, then the flow is underexpanded, as it needs to be expanded to have the flow pressure equal the back pressure.

Let's compute $p_{e,6}$ first. We have that $A_e = \pi \cdot d^2/4 = \pi \cdot 3.76^2/4 = 11.10\,\text{m}^2$, and thus

$$\frac{A_e}{A^*} = \frac{11.10}{0.694} = 16$$

From appendix A, we then have $p_0/p = 271.9$, or $p/p_0 = 0.003678$, using the supersonic part of the appendix. Now, computing $p_{e,5}$. For $A_e/A^* = 16$, the Mach number at the exit equals 4.45 (appendix A). If there'd be a NSW located at the exit, then the pressure drop over this NSW would equal $p_2/p_1 = 22.94$, and thus we have

$$\frac{p_{e,5}}{p_0} = \frac{p_{e,5}}{p_1}\frac{p_1}{p_0} = 22.94 \cdot \frac{1}{271.9} = 0.0844$$

At sea level, we have that

$$\frac{p_b}{p_0} = \frac{1.01 \cdot 10^5}{70 \cdot 10^5} = 0.0144$$

Meaning that $p_{e,6} < p_b < p_{e,5}$, meaning that the flow is overexpanded. The flow field at the exit then looks like an exit with two OSWs originating from the endpoints of the nozzle, similar to figure 10.9d on page 73 of the summary.

At a height of 60 km, we have that

$$\frac{p_b}{p_0} = \frac{20}{70 \cdot 10^5} = 2.86 \cdot 10^{-6}$$

meaning that $p_b < p_{e,6}$. This means that the flow is underexpanded. This leads to a flow field similar to figure 10.9f on page 73 of the summary, with PMEWs originating from the endpoints of the nozzle.

Part III

Exam January 2015

1.11. Problem 1

1.11.1. Question i

The assumptions for which supersonic linear theory may be applied are thin airfoils and small angle of attack (so that perturbations are small).

Question ii

First, for the the upper side: for positive values of dy/dx, the flow is turned into itself, hence θ has a positive relation with dy/dx. Furthermore, for positive values of α, the flow is turned away from the surface, meaning that θ has a negative relation with α. Now, we have

$$\begin{aligned} y &= -h\left(\frac{x}{c}\right)^2 \\ \frac{dy}{dx} &= -2\frac{hx}{c^2} \end{aligned}$$

and thus

$$C_{p,u} = \frac{2\theta}{\sqrt{M_\infty^2-1}} = \frac{2\cdot\left(-2\frac{hx}{c^2}-\alpha\right)}{\sqrt{M_\infty^2-1}} = \frac{-2\left(2\frac{hx}{c^2}+\alpha\right)}{\sqrt{M_\infty^2-1}}$$

Now, for the lower surface, we have that the flow is turned away from itself for positive values of dy/dx, so θ has a negative relation with dy/dx. Furthermore, for positive values of α, the flow is turned into itself, meaning that θ has a positive relation with C_p. Thus:

$$C_{p,l} = \frac{2\theta}{\sqrt{M_\infty^2-1}} = \frac{2\cdot\left(--2\frac{hx}{c^2}+\alpha\right)}{\sqrt{M_\infty^2-1}} = \frac{2\left(2\frac{hx}{c^2}+\alpha\right)}{\sqrt{M_\infty^2-1}}$$

Question iii

For the drag coefficient, we have

$$c_d = c_n\alpha + c_a$$

We have:

$$\begin{aligned} c_n &= \frac{1}{c}\int_0^c\left(C_{p,l}-C_{p,u}\right)dx = \frac{1}{c}\int_0^c\left(\frac{2\left(2\frac{hx}{c^2}+\alpha\right)}{\sqrt{M_\infty^2-1}} - \frac{-2\left(2\frac{hx}{c^2}+\alpha\right)}{\sqrt{M_\infty^2-1}}\right)dx = \frac{4}{c\sqrt{M_\infty^2-1}}\int_0^c\left(2\frac{hx}{c^2}+\alpha\right)dx \\ &= \frac{4}{c\sqrt{M_\infty^2}}\left[\frac{hx^2}{c^2}+\alpha x\right]_0^c = \frac{4}{c\sqrt{M_\infty^2-1}}(h+\alpha c) = \frac{4h}{c\sqrt{M_\infty^2-1}} + \frac{4\alpha}{\sqrt{M_\infty^2-1}} = \frac{4\cdot 0.05}{\sqrt{2^2-1}} + \frac{4\cdot 3\cdot\frac{\pi}{180}}{\sqrt{2^2-1}} \\ &= 0.23639 \end{aligned}$$

and

$$\begin{aligned}
c_a &= \frac{1}{c}\int_0^c \left(C_{p,u}\frac{dy_u}{dx} - C_{p,l}\frac{dy_l}{dx}\right)dx = \frac{1}{c}\int_0^c \left[\left(\frac{-2\left(2hx/c^2+\alpha\right)}{\sqrt{M_\infty^2-1}}\cdot -2\frac{hx}{c^2}\right) - \left(\frac{2\left(2hx/c^2+\alpha\right)}{\sqrt{M_\infty^2-1}}\cdot -2\frac{hx}{c^2}\right)\right]dx \\
&= \frac{1}{c}\int_0^c \frac{8\left(2hx/c^2+\alpha\right)}{\sqrt{M_\infty^2-1}}\cdot\frac{hx}{c^2}dx = \frac{8h}{c^3\sqrt{M_\infty^2-1}}\int_0^c \left(2h\frac{x^2}{c^2}+\alpha x\right)dx \\
&= \frac{8h}{c^3\sqrt{M_\infty^2-1}}\left[\frac{2h}{3}\frac{x^3}{c^2}+\frac{\alpha x^2}{2}\right]_0^c = \frac{8h}{c^3\sqrt{M_\infty^2-1}}\left[\frac{2}{3}hc+\frac{\alpha c^2}{2}\right] = \frac{16h^2}{3c^2\sqrt{M_\infty^2-1}} + \frac{4h\alpha}{c\sqrt{M_\infty^2-1}} \\
&= \frac{16\cdot 0.05^2}{3\sqrt{2^2-1}} + \frac{4\cdot 0.05\cdot 3\cdot\frac{\pi}{180}}{\sqrt{M_\infty^2-1}} = 0.01374
\end{aligned}$$

And thus we deduce that

$$c_d = 0.23639\cdot 3\cdot\frac{\pi}{180} + 0.01374 = 0.261$$

1.12. Problem 2

1.12.1. Question i

See sketch in figure 1.10. Note that it is absolutely 100% to apply linearized theory here (so you cannot draw just some random Mach lines, each with the same angle with the freestream), you have to draw the OSWs and PMEWs.

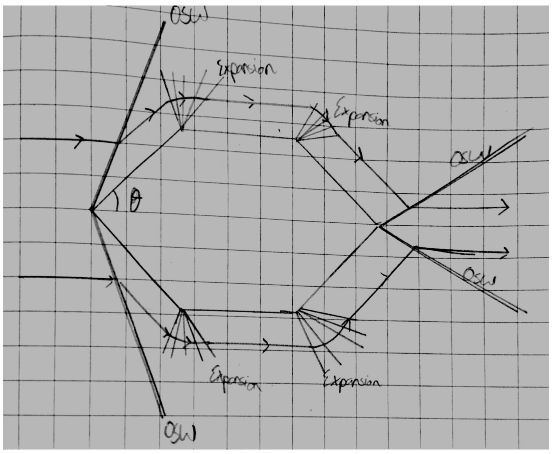

Figure 1.10: Sketch.

Furthermore, note that there is no slip line: due to the perfect symmetry, the flow going over the upper surface will have exactly the same properties as the flow going over the lower surface after the body, so there won't be any differences in the two regions and thus there will be no slip (and also no slip line). Furthermore, some things to keep in mind when drawing it:

- OSWs always point downstream from the local flow direction. So, for example, the topright OSW must point to the right of the normal of the topright surface.
- Similarly, the first wave of a PMEW must point downstream from the local flow direction before the PMEW. The final wave of PMEW will also point downstream from the local flow direction after the PMEW.

- Through a OSW, the streamlines make a discontinuous change in direction; through a PMEW, the change is continuous.

1.12.2. Question ii

Relatively straightforward. First, the pressure coefficient can be written as

$$C_p = \frac{p_2 - p_\infty}{q_\infty} = \frac{p_2 - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\frac{p_2 - p_\infty}{\rho_\infty M_\infty^2 a_\infty^2} = \frac{2}{M_\infty^2}\frac{p_2 - p_\infty}{\rho_\infty \gamma R T_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_2}{p_\infty} - 1\right)$$

Thus we need to know p_2/p_∞. First, for the OSW, for $M_\infty = 3$ and $\theta = 25°$, we have

$$\begin{aligned}
\beta &= 44° \\
M_{n,\infty} &= M_\infty \sin\beta = 3 \cdot \sin 44° = 2.084 \\
M_{n,1} &= 0.5613 \qquad \text{Using appendix B, method of nearest entry} \\
M_1 &= \frac{M_{n,1}}{\sin(\beta - \theta)} = \frac{0.5613}{\sin(44-25)} = 1.724 \\
\frac{p_1}{p_\infty} &= 4.978 \qquad \text{Using appendix B, method of nearest entry} \\
\frac{p_0}{p_1} &= 5.087 \qquad \text{Using appendix A, method of nearest entry} \\
v(M_1) &= 18.4° \\
v(M_2) &= \theta + v(M_1) = 25 + 18.4 = 43.4° \\
M_2 &= 2.7 \qquad \text{Using appendix C, method nearest entry} \\
\frac{p_0}{p_2} &= 23.28 \qquad \text{Using appendix A, method of nearest entry}
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\frac{p_2}{p_\infty} &= \frac{p_2}{p_0}\frac{p_0}{p_1}\frac{p_1}{p_\infty} = \frac{1}{23.28} \cdot 5.087 \cdot 4.978 = 1.0878 \\
C_{p,2} &= \frac{2}{1.4 \cdot 3^2}(1.0878 - 1) = 0.0139
\end{aligned}$$

1.13. Problem 3

1.13.1. Question i

$$\begin{aligned}
h_1 + \frac{u_1^2}{2} &= h_2 + \frac{u_2^2}{2} = h_0 = c_p T_0 \\
c_p T + \frac{u^2}{2} &= c_p T_0 \\
\frac{\gamma R T}{\gamma - 1} + \frac{u^2}{2} &= \frac{\gamma R T_0}{\gamma - 1} \\
\frac{a^2}{\gamma - 1} + \frac{u^2}{2} &= \frac{a_0^2}{\gamma - 1} \\
u^2 &= 2\left(\frac{a_0^2}{\gamma - 1} - \frac{a^2}{\gamma - 1}\right) \\
u &= \sqrt{2\frac{a_0^2 - a^2}{\gamma - 1}}
\end{aligned}$$

Alternatively, to show the eclipseness more, we can write

$$\begin{aligned} \frac{a^2}{\gamma-1}+\frac{u^2}{2} &= \frac{a_0^2}{\gamma-1} \\ \frac{a^2}{a_0^2}+\frac{u^2(\gamma-1)}{2a_0^2} &= 1 \end{aligned}$$

1.13.2. Question ii

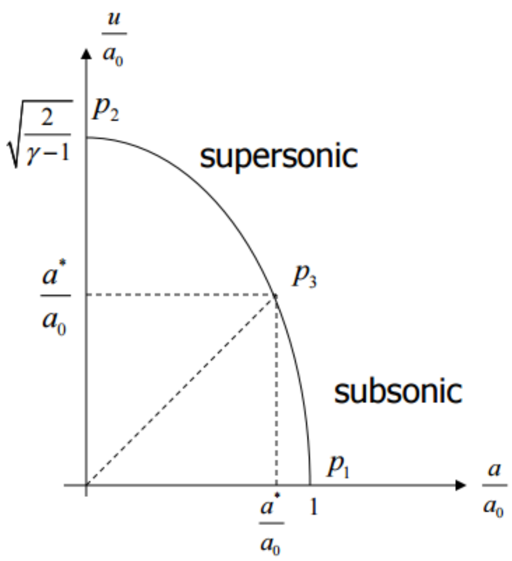

Figure 1.11: Energy eclipse.

See figure 1.11. We have:

a. The limit velocity is the maximum theoretical velocity, reached as $T \to 0$ (i.e. $a \to 0$). The limit velocity requals

$$u = \sqrt{2\frac{a_0^2-0^2}{\gamma-1}} = a_0\sqrt{\frac{2}{\gamma-1}}$$

b. The stagnation conditions are the conditions when a flow is brought adiabatically to a halt, i.e. $u=0$ We then have $a=a_0$, and thus $a/a_0=1$, as seen in the eclipse.

c. The critical conditions are the conditions when a flow is sonic, i.e. when $u=a=a^*$. In this case, we have

$$\begin{aligned} \frac{a^{*2}}{\gamma-1}+\frac{a^{*2}}{2} &= \frac{a_0^2}{\gamma-1} \\ a^{*2}\left(1+\frac{\gamma-1}{2}\right) &= a_0^2 \\ a^{*2}\left(\frac{\gamma+1}{2}\right) &= a_0^2 \\ a^* &= \sqrt{\frac{2}{\gamma+1}}a_0 \end{aligned}$$

1.14. Problem 4

1.14.1. Question i

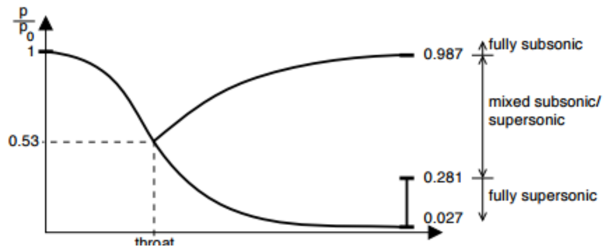

Figure 1.12: Important flow regimes. Numerical values do not correspond to this question. The top graph is associated with $p_e = p_{e,3}$, the middle one with $p_e = p_{e,5}$ and the bottom one with $p_e = p_{e,6}$.

We've done this a few times already. For the fully subsonic solution but sonic at the throat ($p_{e,3}$), we find from the subsonic branch of appendix A that for $A/A^* = 1.5$ that $p_0/p = 1.142$ (method of nearest entry), and thus to have a **fully subsonic solution**, we must have

$$p_0 < \frac{p_0}{p} p = 1.142 \cdot 100 = 114\,\text{kPa}$$

It must be smaller than this, because if we'd increase it more, the pressure ratio p_e/p_0 would go down.

If we want to have an isentropic solution within the entire nozzle, then we either have a fully subsonic solution, or a supersonic solution for which no NSW appears within the nozzle. We already determined the first one, but for the supersonic solution with no NSW, we must know what the pressure ratio should be, had there been a NSW at the exit (as this is the most extreme event). From appendix A, we find for $A/A^* = 1.5$ in the supersonic branch that $M = 1.86$ and $p_0/p_1 = 6.3$. From appendix B, we then have that after the shock wave, $p_2/p_1 = 3.87$. Thus, NSWs appear if the flow is sonic at the throat and

$$p_0 < \frac{p_0}{p_1} \frac{p_1}{p_2} p_2 = 6.3 \cdot \frac{1}{3.87} \cdot 100 = 162.8\,\text{kPa}$$

And thus the flow is **isentropic** for $p_0 < 114\,\text{kPa}$ and $p_0 > 162.8\,\text{kPa}$.

If the flow is overexpanded, then the exit pressure is between $p_{e,5}$ and $p_{e,6}$. The total pressure associated with $p_{e,6}$ is

$$p_0 = \frac{p_0}{p_e} p_e = 6.3 \cdot 100 = 630\,\text{kPa} \qquad \text{Looked up before}$$

And thus the flow is **overexpanded** when $164\,\text{kPa} < p_0 < 630\,\text{kPa}$. The flow is **underexpanded** when $p_0 > 630\,\text{kPa}$.

1.14.2. Question ii

Comparing the exit and sonic throat, we have

$$\begin{aligned} \rho_e A_e u_e &= \rho^* A^* u^* \\ \rho_e A_e M_e a_e &= \rho^* A^* a^* \\ M_e &= \frac{\rho^* A^* a^*}{\rho_e A_e a_e} \end{aligned}$$

Square this because fuck why not:

$$M_e^2 = \left(\frac{\rho^*}{\rho_e}\right)^2 \left(\frac{A^*}{A_e}\right)^2 \left(\frac{a^*}{a_e}\right)^2 = \left(\frac{\rho^*}{\rho_e}\right)^2 \left(\frac{A^*}{A_e}\right)^2 \frac{a^{*2}}{a_e^2}$$

Now, we happened to derive in the previous question that

$$a^* = a_0 \sqrt{\frac{2}{\gamma+1}} = \sqrt{\gamma R T_0} \sqrt{\frac{2}{\gamma+1}}$$

Furthermore, we have for ρ^*:

$$\begin{aligned} \frac{\rho_0}{\rho^*} &= \left(1 + \frac{\gamma-1}{2} M^{*2}\right)^{1/(\gamma-1)} = \left(\frac{\gamma+1}{2}\right)^{1/(\gamma-1)} \\ \rho^* &= \rho_0 \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} \end{aligned}$$

Finally, we have

$$a_e^2 = \sqrt{\gamma R T_e}$$

From this, we can write

$$M_e^2 = \left(\frac{\rho_0}{\rho_e}\right)^2 \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)} \left(\frac{A^*}{A_e}\right)^2 \frac{\gamma R T_0}{\gamma R T_e} \frac{2}{\gamma+1}$$

We can combine the two exponents of $2/(\gamma+1)$ to

$$\frac{2}{\gamma-1} + 1 = \frac{2 + (\gamma-1)}{\gamma-1} = \frac{\gamma+1}{\gamma-1}$$

leading to

$$M_e^2 = \left(\frac{\rho_0}{\rho_e}\right)^2 \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \left(\frac{A^*}{A_e}\right)^2 \frac{T_0}{T_e}$$

Now, we have $\rho = p/(RT)$ and thus we can write

$$M_e^2 = \left(\frac{p_0 R T_e}{p_e R T_0}\right)^2 \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \left(\frac{A^*}{A_e}\right)^2 \frac{T_0}{T_e} = \left(\frac{p_0}{p_e}\right)^2 \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \left(\frac{A^*}{A_e}\right)^2 \frac{T_e}{T_0}$$

Now, we have

$$\begin{aligned} \frac{T_0}{T_e} &= 1 + \frac{\gamma-1}{2} M_e^2 \\ \frac{T_e}{T_0} &= \frac{1}{1 + \frac{\gamma-1}{2} M_e^2} \end{aligned}$$

and thus

$$\begin{aligned} M_e^2 &= \left(\frac{p_0 R T_e}{p_e R T_0}\right)^2 \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \left(\frac{A^*}{A_e}\right)^2 \frac{T_0}{T_e} = \left(\frac{p_0}{p_e}\right)^2 \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \left(\frac{A^*}{A_e}\right)^2 \frac{1}{1 + \frac{\gamma-1}{2} M_e^2} \\ M_e^2 + \frac{\gamma-1}{2} M_e^4 &= \left(\frac{p_0}{p_e}\right)^2 \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \left(\frac{A^*}{A_e}\right)^2 \\ 0 &= \frac{\gamma-1}{2} M_e^4 + M_e^2 - \left(\frac{p_0}{p_e}\right)^2 \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \left(\frac{A^*}{A_e}\right)^2 \end{aligned}$$

Now, applying the quadratic formula, we have that

$$M_e^2 = \frac{-1 \pm \sqrt{1^2 - 4\cdot\frac{\gamma-1}{2}\cdot - \left(\frac{p_0}{p_e}\right)^2\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\left(\frac{A^*}{A_e}\right)^2}}{2\cdot\frac{\gamma-1}{2}}$$

Only taking the positive solution results in

$$M_e^2 = \frac{-1 + \sqrt{1^2 + 2(\gamma-1)\left(\frac{p_0}{p_e}\right)^2\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\left(\frac{A^*}{A_e}\right)^2}}{\gamma-1} = \frac{1}{\gamma-1}\left[\sqrt{1^2 + 2(\gamma-1)\left(\frac{p_0}{p_e}\right)^2\left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)}\left(\frac{A^*}{A_e}\right)^2} - 1\right]$$

Do note: if you had absolutely no idea how to derive this, study it how to do it anyway, cause it reappeared on a different exam. Just remember the steps: use continuity equation, comparing exit with the throat. Rewrite and square. Then, find expressions for ρ^*, a^{*^2} and a_e^2. Substitute all of this in and rewrite a bit. Replace ρ_e and ρ_0 using the equation of state. Replace T_e/T_0 with the formula on the formula sheet, and solve.

1.14.3. Question iii

We can compute M_e directly now:

$$M_e^2 = \frac{1}{1.4-1}\left[\sqrt{1 + 2(1.4-1)\left(\frac{2}{1.4+1}\right)^{(1.4+1)/(1.4-1)}\left(\frac{0.1}{0.15}\frac{140}{100}\right)^2} - 1\right] = 0.2764$$

Meaning that $M_e = 0.5258$. For $M_e = 0.5258$, we have $p_0/p = 1.202$, meaning that $p_{0,2} = 1.202p_e = 1.202\cdot 100 = 120.2\,\mathrm{kPa}$. So, now we know the total pressure ratio over the NSW: $p_{0,2}/p_{0,1} = 120.2/140 = 0.8586$, and from appendix B, we thus have $M = 1.7$, i.e. the Mach number just in front of the NSW is 1.7. From appendix A, we then deduce that at the location of the NSW, we have $A/A^* = 1.34$. Thus, we must solve

$$\begin{aligned} 1.34 &= 0.75\left(1+\frac{x}{L}\right) \\ 0.75\frac{x}{L} &= 1.34 - 0.75 \\ \frac{x}{L} &= 0.787 \end{aligned}$$

1.15. Problem 5

1.15.1. Question i

We have

$$
\begin{aligned}
\hat{\phi}(x,y) &= \frac{10}{\sqrt{1-M_\infty^2}} e^{-2\pi\sqrt{1-M_\infty^2}y} \sin(2\pi x) \\
\hat{u}(x,y) = \frac{\partial\hat{\phi}}{\partial x}(x,y) &= \frac{20\pi}{\sqrt{1-M_\infty^2}} e^{-2\pi\sqrt{1-M_\infty^2}y} \cos(2\pi x) \\
\hat{v}(x,y) = \frac{\partial\hat{\phi}}{\partial y}(x,y) &= -20\pi e^{-2\pi\sqrt{1-M_\infty^2}y} \sin(2\pi x) \\
M_\infty &= \frac{V_\infty}{a_\infty} = \frac{V_\infty}{\sqrt{\gamma R T_\infty}} = \frac{170}{\sqrt{1.4\cdot 287\cdot 300}} = 0.4896 \\
\hat{u}(0.6,0.1) &= \frac{20\pi}{\sqrt{1-0.4896^2}} e^{-2\pi\sqrt{1-0.4896^2}\cdot 0.1} \cos(2\pi\cdot 0.6) = -33.71\,\mathrm{m/s} \\
\hat{v}(0.6,0.1) &= -20\pi e^{-2\pi\sqrt{1-0.4896^2}\cdot 0.1} \sin(2\pi\cdot 0.6) = 21.35\,\mathrm{m/s}
\end{aligned}
$$

And thus

$$
V = \sqrt{(V_\infty+\hat{u})^2+\hat{v}^2} = \sqrt{(170-33.71)^2+21.35^2} = 138\,\mathrm{m/s}
$$

1.15.2. Question ii

For the static temperature, we have

$$
\begin{aligned}
c_p T_\infty + \frac{V_\infty^2}{2} &= c_p T + \frac{V^2}{2} \\
1004\cdot 300 + \frac{170^2}{2} &= 1004T + \frac{138^2}{2} \\
T &= 304.91\,\mathrm{K}
\end{aligned}
$$

1.15.3. Question iii

We have

$$
M = \frac{V}{a} = \frac{V}{\sqrt{\gamma R T}} = \frac{138}{\sqrt{1.4\cdot 287\cdot 304.91}} = 0.394
$$

Part IV

April 2015

1.16. Problem 1

1.16.1. Question i

Unfortunately, there is no exam I can compare this question with. Considering you are allowed to use linear theory, the flow around the geometry is simply a bunch of Mach waves, each with the same angle to the freestream flow, as shown in figure 1.13. On the other hand, if they do mean the actual flow, then the flow is slightly more complex and is shown in figure 1.14.

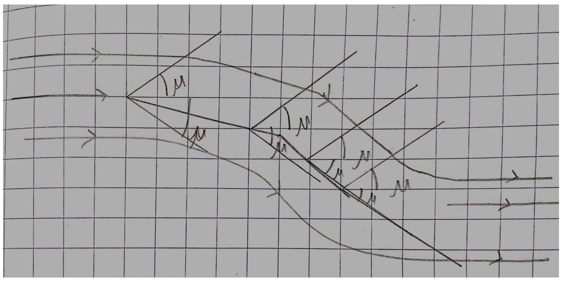

Figure 1.13: Sketch.

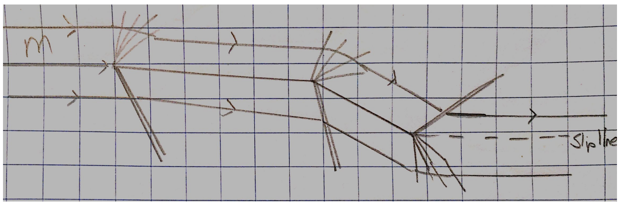

Figure 1.14: Sketch.

1.16.2. Question ii

We have

$$C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}$$

First, for the upper side, for the first part of the airfoil: this part is flat, so $\frac{dy}{dx} = 0$. If we increase α, then along the upper surface, the flow is turned out of itself, meaning there is a negative relation between θ and α there. Thus,

$$C_{p,u} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}} = \frac{-2\alpha}{\sqrt{M_\infty^2 - 1}} = \frac{-2 \cdot 3 \cdot \frac{\pi}{180}}{\sqrt{2^2 - 1}} = -0.0605 \quad \text{for } 0 < x/c < 0.8$$

For the upper surface of the flap, for positive values of δ, the flow is turned out of itself, meaning there is a negative relation between θ and δ. Furthermore, for positive values of α, the flow is turned out of itself, meaning there is a negative relation between θ and α there. Thus,

$$C_{p,u} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}} = \frac{2(-\alpha - \delta)}{\sqrt{M_\infty^2 - 1}} = \frac{2 \cdot \left(-3 \cdot \frac{\pi}{180} - 3 \cdot \frac{\pi}{180}\right)}{\sqrt{2^2 - 1}} = -0.121 \quad \text{for } 0.8 < x/c < 1$$

For the lower surface, we can apply the exact same method, which will end you up with

$$
\begin{aligned}
C_{p,l} &= \frac{2\theta}{\sqrt{M_\infty^2-1}} = \frac{2\alpha}{\sqrt{M_\infty^2-1}} = \frac{2\cdot 3\cdot\frac{\pi}{180}}{\sqrt{2^2-1}} = 0.0605 \quad \text{for } 0 < x/c < 0.8 \\
C_{p,l} &= \frac{2\theta}{\sqrt{M_\infty^2-1}} = \frac{2(\alpha+\delta)}{\sqrt{M_\infty^2-1}} = \frac{2\cdot\left(3\cdot\frac{\pi}{180}+3\cdot\frac{\pi}{180}\right)}{\sqrt{2^2-1}} = 0.121 \quad \text{for } 0.8 < x/c < 1
\end{aligned}
$$

1.16.3. Question iii

We have

$$
\begin{aligned}
c_l &= c_n \\
c_d &= c_n\alpha + c_a
\end{aligned}
$$

with

$$
\begin{aligned}
c_n &= \frac{1}{c}\int_0^c \left(C_{p,l}-C_{p,u}\right)dx = \frac{1}{c}\left(\int_0^{0.8c} 2\cdot 0.0605dx + \int_{0.8c}^{c} 2\cdot 0.121dx\right) = 0.1452 \\
c_a &= \frac{1}{c}\int_0^c \left(C_{p,u}\frac{dy_u}{dx} - C_{p,l}\frac{dy_l}{dx}\right)dx = \frac{1}{c}\int_0^{0.8c}\left(C_{p,u}-C_{p,l}\right)\frac{dy}{dx}dx + \frac{1}{c}\int_{0.8c}^{c}\left(C_{p,u}-C_{p,l}\right)\frac{dy}{dx}dx \\
&= \frac{1}{c}\int_0^{0.8c}(-0.0605-0.0605)\cdot 0dx + \frac{1}{c}\int_{0.8c}^{c}(-0.121-0.121)\cdot -\delta dx = \frac{0.242}{c}\cdot\frac{3\pi}{180}\int_{0.8c}^{c} dx = 0.00253
\end{aligned}
$$

and thus

$$
\begin{aligned}
c_l &= 0.1452 \\
c_d &= 0.1452\cdot\frac{3\pi}{180} + 0.00253 = 0.0101
\end{aligned}
$$

You can also subtract αc_a from c_l but this is not necessary.

1.16.4. Question iv

We have

$$
\begin{aligned}
c_l &= c_n = \frac{1}{c}\int_0^{0.8c}\left(C_{p,l}-C_{p,u}\right)dx + \frac{1}{c}\int_{0.8c}^{c}\left(C_{p,l}-C_{p,u}\right)dx = \frac{1}{c\sqrt{M_\infty^2-1}}\left(\int_0^{0.8c} 4\alpha dx + \int_{0.8c}^{c} 4(\alpha+\delta)dx\right) \\
&= \frac{4}{\sqrt{M_\infty^2-1}}(0.8\alpha+0.2\alpha+0.2\delta) = \frac{4}{\sqrt{M_\infty^2-1}}(\alpha+0.2\delta)
\end{aligned}
$$

and thus $c_l = 0$ when $\delta = -5\alpha = -15°$.

1.17. Problem 2

1.17.1. Question i

Quite similar to problem 2i from August 2012. We have that the coefficient of pressure equals

$$
C_p = \frac{p_3-p_\infty}{q_\infty} = \frac{p_3-p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\frac{p_3-p_\infty}{\rho_\infty M_\infty^2 a_\infty^2} = \frac{2}{M_\infty^2}\frac{p_3-p_\infty}{\rho_\infty\gamma R T_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_3}{p_\infty}-1\right)
$$

Now, for the first OSW, we have

$$\begin{aligned}
\theta &= 7.5^\circ \\
M_{n,1} &= M_1 \sin\beta = 3 \cdot \sin 25^\circ = 1.268 \\
M_{n,2} &= 0.8071 \\
M_2 &= \frac{M_{n,2}}{\sin(\beta-\theta)} = \frac{0.8071}{\sin(25-7.5)} = 2.684 \\
\frac{p_2}{p_\infty} &= 1.686
\end{aligned}$$

Then, after region 2, the flow again needs to be deflected by 7.5° but now at $M = 2.684$, leading to:

$$\begin{aligned}
\beta &= 27.5^\circ \\
M_{n,2} &= M_2 \sin\beta = 2.684 \cdot \sin 27.5^\circ = 1.239 \\
M_{n,3} &= 0.8183 \\
M_3 &= \frac{M_{n,3}}{\sin(\beta-\theta)} = \frac{0.8183}{\sin(27.5-7.5)} = 2.393 \\
\frac{p_3}{p_2} &= 1.627
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{p_3}{p_\infty} &= \frac{p_3}{p_2}\frac{p_2}{p_\infty} = 1.627 \cdot 1.686 = 2.743 \\
C_p &= \frac{2}{1.4 \cdot 3^2}(2.743-1) = 0.28
\end{aligned}$$

Note that for M_3, the difference with the given answer may seem relatively large. However, this difference is simply caused due to the inaccuracy of nearest entry method; as long as you clearly indicate each step you take (by referring to which appendix you use and which entry), you should be fine.

1.17.2. Question ii

We can still use the results of region 2, fortunately. Note that θ_1 was only 7.5°, so we in fact have an expansion wave where the expansion wave happens over a $15-7.5 = 7.5^\circ$ angle. We then have

$$\begin{aligned}
\frac{p_0}{p_2} &= 23.28 \\
v(M_2) &= 43.62^\circ \\
v(M_3) &= \theta + v(M_2) = 7.5 + 43.62 = 51.12^\circ \\
M_3 &= 3.050 \\
\frac{p_0}{p_3} &= 39.59
\end{aligned}$$

We then have

$$\frac{p_3}{p_\infty} = \frac{p_3}{p_0}\frac{p_0}{p_2}\frac{p_2}{p_\infty} = \frac{1}{39.59} \cdot 23.28 \cdot 1.686 = 0.9914$$

$$C_p = \frac{2}{1.4 \cdot 3^2}(0.9914-1) = -0.00136$$

Again, differences may seem rather large, but that's just due to rounding. As 0.9914 is very close to 1, rounding makes it relatively more inaccurate than it actually is.

1.18. Problem 3

1.18.1. Question i

A pitot tube measures $p_{0,2}$, behind the shock wave, and p_1, in front of the NSW. We have

$$\frac{p_{0,2}}{p_1} = \frac{p_{0,2}}{p_2}\frac{p_2}{p_1} = \left(1+\frac{\gamma-1}{2}M_2^2\right)^{\gamma/(\gamma-1)} \cdot \left[1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)\right]$$

where

$$M_2^2 = \frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2-\frac{\gamma-1}{2}}$$

and thus

$$\begin{aligned}
\frac{p_{0,2}}{p_1} &= \left(1+\frac{\gamma-1}{2}\frac{1+\frac{\gamma-1}{2}M_1^2}{\gamma M_1^2-\frac{\gamma-1}{2}}\right)^{\gamma/(\gamma-1)} \cdot \left[1+\frac{2\gamma}{\gamma+1}\left(M_1^2-1\right)\right] \\
&= \left(1+(\gamma-1)\frac{1+\frac{\gamma-1}{2}M_1^2}{2\gamma M_1^2-(\gamma-1)}\right)^{\gamma/(\gamma-1)}\left[\frac{\gamma+1+2\gamma M_1^2-2\gamma}{\gamma+1}\right] \\
&= \left(1+\frac{2(\gamma-1)+(\gamma-1)^2M_1^2}{4\gamma M_1^2-2(\gamma-1)}\right)^{\gamma/(\gamma-1)}\left[\frac{1-\gamma+2\gamma M_1^2}{\gamma+1}\right] \\
&= \left(\frac{4\gamma M_1^2-2(\gamma-1)+2(\gamma-1)+\gamma^2M_1^2-2\gamma M_1^2+M_1^2}{4\gamma M_1^2-2(\gamma-1)}\right)^{\gamma/(\gamma-1)}\left[\frac{1-\gamma+2\gamma M_1^2}{\gamma+1}\right] \\
&= \left(\frac{\gamma^2M_1^2+2\gamma M_1^2+M_1^2}{4\gamma M_1^2-2(\gamma-1)}\right)^{\gamma/(\gamma-1)}\left[\frac{1-\gamma+2\gamma M_1^2}{\gamma+1}\right] = \left(\frac{(\gamma+1)^2M_1^2}{4\gamma M_1^2-2(\gamma-1)}\right)^{\gamma/(\gamma-1)}\left[\frac{1-\gamma+2\gamma M_1^2}{\gamma+1}\right]
\end{aligned}$$

1.18.2. Question ii

We have $p_{0,2}/p_1 = 230/41 = 5.610$. This leads to:

$$\begin{aligned}
M &= 2.00 \qquad \text{Appendix B for } p_{0,2}/p_1 = 5.610\text{, using method of nearest entry} \\
a &= \sqrt{\gamma RT} = \sqrt{1.4\cdot287\cdot243} = 312.47\,\text{m/s} \\
V &= Ma = 2\cdot312.47 = 625\,\text{m/s} \\
T_0 &= \frac{V^2}{2c_p}+T = \frac{625^2}{2\cdot1004}+243 = 437.5\,\text{K}
\end{aligned}$$

For the pressure at stagnation point, do *not* calculate p_0/p (or look it up in appendix A). As aircraft flies at supersonic speeds, there *will* be a NSW in front of the stagnation point, and thus we can use the Pitot pressure, so $p_0 = 230\,\text{kPa}$.

1.19. Problem 4

1.19.1. Question i

For this question, you have to wonder, how can we, for a given p_e/p_0, maximize the Mach number in the wind tunnel, if we don't even know anything yet about A_e/A^* or something? Naturally, a lower

p_e/p_0 ratio is beneficial for achieving higher Mach numbers at the exit. However, if p_e/p_0 is fixed (in our case, it's equal to 0.278), then the best thing you can do is having a NSW at the exit. Think back to the figure showing all of $p_{e,2}$, $p_{e,3}$ etc. If we design our wind tunnel (that is, A_e/A^*) in such a way that $p_{e,5} = p_0 = 0.278$, then we have a NSW at the exit. Just in front of the exit, the flow will thus be supersonic with a certain Mach number. It is obvious that moving the NSW forward (towards the throat) will not increase this maximum Mach number. Going for an overexpanded, perfectly expanded or underexpanded solution is also stupid: it'd require a lower p_e/p_0 but you'd end up getting the same Mach number just before the exit; in other words, to keep p_e/p_0 fixed you'd end up getting a lower Mach number just before the exit. Hence, having a NSW just before the exit is beneficial.

How can we find what Mach number we can achieve with this pressure ratio of 0.278? There are two ways basically:

- Assume the maximum Mach number is a certain value, say $M = 2$. Look up in appendix A the ratio p_0/p for $M = 2$ (equal to 7.824). Then use appendix B to look up p_2/p_1 for $M = 2$ (equal to 4.5) and calculate

  $$\frac{p_2}{p_0} = \frac{p_2}{p_1}\frac{p_1}{p_0} = 4.5 \cdot \frac{1}{7.824} = 0.5751$$

  If this ratio is higher than 0.278, increase M; if this ratio is lower than 0.278, decrease it. So, let's go for $M = 4$. There:

  $$\frac{p_2}{p_0} = 18.50 \cdot \frac{1}{162.3} = 0.1140$$

  Let's now guess $M = 3$:

  $$\frac{p_2}{p_0} = 10.33 \cdot \frac{1}{36.73} = 0.281$$

  which comes reasonably close, so the maximum Mach number in the section is $M = 3$. This method can take some time, but each iteration goes rather quickly.

- Alternatively, we have

  $$\frac{p_2}{p_0} = \frac{p_2}{p_1}\frac{p_1}{p_0}$$

  With

  $$\begin{aligned} \frac{p_2}{p_1} &= 1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right) \\ \frac{p_1}{p_0} &= \left(1 + \frac{\gamma-1}{2}M_1^2\right)^{-\gamma/(\gamma-1)} \end{aligned}$$

  And thus

  $$\left(1 + \frac{2\gamma}{\gamma+1}\left(M_1^2 - 1\right)\right)\left(1 + \frac{\gamma-1}{2}M_1^2\right)^{-\gamma/(\gamma-1)} = 0.278$$

  Solving for M_1 (with your graphical calculator) $M = 3.015$.

1.19.2. Question ii

We have at the throat

$$\dot{m} = \rho A_t u = \rho A_t a$$

We have

$$
\begin{aligned}
\rho &= \frac{\rho}{\rho_0}\rho_0 = \frac{1}{1.577} \qquad \text{Using appendix A for } M = 1 \\
\rho_0 &= \frac{p_0}{RT_0} = \frac{360 \cdot 10^3}{287 \cdot 300} = 4.181\,\mathrm{kg/m^3} \\
\rho &= \frac{1}{1.577} \cdot 4.181 = 2.651\,\mathrm{kg/m^3} \\
a &= \sqrt{\gamma RT} \\
T &= \frac{T}{T_0}T_0 = \frac{1}{1.2} \cdot 300 = 250\,\mathrm{K} \\
a &= \sqrt{1.4 \cdot 287 \cdot 250} = 316.94\,\mathrm{m/s} \\
A_t &= \frac{\dot{m}}{\rho a} = \frac{2}{2.651 \cdot 316.94} = 0.00238\,\mathrm{m^2} = 2.38 \times 10^{-3}\,\mathrm{m^2}
\end{aligned}
$$

We then have for the exit for $M = 3$ that $A/A^* = 4.235$, and thus

$$
A_e = \frac{A}{A^*}A^* = 4.235 \cdot 2.38 \cdot 10^{-3} = 0.0101\,\mathrm{m^2} = 10.1 \times 10^{-3}\,\mathrm{m^2}
$$

1.19.3. Question iii

We have

$$
\frac{A_{t,2}}{A_{t,1}} = \frac{p_{0,1}}{p_{0,2}}
$$

And $A_{t,2}$ may not decreased below this, or the diffuser will choke and the tunnel unstarted. If you don't remember this, just read the sentence below equation 10.38 on the formula sheet. In the worst case scenario, we have a NSW (that's what we designed for actually) at $M = 3$, and from appendix B, we then have $p_{0,2}/p_{0,1} = 0.3283$, and thus

$$
A_{t,2} = A_{t,1} \cdot \frac{p_{0,1}}{p_{0,2}} = 2.38 \cdot 10^{-3} \cdot \frac{1}{0.3283} = 7.26 \times 10^{-3}\,\mathrm{m^2}
$$

1.20. Problem 5

1.20.1. Question i

We have

$$
\begin{aligned}
\hat{\phi}(x,y) &= \frac{20}{\sqrt{1 - M_\infty^2}} e^{-2\pi\sqrt{1-M_\infty^2}\,y} \sin(2\pi x) \\
\hat{u}(x,y) = \frac{\partial \hat{\phi}}{\partial x}(x,y) &= \frac{40\pi}{\sqrt{1 - M_\infty^2}} e^{-2\pi\sqrt{1-M_\infty^2}\,y} \cos(2\pi x) \\
\hat{v}(x,y) = \frac{\partial \hat{\phi}}{\partial y}(x,y) &= -40\pi e^{-2\pi\sqrt{1-M_\infty^2}\,y} \sin(2\pi x) \\
M_\infty &= \frac{V_\infty}{a_\infty} = \frac{V_\infty}{\sqrt{\gamma RT_\infty}} = \frac{200}{\sqrt{1.4 \cdot 287 \cdot 250}} = 0.63104 \\
\hat{u}(0.8, 0.2) &= \frac{40\pi}{\sqrt{1 - 0.63104^2}} e^{-2\pi\sqrt{1-0.63104^2}\cdot 0.2} \cos(2\pi \cdot 0.8) = 18.89\,\mathrm{m/s} \\
\hat{v}(0.8, 0.2) &= -40\pi e^{-2\pi\sqrt{1-0.63104^2}\cdot 0.2} \sin(2\pi \cdot 0.8) = 14.35\,\mathrm{m/s}
\end{aligned}
$$

And thus

$$\begin{aligned}
V &= \sqrt{(V_\infty + \hat{u})^2 + \hat{v}^2} = \sqrt{(200+18.89)^2 + 14.35^2} = 219.36\,\mathrm{m/s} \\
c_p T_\infty + \frac{V_\infty^2}{2} &= c_p T + \frac{V^2}{2} \\
T &= \frac{1004 \cdot 250 + 200^2/2 - 219.36^2/2}{1004} = 246\,\mathrm{K} \\
a &= \sqrt{\gamma R T} = \sqrt{1.4 \cdot 287 \cdot 246} = 314.37\,\mathrm{m/s} \\
M &= \frac{V}{a} = \frac{219.36}{314.37} = 0.70
\end{aligned}$$

I'm gonna go with rounding errors that this answer deviates from the 0.71 given in the answers.

1.20.2. Question ii

We have

$$C_p = \frac{p - p_\infty}{q_\infty} = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\frac{p - p_\infty}{\rho_\infty M_\infty^2 a_\infty^2} = \frac{2}{M_\infty^2}\frac{p - p_\infty}{\rho_\infty \gamma R T_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p}{p_\infty} - 1\right)$$

We must calculate p based on the Mach number at the wall (the point with coordinates (0.8,0). Redoing all of the previous calculations:

$$\begin{aligned}
\hat{u}(0.8,0) &= \frac{40\pi}{\sqrt{1-0.63104^2}} e^{-2\pi\sqrt{1-0.63104^2}\cdot 0} \cos(2\pi \cdot 0.8) = 50.06\,\mathrm{m/s} \\
\hat{v}(0.8,0) &= -40\pi e^{-2\pi\sqrt{1-0.63104^2}\cdot 0} \sin(2\pi \cdot 0.8) = 119.51\,\mathrm{m/s} \\
V &= \sqrt{(200+50.06)^2 + 119.51^2} = 277.15\,\mathrm{m/s} \\
T &= \frac{1004 \cdot 250 + 200^2/2 - 277.15^2/2}{1004} = 231.67\,\mathrm{K} \\
a &= \sqrt{\gamma R T} = \sqrt{1.4 \cdot 287 \cdot 231.67} = 305.10\,\mathrm{m/s} \\
M &= \frac{V}{a} = \frac{277.15}{305.10} = 0.9084
\end{aligned}$$

We have

$$\begin{aligned}
\frac{p}{p_\infty} &= \frac{p_0/p_\infty}{p_0/p} = \left(\frac{1 + \frac{\gamma-1}{2}M_\infty^2}{1 + \frac{\gamma-1}{2}M^2}\right)^{\gamma/(\gamma-1)} = \left(\frac{1 + \frac{1.4-1}{2}0.63104^2}{1 + \frac{1.4-1}{2}0.9084^2}\right)^{1.4/(1.4-1)} = 0.766 \\
C_p &= \frac{2}{1.4 \cdot 0.63104^2} \cdot (0.766 - 1) = -0.839
\end{aligned}$$

Alternatively, we have

$$C_p = \frac{-2\hat{u}}{V_\infty} = \frac{-2 \cdot 50.06}{200} = -0.50$$

Part V

July 2013

1.21. Problem 1

1.21.1. Question i

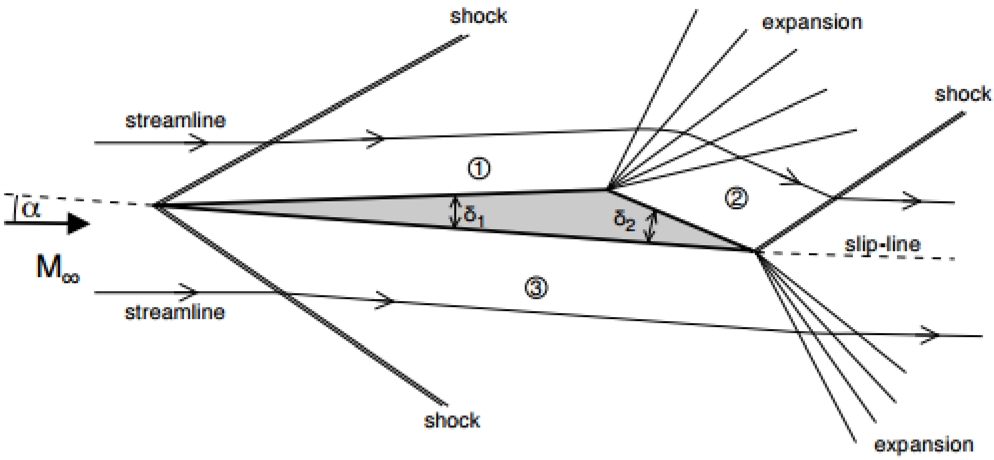

Figure 1.15: Sketch of flow.

See figure 1.15.

1.21.2. Question ii

Let's first determine the pressure coefficient everywhere. First, the upper surface, the first part: there, we have that for positive values of dy/dx that the flow is turned into itself, meaning θ has a positive relation with dy/dx. If α is increased, then the flow is turned out of itself, meaning C_p has a negative relation with α. Thus, for the first region, we have

$$C_{p,u,1} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}} = \frac{2\cdot\left(\frac{0.04c}{0.6c} - 2\cdot\frac{\pi}{180}\right)}{\sqrt{2^2-1}} = 0.0367$$

For the second part, the same relation ships hold, and thus

$$C_{p,u,2} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}} = \frac{2\cdot\left(\frac{-0.04c}{0.4c} - 2\cdot\frac{\pi}{180}\right)}{\sqrt{2^2-1}} = -0.1558$$

Now, for the lower surface, $dy/dx = 0$ and so we don't care about that. If α is increased, the flow is turned into itself, meaning θ has a positive relation with α. Thus:

$$C_{p,l} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}} = \frac{2\cdot 2\cdot\frac{\pi}{180}}{\sqrt{2^2-1}} = 0.0403$$

We have

$$c_d = c_n\alpha + c_a$$

with

$$c_n = \frac{1}{c}\int_0^c \left(C_{p,l} - C_{p,u}\right)dx = \frac{1}{c}\left(\int_0^{0.6c}(0.0403-0.0367)\,dx + \int_{0.6c}^{c}(0.0403 - -0.1558)\,dx\right) = 0.00216 + 0.07844 = 0.0806$$

Alternatively, you can use

$$c_n = \frac{4\alpha}{\sqrt{M_\infty^2}} = \frac{4 \cdot \frac{2\pi}{180}}{\sqrt{2^2-1}} = 0.0806$$

For the axial force coefficient, we get

$$c_a = \frac{1}{c}\int_0^c \left(C_{p,u}\frac{dy_u}{dx} - C_{p,l}\frac{dy_l}{dx}\right)dx$$

Performing this integration should be relatively straightforward, as you simply need to split the integral in half; furthermore, dy_u/dx and dy_l/dx are simply constants. However, you can also draw it as shown in figure 1.16, from which we easily deduce that

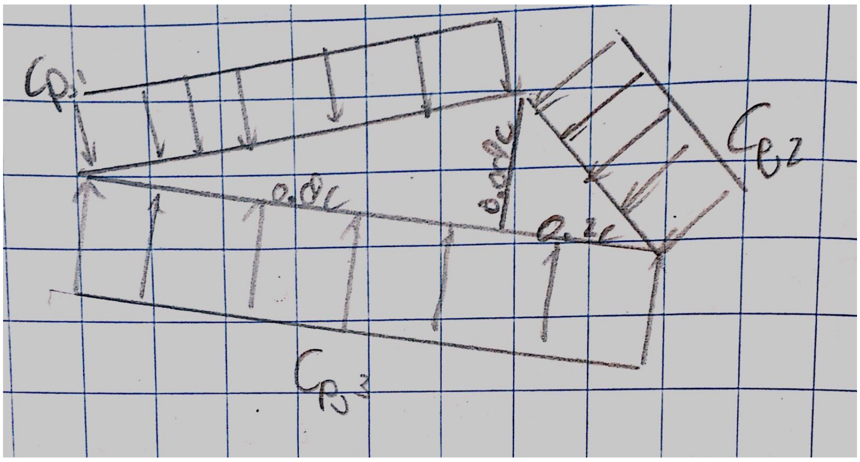

Figure 1.16: A sketch. The vertical distance is now 0.04c, the horizontal ones 0.6c and 0.4c, respectively.

$$c_a = C_{p,1} \cdot 0.04 - C_{p,2} \cdot 0.04 = 0.0367 \cdot 0.04 - -0.1558 \cdot 0.04 = 0.0077$$

and thus

$$c_d = c_n\alpha + c_a = 0.0806 \cdot \frac{2\pi}{180} + 0.0077 = 0.0105$$

1.21.3. Question iii

For the first one, we have a flow deflection of

$$\theta = \frac{dy}{dx} - \alpha = \frac{0.04c}{0.6c} - 2 \cdot \frac{\pi}{180} = 0.03176\,\text{rad} = 1.820°$$

We then find from the $\theta-\beta-M$ relation that $\beta = 31.5°$. Then, to find the total pressure in region 1:

$$\begin{aligned}
M_{n,1} &= M_1 \sin\beta = 2 \cdot 31.5° = 1.045 \\
M_{n,2} &= 0.9620 \\
M_2 &= \frac{M_{n,2}}{\sin(\beta-\theta)} = \frac{0.9620}{\sin(31.5-1.820)} = 1.943 \\
\frac{p_1}{p_\infty} &= 1.095 \\
\frac{p_{0,1}}{p_1} &= 7.128
\end{aligned}$$

For the second region, we have a PMEW:

$$\begin{aligned}
v(M_2) &= 24.71^\circ \\
v(M_3) &= v(M_2)+\theta = 24.71^\circ + \left(\frac{0.04}{0.6}+\frac{0.4}{0.4}\right)\cdot\frac{180}{\pi} = 34.26^\circ \\
M_3 &= 2.3 \\
\frac{p_{0,2}}{p_2} &= 12.50 \\
\frac{p_2}{p_\infty} &= \frac{p_2}{p_{0,2}}\cdot\frac{p_{0,1}}{p_1}\cdot\frac{p_1}{p_\infty} = \frac{1}{12.50}\cdot 7.128\cdot 1.095 = 0.6244
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
C_p &\equiv \frac{p_2-p_\infty}{q_\infty} = \frac{p_2-p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\frac{p_2-p_\infty}{\rho_\infty a_\infty^2 M_\infty^2} = \frac{2}{M_\infty^2}\frac{p_2-p_\infty}{\rho_\infty(\gamma R T_\infty)} = \frac{2}{\gamma M_\infty^2}\frac{p_2-p_\infty}{p_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_2}{p_\infty}-1\right) \\
&= \frac{2}{1.4\cdot 2^2}(0.6244-1) = -0.13414
\end{aligned}$$

1.22. Problem 2

1.22.1. Question i

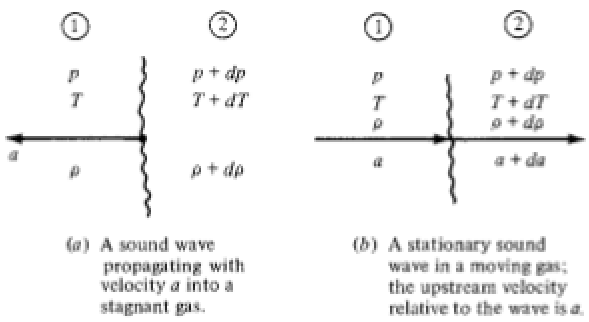

Figure 1.17: Control volume.

See the right part of figure 1.17. Draw a control volume around it with equal areas on both sides (so a rectangle). We then have from the continuity equation

$$\frac{\partial}{\partial t}\iiint_{\mathcal{V}} \rho d\mathcal{V} + \oiint_S \rho \mathbf{V}\cdot d\mathbf{S} = 0$$

For steady flow, $\frac{\partial}{\partial t}=0$. We then have

$$-\rho a A + (\rho+d\rho)(a+da)A = 0$$
$$\rho a = (\rho+d\rho)(a+da) = \rho a + a d\rho + \rho da + da\, d\rho$$

The last term can be neglected as it's the product of two differentials, so that we can write

$$\begin{aligned}
0 &= a d\rho + \rho da \\
da &= -a\frac{d\rho}{\rho}
\end{aligned}$$

From the momentum equation, we have

$$\frac{\partial}{\partial t}\oiiint_{\mathcal{V}}\rho\mathbf{V}d\mathcal{V}+\oiint_{S}(\rho\mathbf{V}\cdot d\mathbf{S})\mathbf{V}=-\oiint_{S}p\,d\mathbf{S}+\oiiint_{\mathcal{V}}\rho\mathbf{f}d\mathcal{V}$$

Assuming steady flow and no body forces, and realizing that $\mathbf{V}\cdot d\mathbf{S}$ is negative for the flow in front of the sound wave and positive for the flow behind the shock wave (as $d\mathbf{S}$ points outwards), and realizing that $d\mathbf{S}$ is negative for the flow in front of the sound wave and positive for the flow behind the shock wave, we can write

$$\begin{aligned}-\rho aAa+(\rho+d\rho)(a+da)A(a+da) &= pA-(p+dp)A\\ p+\rho a^2 &= p+dp+(\rho a+ad\rho+\rho da+da\,d\rho)(a+da)\end{aligned}$$

Again, $da\,d\rho$ can be neglected. Further expanding it yields

$$\rho a^2 = dp+\rho a^2+\rho ada+a^2d\rho+ad\rho da+\rho ada+\rho(da)^2$$

and again, neglecting all differentials of second order or higher:

$$\begin{aligned}0 &= dp+\rho ada+a^2d\rho+\rho ada=dp+2\rho ada+a^2d\rho\\ a^2 &= -\frac{2\rho ada+dp}{d\rho}\end{aligned}$$

Substituting $da=-a\frac{d\rho}{\rho}$

$$\begin{aligned}a^2 &= -\frac{2\rho\cdot -a^2\frac{d\rho}{\rho}+dp}{d\rho}=2a^2-\frac{dp}{d\rho}\\ a^2 &= \frac{dp}{d\rho}\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\frac{p_1}{p_2} &= \left(\frac{\rho_1}{\rho_2}\right)^{\gamma} \qquad \text{As we have isentropic flow over a sound wave}\\ \frac{p}{\rho^{\gamma}} &= \text{constant}=c\\ p &= c\rho^{\gamma}\\ \left(\frac{\partial p}{\partial\rho}\right) &= c\gamma\rho^{\gamma-1}=\frac{p}{\rho^{\gamma}}\gamma\rho^{\gamma-1}=\left(\frac{p}{\rho^{\gamma}}\right)\gamma\left(\frac{\rho^{\gamma}}{\rho}\right)=\frac{\gamma p}{\rho}\\ a &= \sqrt{\frac{\gamma p}{\rho}}\\ a &= \sqrt{\gamma RT} \qquad \text{From } \frac{p}{\rho}=RT\end{aligned}$$

1.22.2. Question ii

For the Mach number, we have

$$\begin{aligned}
M &= \frac{V}{a} = \frac{700}{a} \\
a &= \sqrt{\gamma RT} \\
c_p T + \frac{V^2}{2} &= c_p T_0 \\
T &= T_0 - \frac{V^2}{2c_p} = 400 - \frac{700^2}{2 \cdot 1004} = 156\,\mathrm{K} \\
a &= \sqrt{1.4 \cdot 287 \cdot 156} = 250\,\mathrm{m/s} \\
M &= \frac{700}{250} = 2.8
\end{aligned}$$

For the characteristic Mach number:

$$\begin{aligned}
M^* &= \frac{V}{a^*} = \frac{700}{a^*} \\
\frac{a^2}{\gamma - 1} + \frac{V^2}{2} &= \frac{\gamma + 1}{2(\gamma - 1)} a^{*2} \\
\frac{250^2}{1.4 - 1} + \frac{700^2}{2} &= \frac{1.4 + 1}{2(1.4 - 1)} a^{*2} \\
a^* &= 366\,\mathrm{m/s} \\
M^* &= \frac{700}{366} = 1.91
\end{aligned}$$

1.22.3. Question iii

Exactly the same calculations, but now C_p, R and γ are different. We have:

$$\begin{aligned}
\gamma &= \frac{3+2}{3} = 1.67 && \text{Mono-atomic, thus three DoFs} \\
R &= \frac{R_0}{\mu} = \frac{8314}{4} = 2078.5\,\mathrm{J/kg/K} \\
C_p &= \frac{\gamma R}{\gamma - 1} = \frac{1.67 \cdot 2078.5}{1.67 - 1} = 5196.25\,\mathrm{J/kg/K}
\end{aligned}$$

Then:

$$\begin{aligned}
M &= \frac{V}{a} = \frac{700}{a} \\
a &= \sqrt{\gamma RT} \\
c_p T + \frac{V^2}{2} &= c_p T_0 \\
T &= T_0 + \frac{V^2}{2c_p} = 400 - \frac{700^2}{2 \cdot 5196.25} = 352.85\,\mathrm{K} \\
a &= \sqrt{1.67 \cdot 2078.5 \cdot 352.85} = 1106\,\mathrm{m/s} \\
M &= \frac{700}{1106} = 0.63
\end{aligned}$$

For the characteristic Mach number:

$$\begin{aligned}
M^* &= \frac{V}{a^*} = \frac{700}{a^*} \\
\frac{a^2}{\gamma-1} + \frac{V^2}{2} &= \frac{\gamma+1}{2(\gamma-1)}a^{*2} \\
\frac{1106^2}{1.67-1} + \frac{700^2}{2} &= \frac{1.67+1}{2(1.67-1)}a^{*2} \\
a^* &= 1020\,\mathrm{m/s} \\
M^* &= \frac{700}{1020} = 0.68
\end{aligned}$$

1.23. Problem 3

1.23.1. Question i

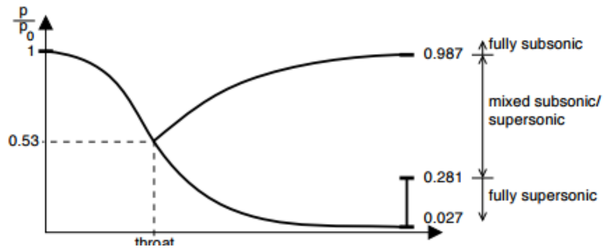

Figure 1.18: Important flow regimes. Numerical values do not correspond to this question.

For this question, it is helpful to determine the three ranges as shown in figure 1.18. For the fully subsonic one, we see that for $A^*/A = 2.5$ in the subsonic part of appendix A, $p_0/p = 1.041$.

For the second range, we use the fully isentropic solution, for which the pressure ratio at the exit, for $A^*/A_e = 2.5$, equals 15.81, and the Mach number 2.45. If there was a NSW at the exit (the most extreme case for the second range), then pressure ratio over this NSW would be $p_2/p_1 = 6.836$, so that

$$\frac{p_0}{p} = \frac{p_0}{p_1}\frac{p_1}{p_2} = 15.81 \cdot \frac{1}{6.836} = 2.313$$

Higher than this pressure ratio, the flow will be isentropic up until the exit, and it'll be isentropic after the exit as well if and only if $p_0/p = 15.81$.

Now, let's do the total pressures:

a. We have $p_0/p = 1.02$. This is clearly smaller than 1.041, and thus we have a fully subsonic flow here. Use appendix A, column p_0/p (*not* A/A^*) to find $M_e = 0.16$ (nearest entry method). Calculating the mass flow:

$$
\begin{aligned}
\dot{m} &= \rho A V \\
\rho &= \frac{\rho}{\rho_0}\rho_0 \\
\frac{\rho_0}{\rho} &= 1.013 \qquad \text{Appendix A for } M = 0.16 \\
\rho_0 &= \frac{p_0}{RT_0} = \frac{102000}{287 \cdot 300} = 1.185\,\mathrm{kg/m^3} \\
\rho &= \frac{1}{1.013} \cdot 1.185 = 1.1695\,\mathrm{kg/m^3} \\
A &= 0.5\,\mathrm{m^2} \\
V &= Ma = M\sqrt{\gamma RT} \\
T &= \frac{T}{T_0}T \\
\frac{T_0}{T} &= 1.005 \\
T &= \frac{1}{1.005} \cdot 300 = 298.51\,\mathrm{K} \\
V &= 0.16\sqrt{1.4 \cdot 287 \cdot 298.51} = 55.41\,\mathrm{m/s} \\
\dot{m} &= 1.1695 \cdot 0.5 \cdot 55.41 = 32.40\,\mathrm{kg/s}
\end{aligned}
$$

Alternatively, you can calculate the mass flow using the sonic throat area (so not $0.2\,\mathrm{m}^2$, but $A \cdot A^*/A = 0.5/3.673 = 0.1361\,\mathrm{m}^2$, and then doing the computations based on $M = 1$. It's whatever floats your boat.

b. We have $p_0/p_e = 170/100 = 1.7$. This is clearly higher than 1.041, but lower than 2.313, meaning we have a NSW, somewhere. Finding the location of the shock wave is analogous to example 3c on page 75 of the summary. Alternatively, you may rederive the equation used in problem 4ii of January 2015. I personally prefer the latter, so I'll do the latter again. Remember the steps: continuity equation between exit and throat and square it, find expression for ρ^* as function of ρ_0,

a^* as function of a_0, the apply $a = \sqrt{\gamma RT}$ and $\rho = p/RT$. Then use the expression for T_e/T_0:

$$\begin{aligned}
\rho^* A^* a^* &= \rho_e A_e M_e a_e \\
M_e^2 &= \left(\frac{\rho^*}{\rho_e}\right)^2 \left(\frac{A^*}{A_e}\right)^2 \cdot \frac{a^{*2}}{a_e^2} \\
\frac{\rho_0}{\rho_e} &= \left(1 + \frac{\gamma-1}{2} M_t^2\right)^{1/(\gamma-1)} = \left(1 + \frac{\gamma-1}{2} \cdot 1^2\right)^{1/(\gamma-1)} = \left(\frac{\gamma+1}{2}\right)^{1/(\gamma-1)} \\
\rho_e &= \rho_0 \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)} \\
\frac{\gamma+1}{2(\gamma-1)} a^{*2} &= \frac{a^2}{\gamma-1} + \frac{u^2}{2} = \frac{a_0^2}{\gamma-1} \\
a^{*2} &= \frac{2}{\gamma+1} a_0^2 = \frac{2}{\gamma+1} \gamma R T_0 \\
a_e^2 &= \gamma R T_e \\
M_e^2 &= \left(\frac{\rho_0}{\rho_e}\right)^2 \cdot \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)} \left(\frac{A^*}{A_e}\right)^2 \cdot \frac{2}{\gamma+1} \frac{\gamma R T_0}{\gamma R T_e} \\
&= \left(\frac{p_0}{p_e}\right)^2 \cdot \left(\frac{RT_e}{RT_0}\right)^2 \cdot \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \left(\frac{A^*}{A_e}\right)^2 \cdot \frac{T_0}{T_e} \\
&= \left(\frac{p_0}{p_e}\right)^2 \cdot \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \left(\frac{A^*}{A_e}\right)^2 \cdot \frac{T_e}{T_0} = \left(\frac{p_0}{p_e}\right)^2 \cdot \left(\frac{2}{\gamma+1}\right)^{(\gamma+1)/(\gamma-1)} \left(\frac{A^*}{A_e}\right)^2 \cdot \frac{1}{1 + \frac{\gamma-1}{2} M_e^2} \\
&= 1.7^2 \cdot \left(\frac{2}{1.4+1}\right)^{(1.4+1)/(1.4-1)} \cdot \left(\frac{0.2}{0.5}\right)^2 \cdot \frac{1}{1 + \frac{1.4-1}{2} M_e^2}
\end{aligned}$$

You can either solve this equation algebraically, or by use of your graphical calculator, yielding $M_e = 0.3877$. Now, onto the mass flow. We can compute that the easiest at the throat:

$$\begin{aligned}
\dot{m} &= \rho A a \\
\rho &= \frac{\rho}{\rho_0} \rho_0 \\
\rho_0 &= \frac{p_0}{RT_0} = \frac{170 \cdot 10^3}{287 \cdot 300} = 1.974\,\mathrm{kg/m^3} \\
\rho &= \frac{\rho}{\rho_0} \rho_0 = \frac{1}{1.577} \cdot 1.974 = 1.252\,\mathrm{kg/m^3} \qquad \text{Using appendix A for } M = 1 \\
A &= 0.2\,\mathrm{m^2} \\
a &= \sqrt{\gamma RT} \\
T &= \frac{T}{T_0} T_0 = \frac{1}{1.2} \cdot 300 = 250\,\mathrm{K} \qquad \text{Using appendix A for } M = 1 \\
a &= \sqrt{1.4 \cdot 287 \cdot 250} = 316.94\,\mathrm{m/s} \\
\dot{m} &= 1.2517 \cdot 0.2 \cdot 316.94 = 79.35\,\mathrm{kg/s}
\end{aligned}$$

c. We have $p_0/p = 10000/100 = 100$. As this is clearly higher than 15.81, the flow will be isentropic until the exit, and thus the isentropic solution holds for the exit. For $A/A^* = 2.5$ we have $M_e = 2.45$. Computing the new total density to be $10 \cdot 10^6/287/300 = 116.14\,\mathrm{kg/m^3}$, and thus the density at the throat to be $73.649\,\mathrm{kg/m^3}$, this leads to a mass flow of $4668.44\,\mathrm{kg/s}$.

1.23.2. Question ii

Case 1) and 2) are neither overexpanded nor underexpanded as the flow is not supersonic as it leaves the exit. Case 3) is underexpanded, as $p_0/p_e = 100 > 15.81$, meaning that the pressure of the flow as it leaves the exit (which equals $1/15.81 p_0 = 0.06325 p_0$) will be higher than the back pressure (which equals $0.01 p_0$), meaning it needs to be expanded to reduce the pressure of it.

1.23.3. Question iii

The exit Mach number is 2.45 currently. Thus, assuming the worst case scenario, which is a NSW, the area of the second throat must at least equal

$$A_{t,2} = A_{t,1} \frac{p_{0,1}}{p_{0,2}}$$

where $p_{0,1}/p_{0,2}$ is the total pressure ratio over a NSW, which for $M_e = 2.45$ equals 0.5193. Thus, we must have

$$A_{t,2} = 0.2 \cdot \frac{1}{0.5193} = 0.385\,\mathrm{m}^2$$

If $A_{t,2}$ is reduced below this value, the tunnel will unstart.

1.24. Problem 4

1.24.1. Question i

For $M_\infty = 0.7$, we have $c_{l,0} = 2\pi \cdot 2/180 \cdot \pi = 0.219$. Then applying Prandlt-Glauert:

$$c_l = \frac{0.219}{\sqrt{1-0.7^2}} = 0.307$$

For $M_\infty = 2.3$, we have

$$c_l = \frac{4 \cdot 2 \cdot \frac{180}{\pi}}{\sqrt{2.3^2 - 1}} = 0.0674$$

1.24.2. Question ii

The critical Mach number is the lowest freestream Mach number where for the first time anywhere along the airfoil $M = 1$ is reached.

1.24.3. Question iii

- At low subsonic speeds, the drag coefficient is more or less constant.
- Once M_{cr} is reached, the drag starts to gradually increase. Beyond this point, there are sonic regions over the airfoil, which produce weak shock waves, creating a small amount of wave drag.
- Shortly thereafter, after M_{dd}, an extensive region of supersonic flow over the airfoil appears, increasing the wave drag significantly.
- It increases until about the Mach number reaches unity, when the sound barrier is reached and the drag coefficient is maximum.

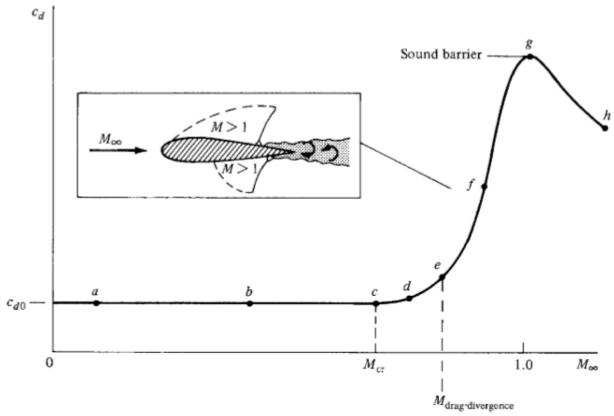

Figure 1.19: Drag divergence and stuff.

- It decreases thereafter as the shock waves become weaker with increasing Mach number, as the shock wave angle decreases and thus the shock waves become weaker. Furthermore, directly after $M = 1$, the flow around the entire airfoil is supersonic, except possibly just in front of the airfoil if a detached shock wave is formed. There will either be a detached shock wave in front of the airfoil, or an oblique shock wave at the leading edge of the airfoil. Both of these do not cause boundary layer separation (as there is simply no boundary layer at that point yet), contrary to the shock waves which formed at $M_\infty < 1$, thus also reducing the drag due to boundary layer separation due to shock waves.

Part VI

January 2016

1.25. Problem 1

1.25.1. Question i

Well this looks familiar doesn't it. Just draw it like shown in figure 1.20.

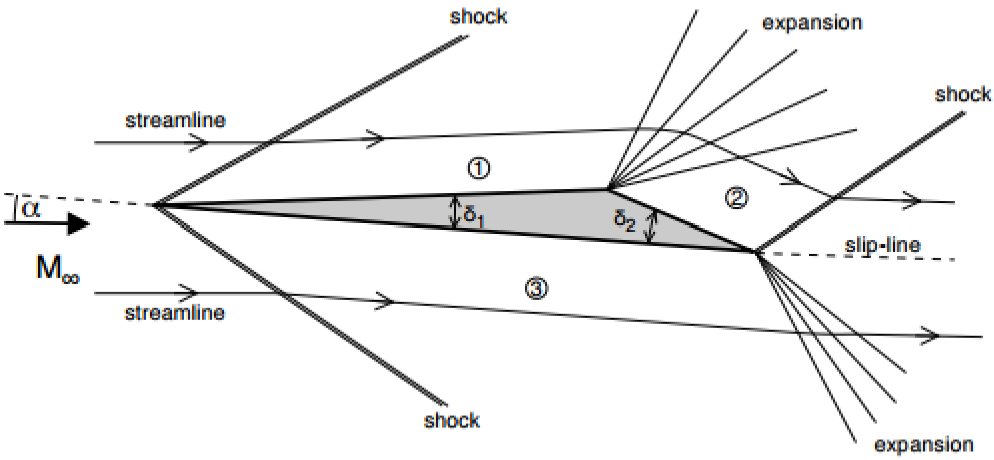

Figure 1.20: Sketch of flow.

1.25.2. Question ii

You know how to do this by know. In region 1, we have that θ has a positive relation with dy/dx and negative with α and the same holds for region 2. We then have

$$\begin{aligned} C_{p,1} &= \frac{2\theta}{\sqrt{M_\infty^2-1}} = \frac{2\left(\frac{0.05c}{0.7c} - 2\cdot\frac{\pi}{180}\right)}{\sqrt{3^2-1}} = 0.0258 \\ C_{p,2} &= \frac{2\theta}{\sqrt{M_\infty^2-1}} = \frac{2\left(\frac{-0.05c}{0.3c} - 2\cdot\frac{\pi}{180}\right)}{\sqrt{3^2-1}} = -0.1425 \end{aligned}$$

For region 3, $dy/dx = 0$ and there is a positive relation with α, and thus we have

$$C_{p,3} = \frac{2\theta}{\sqrt{M_\infty^2-1}} = \frac{2\cdot\frac{2\pi}{180}}{\sqrt{3^2-1}} = 0.0247$$

Now, we have

$$c_d = c_n\alpha + c_a$$

with

$$c_n = \frac{4\alpha}{\sqrt{M_\infty^2-1}} = \frac{4\cdot\frac{2\pi}{180}}{\sqrt{3^2-1}} = 0.04937$$

Furthermore, c_a is most easily deduced from figure 1.21. From there, we realize that

$$c_a = 0.05\cdot C_{p,1} - 0.05\cdot C_{p,2} = 0.05\cdot 0.0258 - 0.05\cdot -0.1425 = 0.008415$$

and thus

$$c_d = c_n\alpha + c_a = 0.04937\cdot\frac{2\pi}{180} + 0.008415 = 0.0101$$

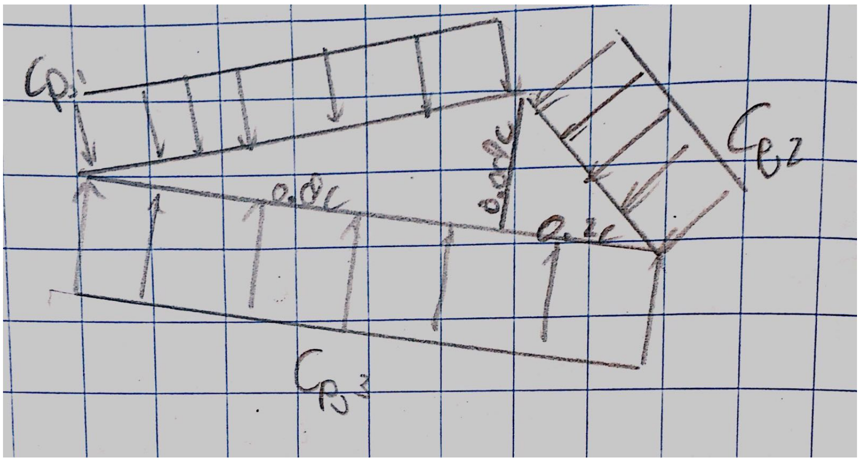

Figure 1.21: A sketch. The vertical distance is now 0.05c, the horizontal ones 0.7c and 0.3c, respectively.

1.25.3. Question iii

Again, just like we always did. First the deflection angle for region 1 is

$$\theta = \frac{0.05c}{0.7c} \cdot \frac{180}{\pi} - 2 = 2.09^\circ$$

For $M_\infty = 3.0$, we have

$$\begin{aligned}
\beta &= 21^\circ \\
M_{n,\infty} &= M_\infty \sin\beta = 3.0 \cdot \sin 21^\circ = 1.0751 \\
M_{n,1} &= 0.9277 \\
M_1 &= \frac{M_{n,1}}{\sin(\beta-\theta)} = \frac{0.9277}{\sin(21-2.09)} = 2.8625 \\
\frac{p_1}{p_\infty} &= 1.194 \\
\frac{p_{0,1}}{p_1} &= 29.29 \\
v(M_1) &= 46.78^\circ \\
v(M_2) &= v(M_1) + \theta = 46.78 + \left(\frac{0.05c}{0.7c} - -\frac{0.05c}{0.3c}\right) \cdot \frac{180}{\pi} = 60.42^\circ \\
M_2 &= 3.60 \\
\frac{p_{0,2}}{p_2} &= 87.84 \\
\frac{p_2}{p_\infty} &= \frac{p_2}{p_{0,2}} \frac{p_{0,1}}{p_1} \frac{p_1}{p_\infty} = \frac{1}{87.84} \cdot 29.29 \cdot 1.194 = 0.398
\end{aligned}$$

Due to inaccuracies of nearest entry method, this is slightly different from the actual answer. We can find the pressure ratio for the linear theory by applying Furthermore, as before, we simply have

$$\begin{aligned}
C_p &= \frac{p_2 - p_\infty}{q_\infty} = \frac{p_2 - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\frac{p_2 - p_\infty}{\rho_\infty M_\infty^2 a_\infty^2} = \frac{2}{M_\infty^2} \frac{p_2 - p_\infty}{\rho_\infty \gamma R T_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_2}{p_\infty} - 1\right) \\
\frac{p_2}{p_\infty} &= \frac{\gamma M_\infty^2 C_p}{2} = 0.102
\end{aligned}$$

This discrepancy is caused by the the inaccuracies of linear theory, which assumes small angle of attack and small perturbations. For region 2, the angle of attack of the surface is arguably rather large, raising doubts over the accuracy of linear theory.

1.26. Problem 2

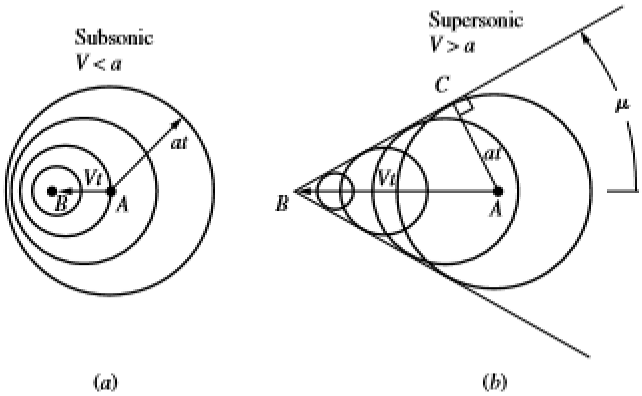

Figure 1.22: Waves.

Look at figure 1.22. When the source of sound, starting at A has travelled for a time t at a velocity v, then it will be at a place B. Meanwhile, the sound wave originating from A will be in a circle with radius at. Then we clearly have

$$\begin{aligned} \sin\mu &= \frac{at}{Vt} = \frac{a}{V} = \frac{1}{M} \\ \mu &= \arcsin\frac{1}{M} \end{aligned}$$

1.27. Problem 3

1.27.1. Question i

For air, we have

$$\begin{aligned} h_1 + \frac{u_1^2}{2} &= h_2 + \frac{u_2^2}{2} \\ c_pT_1 + \frac{u_1^2}{2} &= c_pT_0 \\ \frac{\gamma RT}{\gamma-1} + \frac{V^2}{2} &= \frac{\gamma RT_0}{\gamma-1} \\ \frac{a^2}{\gamma-1} + \frac{V^2}{2} &= \frac{\gamma RT_0}{\gamma-1} \end{aligned}$$

If we have $V_{\text{lim}} = 1300\,\text{m/s}$ and $M = \infty$, then we must have that $a = 0$. Thus, we simply have

$$\begin{aligned} \frac{1300^2}{2} &= \frac{1.4\cdot 287}{1.4-1}T_0 \\ T_0 &= 842\,\text{K} \end{aligned}$$

1.27.2. Question ii

For helium, it's exactly the same, ending up at

$$\frac{V^2}{2} = \frac{\gamma R T_0}{\gamma - 1}$$

But now we have, because we have three degrees of freedom and a different molecular mass:

$$\begin{aligned}\gamma &= \frac{3+2}{3} = 1.667 \\ R &= \frac{R_0}{\mu} = \frac{8314}{4} = 2078.5\,\mathrm{J/kg/K}\end{aligned}$$

and thus

$$\begin{aligned}\frac{1300^2}{2} &= \frac{1.67 \cdot 2078.5 T_0}{1.67-1} \\ T_0 &= 163\,\mathrm{K}\end{aligned}$$

1.28. Problem 4

1.28.1. Question i

This means that $p_{e,6}/p_0 = 0.0585$, or $p_0/p_{e,6} = 17.1$. From appendix A, we then have $A_e/A^* = 2.64$ and $M_e = 2.5$.

1.28.2. Question ii

It is overexpanded when $p_{e,6}/p_0 < p_e/p_0 < p_{e,5}/p_0$. We already found that $p_{e,6}/p_0 = 0.0585$. For $p_{e,5}/p_0$, we place a NSW at the exit; in this case, we have

$$\frac{p_{e,5}}{p_0} = \frac{p_2}{p_1}\frac{p_1}{p_0} = 7.125 \cdot \frac{1}{17.1} = 0.4167$$

However, p_e is fixed at 0.585 bar, so the associated $p_0 = 0.585/0.4167 = 1.39\,\mathrm{bar}$. Thus, if $1.39\,\mathrm{bar} < p_0 < 10\,\mathrm{bar}$, then the flow is overexpanded.

The flow is underexpanded when $p_e/p_0 < p_{e,6}/p_0 = 0.0585$. As p_e is fixed at 0.585 bar, for $p_0 > 10\,\mathrm{bar}$, the flow is underexpanded.

I do not understand how this problem is worth 23 points, honestly.

1.29. Problem 5

1.29.1. Question i

At infinity, we have $\hat{\phi} = \text{constant}$, so that $\hat{u} = \hat{v} = 0$. At the airfoil surface, we must have

$$\tan\theta = \frac{\hat{v}}{V_\infty + \hat{u}} \approx \frac{\hat{v}}{V_\infty}$$

where θ is angle between the airfoils surface and the freestream. This can be explicitly written as

$$\hat{v} = \frac{\partial \hat{\phi}}{\partial y} = V_\infty \tan\theta$$

1.29.2. Question ii

For a), we have

$$c_p = \frac{c_{p,0}}{\sqrt{1-M_\infty^2}}$$

First converting to $c_{p,0}$ gives

$$c_{p,0} = c_p\sqrt{1-M_\infty^2} = -0.3 \cdot \sqrt{1-0.5^2} = -0.2598$$

then to $M = 0.7$:

$$c_p = \frac{-0.2598}{\sqrt{1-0.7^2}} = -0.3638$$

For b), we have that the pressure coefficient for a point A where $M = 1$ equals

$$c_p = \frac{p_2 - p_\infty}{q_\infty} = \frac{p_A - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\frac{p_A - p_\infty}{\rho_\infty M_\infty^2 a_\infty^2} = \frac{2}{M_\infty^2}\frac{p_A - p_\infty}{\rho_\infty \gamma R T_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_A}{p_\infty} - 1\right)$$

where the ratio p_A/p_∞ can be written as

$$\frac{p_A}{p_\infty} = \frac{p_0/p_\infty}{p_A/p_\infty} = \left(\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}M_A^2}\right)^{\gamma/(\gamma-1)} = \left(\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}}\right)^{\gamma/(\gamma-1)}$$

and thus

$$c_{p,cr} = \frac{2}{\gamma M_\infty^2}\left(\left(\frac{1+\frac{\gamma-1}{2}M_\infty^2}{1+\frac{\gamma-1}{2}}\right)^{\gamma/(\gamma-1)} - 1\right)$$

setting this equal to

$$c_p = \frac{-0.2598}{\sqrt{1-M_\infty^2}}$$

and solving by use of your graphical calculator results in $M_{cr} = 0.8004$ (and $c_{p,cr} = -0.433$, but it's not part of the question).

Part VII

June 2010

1.30. Problem 1a

1.30.1. Question i

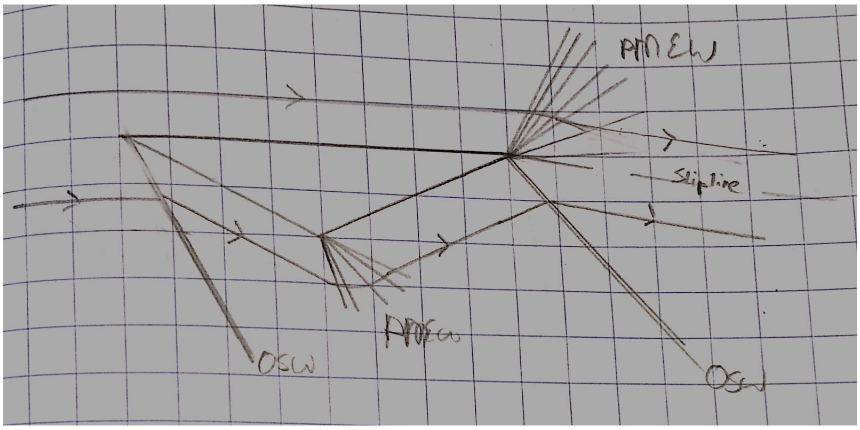

Figure 1.23: A shockingly ugly sketch.

See figure 1.23. Do note: the slip line must not be drawn parallel to the freestream flow, as this is very unlikely to happen. How can we reason our way into that it points downward? Over an OSW, the static pressure increases (it is compressed after all); over a PMEW, the static pressure decreases as the flow is accelerated. However, the OSWs also lower the total pressure of a flow (whereas PMWEs don't increase it). Therefore, it is reasonable to assume that the flow after the second OSW will have a lower static pressure than the flow that passed over the upper surface. Therefore, the flow going over the upper surface must be expanded, and the flow going over the lower surface must be compressed more. This causes the slip line to be deflected downward.

1.30.2. Question ii

According to linear theory, we have

$$C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}$$

For the upper surface, $\theta = 0$, and thus

$$C_{p,u} = \frac{2\theta}{\sqrt{M_\infty^2 - 1}} = \frac{2 \cdot 0}{\sqrt{3^2 - 1}} = 0$$

For the lower surface, we have for the first part that for positive values of $\frac{dy}{dx}$, the flow is turned away from itself (okay if it'd be positive it'd look very strange, but you get the point hopefully). Thus, θ has a negative relation with dy/dx, which equals $\text{atan}(0.12c/0.5c) = \text{atan}(0.24)$, and thus

$$C_{p,l,1} = \frac{2 \cdot \text{atan}(0.24)}{\sqrt{3^2 - 1}} = 0.16655$$

For the second part, the relation is simply inversed:

$$C_{p,l,2} = \frac{2 \cdot -\text{atan}(0.24)}{\sqrt{3^2 - 1}} = -0.16655$$

The lift coefficient follows directly from

$$c_l = c_n - c_a\alpha = c_n - c_a \cdot 0 = c_n = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} = \frac{4 \cdot 0}{\sqrt{3^2 - 1}} = 0$$

1.30.3. Question iii

First, we derive that

$$C_p = \frac{p_2 - p_\infty}{q_\infty} = \frac{p_A - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = 2\frac{p_A - p_\infty}{\rho_\infty M_\infty^2 a_\infty^2} = \frac{2}{M_\infty^2}\frac{p_A - p_\infty}{\rho_\infty \gamma R T_\infty} = \frac{2}{\gamma M_\infty^2}\left(\frac{p_A}{p_\infty} - 1\right)$$

where A is any location along the airfoil. For the upper surface, there is neither an expansion wave nor OSW, so the pressure along the upper surface p_u is equal to p_∞ and thus

$$C_{p,u} = \frac{2}{1.4 \cdot 3^2}(1-1) = 0$$

Now, for region 1, i.e. the first part of the lower surface. There, we have an OSW with a deflection angle $\text{atan}(0.24) = 13.5°$. For $M = 3$, we then have $\beta = 31°$, and thus

$$\begin{aligned}
M_{n,\infty} &= M_\infty \sin\beta = 3 \cdot 31° = 1.545 \\
M_{n,1} &= 0.6874 \\
M_1 &= \frac{M_{n,1}}{\sin(\beta - \theta)} = \frac{0.6874}{\sin(31 - 13.5)} = 2.286 \\
\frac{p_1}{p_\infty} &= 2.60 \\
\frac{p_0}{p_1} &= 12.50 \\
C_{p,l,1} &= \frac{2}{1.4 \cdot 3^2}(2.60 - 1) = 0.2540
\end{aligned}$$

For the second part, we have

$$\begin{aligned}
v(M_1) &= 34.28° \\
v(M_2) &= 34.28 + 2 \cdot 13.5 = 61.28° \\
M_2 &= 3.70 \\
\frac{p_0}{p_2} &= 101.0 \\
\frac{p_2}{p_\infty} &= \frac{p_2}{p_0}\frac{p_0}{p_1}\frac{p_1}{p_\infty} = \frac{1}{101.0} \cdot 12.50 \cdot 2.60 = 0.3218 \\
C_{p,l,2} &= \frac{2}{1.4 \cdot 3^2}(0.3218 - 1) = -0.108
\end{aligned}$$

From figure 1.24, we then rather straightforwardly have

$$c_l = c_n - c_a\alpha = c_n - c_a \cdot 0 = c_n = \left(C_{p,l} - C_{p,u}\right) = 0.5 \cdot \left(C_{p,l,1} - C_{p,u}\right) + 0.5 \cdot \left(C_{p,l,2} - C_{p,u}\right) = 0.5 \cdot (0.2540 - 0) + 0.5 \cdot (-0.108) = 0.073$$

1.30.4. Question iv

Linear theory predicted zero lift coefficient even though the shock-expansion predicted a c_l of 0.073. This difference is due to the fact that linear theory assumes small angles and small perturbations. An airfoil of $0.12c$ thickness is arguably no longer thin enough to apply linear theory.

1.31. Problem 1b

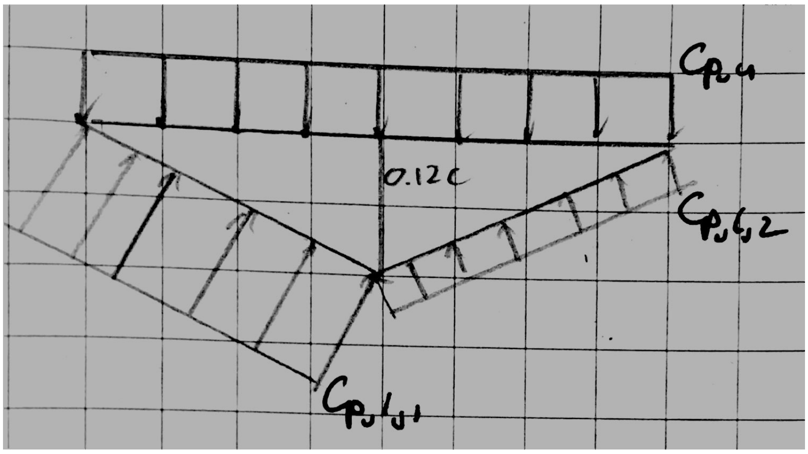

Figure 1.24: A sketch.

1.31.1. Question i

a) For a supersonic flow without shock waves in a convergent channel, A/A^* is continuously decreasing. From the area-Mach relation, we then have that the Mach number decreases (and thus also the flow velocity decreases). Furthermore, as the flow is isentropic (there are no shocks), p_0 and T_0 are constant, and from the isentropic flow relations, we see that for constant T_0 and p_0, T and p must increase if M decreases.

b) When a flow is turned into itself, an OSW is created. Over an OSW, the flow is compressed, decreasing the flow velocity, increasing the static temperature and pressure. Total temperature is always constant however, as long as the flow is adiabatic. Total pressure also decreases over a shock wave, due to the entropy generated.

c) When a flow is turned away from itself, the flow is expanded, increasing the flow velocity and decreasing the static pressure and static temperature. However, as the flow is isentropic, total temperature and total pressure remain constant.

1.31.2. Question ii

From the energy equation, we have

$$\begin{aligned}
h_1 + \frac{u_1^2}{2} &= h_2 + \frac{u_2^2}{2} = h_0 \\
c_p T + \frac{V^2}{2} &= c_p T_0 \\
T + \frac{V^2}{2c_p} &= T_0 \\
T + \frac{M^2 a^2}{2\frac{\gamma R}{\gamma - 1}} &= T_0 \\
T + \frac{M^2 \gamma R T}{2\frac{\gamma R}{\gamma - 1}} &= T_0 \\
T + \frac{\gamma - 1}{2} M^2 T &= T_0 \\
T\left(1 + \frac{\gamma - 1}{2} M^2\right) &= T_0 \\
\frac{T_0}{T} &= 1 + \frac{\gamma - 1}{2} M^2 \\
\frac{p_0}{p} &= \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma-1)}
\end{aligned}$$

1.32. Problem 2a

1.32.1. Question i

First computing the Mach number, we have

$$\begin{aligned}
M &= \frac{V}{a} \\
a &= \sqrt{\gamma R T} = \sqrt{1.4 \cdot 287 \cdot 289} = 340.76\,\mathrm{m/s} \\
M &= \frac{170.5}{340.76} = 0.500
\end{aligned}$$

For $M = 0.50$, we find from appendix A that $A_1/A^* = 1.340$. With A_2/A_1 equalling 0.76, this means that

$$\frac{A_2}{A^*} = \frac{A_2}{A_1}\frac{A_1}{A^*} = 0.76 \cdot 1.340 = 1.0184$$

and thus $A_2 > A^*$. This means that the throat is *not* choked and thus that the flow is fully subsonic throughout the entire wind tunnel. We find for A_4/A^* that

$$\frac{A_4}{A^*} = \frac{A_4}{A_1}\frac{A_1}{A^*} = 0.785 \cdot 1.34 = 1.0519$$

for which $M_4 = 0.78$. Furthermore, as the flow is fully isentropic (as it's fully subsonic), we can simply compute the total pressure for station 1:

$$\begin{aligned}
\frac{p_0}{p_1} &= 1.186 \\
p_0 &= 1.186 p_1 = 1.186 \cdot 10^5 = 1.186 \times 10^5\,\mathrm{N/m^2}
\end{aligned}$$

which will also be the total pressure at station 2. Finally, computing the velocity:

$$\begin{aligned}
\rho_1 A_1 V_1 &= \rho_4 A_4 V_4 \\
V_4 &= \frac{\rho_1}{\rho_4}\frac{A_1}{A_4}V_1 \\
\frac{\rho_0}{\rho_1} &= 1.130 \\
\frac{\rho_0}{\rho_4} &= 1.314 \\
\frac{\rho_1}{\rho_4} &= \frac{\rho_1}{\rho_0}\frac{\rho_0}{\rho_4} = \frac{1}{1.130}\cdot 1.314 = 1.1628 \\
V_4 &= 1.1628\cdot\frac{1}{0.785}\cdot 170.5 = 252.56\,\mathrm{m/s}
\end{aligned}$$

Alternatively you can compute the speed of sound and use that.

1.32.2. Question ii

From (i), we saw that $A_1/A^* = 1.340$. So, A_2/A_1 must be 0.7463 at most. Don't know why they say minimum tho.

1.32.3. Question iii

If $M = 1.2$ directly in front of the shock, we must have $A_3/A_2 = 1.030$. We already computed the total pressure before the shock; this equalled $1.186\times 10^5\,\mathrm{N/m^2}$. We can look up the total pressure loss in appendix B:

$$\begin{aligned}
\frac{p_{0,3}}{p_{0,2}} &= 0.9928 \\
p_{0,3} &= 0.9928\cdot 1.186\cdot 10^5 = 1.177\times 10^5\,\mathrm{N/m^2}
\end{aligned}$$

As the flow is subsonic after a NSW and thus isentropic, we have $p_{0,4} = p_{0,3} = 1.177\times 10^5\,\mathrm{N/m^2}$. Furthermore, the Mach number can be computed by use of appendix A. After the normal shock, $M_3 = 0.8422$, and thus

$$\frac{A_3}{A^*} = 1.024$$

Please note, A^* now corresponds to the sonic throat area of the flow *after* the NSW, which is smaller than A_2 (which was the sonic throat area of the flow before the NSW). We can now compute

$$\frac{A_4}{A^*} = \frac{A_4}{A_2}\frac{A_2}{A_3}\frac{A_3}{A^*} = \frac{A_4}{A_1}\frac{A_1}{A_2}\frac{A_2}{A_3}\frac{A_3}{A^*} = 0.785\cdot\frac{1}{0.76}\cdot\frac{1}{1.030}\cdot 1.024 = 1.0269$$

For some very odd reason, the solutions got 1.047 but they do exactly the same steps as I do. Anyway, for $A_4/A^* = 1.0269$ we have $M = 0.82$. The velocity can be calculated using the speed of sound (using the continuity equation is now much harder, as the total density is not the same). For station 1, the Mach number has changed from before (as the flow now is choked); instead, we must look up T_0/T from $A_1/A_2 = 1.316$, for which $T_0/T = 1.054$, and thus

$$T_0 = 1.054\cdot T_1 = 304.606\,\mathrm{K}$$

For station 4, where $A_4/A^* = 1.0269$, we find $T_0/T = 1.134$, and thus

$$V = Ma = 0.82\sqrt{1.4\cdot 287\cdot\frac{304.606}{1.134}} = 269.39\,\mathrm{m/s}$$

The difference is rather large compared to the answer of the professors (which is 257 m/s), but this is purely because of the different values for A_4/A^*.

1.33. Problem 2b

1.33.1. Question i

$$
\begin{aligned}
M &= \frac{V}{a} \\
V &= 2156\,\mathrm{km/h} = 598.89\,\mathrm{m/s} \\
a &= \sqrt{\gamma RT} = \sqrt{1.4 \cdot 287 \cdot 213.15} = 292.65\,\mathrm{m/s} \\
M &= \frac{598.89}{292.65} = 2.046
\end{aligned}
$$

1.33.2. Question ii

First, the total temperature. As this is constant for an adiabatic flow, we can simply compute it for the freestream conditions:

$$
\begin{aligned}
\frac{T_0}{T} &= 1 + \frac{\gamma - 1}{2} M^2 \\
T_0 &= 213.15 \cdot \left(1 + \frac{1.4-1}{2} \cdot 2.046^2\right) = 391.60\,\mathrm{K}
\end{aligned}
$$

For the total pressure, you *must* use the Rayleigh-Pitot formula:

$$
\begin{aligned}
\frac{p_{0,2}}{p_1} &= \left[\frac{(\gamma+1)^2 M_1^2}{4\gamma M_1^2 - 2(\gamma-1)}\right]^{\gamma/(\gamma-1)} \frac{1-\gamma+2\gamma M_1^2}{\gamma+1} \\
p_{0,2} &= 2.27 \cdot 10^4 \cdot \left[\frac{(1.4+1)^2 \cdot 2.045^2}{4 \cdot 1.4 \cdot 2.045^2 - 2 \cdot (1.4-1)}\right]^{1.4/(1.4-1)} \frac{1-1.4+2 \cdot 1.4 \cdot 2.045^2}{1.4+1} = 1.33 \times 10^5\,\mathrm{N/m^2}
\end{aligned}
$$