Instrumentation and signals: solution manual - 2018-2019 edition

Sam van Elsloo

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Preface

Okay so in my year we were guaranteed to get a complex Fourier series on the exam, so that's why I made all the solutions be complex Fourier series even though they asked for real Fourier series. However, it is relatively straightforward to convert the complex Fourier series to real Fourier series in case you want to practice with those so that you can check your final answer: note that if we had a real Fourier series given by

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

then the complex coefficient X_n related to this by

$$X_n = \begin{cases} \frac{1}{2}(a_n - jb_n) & : n > 0 \\ a_n & : n = 0 \\ \frac{1}{2}(a_{-n} + jb_{-n}) & : n < 0 \end{cases}$$

and from this it will be possible to determine a_n and b_n corresponding to X_n.

Part I

Exam April 2013

1.1. Question 1

Answer A is correct. Let's go through the statements one by one:

- Statement I is clearly false: X_0 is the average value, which is 0 in this case.
- Statement II is also false: just absolutely wrong. $x(t)$ is not even half-wave odd, and even if it was, it would have meant all *even*-indexed would have been zero, but the odd-indexed coefficients would have remained nonzero. Only if the wave would have been half-wave even, the odd-indexed coefficients would have been zero.
- Statement III is true: if $b = 0$, the function becomes odd, and thus all coefficients X_k will be purely imaginary.

1.2. Question 2

Answer D is correct. The width of a regular pulse is 1; thus, the pulse now expanded, meaning time goes slower. This means we get $\frac{1}{2}$ in there. Furthermore, it is now centered at $t = 4$, meaning we end up at

$$\Pi\left(\frac{t-4}{2}\right)$$

and thus answer D is correct.

1.3. Question 3

Answer C is correct. Simply use pairs 12 and 13 of the formula sheet. We then have

$$\begin{aligned} x(t) &= \cos(2\pi f_1 t) + \sin(2\pi 2 f_1 t) \\ X(f) &= \frac{1}{2}\delta(f - f_1) + \frac{1}{2}\delta(f + f_1) + \frac{1}{2j}\delta(f - 2f_1) - \frac{1}{2j}\delta(f + 2f_1) \end{aligned}$$

Note that $1/j = -j$, and thus we get

$$\begin{aligned} X(f) &= \frac{1}{2}\delta(f - f_1) + \frac{1}{2}\delta(f + f_1) - \frac{j}{2}\delta(f - 2f_1) + \frac{j}{2}\delta(f + 2f_1) \\ X(f) &= \frac{1}{2}\left[\delta(f - f_1) - j\delta(f - 2f_1)\right] + \frac{1}{2}\left[\delta(f + f_1) + j\delta(f + 2f_1)\right] \end{aligned}$$

and thus answer C is correct.

1.4. Question 4

Answer B is correct. Once again, it's key to sketch the signal. This is done in figure 1.1.

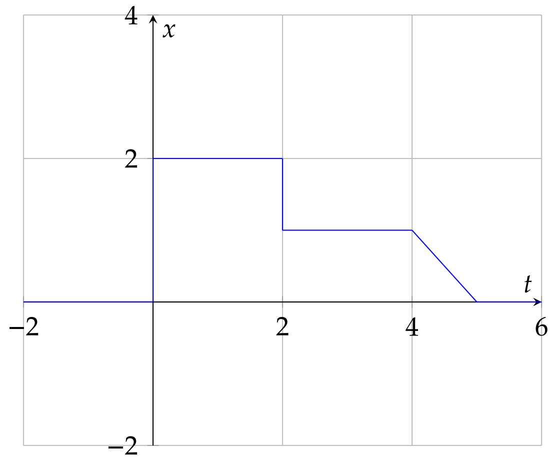


Figure 1.1: Sketch of signal.

Clearly, this is an energy signal. Its energy is found by integrating the square of the absolute value of x. This means that between $t = 0$ and $t = 2$, we get an energy of $2 \cdot 2^2 = 8\,\mathrm{J}$; between $t = 2$ and $t = 4$ we get an energy of $2 \cdot 1^2 = 2\,\mathrm{J}$; between $t = 4$ and $t = 5$ we get an energy of

$$\int_0^1 (1-x)^2\,dx = \frac{1}{3}$$

Thus, the total energy is $8 + 2 + \frac{1}{3} = 10\frac{1}{3}$ Joule and thus answer B is correct.

1.5. Question 5

The correct answer is B. System 3 is *not* causal: it is dependent on the value of x in the future (to be precise: the output of y at a time t is dependent on the input at a time $t + 10$). This means that S3 is not causal (meaning answer B is the only one that's possibly correct). All systems are dynamic, however, as they all derivatives in them.

1.6. Question 6

The correct answer is B. First, we compute the Fourier transform of $x(t)$; again, we need to get rid of $\cos^2$, using $\cos^2(u) = 1/2 + 1/2 \cdot \cos(2u)$, meaning we get

$$\begin{aligned} x(t) &= 2\mathrm{sinc}(3t)\cos^2(12\pi t) = 2\mathrm{sinc}(3t)\left(\frac{1}{2} + \frac{1}{2}\cos(24\pi t)\right) = \mathrm{sinc}(3t) + \mathrm{sinc}(3t)\cos(24\pi t) \\ &= \frac{1}{3} \cdot 3\mathrm{sinc}(3t) + \frac{1}{3} \cdot 3\mathrm{sinc}(3t)\cos(24\pi t) \end{aligned}$$

I include the $\frac{1}{3} \cdot 3$ to make it clearer how we do the Fourier transform. For the first term, it's really straightforward; use pair 2 on the formula sheet; we have $2W = 3$. For the second term, it's almost as straightforward, although we need to keep into account the modulation theorem with frequency $12\,\mathrm{Hz}$, meaning we get

$$X(f) = \frac{1}{3} \cdot \Pi\left(\frac{f}{3}\right) + \frac{1}{6} \cdot \Pi\left(\frac{f+12}{3}\right) + \frac{1}{6} \cdot \Pi\left(\frac{f-12}{3}\right)$$

We can plot this as shown in figure 1.2; remember that all the pulses are of width $3\,\mathrm{Hz}$:

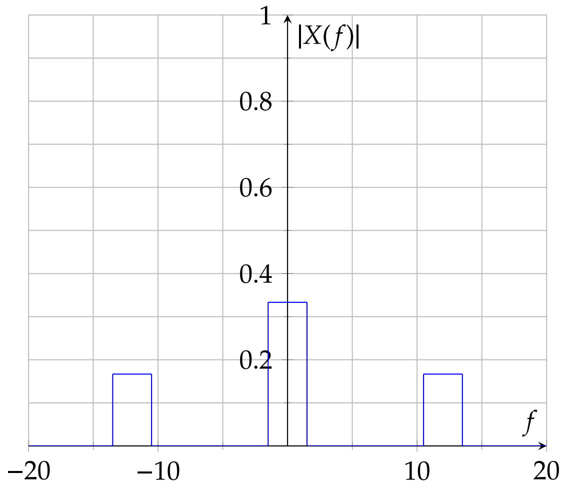


Figure 1.2: Plot of input signal.

We can compare this with the output signal. Again, using the modulation theorem and transform pair 2, we get

$$\begin{aligned} y(t) &= 12\text{sinc}(3t)\cos(24\pi t) = 4 \cdot 3\text{sinc}(3t)\cos(24\pi t) \\ Y(f) &= 2\Pi\left(\frac{f+12}{3}\right) + 2\Pi\left(\frac{f-12}{3}\right) \end{aligned}$$

which in the amplitude spectrum looks like as shown in figure 1.3:

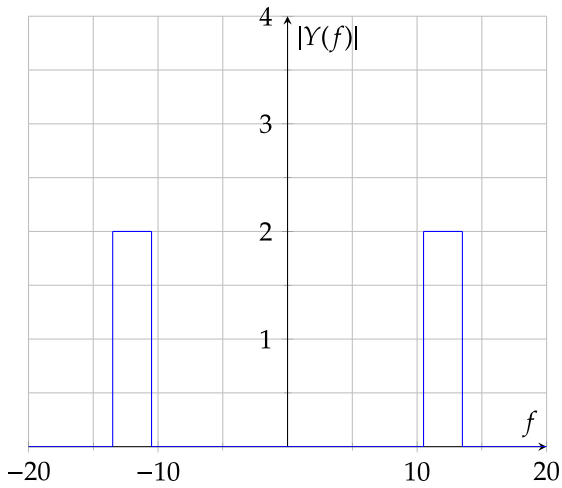


Figure 1.3: Plot of output signal.

Comparing the output with input, we quite clearly see that we have an ideal band-pass filter, centered around 12 Hz and with a bandwidth of 6 Hz (although the bandwidth could also have been 3 Hz or 9 Hz even) (and a gain of 12). If you don't immediately see why, consider plots of all filters: in blue the low-pass filter; in red the high-pass filter and in green the band-pass filter:

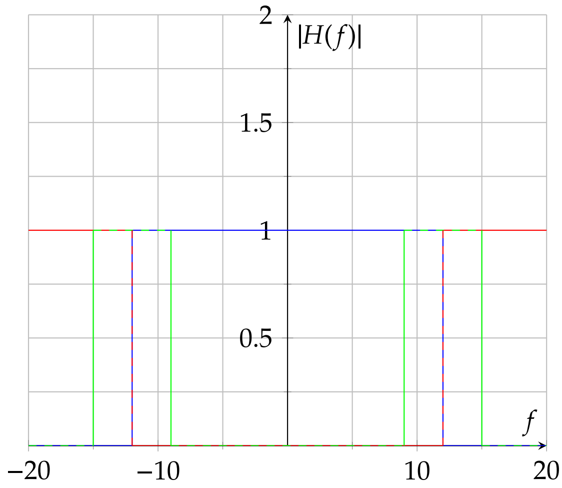


Figure 1.4: Plot of the three filters. The blue is the low-pass filter; the red the high-pass filter and the green the band-pass filter.

Now, if we'd use the blue low-pass filter, we'd keep the pulse in the middle of $X(f)$ in there, even though $Y(f)$ doesn't contain that pulse any more. Thus, it's not a low-pass filter. It's not a high-pass filter either; in that case, the pulses centred at $f = -12$ Hz and $f = 12$ Hz would have been partly filtered out (since the parts where $|f| < 12$ would have been filtered out), so it's not a high-pass filter either. On the contrary, a band-pass filter centred at 12 Hz with bandwidth of 6 Hz would completely preserve the pulses centred at ± 12 Hz but get rid of the one in the middle. If we'd impose a gain of 12, we make sure the amplitudes are correct too (although it's not asked here). In any case, answer B is correct.

1.7. Question 7

The correct answer is answer D. Note that we must rewrite the $\sin^2$ using the substitution $\sin^2(u) = \frac{1}{2} - \frac{1}{2}\cos(2u)$, so that we get

$$x(t) = 2 + 4\cos(10\pi t) + 16\sin^2(20\pi t) = 2 + 4\cos(10\pi t) + 8 - 8\cos(40\pi t)$$

The frequency of the first cosine is 5 Hz; of the second cosine it is 20 Hz. Thus, the harmonic frequency of the signal is 20 Hz and thus the Nyquist *rate* is 40 Hz, and thus answer A is definitely false. Furthermore, since we are sampling at 40 Hz, the Nyquist *frequency* is 20 Hz. Thus, answer B is definitely wrong as well, leaving only answer C and D being possible. The question remains: will there be aliasing? If we sample at 40 Hz, there will be a copy at 0 Hz that produces a peak at 20 Hz (due to the cosine with of 20 Hz, and a copy at 40 Hz that also produces a peak at 20 Hz (due to the cosine with frequency 20 Hz again). Thus, the spectra of neighbouring copies interfere and thus there is aliasing. Answer D is correct.

1.8. Question 8

The correct answer is B. Note that the cosine has a frequency of 40 Hz. This means that the copy at 0 Hz will produce a peak at 40 Hz. Furthermore, as the Fourier transform is

$$\begin{aligned} x(t) &= 3\cos(80\pi t) \\ X(f) &= \frac{3}{2}\delta(f-40) + \frac{3}{2}\delta(f+40) \end{aligned}$$

this will produce a peak with weight $80 \cdot \frac{3}{2} = 120$ (since the amplitudes are all multiplied by a factor f_s). However, there'll be another peak due to the copy at 80 Hz located at 40 Hz: this will double the weight to 240. Thus, answer B is correct.

1.9. Question 9

The correct answer is B. If we set $\alpha = 0.05$, then from the formula sheet, we see that $k_{\alpha_z} = 1.645$, so that

$$k_\alpha = \sigma \cdot k_{\alpha_z} + x_0 = 2 \cdot 1.645 - 2 = 1.29$$

Thus, we must find the probability

$$\beta = P(\mathbf{y} < 1.29 | H_1 \text{ is true})$$

If H_1 is true, that means that $\mu = 2$ and $\sigma = 2$, thus we get

$$\beta = P\left(\frac{\mathbf{y}-2}{2} < \frac{1.29-2}{2}\right) = P\left(\frac{\mathbf{y}-2}{2} < -0.355\right)$$

This probability can be easily looked up from the formula sheet due to symmetry; we find $\beta = 0.3613$ (look up $r_\alpha = 0.35$ and $r_\alpha = 0.36$ and take the average). Thus, answer B is correct.

1.10. Question 10

The correct answer is D. If we have

$$P(E) = 1 - \Phi\left(\sqrt{\frac{A^2 T_b}{2N_0}}\right)$$

then if we double A, then the argument of Φ becomes larger. As $\Phi()$ is the probability that a standard normal distribution ($\mu = 0$, $\sigma = 1$) is smaller than the thing between brackets, this means that this probability increases, and thus $P(E)$ decreases. However, nothing can be said about how much $P(E)$ decreases. Thus, answer D is correct. Note that this also makes sense: if we make the signal more powerful, it'll be easier to distinguish between 1s and 0s.

1.11. Question 11

The correct answer is B. We get the following values in decibels:

$$\begin{aligned}
(P_{T_x})_{\mathrm{dB}} &= 10\log_{10}(450) = 26.53\,\mathrm{dBW} \\
(G_{FS})_{\mathrm{dB}} &= 10\log_{10}\cdot\left(\frac{\frac{3\cdot 10^8}{1090\cdot 10^6}}{4\pi\cdot 100\cdot 1852}\right)^2 = -138.54\,\mathrm{dB}
\end{aligned}$$

Thus, answers C and D are definitely false. The power at the receiver's end is $26.53 - 138.54 = -112.01\,\mathrm{dBW}$, and thus answer B is correct.

1.12. Question 12

The correct answer is C. Per 4 meters, we have a gain of $10\log(0.98) = -0.08774$, i.e. $-0.08774/4 = -0.02193$ per meter. We need to get rid of 4 dB, so $-4/-0.02193 = 182.3585\,\mathrm{m}$ of cable is the maximum allowed.

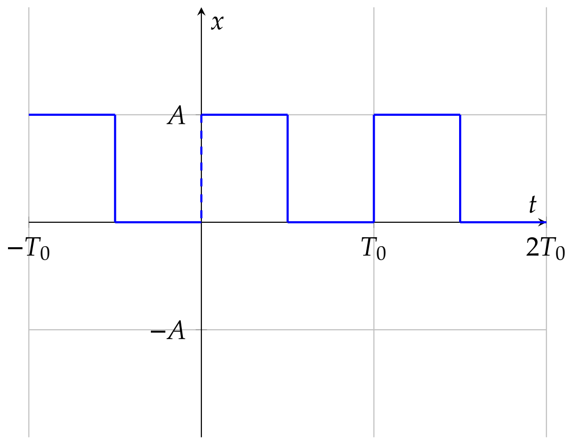


Figure 1.5: Square wave.

1.13. Question 13

We can sketch the wave as shown in figure 1.5. From a purely mathematical point of view, this wave is neither odd, even or half-wave odd. However, if we subtract the average value (the average is equal to $A/2$), from a practical point of view, the wave does become odd (as it becomes point symmetric with respect to the origin) and half-wave odd (when the wave is shifted half a period, the resultant wave is the negative of the original wave).

The average value can be computed as

$$X_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{T_0}\int_0^{T_0/2} A dt = \frac{A}{2}$$

Now, obtaining the complex Fourier series:

$$x(t) = \sum_{n=-\infty}^{n=\infty} X_n e^{nj\omega_0 t}$$

with $\omega_0 = 2\pi f = \frac{2\pi}{T}$. The X_n are given by

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)\,e^{-jn\omega_t}dt = \frac{1}{T_0}\int_0^{T_0/2} Ae^{-jn\omega_0 t}dt$$

For $n=0$, it was already computed that $X_0 = \frac{A}{2}$. For $n \neq 0$, we get

$$\begin{aligned} X_n &= \frac{1}{T_0}\int_0^{T_0/2} Ae^{-jn\omega_0 t}dt = \frac{1}{T_0}\left[\frac{A}{-jn\omega_0}e^{-jn\omega_0 t}\right]_0^{T_0/2} \\ &= \frac{jA}{2\pi n}\left(e^{-jn\frac{2\pi}{T_0}\frac{T_0}{2}} - 1\right) = \frac{jA}{2\pi n}\left(e^{-jn\pi} - 1\right) \end{aligned}$$

Here, for even n, $e^{-jn\pi} = 1$ so that it reduces to zero; for odd n, $e^{-jn\pi} = -1$, so that for odd n, we have

$$X_n = \frac{jA}{2\pi n}\cdot -2 = -\frac{jA}{\pi n}$$

Thus, to summarize: $X_0 = A/2$; $X_n = 0$ for even n and $X_n = -jA/(\pi n)$ for odd n.

1.14. Question 14

1.14.1. Part a)

For part a), we realize that the convolution of two unit pulses of width τ results in a triangle of width τ and height τ; this was shown during the lectures. Thus, to get the shown triangular signal (with width τ but height 1), we must have

$$\Lambda\left(\frac{t}{\tau}\right) = \frac{1}{\tau}\Pi\left(\frac{t}{\tau}\right)\star\Pi\left(\frac{t}{\tau}\right)$$

Convolution in the time domain becomes multiplication in the frequency domain; the Fourier transform of $\Pi\left(\frac{t}{\tau}\right)$ is simply

$$X(f) = \tau\text{sinc}(f\tau)$$

so that the Fourier transform of $\Lambda\left(\frac{t}{\tau}\right)$ becomes

$$Y(f) = \frac{1}{\tau}\cdot\tau\text{sinc}(f\tau)\cdot\tau\text{sinc}(f\tau) = \tau\text{sinc}^2(f\tau)$$

1.14.2. Part b)

The derivative of $\Lambda\left(\frac{t}{\tau}\right)$ can be described by

$$\frac{d\Lambda\left(\frac{t}{\tau}\right)}{dt} = \frac{1}{\tau}\Pi\left(\frac{t+\tau/2}{\tau}\right)-\frac{1}{\tau}\Pi\left(\frac{t-\tau/2}{\tau}\right)$$

We can use the fact the differentiation theorem to now compute the Fourier transform of $\Lambda\left(\frac{t}{\tau}\right)$; after all, the Fourier transform of the derivative of $\Lambda\left(\frac{t}{\tau}\right)$ will equal $j2\pi fX(f)$ with $X(f)$ the Fourier transform of $\Lambda\left(\frac{\tau}{t}\right)$. Thus, let us compute the Fourier transform of the derivative and let's denote it with $Y(f)$:

$$\begin{aligned}
y(t) &= \frac{1}{\tau}\Pi\left(\frac{t+\tau/2}{\tau}\right)-\frac{1}{\tau}\Pi\left(\frac{t-\tau/2}{\tau}\right)\\
Y(f) &= \frac{1}{\tau}\cdot\text{sinc}(\tau f)\cdot e^{-j2\pi f\cdot\frac{-\tau}{2}}-\frac{1}{\tau}\cdot\tau\text{sinc}(\tau f)\cdot e^{-j2\pi f\cdot\frac{\tau}{2}}\\
&= \text{sinc}(\tau f)e^{j\pi f\tau}-\text{sinc}(\tau f)e^{-j\pi f\tau} = \text{sinc}(\tau f)\cdot\sin(\pi f t)\cdot 2j = 2j\text{sinc}(\tau f)\cdot\frac{\sin(\pi f\tau)}{\pi f\tau}\cdot\pi f\tau\\
&= 2j\pi f\tau\text{sinc}^2(f\tau)
\end{aligned}$$

Thus, we must have

$$\begin{aligned}
Y(f) &= j2\pi fX(f)\\
2j\pi f\tau\text{sinc}^2(f\tau) &= j2\pi fX(f)
\end{aligned}$$

so that $X(f) = \tau\text{sinc}^2(f\tau)$.

1.15. Question 15

1.15.1. Part a)

The average power of the signal is

$$P_x = \frac{1}{2}\cdot\left(-3\cdot 10^{-7}\right)^2+\frac{1}{2}\cdot\left(3\cdot 10^{-7}\right)^2 = 9\times 10^{-14}\,\text{W}$$

The noise power is

$$P_e = k \cdot T \cdot B = 1.38 \cdot 10^{-23} \cdot 326 \cdot 5 \cdot 10^{6} = 2.2494 \times 10^{-14}\,\mathrm{W}$$

In decibels, this is equal to

$$\begin{aligned}
(P_x)_{\mathrm{dB}} &= 10\log_{10}\left(9 \cdot 10^{-14}\right) = -130.46\,\mathrm{dB} \\
(P_e)_{\mathrm{dB}} &= 10\log_{10}\left(2.2494 \cdot 10^{-14}\right) = -136.48\,\mathrm{dB}
\end{aligned}$$

Thus, the signal to noise ratio is equal to SNR $= -130.46 - -136.48 = 6.022\,\mathrm{dB}$.

1.15.2. Part b)

If you're interested in the whole derivation: we have

$$\text{reject } H_0 \text{ if } \frac{f_{\mathbf{y}}(y|H_0)}{f_{\mathbf{y}}(y|H_a)} < a$$

Setting $a = 1$ as α (false alarm) and β (missed detection) are equally valuable:

$$\begin{aligned}
\frac{f_{\mathbf{y}}(y|H_0)}{f_{\mathbf{y}}(y|H_a)} &< a \\
\frac{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-x_0)^2}}{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y-x_a)^2}} &< 1 \\
e^{-\frac{1}{2\sigma^2}\left((y-x_0)^2-(y-x_a)^2\right)} &< 1 \\
-\frac{1}{2\sigma^2}\left(y^2 - 2yx_0 + x_0^2 - y^2 + 2yx_a - x_a^2\right) &< 0 \\
2y(x_a - x_0) &> x_a^2 - x_0^2 \\
y &> \frac{(x_a - x_0)(x_a + x_0)}{2(x_a - x_0)} \\
y &> \frac{x_a + x_0}{2} \\
\frac{y - x_0}{\sigma} &> \frac{x_a - x_0}{2\sigma} = \frac{\nabla}{2\sigma}
\end{aligned}$$

Looking at the second-to-last equation, this decision rule makes sense: if the observation is larger than the average value of the amplitudes, it is decided that it is a '1'; if it is lower, it is decided that it is a '0'.

1.15.3. Part c)

So, we reject H_0 if

$$\frac{y - x_0}{\sigma} > \frac{\nabla}{2\sigma}$$

Then, the bit-error-probability is

$$P(E) = P(E|H_0)P(H_0) + P(E|H_1)P(H_1) = \alpha P(H_0) + \beta P(H_1)$$

Our test was based upon valuing α and β equally, thus $\alpha = \beta$. Furthermore, the probability that a bit is a 0 is the same as the probability that a bit is a 1, thus $P(H_0) = P(H_1) = \frac{1}{2}$. Thus,

$$P(E) = \alpha \cdot \frac{1}{2} + \alpha \cdot \frac{1}{2} = \alpha$$

α is equal to

$$P\left(\frac{y-x_0}{\sigma}>\frac{\nabla}{2\sigma}\right)=1-\Phi\left(\frac{\nabla}{2\sigma}\right)$$

In this case, $\nabla=||x_a-x_0||=0.3--0.3=0.6\mu\mathrm{V}$. With $\sigma=\sqrt{P_e}=\sqrt{kTB}=\sqrt{1.38\cdot10^{-23}\cdot326\cdot5\cdot10^6}=1.5\times10^{-7}\,\mathrm{V}$. Thus, we get

$$P(E)=\alpha=1-\Phi\left(\frac{0.3\cdot10^{-6}}{1.5\cdot10^{-7}}\right)=1-\Phi(2)$$

Then (from formula sheet)

$$\Phi(2)\approx1-0.0228=0.9772$$

so that

$$P(E)=1-0.9972=0.0228$$

Part II

Exam August 2014

1.16. Question 1

Answer B is correct. All a_m are zero; this means that the signal is *odd* and not even, thus statement I is false. As $a_0 = 0$, the average of $x(t)$ is zero as well, so statement II is true. Furthermore, if $x(t)$ would have half-wave odd symmetry, then all even-indexed coefficients would have been zero; this is not the case so statement III is false.

1.17. Question 2

Answer A is correct. We first rewrite it to

$$x(t) = 4\Pi(4(t+2.5))$$

The Fourier transform of $\Pi(4t)$ is $\frac{1}{4}\mathrm{sinc}\left(\frac{f}{4}\right)$. Using the time-delay theorem, we then get for $\Pi(4(t+2.5))$ that the Fourier transform becomes

$$\frac{1}{4}\mathrm{sinc}\left(\frac{f}{4}\right)e^{-j2\pi f\cdot -2.5} = \frac{1}{4}\mathrm{sinc}\left(\frac{f}{4}\right)e^{j5\pi f}$$

So, in the end, we have

$$\begin{aligned} x(t) &= 4\Pi(4(t+2.5)) \\ X(f) &= \mathrm{sinc}\left(\frac{f}{4}\right)e^{j5\pi f} \end{aligned}$$

1.18. Question 3

Answer D is correct. $x(t)$ is clearly odd, so you need sines and thus purely imaginary components.

1.19. Question 4

Answer X is correct. Again, it's fundamental to sketch the signal, as done in figure 1.10. Note that we must rewrite the step function to $u(-(t+2))$; i.e. it is a step that starts at $t=-2$ and then progresses towards minus infinity (so the opposite direction).

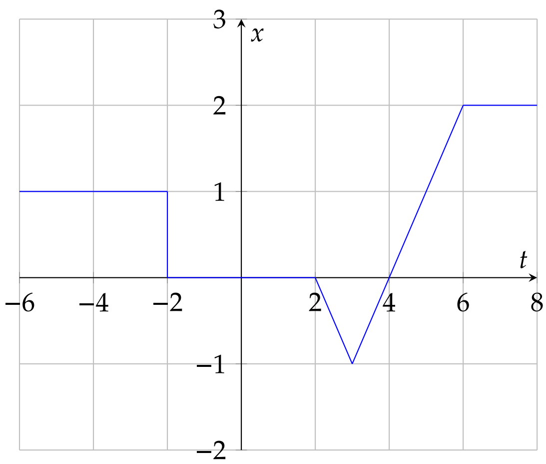


Figure 1.6: Sketch of signal.

This is a power signal: its value remains bounded as $t \to \pm\infty$ (but does not equal zero for both of these extremes). Half of time, the signal strength is $x = 1$; the other half of the time the signal strength is $x = 2$; thus, the average power is

$$P_{av} = \frac{1}{2} \cdot 1^2 + \frac{1}{2} \cdot 2^2 = 2.5\,\mathrm{W}$$

1.20. Question 5

Answer C is correct: it's non-linear due to the $y(t)d^2y/dt^2$; it's time varying due to the $3t \cdot dy/dt$ and it's non-causal as we require the input at a time $t+5$.

1.21. Question 6

Answer D is correct. We first rewrite the $\cos^2$ using $\cos^2(u) = \frac{1}{2} + \cos(2u)$ to (we can completely ignore the phases)

$$x(t) = 5 + 5\cos(64\pi t) - 5\sin(100\pi t)$$

The first term will produce a peak of weight 5 at $f = 0$; the cosine term will produce a peak of weight $5/2 = 2.5$ at $f = -32\,\mathrm{Hz}$ and $f = 32\,\mathrm{Hz}$; the sine term will produce a peak of weight $5/2 = 2.5$ at $f = -50\,\mathrm{Hz}$ and $f = 50\,\mathrm{Hz}$. Thus, the power of the input signal will be

$$2.5^2 + 2.5^2 + 5^2 + 2.5^2 + 2.5^2 = 50\,\mathrm{W}$$

Thus, answer C is incorrect. With the band-pass filter, only the peaks at $-32\,\mathrm{Hz}$ and $32\,\mathrm{Hz}$ remain, so the power of the output signal will be

$$2.5^2 + 2.5^2 = 12.5\,\mathrm{W}$$

Thus, answer D is correct.

1.22. Question 7

Answer D is correct. Since we're sampling at $f_s = 40\,\mathrm{Hz}$, we get copies at $40\,\mathrm{Hz}$, $80\,\mathrm{Hz}$, $-40\,\mathrm{Hz}$, etc. So, spectrum (I) is definitely wrong. These copies also have their amplitudes *multiplied* by a factor $f_c = 40$; thus, the peaks should now be $40 \cdot 5 = 200$. Thus, neither spectrum (II) nor (III) are correct. Thus, answer D is correct.

1.23. Question 8

Answer A is correct. Again, we can just totally disregard the phase. The cosine has a frequency of 3 Hz, the sine a frequency of 12 Hz. If we sample at 8 Hz, then the copy at 0 Hz will yield peaks at -12 Hz, -3 Hz, 3 Hz and 12 Hz. The copy at -8 Hz, will yield peaks at -20 Hz, -11 Hz, -5 Hz and 4 Hz. The copy at -16 Hz will not produce any peaks between 0 and 10 Hz.

The copy at 8 Hz will yield peaks at -4 Hz, 5 Hz, 11 Hz and 20 Hz. The copy at 16 Hz will yield peaks at 4 Hz, 13 Hz, 19 Hz and 28 Hz. The copy at 24 Hz will no longer yield peaks between 0 and 10 Hz.

In total, we will get peaks at 3, 4 and 5 Hz, so answer A is correct.

1.24. Question 9

The correct answer is C. If $\beta = 0.4840$, then k_β (this is also equal to k_α must have been such that

$$P\left(\mathbf{y} < k_\beta | H_1 \text{ is true}\right) = 0.4840$$

If H_1 is true, then $\mu = 2$ and $\sigma = 1$, and thus we have to solve

$$P\left(\frac{\mathbf{y}-2}{1} < \frac{k_\beta - 2}{1}\right) = 0.4840$$

We find $k_{\beta_z} = 0.04$. Now, to compute k_β, you have to compute that this must be located to the *left* of the mean (this makes sense looking at figure **??**). Thus, we now get $k_\beta = \mu - \sigma k_{\beta_z} = 2 - 1 \cdot 0.04 = 1.96 = k_\alpha$. Then,

$$\alpha = P\left(\mathbf{y} > k_\alpha | H_0 \text{ is true}\right) = P\left(\frac{\mathbf{y}-0}{1} > \frac{1.96-0}{1}\right) = P\left(\mathbf{y} > 1.96\right)$$

since H_0 means $\mu = 0$ and $\sigma = 1$. This probability can be straightforwardly looked up to equal 0.0250, and thus answer C is correct.

1.25. Question 10

The correct answer is C. Again, we find $x_a - x_0$:

$$x_a - x_0 = \begin{bmatrix} A \\ \vdots \\ A \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ \vdots \\ A \end{bmatrix} = A\vec{u}_m$$

where $\vec{u}_m$ is a vector of m entries, containing only ones. Then,

$$||x_a - x_0|| = \sqrt{A^2 + A^2 + ... + A^2} = \sqrt{m}A$$

so that

$$c = \frac{A\vec{u}_m}{A\sqrt{m}} = \frac{1}{\sqrt{m}}\vec{u}_m$$

Furthermore,

$$y - x_0 = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ v_m \end{bmatrix}$$

Then we get

$$\begin{aligned} \frac{c^T\left(y - x_0\right)}{\sigma} &> k_\alpha \\ \frac{1}{\sqrt{m}\sigma}\vec{u}_m^T\left(y - x_0\right) = \frac{1}{\sqrt{m}\sigma}\left(1 \cdot y_1 + 1 \cdot y_2 + ... + 1 \cdot y_m\right) &> k_\alpha \\ \frac{1}{\sqrt{m}\sigma}\sum_{i=1}^{m} y_i &> k_\alpha \end{aligned}$$

Since $\sigma = 1$, this means that answer C is correct.

1.26. Question 11

The correct answer is A. In the case A is halved, the argument

$$\sqrt{\frac{A^2 T_b}{2N_0}}$$

decreases as well. This means that $\Phi\left(\sqrt{\frac{A^2 T_b}{2N_0}}\right)$ decreases (as there is a smaller probability that a standard normal distribution will be smaller than this value), thus $P(E)$ increases.

1.27. Question 12

The correct is answer C. Per 2 meters, we lose $10\log(0.98) = -0.0877$ decibels, or -0.0439 dB per meter. As we are allowed to lose 5 dB, this means the maximum length is $5/0.0439 = 113.97$ m. Thus, answer C is correct.

1.28. Question 13

1.28.1. Part a)

Sketching the wave yields figure 1.7. Clearly, from inspection, the average value is $1 + A/2$. It can also be computed:

$$X_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{T_0}\int_0^{T_0/2} A\,dt + \frac{1}{T_0}\int_0^{T_0} 1\,dt = \frac{A}{2} + 1$$

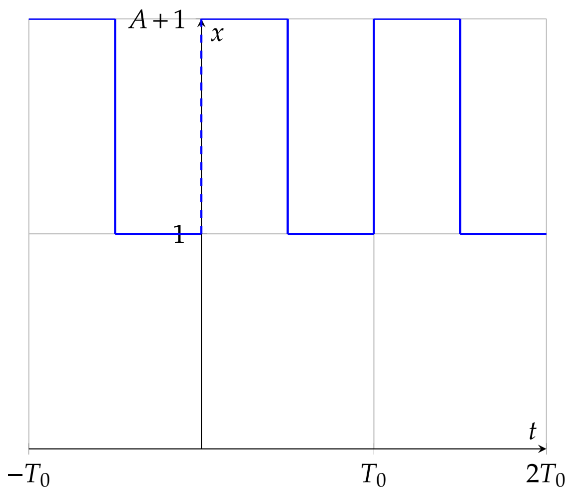


Figure 1.7: Square wave.

1.28.2. Part b)

Looking at figure 1.7, the wave is neither odd, even, nor half-wave odd. However, if we subtract the average value $(1 + A/2)$, the wave becomes both odd (as the wave is point symmetric around the origin) and half-wave odd (if the wave is shifted half a period, the new wave is the negative of the old wave).

1.28.3. Part c)

We get

$$x(t) = \sum_{n=-\infty}^{n=\infty} X_n e^{jn\omega_0 t}$$

where

$$X_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-jn\omega_0 t}\,dt, \qquad n = \ldots,-3,-2,-1,0,1,2,3,\ldots$$

For $n = 0$, this reduces to

$$X_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{T_0}\int_0^{T_0/2} A\,dt + \frac{1}{T_0}\int_0^{T_0} 1\,dt = \frac{A}{2} + 1$$

For $n \neq 0$, we get

$$X_n = \frac{1}{T_0}\int_0^{T_0/2} Ae^{-jn\omega_0 t}dt + \frac{1}{T_0}\int_0^{T_0} e^{-jn\omega_0 t}dt$$

The second integral decreases to 0 as

$$e^{-jn\omega_0 t} = \cos(-\omega_0 t) + j\sin(-\omega_0 t) = \cos(\omega_0 t) - j\sin(\omega_0 t)$$

and integrating this over a full period yields 0. Then:

$$\begin{aligned} X_n &= \frac{1}{T_0}\int_0^{T_0/2} Ae^{-jn\omega_0 t}dt = \frac{1}{T_0}\left[\frac{A}{-jn\omega_0}e^{-jn\omega_0 t}\right]_0^{T_0/2} \\ &= \frac{jA}{2\pi n}\left(e^{-jn\frac{2\pi}{T_0}\frac{T_0}{2}} - 1\right) = \frac{jA}{2\pi n}\left(e^{-jn\pi} - 1\right) \end{aligned}$$

Here, for even n, $e^{-jn\pi} = 1$ so that it reduces to zero; for odd n, $e^{-jn\pi} = -1$, so that for odd n, we have

$$X_n = \frac{jA}{2\pi n}\cdot -2 = -\frac{jA}{\pi n}$$

Thus, to summarize: $X_0 = A/2 + 1$; $X_n = 0$ for even n and $X_n = -jA/(\pi n)$ for odd n.

1.29. Question 14

For a), we have

$$X(f) = 4\Pi\left(\frac{f+750}{500}\right) + 4\Pi\left(\frac{f-750}{500}\right)$$

Why? For the left block, we have an amplitude of 4, the width is 500 Hz and it is shifted 750 Hz to the left. For the right block, we get the same reasoning, although it is now shifted 750 Hz to the right. Using the modulation theorem, we see that this is akin to the modulation theorem, which states that

$$y(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}Y(f - f_0) + \frac{1}{2}Y(f + f_0)$$

Thus, $y(t)$ would be the inverse transform of $8\Pi\left(\frac{f}{500}\right)$, and $\omega_0 = 2\pi\cdot 750$. The inverse transform of $8\Pi\left(\frac{f}{500}\right)$ is, per pair 2:

$$y(t) = 8\cdot 500\,\mathrm{sinc}(500t) = 4000\,\mathrm{sinc}(500t)$$

Thus, we have

$$x(t) = y(t)\cos\omega_0 t = 4000\,\mathrm{sinc}(500t)\cdot\cos(2\pi\cdot 750t)$$

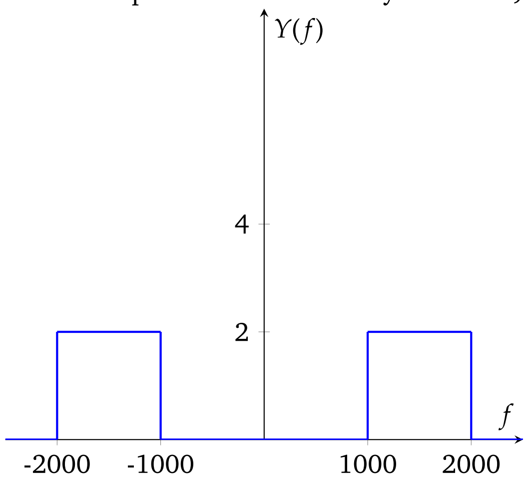
For b), it's much much easier to use Parseval's theorem for this:

$$E = \int_{-\infty}^{\infty} |X(f)|^2 \, df$$

We see that $|X(f)| = 0$ for almost all f, except for $-1000 < f < -500$ and $500 < f < 1000$, where $|X(f)| = 4$, and thus $|X(f)|^2 = 16$. We then have that the energy equals

$$E = 500 \cdot 16 + 500 \cdot 16 = 16000 \, \text{J}$$

For c), we get the graph shown below: according to the scale change theorem, if we compress the signal in time (with a factor 2 in this case), the signal is expanded in time (meaning that the frequencies are 'stretched out' by a factor 2, and the amplitude is decreased by a factor 2).



For d), we now see that $Y(f)$ is equal to zero almost everywhere, except between $-2000 < f < -1000$ and $1000 < f < 2000$, where $|Y(f)| = 2$ and thus $|Y(f)|^2 = 4$. Thus, the energy equals

$$E = 1000 \cdot 4 + 1000 \cdot 4 = 8000 \, \text{J}$$

1.30. Question 15

1.30.1. Part a)

In decibels, we get the following quantities:

$$\begin{aligned}
(P_{T_x})_{\text{dB}} &= 10 \log(250) = 23.979 \, \text{dBW} \\
(G_{T_x})_{\text{dB}} &= 10 \log(10) = 10 \, \text{dB} \\
(G_{AT})_{\text{dB}} &= 10 \log(0.5) = -3.0103 \, \text{dB} \\
(G_{AR})_{\text{dB}} &= 10 \log(0.5) = -3.0103 \, \text{dB} \\
(G_{FS})_{\text{dB}} &= 10 \log\left(\left(\frac{\frac{3 \cdot 10^8}{1090 \cdot 10^6}}{4\pi \cdot 670000}\right)^2\right) = -149.71 \, \text{dB}
\end{aligned}$$

Thus, the signal power received is

$$(P_{R_x})_{\text{dB}} = 23.979 + 10 - 3.0103 - 3.0103 - 149.71 = -121.75 \, \text{dBW}$$

1.30.2. Part b)

Simply

$$(P_e)_{\text{dB}} = 10\log(kTB) = 10\log\left(1.38\cdot10^{-23}\cdot362\cdot2\cdot10^{6}\right) = -140\,\text{dBW}$$

1.30.3. Part c)

The signal-to-noise ratio is thus simply $-121.75 - -140 = 18.25\,\text{dB}$. The required $18\,\text{dB}$ is thus met.

1.30.4. Part d)

We must then have

$$(P_{T_x})_{\text{dB}} + (G_{T_x})_{\text{dB}} + (G_{AT})_{\text{dB}} + (G_{AR})_{\text{dB}} + (G_{FS})_{\text{dB}} - (P_e)_{\text{dB}} = 18\,\text{dB}$$

Thus:

$$\begin{aligned} 23.9797 + 10 - 3.0103 - 3.0103 + (G_{FS})_{\text{dB}} - -140 &= 18 \\ G_{FS} &= -149.9591\,\text{dB} \end{aligned}$$

Thus, we get

$$\begin{aligned} \left(\frac{\frac{3\cdot10^{8}}{1090\cdot10^{6}}}{4\pi d}\right)^{2} &= 10^{-149.9591/10} \\ \frac{4.797\cdot10^{-4}}{d^{2}} &= 1.009\cdot10^{-15} \\ d &= 689\,350\,\text{m} = 689.350\,\text{km} \end{aligned}$$

Part III

January 2013

1.31. Question 1

The correct answer is b). Let's go through the statements one by one. Statement I is obviously false: X_0 is the average value of the signal, which is rather clearly equal to 0. Statement II is false: if $b=0$, then the function is neither odd nor even, thus the coefficients won't be purely imaginary. Statement III is false: the function is half-wave odd (shift the wave a distance $\frac{T_0}{2}$; the resulting wave is simply the negative of the original signal), which means that all *even*-indexed coefficients will be zero. Thus, b) is correct.

1.32. Question 2

The correct answer is C. First, $X(f)$ is simply $6\cdot 2\mathrm{sinc}(2f) = 12\mathrm{sinc}(2f)$ (see pair 2 of Fourier Transform Pairs). Then, $Z(t) = \frac{1}{2}x(t)$ would become $Z(f) = 6\mathrm{sinc}(2f)$. However, the scale change makes it (see theorem 3a, with $a = -\frac{1}{2}$)

$$Y(f) = \frac{1}{\left|-\frac{1}{2}\right|}\cdot Z\left(\frac{f}{-\frac{1}{2}}\right) = 2\cdot 6\mathrm{sinc}(2f\cdot -2) = 12\mathrm{sinc}(4f)$$

where the minus sign disappears due to the fact that sinc is even (i.e. $\mathrm{sinc}(-4f) = \mathrm{sinc}(4f)$.

1.33. Question 3

The correct answer is A. $x(t)$ is as follows:

$$x(t) = 2\Pi\left(\frac{t}{3}\right) - \Pi(t)$$

as we have a positive pulse of amplitude 2 with width 3, and a negative pulse with amplitude 1 with width 1. Using pair 1 we simply get

$$X(f) = 2\cdot 3\mathrm{sinc}(3f) - \mathrm{sinc}(f) = 6\mathrm{sinc}(3f) - \mathrm{sinc}(f)$$

and thus answer A is correct.

1.34. Question 5

The correct answer is C. Using almost exactly the same derivation as above, you can show that S1 and S3 are both linear. Now, how do we prove S2 is non-linear? Well, we have

$$\begin{aligned}
\frac{dy_1}{dt} + y_1 + 1 &= x_1 \\
\frac{dy_2}{dt} + y_2 + 2 &= x_2 \\
\alpha_1\frac{dy_1}{dt} + \alpha_2\frac{dy_2}{dt} + \alpha_1 y_1 + \alpha_2 y_2 + \alpha_1 + \alpha_2 &= \alpha_1 x_1 + \alpha_2 x_2 \\
\frac{d}{dt}(\alpha_1 y_1 + \alpha_2 y_2) + (\alpha_1 y_1 + \alpha_2 y_2) + (\alpha_1 + \alpha_2) &= (\alpha_1 x_1 + \alpha_2 x_2)
\end{aligned}$$

Now there's something messed up: we no longer get the 1 there, but $\alpha_1 + \alpha_2$. Had you plugged in $y = \alpha_1 y_1 + \alpha_2 y_2$ directly, you would have just gotten 1 there, so the system is non-linear. There, answer C is correct.

1.35. Question 6

The correct answer is A. Again, we can safely ignore the phases. We must then rewrite $\sin^2(60\pi t)$ to $1/2 - 1/2\cos(120\pi t)$, so that we get

$$x(t) = 2 + 5\cos(40\pi t) - 1 + \cos(120\pi t) = 1 + 5\cos(40\pi t) + \cos(120\pi t)$$

The first cosine has a frequency of 20 Hz; the second has a frequency of 60 Hz; thus, the Fourier transform becomes

$$X(f) = 1 + \frac{5}{2}\delta(f+20) + \frac{5}{2}\delta(f-20) + \frac{1}{2}\delta(f+60) + \frac{1}{2}\delta(f-60)$$

so that the amplitude spectrum simply looks like

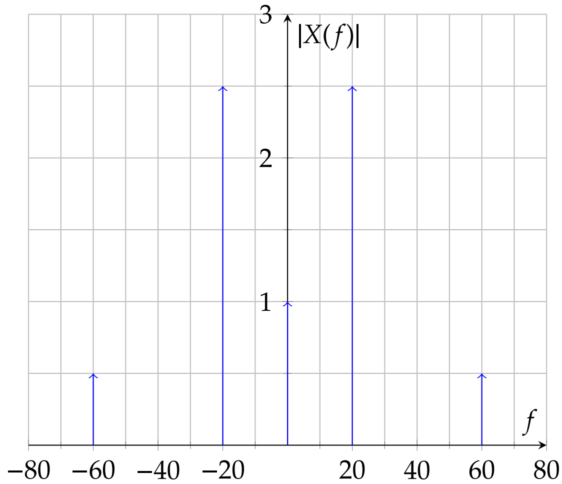


Figure 1.8: Signal in frequency domain.

Thus, the power of the input signal is

$$P_x = 0.5^2 + 2.5^2 + 1^2 + 2.5^2 + 0.5^2 = 14\,\mathrm{W}$$

Thus, answer B is incorrect. We can also plot the filter:

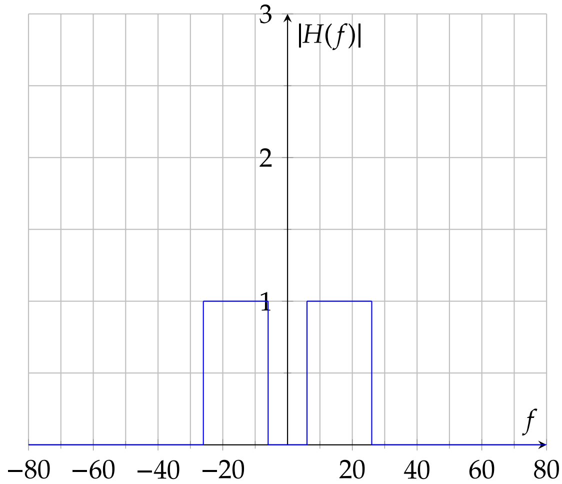


Figure 1.9: Plot of filter.

Multiplying figure 1.8 with 1.9 leads to the following output for $Y(f)$:

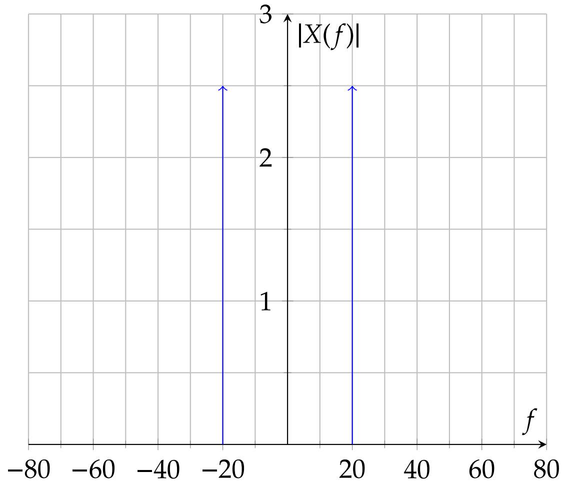


Figure 1.10: Output signal in frequency domain.

So the power of the output signal is

$$P_y = 2.5^2 + 2.5^2 = 12.5\,\text{W}$$

This means that neither C or D are correct either, and thus answer A is correct. Just goes to show that "None of the other answers is correct" is occasionally the correct one too.

1.36. Question 7

The correct answer A. For this, it is helpful to first draw the Fourier domain of the original signal. For this, we must first compute the Fourier transform. Using pair 2 and the modulation theorem, we simply get

$$\begin{aligned} x(t) &= \frac{5}{2} \cdot 2\text{sinc}(2t)\cos(10\pi t) \\ X(f) &= \frac{5}{2}\Pi\left(\frac{f-5}{2}\right) + \frac{5}{2}\Pi\left(\frac{f+5}{2}\right) \end{aligned}$$

We can sketch this as shown below (remember that the pulse width is 2 Hz):

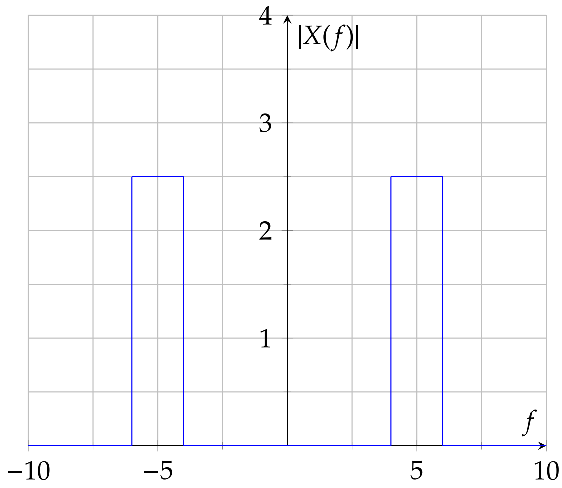


Figure 1.11: Plot of input signal.

Now, you can clearly see that the bandwidth of this signal is $f_h = 6\,\text{Hz}$; this means that the Nyquist rate of this signal is 12 Hz and thus answer A is correct.

1.37. Question 8

The correct answer is C: note that the frequency of the first cosine is 5 Hz, and the frequency of the second cosine is 10 Hz. Since we're sampling at 40 Hz, we can ignore aliasing. Thus, the first peak will already appear at 5 Hz and thus answer A is wrong. The second peak does indeed appear at 10 Hz, but its weight is not equal to 20: as we're transforming to the frequency domain, the 2 would normally be halved to 1 (see pair 12 of the formula sheet). As we sample at 40 Hz, the weight would thus become $1 \cdot 40 = 40$, and not 20. Thus, answer B is wrong.

For C, let's just count. Due to the copy at 0 Hz, we have a peak at 5 Hz and 10 Hz. Due to the copy at 40 Hz, we have a peak at 30 Hz, 35 Hz, 45 Hz and 50 Hz (since (co)sines also produce peaks at their negative frequencies). Due to the copy at 80 Hz, we have a peak at 70 Hz, 75 Hz, 85 Hz and 90 Hz. The first peak due to the copy at 120 Hz only occurs at 110 Hz, thus indeed, we have ten peaks between 0 and 100 Hz, and thus answer C is correct.

1.38. Question 9

The correct answer is D. This equation is very similar to the example calculations of β I showed before. The first step is to determine the critical value. We must have

$$P(\mathbf{y} > k_\alpha | H_0 \text{ is true}) = 0.05$$

This assume H_0 is true, i.e. $\mu = -1$ and $\sigma = 2$. This means that we get the calculation

$$P(\mathbf{y} > k_\alpha) = P\left(\frac{\mathbf{y}\mathbf{-1}}{2} > \frac{k_\alpha - -1}{2}\right) = P\left(\frac{\mathbf{y+1}}{2} > \frac{k_\alpha + 1}{2}\right) = 0.05$$

From the table, we get $k_{\alpha_z} = 1.645$; this means that $k_\alpha = k_{z_\alpha}\sigma + \mu = 1.645 \cdot 2 - 1 = 2.29$. Then, computing β amounts to

$$P(\mathbf{y} < k_\alpha | H_1 \text{ is true})$$

where $k_\alpha = 2.29$; since H_1 is now true, $\mu = 1$ and $\sigma = 2$. Thus, we get

$$P(\mathbf{y} < 2.29) = P\left(\frac{\mathbf{y\text{-}1}}{2} < \frac{2.29-1}{2}\right) = P\left(\frac{\mathbf{y}-1}{2} < 0.645\right) = 1 - P\left(\frac{\mathbf{y}-1}{2} \geq 0.645\right)$$

From the table, we see that the latter probability is 0.2595 (averaging between $r_\alpha = 0.64$ and $r_\alpha = 0.65$), and thus $\beta = 1 - 0.2595 = 0.7406$ and thus answer D is correct.

1.39. Question 10

The correct answer is A. In the case T_b is halved, the argument

$$\sqrt{\frac{2A^2 T_b}{N_0}}$$

decreases as well. This means that $\Phi\left(\sqrt{\frac{2A^2T_b}{N_0}}\right)$ decreases (as there is a smaller probability that a standard normal distribution will be smaller than this value), thus $P(E)$ increases.

1.40. Question 11

The correct answer is B. Like I said, they give you the decision rule, which is nice. Then you just have to see it in practice once, so that you have an idea what to do. First, we focus on finding c^T: for that, we need

$$x_a - x_0 = \begin{bmatrix} A \\ \vdots \\ A \end{bmatrix} - \begin{bmatrix} -A \\ \vdots \\ -A \end{bmatrix} = \begin{bmatrix} 2A \\ \vdots \\ 2A \end{bmatrix} = 2A\vec{u}_m$$

where $\vec{u}_m$ is a vector containing only 1s, of size $m \times 1$. Then,

$$||x_a - x_0|| = \sqrt{(2A)^2 + (2A)^2 + ... + (2A)^2} = 2A \cdot \sqrt{m}$$

since we have m such entries. Thus,

$$c = \frac{2A\vec{u}_m}{2A\sqrt{m}} = \frac{1}{\sqrt{m}}\vec{u}_m$$

Then, $y - x_0$ is given by

$$y - x_0 = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} - \begin{bmatrix} -A \\ \vdots \\ -A \end{bmatrix} = \begin{bmatrix} y_1 + A \\ \vdots \\ y_m + A \end{bmatrix}$$

Then, we get

$$\begin{aligned} \frac{c^T(y-x_0)}{\sigma} &> k_\alpha \\ \frac{1}{\sqrt{m}\sigma}\left(u_m^T(y-x_0)\right) = \frac{1}{\sqrt{m}\sigma}(1\cdot(y_1+A) + 1\cdot(y_2+A) + ... + 1\cdot(y_m+A)) &> k_\alpha \end{aligned}$$

since $u_m^T(y - x_0)$ is simply the dotproduct and u_m contains only 1s. We can rewrite this to

$$\begin{aligned}
\frac{1}{\sqrt{m}\sigma}\left(1\cdot(y_1+A)+1\cdot(y_2+A)+...+1\cdot(y_m+A)\right) &> k_\alpha \\
\frac{1}{\sqrt{m}\sigma}\left(\sum_{i=1}^{m} y_i + m\cdot A\right) &> k_\alpha \\
\sum_{i=1}^{m} y_i + m\cdot A &> k_\alpha\sigma\sqrt{m} \\
\sum_{i=1}^{m} y_i &> k_\alpha\sigma\sqrt{m} - mA
\end{aligned}$$

and thus answer B is correct.

1.41. Question 12

The correct answer is C. Per 2 meters, we have loss of $10\log(0.99) = -0.0436$ dB, or a loss of 0.0218 dB per meter. This means that to lose 2 dB, we can have a length of $2/0.0218 = 91.642\,\mathrm{m}$ and thus answer C is correct.

1.42. Question 13

The wave looks like as shown in figure 1.12.

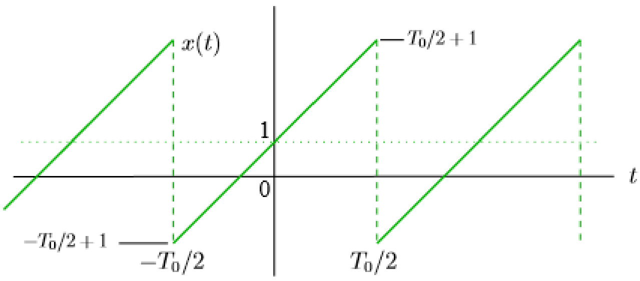


Figure 1.12: Saw-tooth wave.

As the average is not equal to zero, from a strictly mathematical point of view, this wave is neither odd, even nor half-wave odd (even though it becomes odd when you shift it down a distance 1). Furthermore, clearly from the sketch the signal is not even, as $x(t) \neq x(-t)$ (note: you should *always* use the strict mathematical definitions to explain whether it is odd, half-wave odd or even).

Now, note that for this particular signal, if you subtract the average, the signal becomes odd (but does not become halfwave odd). It is clear from the sketch that the average is 1 (if you draw a clear sketch, it is allowed to deduce the average from the sketch, otherwise you have to compute it using the integral).

Furthermore, since a_0 equals the average, we have that $a_0 = 1$. Furthermore, since a_0 takes into account the average value of the function, we can act as if the signal is odd for the remaining calculations (as the other coefficients work on capturing the signal oscillation around the average). Now, for b_m, we have

$$\begin{aligned}
b_m &= \frac{2}{T_0}\int_{T_0} x(t)\sin(m\omega_0 t)\,dt = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2}(1+t)\sin(m\omega_0 t)\,dt \\
&= \frac{2}{T_0}\int_{-T_0/2}^{T_0/2}\sin(m\omega_0 t)\,dt + \int_{-T_0/2}^{T_0/2} t\sin(m\omega_0 t)\,dt
\end{aligned}$$

Now, before you're like, ohh noooo now I have to do integration by parts, good time to take a break, note that your formula sheet is 8 pages long and contains actually useful formulas. On page 5, a list of standard integrals is given, among which

$$\int x\sin(ax)\,dx = \left[\sin(ax) - ax\cos(ax)\right]/a^2$$

Thus, we straightforwardly get

$$\begin{aligned}
b_m &= \frac{2}{T_0}\left(\int_{-T_0/2}^{T_0/2}\sin(m\omega_0 t)\,dt + \int_{-T_0/2}^{T_0/2} t\sin(m\omega_0 t)\,dt\right) \\
&= \frac{2}{T_0}\left(\left[-\frac{\cos(m\omega_0 t)}{m\omega_0}\right]_{-T_0/2}^{T_0/2} + \frac{\left[\sin(m\omega_0 t) - m\omega_0 t\cos(m\omega_0 t)\right]_{-T_0/2}^{T_0/2}}{m^2\omega_0^2}\right)
\end{aligned}$$

which is still rather ugly. Substituting in $\omega_0 = \frac{2\pi}{T_0}$ gives us

$$b_m = \frac{2}{T_0}\left(\left[-\frac{T_0\cos\left(\frac{2m\pi t}{T_0}\right)}{2m\pi}\right]_{-T_0/2}^{T_0/2} + \left[\frac{T_0^2}{4m^2\pi^2}\sin\left(\frac{2m\pi t}{T_0}\right)\right]_{-T_0/2}^{T_0/2} - \left[\frac{tT_0}{2m\pi}\cos\left(\frac{2m\pi t}{T_0}\right)\right]_{-T_0/2}^{T_0/2}\right)$$

Now, if we work that out, we get:

$$\left[-\frac{T_0\cos\left(\frac{2m\pi t}{T_0}\right)}{2m\pi}\right]_{-T_0/2}^{T_0/2} = -\frac{T_0}{2m\pi}\left[\cos\left(\frac{2m\pi}{T_0}\cdot\frac{T_0}{2}\right) - \cos\left(\frac{2m\pi}{T_0}\cdot -\frac{T_0}{2}\right)\right] = 0$$

as $\cos(x) = \cos(-x)$. Furthermore,

$$\begin{aligned}
\left[\frac{T_0^2}{4m^2\pi^2}\sin\left(\frac{2m\pi t}{T_0}\right)\right]_{-T_0/2}^{T_0/2} &= \frac{T_0^2}{4m^2\pi^2}\left(\sin\left(\frac{2m\pi}{T_0}\cdot\frac{T_0}{2}\right) - \sin\left(\frac{2m\pi}{T_0}\cdot -\frac{T_0}{2}\right)\right) \\
&= \frac{T_0^2}{4m^2\pi^2}\left(\sin(m\pi) - \sin(-m\pi)\right) = 0
\end{aligned}$$

as $\sin(m\pi)$ and $\sin(-m\pi)$ are both zero for integer values of m. Finally, we have

$$\begin{aligned}
\left[\frac{tT_0}{2m\pi}\cos\left(\frac{2m\pi t}{T_0}\right)\right]_{-T_0/2}^{T_0/2} &= \frac{T_0}{2m\pi}\left(\frac{T_0}{2}\cos\left(\frac{2m\pi}{T_0}\cdot\frac{T_0}{2}\right) - -\frac{T_0}{2}\cos\left(\frac{2m\pi}{T_0}\cdot\frac{-T_0}{2}\right)\right) \\
&= \frac{T_0^2}{4m\pi}\cdot\left(\cos(m\pi) + \cos(-m\pi)\right) = \frac{T_0^2}{2m\pi}\cdot\cos(m\pi) = \frac{T_0^2}{2m\pi}\cdot(-1)^m
\end{aligned}$$

since $\cos(m\pi) = \cos(-m\pi)$ and $\cos(m\pi)$ is -1 for odd m and 1 for even m. Thus, we can write

$$\begin{aligned} b_m &= \frac{2}{T_0}\left(\left[-\frac{\cos(m\omega_0 t)}{m\omega_0}\right]_{-T_0/2}^{T_0/2} + \frac{\left[\sin(m\omega_0 t) - m\omega_0 t\cos(m\omega_0 t)\right]_{-T_0/2}^{T_0/2}}{m^2\omega_0^2}\right) \\ &= -\frac{2}{T_0}\cdot\frac{T_0^2}{2m\pi}\cdot(-1)^m = \frac{T_0}{m\pi}(-1)^{m+1} \end{aligned}$$

1.43. Question 14

1.43.1. Part a)

We have the following proof:

$$\begin{aligned} Y(f) &= \int_{-\infty}^{\infty} x(t)\cos(2\pi f_0 t)e^{-j2\pi f t}dt = \int_{-\infty}^{\infty} x(t)\cdot\left(\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right)e^{-j2\pi f t}dt \\ &= \frac{1}{2}\int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-f_0)t}dt + \frac{1}{2}\int_{-\infty}^{\infty} x(t)e^{-j2\pi(f+f_0)t}dt \\ &= \frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0) \end{aligned}$$

1.43.2. Part b)

We now simply have that the transform of $15\text{sinc}(3t) = \frac{15}{3}\Pi\left(\frac{f}{3}\right) = 5\Pi\left(\frac{f}{3}\right)$. With the modulation with $f_0 = \frac{40\pi}{2\pi} = 20\,\text{Hz}$, this means that we have

$$A(f) = \frac{1}{2}\cdot 5\Pi\left(\frac{f-20}{3}\right) + \frac{1}{2}\cdot 5\Pi\left(\frac{f+20}{3}\right)$$

1.43.3. Part c)

This is an energy signal. $\cos(40\pi t)$ remains bounded as $t \to \pm\infty$, but $\text{sinc}(3t) \to 0$ as $t \to \pm\infty$. This means that $x(t) \to 0$ as $t \to \pm\infty$, which means that the power will be zero and the energy a finite non-zero number.

1.43.4. Part d)

The energy is easily computed by use of the Fourier transform. We have

$$A(f) = \frac{1}{2}\cdot 5\Pi\left(\frac{f-20}{3}\right) + \frac{1}{2}\cdot 5\Pi\left(\frac{f+20}{3}\right)$$

so the amplitude is $\frac{5}{2}$ for two periods of $3\,\text{Hz}$. This means that the energy is

$$2\cdot 3\cdot\left(\frac{5}{2}\right)^2 = 37.5\,\text{J}$$

1.44. Question 15

1.44.1. Part a)

I prefer to do everything in decibels (for question d) this helps a bit), so I'll only show the calculations for decibels. We get:

$$
\begin{aligned}
(P_{T_x})_{\text{dB}} &= 10\log_{10}(425) = 26.28\,\text{dBW} \\
(G_{AT})_{\text{dB}} &= 10\log_{10}(0.75) = -1.2494\,\text{dB} \\
(G_{FS})_{\text{dB}} &= 10\log_{10}\cdot\left(\frac{\frac{3\cdot 10^8}{1090\cdot 10^6}}{4\pi\cdot 180\cdot 1852}\right)^2 = -143.65\,\text{dB} \\
(G_{AR})_{\text{dB}} &= 10\log_{10}(0.9) = -3.0103\,\text{dB}
\end{aligned}
$$

Thus, the power received is $(P_{R_x})_{\text{dB}} = 26.28 - 1.2494 - 143.65 - 3.0103 = -121.62\,\text{dBW}$.

1.44.2. Part b)

We simply get that the noise power is

$$P_e = kTB = 1.38\cdot 10^{-23}\cdot 525\cdot 4\cdot 10^6 = 2.898\times 10^{-14}\,\text{W}$$

which translates to $(P_e)_{\text{dB}} = 10\log\left(2.898\cdot 10^{-14}\right) = -135.38\,\text{dBW}$.

1.44.3. Part c)

We have a SNR of $-121.62 - -135.38 = 13.76\,\text{dB}$, so the requirement is met.

1.44.4. Part d)

To get an SNR of 10 dB, we need to have a space loss of

$$(P_{T_x})_{\text{dB}} + (G_{AT})_{\text{dB}} + (G_{FS})_{\text{dB}} + (G_{AR})_{\text{dB}} - (P_e)_{\text{dB}} = 26.28 - 1.2494 - (G_{FS})_{\text{dB}} - 3.0103 - -135.58 = 10\,\text{dB}$$

This requires a space gain of $-147.55\,\text{dB}$. This corresponds to solving

$$
\begin{aligned}
\left(\frac{\frac{3\cdot 10^8}{1090\cdot 10^6}}{4\pi\cdot d\cdot 1852}\right)^2 &= 10^{-147.55/10} \\
\frac{1.3986\cdot 10^{-10}}{d^2} &= 1.758\cdot 10^{-15} \\
d &= 282.06\,\text{NM}
\end{aligned}
$$

Part IV

June 2014

1.45. Question 1

The correct answer is B. Let's first consider $x_1(t)$. Note that for the single-sided spectrum, we need to rewrite this to

$$x_1(t) = 3e^{j(10\pi t-\pi/2)} + 3e^{-j(10\pi t-\pi/2)} = 6\cos(10\pi t - \pi/2)$$

This means that the frequency is $\frac{10\pi}{2\pi} = 5\,\text{Hz}$ and the amplitude 6 and the phase $-\pi/2$. Thus, the spectrum of $x_1(t)$ is correct.

Now, for $x_2(t)$; you can write it as exponential functions but you can also just use your brains. For the first term, the frequency will be $\frac{8\pi}{2\pi} = \pm 4\,\text{Hz}$; both of which will have an amplitude of $2/2 = 1$. Furthermore, the positive frequency will have a phase of $\pi/4$; the negative frequency will have a phase of $-\pi/4$ (should be pretty obvious). For the second term, we need to rewrite the sin to a cos:

$$6\sin(20\pi t - \pi/8) = 6\cos(20\pi t - \pi/8 - \pi/2) = 6\cos\left(20\pi t - \frac{5\pi}{8}\right)$$

Thus, the frequency is $\pm\frac{20\pi}{2\pi} = \pm 10\,\text{Hz}$, and the amplitude of both will be 3. The phase of the positive frequency will be $-\frac{5\pi}{8}$, the phase of the negative frequency will be $\frac{5\pi}{8}$. Thus, clearly the spectrum if $x_2(t)$ is incorrect as the phases are wrong.

1.46. Question 2

The correct answer is B. The average of $x(t)$ is always equal to a_0, which is not equal to 0, so the average of $x(t)$ is not zero, thus statement I is false.

The coefficients a_m for $m \geq 1$ are all 0, i.e. the coefficients of the cosines (even functions) are all zero. On the other hand, some of the coefficients of the sines (odd functions) are nonzero. This would make us believe that $x(t)$ is an odd signal. However, note that since the average is not zero, the signal is also not odd in the strict mathematical sense (and for these kind of questions, it's the strict mathematical sense that counts). Nevertheless, $x(t)$ is definitely not even, so statement II is false.

Again, the fact that all b_m coefficients for even m are zero would make you believe that $x(t)$ has half-wave odd symmetry. However, again, the fact that the average is not zero means that from a strict mathematical point of view, this is not the case, and *in this course, it's this point of view that counts*. Thus, $x(t)$ is not half-wave odd, and thus statement III is false as well. Thus, answer B is correct.

1.47. Question 3

The correct answer is C. If you look carefully at the formula sheet, you see that this is pair 4 of the Fourier transform pairs, but with a time delay (theorem 2 of the Fourier Transform Theorems) of $t_0 = 1$. What does this mean? Well, according to the time delay theorem,

$$x(t-t_0) \to X(f)e^{-j2\pi f t_0}$$

In other words: we find the transform of the "unshifted" function $x(t)$ and then multiply this with that exponent. In this case, the transform of the unshifted function is (since $a = 2$)

$$X(f) = \frac{1}{2+j2\pi f}$$

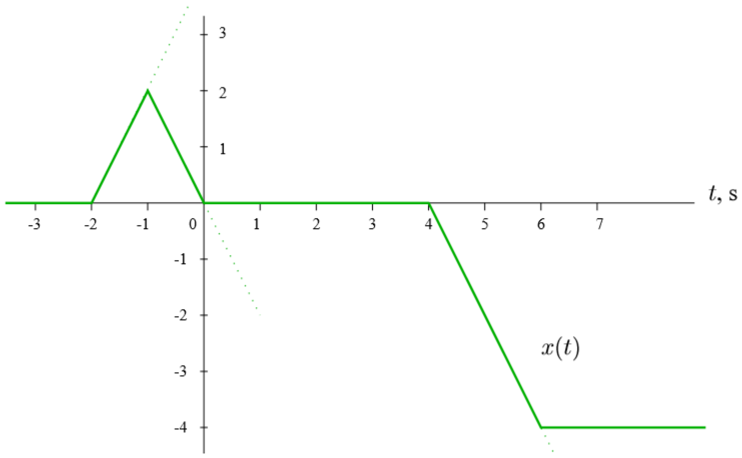
and thus we get that the transform becomes

$$\mathcal{F}\left\{x(t) = e^{-2(t-1)}u(t-1)\right\} = \frac{e^{-j2\pi f\cdot 1}}{2+j2\pi f}$$

Thus, answer C is clearly correct.

1.48. Question 4

The correct answer is D. Again, just make the sketch:



We see that $x(t)$ remains bounded, but not zero, as $t \to \infty$, so it is a power signal, meaning it's definitely not B or C. Furthermore, in the grand scheme of things, the signal is essentially equal to 0 for half of the time, and equal to -4 for the other half of the time (if you zoom ridiculously far out, the graph would look like that, at least). Thus, the signal power equals $\frac{1}{2} \cdot |-4|^2 = \frac{1}{2} \cdot 16 = 8\,\text{W}$, and thus answer D is correct.

1.49. Question 5

The correct answer is B. Don't forget we need to get rid of the integrals, thus we need to differentiate with respect to time another time:

$$\begin{aligned} \frac{d}{dt}\left(y(t)\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} - 3y(t)\right) &= x(t-1) \\ \frac{dy}{dt}\frac{d^2y}{dt^2} + y\frac{d^3y}{dt^3} - 3\frac{dy}{dt} &= x(t-1) \end{aligned}$$

and thus the order is three, meaning it's either A or B. Now, I like to point out: proving the system is either dynamic or causal is *not* enough to conclude whether it's then A or B. After all, the system is dynamic (since there are derivatives in there), but answer B does not state that it is *not* dynamic, and thus it is not enough to conclude that A is correct. Similarly, although the system is causal (it does not depend on future values of x; had the integral run to $t+1$ instead of $t-1$, it would have been non-causal), answer A does not deny that it is non-causal so it is insufficient evidence to conclude that B is correct. Indeed, we must either prove that it is (non)-linear, or that it is time-varying/fixed.

It should be rather clear that the system is fixed, as the ODE does not change over time. Therefore, answer B is correct. You can also prove that it is non-linear, however, writing this out can get a bit messy so that's why I chose to prove that it's fixed.

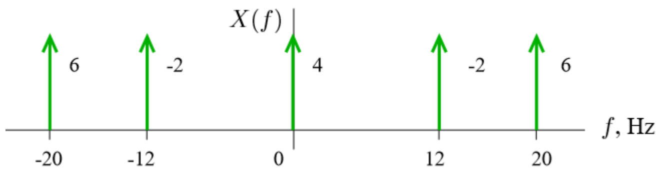
I should note that you may have some intuition that the system is non-linear by just looking at it, however, I'd like to give some advice regarding intuition about (non-)linearity: if your gut tells you the system is non-linear, then it's probably non-linear (assuming you're not a total dumbass). If your gut tells you the system is linear, please verify because for example system S2 of January 2013: question 5 may make you believe that it is linear (at least to me it did), but it's actually not. So only trust your gut when it tells you a system is non-linear, not when it tells you a system is linear.

1.50. Question 6

The correct answer is A. First, let's calculate the power of the input signal. For this, we must get rid of the $\sin^2$: we have $\sin^2(u) = \frac{1}{2} - \frac{1}{2}\cos(2u)$, and thus

$$\begin{aligned} x(t) &= 8\sin^2(12\pi t) + 12\cos(40\pi t + \pi/3) = 8\left(\frac{1}{2} - \frac{1}{2}\cos(24\pi t)\right) + 12\cos(40\pi t + \pi/3) \\ &= 4 - 4\cos(24\pi t) + 12\cos(40\pi t + \pi/3) \end{aligned}$$

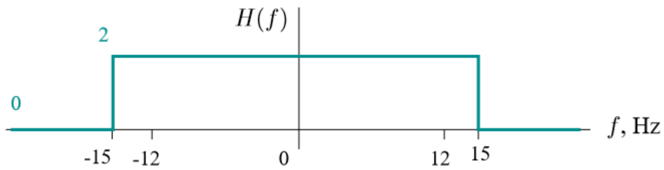
As the phase (rather logically) has no influence of the average power, we can safely ignore the phase (the official solutions literally says it's okay to do, so even if this would have been an open question, you can say you'll ignore the phase). The Fourier transform of $x(t)$, $X(f)$, is shown below:



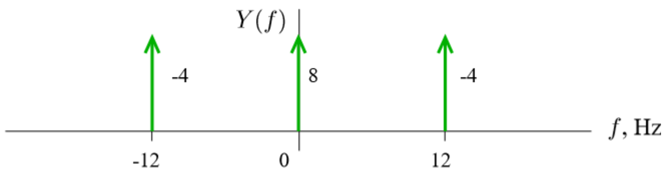
The power is then easily computed by just summing the squares of the heights of the Dirac impulses:

$$P_x = 6^2 + (-2)^2 + 4^2 + (-2)^2 + 6^2 = 96\,\text{W}$$

Thus, answer B is incorrect. Now we sketch the ideal low-pass filter (don't forget the gain of 2):



We multiply this with $X(f)$ to get the Fourier transform of the output, $Y(f)$:



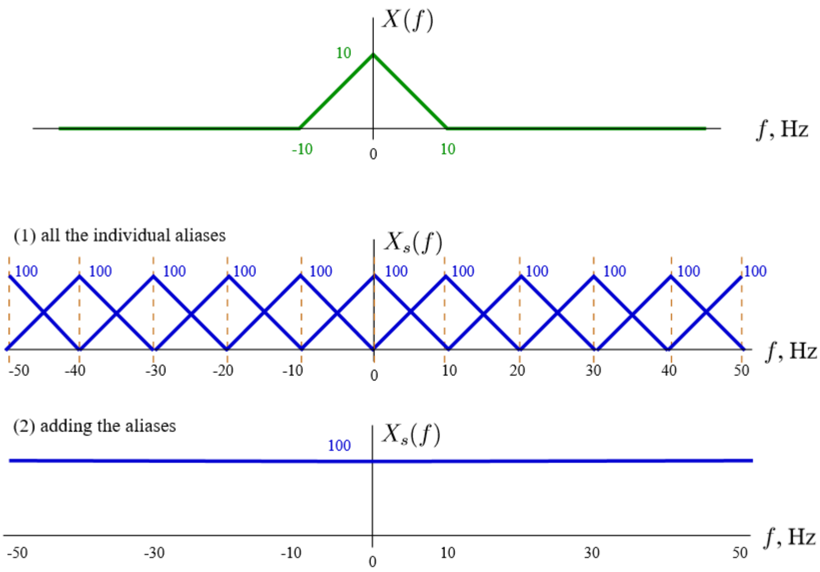
Note that the peaks at frequencies higher than $|f| > 15\,\text{Hz}$ have completely vanished; the peaks associated with $|f| < 15\,\text{Hz}$ have doubled in size. The average power is now simply

$$P_y = (-4)^2 + 8^2 + (-4)^2 = 96\,\text{W}$$

It is pure coincidence that this is the same as the input average power, in case you were wondering. The correct answer is A, at least.

1.51. Question 7

The correct answer is B. Since we sample at 10 Hz, we return copies of the signal at 10 Hz, 20 Hz, −10 Hz, etc. Furthermore, the amplitude of each copy is multiplied by 10. Thus, we get the process shown in the plot below.



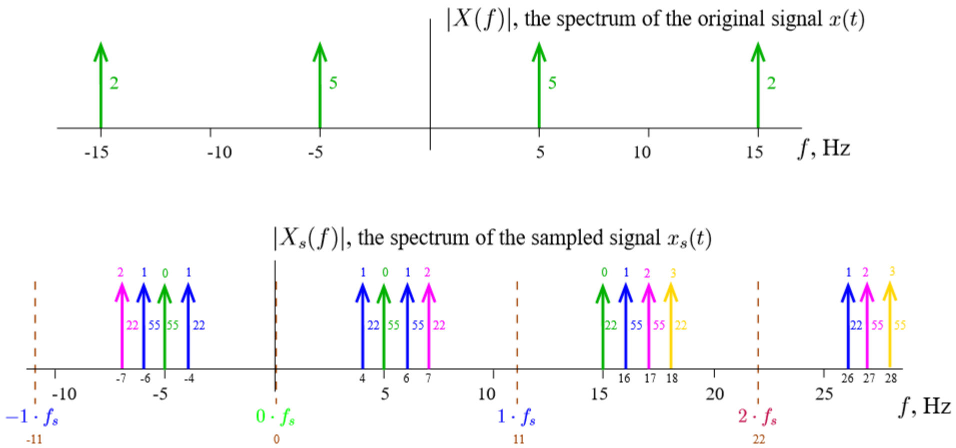
Clearly, answer B is correct.

1.52. Question 8

The correct answer is D. Note that the frequency of the cosine is 5 Hz, and the frequency of the sine is 15 Hz. Then, the copy centered at 0 Hz will return peaks at −15 Hz, −5 Hz, 5 Hz and 15 Hz. The copy centered at −11 Hz will return peaks at −26 Hz, −16 Hz, −6 Hz and 4 Hz. The copy centered at −22 Hz won't return any peaks at positive frequencies any more.

The copy centered at 11 Hz will return peaks at −4 Hz, 6 Hz, 16 Hz and 26 Hz. The copy centered at 22 Hz will return peaks at 7 Hz, 17 Hz, 27 Hz and 37 Hz. The copy at 33 Hz won't return any copies between 0 and 10 Hz.

So, in total, we count a peak at 4 Hz, 5 Hz, 6 Hz and 7 Hz, and thus answer D is correct. We can visualize as shown below:



Note that you should not forget to include the peaks caused by copies centered at negative frequencies!

1.53. Question 9

The correct answer is B. We need to have $6.4/0.2 = 32$ steps. That is achieved by having $\log_2 32 = 5$ bits. Note that this time, you got a nice number of bits; if it needs to be rounded, *always* round up!

1.54. Question 10

The correct answer is D. This question is almost exactly the same as before. The only caveat is in that $\beta > 0.5$, which requires you to think a bit. However, the solution to this caveat is very simple: looking at figure **??**, k_β will now be located to the right of the peak (i.e. to the right of μ_{H_1}). Finding the location of it can then be done by solving

$$P\left(\mathbf{y} < k_\beta | H_1 \text{ is true}\right) = 1 - 0.5636 = 0.4364$$

If H_1 is true, then $\mu = 1$ and $\sigma = 2$, and thus we have to solve

$$P\left(\frac{\mathbf{y}-1}{2} < \frac{k_\beta - 1}{2}\right) = 0.4364$$

We find $k_{\beta_z} = 0.16$. Now, to compute k_β, you have to compute that this must be located to the *right* of the mean (this makes sense looking at figure **??**). Thus, we now get $k_\beta = \mu + \sigma k_{\beta_z} = 1 + 2 \cdot 0.16 = 1.32 = k_\alpha$. Then,

$$\alpha = P\left(\mathbf{y} > k_\alpha | H_0 \text{ is true}\right) = P\left(\frac{\mathbf{y}--1}{2} > \frac{1.32--1}{2}\right) = P\left(\frac{\mathbf{y+1}}{2} > 1.16\right)$$

since H_0 means $\mu = -1$ and $\sigma = 2$. This probability can be straightforwardly looked up to equal 0.1230, and thus answer D is correct.

1.55. Question 11

The correct answer is B. We have $m = 1$, and P_x and P_e both need to converted to Watt: $P_x = 10^{-158/10} = 1.585 \times 10^{-16}\,\text{W}$ (outdoor) and $P_e = 10^{-187/10} = 1.995 \times 10^{-19}\,\text{W}$. This means that for the outdoor use, we have to compute

$$P(E) = 1 - \Phi\left(\sqrt{1 \cdot \frac{1.58 \cdot 10^{-16}}{1.995 \cdot 10^{-19}}}\right) = 1 - \Phi(28.1)$$

I can assure you, the probability that a standard normal distribution, $N(0,1)$, is smaller than 28.1 is almost exactly 1. Thus, $P(E)$ outdoor use is almost equal to 0. Then, to check what $P(E)$ of indoor use will be, we divide P_x by 250 so that we get

$$P(E) = 1 - \Phi\left(\sqrt{1 \cdot \frac{1.58 \cdot 10^{-16}/250}{1.995 \cdot 10^{-19}}}\right) = 1 - \Phi(1.780)$$

This can be looked up from the formula sheet; look up $r_\alpha = 1.78$, then we see that $1 - \Phi(1.78) \approx 0.0375$. Thus, professor III is indeed correct and thus answer B is correct.

1.56. Question 12

The correct answer is C. You can do this in normal units (in Watts), and then convert the answer to decibels at the end, or immediately convert to decibels, whichever you prefer. I'll show both methods. In normal units, we get

$$P_{R_x} = P_{T_x} \cdot G_{AT} \cdot \left(\frac{\frac{c}{f}}{4\pi d} \right)^2 \cdot G_{AR} = 300 \cdot 0.9 \cdot \left(\frac{\frac{3 \cdot 10^8}{1090 \cdot 10^6}}{4\pi \cdot 120 \cdot 1852} \right)^2 \cdot 0.9 = 2.4305 \times 10^{-12}\,\text{W}$$

In decibels, this is $(P_{R_x})_{\text{dB}} = 10 \log_{10} \left(2.4305 \cdot 10^{12} \right) = -116.27\,\text{dBW}$ and thus answer C is correct.

In decibels, the calculation would be as follows:

$$\begin{aligned}
(P_{T_x})_{\text{dB}} &= 10 \log_{10}(300) = 24.77\,\text{dBW} \\
(G_{AT})_{\text{dB}} &= 10 \log_{10}(0.9) = -0.4576\,\text{dB} \\
(G_{FS})_{\text{dB}} &= 10 \log_{10} \cdot \left(\frac{\frac{3 \cdot 10^8}{1090 \cdot 10^6}}{4\pi \cdot 120 \cdot 1852} \right)^2 = -140.13\,\text{dB} \\
(G_{AR})_{\text{dB}} &= 10 \log_{10}(0.9) = -0.4576\,\text{dB}
\end{aligned}$$

Thus, our signal power becomes $24.77 - 0.4576 - 140.13 - 0.4576 = -116.27\,\text{dBW}$ and thus answer C is correct.

1.57. Question 13

1.57.1. Part a)

Note that the frequency of the cosine is $\frac{2\pi}{2\pi} = 1\,\text{Hz}$, and the frequency of the sine is $\frac{6\pi}{2\pi} = 3\,\text{Hz}$. Thus, the fundamental frequency (the largest frequency of which both frequencies are an integer multiple) is 1 Hz, and thus the period of the signal is $T_0 = \frac{1}{f_0} = \frac{1}{1} = 1\,\text{s}$.

1.57.2. Part b)

The average value of this signal is simply 1. Why? Over a full period, the average value of cosines and sines is zero, so the average value of $x(t)$ is simply equal to the constant in the function, 1.

1.57.3. Part c)

You can do all of the hard calculations, but that'll take you extremely long for a question that's supposed to take you only 16 minutes. Instead, what you have to realize is that we essentially have $\omega_0 = 2\pi$, and that the Fourier series will be very similar to

$$x(t) = a_0 + a_1 \cos(\omega_0 t) + b_3 \sin(3\omega_0 t)$$

with all other coefficients equal to 0. Now, note from comparison that $a_0 = 1$ and $a_1 = 2$. However, b_3 is slightly more work. After all, we are supposed to use $\sin(3\omega_0 t)$ but we are given $\sin(3\omega_0 t - \pi/6)$, so how do we deal with that? Well, we simply look at our formula sheet: on page 6, we see a list of trigonometric identities, among which $\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v$. Thus, we simply get

$$\sin(3\omega_0 t - \pi/6) = \sin(3\omega_0 t)\cos(\pi/6) - \cos(3\omega_0 t)\sin(\pi/6) = \frac{\sqrt{3}}{2}\sin(3\omega_0 t) - \frac{1}{2}\cos(3\omega_0 t)$$

Thus, we can write the function as

$$x(t) = 1 + 2\cos(2\pi t) + 4\sin(6\pi t - \pi/6) = 1 + 2\cos(\omega_0 t) - 2\cos(3\omega_0 t) + 2\sqrt{3}\sin(3\omega_0 t)$$

Thus, comparing it with the Fourier series

$$x(t) = a_0 + a_1\cos(\omega_0 t) + a_3\cos(3\omega_0 t) + b_3\sin(\omega_0 t)$$

we see that the Fourier series coefficients are $a_0 = 1$, $a_1 = 2$, $a_3 = -2$ and $b_3 = 2\sqrt{3}$.

1.58. Question 14

1.58.1. Part a)

This one is pretty straightforward: we simply have to calculate the Fourier transform ourselves:

$$\begin{aligned}
B(f) &= \int_{-\infty}^{\infty} b(t)e^{-j2\pi ft}dt = \int_{-\tau/2}^{\tau/2} e^{-j2\pi ft}dt = \left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_{-\tau/2}^{\tau/2} = \frac{-1}{j2\pi f}\left[e^{-j2\pi f\cdot\frac{\tau}{2}} - e^{-j2\pi f\cdot\frac{-\tau}{2}}\right] \\
&= \frac{1}{\pi f}\cdot\frac{e^{j\pi f\tau} - e^{-j\pi f\tau}}{2j} = \frac{1}{\pi f}\sin(\pi f\tau) = \tau\cdot\frac{\sin(\pi f\tau)}{\pi f\tau} = \tau\mathrm{sinc}(f\tau)
\end{aligned}$$

1.58.2. Part b)

You either know it or you don't. If you looked at the video I told you to watch, you can recall that $\Pi(t)$ with $\Pi(t)$ (i.e. two pulses with magnitude 1 and width 1) resulted in a triangle with height 1 and width 1). Now, if you think about it, if we convolve $\Pi(t/\tau)$ with $\Pi(t/\tau)$ (i.e. two pulses with magnitude 1 and width τ), then we'd get a triangle with height τ and width τ as well. So, to get a triangular signal with width τ and height 1 instead of τ from two unit pulses, we need to convolve two unit pulses with width τ, and then divide by τ (to get the correct height). In other words,

$$\Lambda\left(\frac{t}{\tau}\right) = \frac{1}{\tau}\Pi\left(\frac{t}{\tau}\right)\star\Pi\left(\frac{t}{\tau}\right)$$

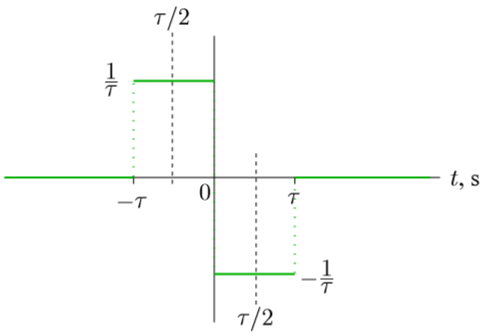
Now, convolution in the time domain becomes multiplication in the frequency domain, thus we get simply

$$X(f) = \frac{1}{\tau}\cdot\tau\mathrm{sinc}(f\tau)\cdot\tau\mathrm{sinc}(f\tau) = \tau\mathrm{sinc}^2(f\tau)$$

and that's the proof. As long as you remember what convolution is and watched the video, you can do it (maybe you'd forgot to divide by τ, but at least you'd get partial points).

1.58.3. Part c)

We have to use the differentiation theorem. Now, you may wonder, how can we apply the differentiation theorem for this? Well, look at the signal below; this is clearly the derivative of the triangular signal (note that the height is only $1/\tau$; again, this is because the slope of the triangular function is only $1/\tau$.



This function is given by

$$y(t) = \frac{1}{\tau}\Pi\left(\frac{t+\tau/2}{\tau}\right) + \frac{-1}{\tau}\Pi\left(\frac{t-\tau/2}{\tau}\right)$$

Why? First we have a pulse with amplitude $\frac{1}{\tau}$ (thus the $1/\tau$ in front), with width τ (thus the argument is divided by τ). Furthermore, it is shifted $\tau/2$ to the left, so we add $\tau/2$ to t. After that, we have a pulse with amplitude $\frac{-1}{\tau}$ (thus the $-1/\tau$ in front), with width τ (thus the argument is divided by τ). Furthermore, it is shifted $\tau/2$ to the right, so we subtract $\tau/2$ from t. Since $y(t)$ is the derivative of $x(t)$, we expect

$$Y(f) = j2\pi f X(f)$$

where $X(f)$ will be our final answer. We can compute $Y(f)$ directly, using pair 1 of the Fourier Transform Pairs and the time delay theorem. We then straightforwardly get

$$Y(f) = \frac{1}{\tau}\cdot\tau\text{sinc}(f\tau)\cdot e^{-j2\pi f\cdot -\tau/2} + \frac{-1}{\tau}\cdot\tau\text{sinc}(f\tau)\cdot e^{-j2\pi f\cdot\tau/2}$$

Now, this can be rewritten as follows:

$$\begin{aligned} Y(f) &= \frac{1}{\tau}\cdot\tau\text{sinc}(f\tau)\cdot e^{-j2\pi f\cdot -\tau/2} + \frac{-1}{\tau}\cdot\tau\text{sinc}(f\tau)\cdot e^{-j2\pi f\cdot\tau/2} \\ &= \text{sinc}(f\tau)\cdot\left(e^{j\pi f\tau} - e^{-j\pi f\tau}\right) = \text{sinc}(f\tau)\cdot\left(e^{j\pi f\tau} - e^{-j\pi f\tau}\right)\cdot\frac{2j}{2j} \\ &= 2j\cdot\text{sinc}(f\tau)\sin(\pi f\tau) = 2j\cdot\text{sinc}(f\tau)\sin(\pi f\tau)\cdot\frac{\pi f\tau}{\pi f\tau} \\ &= j2\pi f\tau\cdot\text{sinc}(f\tau)\cdot\frac{\sin(\pi f\tau)}{\pi f\tau} = j2\pi f\tau\cdot\text{sinc}^2(f\tau) \end{aligned}$$

As we have $Y(f) = j2\pi f X(f)$ per the differentiation theorem, we have indeed $X(f) = \tau\text{sinc}^2(f\tau)$.

1.59. Question 15

1.59.1. Part a)

Very simple: half the time, the amplitude is equal to A; the other half it's equal to $-A$; thus, the average power becomes

$$P = \frac{1}{2}\cdot A^2 + \frac{1}{2}\cdot(-A)^2$$

1.59.2. Part b)

The question seems a lot harder than it is. We must simply find the expression for ∇. For polar signalling, we have

$$\begin{aligned} x_a - x_0 &= \begin{bmatrix} A \\ \vdots \\ A \end{bmatrix} - \begin{bmatrix} -A \\ \vdots \\ -A \end{bmatrix} = \begin{bmatrix} 2A \\ \vdots \\ 2A \end{bmatrix} \\ \nabla = \|x_a - x_0\| &= \sqrt{(2A)^2 + (2A)^2 + \ldots + (2A)^2} = 2A\sqrt{m} \end{aligned}$$

Thus,

$$P(E) = 1 - \Phi\left(\frac{\nabla}{2\sigma}\right) = 1 - \Phi\left(\frac{2A\sqrt{m}}{2\sigma}\right) = 1 - \Phi\left(\frac{A\sqrt{m}}{\sigma}\right)$$

1.59.3. Part c)

We get similar derivations as before. First, we have

$$\begin{aligned} c &= \frac{x_a - x_0}{\nabla} = \frac{2A\vec{u}_m}{2A\sqrt{m}} = \frac{1}{\sqrt{m}}\vec{u}_m \\ y - x_0 &= \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} - \begin{bmatrix} -A \\ \vdots \\ -A \end{bmatrix} = \begin{bmatrix} y_1 + A \\ \vdots \\ y_m + A \end{bmatrix} \end{aligned}$$

Thus, we get

$$\begin{aligned} \frac{c^T(y - x_0)}{\sigma} &> \frac{\nabla}{2\sigma} \\ \frac{1}{\sqrt{m}\sigma}\vec{u}_m^T(y - x_0) = \frac{1}{\sqrt{m}\sigma}(1 \cdot (y_1 + A) + \ldots + 1 \cdot (y_m + A)) &> \frac{2A\sqrt{m}}{2\sigma} \\ \frac{1}{m}\left(\sum_{i=1}^{m} y_i + m \cdot A\right) &> A \\ \frac{1}{m}\sum_{i=1}^{m} y_i &> 0 \end{aligned}$$

This makes a lot of sense: the 'half-way' value between the signal levels lies at exactly 0 V. So when the average of m samples per bit lies above the 0 Volt we decide that the bit being sent is a '1'. If the average lies below 0 Volt, we decide that the bit being sent is a '0'.

1.59.4. Part d)

This one is more of a bitch. You have to realize that in the time domain, a pulse looks like

$$x(t) = \pm A\Pi\left(\frac{t}{T_b}\right)$$

of which the Fourier transform is (use pair 1)

$$X(f) = \pm AT_b \text{sinc}(T_b f)$$

With the modulation theorem, we thus get that the Fourier transform is

$$V(f) = \pm\frac{AT_b}{2}\left(\text{sinc}\left((f + f_c)T_b\right) + \text{sinc}\left((f - f_c)T_b\right)\right)$$

Why is it okay to only consider a single pulse? Remember that the average power will be the total energy of all bits divided by the total time it takes for all these bits to happen. However, you'll agree with me that the energy of a single will be the same for every bit; after all, the amplitude is either A or $-A$ so the energy should be the same per bit. Thus, to calculate the average power, we can also calculate just calculate the energy of one bit, centered at $t=0$, and divide it by the time of one bit. Much, much easier.

Now, let's calculate the energy of this one bit. We have

$$\begin{aligned}
E_v &= \int_{f=-\infty}^{f=\infty} V^2(f)\,df = \int_{f=-\infty}^{f=\infty} \left(\pm\frac{AT_b}{2}\left(\operatorname{sinc}\left((f+f_c)T_b\right)+\operatorname{sinc}\left((f-f_c)T_b\right)\right)\right)^2 df \\
&= \frac{A^2T_b^2}{4}\int_{-\infty}^{\infty}\left[\operatorname{sinc}^2\left((f+f_c)T_b\right)+2\operatorname{sinc}\left((f+f_c)T_b\right)\operatorname{sinc}\left((f-f_c)T_b\right)+\operatorname{sinc}^2\left((f-f_c)T_b\right)\right]df
\end{aligned}$$

Now, since $f_c >> 1/T_b$, or $f_cT_b >> 1$, this means that if you plot $\operatorname{sinc}((f+f_c)T_b)$ and $\operatorname{sinc}((f-f_c)T_b)$, the peaks will be located very far from each other (symmetric w.r.t. to vertical axis, one located at $-f_c$ and the other at f_c. This means that if you multiply the one with the other, then consider what happens around $f=f_c$: one of the sinc has a peak there, but the other one will be pretty much 0 there, meaning the product is pretty much zero as well. The same can be said about the other peak; then the other sinc will be almost 0 there, meaning the product is pretty much zero as well. In other words,

$$\int_{-\infty}^{\infty}\left[2\operatorname{sinc}\left((f+f_c)T_b\right)\operatorname{sinc}\left((f-f_c)T_b\right)\right]df$$

will reduce to zero. Furthermore, note that the remaining squared sincs are just copies of each other at different frequencies; thus, it suffices to merely calculate one of them, and them multiply the result by two. In other words:

$$E_v = \frac{A^2T_b^2}{4}\cdot 2\int_{-\infty}^{\infty}\left[\operatorname{sinc}^2\left((f+f_c)T_b\right)\right]df$$

This integral can be reduced by noting that this is symmetric around $f=-f_c$; this means we can essentially write

$$E_v = \frac{A^2T_b^2}{2}\cdot 2\int_{0}^{\infty}\left[\operatorname{sinc}^2\left(f'T_b\right)\right]df'$$

with $f' = f+f_c$ (basically, we shifted the sinc function back to be on centered on the vertical axis). This integral can easily be computed using the top entry on page 6 of the formula sheet:

$$E_v = A^2T_b^2\cdot\int_0^{\infty}\frac{\sin^2\left(\pi f'T_b\right)}{\left(\pi f'T_b\right)^2}df'$$

We have $x=\pi f'T_b$, or $df' = dx/(\pi T_b)$, and thus

$$\begin{aligned}
E_v &= A^2T_b^2\cdot\int_0^{\infty}\frac{\sin^2(x)}{x^2}\frac{dx}{\pi T_b} \\
&= \frac{A^2T_b}{\pi}\cdot\frac{\pi}{2} = \frac{A^2T_b}{2}
\end{aligned}$$

The power is then simply the energy divided by T_b, i.e.

$$P_{av} = \frac{E_v}{T_b} = \frac{A^2}{2}$$

Part V

Example exam

1.60. Question 1

The correct answer is answer C. Although the function may look odd to you, note that it is nowhere specified where the pulses are located (if you look closely, you see that it's not symmetric with respect to the y-axis). Thus, it is neither odd nor even. Furthermore, the average, $X(0)$, will be 0, so answer B is false too. This means that answer C is correct.

1.61. Question 2

The correct answer is answer B and D. The Fourier transform of $x(t)$ is rather clearly

$$X(f) = 12\text{sinc}(4f)$$

For $Y(f)$, we first multiply this with 2:

$$Y'(f) = 24\text{sinc}(4f)$$

after which we apply the scale change theorem, meaning we get

$$Y(f) = \frac{24}{|2|}Y'\left(\frac{f}{2}\right) = 12\text{sinc}(2f)$$

Thus, answer B and D are both correct.

1.62. Question 3

The correct answer is D. A sum of sines and cosines is rather clearly a power signal (it will not tend to zero as $t \to \pm\infty$, and it'll also remain bounded). The Fourier transform is then (disregarding the phases as we don't give a fuck about that):

$$X(f) = 4\delta(f-5) + 4\delta(f+5) + \frac{3}{2}\delta(f-10) + \frac{3}{2}\delta(f+10)$$

The corresponding power is thus

$$P_x = 4^2 + 4^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = 36.5\,\text{W}$$

The power between 9 and 15 Hz is equal to

$$P_x = \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = 4.5\,\text{W}$$

and thus answer D is correct.

1.63. Question 4

Yeah I'm pretty sure the pre-2013 exams do not use the same definition for the order of the system. This system is third order (you need to differentiate once to get rid of the integral); it's linear, dynamic, time-varying and causal. There's no answer that meets that requirement. Therefore, I think the answer was intended to be C (where they used the wrong definition for the system). The 'most' correct answer would be:

- Linear, time-varying, causal, dynamic, and its order is three.

1.64. Question 5

The correct answer is C. The frequency of the first cosine is 4 Hz; of the second it is 7 Hz; of the third it is 12 Hz. Only the frequencies below 10 Hz are preserved; thus, the Fourier transform of the filtered signal is

$$X(f) = 10\delta(f-4) + 10\delta(f+4) + 6\delta(f-7) + 6\delta(f+7)$$

so that the power is $10^2 + 10^2 + 6^2 + 6^2 = 272\,\mathrm{W}$. Thus, answer C is correct.

1.65. Question 6

The correct answer is C. The cosine has a frequency of 40 Hz; thus, if we sample at 60 Hz, the copy at 0 Hz will produce a peak at 40 Hz and the copy at 60 Hz will produce a peak at 20 Hz. Thus, answer A is definitely false, and answer C is definitely correct. Proving that answer D is false is also easy: the weight of each and every peak will be $\frac{3}{2} \cdot 60 = 90$ and not 180.

1.66. Question 7

The correct answer is B. For this, it is helpful to first draw the Fourier domain of the original signal. For this, we must first compute the Fourier transform. Using pair 2 and the modulation theorem, we simply get

$$\begin{aligned} x(t) &= \frac{5}{2} \cdot 2\mathrm{sinc}(2t)\cos(10\pi t) \\ X(f) &= \frac{5}{2}\Pi\left(\frac{f-5}{2}\right) + \frac{5}{2}\Pi\left(\frac{f+5}{2}\right) \end{aligned}$$

We can sketch this as shown below (remember that the pulse width is 2 Hz):

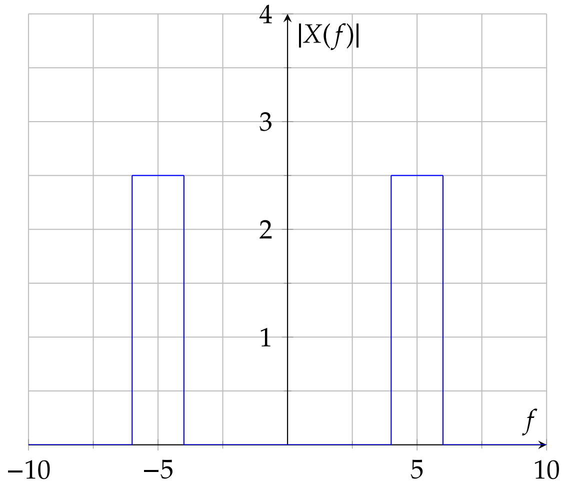


Figure 1.13: Plot of input signal.

Now, you can clearly see that the bandwidth of this signal is $f_h = 6\,\mathrm{Hz}$; this means that the Nyquist rate of this signal is 12 Hz and thus answer B is correct.

1.67. Question 8

Answer B is correct. We require 8 steps in total. This requires $\log_2(8) = 3$ bits. Thus, answer B is correct.

1.68. Question 9

Answer C is correct. If we have an α of 0.2266, we find $k_{\alpha_z} = 0.75$, so that $k_\alpha = 1 \cdot 0.75 + 0 = 0.75$. Then, we must find β by computing

$$P(\mathbf{y} < 0.75 | H_1 \text{ is true}) = P\left(\frac{\mathbf{y}-1}{1} < \frac{0.75-1}{1}\right) = P\left(\frac{\mathbf{y}-1}{1} < -0.25\right) = 0.4013$$

Thus, answer C is correct.

1.69. Question 10

The correct answer is B. When T_b is doubled, the argument of Φ increases, so Φ will increase and $P(E)$ will decrease. It cannot be determined how much it'll decrease though. Note that this also makes sense: if we make our bits last longer, surely it'll be easier for the receiver to understand what's trying to be said.

1.70. Question 11

The complex Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{n=\infty} X_n e^{jn\omega_0 t}$$

with $\omega_0 = \frac{2\pi}{T_0}$. Here, X_n are given by

$$X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

For $n = 0$, this reduces to

$$X_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} At dt = \frac{1}{T_0} \left[\frac{At^2}{2}\right]_{-T_0/2}^{T_0/2} = 0$$

For $n \neq 0$, we get

$$X_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} Ate^{-jn\omega_0 t} dt = \frac{A}{T_0} \left(\left[\frac{te^{-jn\omega_0 t}}{-jn\omega_0}\right]_{-T_0/2}^{T_0/2} - \frac{1}{-jn\omega_0} \int_{-T_0/2}^{T_0/2} e^{-jn\omega_0 t} dt \right)$$

The second integral reduces to zero: integrating the complex exponential function over a full period reduces to 0. The term of the left can be easily worked out, using $\omega_0 = \frac{2\pi}{T_0}$:

$$\begin{aligned} X_n &= \frac{A}{T_0} \left[\frac{\frac{T_0}{2} \cdot e^{-jn\frac{2\pi}{T_0}\frac{T_0}{2}}}{-jn\frac{2\pi}{T_0}} - \frac{-\frac{T_0}{2} \cdot e^{-jn\frac{2\pi}{T_0}\cdot\frac{-T_0}{2}}}{-jn\frac{2\pi}{T_0}} \right] \\ &= \frac{A}{T_0} \left[\frac{jT_0^2}{4\pi} \cdot (-1)^n - \frac{-jT_0^2}{4\pi} \cdot (-1)^n \right] = \frac{jAT_0}{2\pi} \cdot (-1)^n \end{aligned}$$

where we used $1/(-j) = j$ and $e^{-jn\pi} = (-1)^n$. So, the result is that $X_0 = 0$ and for $n \neq 0$, $X_n = jAT_0/(2\pi) \cdot (-1)^n$.

1.71. Question 12

1.71.1. Part a)

Easy peasy lemon squeezy. We first use the substitution $\cos^2(u) = \frac{1}{2} + \frac{1}{2}\cos(2u)$, so that we get

$$x(t) = \text{sinc}(3t) + \text{sinc}(3t)\cos(24\pi t)\frac{1}{3}\cdot 3\text{sinc}(3t) + \frac{1}{3}\cdot 3\text{sinc}(3t)\cos(24\pi t)$$

Using pair 2, the Fourier transform of $3\text{sinc}(3t)$ is

$$\Pi\left(\frac{f}{3}\right)$$

Using the modulation theorem, we thus get

$$X(f) = \frac{1}{3}\Pi\left(\frac{f}{3}\right) + \frac{1}{3}\Pi\left(\frac{f-12}{3}\right) + \frac{1}{3}\Pi\left(\frac{f+12}{3}\right)$$

1.71.2. Part b)

Quite a different question from anything you've seen before. This question is asking you to draw how filters look like in the Bode plot. Note that by comparing $y(t)$ with $x(t)$, we see that we used a filter that 'deleted' the sinc($3t$) term that occurred at $f = 0$. Instead (in the frequency domain), we are merely left with two pulses that extend between $-13.5 < f < -10.5$ Hz and $10.5 < f < 13.5$ Hz. Thus, one possible filter would be a bandpass filter, centred at 12 Hz, and for example a bandwidth of 3 Hz. This filter would also need to have a gain of 12, clearly. The Bode plot is sketched in figure 1.14.

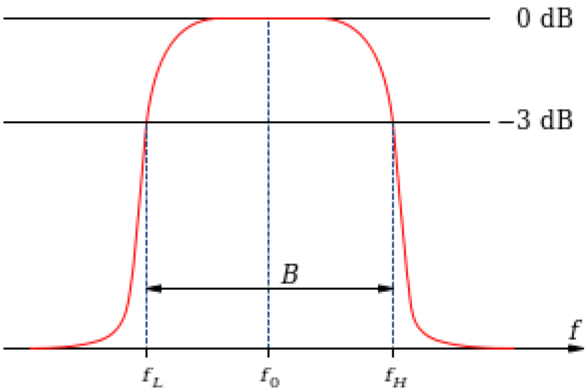


Figure 1.14: Bandpass filter.

1.71.3. Part c)

Yes, multiple filters were possible, e.g. bandapss filters that are centred at a different frequency or different bandwidth. Highpass filters would also be allowed, if they for example have a cut-off frequency of 6 Hz and a gain of 12.

1.72. Question 13

1.72.1. Part a)

Using pair 1 of the formula sheet, we simply have

$$M(f) = \pm AT_b\text{sinc}(T_b f)$$

1.72.2. Part b)

Using the modulation theorem, this can easily be computed:

$$\begin{aligned} x(t) &= m(t)\cos(2\pi f_c t) \\ X(f) &= \frac{1}{2}M(f-f_c)+\frac{1}{2}M(f+f_c) = \pm\left(\frac{AT_b}{2}\text{sinc}(T_b(f-f_c))+\frac{AT_b}{2}\text{sinc}(T_b(f+f_c))\right) \end{aligned}$$

1.72.3. Part c)

We have $G(f)=|X(f)|^2$. Thus, we get

$$\begin{aligned} G(f) &= \left|X(f)\right|^2 = \left(\pm\frac{AT_b}{2}\cdot(\text{sinc}(T_b(f-f_c))+\text{sinc}(T_b(f+f_c))\right)^2 \\ &= \frac{A^2T_b^2}{4}\left[\text{sinc}^2(T_b(f-f_c))+2\text{sinc}(T_b(f-f_c))\cdot\text{sinc}(T_b(f+f_c))+\text{sinc}^2(T_b(f+f_c))\right] \end{aligned}$$

Now, as $f_c >> 1/T_b$, the copies of the sine cardinal functions will be placed very far away in the frequency domain relative to how wide the peaks are. This means that at $f=f_c$, when $\text{sinc}(f-f_c)$ has its peak, $\text{sinc}(f+f_c)$ will be almost zero; at $f=-f_c$, when $\text{sinc}(f+f_c)$ has its peak, $\text{sinc}(f-f_c)$ will be almost zero. Thus, the product $\text{sinc}(T_b(f-f_c))\text{sinc}(T_b(f+f_c))$ will reduce to zero, and thus we get

$$G(f)=\frac{A^2T_b^2}{4}\left[\text{sinc}^2(T_b(f-f_c))+\text{sinc}^2(T_b(f+f_c))\right]$$

1.72.4. Part d)

We integrate the energy spectral density over the frequency to get the energy in Joules. This means that the unit of energy spectral density must be J/Hz.

1.72.5. Part e)

Let's focus on $\text{sinc}^2(T_b(f-f_c))$ for this (the other one is simply a copy of this). This can be written as

$$\text{sinc}^2(T_b(f-f_c))=\left(\frac{\sin(\pi T_b(f-f_c))}{\pi T_b(f-f_c)}\right)^2$$

This has its main peak at $f=f_c$. Its first null to the right of this peak is encountered at $f=f_c+1/(T_b)$, and its first null to the left of it is encountered at $f=f_c-1/(T_b)$. Thus, the bandwidth is $1/(T_b)+1/(T_b)=2/T_b$.

1.72.6. Part f)

The total energy is found by integrating $G(f)$ from minus infinity to plus infinity:

$$E=\int_{-\infty}^{\infty}\frac{A^2T_b^2}{4}\left[\text{sinc}^2(T_b(f-f_c))+\text{sinc}^2(T_b(f+f_c))\right]df$$

There are two shortcuts to make here: as $\text{sinc}^2(T_b(f-f_c))$ and $\text{sinc}^2(T_b(f+f_c))$ are identical copies of each other, it suffices to compute

$$E=2\cdot\int_{-\infty}^{\infty}\frac{A^2T_b^2}{4}\text{sinc}^2(T_b(f-f_c))df$$

Additionally, this integral is more easily computed by making the substitution $f' = f - f_c$, $df = df'$ (essentially, you shift it back to the vertical axis).

$$E = \frac{A^2 T_b^2}{2} \int_{-\infty}^{\infty} \text{sinc}^2 (T_b f') df'$$

This integral is then most easily computed by realizing that this is equal to (due to symmetry)

$$E = A^2 T_b^2 \int_{0}^{\infty} \text{sinc}^2 (T_b f') df' = A^2 T_b^2 \int_{0}^{\infty} \frac{\sin^2 (\pi T_b f')}{(\pi T_b f')^2} df'$$

We can use the top equation on page 6 of the formula sheet: $x = \pi T_b f'$, so $df' = dx/(\pi T_b)$, and thus we get

$$E = A^2 T_b^2 \int_{0}^{\infty} \frac{\sin^2 (x)}{x^2} \frac{dx}{\pi T_b} = \frac{A^2 T_b^2}{\pi T_b} \cdot \frac{\pi}{2} = \frac{A^2 T_b}{2}$$

Part VI

Exam June 2015

1.73. Question 1

Answer B is correct. When $b \neq 0$, the signal is not odd any more (and it's also not even). Thus, all coefficients X_k will be complex. Thus, statement 1 is true. Statement 2 is false, as the average value X_0 is clearly 0, not $\frac{1}{2}AT_0$.

1.74. Question 2

Answer B is correct. As the direction of the signal is reversed (normally the step input would go in the opposite way if you get what I mean), there should be a minus sign in front of t, and thus σ should be positive. Furthermore, the switching 'on/off' should happen at a positive value of t. As σ is positive, this requires that $\rho > 0$. For example, $u(3-2t)$ would be suitable; its switch would be at $t = 1.5\,\mathrm{s}$.

1.75. Question 3

Answer B is correct. The real part is solely determined by the integral of the cosine, the imaginary part is solely determined by the integral of the sine. Thus, answer B is obviously bullshit. Answer C is also just bullshit. Answer B is correct: yes the cosine is an even function of both f and t, but you integrate over time so you complete get rid of time. This means that t does not matter anymore, in the end only f matters. Thus, the reason is answer B.

1.76. Question 4

Answer C is correct. As sinc$(8t)$ goes to 0 as $t \to \pm\infty$, this is quite clearly an energy signal. Its energy is most easily found by computing the Fourier transform:

$$\begin{aligned} x(t) &= 5\operatorname{sinc}(8t)\cos(20\pi t) = \frac{5}{8}\cdot 8\operatorname{sinc}(8t)\cos(20\pi t) \\ X(f) &= \frac{5}{16}\Pi\left(\frac{f-10}{8}\right) + \frac{5}{16}\Pi\left(\frac{f+10}{8}\right) \end{aligned}$$

So, during two time periods of 8 Hz, the amplitude will be 5/16; thus we get

$$E_x = 2\cdot\left(\frac{5}{16}\right)^2\cdot 8 = \frac{25}{16}\mathrm{J}$$

1.77. Question 5

Answer D is correct. The system is linear (you can verify it); the system is clearly time-varying due to the $\cos(50\pi t)$; the system is not dynamic (thus instantaneous) as the output $y(t)$ does not depend on any integrals/derivatives of $x(t)$; the system is causal as we don't need any information about the input at a time in the future.

1.78. Question 6

Answer C is correct. We first rewrite $\cos^2(u)$ to $\frac{1}{2}+\frac{1}{2}\cos(2u)$, and $\sin^2(2u) = \frac{1}{2}-\frac{1}{2}\cos(u)$. Thus, we get

$$x(t) = 6+6\cos(16\pi t)-3+3\cos(24\pi t)+12\cos(14\pi t+\pi/6) = 3+12\cos(14\pi t+\pi/6)+6\cos(16\pi t)+3\cos(24\pi t)$$

The Fourier transform of this is (just totally ignore the phases):

$$X(f) = 3+6\delta(f-7)+6\delta(f+7)+3\delta(f-8)+3\delta(f+8)+\frac{3}{2}\delta(f-12)+\frac{3}{2}\delta(f+12)$$

Thus, the power of the input signal is

$$P_x = 3^2 + 6^2 + 6^2 + 3^2 + 3^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 = 67.5\,\mathrm{W}$$

After the filter, the resulting signal will merely have as Fourier transform

$$Y(f) = 3$$

of which the power is $P_y = 3^2 = 9\,\mathrm{W}$. This means that none of the other answers are correct and thus answer C is correct.

1.79. Question 7

The correct answer is B. We first rewrite $\cos^2(20\pi t) = \frac{1}{2} + \frac{1}{2}\cos(40\pi t)$, so that we get

$$x(t) = \mathrm{sinc}(6t) + \mathrm{sinc}(6t)\cos(40\pi t) = \frac{1}{6}\cdot 6\mathrm{sinc}(6t) + \frac{1}{6}\cdot 6\mathrm{sinc}(6t)\cos(40\pi t)$$

The Fourier transform of this is

$$X(f) = \frac{1}{6}\Pi\left(\frac{f}{6}\right) + \frac{1}{12}\Pi\left(\frac{f-20}{6}\right) + \frac{1}{12}\Pi\left(\frac{f+20}{6}\right)$$

This looks like shown below:

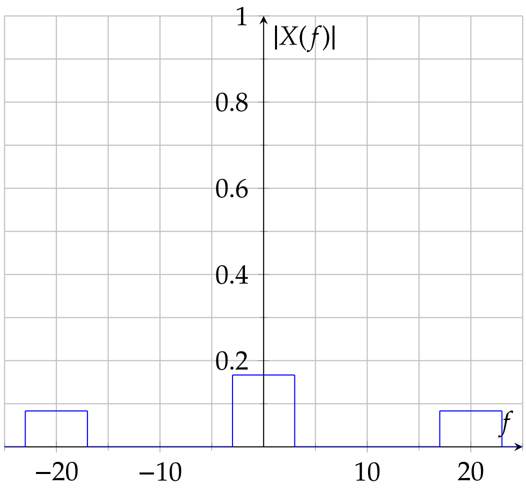


Figure 1.15: Plot of input signal.

Thus, the bandwidth of the input signal is 23 Hz. This means that the Nyquist rate is $2 \cdot 23 = 46\,\mathrm{Hz}$. The signal would need to be sampled at at least 46 Hz in order to be able to reconstruct the original signal from its samples. Thus, none of the answer is correct and thus answer B is correct.

1.80. Question 8

Answer B is correct. Note that $\cos(6\pi t - \pi/7)$ will cause peaks at -3 Hz and +3 Hz. Thus, the peak at 2 Hz must be caused by a copy located at +5 Hz, thus the sampling frequency is 5 Hz and thus answer B is correct.

1.81. Question 9

Answer A is correct. $x(t)$ ranges between -2.5 and 2.5, i.e. a range of 5 V. If we want resolution of 0.2 V, we need 5/0.2=25 steps, which corresponds to $\log_2(25) \approx 5$ bits (*always* round up).

1.82. Question 10

Answer B is correct. Note that they give γ and not β. Thus, apparently we have the probability

$$P\left(\mathbf{y} > k_\gamma | H_a \text{ is true}\right) = 0.1492$$

Looking this up from the table, this requires $k_{\gamma_z} = 1.04$, i.e. $k_\gamma = \sigma k_{\gamma_z} + \mu = 2 \cdot 1.04 + 2 = 4.08$. Then, we must compute

$$\alpha = P\left(\mathbf{y} > k_\gamma | H_0 \text{ is true}\right)$$

which becomes

$$\alpha = P\left(\frac{\mathbf{y{-}2}}{2} > \frac{4.08 - -2}{2}\right) = P\left(\frac{\mathbf{y{+}2}}{2} > 3.04\right) = 0.0012$$

as can be looked up from the formula sheet.

1.83. Question 11

Answer C is correct. If A is doubled, the argument of Φ increases as well, meaning that $\Phi()$ becomes larger too. This decreases $P(E)$. This also makes sense from a physical point of view: if A is increased, it becomes easier to distinguish between 0s and 1s, and thus the bit error probability decreases. Nothing can be said about how much it decreases though.

1.84. Question 12

Answer A is correct. We have:

$$\begin{aligned}
(P_{T_x})_{\text{dB}} &= 10\log_{10}(250) = 23.979\,\text{dBW} \\
(G_{FS})_{\text{dB}} &= 10\log_{10}\cdot\left(\frac{\frac{3\cdot 10^8}{1090\cdot 10^6}}{4\pi\cdot 150\cdot 1852}\right)^2 = -142.06\,\text{dB}
\end{aligned}$$

Thus, the power at the receiver's end is $23.979 - 142.06 = -118.086\,\text{dBW}$.

1.85. Question 13

1.85.1. Part a)

The signal looks like shown below, but shifted down 3 units (can't be bothered to make a new sketch). The average of this signal is rather clearly -2; this is confirmed by computing

$$X_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2}(t-2)\,dt = \left[\frac{t^2}{2} - 2t\right]_{-T_0/2}^{T_0/2} = -2$$

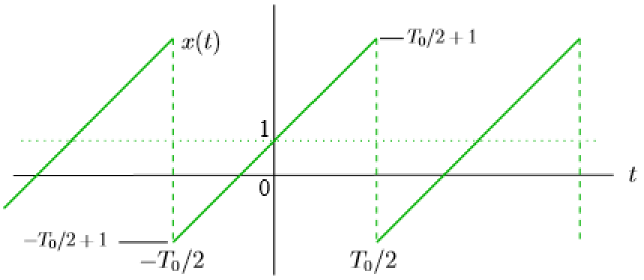


Figure 1.16: Saw-tooth wave.

1.85.2. Part b)

From a mathematical point of view, this signal is neither odd, even nor half-wave odd. However, if one subtracts the average value of -2 (i.e. shift it up two units), the graph becomes odd, although it remains neither even nor half-wave odd.

1.85.3. Part c)

The complex Fourier series is given by

$$x(t) = \sum_{n=-\infty}^{n=\infty} X_n e^{jn\omega_0 t}$$

with $\omega_0 = \frac{2\pi}{T_0}$ and the X_n are given by

$$X_n = \frac{1}{T_0} \int_{T_0} x(t) e^{-jn\omega_0 t} dt$$

For $n = 0$, this reduces to

$$X_0 = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} (t-2)\, dt = -2$$

as calculated previously. For $n \neq 0$, the computation becomes

$$X_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} te^{-jn\omega_0 t} dt + \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} -2e^{-jn\omega_0 t} dt = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} te^{-jn\omega_0 t} dt - \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} e^{-jn\omega_0 t} dt$$

The second integral reduces to zero as integrating the complex exponential over an entire period reduces to zero. For the other integral, we can use the standard integral given on the formula sheet:

$$X_n = \frac{1}{T_0} te^{-jn\omega_0 t} dt = \frac{1}{T_0} \left(\left[\frac{te^{-jn\omega_0 t}}{-jn\omega_0} \right]_{-T_0/2}^{T_0/2} - \frac{1}{-jn\omega_0} \int_{-T_0/2}^{T_0/2} e^{-jn\omega_0 t} dt \right)$$

The second integral again reduces to zero as it is integrating a complex exponential over a full period. Then, the term of the left can be worked out using $\omega_0 = \frac{2\pi}{T_0}$:

$$\begin{aligned} X_n &= \frac{1}{T_0}\left[\frac{\frac{T_0}{2}\cdot e^{-jn\frac{2\pi}{T_0}\frac{T_0}{2}}}{-jn\frac{2\pi}{T_0}} - \frac{-\frac{T_0}{2}e^{-jn\frac{2\pi}{T_0}\frac{-T_0}{2}}}{-jn\frac{2\pi}{T_0}}\right] \\ &= \frac{1}{T_0}\left[\frac{jT_0^2}{4\pi}\cdot(-1)^n - \frac{-jT_0^2}{4\pi}\cdot(-1)^n\right] = \frac{jAT_0}{2\pi}\cdot(-1)^n \end{aligned}$$

where we used $1/(-j) = j$ and $e^{-jn\pi} = (-1)^n$. So, the result is $X_0 = -2$ and for $n \neq 0$, $X_n = jAT/(2\pi)\cdot(-1)^n$.

1.86. Question 14

1.86.1. Part a)

Easy peasy lemon squeezy. We use $\sin^2(u) = \frac{1}{2} - \frac{1}{2}\cos(2u)$ to write

$$x(t) = 2 + 4\text{sinc}(4t) - 4\text{sinc}(4t)\cos(14\pi t)$$

Using pairs 9, 2 and the modulation theorem, the Fourier transform of this becomes

$$X(f) = 2\delta(f) + \Pi\left(\frac{f}{4}\right) - \frac{1}{2}\Pi\left(\frac{f-7}{4}\right) - \frac{1}{2}\Pi\left(\frac{f+7}{4}\right)$$

1.86.2. Part b)

Note that this signal is a *power* signal: as $t \to \pm\infty$, $x(t) \to 2$. So, the energy is infinite.

1.86.3. Part c)

We need to from

$$x(t) = 2 + 4\text{sinc}(4t) - 4\text{sinc}(4t)\cos(14\pi t)$$

to

$$y(t) = 6\text{sinc}(4t)\cos(14\pi t)$$

This is achieved by making imposing a filter. To see what filter needs to be applied, let's consider the Fourier transforms. We already established:

$$X(f) = 2\delta(f) + \Pi\left(\frac{f}{4}\right) - \frac{1}{2}\Pi\left(\frac{f-7}{4}\right) - \frac{1}{2}\Pi\left(\frac{f+7}{4}\right)$$

The Fourier transform of the response is

$$Y(f) = \frac{3}{4}\Pi\left(\frac{f+7}{4}\right) + \frac{3}{4}\Pi\left(\frac{f-7}{4}\right)$$

These pulses are located between $-9 < f < -5$ Hz and $5 < f < 9$ Hz. Thus, we need a filter that filters out all components below 5 Hz. For example, a bandpass filter, centered at 7 Hz with bandwidth 2.5 Hz would succeed in getting rid of the other two terms in $x(t)$ without losing the remaining pulse. The gain would have to be -1.5, as the amplitudes of the remaining components are multiplied by -1.5 (from comparison between $X(f)$ and $Y(f)$). The bode plot is shown below; $f_L = 4.5$ Hz, $f_0 = 7$ H and $f_H = 10.5$ Hz. Note that the amplitude should be $20\log(1.5) = 3.522$ dB.

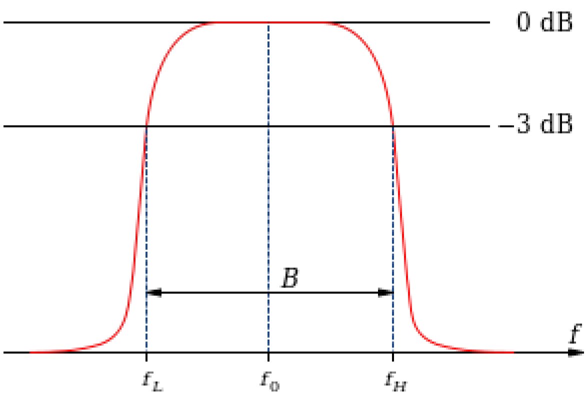


Figure 1.17: Bode plot of bandpass filter.

1.86.4. Part d)

Note that the output signal *is* an energy signal this time. The Fourier transform aws

$$Y(f) = \frac{3}{4}\Pi\left(\frac{f+7}{4}\right) + \frac{3}{4}\Pi\left(\frac{f-7}{4}\right)$$

so the energy is

$$E_y = \left(\frac{3}{4}\right)^2 \cdot 4 \cdot 2 = 4.5\text{J}$$

1.86.5. Part e)

There are multiple frequency response functions possible; a high-pass filter would have worked for example (if it only passed through frequencies that are higher than 4.5 Hz, for example).

1.87. Question 15

1.87.1. Part a)

The PSD $S(f)$ is defined as

$$S(f) = \frac{|X(f)|^2}{T}$$

We can use an heuristic approach to determine this: if we consider one bit, then $T = T_b$, then the signal is described by

$$x(t) = \pm A\Pi\left(\frac{t}{T_b}\right)$$

of which the Fourier transform is

$$X(f) = \pm AT_b\text{sinc}(T_b f)$$

so that

$$S(f) = \frac{A^2T_b^2\text{sinc}(T_b f)}{T_b} = A^2T_b\text{sinc}(T_b f)$$

1.87.2. Part b)

The power of a polar signal is

$$P_x = (-A)^2 \cdot \frac{1}{2} + A^2 \cdot \frac{1}{2} = A^2$$

so that the power over merely the positive frequencies will be $A^2/2$. The equivalent bandwidth is such that when multiplied with the value of the highest peak in the PSD (i.e. $S(0)$, the area of th rectangle equals the power over the positive frequencies. $S(0) = A^2 T_b$, so we must have

$$A^2/2 = A^2 T_b B_{eq}$$

so that $B_{eq} = 1/(2T_b)$.

1.87.3. Part c)

As computed, $P_x = A^2 = \left(0.3 \cdot 10^{-6}\right)^2 = 9 \times 10^{-14}\,\text{W}$. The bandwidth is

$$B = B_{eq} = \frac{1}{2T_b} = \frac{1}{2 \cdot 0.1 \cdot 10^{-6}} = 5 \times 10^6\,\text{Hz}$$

so that

$$P_e = kTB = 1.38 \cdot 10^{-23} \cdot 326 \cdot 5 \cdot 10^6 = 2.2494 \times 10^{-14}\,\text{W}$$

Converting both to decibels and subtracting yields

$$\text{SNR} = 10 \log\left(9 \cdot 10^{-14}\right) - 10 \log\left(2.2 \cdot 10^{-14}\right) = 6.022\,\text{dB}$$

1.87.4. Part d)

Regarding questions d) and e), I'm not 100% sure where to start derivations. I emailed prof. Mulder about this, but his reply was basically that it never hurts to show off your know knowledge, and that he preferred a mathematical derivation but if you come up with a logical reasoning, it's also fine. However, he did not explicitly state that it was fine to directly give the end-result.

To start off, note that this is actually *single*-observation: the maximum sample rate at which independence of observations can be achieved is $1/(2B)$, so the largest number of observations that can be made during a bit duration T_b is

$$m = \frac{T_b}{\frac{1}{2B}} = 2T_b B = 2 \cdot 0.1 \cdot 10^{-6} \cdot 5 \cdot 10^6 = 1$$

Thus, it's actually one-dimensional, for which the testing principle is (relatively) straightforwardly determined:

$$\text{reject } H_0 \text{ if } \frac{f_{\mathbf{y}}(y|H_0)}{f_{\mathbf{y}}(y|H_a)} < \frac{c_a}{c_0}$$

Set $c_a = c_0$ as α and β are equally valuable. We then have

$$\begin{aligned}
\frac{\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(y-x_0)^2}}{\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(y-x_a)^2}} &< 1 \\
e^{-\frac{1}{2\sigma^2}\left((y-x_0)^2-(y-x_a)^2\right)} &< 1 \\
-\frac{1}{2\sigma^2}\left((y-x_0)^2-(y-x_a)^2\right) &< 0 \\
(y-x_0)^2-(y-x_a)^2 &> 0 \\
y^2-2x_0y+x_0^2-y^2+2yx_a-x_a^2 &> 0 \\
2y(x_a-x_0) &> x_a^2-x_0^2=(x_a-x_0)(x_a+x_0) \\
y &> \frac{x_a+x_0}{2} \\
\frac{y-x_0}{\sigma} &> \frac{x_a-x_0}{2\sigma}
\end{aligned}$$

Thus, looking at the second-to-last row, the best threshold is to set the threshold at the average of x_a and x_0. This makes sense: since both distributions are the same, just shifted, this will make α and β equally likely. We say it's a 1-bit if the observation $y>0$; we say it's a 0-bit if the observation $y<0$.

1.87.5. Part e)

From the previous question, we see that we reject H_0 if

$$\frac{y-x_0}{\sigma} > \frac{x_a-x_0}{2\sigma}$$

Furthermore, we have

$$P(E) = P(E|H_0)P(H_0)+P(E|H_1)P(H_1) = \alpha\cdot\frac{1}{2}+\beta\cdot\frac{1}{2}$$

Since we set our threshold such that $\alpha=\beta$, this leads to

$$P(E)=\alpha$$

Furthermore, we have

$$\alpha = P\left(\frac{y-x_0}{\sigma} > \frac{x_a-x_0}{2\sigma}\right) = 1-\Phi\left(\frac{x_a-x_0}{2\sigma}\right)$$

Here, $x_a = 3\times10^{-7}$ V, $x_0 = -3\times10^{-7}$ V and $\sigma = \sqrt{P_e} = \sqrt{kTB} = \sqrt{1.38\cdot10^{-23}\cdot326\cdot5\cdot10^6} = 1.5\times10^{-7}$ V, so that we get

$$P(E) = \alpha = 1-Phi\left(\frac{3\cdot10^{-7}--3\cdot10^{-7}}{2\cdot1.5\cdot10^{-7}}\right) = 1-\Phi(2) = 0.0228$$

as easily visible from the formula sheet.

1.87.6. Part f)

If A is increased, then the detection error probability will decrease: a higher A makes it easier to distinguish between signal values A and $-A$, and will thus reduce the detection error probability.

Part VII

July 2016

1.88. Question 1

Correct is C.

- Statement 1 is false: for it to be even, all b_m would need to be 0.
- Statement 2 is false: for it to be half-odd, *all* even coefficients would need to be zero.
- Statement 3 is correct: $a_0 = 2$.

1.89. Question 2

Correct is D. Note that the second term is simply the time reversal of the first term: thus, we only need to compute $X'(f)$ of the left term; for the right term it'll then be $X'(-f)$. From the formula sheet, we see that

$$\begin{aligned} x'(t) &= 4e^{-2t}u(t) \\ X'(f) &= \frac{4}{2+j2\pi f} \end{aligned}$$

Thus, our Fourier transform is

$$\begin{aligned} X(f) &= X'(f)+X'(-f) = \frac{4}{2+j2\pi f}+\frac{4}{2-j2\pi f} = \frac{4(2-j2\pi f)+4(2+j2\pi f)}{(2+j2\pi f)(2-j2\pi f)} \\ &= \frac{8-8j\pi f+8+8j\pi f}{4+(2\pi f)^2} = \frac{16}{4+(2\pi f)^2} \end{aligned}$$

1.90. Question 3

Correct is answer D. The frequency of the left term is $\frac{3}{8}$ Hz; the frequency of the right term is 2 Hz. Then, the fundamental frequency is the largest frequency of which both frequency are an integer multiple; this is $\frac{1}{8}$ Hz.

1.91. Question 4

Correct is answer D. If you sketch the signal, you'll see that for $t \leq 0$, its value is 0. After $t = 6$, its value is 3 (and never changes after that). This means that the energy is infinite; the power is

$$P_x = \frac{1}{2}\cdot 0^2 + \frac{1}{2}\cdot 3^2 = 4.5\,\mathrm{W}$$

1.92. Question 5

Answer C is correct. Rather clearly, it is linear; it is time varying due to $\cos(2\pi f_0 t)$ and it is instantaneous as there are no derivatives or integrals.

1.93. Question 6

Answer A is correct. We first rewrite it to (totally ignoring phase)

$$x(t) = 8+8\cos(32\pi t)+8\mathrm{sinc}(4t)\cos(16\pi t)$$

The Fourier transform of this is

$$X(f) = 8\delta(f) + 4\delta(f-16) + 4\delta(f+16) + \Pi\left(\frac{f-8}{4}\right) + \Pi\left(\frac{f+8}{4}\right)$$

Thus, the power of the input signal is $8^2 + 4^2 + 4^2 = 96\,\text{W}$ and thus answer B is false. If we have a bandpass filter centered at 9 Hz with bandwidth 8 Hz, all components outside of $-13 < f < -5$ and $5 < f < 13$ are discarded. This means we will be merely left with

$$X(f) = \Pi\left(\frac{f-8}{4}\right) + \Pi\left(\frac{f+8}{4}\right)$$

as these only range from $-10 < f < -6$ and $6 < f < 10$. Note that these are amplified by a factor 2, so we actually get

$$Y(f) = 2\Pi\left(\frac{f-8}{4}\right) + 2\Pi\left(\frac{f+8}{4}\right)$$

The energy of this signal is

$$E = 2 \cdot 2^2 \cdot 4 = 32\,\text{J}$$

1.94. Question 7

The correct answer is D. We first rewrite it to

$$x(t) = \text{sinc}(6t) + \text{sinc}(6t) \cdot \cos(40\pi t)$$

The Fourier transform of this is

$$X(f) = \frac{1}{6}\Pi\left(\frac{f}{6}\right) + \frac{1}{12}\Pi\left(\frac{f-20}{6}\right) + \frac{1}{12}\Pi\left(\frac{f+20}{6}\right)$$

Thus, this has frequency components ranging from $-23 < f < 23$ Hz. Thus, the Nyquist rate of this signal is 46 Hz.

1.95. Question 8

Correct is answer A. If you place a copy at 30 Hz, you see that you get the shown graph: the upwards slope of the left part of the left triangle would cancel exactly out with the downward slope right part of the right triangle.

1.96. Question 9

Correct is answer B. x ranges between -1 and 3, i.e. a range of 4. Then, we need $4/0.015 = 266.67$ steps. So we need $\log_2(266.67) \approx 9$ bits (round *up*wards).

1.97. Question 10

Correct is A. We have $\beta = 0.0668$, so we have $k_{\beta_z} = 1.5$, meaning we have

$$k_\beta = \mu - k_{\beta_z} \cdot \sigma = 2 - 1.5 \cdot 2 = -1$$

Then, as the null-hypothesis is *centered* at -1, the probability of false alarm 0.5 (the critical value is exactly at the middle of the normal distribution).

1.98. Question 11

Correct is D. Per 20 meter, we lose $10\log(0.99) = -0.0436$ dB, or 0.002182 dB per meter. We can lose 2 dB, we get $2/0.002182 = 916.421\,\text{m}$.

1.99. Question 12

Answer A is correct. We have:

$$\begin{aligned}(P_{T_x})_{\text{dB}} &= 10\log_{10}(300) = 24.771\,\text{dBW}\\ (G_{FS})_{\text{dB}} &= 10\log_{10}\cdot\left(\frac{\frac{3\cdot 10^8}{1090\cdot 10^6}}{4\pi\cdot 200\cdot 1852}\right)^2 = -144.56\,\text{dB}\end{aligned}$$

Thus, the power at the receiver's end is $24.771 - 144.56 = -119.79\,\text{dBW}$.

1.100. Question 13

1.100.1. Part a)

The average is clearly 2; the average value of sines and cosines are both 0, so the average value is 2.

1.100.2. Part b)

The frequency of the first term is 1.5 Hz, of the second signal it is 4.5 Hz. Thus, the highest common frequency is 1.5 Hz (both frequencies are an integer multiple of this frequency). This means one period lasts for $1/1.5 = \frac{2}{3}$ seconds.

1.100.3. Part c)

We have

$$x(t) = \cos(3\pi t) - 2\sin(9\pi t + \pi/3) + 2$$

We get of the phase by use of

$$\sin(u+v) = \sin(u)\cos(v) + \cos(u)\sin(v)$$

i.e.

$$\sin(9\pi t + \pi/3) = \sin(9\pi t)\cos(\pi 3) + \cos(9\pi t)\sin(\pi 3) = \frac{1}{2}\sin(9\pi t) + \frac{\sqrt{3}}{2}\cos(9\pi t)$$

so that we get

$$x(t) = \cos(3\pi t) - \sin(9\pi t) - \sqrt{3}\cos(9\pi t) + 2$$

Thus, since $\omega_0 = 3\pi$, the Fourier series coefficients are $a_0 = 2$, $a_1 = 1$, $a_3 = -\sqrt{3}$, $b_3 = -1$ (all others are zero).

1.100.4. Part d)

For $c = \pi/6 + k\pi$ with k an integer: in that case, the sine is shifted exactly a phase of $\pi/2$: this makes it a cosine: if you shift it another phase π, it's again a cosine etc., etc. This means that your original signal consists of only cosines, and thus the Fourier series will also only consist of cosines.

With a radial velocity of 9π rad/s, this $c = \pi/6$ corresponds to

$$\frac{\frac{\pi}{6}}{9\pi} = \frac{1}{54}\text{seconds}$$

Every k then adds another $\pi/(9\pi) = 1/9$ seconds to it.

1.101. Question 14

1.101.1. Part a)

The Fourier transform is simply

$$X(f) = 2\Pi\left(\frac{f}{12}\right)$$

The energy of this is simply (it's a pulse of width 12 Hz with amplitude 2)

$$E = 2^2 \cdot 12 = 48\,\mathrm{J}$$

1.101.2. Part b)

The Fourier transform of

$$z(t) = 24\mathrm{sinc}(12t)\cos(2\pi f_c t) = \Pi\left(\frac{f-f_c}{12}\right) + \Pi\left(\frac{f+f_c}{12}\right)$$

The energy of this will be

$$E = 2 \cdot 1^2 \cdot 12 = 24\,\mathrm{J}$$

1.101.3. Part c)

As only between 2 to 8 Joules of energy may be left over, this means that 1/12 to 1/3 of the energy must be filtered out. This means that 1/12 to 1/3 from both pulses must be outside the filter. Now, the filter is centered at 100 Hz and has bandwidth 10 Hz the width of the pulses themselves is 12 Hz. In the case that only 1/12 would be preserved, only 1 Hz of this pulse would be saved. This means that the pulse would range from 84 Hz to 96 Hz, so centered at 90 Hz. Alternatively, it ranges from 104 Hz to 116 Hz, so centered at 110 Hz.

In the case 1/3 of the pulse is preserved, only 4 Hz of the pulse would be within the bandwidth of the filter. Thus, it would range from 87 to 99 Hz (i.e. centered at 93 Hz), or alternatively, from 101 Hz to 113 Hz (i.e. centered at 107 Hz). Thus, valid frequencies modulation frequencies would be from $90 < f_c < 93$ and $107 < f_c < 110$ Hz.

1.102. Question 15

1.102.1. Part a)

Half of the time, the amplitude of the signal is 0; the other half of the time, it's A; thus, the power is

$$P_x = A^2 \cdot \frac{1}{2} + 0^2 \cdot \frac{1}{2} = \frac{A^2}{2}$$

1.102.2. Part b)

For unipolar signalling,

$$x_a - x_0 = \begin{bmatrix} A \\ \vdots \\ A \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} A \\ \vdots \\ A \end{bmatrix}$$

so that

$$\nabla = ||x_a - x_0|| = \sqrt{A^2 + A^2 + \ldots + A^2} = \sqrt{m}A$$

Thus,

$$P(E) = 1 - \Phi\left(\frac{\nabla}{2\sigma}\right) = 1 - \Phi\left(\frac{\sqrt{m}A}{2\sigma}\right)$$

1.102.3. Part c)

We have

$$x_a - x_0 = \begin{bmatrix} A \\ \vdots \\ A \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = A\vec{u}_m$$

where $\vec{u}_m$ is a vector containing only 1s, consisting of m entries. Then,

$$c = \frac{x_a - x_0}{\nabla} = \frac{A\vec{u}_m}{\sqrt{m}A} = \frac{1}{\sqrt{m}}\vec{u}_m$$

Thus, we get

$$\begin{aligned} \frac{c^T(y - x_0)}{\sigma} &> \frac{\nabla}{2\sigma} \\ \frac{\vec{u}_m^T(y - x_0)}{\sqrt{m}} &> \frac{\sqrt{m}A}{2} \end{aligned}$$

Here,

$$y - x_0 = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} - \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

so

$$\begin{aligned} \frac{\vec{u}_m^T(y - x_0)}{\sqrt{m}} &> \sqrt{m}\frac{A}{2} \\ \frac{1}{m}(1 \cdot y_1 + 1 \cdot y_2 + \ldots + 1 \cdot y_m) &> \frac{A}{2} \\ \frac{1}{m}\sum_{i=1}^{m} y_i &> \frac{A}{2} \end{aligned}$$

Thus, when the average value of the observations is larger than half the amplitude, it will be concluded that the incoming bit is a 1; if it's less than half the amplitude, it'll be concluded it's a 0. This makes perfect sense: the probabilities of 0 and 1s is equal, so it only makes sense to put the threshold in the middle.

1.102.4. Part d)

For OOK, half the time the amplitude is 0, and the other half the time the value is $A\cos(2\pi f_c t)$. This means that per bit, the power is

$$P = \frac{1}{T_b}\int_0^{T_b}\left[\frac{1}{2}\cdot 0^2 + \frac{1}{2}\cdot(A\cos(2\pi f_c t))^2\right]dt$$

We substitute $\cos^2(u) = \frac{1}{2} + \frac{1}{2}\cos(2u)$, so that we get

$$\begin{aligned} P &= \frac{1}{T_b}\int_0^{T_b}\left[\frac{1}{2}\cdot 0^2 + \frac{1}{2}\cdot\left(\frac{A^2}{2} + \frac{A^2}{2}\cos(4\pi f_c t)\right)\right]dt \\ &= \frac{1}{T_b}\int_0^{T_b}\frac{A^2}{4}dt + \frac{1}{T_b}\int_0^{T_b}\frac{A^2}{4}\cos(4\pi f_c t)\,dt \end{aligned}$$

The first integral is simply $A^2T_b/4$; the second integral reduces to 0 as we are integrating over a whole period. Thus, the power is

$$P = \frac{1}{T_B}\cdot\frac{A^2T_b}{4} = \frac{A^2}{4}$$

1.102.5. Part e)

We have

$$P(E) = P(E|H_0)P(H_0) + P(E|H_1)P(H_1) = \alpha\frac{1}{2} + \beta\frac{1}{2}$$

with $\alpha = \beta$, this becomes

$$P(E) = \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$$

Furthermore,

$$\alpha = P\left(\frac{c^T(y-x_0)}{\sigma}\right) = 1-\Phi\left(\frac{\nabla}{2\sigma}\right)$$

Now, ∇ is not $\sqrt{m}A$ as calculated previously (that was for unipolar); for unipolar, it'll be:

$$\begin{aligned}
x_a - x_0 &= \begin{bmatrix} A\cos(2\pi f_c T_s) \\ A\cos(2\pi f_c 2T_s) \\ \vdots \\ A\cos(2\pi f_c m T_s)\end{bmatrix} - \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} = \begin{bmatrix} A\cos(2\pi f_c T_s) \\ A\cos(2\pi f_c 2T_s) \\ \vdots \\ A\cos(2\pi f_c m T_s)\end{bmatrix} \\
\nabla &= \sqrt{A^2\cos^2(2\pi f_c T_s) + A^2\cos^2(2\pi f_c 2T_s) + \ldots + A^2\cos^2(2\pi f_c m T_s)} = A\sqrt{\sum_{i=1}^{m}\cos^2(2\pi f_c i T_s)}
\end{aligned}$$

Making use of $\cos^2(u) = \frac{1}{2} + \frac{1}{2}\cos(2u)$, we get

$$\nabla = A\sqrt{\sum_{i=1}^{m}\frac{1}{2} + \sum_{i=1}^{m}\cos(4\pi f_c i T_s)}$$

The first sum is obviously $\frac{m}{2}$; the second sum is equal to 0: the average of a cosine is 0 after all. Thus, we get

$$\nabla = A\sqrt{\frac{m}{2}}$$

Furthermore $\sigma = \sqrt{P_e} = \sqrt{kTB} = \sqrt{N_0B}$, meaning we get

$$P(E) = \alpha = 1-\Phi\left(\frac{\sqrt{m/2}A}{2\sqrt{N_0B}}\right) = 1-\Phi\left(\sqrt{\frac{mA^2}{8N_0B}}\right)$$

Now, what is the maximum number of m (as this would lead to the minimum probability error)? It was shown in the reader that the maximum rate at which can be sampled to not cause correlated observations equals $1/(2B)$. Thus, the maximum number observations in a time period T_b is

$$m = \frac{T_b}{\frac{1}{2B}} = 2BT_b$$

Thus, we get

$$P(E) = 1-\Phi\left(\sqrt{\frac{2BT_bA^2}{8N_0B}}\right) = 1-\Phi\left(\sqrt{\frac{A^2T_b}{4N_0}}\right)$$

1.102.6. Part f)

It decreases: the larger the A, the easier it is to distinguish between signal values, thus the smaller the detection error probability.