# Question 1

The heat equation, $\frac{\partial u}{\partial t} = \upsilon \frac{\partial^2 u}{\partial x^2}$ is discretised with the following finite-difference method on a uniform mesh ($\Delta x$ is constant):

$$u_i^{n+1} = u_i^n + \upsilon \Delta t \frac{2u_i^n - 5u_{i+1}^n + 4u_{i+2}^n - u_{i+3}^n}{\Delta x^2}$$

Derive the modified equation. What statement is correct?

- The method is inconsistent.
- The method is consistent; first order accurate in space and first order accurate in time.
- The method is consistent; first order accurate in space and second order accurate in time.
- The method is consistent; second order accurate in space and first order accurate in time.
- The method is consistent; second order accurate in space and second order accurate in time.

# Question 2

Compute the expression for the amplification factor $\rho$ for the following implicit differentiation scheme for the linear advection equation, $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$:

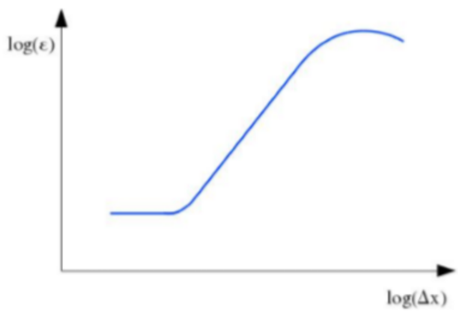
$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + c\frac{u_{i+1}^{n+1} - u_{i-1}^{n+1}}{2\Delta x} = 0$$

# Question 3

A discretisation for the following PDE, $\frac{\partial^3 u}{\partial x^3} + \frac{\partial u}{\partial t} = 0$, is designed to have the truncation error of the form

$$\epsilon = \mathcal{O}(\Delta x) + \mathcal{O}\left(\Delta t^2\right)$$

A mesh refinement study is performed, where $\Delta x$ is decreased by a factor 4 at each refinement, and $\Delta t$ by a factor 2 at each refinement. This produces the plot below.




Based on this information, what is the *most* likely cause of the levelling off in the curve shown above?

- Discretisation errors in space
- Discretisation errors in time
- Solution iteration errors
- Rounding errors

# Question 4

Consider the PDE

$$u_{xx} = f, \qquad \text{on} \quad 0 < x < 1, u_x(0) = 3, u(1) = 2t$$

## Part 1)

Indicate which of the following statements are correct when using the integrated-by-parts weak form of the solution, using the spectral approach:

- All basis functions of the discrete solution will be zero at $x = 1$.
- As $u_{xx}$ appears in the equation, all suitable basis functions must be at least twice differentiable without reducing to zero.

## Part 2)

For this problem, indicate which of the following functions would be suitable weighting functions:

- $w(x) = 1 - x$
- $w(x) = \ln(x)$
- $w(x) = \cos(\pi x)$
- $w(x) = x$
- $w(x) = 1/x$

## Part 3)

Consider now that instead of a Neumann boundary at $x = 0$, one would impose a Robin boundary at $x = 0$, so that the problem statement becomes

$$u_{xx} = f, \qquad \text{on} \quad 0 < x < 1, u(0) + u_x(0) = q,\ u(1) = 2t$$

Integration by parts will give an equation similar to

$$A - \int_0^1 B dx = \int_0^1 C dx + D$$

where $A$, $B$, $C$ and $D$ may or may not be zero. Give the expressions for all these quantities.

# Question 5

Consider the PDE

$$u_{xx} + 2u_x + u = 3x^2, \qquad \text{on} \quad -1 < u < 1, u_x(-1) = -18, u_x(1) = -6$$

The following basis functions are used to compute the discrete solution:

$$\begin{aligned}
T_0(x) &= \frac{\sqrt{2}}{2} \\
T_1(x) &= \frac{\sqrt{6}}{2} x \\
T_2(x) &= \frac{\sqrt{90}}{4} \left(2x^2 - 1\right) \\
T_3(x) &= \frac{\sqrt{1190}}{34} \left(4x^3 - 3x\right)
\end{aligned}$$

Find the expression for the discrete solution. Also, bonus points if you remember correctly how these polynomials are called (note that they are the normalised versions of a certain set of polynomials.

## Question 6

Consider the following problem on the domain $0 < x < 1$:

$$u_x - u_{xx} = f, \quad u(0) = u(1) = 0$$

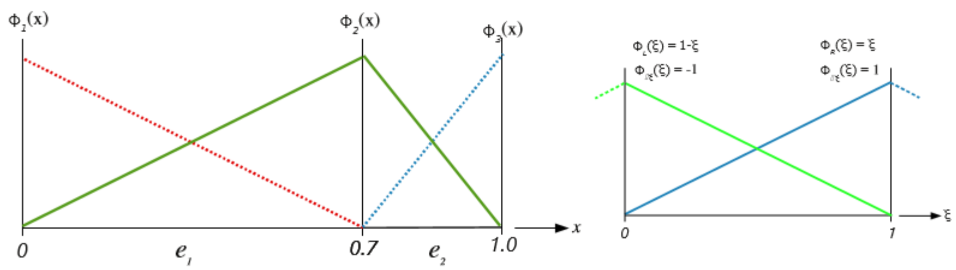
Integration by parts leads to

$$\int_0^1 (w + w_x)\, u_x dx - [w u_x]_0^1 = \int_0^1 w f dx$$

This is to be solved using a finite-element methods using a mesh with two elements and piecewise linear basis functions, as shown below (left). For the master element, the basis functions are:

$$\begin{aligned} \phi_L &= 1 - \xi \\ \phi_R &= \xi \end{aligned}$$

and the general transformation is $\xi = (x - x_0)/h$, where $x_0$ is the element starting coordinate and $h$ is its length.




This leads to a matrix equation

$$K\mathbf{a} = \mathbf{f}$$

where $K$ is the stiffness matrix of size $3 \times 3$.

### Part 1)

Find entry $K_{21}$, i.e. the entry in $K$ in the second row, first column.

### Part 2)

Find entry $K_{22}$, i.e. the entry in $K$ in the second row, second column.