Part I

Practice extra

1.1. Question 1

a) We simply apply the problem solving guide of the summary for this one. We get

$$\begin{aligned}
\frac{u_i^{n+1}-u_i^n}{\Delta t}+c\left(\frac{u_{i-2}^n-4u_{i-1}^n+3u_i^n}{2\Delta x}\right) &= 0\\
e^{a\Delta t}-1+\frac{c\Delta t}{2\Delta x}\cdot\left(e^{-I2k_m\Delta x}-4e^{-Ik_m\Delta x}+3\right) &= 0\\
\rho-1+m\cdot\left(e^{-2Ib}-4e^{-Ib}+3\right) &= 0\\
\rho &= 1-m\cdot\left(\cos(-2b)+I\sin(-2b)-4\cos(-b)-4I\sin(-b)+3\right)\\
&= 1-m\cdot\left(\cos(2b)-I\sin(2b)-4\cos(b)+4I\sin(b)+3\right)
\end{aligned}$$

Note that that it is actually not required to write out the exponentials as sines and cosines; you may also answer

$$\rho = 1-m\cdot\left(e^{-2Ib}-4e^{-Ib}+3\right)$$

b) Really basic question. We require $|\rho|\leq 1$ for the range $b=0$ to π.

1.2. Question 2

1.2.1. Part 1

Pretty obvious: we are constantly decreasing Δx, i.e. the mesh spacing in spatial direction. Thus, the discretisation errors in space will be constantly decreased, and not level off.

1.2.2. Part 2

We simply have

$$p_0=\frac{\ln\left(\frac{f_3-f_2}{f_2-f_1}\right)}{\ln(r)}=\frac{\ln\left(\frac{1.322-1.080}{1.080-1.076}\right)}{\ln(4)}=2.959$$

1.2.3. Part 3

We have

$$f_e=f_1+\epsilon=f_1+\frac{f_1-f_2}{r^p-1}=1.076+\frac{1.076-1.080}{4^3-1}=1.07594$$

1.3. Question 3

a) An unsuitable weighting function would be $1/x$: the derivative of $1/x$ is $-1/x^2$; we'd then would have to integrate

$$\int_0^1\left(-\frac{1}{x^2}\right)^2dx=\int_0^1\frac{1}{x^4}dx$$

which goes to infinity (and is thus not suitable). All other functions do not go to infinitely if their derivatives squared are integrated.

b) A suitable weighting function w would then be $1-x$, as that one is the only one to go to zero at the boundary (for Dirichlet boundary conditions, it is required that the weighting functions are zero at the boundaries).

1.4. Question 4

a) That entry is equal to

$$\begin{aligned}\int_0^\pi \phi_1(x)\sin(x)\,dx &= \int_0^\pi \sin^2(x)\,dx = \left[\frac{x}{2}-\frac{\sin(x)\cos(x)}{2}\right]_0^\pi \\ &= \frac{\pi}{2}-\frac{\sin(\pi)\cos(\pi)}{2}-\frac{0}{2}+\frac{\sin(0)\cos(0)}{2}=\frac{\pi}{2}\end{aligned}$$

b) You can do this in a horrendously complicated way, or you can do this by just being smart. Due to the Dirichlet boundary conditions, we see that ϕ_2 and ϕ_3 will both not be used in the final solution, as they are not equal to zero at the boundaries. Thus, essentially, our solution will exist of only one basis function, $\phi_1(x)=\sin(x)$. This means we only have one equation to satisfy:

$$\int_0^\pi wu\,dx+\int_0^\pi w_x u_x\,dx=\int_0^\pi w\sin(x)\,dx$$

with $w=\sin(x)$, $u=a_1\sin(x)$ and $u_x=a_1\cos(x)$ (note that the term in the middle disappears due to the Dirichlet boundary conditions, due to which $w(0)=w(\pi)=0$). This leads to

$$\begin{aligned}\int_0^\pi \sin(x)a_1\sin(x)\,dx+\int_0^\pi \cos(x)a_1\cos(x)\,dx &= \frac{\pi}{2}\\ a_1\int_0^\pi \sin^2(x)\,dx+a_1\int_0^\pi \cos^2(x)\,dx &= \frac{\pi}{2}\\ a_1\int_0^\pi\left(\sin^2(x)+\cos^2(x)\right)dx &= \frac{\pi}{2}\\ a_1\int_0^\pi 1\,dx=\pi a_1 &= \frac{\pi}{2}\end{aligned}$$

so $a_1=\frac{1}{2}$, and thus we get

$$u(x)=\frac{\sin(x)}{2}$$

1.5. Question 5

a) If we have $\xi=2\pi x$, then $x=\frac{\xi}{2\pi}$ and thus $dx=\frac{d\xi}{2\pi}$.

b) This one is a bit ugly. We have

$$c = \int_0^{1/2} \phi\phi_{xx} dx = \int_0^{\pi} \phi\phi_{xx} \frac{d\xi}{2\pi} = \frac{1}{2\pi}\int_0^{\pi} \phi\phi_{xx} d\xi$$

Now, $\phi = \sin(\xi)$, but note that $\phi_{xx} \neq -\sin(\xi)$. Instead, we have

$$\begin{aligned}
\phi &= \sin(\xi) \\
\phi_x &= \frac{d\phi}{d\xi}\frac{d\xi}{dx} = \cos(\xi)\cdot 2\pi \\
\phi_{xx} &= \frac{d\phi_x}{d\xi}\frac{d\xi}{dx} = -\sin(\xi)\cdot 2\pi\cdot 2\pi = -\sin(\xi)\cdot 4\pi^2
\end{aligned}$$

Thus, we get

$$c = \frac{1}{2\pi}\int_0^{\pi} \phi\phi_{xx} d\xi = \frac{1}{2\pi}\int_0^{\pi} \sin(\xi)\cdot -\sin(\xi)\cdot 4\pi^2 d\xi = -2\pi\int_0^{\pi} \sin^2(\xi)\, d\xi = -2\pi\cdot\frac{\pi}{2} = -\pi^2$$

Part II

Tutorial problems

1.6. Question 1

Using my problem solving guide, we get

$$e^{a\Delta t} = \frac{e^{Ik_m\Delta x}+e^{-Ik_M\Delta x}}{2} - r\frac{e^{Ik_m\Delta x}-e^{-Ik_m\Delta x}}{2} = \frac{e^{Ib}+e^{-Ib}}{2} - r\frac{e^{2Ib}-e^{2Ib}}{2} = \cos(b) - r\cdot I\cdot\sin(2b)$$

making use of the properties

$$\begin{aligned} \frac{e^x+e^{-x}}{2} &= \cos(x) \\ \frac{e^x-e^{-x}}{2} &= I\sin(x) \end{aligned}$$

1.7. Question 2

Let's write out the Taylor series:

$$\begin{aligned} u_i &= u_i \\ u_{i-1} &= u_i - \Delta x\frac{\partial u}{\partial x} + \frac{\Delta x^2}{2}\frac{\partial^2 u}{\partial x^2} + \ldots \\ u_{i-2} &= u_i - 2\Delta x\frac{\partial u}{\partial x} + \frac{(2\Delta x)^2}{2}\frac{\partial^2 u}{\partial x^2} + \ldots = u_i - 2\Delta x\frac{\partial u}{\partial x} + 2\Delta x^2\frac{\partial^2 u}{\partial x^2} + \ldots \end{aligned}$$

Plugging this in leads to

$$\begin{aligned} \frac{5u_i - 8u_{i-1} + 3u_{i-2}}{2\Delta x} &= 0 \\ \frac{5u_i - 8u_i + 8\Delta x\frac{\partial u}{\partial x} - 4\Delta x^2\frac{\partial^2 u}{\partial x^2} + \ldots + 3u_{i-2} - 6\Delta x\frac{\partial u}{\partial x} + 6\Delta x^2\frac{\partial^2 u}{\partial x^2} + \ldots}{2\Delta x} &= 0 \end{aligned}$$

Rewriting this leads to

$$\frac{\partial u}{\partial x} + \Delta x\frac{\partial^2 u}{\partial x^2} + \ldots = 0$$

Comparing this with the original equation we want to solve for, $\frac{\partial u}{\partial x} = 0$, we see that we have introduced a first order error (since Δx is first order; the $\frac{\partial^2 u}{\partial x^2}$ doesn't count for anything her). Thus, the correct answer is **Consistent, first order accurate**.

1.8. Question 3

We have

$$\begin{aligned} |\rho| &= \sqrt{(\mathrm{Re}(\rho))^2 + (\mathrm{Im}(\rho))^2} = \sqrt{\left(\frac{2c\Delta t}{\Delta x}\cos\beta\right)^2 + \left(\frac{2c\Delta t}{\Delta x}\sin\beta\right)^2} \\ &= \frac{2c\Delta t}{\Delta x}\sqrt{\cos^2\beta + \sin^2\beta} = \frac{2c\Delta t}{\Delta x} \end{aligned}$$

Remember that $|\rho| \leq 1$ for all $\beta \in [0,\pi]$, thus this method is stable provided $\frac{c\Delta t}{\Delta x} < \frac{1}{2}$ (it is also still stable if $\frac{c\Delta t}{\Delta x} \leq \frac{1}{2}$).

1.9. Question 4

1.9.1. Part 1

We have

$$\begin{aligned} u(x,t) &= \sin(t)\cos(x) \\ \frac{\partial u}{\partial t} &= \cos(t)\cos(x) \\ \frac{\partial^2 u}{\partial x^2} &= -\sin(t)\cos(x) \end{aligned}$$

so that we get

$$\frac{\partial u}{\partial t} - u\frac{\partial^2 u}{\partial x^2} = \cos(t)\cos(x) - \sin(t)\cos(x)\cdot -\sin(t)\cos(x) = \cos(t)\cos(x) + \sin^2(t)\cos^2(x)$$

i.e. $S(x,t) = \cos(t)\cos(x) + \sin^2(t)\cos^2(x)$.

1.9.2. Part 2

We have

$$\begin{aligned} u(0,t) &= \sin(t)\cos(0) = \sin(t) \\ u\left(\frac{\pi}{2},t\right) &= \sin(t)\cos\left(\frac{\pi}{2}\right) = 0 \end{aligned}$$

1.10. Question 5

1.10.1. Part 1

We simply have

$$p_0 = \frac{\ln\left(\frac{1.10-0.40}{0.40-0.18}\right)}{\ln(2)} = 1.670$$

1.10.2. Part 2

We get

$$f_e = f_1 + \epsilon = f_1 + \frac{f_1 - f_2}{r^p - 1} = 0.12 + \frac{0.12-0.18}{2^2-1} = 0.10$$

1.11. Question 6

First of all, if we perform the integration by parts ourselves, we get

$$[w(x)u(x)]_0^1 - \int_0^1 w_x u dx = \int_0^1 w f dx$$

Now, since we have a numerical boundary at $x=1$, $w(1)u(1)$ appears in the weak form (should be pretty obvious). Now, why doesn't something like $w(0)\cos(t)$ appear? Because for Dirichlet boundaries, we make sure in our own choosing of the weighting functions, that w is zero at those boundaries. Thus, $w(0)=0$ and $w(0)\cos(t)$ disappears from the equation. In short, correct is

$$w(1)u(1) - \int_0^1 w_x u dx = \int_0^1 w f dx$$

1.12. Question 7

Again, you can fully set up the matrix equation, but you can use a short-cut. Since $u(0)=0$, we recognize that $\phi_1(x)$ won't be part of the final solution: it coefficient a_1 would need to be zero, so we can just ignore that. This means that we merely get $u(x)=a_2x$, so that

$$\begin{aligned} w &= x \\ u_x &= a_2 \end{aligned}$$

Plugging that in, we get

$$\begin{aligned} \int_0^1 w(u_x+u)\,dx &= \int_0^1 w\,dx \\ \int_0^1 x(a_2+a_2x)\,dx &= \int_0^1 x\,dx \\ a_2\int_0^1 \left(x+x^2\right)dx &= \int_0^1 x\,dx \\ a_2\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_0^1 &= \left[\frac{x^2}{2}\right]_0^1 \\ a_2\left[\frac{1}{2}+\frac{1}{3}\right] &= \frac{1}{2} \\ a_2 &= \frac{3}{5} \end{aligned}$$

Thus, our discrete solution is

$$u(x)=\frac{3x}{5}$$

1.13. Question 8

Note that

$$w_2=\phi_2(x)=\begin{cases} x & :0\leq x\leq 1 \\ 2-x & :1\leq x\leq 2 \end{cases}$$

Thus, we have to perform the integral

$$\begin{aligned} \int_0^2 w_2(x)\,f\,dx &= \int_0^2 w_2\cdot 1dx=\int_0^2 w_2dx \\ &= \int_0^1 xdx+\int_1^2 (2-x)\,dx=\left[\frac{x^2}{2}\right]_0^1+\left[2x-\frac{x^2}{2}\right]_1^2=\left[\frac{1^2}{2}-\frac{0^2}{2}\right]+\left[2\cdot 2-\frac{2^2}{2}-2\cdot 1+\frac{1^2}{2}\right]=1 \end{aligned}$$