Part I

Tutorial

1.1. Question 1

The correct answer is **epistemic uncertainty**: these were uncertainties associated with deviations of the model from the real system due to things you didn't expect to happen. Clearly, the description in the question matches this.

1.2. Question 2

The correct answer is **stable, but not necessarily consistent.** The engineer only proves that the solution grows towards a single outcome (which is stability). However, no where it is stated that the model produces the *correct* result, so it is not necessarily consistent.

1.3. Question 3

The highest order derivative in y-direction is $\frac{\partial u}{\partial y}$; the highest order derivative in x-direction is $\frac{\partial u}{\partial x}$ and we have the cross derivative $\frac{\partial^2 u}{\partial x \partial y}$. One of these terms ($\frac{\partial u}{\partial x}$) appears squared, so the equation is non-linear.

1.4. Question 4

We have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} = 0$$

so that $\alpha = 1$ and $\beta = -1$, and thus $r_1 = \frac{1+\sqrt{1^2-4\cdot -1}}{2} = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{1^2-4\cdot -1}}{2}$ and thus the factorization is

$$\left(\frac{\partial}{\partial x} + \frac{1+\sqrt{5}}{2}\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + \frac{1-\sqrt{5}}{2}\frac{\partial}{\partial y}\right) = 0$$

From the left one, we have

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{1+\sqrt{5}}{2}\frac{\partial u}{\partial y} &= 0 \\ \frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds} &= 0 \end{aligned}$$

so from comparison, we have $\frac{dx}{ds} = 1$ and $\frac{dy}{ds} = \frac{1+\sqrt{5}}{2}$. Thus, we get

$$\begin{aligned} x &= s + C_1 \\ y &= \frac{1+\sqrt{5}}{2}s + C_2 \end{aligned}$$

We set $x = 0$ when $s = 0$, so that $C_1 = 0$. Furthermore, we set $y = y_0$ when $s = 0$, so that we get $C_2 = y_0$, so that we get (since $s = x$ when $C_1 = 0$)

$$y = \frac{1+\sqrt{5}}{2}x + y_0$$

It should be logical that this would lead to

$$y = \frac{1-\sqrt{5}}{2}x + y_0$$

for the other set of characteristic lines. Thus, the correct answer is

$$A = \frac{1+\sqrt{5}}{2}$$

1.5. Question 5

Note that the characteristic lines defined by the lines $y = x + \text{constant}$ are slightly less inclined than the boundary of B (which has a slope $dy/dx = 2$). This means that are three ways a characteristic lines crosses the boundaries:

- A characteristic line enters via A, leaves via C from $y = 1$ to $y = 2$.
- A characteristic line enters via B, leaves via C from $y = 2$ to $y = 3$.
- A characteristic line enters via the x-axis, leaves via C from $y = 0$ to $y = 1$.

So, if you impose a boundary condition on B and a boundary condition on C from $y = 0$ to $y = 1$, *all* of the characteristic lines cross a boundary where a boundary condition is imposed: remember that along a characteristic, you only needed to know the value of the solution at one point and then you could automatically calculate the value of the solution at all other points of that characteristic. This is why every characteristic needs to cross one boundary condition. However, you may not impose more boundary conditions than this, because then a characteristic line would become overdetermined.

Furthermore, note that we are currently dealing with x and y coordinates: this means that there is no clear 'future' and 'past' in the domain: in the wave equation, we clearly saw a future (which was where t increases) which meant we were not allowed to impose boundary conditions there as that'd led to an ill-posed problem. However, if you are only talking about x and y, then you don't have this problem, as both are spatial coordinates.

Finally, keep in mind that this is not *always* true: for example, for the supersonic flow of the work session, you would still not be allowed to specify boundary conditions on the right boundary of the domain: in supersonic flow, there is still a clear 'future': disturbances can't propagate upstream for a supersonic flow, so it's wrong to impose boundary conditions downstream (but this doesn't hold for subsonic flow, where disturbances *do* propagate upstream).

1.6. Question 6

My values for α and β were $\alpha = 1$ and $\beta = 3$.

1.6.1. Part a

The entry for C_2 is straightforward: the distance from u_i to u_i is zero, so for the third row of the Taylor table, the entry is

$$b \cdot \frac{0^2}{2!} = 0$$

so the correct entry is 0.

1.6.2. Part b

The entry for C_4 follows easily as well: the distance from u_{i+2} to u_i is $\alpha + \beta$. Thus, for the third row, we get

$$d \cdot \frac{(\alpha + \beta)^2}{2!}$$

With $\alpha = 1$ and $\beta = 3$, I get the entry $8d$. However, note that your numbers may differ. Furthermore, don't forget to work out the fraction as much as possible (don't write $b\frac{(1+3)^2}{2!}$, for example).

1.6.3. Part c

If we use four nodes to approximate the second derivative, then the minimum order of accuracy will be $4-2=2$. In general, if we use n nodes to approximate the pth derivative, where the maximum value of p is $n-1$, the order of accuracy is $n-p$. It may be higher when the nodes are located symmetrically around the center of the nodes:

- If p is even, then the order of accuracy will be $n-p+1$.
- If p is odd, the order of accuracy remains $n-p$.

1.7. Question 7

The linear convection equation is

$$\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$$

We only need to consider the final equation of the matrix equation for this:

$$vu_3^{n+1}-4vu_4^{n+1}+u_5^{n+1}+3vu_5^{n+1}=u_5^n$$

Substituting $v=\frac{c\Delta t}{2\Delta x}$, this can be rewritten to

$$\begin{aligned}\frac{c\Delta t}{2\Delta x}u_3^{n+1}-4\frac{c\Delta t}{2\Delta x}u_4^{n+1}+u_5^{n+1}+\frac{3c\Delta t}{2\Delta x}u_5^{n+1} &= u_5^n\\ c\frac{u_3^{n+1}-4u_3^{n+1}+3u_5^{n+1}}{2\Delta x}+\frac{u_5^{n+1}-u_5^n}{\Delta t} &= 0\end{aligned}$$

which you should recognize as a discretization of $\frac{\partial u}{\partial t}+c\frac{\partial u}{\partial x}=0$. Thus, the condition for the final node is simply a **numerical condition**,m as we use the same PDE as before, but just a different discretization.

1.8. Question 8

1.8.1. Part a

We must compute

$$J=\left(x_\xi y_\eta-y_\xi x_\eta\right)$$

Here, the derivatives are each straightforwardly computed: for example, for x_ξ, we take the x-values corresponding to the points adjacent to E in horizontal direction, i.e. points F and D, and divide the difference by $2\Delta\xi=2$, i.e. we obtain

$$x_\xi=\frac{5-0}{2}=2.5$$

For y_η, we take the y-values for the points adjacent to E in vertical direction, i.e. H and B, and again divide by $2\Delta\eta=2$, so we obtain

$$y_\eta=\frac{2-0}{2}=1$$

For y_ξ, we take the difference in the y-values of the points adjacent to E in horizontal direction, i.e. points D and F, and divide by $2\Delta\xi=2$, so we obtain

$$y_\xi=\frac{1-1}{2}=0$$

Finally, for x_η, we take the difference in x-values of the points adjacent to E in vertical direction, i.e. points H and B, and divide by $2\Delta\eta = 2$, so we obtain

$$x_\eta = \frac{2-3}{2} = -\frac{1}{2}$$

Thus,

$$J = x_\xi y_\eta - y_\xi x_\eta = 2.5 \cdot 1 - 0 \cdot \frac{1}{2} = 2.5$$

1.8.2. Part b

We must obtain

$$\frac{\partial u}{\partial y} = u_\xi \xi_y + u_\eta \eta_y$$

Here, ξ_y and η_y may be obtained from computing the matrix equation they give:

$$\begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix} = J^{-1} \begin{bmatrix} y_\eta & -x_\eta \\ -y_\xi & x_\xi \end{bmatrix} = \frac{1}{2.5} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 2.5 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} \\ 0 & 1 \end{bmatrix}$$

Thus, from comparison, $\xi_y = 1/5$ and $\eta_y = 1$. Furthermore, for u_ξ we take the difference in values of u for the two nodes adjacent to E in horizontal direction, i.e. nodes D and F, and divide by $2\Delta\xi = 2$, so we obtain

$$u_\xi = \frac{2-1}{2} = \frac{1}{2}$$

Similarly, for u_η we take the difference in values of u for two nodes adjacent to E in vertical direction, i.e. nodes B and H, so we obtain

$$u_\eta = \frac{6-2}{2} = 2$$

Thus, we get

$$\frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{1}{5} + 2 \cdot 1 = \frac{21}{10} = 2.1$$

Part II

Practice extra

Please note that I'm pretty sure dr. Hulshoff said that especially these questions were very similar to old exam questions, or were in fact old exam questions.

1.9. Question 1

The correct answer is **not well posed**. A well posed problem requires that small changes in the boundary conditions lead to small changes in the solution. Clearly, this is not the case. It has nothing to do with elliptic, hyperbolic, parabolic, and nowhere is mentioned that it has to do with the discretization, so consistency is not the problem.

1.10. Question 2

1.10.1. Part 1

The equation is clearly **linear**; it's linear in all its terms.

1.10.2. Part 2

Remember that if the PDE is of the form

$$a\frac{\partial^2 u}{\partial x^2}+b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}=0$$

then the equation is

- hyperbolic if $b^2-4ac>0$;
- parabolic if $b^2-4ac=0$;
- elliptic if $b^2-4ac<0$.

In this case, we can rewrite the equation to

$$\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}-A\frac{\partial^2 u}{\partial y^2}=0$$

so that we have

$$(-2)^2-4\cdot 1\cdot -A=4+4A$$

We see that the equation is hyperbolic if $A>-1$, parabolic if $A=-1$ and elliptic if $A<-1$.

1.10.3. Part 3

For part a), we have the hyperbolic equation (since $A>-1$)

$$\frac{\partial^2 u}{\partial x^2}-2\frac{\partial^2 u}{\partial x\partial y}-3\frac{\partial^2 u}{\partial y^2}=0$$

which can be factorized as

$$\left(\frac{\partial}{\partial x}-3\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right)u=0$$

so that we must have

$$\begin{aligned}\frac{\partial}{\partial x} - 3\frac{\partial}{\partial y} &= 0\\ \frac{\partial}{\partial x} + \frac{\partial}{\partial y} &= 0\end{aligned}$$

From comparison, we then see that $B_{\max} = 1$.

For part b), we get an elliptic equation (since $A < -1$). This means that there are no characteristic lines, and thus B does not exist, and thus you need to plug in $B_{\max} = 0$.

1.11. Question 3

If we have

$$(1-N)\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial y^2} = 0$$

then to determine whether it's hyperbolic, parabolic or elliptic, we get

$$0^2 - 4\cdot(1-N)\cdot 3 = -12\cdot(1-N)$$

If we have $0 < N < 1$, then this will always be negative, meaning the equation is elliptic. For elliptic equations, boundary conditions must be specified on all boundaries, so the correct answer is 4. Note that if the equation had been hyperbolic, you'd could have calculated the characteristic lines and find that you need boundary conditions on only 3 boundaries (although you need 2 boundary conditions on one of those 3). On elliptic equations, it depends how many you need.

1.12. Question 4

An artificial boundary can have **both physical and numerical** boundary conditions. True boundaries can also have both physical and numerical boundary conditions, for your information.

1.13. Question 5

1.13.1. Part a

Since we use four nodes, we can at most approximate the third derivative: four nodes yields four coefficients to solve for: the first equation will be for the zeroth derivative, the second for the first derivative, the third for the second derivative, the fourth for the third derivative, and then you've run out of equations.

1.13.2. Part b

If we approximate the first derivative with four nodes, we can get order of accuracy of at least $4-1=3$.

1.13.3. Part c

Since we're interested in the second derivative, only entry C_5 will have a nonzero value. B_5, however, will simply be zero.

1.13.4. Part d

The distance from u_{i-3} to the node at which we're approximating (u_i) is $6\Delta x$ in total. Thus, we simply have

$$B_1 = \frac{(-6)^1}{1!} \cdot a = -6a$$

1.14. Question 6

1.14.1. Part a

$A_{13,13}$ is the coefficient corresponding the 13th node in the 13th row of matrix A,[1] where row 13 of matrix A corresponds to the equation you get for the 13th node in the domain. We count from left-to-right, then going from bottom to the top: the 13th node is $u_{3,3}$. The corresponding equation is

$$\frac{u_{3,4} - 2u_{3,3} + u_{3,2}}{h^2} - \frac{u_{3,3} - u_{2,3}}{2h} = 1$$

which can be rewritten to

$$\frac{u_{3,2}}{h^2} + \frac{u_{2,3}}{2h} + u_{3,3} \cdot \left(\frac{-2}{h^2} - \frac{1}{2h}\right) + \frac{u_{3,4}}{h^2} = 1$$

So, the coefficient corresponding to the 13th node ($u_{3,3}$) in the 13th row will be $-\frac{2}{h^2} - \frac{1}{2h}$.

1.14.2. Part b

Just wanna point out that you must be ridiculously stupid if you answer either "rotate" or "translate". That aside, the correct answer is that the bandwidth would **not change**. The bandwidth was previously determined by the distance between $u_{3,2}$ and $u_{3,4}$; the bandwidth was 11 (since we travel from left to right, then from bottom to top and you then encounter 11 nodes along the way: $u_{3,2}$, $u_{4,2}$, $u_{5,2}$, $u_{1,3}$, $u_{2,3}$, $u_{3,3}$, $u_{4,3}$, $u_{5,3}$, $u_{1,4}$, $u_{2,4}$ and $u_{3,4}$). If you add $u_{1,3}$, $u_{4,3}$ and $u_{5,3}$ to the stencil you are using to approximate the PDE, the bandwidth will still be the 'distance' between $u_{3,2}$ and $u_{3,4}$, so it won't have changed.

1.15. Question 7

Artificial dissipation was achieved by making the PDE

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} - k\Delta x^2 \frac{\partial^2 u}{\partial x^2} = 0$$

In other words, we include artificial viscosity by multiplying an approximation of the **second derivative of u w.r.t. x** with a function of the **mesh spacing**.

1.16. Question 8

1.16.1. Part a

Let's consider the second row of the matrix, as that one is the only non-boundary equation. We have

$$\frac{u_1}{\Delta x^2} - \frac{2u_2}{\Delta x^2} + \frac{\Delta x^2}{\Delta x^2} u_2 + \frac{u_3}{\Delta x^2} = 0$$

[1] To be clear, $A_{x,y}$ corresponds to the $x$node in the yth row of matrix A.

which can be rewritten to

$$\frac{u_1 - 2u_2 + u_3}{\Delta x^2} + u_2 = 0$$

You should recognize that the first term corresponds to u_{xx} (i.e. $\frac{\partial^2 u}{\partial x^2}$, and that the second term simply comes from u. Thus, the correct PDE is

$$u_{xx} + u = 0$$

1.16.2. Part b

For the boundary conditions, consider the first and third row of the matrix equation. For the first row, we have

$$\frac{\Delta x^2}{\Delta x^2} u_1 = u_1 = 1$$

This clearly corresponds to $u(0) = 1$. For the third row, we have

$$\frac{-\Delta x}{\Delta x^2} u_2 + \frac{\Delta x}{\Delta x^2} u_3 = \frac{u_3 - u_2}{\Delta x} = 1$$

This corresponds to $u_x(1) = 1$. This, the correct answer is

$$u(0) = 1 \quad \text{and} \quad u_x(1) = 1$$

$$\phi_{\text{analytical}}$$