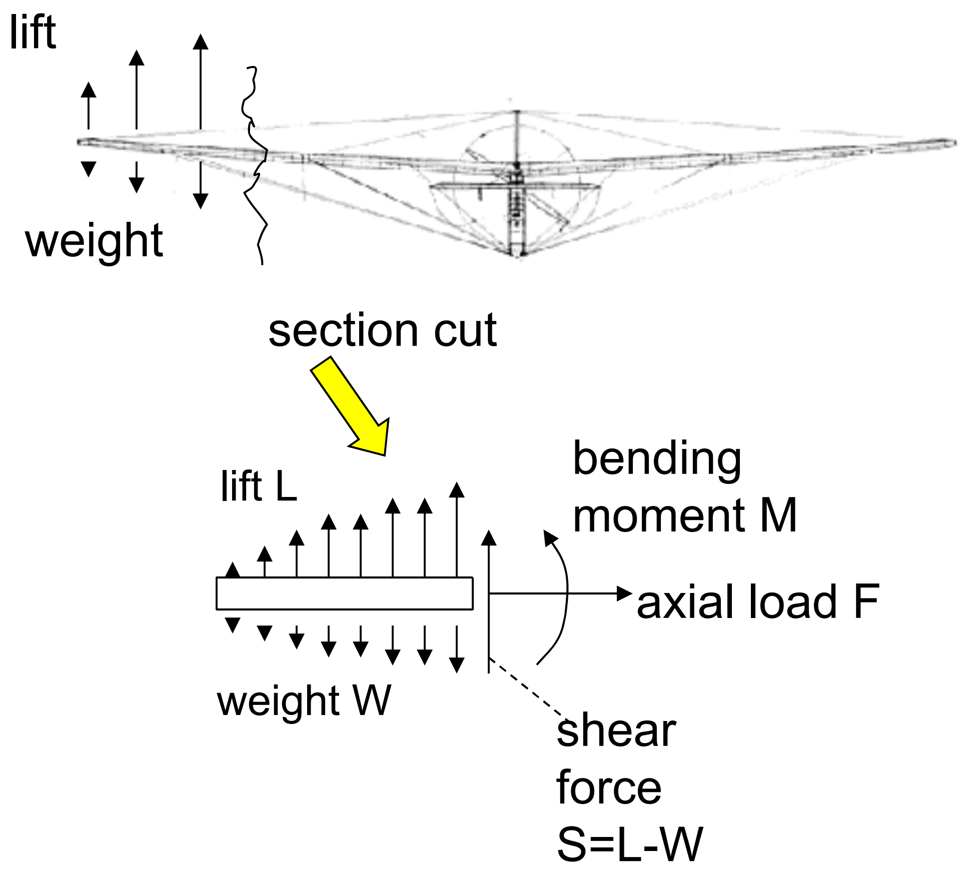
Shear of Beams

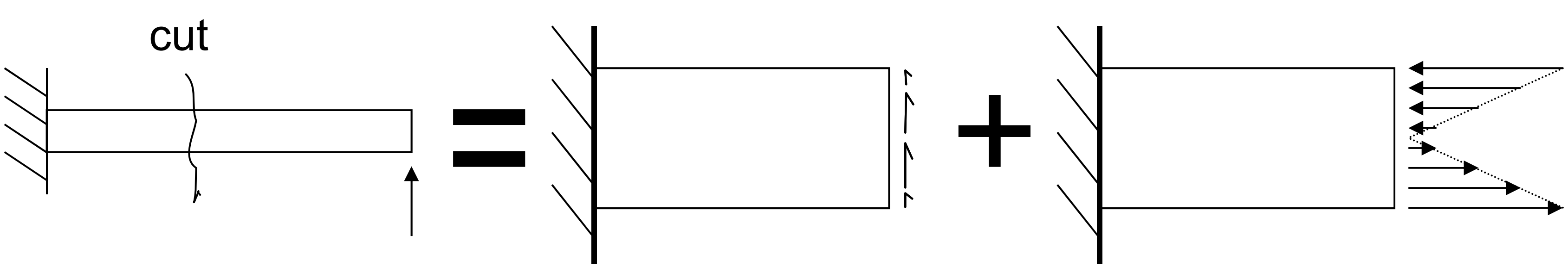
Megson: 17.1 - 17.2.1
17.3 - 17.3.2,
18.1.1(t/m eq 18.4)



Shear vs Moment

• While a bending moment causes only direct stresses on a beam cross section, a shear force causes both normal and shear stresses. The shear stresses are the direct result of the shear force applied.

• The direct stresses come from the bending moment that the shear force creates



beam with rectangular cross-section under shear

shear stresses are zero at top and bottom and max at center of beam (quadratic distribution)

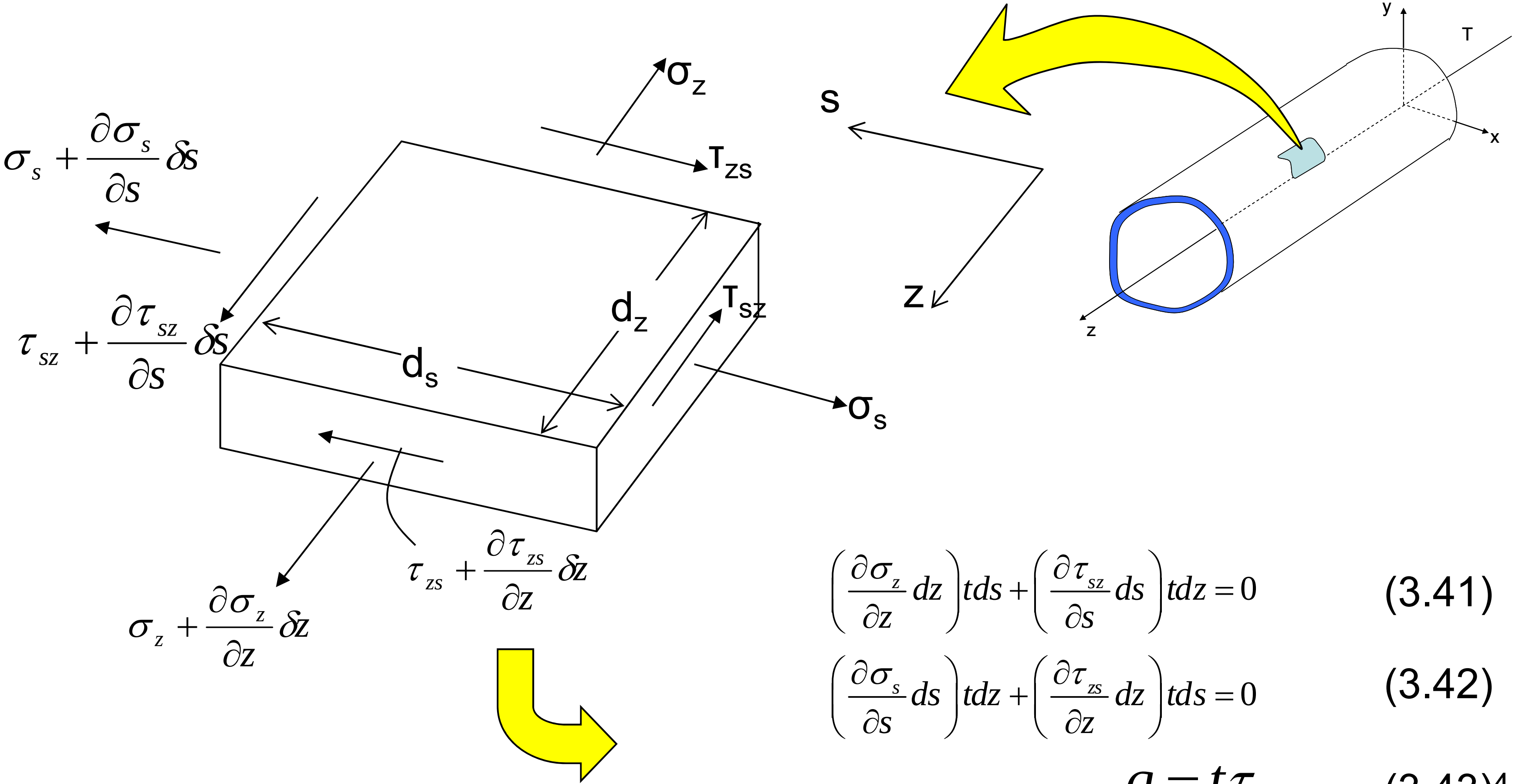
normal stresses are zero at center and max at top and bottom of beam (linear distribution)

Shear vs Moment

- if the shear force is constant along the beam length (e.g. cantilevered beam under shear at the end) the shear stress distribution is the same everywhere along the beam but the normal stress changes along the length because the moment changes (V x distance)

Thin-walled beams under shear – Governing Equations

- back in lecture 5, we placed an element of beam cross-section in equilibrium:



$$\left(\frac{\partial \sigma_z}{\partial z}dz\right)tds + \left(\frac{\partial \tau_{sz}}{\partial s}ds\right)tdz = 0 \qquad (3.41)$$

$$\left(\frac{\partial \sigma_s}{\partial s}ds\right)tdz + \left(\frac{\partial \tau_{zs}}{\partial z}dz\right)tds = 0 \qquad (3.42)$$

$$q = t\tau \qquad (3.43)$$

Thin-walled beams under shear – Governing Equations

$$\left.\begin{array}{r}\left(\frac{\partial \sigma_z}{\partial z} dz\right) tds + \left(\frac{\partial \tau_{sz}}{\partial s} ds\right) tdz = 0 \\ \left(\frac{\partial \sigma_s}{\partial s} ds\right) tdz + \left(\frac{\partial \tau_{zs}}{\partial z} dz\right) tds = 0 \\ q = t\tau\end{array}\right\}$$

$$t\frac{\partial \sigma_z}{\partial z} + \frac{\partial q}{\partial s} = 0 \qquad (5.1)$$

$$t\frac{\partial \sigma_s}{\partial s} + \frac{\partial q}{\partial z} = 0 \qquad (5.2)$$

- now from bending theory, eq (2.5),

$$\sigma_z = \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2}y \qquad (2.5)$$

- we can substitute for σ_z in eq. (5.1) to get:

$$\frac{\partial q}{\partial s} = -\frac{I_{xx}\frac{\partial M_y}{\partial z} - I_{xy}\frac{\partial M_x}{\partial z}}{I_{xx}I_{yy} - I_{xy}^2}tx - \frac{I_{yy}\frac{\partial M_x}{\partial z} - I_{xy}\frac{\partial M_y}{\partial z}}{I_{xx}I_{yy} - I_{xy}^2}ty \qquad (5.3)$$

Thin walled beams under shear – Governing Equations

$$\frac{\partial q}{\partial s}=-\frac{I_{xx}\frac{\partial M_y}{\partial z}-I_{xy}\frac{\partial M_x}{\partial z}}{I_{xx}I_{yy}-I_{xy}^2}tx-\frac{I_{yy}\frac{\partial M_x}{\partial z}-I_{xy}\frac{\partial M_y}{\partial z}}{I_{xx}I_{yy}-I_{xy}^2}ty \tag{5.3}$$

- recall now eq (2.10) which related moment to shear force:

$$S_y=\frac{\partial M_x}{\partial z} \tag{2.10}$$

- can substitute in (5.3) to obtain:

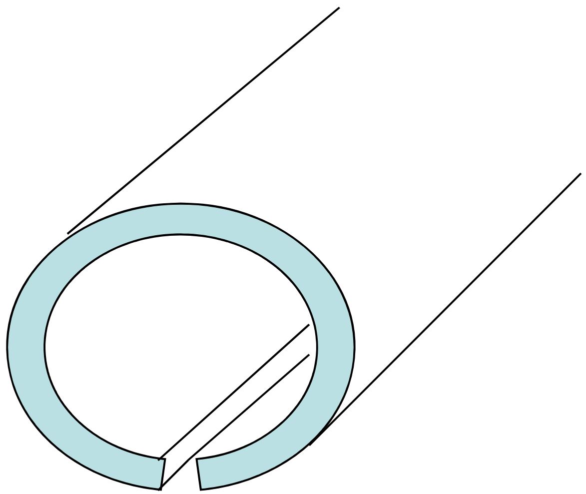
$$\boxed{\frac{\partial q}{\partial s}=-\frac{I_{xx}S_x-I_{xy}S_y}{I_{xx}I_{yy}-I_{xy}^2}tx-\frac{I_{yy}S_y-I_{xy}S_x}{I_{xx}I_{yy}-I_{xy}^2}ty} \tag{5.4}$$

- to solve for q, pick an origin for s, and integrate w.r.t. s:

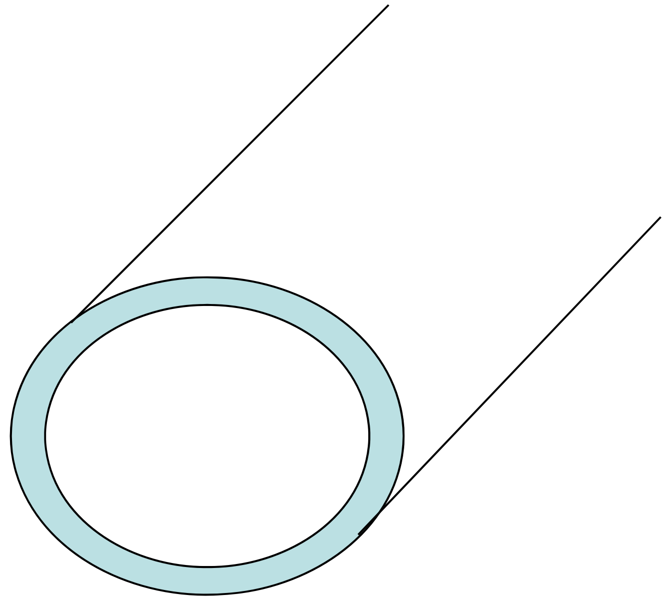
$$\int_0^s\frac{\partial q}{\partial s}ds=-\frac{I_{xx}S_x-I_{xy}S_y}{I_{xx}I_{yy}-I_{xy}^2}\int_0^s txds-\frac{I_{yy}S_y-I_{xy}S_x}{I_{xx}I_{yy}-I_{xy}^2}\int_0^s tyds \tag{5.5}$$

Thin walled beams under shear – Governing Equations

- there are now two possibilities:
  - open section beam
  - closed section beam

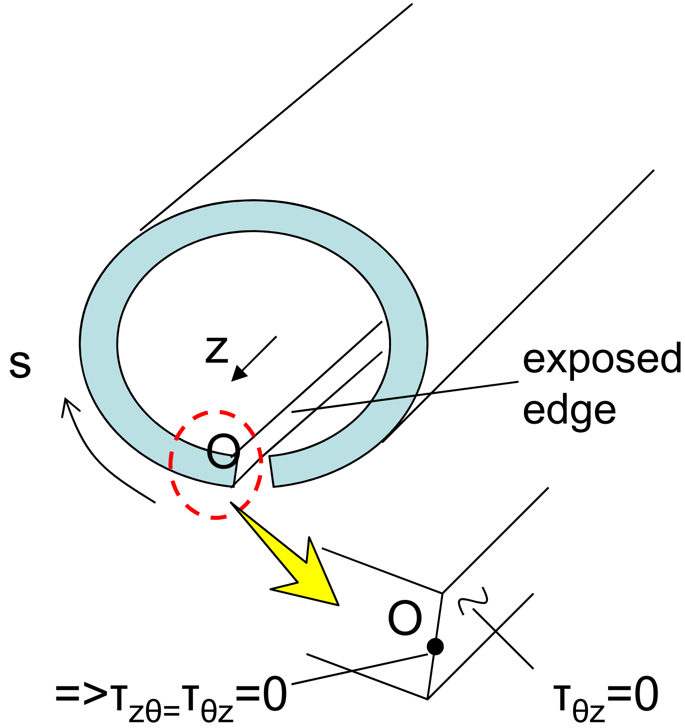


open section



closed section

Thin-walled open section beams



if the shear stress at the exposed edge of the open section is zero, then the corresponding shear flow q =τt is zero at that edge; therefore, it is zero at point O; thus, it is convenient to set the origin s=0 at that location

- eq (5.5)

$$\int_0^s \frac{\partial q}{\partial s} ds = -\frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s txds - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s tyds$$

- then becomes:

$$q(s) - \cancel{q(0)} = -\frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s txds - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s tyds$$

and letting q(s)=q_s :

$$q_s = -\frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s txds - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s tyds \qquad (5.6)$$

Thin-walled open section beams

$$\boxed{q_s = -\frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s tx\,ds - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s ty\,ds}$$

open section (5.6)

• note that, depending on the cross-section, x and y may be functions of s (that's why they are kept inside the integral sign)

• also, if the thickness is not constant, t is a function of s

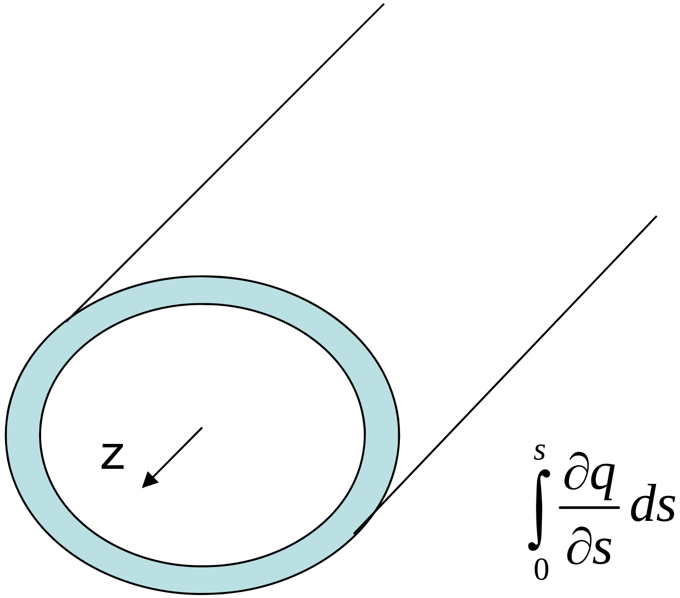
• the problem then boils down to determining x(s), y(s), and t(s) and evaluating the above integrals

• for the special case where either the x or the y axis is an axis of symmetry, I_{xy}=0 and (5.6) simplifies to:

$$\boxed{q_s = -\frac{S_x}{I_{yy}}\int_0^s tx\,ds - \frac{S_y}{I_{xx}}\int_0^s ty\,ds}$$

open section, one axis of symmetry (5.7)

Thin-walled closed section beams



Eq (5.5) is still valid but now there is no obvious location (such as the cut) at which q(s)=0

$$\int_0^s \frac{\partial q}{\partial s} ds = -\frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s txds - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s tyds \qquad (5.5)$$

• pick an (arbitrary) origin for s and let the unknown shear flow there be q_{s0}; then, integrating (5.5)

$$q_s - q_{s0} = -\frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s txds - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s tyds \qquad (5.8)$$

Thin-walled closed section beams

$$q_s - q_{s0} = -\frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s txds - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s tyds \qquad (5.8)$$

• comparing (5.8) with the general equation (5.6) we derived for open section beams:

$$q_s = -\frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s txds - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s tyds \qquad (5.6)$$

• we see that the only difference is the unknown shear flow q_{s0}

• note that, depending on where we take the origin of s for the closed section, the form of eq (5.8) does not change but the value of q_{s0} changes

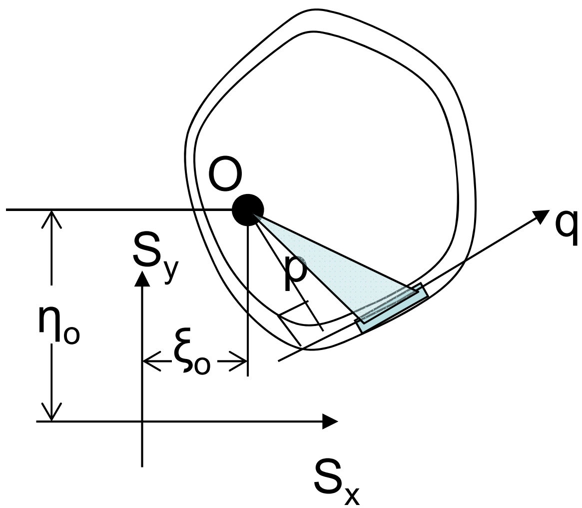
• also, the right hand side of eq. (5.8) is identical to the right hand side of eq (5.6) for open cross-sections

Thin-walled closed section beams

$$q_s - q_{s0} = \underbrace{-\frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s txds - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s tyds}_{q_b}$$

• in order to determine q_{s0} apply moment equivalence: the applied moments about any convenient point are equal to the internal moments caused by the resulting shear flows

$$S_x\eta_o - S_y\xi_o = \oint_s pqds = \oint_s pq_b ds + \oint_s pq_{s0} ds$$



but for a given origin for s, q_{s0} is a fixed (constant) value so it can come out of the integration

also, $pds = 2(Area \quad of \quad shaded \quad triangle)$

so $\oint_s pds = 2A$ where A is enclosed area of beam cross-section

• therefore:

$$S_x\eta_o - S_y\xi_o = \oint_s pq_b ds + 2Aq_{s0} \qquad (5.9)$$

Thin-walled closed section beams

$$S_x\eta_o - S_y\xi_o = \oint_s pq_b ds + 2Aq_{s0} \tag{5.9}$$

• if now the origin O, about which moment equivalence was taken, lies on the line of action of S_x and S_y, the left hand side of (5.9) is zero and we get:

$$\boxed{\oint_s pq_b ds + 2Aq_{s0} = 0} \tag{5.10}$$

• the unknown shear flow q_{s0} is determined from eq (5.9) or eq (5.10)

Thin-walled closed section beams - Observations

• we showed that the shear flow in a closed section beam is obtained as the sum two contributions: (a) an unknown (constant) shear flow q_{s0} which is the shear flow at the origin of s, and (b) a contribution from q_b in the form:

$$q_s = q_{s0} \underbrace{- \frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2} \int_0^s txds - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2} \int_0^s tyds}_{q_b} \qquad (5.8)$$

• but if we compare (5.8) with (5.6), the equation for open section beams, we see that q_b is the shear flow that we would obtain if the beam were cut at the location s=0

$$q_s = - \frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2} \int_0^s txds - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2} \int_0^s tyds = q_b \qquad (5.6)$$

Thin-walled closed section beams - Observations

• in addition, the contribution from q_{s0} (see derivation for eq 5.9) to the total moment was shown to correspond to the shear flow of a constant torque T (recall eq. 3.44)

$$S_x\eta_o - S_y\xi_o = \oint_s p q_b ds + 2Aq_{s0} \tag{5.9}$$

$$T = 2Aq \tag{3.44}$$

• therefore, to determine the shear flow in a closed-section beam:

− cut the beam at a (convenient) location

− determine the shear flow in the resulting open section using open section beam theory

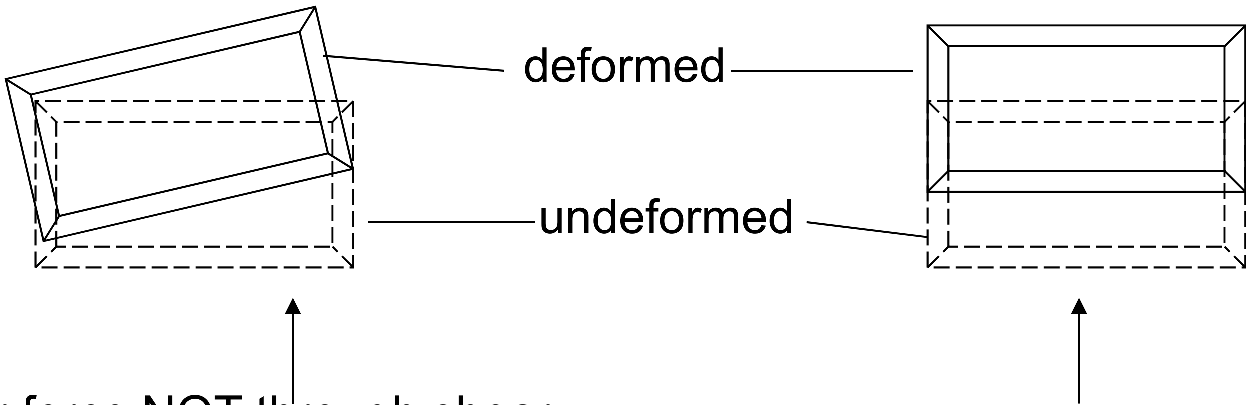
− add to that a constant shear flow that balances the torque around the section

Thin-walled open and closed section beams - Implications

• it is important to recognize that our solution for the open section beams assumed that the applied shear forces to the cross-section **only bent it and did not twist it**

• in general, a shear force acting on a cross-section would bend and twist it at the same time

• only if the shear force acts through **the shear center** is there only bending and no twisting



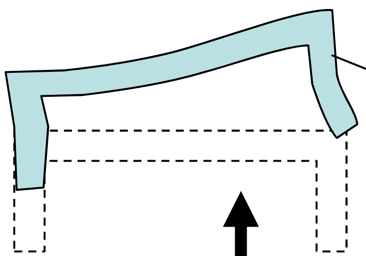
shear force NOT through shear center

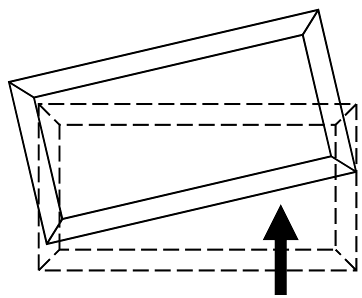
shear force through shear center

Thin-walled open and closed section beams - Implications

- this means that the shear stresses caused by a shear force acting through the shear center of an open cross section **are of the same form** as those of a closed cross section caused by the same shear force and a torque about the shear center

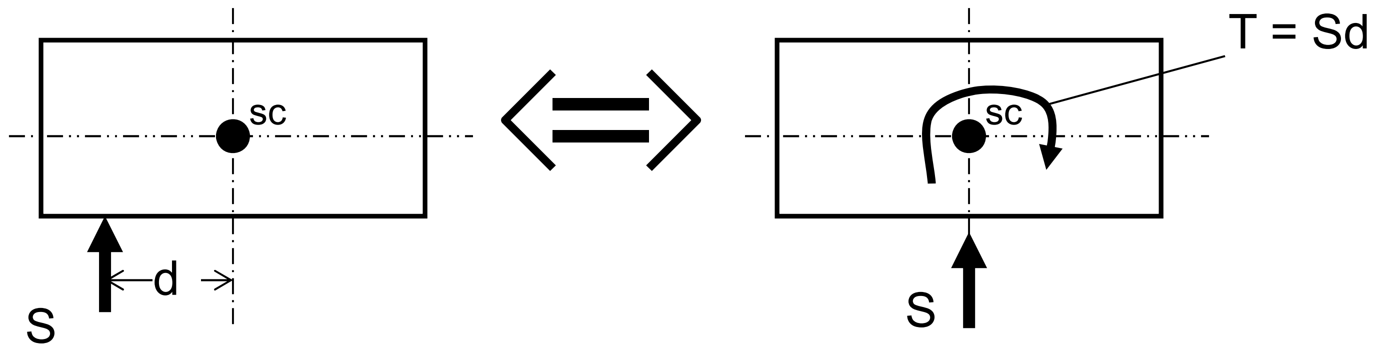
- however, a shear force not acting through the shear center of an open cross-section causes a very different stress distribution than the combination of a shear and a torque in a closed cross section; the reason is that the open cross-section now goes into differential bending





To summarize…

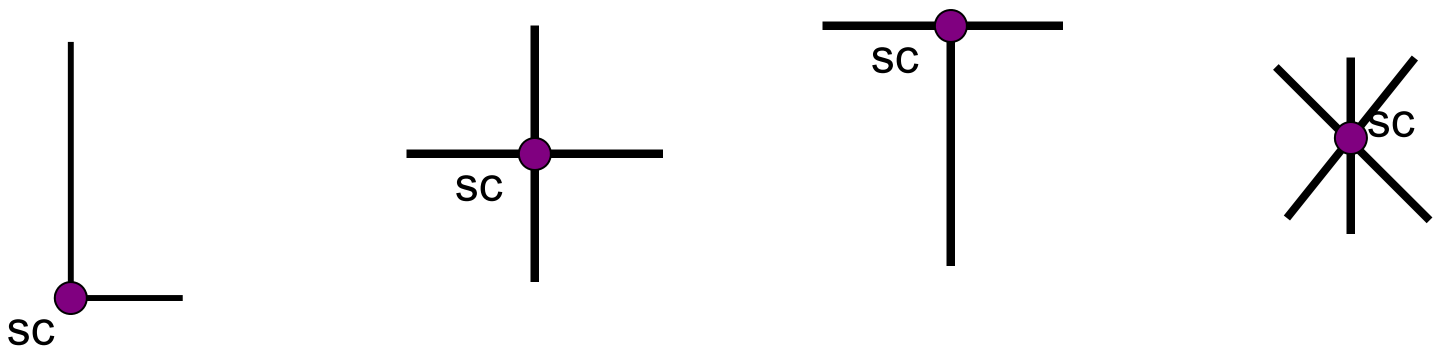
• a shear load S acting at distance d from the shear center can be replaced by a combination of the same shear load acting through the shear center and a torque (equal to S x d)



• this means that knowing where the shear center of a cross-section is, can be crucial for the subsequent analysis

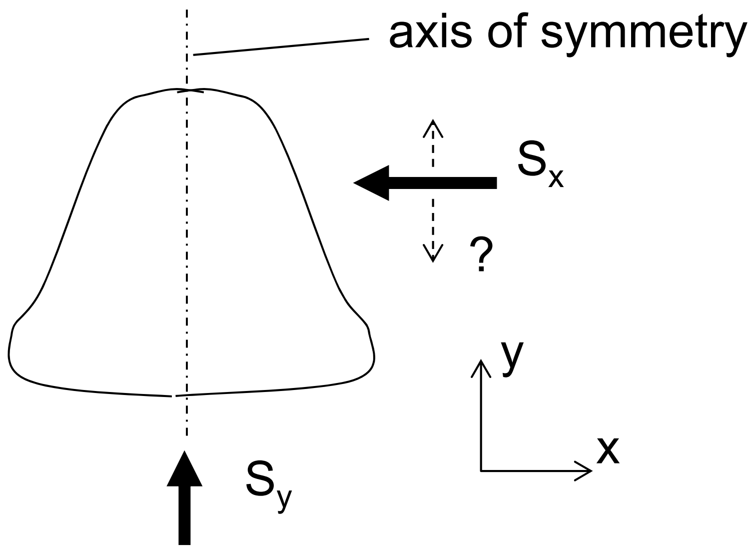
Shear Center

• for the case where all resultant internal shear flows pass through the same point, the sc is **that** intersection point (because they cause no twisting about that point)



Shear Center

• for the case where there is one axis of symmetry, the shear center lies on that axis

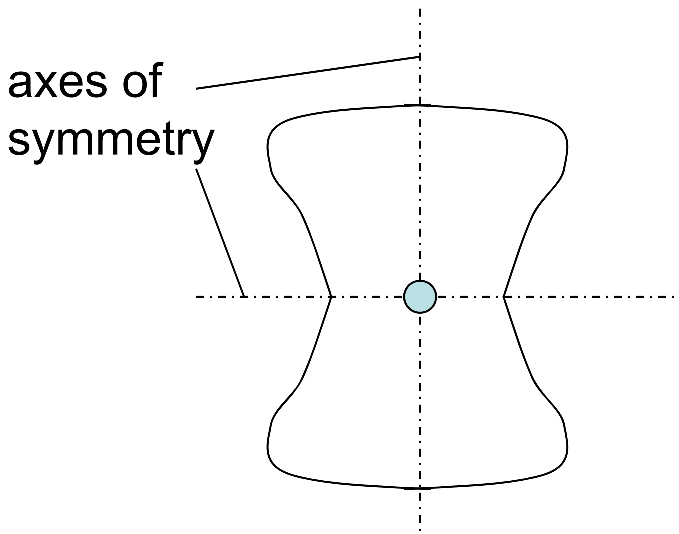


-by symmetry, if S_y acts along the axis of symmetry, the beam simply bends and does not twist; therefore, the sc must lie somewhere on the axis of symmetry

- the exact location of the sc is determined such that when S_x acts, the beam simply bends (to the left) and does not twist

Shear Center

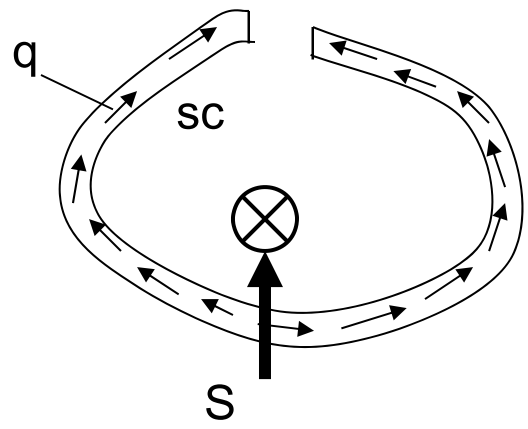
• if there are two axes of symmetry the shear center lies at their intersection



Shear center

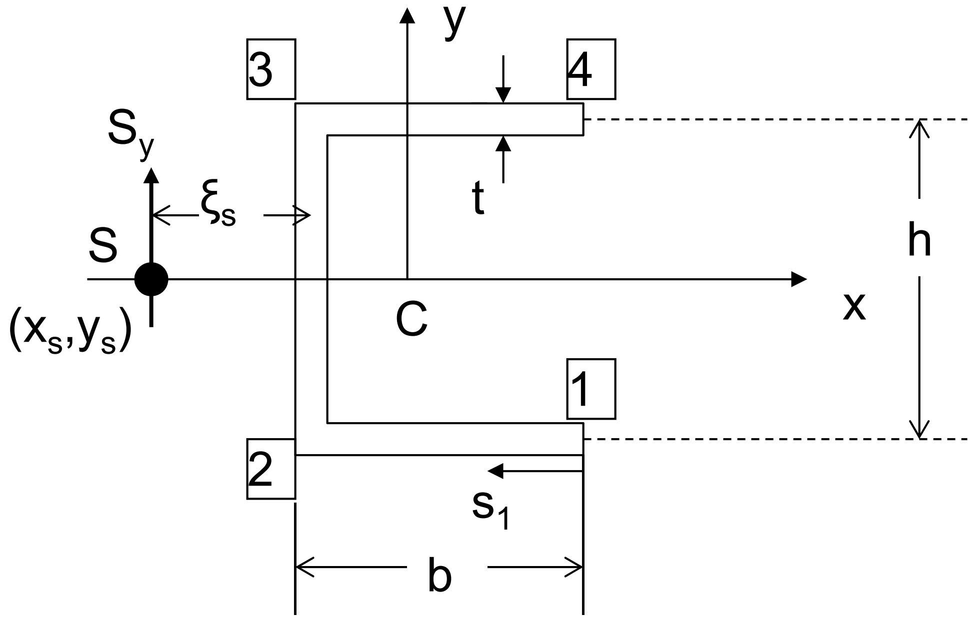
• how do we determine the shear center location for general cross-sections?

• if a shear load acted through the sc, the shear flows created would cause torques (for example shear flows to the left and right of sc below) that balance each other. The net torque is zero so the cross-section does not rotate



Shear center - Example

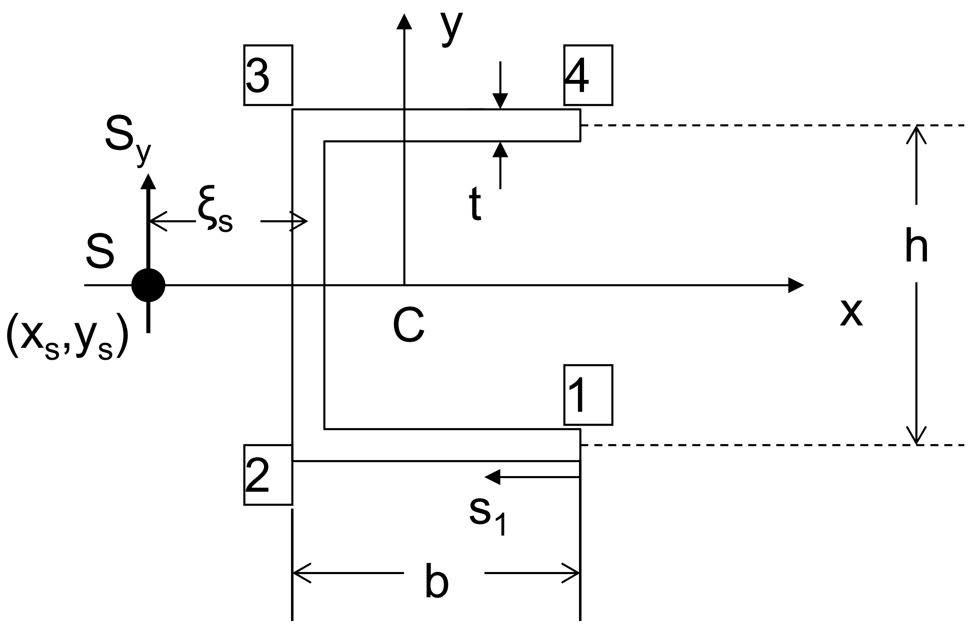
Example: Find shear center of the beam cross-section below



constant thickness t

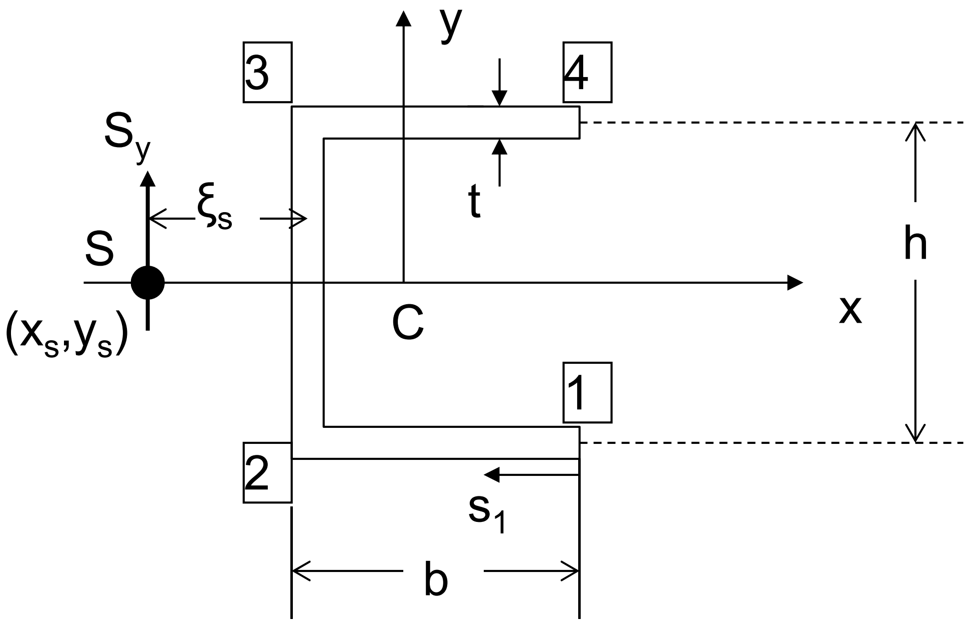
• since axis Cx is axis of symmetry, the sc lies somewhere on it

Shear center – Example calculation



- pick a point S at random on this axis and apply a shear force S_y (parallel to y)
- the torque caused by S_y must be equal to the shear flows created by S_y in 1-2, 2-3, and 3-4

Shear center – Example calculation



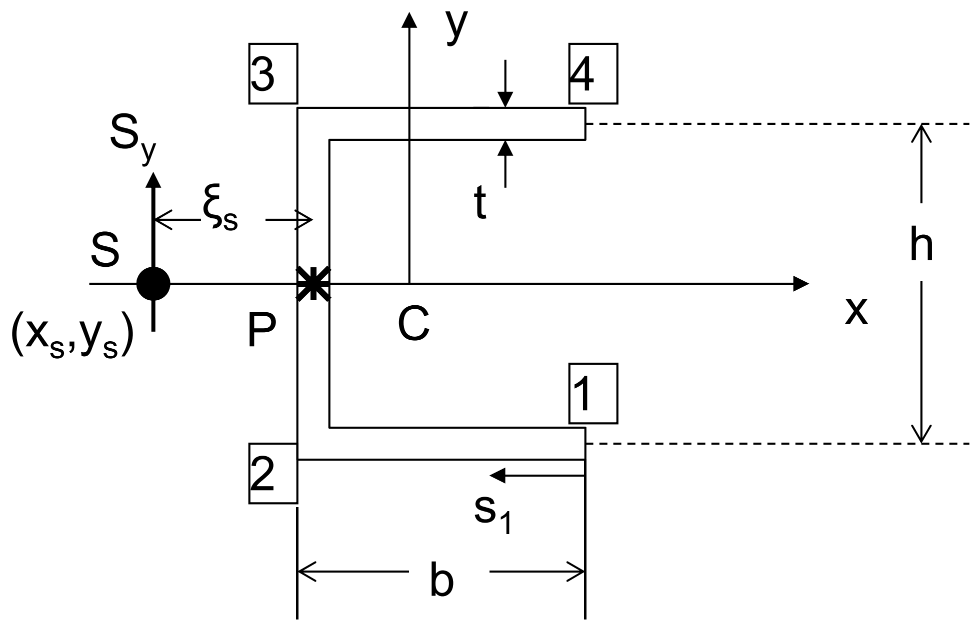
- recall the shear flow equation (5.6) for open sections

$$q=-\left(\frac{S_x I_{xx}-S_y I_{xy}}{I_{xx}I_{yy}-I_{xy}^2}\right)\int_0^s tx\,ds-\left(\frac{S_y I_{yy}-S_x I_{xy}}{I_{xx}I_{yy}-I_{xy}^2}\right)\int_0^s ty\,ds \qquad (5.6)$$

Shear center – Example calculation

- since the cross-section is symmetric, $I_{xy}=0$
- equation reduces to

$$q = -\left(\frac{S_y}{I_{xx}}\right)\int_0^s tyds$$



with

$$I_{xx} = \frac{th^3}{12} + 2bt\left(\frac{h}{2}\right)^2 + \text{ H.O.T.}$$

substituting,

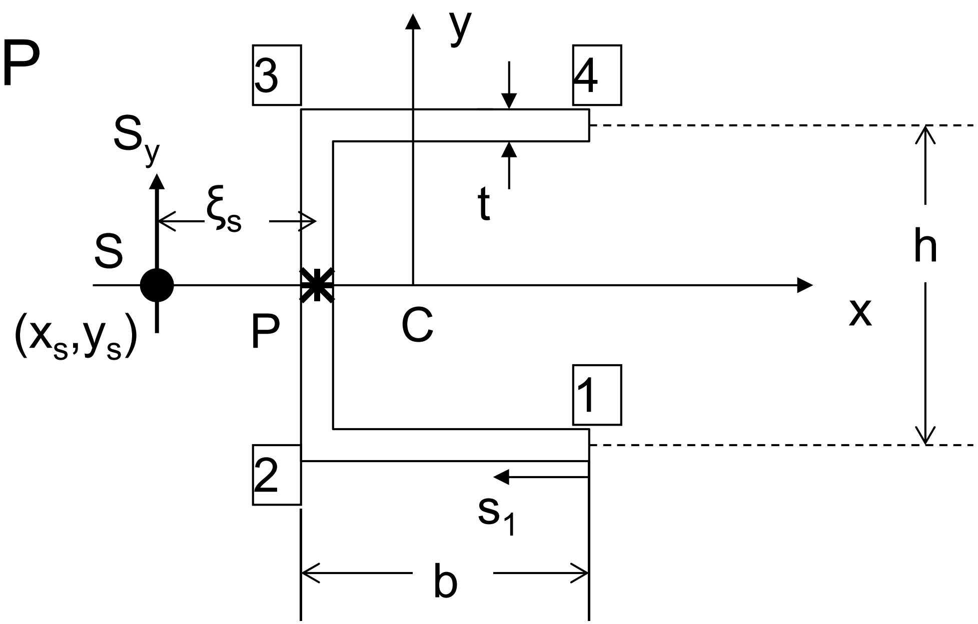
$$q_{12} = -\frac{S_y}{\frac{h^2t}{2}\left(\frac{h}{6}+b\right)}\int_0^{s_1} t\left(-\frac{h}{2}\right)ds = \frac{6S_y s_1}{h(h+6b)}$$

Shear center – Example calculation

- equate moments about pt P

- torque caused by S_y= moment caused by q

note that the shear q_{23} does not contribute to the moment because it goes through P; also, by symmetry, q_{12}=q_{34}



- substituting,

top+bottom flange

moment arm

$$S_y \xi_s = 2\int_0^b q_{12} \frac{h}{2} ds_1 = 2\int_0^b \frac{6 S_y s_1}{h(h+6b)} \frac{h}{2} ds_1$$

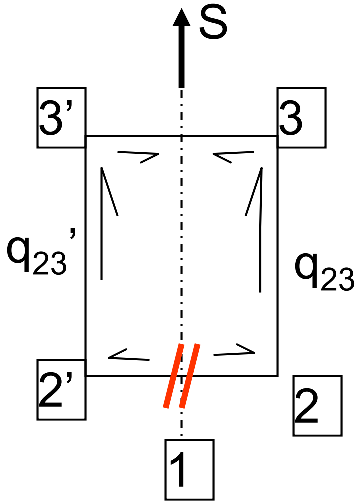
force

always cancels out!

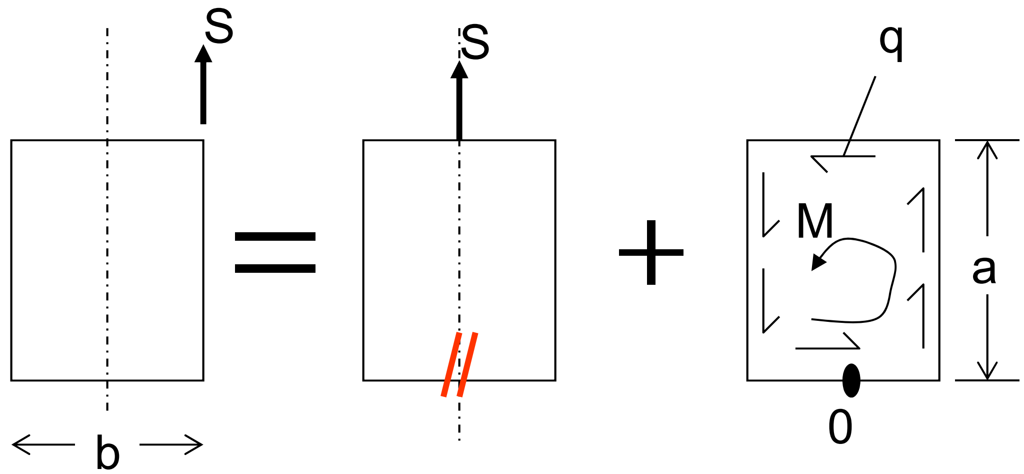
which leads to $$\xi_s = \frac{3b^2}{(h+6b)}$$

Some guidelines on where to cut the closed section

- cut at the intersection of axis of symmetry aligned with the load (if there is one) with the boundary; because q=0 at the cut and it is convenient to start s there



then



$q_{s,1}=0$

$q_{12}=q_{12}'$

$q_{23}=q_{23}'$

…

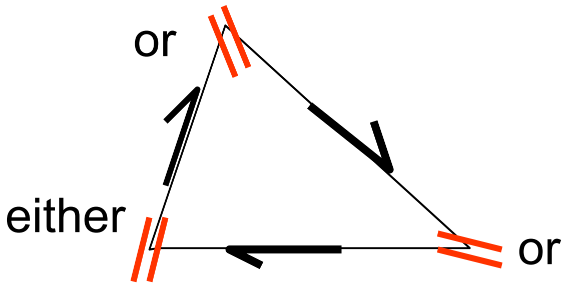
torque caused by S_y= moment caused by q

$$M = S\frac{b}{2}$$

$$M = M about \quad O = 2qa\frac{b}{2} + qba = 2qab = 2qA_{enclosed}$$

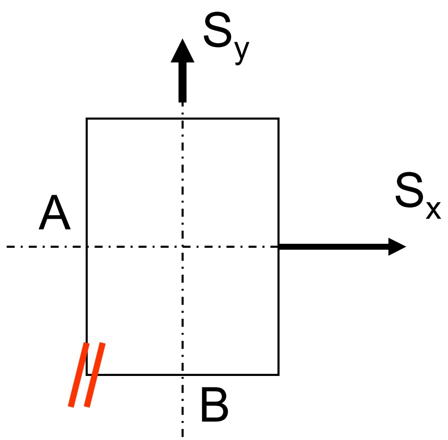
Some guidelines on where to cut the closed section

• try to use locations where more than one shear flows go through so their contribution to the torque is zero when the cut is closed (less calculations)



choosing any of the three vertices as the location to cut the beam ensures that the shear flows through the adjacent sides do not contribute to the torque since they go through the cut

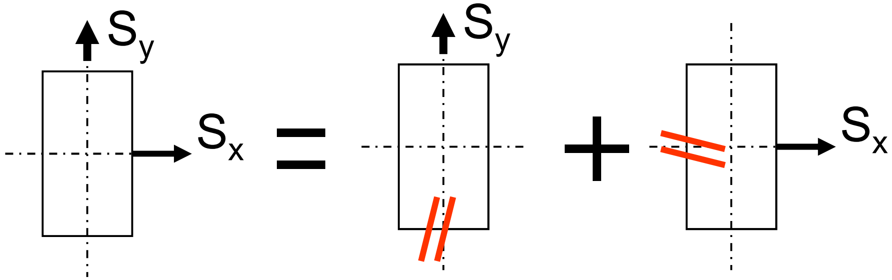
Some guidelines on where to cut the closed section



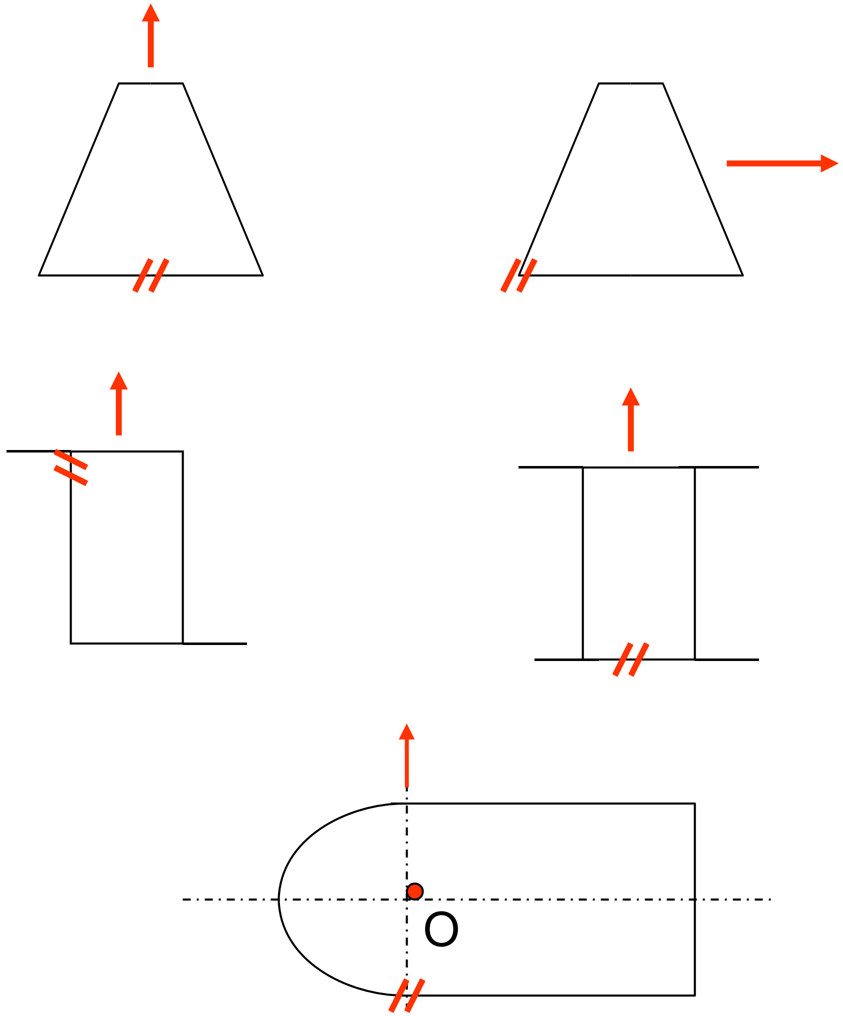
- if point A or B were selected to make the cut, the shear flow at that location would NOT be zero anymore because of the contribution from the corresponding applied shear load (S_y makes q at cut A nonzero and S_x makes q at cut B nonzero)

- instead, the cut at the location shown is better

- alternatively, can solve as the superposition of two problems each of which we saw earlier



Some guidelines on where to cut the closed section



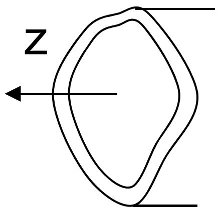
Note that moments about O are convenient because some of the contributions are zero

Shear center of closed cross-sections

• the determination of the shear center of a closed cross-section beam is analogous to that for open cross-sections

• with one difference: in the closed section calculations, a “reference” shear flow $q_{s,0}$ at the selected location of the cut appears (see for example eq 5.8)

• one more equation is needed to determine $q_{s,0}$:



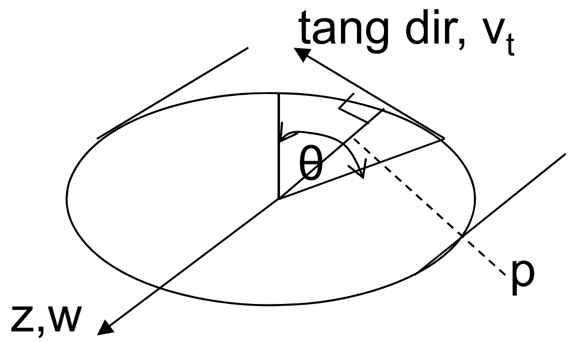
– since the shear force is assumed to be applied through the sc (the location of which we are looking for), there is no twist of the cross section

– this means that the rate of change of the twist angle θ with respect to axial coordinate z is zero:

$$\frac{d\theta}{dz} = 0 \qquad (5.11)$$

Shear center of closed cross-sections

• to determine dθ/dz, relate the total rotation around the circumference of the beam to the shear strain (note: Megson has a more elaborate derivation)



accounting only for contributions that contribute to rotation of the cross-section,

$$v_t = p\theta$$

and $$\frac{\partial v_t}{\partial z} = p\frac{d\theta}{dz}$$

• now the shear strain around the circumference is

$$\gamma_{sz} = \gamma = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z}$$

• but also, from the stress-strain equations

$$\gamma = \frac{\tau}{G} = \frac{q}{tG}$$

Shear center of closed cross-sections

- combining,

$$\frac{q}{Gt} = \frac{\partial w}{\partial s} + \frac{\partial v_t}{\partial z}$$

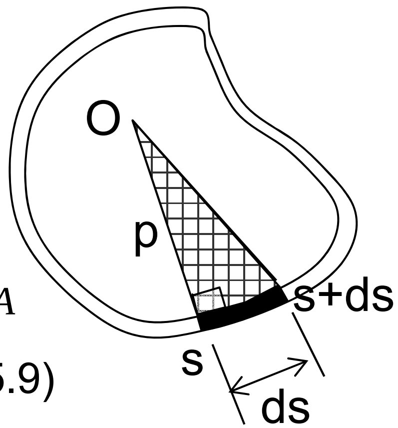
- and using the expression for v_t:

$$\frac{q}{tG} = \frac{\partial w}{\partial s} + p\frac{d\theta}{dz}$$

Note: there are a couple more terms (see Megson) but they do not contribute to the twisting of the cross-section which is what we are interested in here

- integrating around the whole circumference:

$$\oint_s \frac{q}{tG} ds = \underbrace{\oint_s \frac{\partial w}{\partial s} ds}_{=0} + \oint_s p\frac{d\theta}{dz} ds$$



- but dθ/dz is independent of s and $\oint_s p ds = 2A$

(see also eq 5.12)

- therefore: $\boxed{\frac{d\theta}{dz} = \frac{1}{2A}\oint \frac{q_s}{tG} ds}$ (5.12)

Shear center of closed cross-sections

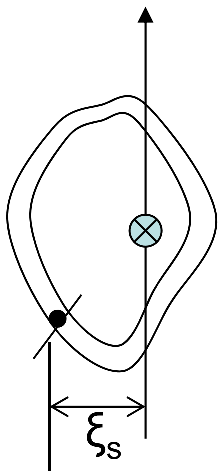
- combining (5.11) and (5.12)

$$\left.\begin{array}{l} \dfrac{d\theta}{dz} = 0 : \\ \dfrac{d\theta}{dz} = \dfrac{1}{2A}\oint \dfrac{q_s}{tG} ds \end{array}\right\} \Rightarrow \oint \frac{q_s}{tG} ds = 0 \qquad (5.13)$$

- and using the fact that, from (5.8) $q_s = q_b + q_{s0}$

$$\oint \frac{q_b}{tG} ds + \oint \frac{q_{s0}}{tG} ds = 0 \qquad (5.14)$$

Shear center



• solving eq (5.14) for q_{s0} (which is constant and comes out of the integration)

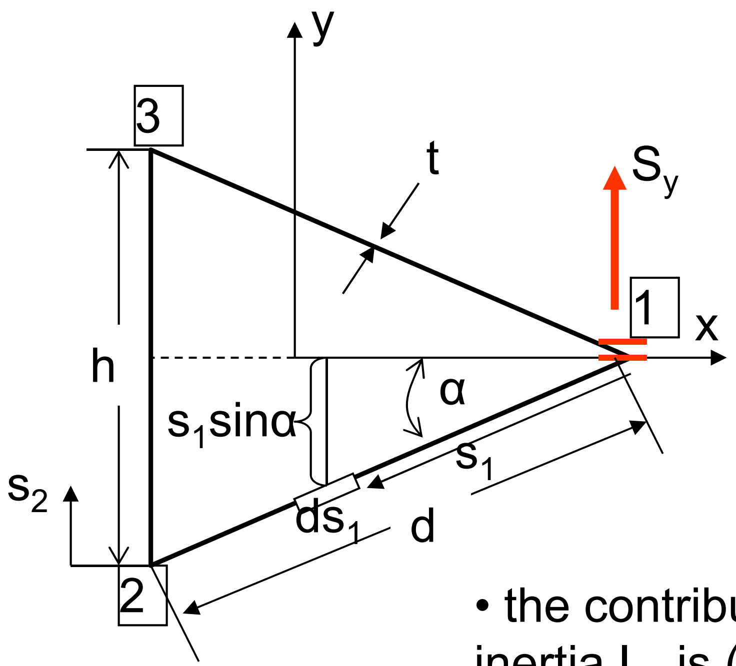
$$\boxed{q_{s,0} = \frac{-\oint \frac{q_b}{Gt} ds}{\oint \frac{1}{Gt} ds}} \qquad (5.15)$$

• if Gt is constant, eq. (5.15) simplifies to

$$\boxed{q_{s,0} = \frac{-\oint q_b ds}{\oint ds}} \qquad (5.16)$$

• eq (5.15) or (5.16) provides the additional equation needed to determine the shear center; the other equation comes from equating the applied torque to the moment caused by the shear flows

Example: Shear flows in triangular beam



Shear force S_y applied at point 1

Constant thickness t all around

Since the cross-section is symmetric, $I_{xy}=0$

Since the loading is in the y direction, bending occurs only about x axis so I_{yy} is of no interest

- the contribution of an element ds_1 to the moment of inertia I_{xx} is $(ds_1)\,(t)\,(s_1\sin\alpha)^2$; So the contribution of one side of length d to I_{xx} is:

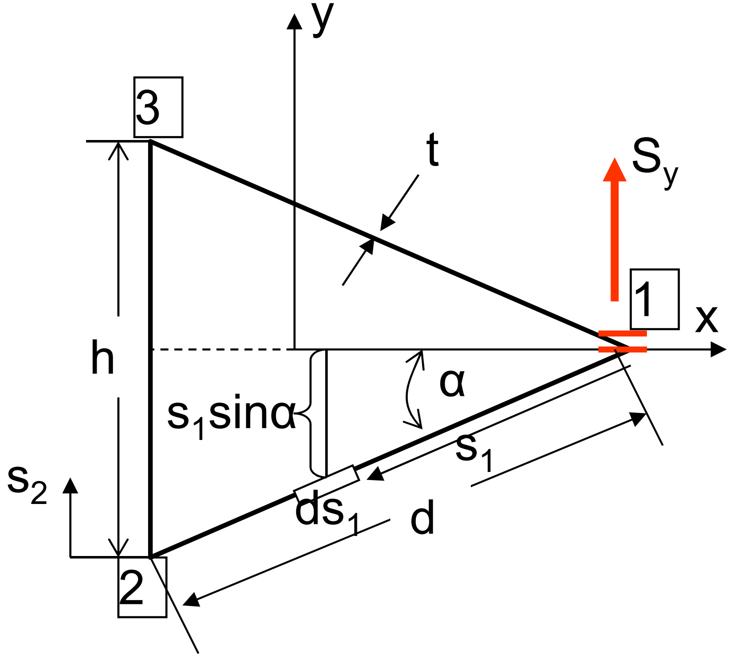
$$\int_0^d (s_1 \sin\alpha)^2\, t ds$$

- then, adding the contribution of all three sides, I_{xx} is

$$I_{xx} = \frac{th^3}{12} + 2\int_0^d (s_1 \sin\alpha)^2\, t ds = \frac{th^2}{12}(h+2d)$$

where the fact that $\sin\alpha = \dfrac{h}{2d}$ was used

Example: Shear flows in triangular beam



- make cut at location 1 to preserve symmetry
- from eq (5.6), with I_{xy}=0 and S_x=0,

$$q_{12} = -\left(\frac{S_y}{I_{xx}}\right)\int_0^d tyds$$

note no q_1 because q_1=0 at the cut

with $y=-s_1\sin\alpha$ leading to

$$q_{12} = -\frac{12S_y}{th^2(h+2d)}t\left(-\frac{h}{2d}\right)\left[\frac{s_1^{\ 2}}{2}\right] = \frac{3S_y}{hd(h+2d)}s_1^{\ 2}$$

- evaluating this expression at s_1=d

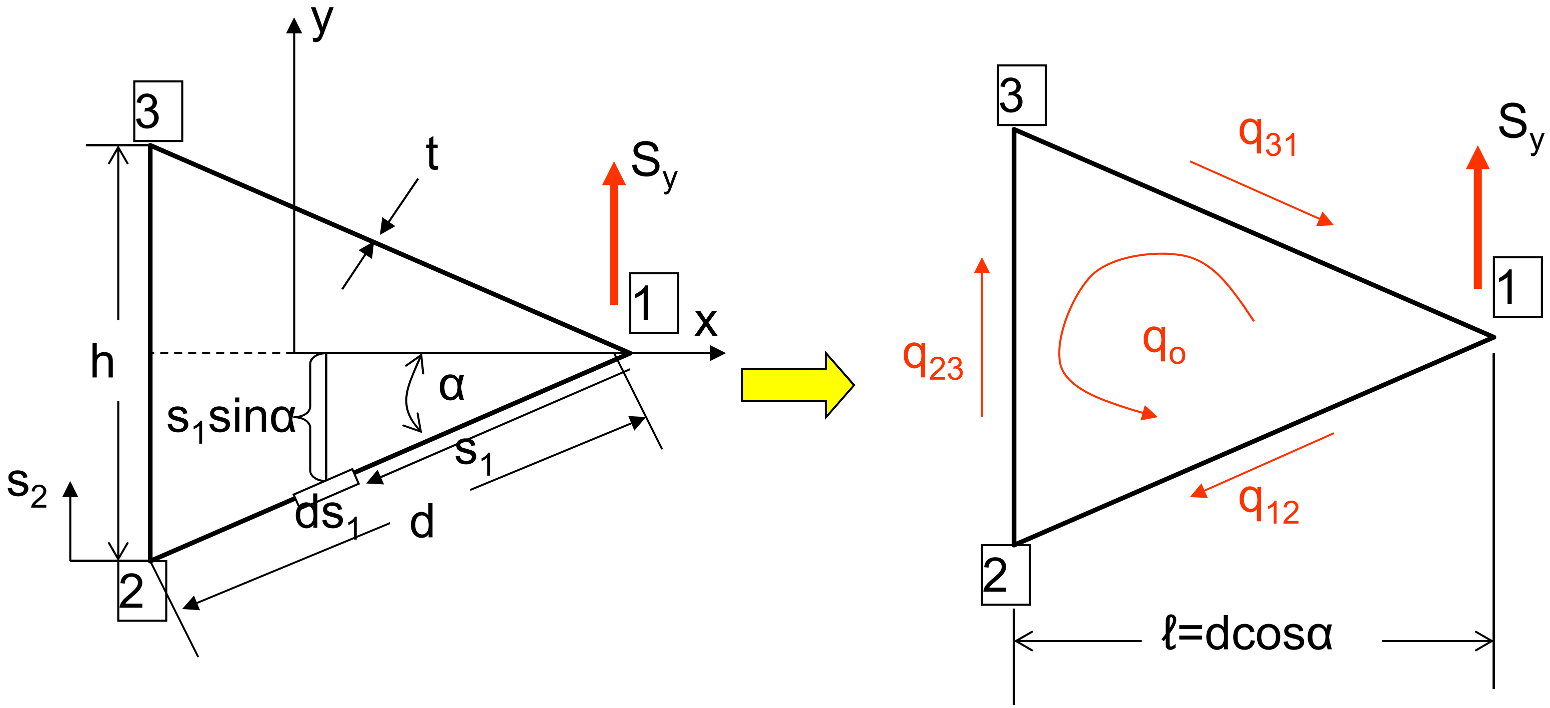
$$q_2 = q_{12}(s_1 = d) = \frac{3S_y d}{h(h+2d)}$$

- repeating the procedure for web 23 and noting that $y=-(h/2)+s_2$

$$q_{23} = q_2 - \frac{S_y}{I_{xx}}\int_0^{s_2} tyds = -\frac{S_y}{I_{xx}}\int_0^{s_2} t\left(-\frac{h}{2}+s_2\right)ds_2 = \frac{3S_y d}{h(h+2d)} - \frac{6S_y}{h^2(h+2d)}\left(-hs_2+s_2^{\ 2}\right)$$

by the way, what is q_{31}?

Example: Shear flows in triangular beam



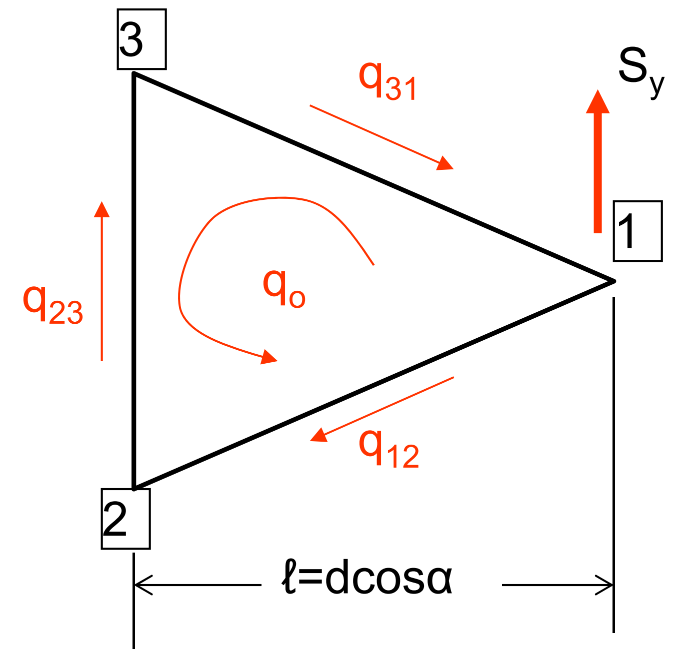
• if we now close the cut at 1, a constant shear flow q_o appears in the beam walls as shown

• take moments about point 1: S_y, q_{12}, and q_{31} do not contribute because their line of action is through pt 1

$$M_{about\ 1} = 0 \Rightarrow \int_0^h q_{23} \ell \, ds_2 = q_o 2A$$

A is area enclosed by the triangle: $A = h\ell/2$

Example: Shear flows in triangular beam



- substituting, carrying out the integrations and solving for q_o

$$q_o = \frac{h+3d}{h(h+2d)} S_y$$

- finally,

$$\boxed{\begin{aligned} q_{12total} &= q_{12} - q_o \\ q_{23total} &= q_{23} - q_o \end{aligned}}$$

note – sign!