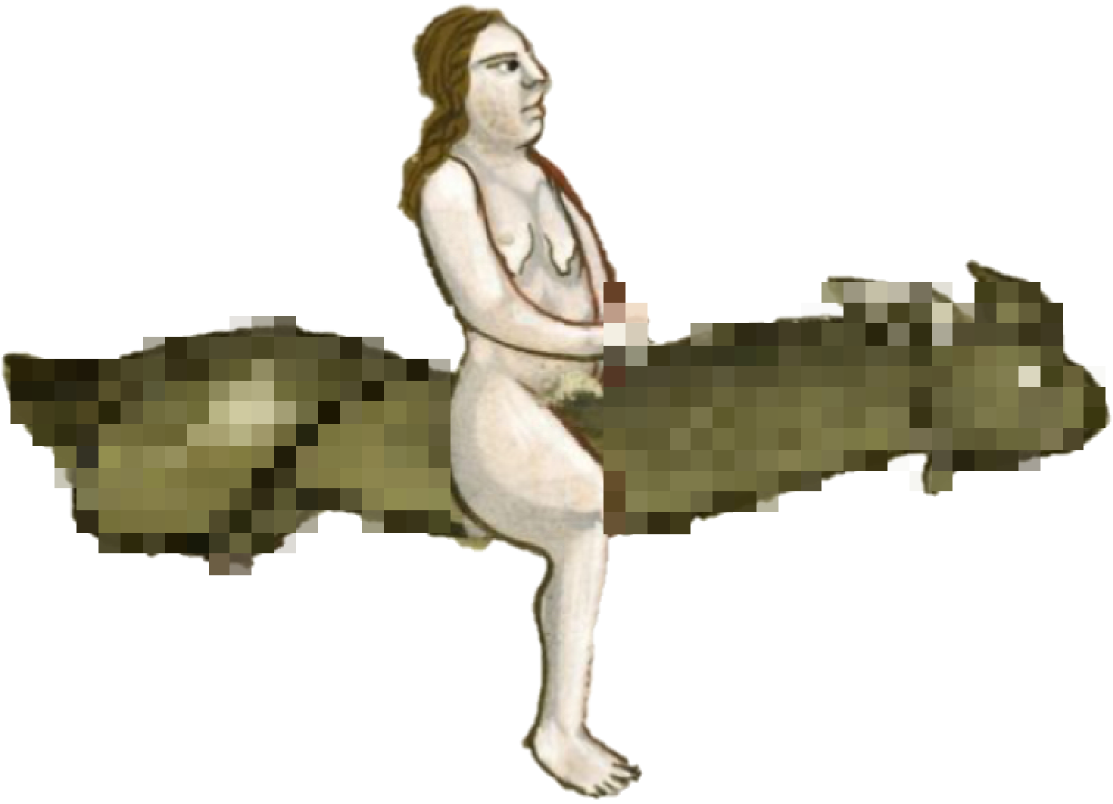
Flight Dynamics summary III: 2019-2020 edition

Based on *Lecture Notes Flight Dynamics* by J.A. Mulder et al.



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Preface

This part is a bit like a buttplug to be honest: no matter what you think you can handle, you have to start small and then work your way up. If you try to look at the final chapter then yes it looks like a pain in the ass, but if you just patiently work through the chapters before, you'll be allright. Furthermore, you'll really only get exam questions about chapters 2-4 and basically nothing about chapter 5, so that chapter is mostly just for shits and giggles.

1 Introduction to dynamic stability and state-space

In the previous two parts, we only discussed static stability. **Static stability** is whether the initial reaction of the system is to return to its equilibrium. For example, a pendulum should initially swing back in the direction of the equilibrium condition. This is not everything however: it does not say anything about how the system will behave after being disturbed: it may be that the pendulum swings back, but then oscillates around the equilibrium condition, with increasingly amplified oscillations. Evidently, it's no longer stable. This kind of stability, how it behaves after the initial reaction, is called **dynamic stability**. A system that is dynamically stable is automatically statically stable, although the other way around does not hold (static stability does not imply dynamic stability).

Part III and part IV cover dynamic stability, and is much more mathematical and abstract than the previous parts. Personally, I'm happy with this, but tastes differ of course. As such, and to help you prepare for the flight test, I'll give you a short introduction to what we'll discuss in the next parts.

1.1 Introduction to eigenmotions

Eigenmotions are the motions displayed by an aircraft absent of external influences. They are the responses to short disturbances. Eigenmotions can be divided in two kinds: the ones associated with longitudinal motion, or also called symmetric eigenmotions (as they corresponds to symmetric flight), and those associated with lateral motion, also called asymmetric eigenmotions (as they correspond to asymmetric flight. This way, we can distinguish five eigenmotions:

- Symmetric eigenmotions:
  - Short period motion
  - Phugoid
- Asymmetric eigenmotions:
  - A-periodic roll
  - Spiral
  - Dutch roll

1.1.1 Short period motion

Suppose we very briefly deflect our elevator a lot? Well, this will increase our angle of attack: the elevator would produce negative lift, causing pitch up of our aircraft. To keep C_m zero, the angle of attack will also increase. The effects on velocity, angle of attack, pitch angle and pitch rate are visualised in figure 1.1.

We see that the response is very rapid: only in the first two seconds there is a sudden increase in angle of attack and pitch rate. Indeed, as this is such a short period, we call this short period motion:

SHORT PERIOD MOTION

Short period motion is the motion that is characterised by a short period (less than 3 seconds). It is heavily damped, is predominantly visible in the pitch rate q, and the velocity is almost not affected at all.

1.1.2 Phugoid

Now, you may have looked at figure 1.1, but doesn't stuff still change after 3 seconds still? The velocity goes down after all, so why is not that part of the short period motion as well?

Well, the reason is as follows: if you disturb it a little by very briefly deflecting the elevator a lot, then nature will automatically try to go to an equilibrium state as quick as possible: in this case by increasing the angle of

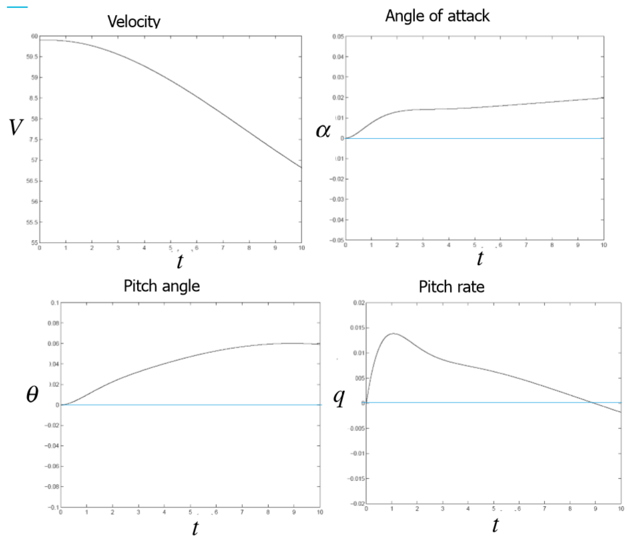


Figure 1.1: Effects on velocity, angle of attack, pitch angle and pitch rate due to a sudden impulse of elevator deflection angle. Note that the pitch rate is simply the time derivative of the pitch angle.

attack as quickly as possible, resulting in an every increasing pitch rate as shown in the right plots of figure 1.1. However, once it reaches this equilibrium state, nature is like oh fuck bitches I still have a pitch rate, so it overshoots the equilibrium state. Consequently, the angle of attack increases further, and nature is like fuck me gotta go back asap, so it slows down the pitch rate and changes the sign of the pitch rate. But when it again reaches the equilibrium state, it again is stupid enough to not slow down properly so it overshoots it once more. This is graphically shown in figure 1.2: it simply reaches an oscillating state.

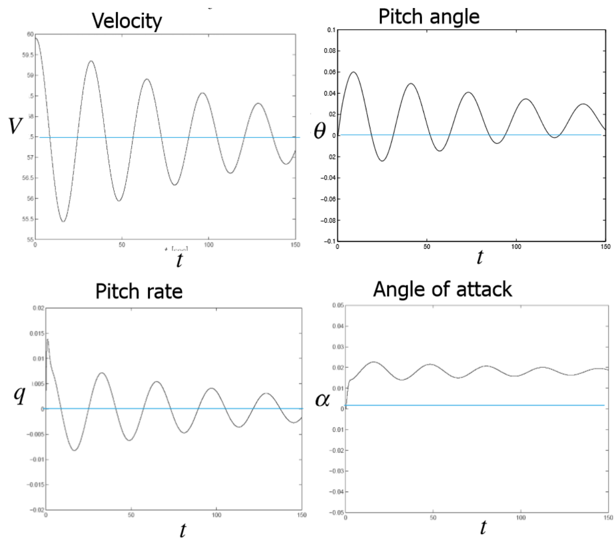


Figure 1.2: Effects on velocity, angle of attack, pitch angle and pitch rate due to a sudden impulse of elevator deflection angle.

The velocity behaviour can be straightforwardly explained: when the pitch angle increases, the altitude increases. As a result, kinetic energy is interchanged with potential energy, meaning kinetic energy is lost (and thus velocity).

In any case, we see that the short-period motion is sort of the introduction to a longer-period motion: this is why we say that the short-period motion only effectively is present for the first three seconds; after that it has been dampened so well that it's not present anymore. The long-period motion is called a phugoid.

PHUGOID

Phugoids are long-period (one period is longer than 25 seconds) eigenmotions, in longitudinal direction. Its motion is depicted in figure 1.3. It is predominantly visible in the pitch q, V and θ; the angle of attack is pretty much constant during a phugoid, on the other hand.

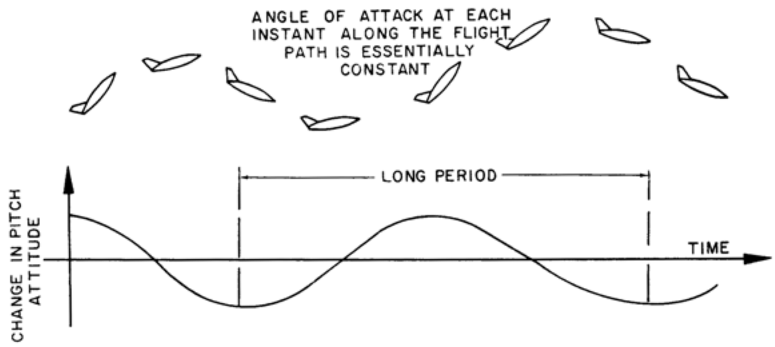


Figure 1.3: A phugoid.

I hope the difference between short-period motion and phugoids is clear: a more mathematical approach will follow later on in the course, you only have to understand the qualitative difference between them. Furthermore, it should be clear what eigenmotions are: we've now discussed two eigenmotions in longitudinal direction; later we'll also see eigenmotions in other directions.

Multiple choice questions: part (1)

Decide for every statement whether it is True or False. No explanation required!

1. When an aircraft is dynamically unstable it must be statically unstable.
2. In the phugoid motion, the pitch rate and airspeed vary in particular.

The correct answers are:

1. No, it's only the other way around: statically unstable aircraft are automatically dynamically unstable. The other way around does not hold. So, this statement is **false**.
2. Although it's true that the pitch rate and airspeed vary in particular, also the altitude and pitch angle vary. Therefore, this statement is both **true** and **false** (according to the official answers).

1.1.3 *A-periodic roll*

A-periodic roll is pretty easy to understand. Remember that roll damping is inevitable: if you roll in one direction, a counteracting moment due to difference in lift is produced. So, what may happen is that you deflect your ailerons (which will generate a moment in one way), and after a while, you obtain a rolling velocity that's so high that the counteracting moment is equal to the moment induced by the ailerons. At this point, you'll keep on rolling forever, as there's no acceleration acting any more.

1.1.4 *Spiral*

Now, suppose, for some reason, you are able to roll the aircraft a small bit. You may remember the stability derivative C_{n_p}, which represents what happens to the yawing when we roll: this is always negative. Thus, suppose we roll to the right (positive), then we start yawing in negative direction (our nose will also start pointing to the right relative to the velocity), and thus we get a negative β. Now, consider C_{l_β}: what happens to the rolling moment if β is changed (weathervane stability)? Well, suppose this is negative as well: as β becomes negative, the rolling velocity would become more positive, which makes β more negative, etc. In other words, you get into a spiral and everyone dies.

1.1.5 Dutch roll

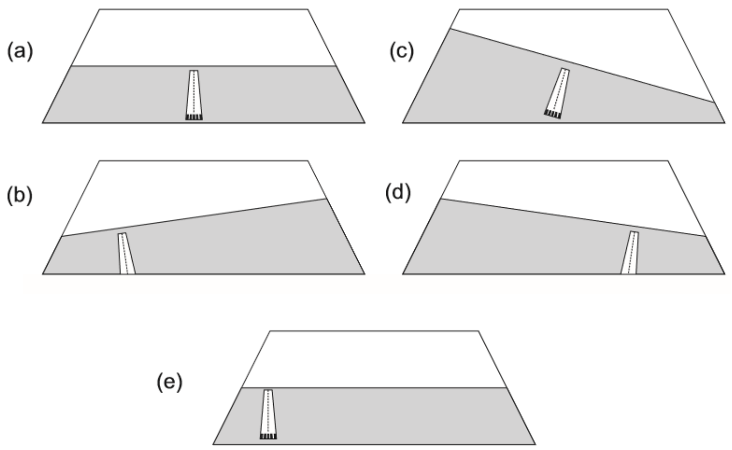
Now, suppose, for some reason, you are able to *yaw* the aircraft a bit (in positive direction, so the nose starts pointing to the left). As C_{l_r}, the change in rolling velocity due to a positive yawing velocity, is positive, this causes a positive rolling velocity (to the right). However, C_{n_p} is negative, so it starts yawing in the negative direction, but then it'll roll in negative direction as well, so it starts yawing in positive direction again, etc. etc. until the end of time.

If you're looking at the aircraft from outside from the right, moving along with the same velocity as the aircraft, then you see the following: first, due to the positive yaw you put in, you'll see the tip of the wing moving forward. Due to the induced rolling moment, it'll start moving upward, but due to this rolling velocity, it'll move backward simultaneously as well. When the tip moves behind its original position, the rolling velocity will become negative and it'll start moving down again. Then, when the tip moves down of its original position, it'll start moving forward again, so that it essentially starts describing a circle.

For all of these eigenmotions, I put videos on the dropbox so that you can visualise them without having to bother going to Brightspace.

Problem 6a August 2008 (5p)

Assume that you as a pilot are introducing the Dutch Roll motion by pressing the left pedal. Give the correct sequence of figures a to e which represents the views of the outside world seen by you during the first cycle. Begin the sequence with figure a which represents the view a time 0 (just before you move the rudder).



Not so hard if you think about it. You'll first yaw in positive direction, meaning your nose will point to the left, and thus the runway will appear to be more to the right. You'll also roll a bit to the right, meaning the horizon seems inclined downward. Thus, D is the second picture (with A being the first). Then you roll some more to the right, meaning C is the third. You then roll to the other side, first going through view E before going through view B. Thus, the correct sequence is A-D-C-E-B.

1.2 State space representation

State space may be a term that sounds familiar. If you're unsure where you heard it before, we discussed state space systems relatively extensively already in the course Aerospace Systems and Control Theory (the course in 4th quarter in 2nd year, with the E-lectures). What exactly were they?

It is rather logical to say that we can put all the information needed to determine the future system behaviour without reference to the derivaties of input and output variables. For example, for an aircraft, we could have a **system state** given by the vector $\mathbf{x}(t)$:

$$\mathbf{x} = \begin{bmatrix} \alpha(t) \\ M(t) \\ h(t) \\ \vdots \end{bmatrix}$$

where α is the angle of attack, M the Mach number and h the altitude. Of course, there are many more variables that would describe the state of the aircraft, but note that we're not including any of the derivatives of those variables in this vector.

Now, some terminology, looking at figure 1.4:

ELEMENTS OF A STATE-SPACE MODEL

The elements of a **state-space** model are

- **State variables**: the set of all variables which combine all necessary knowledge of the system at $t = t_0$ such that behaviour of the system can be determined for $t \geq t_0$.
- **State vector x**: an n-dimensional vector containing all state variables.
- **State space**: n-dimensional space whose axes are the state variables.
- **State equations**: a set of first order differential equations in terms of the state variables:

STATE EQUATION

The state equation describes the dynamics of $\mathbf{x}(t)$ and is given by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1.1}$$

where

- $\mathbf{x}(t)$ is the **state vector**;
- $\mathbf{u}(t)$ is the **input vector**;
- A is the **state matrix**;
- B is the **input matrix**

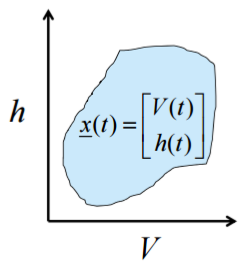


Figure 1.4: Elements of state-space model.

Note that the new state can be calculated (using computers) by integrating the state equation:

$$\mathbf{x}(\Delta t) = \mathbf{x}(0) + \int_0^{\Delta t} \dot{\mathbf{x}}(t)\, dt = \mathbf{x}(0) + \int_0^{\Delta t} (A\mathbf{x}(t) + B\mathbf{u}(t))\, dt$$

The fact that it's linear algebra means it's very straightforward to use computers for this, making it very convenient for us.

Let's do an example to see what we have to do (please note, a very helpful problem solving guide will follow after, cause the steps are literally the same each time, so don't worry if you wouldn't have come up with this yourself):

Example 1

Suppose we have the pendulum shown in figure 1.5. Determine an adequate state equation of the form $\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t)$.

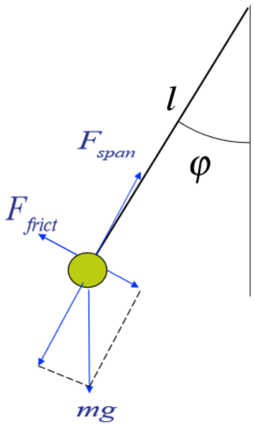


Figure 1.5: Pendulum. Please note: the friction force should point to the bottom-right.

The first step is to obtain a linear, ordinary differential equation that governs the system. From figure 1.5, we simply obtain (ϕ is positive in clockwise direction, and noting that the friction force actually points in opposite direction of what's drawn) in tangential direction:

$$ma_t = -mg\sin\phi - F_{\text{frict}}$$

Now, we have $a_t = l\ddot{\phi}$ and $F_{\text{frict}} = v \cdot V_t = v \cdot l\dot{\phi}$, where v is the friction constant (we assume that the friction force depends linearly on the velocity). Thus, we obtain the differential equation

$$ml\ddot{\phi} + vl\dot{\phi} + mg\sin\phi = 0$$

Assuming small angles we get

$$ml\ddot{\phi} + vl\dot{\phi} + mg\phi = 0$$

Now, what do we do next? We define the state vector. We have two states in this problem: if we know the angle ϕ and the angular velocity $\dot{\phi}$, we know everything there is to know about it: everything else can be computed from those states. In other words, we have

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \phi(t) \\ \dot{\phi}(t) \end{bmatrix}$$

As a result, we have

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \dot{\phi}(t) \\ \ddot{\phi}(t) \end{bmatrix}$$

Our aim is to find an equation for $\dot{x}_1(t)$ and for $\dot{x}_2(t)$. The first equation is very straightforward: just look carefully, $\dot{x}_1(t)$ is equal to $x_2(t)$! This means that we have our first equation:

$$\dot{x}_1(t) = x_2(t)$$

However, we need an additional equation to solve this system. Where does it come from? Well, simply substitute $\ddot{\phi}(t) = \dot{x}_2(t)$, $\dot{\phi}(t) = x_2(t)$ and $\phi(t) = x_1(t)$ in the governing differential equation; we obtain

$$ml\dot{x}_2(t) + vlx_2(t) + mgx_1(t) = 0$$

which we can rewrite to an explicit expression for $\dot{x}_2$:

$$\dot{x}_2(t) = -\frac{g}{l}x_1(t) - \frac{v}{m}x_2(t)$$

Putting these two equations into the state space equation yields

$$\dot{\mathbf{x}}(t) = \begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{g}{l} & -\frac{v}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \boldsymbol{A}\mathbf{x}$$

There were no inputs which is why there's nothing $\boldsymbol{B}\mathbf{u}$ in there. Soon we'll see an example where there is $\boldsymbol{B}\mathbf{u}$ term present.

REWRITING A LINEAR HIGHER ORDER ODE AS SYSTEM OF FIRST ORDER ODES

Starting with an nth order ordinary differential equation of the form

$$a_0 x + a_1\frac{dx}{dt} + \cdots + a_n\frac{d^n x}{dt^n} = a_n a_0\frac{d^n x}{dt^n} + a_1\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_{n-1}\frac{dx}{dt} = a_n$$

1. Introduce the state vector

$$\mathbf{x} = \begin{bmatrix} x_1 = x\\ x_2 = \frac{dx}{dt}\\ \vdots\\ x_{n-1} = \frac{d^{n-2}x}{dt}\\ x_n = \frac{d^{n-1}x}{dt}\end{bmatrix}$$

2. Introduce the vector

$$\dot{\mathbf{x}} = \begin{bmatrix}\frac{dx_1}{dt}\\ \frac{dx_2}{dt}\\ \vdots\\ \frac{dx_{n-1}}{dt}\\ \frac{dx_n}{dt}\end{bmatrix}$$

3. A linear system would be of the form

$$\dot{x} = \boldsymbol{A}\mathbf{x} + \boldsymbol{B}\mathbf{u}$$

For the first $n-1$ rows of the system (i.e. every row but the last), the equation will simply be $\dot{x}_i = x_{i+1}$. For the last row, you need to introduce the state variables into the differential equation:

$$a_0 x + a_1\frac{dx}{dt} + \cdots + a_{n-1}\frac{d^{n-1}x}{dt} + a_n\frac{d^n x}{dt^n} = u$$

becomes, as $x_1 = x$, $x_2 = dx/dt$, ..., $x_n = d^{n-1}/dt^{n-1}$,

$$\begin{aligned} a_0 x_1 + a_1 x_2 + \cdots + a_{n-1}x_n + a_n\dot{x}_n &= u\\ \dot{x}_n &= -\frac{a_0}{a_n}x_1 - \frac{a_1}{a_n}x_2 - \cdots - \frac{a_{n-1}}{a_n}x_n + \frac{1}{a_n}u\end{aligned}$$

4. Write out your matrices, which will become

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \vdots\\ \dot{x}_{n-1}\\ \dot{x}_n\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0 & \cdots & 0\\ 0 & 0 & 1 & 0 & \cdots & 0\\ \vdots & & & & & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1\\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & -\frac{a_3}{a_n} & \cdots & -\frac{a_{n-1}}{a_n}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ \vdots\\ x_{n-1}\\ x_n\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ \vdots\\ 0\\ \frac{1}{a_n}\end{bmatrix}u \tag{1.2}$$

Note that it may be the case that you get two ODEs at once, e.g.

$$\begin{aligned} m\ddot{h} + K_h h &= 0 \\ I_\theta \ddot{\theta} + K_\theta \theta &= 0 \end{aligned}$$

In that case, our state vector will simply be

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} h \\ \dot{h} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

What exactly do we then? Well, note that we'd again have $\dot{x}_1(t) = x_2(t)$, and also $\dot{x}_3(t) = x_4(t)$: the equations associated with the not-highest order derivatives are easy. For the highest order derivatives, i.e. the ones associated with $\dot{x}_2(t)$ and $\dot{x}_4(t)$, the equations follow again by plugging stuff into the original ODEs:

$$\begin{aligned} m\ddot{h} + K_h h &= 0 \\ m\dot{x}_2(t) + K_h x_1(t) &= 0 \\ \dot{x}_2(t) &= -\frac{K_h}{m} x_1(t) \end{aligned}$$

and similarly

$$\begin{aligned} I_\theta \ddot{\theta} + K_\theta \theta &= 0 \\ I_\theta \dot{x}_4(t) + K_\theta x_3(t) &= 0 \\ \dot{x}_4 &= -\frac{K_\theta}{I_\theta} x_3(t) \end{aligned}$$

Thus, we have the systems of equations

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{K_h}{m} x_1(t) \\ \dot{x}_3(t) &= x_4(t) \\ \dot{x}_4(t) &= -\frac{K_\theta}{I_\theta} x_3(t) \end{aligned}$$

or, in matrix notation,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_h}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{K_\theta}{I_\theta} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

1.2.1 The output equation

Now, the bad news is that sometimes we are interested in other output than just the states. For example, for the pendulum maybe you'd want to know the x- and y-coordinates. You then have to relate the outputs you want to the states you already had:

OUTPUT EQUATION

The **output equation** is given by

$$y(t) = C\mathbf{x}(t) + D\mathbf{u}(t) \tag{1.3}$$

where

- $\mathbf{x}(t)$ is the state vector;
- $\mathbf{u}(t)$ is the input vector;
- C is the **output matrix**;
- D is the **feedthrough or feedforward matrix**.

In above equation, $\boldsymbol{C}$ depends heavily in reality on which sensors we have available and how they behave; $\boldsymbol{D}$ depends on how the input is directly transmitted to the output. Often, $\boldsymbol{D}$ is equal to 0 (as that would mean that what you measure is a directly affected by an input, which does not seem likely). Note that these matrices are very different from $\boldsymbol{A}$ and $\boldsymbol{B}$, as those heavily depend on the physical dynamics of the system.

Now, let me be clear: the output and and state equations are totally different equations: the state equation is used to calculate the change in state variables, $\dot{\mathbf{x}}(t)$, whereas the output equation is used to compute the output of the system, as measured by signals. However, *both* equations use exactly the same state variables in their vector $\mathbf{x}(t)$ and the same input vector (so $\mathbf{x}(t)$ and $\mathbf{u}(t)$ re exactly the same equations; you can't be like, nahhh fack it I'll just use a different vector for the output equation).

Example 1: continued

Formulate the output equation of the state-space model if we want to know the x- and y-coordinates of the pendulum at any time t; see also figure 1.6.

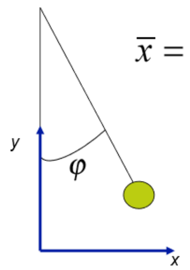


Figure 1.6: Pendulum.

So we have to relate x and y to the states we already had: $x_1(t) = \phi$ and $x_2(t) = \dot{\phi}$. We simply have

$$x = l\sin\phi = l\sin\left[x_1(t)\right]$$
$$y = l - l\cos\phi = l - l\cos\left[x_1(t)\right]$$

Linearising (so that we can put it in matrix form) yields

$$x = lx_1(t)$$
$$y = l - l = 0$$

So, in matrix format, this becomes

$$\mathbf{y}(t) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \phi \\ \dot{\phi} \end{bmatrix} = \boldsymbol{C}\mathbf{x}$$

Once again, as $\mathbf{y}$ does not depend on the input, we don't get a term $\boldsymbol{D}\mathbf{u}(t)$ in there.

Now, a nice definition:

THE COMPLETE STATE-SPACE MODEL

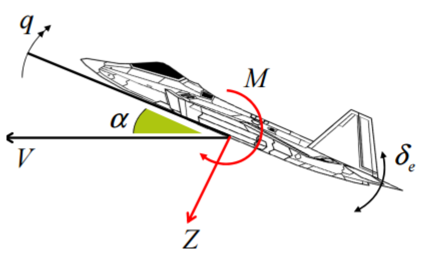
The **complete state-space model** consists of the state equation describing the system dynamics, and the output equation describing which states are measured:

$$\dot{\mathbf{x}}(t) = \boldsymbol{A}\mathbf{x}(t) + \boldsymbol{B}\mathbf{u}(t) \quad (1.4)$$
$$\mathbf{y}(t) = \boldsymbol{C}\mathbf{x}(t) + \boldsymbol{D}\mathbf{u}(t) \quad (1.5)$$

Let's now do a slightly more complicated example, an actual aerospace example:

Example 2

Suppose we investigate the pitch dynamics of an aircraft in state-space form, and we have sensors for angle of attack α and pitch rate q (rate of change of pitch attitude, not the pitch angle itself). Look at figure 1.7 for reference.



Symbol	Description
q	pitch rate
α	angle of attack
V	velocity = **constant!**
δ_e	elevator deflection
Z	Aerodynamic force
M	Aerodynamic moment

Figure 1.7: Aircraft characteristics.

First of all, trust me that we have the following equation of motion (where α is the angle of attack, q the pitch rate (so the change in pitch angle) and δ_e the deflection angle of the elevators).

$$\begin{aligned} z_1\frac{1}{V}\dot{\alpha} + z_2\alpha + z_3\frac{1}{V}q + z_4\delta_e &= 0 \\ m_2\alpha - m_3\frac{1}{V}\dot{q} + m_4\frac{1}{V}q + m_5\delta_e &= 0 \end{aligned}$$

Again, we first define the state vector, input vector and output vector:

$$\mathbf{x} = \begin{bmatrix}\alpha \\ q\end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix}\delta_e\end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix}\alpha \\ q\end{bmatrix}$$

Where does the state vector come from? The highest order derivative of α is first order, so we only include α. The highest order derivative of q is also only first order, so we only include q (if we'd had $\ddot{q}$ appear in one of the equations, we'd have included $\dot{q}$ as well, but not $\dot{\alpha}$ if there had been no $\ddot{\alpha}$).

Then, we rewrite the two equations of motion: the first one can be rewritten to

$$\begin{aligned} z_1\frac{1}{V}\dot{\alpha} + z_2\alpha + z_3\frac{1}{V}q + z_4\delta_e &= 0 \\ \dot{\alpha} &= -\frac{z_2}{z_1}V\alpha - \frac{z_3}{z_1}q - \frac{z_4}{z_1}V\delta_e \end{aligned}$$

and the second one can be rewritten to

$$\begin{aligned} m_2\alpha - m_3\frac{1}{V}\dot{q} + m_4\frac{1}{V}q + m_5\delta_e &= 0 \\ \dot{q} &= \frac{m_2}{m_3}\alpha + \frac{m_4}{m_3}q + \frac{m_5}{m_3}V\delta_e \end{aligned}$$

Thus, we have the system

$$\begin{aligned} \dot{\alpha} &= -\frac{z_2}{z_1}V\alpha - \frac{z_3}{z_1}q - \frac{z_4}{z_1}V\delta_e \\ \dot{q} &= \frac{m_2}{m_3}\alpha + \frac{m_4}{m_3}q + \frac{m_5}{m_3}V\delta_e \\ \dot{\mathbf{x}}(t) &= \boldsymbol{A}\mathbf{x}(t) + \boldsymbol{B}\mathbf{u}(t) \\ \begin{bmatrix}\dot{\alpha} \\ \dot{q}\end{bmatrix} &= \begin{bmatrix}-\frac{z_2}{z_1}V & -\frac{z_3}{z_1} \\ \frac{m_2}{m_3}V & \frac{m_4}{m_3}\end{bmatrix}\begin{bmatrix}\alpha \\ q\end{bmatrix} + \begin{bmatrix}-\frac{z_4}{z_1}V \\ \frac{m_5}{m_3}V\end{bmatrix}\delta_e \end{aligned}$$

Suppose we'd have $z_1 = -1.5$, $z_2 = -1.25$, $z_3 = -4$, $z_4 = -0.15$, $m_2 = 2$, $m_3 = -15$, $m_4 = -0.1$ and $m_5 = -0.025$, then this equation becomes

$$\dot{\mathbf{x}} = \begin{bmatrix}\dot{\alpha} \\ \dot{q}\end{bmatrix} = \begin{bmatrix}-83.33 & -2.67 \\ -5 & -7.5\end{bmatrix}\begin{bmatrix}\alpha \\ q\end{bmatrix} + \begin{bmatrix}-10 \\ -1.25\end{bmatrix}\delta_e$$

For the output equation, we have $\mathbf{y} = \begin{bmatrix}\alpha \\ q\end{bmatrix} = \begin{bmatrix}\alpha \\ q\end{bmatrix}$, thus we simply have

$$\begin{aligned}\mathbf{y} &= C\mathbf{x}(t) + D\mathbf{u}(t) \\ \begin{bmatrix}\alpha \\ q\end{bmatrix} &= \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha \\ q\end{bmatrix} + \begin{bmatrix}0 \\ 0\end{bmatrix}\delta_e\end{aligned}$$

The full state-space model is thus given by:

$$\begin{aligned}\begin{bmatrix}\dot{\alpha} \\ \dot{q}\end{bmatrix} &= \begin{bmatrix}-83.33 & -2.67 \\ -5 & -7.5\end{bmatrix}\begin{bmatrix}\alpha \\ q\end{bmatrix} + \begin{bmatrix}-10 \\ -1.25\end{bmatrix}\delta_e \\ \begin{bmatrix}\alpha \\ q\end{bmatrix} &= \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha \\ q\end{bmatrix} + \begin{bmatrix}0 \\ 0\end{bmatrix}\delta_e\end{aligned}$$

Last year you could do this, so you should still be able to do this.

1.3 *Eigenvalues*

The eigenmotions of a system are greatly related to the eigenvalues of the corresponding state-space system are greatly related to the eigenmotions of a system. Remember that for a system of n first order linear differential equations (and a state-space system is such a system), the solution is given by

$$\mathbf{x}(t) = \sum_{i=1}^{n} e^{\lambda_i t}\boldsymbol{\chi}_i$$

Naturally, if the eigenvalues are all fully real and negative, you'll get a solution that's monotonously decreasing (so it goes to 0); if they're fully real and positive the solution is monotonously increasing and goes to infinity; if they're complex you get an oscillating solution[1]. So, in the remaining two parts, we'll focus on deriving the state-space systems, and then analysing the eigenvalues. That's all we'll do basically.

[1] I hope that's something that's familiar from previous years. But again, we'll step into more detail later in the course, we'll first do something completely different.

2 Reference frames

Before we start our analysis, we have to define reference frames. Now, reference frames should have two properties:

- They should be orthogonal. Seems obvious, but in Dynamics there are still people who draw non-orthogonal coordinate systems so if you're one of those people, it's honestly a surprise you made it so far into your Bachelor.
- They should be right-handed. Honestly, I know a lot of people don't give two flying fucks whether it's right-handed or left-handed (or don't even know the difference), but seriously, *always* draw your coordinate systems to be right-handed. Right-handed means that if you stretch out your hand in the direction of x, then pull your fingers in so that they point in y, the z-axis points in the same direction as your thumb. If z points in opposite direction, it's left-handed, and then all of your signs become fucked up. Honestly, just don't even bother.

In this chapter, we'll discuss some reference frames that are commonly used. You won't need to remember all of them, we'll see the important ones in use later on, and as you do you'll start remembering them. But don't worry too much about learning them by heart now.

2.1 Inertial reference frame F_I

We first define an inertial reference frame, one that does not move at all. It is depicted in figure 2.1a: Z_I points through the North-pole, X_I points through the crossing of the equator and the ecliptic plane[1], and Y_I is orientated such that the reference frame is right-handed. This reference frame is called the **Inertial reference frame**, F_I.

2.2 Earth-centered, Earth-fixed reference frame F_C

Note that the inertial reference frame did not move at all. However, we can also define a reference frame that *does* co-rotate with the Earth. This is depicted in figure 2.1b: the Earth rotates with a velocity $\mathbf{\Omega}$, so after a time t_O, it'll have rotated $\mathbf{\Omega} t_O$. This is called the **Earth-centered, Earth-fixed reference frame**, F_C.

2.3 Vehicle-carried normal Earth reference frame, F_E

This is the reference frame we'll be using to derive our equations of motion and is depicted in figure 2.1c: it is centered at the center of gravity of the vehicle (e.g. aircraft or spacecraft), with the $X_E Y_E$-plane being locally tangent to the Earth's surface. This means that Z_E points downwards, perpendicular to the Earth's surface directly below the aircraft. X_E points to the North, Y_E points to the left[2]. This reference frame is called **vehicle-carried normal Earth reference frame**, F_E.

2.4 Body-fixed reference frame, F_b

In the **body-fixed reference frame** F_b, the origin is centred at the center of gravity, the X_b goes through the nose, the Z_b points downward and Y_b points to the right to make it a right-handed coordinate system, as shown

[1] Remember that the Earth's rotation axis is inclined like 23° with respect to the Z_I-axis. This is where the ecliptic plane comes from, it's the plane perpendicular to the axis of actual rotation.

[2] So to be clear, X_E does not point in the direction of the vehicle or smth. It just always points North, even if the aircraft goes in a totally different direction.

in figure 2.1d. Note that this is not the same as the F_E frame! In the F_E frame, the X_E always points to the North, and the Z_E always perpendicular to the Earth's surface.

RELATION BETWEEN BODY-FIXED AND VEHICLE-CARRIED NORMAL EARTH REFERENCE FRAME

F_b, the body-fixed reference frame, and F_E, the vehicle-carried normal Earth reference frame, are related via

- the **pitch angle** θ (rotation around Y_E-axis)
- the **roll angle** ϕ (rotation around X_E-axis)
- the **yaw angle** ψ (rotation around Z_E-axis)

Note: these angles may be defined a bit weirdly: if your aircraft is flying in a straight line, then all of the angles may be very different from zero, unless the aircraft is flying exactly to the North[3]. Indeed, we define the following angles that actually correspond to the angles that physically make sense:

BODY-FIXED ANGLES

We define the following angles:

- **Azimuth angle**: angle between X_b-axis and its projection on a predefined (local) vertical plane.
- **Climb angle**: angle between X_b-axis and its projection on the (local) horizontal plane.
- **Bank angle**: angle between Y_b-axis and its projection on the (local) horizontal plane.

These angles are how you expect them to be, e.g. if your aircraft is banked at 30°, the bank angle is actually 30° (whereas the roll angle is not necessarily 30°). The azimuth angle is for example taken between the X_b-axis and the direction pointing North, so that it's sort of indicates the heading.

2.4.1 *Stability reference frame*

Now, sometimes we do not let the X-axis go through the nose, but rather let it be aligned with the component of the free-stream velocity in the plane of symmetry; see figure 2.2a. This is called the **stability reference frame** F_S. X_S and Z_S are inclined with the angle of attack α with respect to the X_b and Z_b axes.

2.5 *Vehicle reference frame*

The **vehicle reference frame** is a reference frame that was sometimes used in part I of the summary. We won't use it ever-after, as it's a *left*-handed coordinate system so everyone hates it. It's usefulness was that it was nice to compute distances with it, e.g. $x_w - x_{c.g.}$ etc., as X_r points to the *back* of the aircraft, and Z_r points upward, as shown in figure 2.2b. That makes it convenient to measure distances, but not more than that. Just forget about it already tbh.

2.6 *Aerodynamic (air-path) reference frame, F_a*

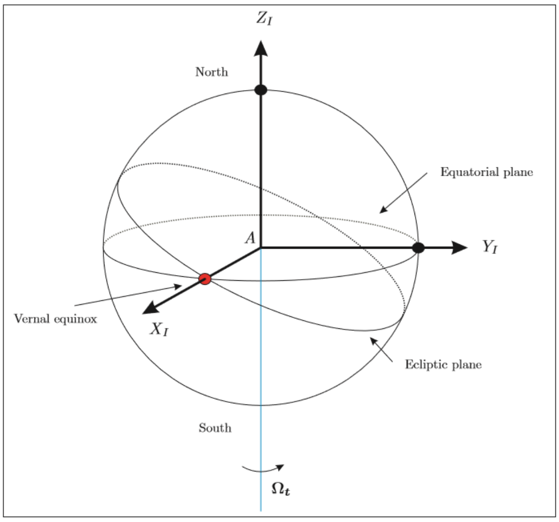
Finally, we have the **aerodynamic (air-path) reference frame**. Here, X_a is aligned with the freestream velocity $\mathbf{V}_a$, including the side-slip out of the symmetry plane. It thus includes both the angle of attack α_a and sideslip angle β_a. See figure 2.2c.

RELATION BETWEEN AERODYNAMIC AND BODY-FIXED REFERENCE FRAME

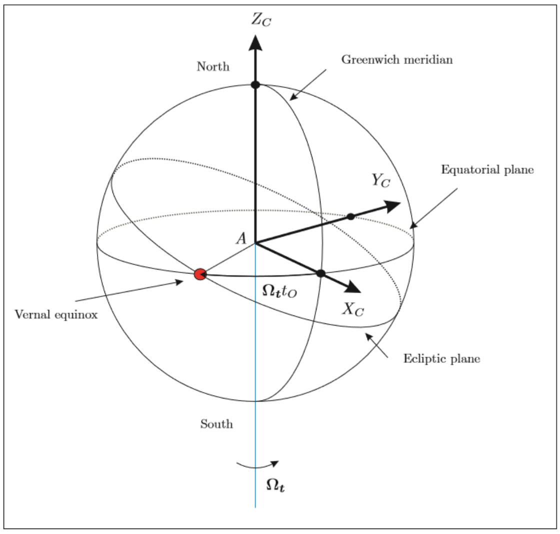
F_a, the aerodynamic reference frame, and F_b, the body-fixed reference frame are related via

- the **angle of attack** α_a around the Y_b-axis
- the **angle of sideslip** β_a around the Z_b-axis

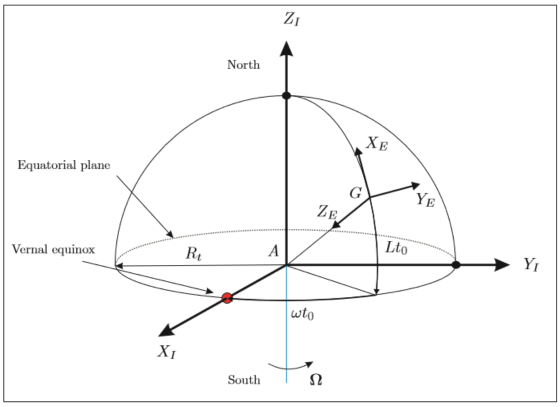
[3]Just consider the yaw angle: unless you fly exactly to the North, you'll always have a yaw angle as it's measured from the X_E-axis, which points to the North, rather than being measured from the body-fixed X_B-axis.



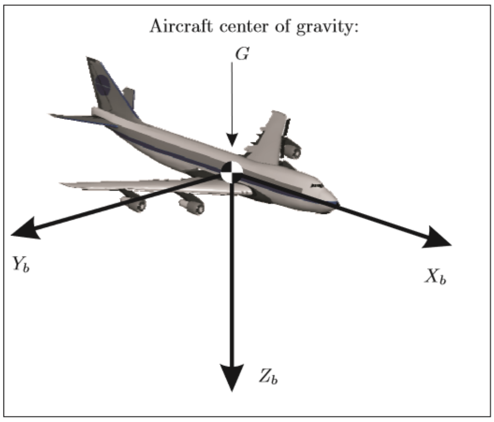
(a) Definition of Earth-centered inertial reference frame.



(b) Definition of Earth-centered, Earth-fixed reference frame.

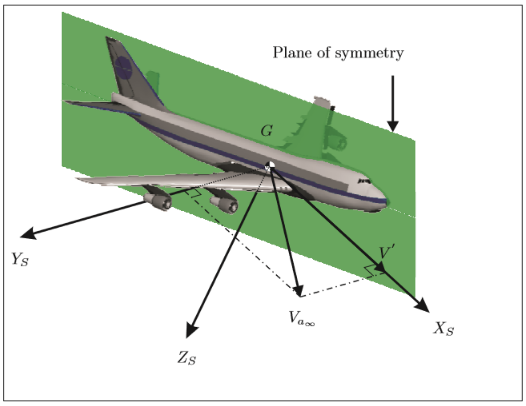


(c) Definition of vehicle-carried normal Earth reference frame.

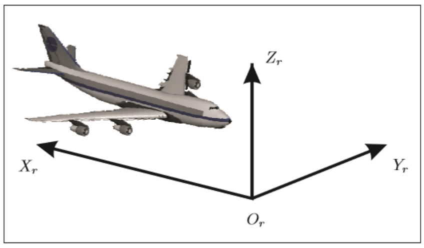


(d) Definition of body-fixed reference frame.

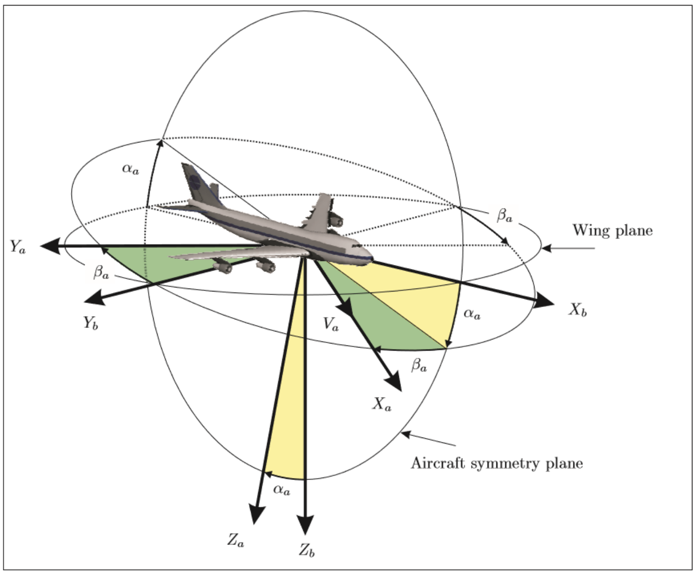
Figure 2.1: Definition of F_I, F_C, F_E and F_b.



(a) Definition of stability reference frame.



(b) Definition of aircraft reference frame.



(c) Aerodynamic reference frame in relation to body-fixed reference frame.

Figure 2.2: Definition of F_S, F_r and F_a.

Relation between aerodynamic and vehicle-carried normal Earth reference frame

F_a, the aerodynamic reference frame, and F_e, the vehicle-carried normal Earth reference frame are related via

- the **aerodynamic heading angle** χ_a about the Z_E-axis
- the **aerodynamic flight-path angle** about the Y_E-axis
- the **aerodynamic bank angle** μ_a about the X_E-axis

Yes this chapter is a boatload of information, and for now you definitely shouldn't learn everything by heart already. However, use it as a reference in case you get lost what each reference frame exactly meant etc.

3 Transformations between reference frames

Sometimes in life, such as in this course, you'll need to be able to transform from one reference frame to another. This is something even first-year you could do however, as we already discussed this in linear algebra. But as you probably forgot about this already, I'll repeat that part first.

3.1 Static transformations

Static transformations are simply rotations involving Euler angles. Here, we rotate one of the planes (e.g. the xy-plane) a certain angle. For example, as seen in figure 3.1, we can rotate a triangle in the xy-plane about the axis going into the paper (which would be the z-axis). Now, if you know the initial coordinates of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$, it's not that hard to graphically find the new coordinates based on the sketch.

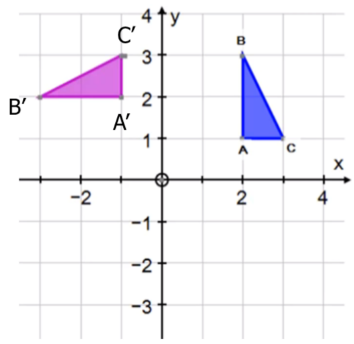


Figure 3.1: 90° rotation of a triangle.

Now, is there a general way to rotate a point, so that we don't have to make a sketch everytime? Fortunately, yes, there is, as there exists a thing named transformation matrix. If you have a point with initial coordinates (x_1, y_1), its coordinates after rotating it an angle θ are given by

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

How do we obtain this matrix? Well, for this we consider the rotation of the unit vectors in x- and y-direction. We see that if we rotate the unit vector in x-direction an angle θ, as shown in figure 3.2, the new vector can be described by $[\cos\theta, \sin\theta]^T$; the unit vector in y-direction can be described by $[-\sin\theta, \cos\theta]^T$. This first vector makes up the first column of the transformation matrix, the second vector makes up the second column. Not that difficult.

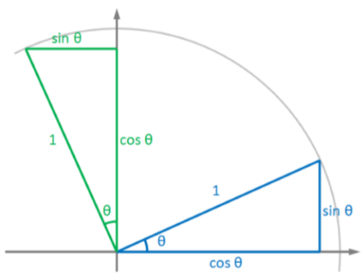


Figure 3.2: Setting up transformation matrix for rotation.

Now, how does it work when we rotate not the point itself, but the *reference frame*, as shown in figure 3.3? Well, if you think about it, rotating the reference frame is like rotating the point, but in opposite direction. In other words, you have to put a minus sign in front of the angles; using the fact that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$, and bearing in mind that the x-coordinate remains unchanged in figure 3.3, this means:

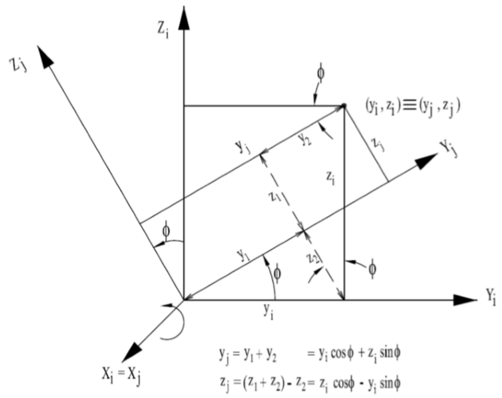


Figure 3.3: Setting up transformation matrix for rotation of coordinate frame.

TRANSFORMATION MATRIX

The transformation matrix to rotate the coordinates of a point with coordinates $[x_i, y_i, z_i]^T$ in the original i-coordinate system to the coordinates $[x_j, y_j, z_j]^T$ in the j-coordinate system, rotated an angle θ around the x-axis in positive direction, is given by

$$\begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{3.1}$$

Similarly, rotations around the y- and z-axes are given by

$$\begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{3.2}$$

$$\begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{3.3}$$

These equations can be written in short-hand notation as

$$\mathbf{X}^j = \mathbb{T}_{ji}(\theta)\,\mathbf{X}^i \tag{3.4}$$

where $\mathbf{X}^k$ means the vector $\mathbf{X}$ expressed in reference frame k. Another way of notation is the following:

$$\mathbf{X}^j = \mathbb{T}_x(\theta)\big|_i\,\mathbf{X}^i$$
$$\mathbf{X}^j = \mathbb{T}_y(\theta)\big|_i\,\mathbf{X}^i$$
$$\mathbf{X}^j = \mathbb{T}_z(\theta)\big|_i\,\mathbf{X}^i$$

Here, $\mathbb{T}_x\big|_i$ indicates a rotation around the original x-axis in the i-reference frame. For both notations, the argument of the transformation is sometimes dropped (i.e. $\mathbb{T}$ is written rather than $\mathbb{T}(\theta)$).

Both notations are used, you need to be kinda flexible in recognizing them (it's not really hard anyway).

Note that the transformation matrices are very similar, just remember that for the one about the y-axis the minus sign is now in the top right corner rather than the bottom left corner.

Now, what would you need to write if you'd first rotate the coordinate frame around the x-axis by an angle θ, and then around the y-axis by an angle ϕ? It actually becomes pretty easy:

$$\mathbf{x}_2 = \mathbb{T}_y(\phi)\Big|_1 \mathbb{T}_x(\theta)\Big|_0 \mathbf{x}_0 = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{3.5}$$

Note the order in which the matrices appear! Matrix multiplication is performed from right to left, so you need to put the first rotation (the one around the x-axis) closest to the original vector, and then the second rotation next to that etc.

Furthermore, care should be taken that you correctly describe the order in which you're doing the transformations! Different orders lead to different outcomes, as shown in figure 3.4. In aerospace industry, it's customary to first describe a rotation around the z-axis, then a rotation around the y-axis then one around the x-axis. However, on the exams this is not the custom and we'll see examples later on where the rotation order is different.

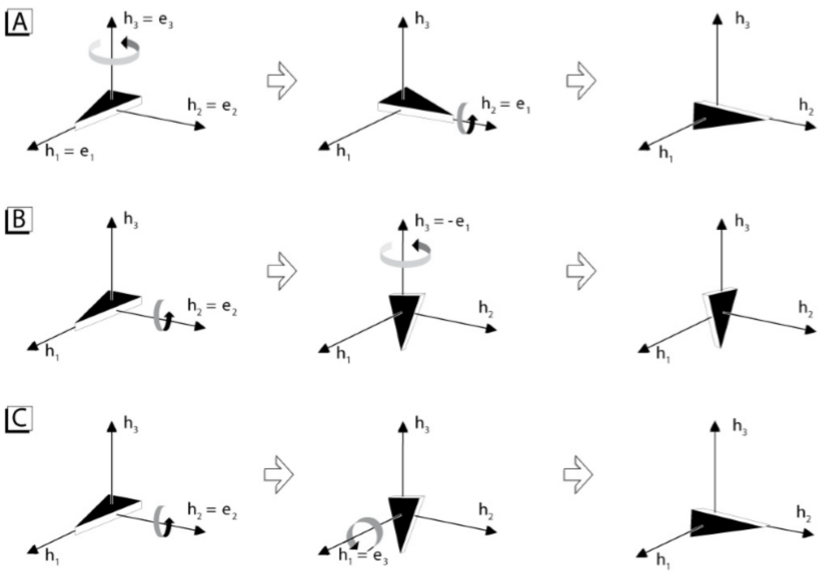


Figure 3.4: Setting up transformation matrix for rotation of coordinate frame.

3.1.1 *Properties of the transformation matrix*

Transformation matrices have some special properties, as they are *orthonormal*. This guarantees two important properties:

PROPERTIES OF ORTHONORMAL MATRICES

The following two properties hold for orthonormal matrices:

$$\mathbb{T}_{ab}^{-1} = \mathbb{T}_{ab}^{t} = \mathbb{T}_{ba} \tag{3.6}$$

$$\mathbb{T}_{ab}^{t}\mathbb{T}_{ab} = \mathbb{T}_{ab}^{-1}\mathbb{T}_{ab} = \mathbb{I} \tag{3.7}$$

These make a fair bit of sense: basically, if you take the inverse of a rotation from b to a, it's equivalent to a rotation from a to b, i.e. $\mathbb{T}^{-1}(\theta) = \mathbb{T}(-\theta)$.

3.2 *Application of reference frame transformations*

Hopefully it's clear now how a single rotation works, but it's probably still a bit vague what you have to do when you have multiple rotations occurring. Therefore, let's do some example transformations between reference frames discussed in chapter 2.

3.2.1 Transformation from F_I to F_C

Let's first discuss the transformation from the inertial reference frame to the Earth-centered reference frame. Consider figure 3.5. To go from the F_I-frame to the F_C-frame, we merely have to rotate an angle $\Omega_t \cdot t_O$ around the Z_I-axis, in positive direction. Thus, we simply have

ROTATION FROM F_I TO F_C

The rotation from F_I to F_C is given by

$$\mathbf{X}^C = \mathbb{T}_{CI}\mathbf{X}^I = \mathbb{T}_z\left(\Omega_t \cdot t_O\right)\Big|_I \mathbf{X}^I \tag{3.8}$$

$$\mathbb{T}_{CI} = \begin{bmatrix} \cos\Omega_t t_O & \sin\Omega_t t_O & 0 \\ -\sin\Omega_t t_O & \cos\Omega_t t_O & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.9}$$

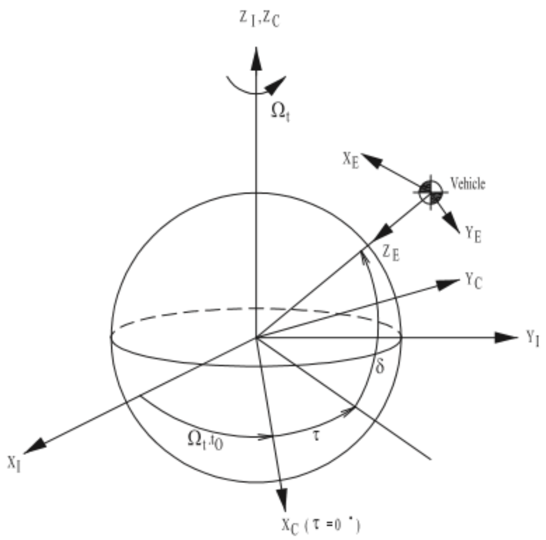


Figure 3.5: Relation between the inertial, the Earth-centerd Earth-fixed and vehicle-carried normal reference frame.

3.2.2 Transformation from F_C to F_E

Let's now consider the transformation between the Earth-centered reference frame and the vehicle-carried normal Earth reference frame. Consider figure 3.5 once more. When we want to rotate from F_C to F_E, we need to rotate an angle τ in positive direction around Z_C. We must then rotate around the Y-axis. However, it is very deceitful (and quite badly drawn): you have to rotate δ plus 90 degrees (in negative direction)! After all, if you only rotate δ, then X_E still points out of the Earth, but you want to make it point parallel to the Earth's surface, so it needs to rotate a further 90 degrees. This means that the transformation matrices become

ROTATION FROM F_C TO F_E

The rotation from F_C to F_E is given by

$$\mathbf{X}^E = \mathbb{T}_{EC}\mathbf{X}^C = \mathbb{T}_y\left(-\delta - \pi/2\right)\Big|_{C'} \mathbb{T}_z\left(\tau\right)\Big|_C \mathbf{X}^C \tag{3.10}$$

$$\mathbb{T}_{EC} = \begin{bmatrix} \cos\left(-\delta - \frac{\pi}{2}\right) & 0 & -\sin\left(-\delta - \frac{\pi}{2}\right) \\ 0 & 1 & 0 \\ \sin\left(-\delta - \frac{\pi}{2}\right) & 0 & \cos\left(-\delta - \frac{\pi}{2}\right) \end{bmatrix} \begin{bmatrix} \cos\tau & \sin\tau & 0 \\ -\sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\sin\delta\cos\tau & -\sin\delta\sin\tau & \cos\delta \\ -\sin\tau & \cos\tau & 0 \\ -\cos\delta\cos\tau & -\cos\delta\sin\tau & -\sin\delta \end{bmatrix} \tag{3.11}$$

where use was made of the fact that $\sin(x - \pi/2) = -\cos(x)$ and $\cos(x - \pi/2) = \sin(x)$. Note that the matrix multiplication is something you need to be able to perform yourself; it's not really difficult though, just remember that for the entry in the ith row and jth column you perform the dot product between the ith row of the left matrix and j column of the right matrix.

3.2.3 Transformation from F_E to F_b

Let's now consider the transformation between the vehicle carried Earth normal reference frame and the body-fixed reference. For this, let's consider figure 3.6, which shows a lot of angles, but stay calm and we'll be able to deal with it. In figure 3.6a, the complete transformation is shown, but it's easier for us if we decompose it in several separate rotations.

First, we rotate the reference to the E'-reference system by rotating it an angle $-\psi$ around the Z_E-axis in negative direction, where ψ is the **yaw angle**. So, in fact we are rotating an angle ψ in positive direction around the Z_E-axis.

Then, in figure 3.6c, we rotate from the E'-reference frame to the E''-reference frame, by rotating an angle θ in *positive* direction around the $Y_{E'}$ axis. In figure 3.6d, we rotate from the E''-reference frame to the F_b-reference by rotating ϕ in positive direction around the $X_{E''}$ axis. Putting all of this together, this means that the transformation matrix becomes, from equations (3.1)-(3.3)

ROTATION FROM F_E TO F_b

The rotation from F_E to F_b is given by

$$\mathbf{X}^b = \mathbb{T}_{bE}\mathbf{X}^E = \mathbb{T}_{bE''}\mathbb{T}_{E''E'}\mathbb{T}_{E'E} = \mathbb{T}_x(\phi)\big|_{E''}\,\mathbb{T}_y(\theta)\Big|_{E'}\,\mathbb{T}_z(\psi)\big|_E\,\mathbf{X}^E \tag{3.12}$$

$$\mathbb{T}_{bE} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.13}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{bmatrix} \tag{3.14}$$

Yes this one was already a bit more fucked up, but basically you had to ask yourself two questions each time:

- Do they write the negative angle (e.g. $-\psi$) or the positive angle (e.g. θ)?
- Is the arrow drawn in negative or positive direction?

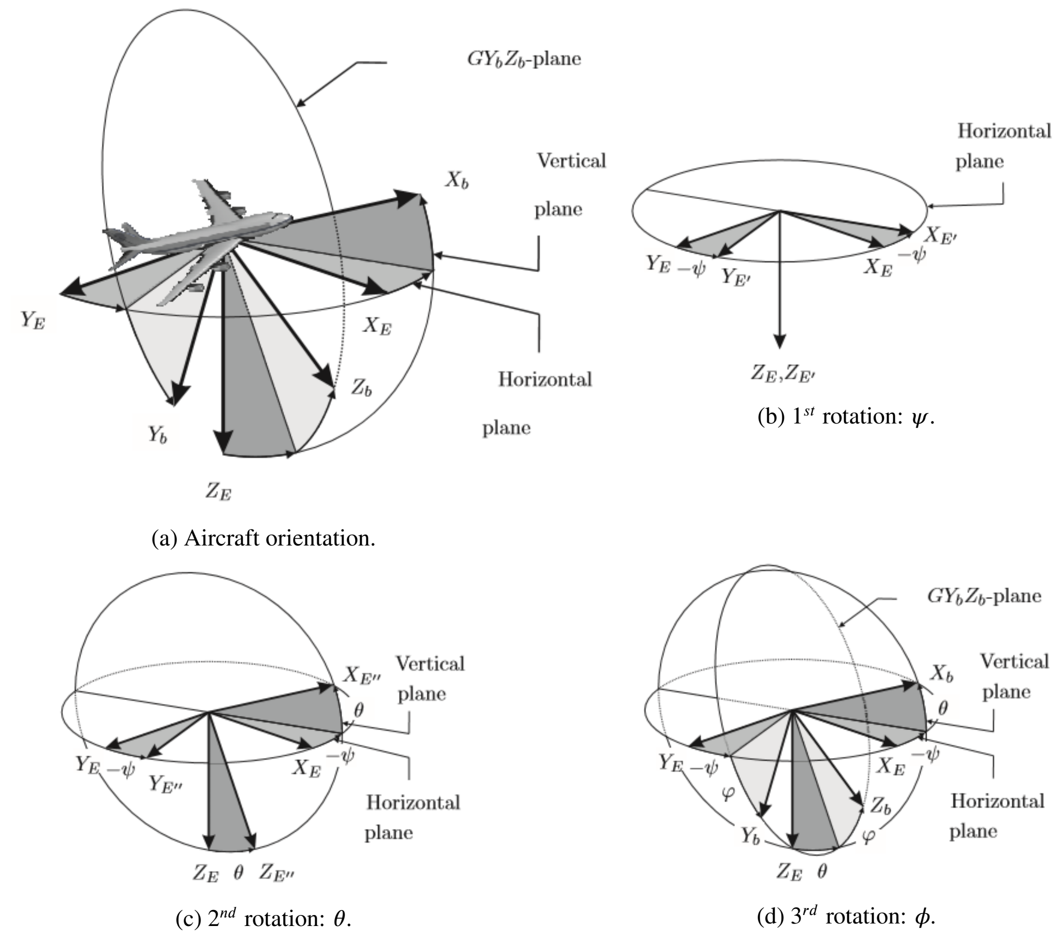
If there should be a minus sign, then the sine in the corresponding transformation matrix given in equations (3.1)-(3.3) should have an (additional) minus sign in front. Two minus signs then obviously cancel out. Furthermore, I hope you see why the subscripts are ordered like they are: we see that $\mathbb{T}_{bE''}\mathbb{T}_{E''E'}\mathbb{T}_{E'E}$ represents the rotation from the very last subscript (E) to the very first subscript (b).

3.2.4 Transformation from F_E to F_a

Let's now discover how to transform from the vehicle carried normal Earth and aerodynamic reference frame. The principle is very similar to the previous subsection. Consider figure 3.7, where all the rotations are shown. How can we then determine the correct order of rotations and the signs? Well, look first at the Y-axis: we see that $-\chi_a$ must happen before $-\mu_a$. Looking at the Z-axis, we see that also γ_a must happen before $-\mu_a$. Looking at the X-axis, we see that $-\chi_a$ must occur before γ_a, so the correct order of rotation is

$$-\chi_a \to \gamma \to -\mu_a$$

Or in other words, first rotation around the Z-axis, then around the Y-axis and then around the X-axis. We see that for the rotation around the Z-axis, the arrow denoting $-\chi_a$ is drawn in *negative* direction around the Z-axis, and thus the minus sign will disappear in the transformation matrix.



(a) Aircraft orientation.

(b) 1^{st} rotation: ψ.

(c) 2^{nd} rotation: θ.

(d) 3^{rd} rotation: ϕ.

Figure 3.6: Transformation from vehicle carried normal Earth reference frame F_E to the body-fixed reference frame F_b.

For the rotation around the Y-axis, we see that γ_a is drawn in positive direction. For rotation around the X-axis, $-\mu_a$ is drawn in positive direction. Thus, the transformation matrices become

Rotation from F_E to F_a

The rotation from F_E to F_a is given by

$$\mathbf{X}^a = \mathbb{T}_{aE}\mathbf{X}^E = \mathbb{T}_{AE''}\mathbb{T}_{E''E'}\mathbb{T}_{E'E} = \mathbb{T}_x\left(-\mu_a\right)\Big|_{E''}\mathbb{T}_y\left(\gamma_a\right)\Big|_{E'}\mathbb{T}_z\left(\chi_a\right)\Big|_{E}\mathbf{X}^E \tag{3.15}$$

$$\mathbb{T}_{aE} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu_a & -\sin\mu_a \\ 0 & \sin\mu_a & \cos\mu_a \end{bmatrix}\begin{bmatrix} \cos\gamma_a & 0 & -\sin\gamma_a \\ 0 & 1 & 0 \\ \sin\gamma_a & 0 & \cos\gamma_a \end{bmatrix}\begin{bmatrix} \cos\chi_a & \sin\chi_a & 0 \\ -\sin\chi_a & \cos\chi_a & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.16}$$

$$= \begin{bmatrix} \cos\gamma_a\cos\chi_a & \cos\gamma_a\sin\chi_a & -\sin\gamma_a \\ -\sin\mu_a\sin\gamma_a\cos\chi_a - \cos\mu_a\sin\chi_a & -\sin\mu_a\sin\gamma_a\sin\chi_a + \cos\mu_a\cos\chi_a & -\sin\mu_a\cos\gamma_a \\ \cos\mu_a\sin\gamma_a\cos\chi_a - \sin\mu_a\sin\chi_a & \cos\mu_a\sin\gamma_a\sin\chi_a + \sin\mu_a\cos\chi_a & \cos\mu_a\cos\gamma_a \end{bmatrix} \tag{3.17}$$

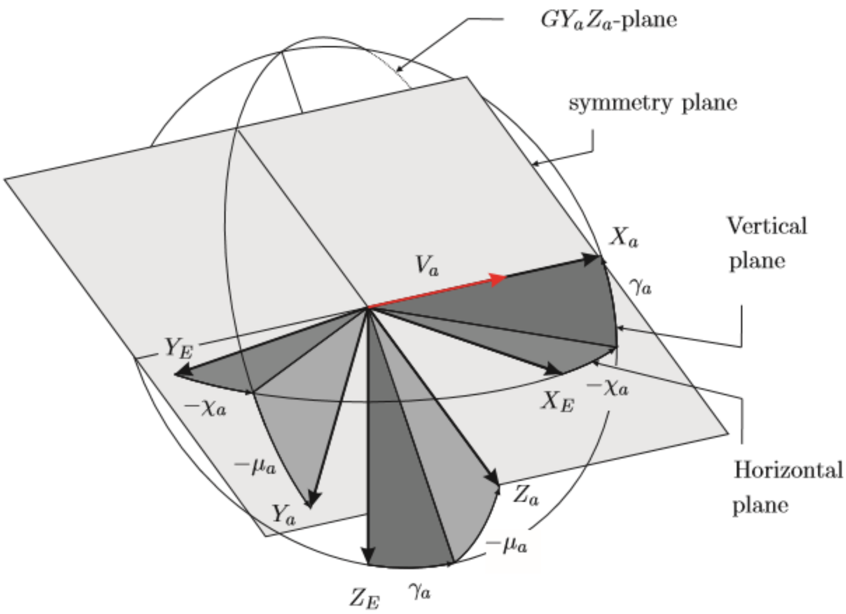


Figure 3.7: Transformation from vehicle carried normal Earth reference frame F_E to the aerodynamic reference frame F_a.

3.2.5 Transformation from F_b to F_a

For the transformation of the body-fixed reference frame F_b to the aerodynamic reference frame F_a consider figure 3.8. We first rotate in the *opposite* direction of the α_0 that's drawn; α_0 is drawn in positive direction so this rotation around the Y-axis will get a minus sign. We then rotate in the direction of β_a, which is drawn in positive direction, around Z. Thus, the transformation matrices look like

ROTATION FROM F_b TO F_a

The rotation from F_b to F_a is given by

$$\mathbf{X}^a = \mathbb{T}_{ab}\mathbf{X}^b = \mathbb{T}_{ab'}\mathbb{T}_{b'b} = \mathbb{T}_z\left(\beta_a\right)\Big|_{b'}\mathbb{T}_y\left(-\alpha_a\right)\Big|_{b}\mathbf{X}^b \tag{3.18}$$

$$\mathbb{T}_{ab} = \begin{bmatrix} \cos\beta_a & \sin\beta_a & 0 \\ -\sin\beta_a & \cos\beta_a & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha_a & 0 & \sin\alpha_a \\ 0 & 1 & 0 \\ -\sin\alpha_a & 0 & \cos\alpha_a \end{bmatrix} = \begin{bmatrix} \cos\beta_a\cos\alpha_a & \sin\beta_a & \cos\beta_a\sin\alpha_a \\ -\sin\beta_a\cos\alpha_a & \cos\beta_a & -\sin\beta_a\sin\alpha_a \\ -\sin\alpha_a & 0 & \cos\alpha_a \end{bmatrix} \tag{3.19}$$

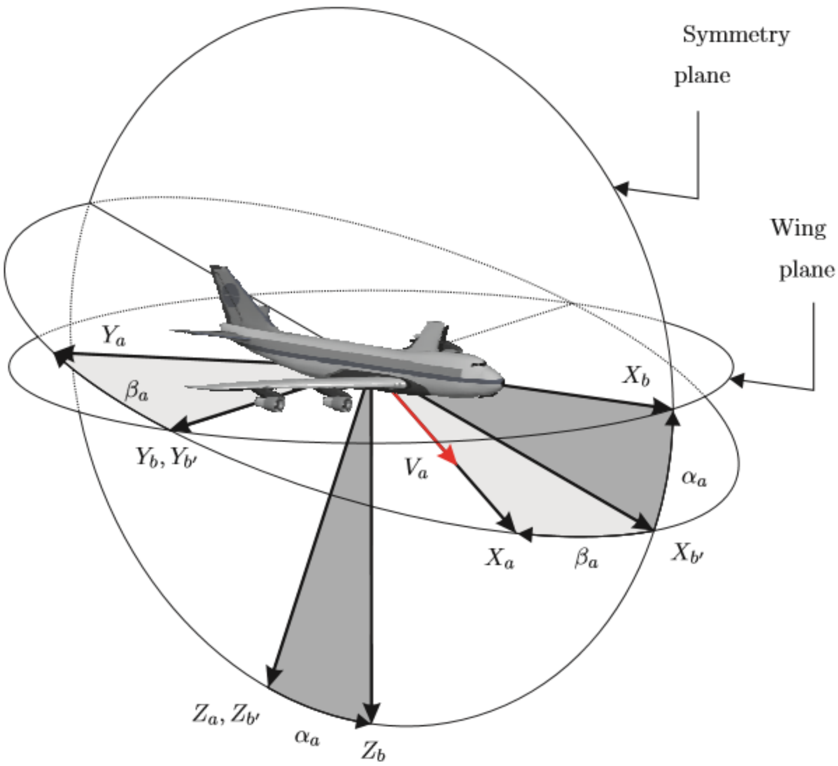


Figure 3.8: Transformation from body-fixed reference frame F_b to the aerodynamic reference frame F_a.

SETTING UP A SEQUENCE OF TRANSFORMATIONS

1. Determine the correct sequence of rotations. Do this by simply following the arrows drawn in the figure.
2. For each arrow, establish whether:
   (a) The angle written near the arrow has a minus sign in front of it;
   (b) The arrow points in positive or negative direction;
   (c) The arrow is traversed in positive or negative direction to perform the rotation.
3. Based on the number of negatives found in step 2., add a minus sign to the sines in the transfor-

mation following transformation matrices:

$$\begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & \sin\theta_x \\ 0 & -\sin\theta_x & \cos\theta_x \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{3.20}$$

$$\begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{3.21}$$

$$\begin{bmatrix} x_j \\ y_j \\ z_j \end{bmatrix} = \begin{bmatrix} \cos\theta_z & \sin\theta_z & 0 \\ -\sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} \tag{3.22}$$

3.2.6 *Exam question*

Exam October 2011: problem 2 (20p)

In flight simulation packages it is common practice to propagate the state in the inertial reference frame, index I, whereas the forces are known in the body frame, index B. Therefore, transformations are involved between the body and inertial frame. You are required to derive the following.

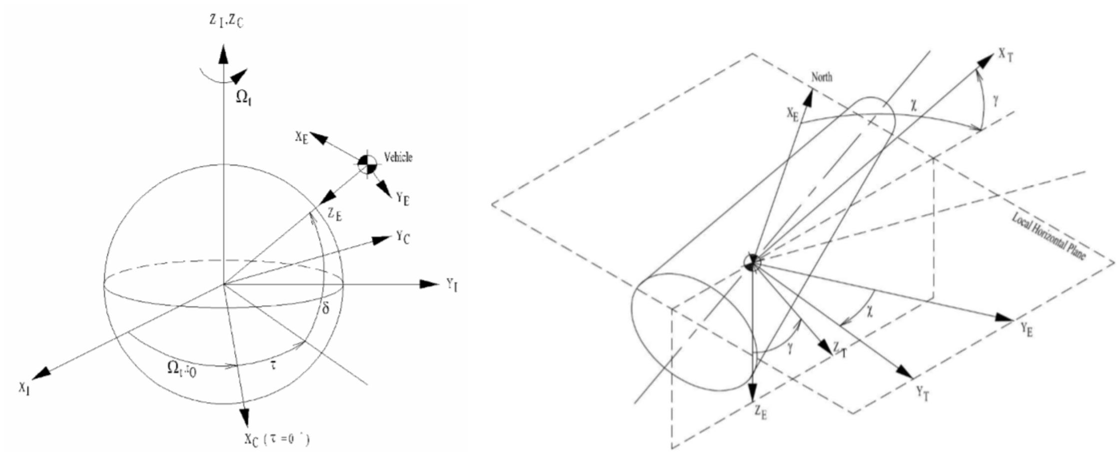
(a) (**5 points**) Derive the unit-axis transformation matrix for a rotation about the X-axis, $\mathbb{T}_x$. Provide a clear drawing to support the derivation.

(b) (**2 points**) State the two remaining unit-axis transformation matrices for rotations about the Y- and Z-axes.

(c) (**5 points**) Set up the sequence of rotations to go from body do inertial frame, $\mathbb{T}_{I,B}$, see also Figure 3.9 and 3.10. Note: only the right sequence of unit-axis transformations should be given (not the final transformation matrix).

(d) (**5 points**) Provide the inverse transformation matrix $\mathbb{T}_{B,I}$ in terms of the sequence of unit-axis transformations.

(e) (**3 points**) If you would compare the final result $\mathbb{T}_{B,I}$ with $\mathbb{T}_{I,B}$, what can you say about the individual matrix elements in each row (and column)?



(a) I-, C- and E-frames (b) E and T-frames

Figure 3.9: Reference frames involved in the transformation between I and E-frames. Angles are all positive.

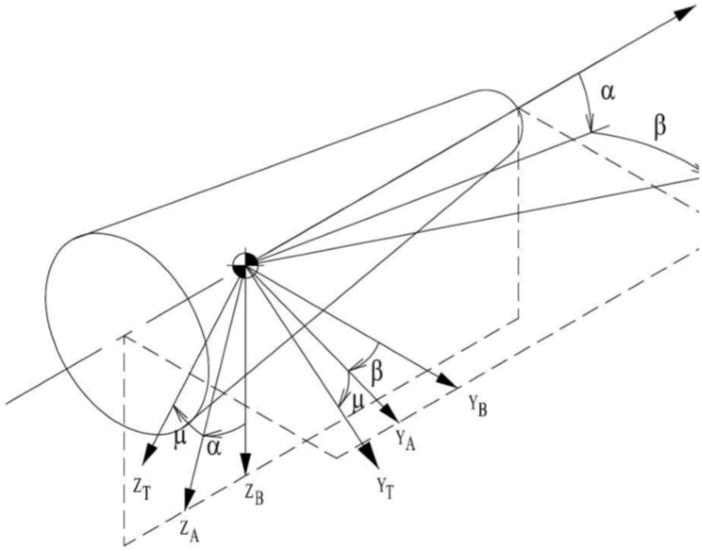


Figure 3.10: Reference frames involved in the transformation between B and T-frames. Angles are all positive.

For a), see figure 3.11. We see that for a point with coordinates (y_i, z_i) in reference system i, the new coordinates in the reference system j are given by (y_j, z_j), with $x_i = x_j$. Here, y_j is given by $y_j = y_1 + y_2$, where $y_1 = y_i \cos\phi$, and $y_2 = z_i \sin\phi$, such that

$$y_j = y_i \cos\phi + z_i \sin\phi$$

Similarly, $z_j = z_1 = \left(z_1 + z_2\right) - z_2$, where $\left(z_1 + z_2\right) = z_i \cos\phi$, and $z_2 = y_i \sin\phi$, such that

$$z_j = z_i \cos\phi - y_i \sin\phi$$

Putting all of this in a matrix leads to

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}^i$$

Evidently, the transformation matrix is

$$\mathbb{T}_x(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

Yes the drawing is a bit ugly, but you simply have to learn it by heart, unfortunately.

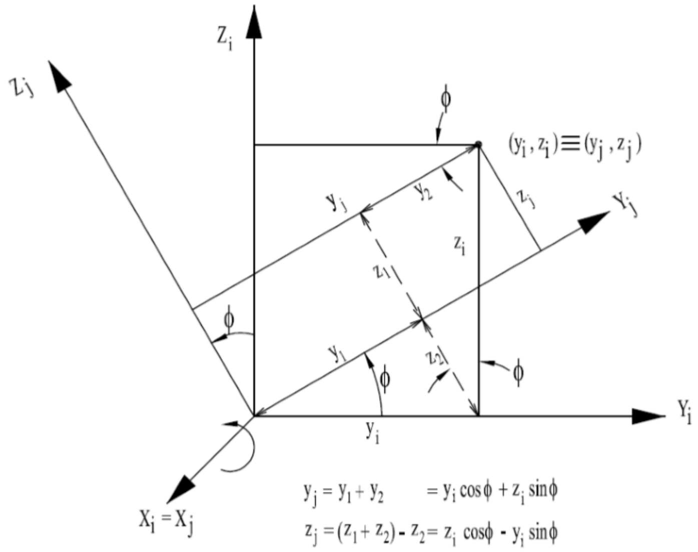


Figure 3.11: Unit-axis transformation.

For b), the remaining ones are

$$T_y(\phi) = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}$$
$$T_z(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Just remember them by heart, honestly.

For c), yes this question sucks, but if you just stay calm, it's very doable. We have to go from B to I. We'll decompose this into several rotations:

$$\mathbb{T}_{I,B} = \mathbb{T}_{I,C}\mathbb{T}_{C,E}\mathbb{T}_{E,T}\mathbb{T}_{T,B}$$

based on the figures they provided. Remember that the first rotation always occurs at the end of the sequence, as matrix multiplication happens from right to left.

Let's first focus on $\mathbb{T}, \mathbb{B}$, as this is the first rotation that will be performed. Consider figure 3.10 again: we see that to go from B to T, we must follow the order $\alpha - \beta - \mu$ (α must happen before β as seen from the X-axes, β must occur before μ as seen from the Y-axes). In more detail, we first have to rotate an angle α in the *negative* direction around the Y_B-axis, then an angle β in *positive* direction around the Z_A-axis, and then an angle μ in *positive* direction around the X_A-axis. Thus, we have

$$\begin{aligned}
\mathbb{T}_{T,B} &= \mathbb{T}_x(\mu)\ \mathbb{T}_z(\beta)\big|_A\ \mathbb{T}_y(-\alpha)\Big|_B \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & \sin\mu \\ 0 & -\sin\mu & \cos\mu \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-\alpha) & 0 & \sin(-\alpha) \\ 0 & 1 & 0 \\ -\sin(-\alpha) & 0 & \cos(-\alpha) \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\mu & \sin\mu \\ 0 & -\sin\mu & \cos\mu \end{bmatrix} \begin{bmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}
\end{aligned}$$

Then, consider figure 3.10b for $\mathbb{T}_{E,T}$: we see that to go from T to E, we must first transverse γ in *opposite* direction to the direction shown; this direction of γ in itself is in positive direction around the Y_T-axis, so we get a minus sign. We then have to transverse χ in *opposite* direction to the direction shown, where χ is in positive direction around the Z_T-axis, thus we get a minus sign here as well. Thus:

$$\begin{aligned}\mathbb{T}_{E,T} &= \mathbb{T}_z(-\chi)\,\mathbb{T}_y(-\gamma)\Big|_T \\ &= \begin{bmatrix}\cos\chi & -\sin\chi & 0\\ \sin\chi & \cos\chi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\gamma & 0 & \sin\gamma\\ 0 & 1 & 0\\ -\sin\gamma & 0 & \cos\gamma\end{bmatrix}\end{aligned}$$

Then, finally, consider figure 3.9 for the final two rotation matrices. First we have to go from E to C, i.e. $\mathbb{T}_{C,E}$. We see that we must first rotate an angle $\delta + \pi/2$ in the *opposite* direction of how δ is drawn in the figure; δ itself is drawn in negative direction around the Y_E-axis so the minus sign drops out. Afterwards we traverse τ in negative direction; τ is in positive direction around the Z-axis. Thus, our transformation matrix becomes

$$\mathbb{T}_{C,E} = \mathbb{T}_z(-\tau)\,\mathbb{T}_y\left(\frac{\pi}{2}+\delta\right)\Big|_E$$

Finally, we need to go from C to I, which is done by traversing $\Omega_t \cdot t_O$ in *opposite* direction of how it's drawn, which is in positive direction, so we need a minus sign. Consequently, we simply have

$$\mathbb{T}_{I,C} = \mathbb{T}_z\left(-\Omega_t \cdot t_O\right)\Big|_C$$

Combining $\mathbb{T}_{I,C}\mathbb{T}_{C,E}$:

$$\begin{aligned}\mathbb{T}_{I,E} = \mathbb{T}_{I,C}\mathbb{T}_{C,E} &= \mathbb{T}_z\left(-\Omega_t\cdot t_O\right)\Big|_C\,\mathbb{T}_z(-\tau)\,\mathbb{T}_y\left(\frac{\pi}{2}+\delta\right)\Big|_E = \mathbb{T}_z\left(-\Omega_t\cdot t_O-\tau\right)\mathbb{T}_y\left(\frac{\pi}{2}+\delta\right)\Big|_E \\ &= \begin{bmatrix}\cos\left(-\Omega_t\cdot t_O-\tau\right) & \sin\left(-\Omega_t\cdot t_O-\tau\right) & 0\\ -\sin\left(-\Omega_t\cdot t_O-\tau\right) & \cos\left(-\Omega_t\cdot t_O-\tau\right) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\left(\frac{\pi}{2}+\delta\right) & 0 & -\sin\left(\frac{\pi}{2}+\delta\right)\\ 0 & 1 & 0\\ \sin\left(\frac{\pi}{2}+\delta\right) & 0 & \cos\left(\frac{\pi}{2}+\delta\right)\end{bmatrix} \\ &= \begin{bmatrix}\cos\left(\Omega_t\cdot t_O+\tau\right) & -\sin\left(\Omega_t\cdot t_O+\tau\right) & 0\\ \sin\left(\Omega_t\cdot t_O+\tau\right) & \cos\left(\Omega_t\cdot t_O+\tau\right) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-\sin\delta & 0 & -\cos\delta\\ 0 & 1 & 0\\ \cos\delta & 0 & -\sin\delta\end{bmatrix}\end{aligned}$$

Thus, the complete sequence of unit axis transformations is given by

$$\begin{aligned}\mathbb{T}_{I,B} =& \mathbb{T}_{I,C}\mathbb{T}_{C,E}\mathbb{T}_{E,T}\mathbb{T}_{T,B} \\ =& \mathbb{T}_z\left(-\Omega_t\cdot t_O-\tau\right)\mathbb{T}_y\left(\frac{\pi}{2}+\delta\right)\Big|_E\,\mathbb{T}_z(-\chi)\,\mathbb{T}_y(-\gamma)\Big|_T\,\mathbb{T}_x(\mu)\,\mathbb{T}_z(\beta)\Big|_A\,\mathbb{T}_y(-\alpha)\Big|_B \\ =& \begin{bmatrix}\cos\left(\Omega_t\cdot t_O+\tau\right) & -\sin\left(\Omega_t\cdot t_O+\tau\right) & 0\\ \sin\left(\Omega_t\cdot t_O+\tau\right) & \cos\left(\Omega_t\cdot t_O+\tau\right) & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-\sin\delta & 0 & -\cos\delta\\ 0 & 1 & 0\\ \cos\delta & 0 & -\sin\delta\end{bmatrix} \\ & \begin{bmatrix}\cos\chi & -\sin\chi & 0\\ \sin\chi & \cos\chi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\gamma & 0 & \sin\gamma\\ 0 & 1 & 0\\ -\sin\gamma & 0 & \cos\gamma\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\mu & \sin\mu\\ 0 & -\sin\mu & \cos\mu\end{bmatrix}\begin{bmatrix}\cos\beta & \sin\beta & 0\\ -\sin\beta & \cos\beta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & 0 & \sin\alpha\\ 0 & 1 & 0\\ -\sin\alpha & 0 & \cos\alpha\end{bmatrix}\end{aligned}$$

For d), all transformation matrices it holds that $\mathbb{T}^{-1}(\phi) = \mathbb{T}(-\phi)$. Consequently, we have that

$$\begin{aligned}\mathbb{T}_{B,I} &= \left(\mathbb{T}_z\left(-\Omega_t\cdot t_O-\tau\right)\mathbb{T}_y\left(\frac{\pi}{2}+\delta\right)\Big|_E\,\mathbb{T}_z(-\chi)\,\mathbb{T}_y(-\gamma)\Big|_T\,\mathbb{T}_x(\mu)\,\mathbb{T}_z(\beta)\Big|_A\,\mathbb{T}_y(-\alpha)\Big|_B\right)^{-1} \\ &= \mathbb{T}_y^{-1}(-\alpha)\Big|_B\,\mathbb{T}_z^{-1}(\beta)\,\mathbb{T}_x^{-1}(\mu)\Big|_A\,\mathbb{T}_y^{-1}(-\gamma)\,\mathbb{T}_z^{-1}(-\chi)\Big|_T\,\mathbb{T}_y^{-1}\left(\frac{\pi}{2}+\delta\right)\mathbb{T}_z^{-1}\left(-\Omega_t\cdot t_O-\tau\right)\Big|_E \\ &= \mathbb{T}_y(\alpha)\Big|_B\,\mathbb{T}_z(-\beta)\,\mathbb{T}_x(-\mu)\Big|_A\,\mathbb{T}_y(\gamma)\,\mathbb{T}_z(\chi)\Big|_T\,\mathbb{T}_y\left(-\frac{\pi}{2}-\delta\right)\mathbb{T}_z\left(\Omega_t\cdot t_O+\tau\right)\Big|_E\end{aligned}$$

For e), for orthonormal matrices it holds that $A^{-1} = A^T$. Thus, $\mathbb{T}_{B,I}$ will be the transpose of $\mathbb{T}_{I,B}$, so the rows and columns are simply swapped.

3.3 Dynamic transition and relative velocity

Now, how miserable does our life become when we start including angular velocities in our stuff? Well, we need to refresh our knowledge of Dynamics on that so it's pretty miserable, except dynamics isn't all that hard anymore now that you're a third-year (at least, it shouldn't).

Consider the general case where we have two reference frames that are both translating and rotating. How is then a velocity related between these two reference frames (e.g. we may know the velocity vector in the F_b-system, but we may be interested in the velocity vector expressed in the F_E-system, so how do we relate the two)?

The general expression for this is as follows:

RELATION BETWEEN VELOCITY VECTORS IN DIFFERENT REFERENCE FRAMES

Consider two reference frames A and B. Let $\mathbf{R}|_A$ denote the position vector of the origin of reference B with respect to the origin of reference frame A, and let $d\mathbf{R}/dt|_A$ denote the velocity vector of the origin of B in reference frame A. Let $\mathbf{r}$ denote the position vector of a point P in reference frame B, and $d\mathbf{r}/dt|_B$ the velocity vector of P within reference B. The velocity vector of this point in reference frame A is then given by

$$\left.\frac{d\mathbf{R}_P}{dt}\right|_A = \left.\frac{d\mathbf{R}}{dt}\right|_A + \left.\frac{d\mathbf{r}}{dt}\right|_B + \mathbf{\Omega}_{BA} \times \mathbf{r} \tag{3.23}$$

where $\mathbf{\Omega}_{BA}$ is the angular velocity vector of reference B *relative* to the angular velocity vector of reference A.

All vectors may be expressed in any reference system, even a reference system other than A or B. However, all terms should be expressed in the *same* reference system: be consistent!

Okay I'll agree, this is very formal, and probably you're not really sure what exactly I mean. So, let's do an example.

Example 1

Consider an aircraft with pitch angle $\theta = 10°$ and pitch up rate q (5 deg /s) as depicted in figure 3.12. The aircraft is flying with a speed of 200 m/s in the direction of $\mathbf{I}$ at an altitude of 11 km at a distance of 5 km from point O, i.e. $\mathbf{R}^E = [5 \cdot 10^3, 0, -11 \cdot 10^3]^T$. The location of the pilot (point P) given with respect to the body fixed reference frame is $\mathbf{r}^b = [10, 0, -5]^T$. The vehicle carried normal Earth reference frame is given by the quadruple $(O, \mathbf{I}, \mathbf{J}, \mathbf{K})$ and is fixed. The moving, rotating reference frame F_b is fixed to the aircraft in the center of gravity $(G, \mathbf{i}, \mathbf{j}, \mathbf{k})$. We will derive the velocity of the pilot with respect to F_E.

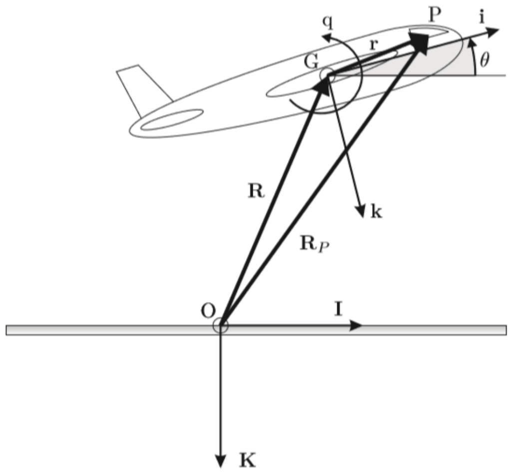


Figure 3.12: Example - derivation of pilot velocity with respect to F_E.

Okay, let's just go through the terms in equation (3.23) one-by-one:

$$\left.\frac{d\mathbf{R}_P}{dt}\right|_E = \left.\frac{d\mathbf{R}}{dt}\right|_E + \left.\frac{d\mathbf{r}}{dt}\right|_b + \boldsymbol{\Omega}_{bE} \times \mathbf{r}$$

Here, $d\mathbf{R}/dt$ is the velocity vector of the origin of the b-frame relative to the E-frame. This is simply equal to $200\mathbf{I}$, as the aircraft is moving to the right at $200\,\mathrm{m/s}$ whereas the Earth is given to be fixed. Then, $d\mathbf{r}/dt$ is the velocity vector of point P with respect to the b-frame. This will be the null-vector! After all, the pilot will not move relative to the aircraft c.g.: yes it'll pitch up together with the aircraft, however, within the aircraft itself, it won't move. The rotation is fully taken up by the next term.

Then, $\boldsymbol{\Omega}_{bE}$ is the angular velocity vector of frame b relative to E. This is simply equal to $q\mathbf{J} = 5\pi/180\mathbf{J}$: the $\mathbf{J}$-vector points out of the paper, and if you use your right-hand rule, you see that q is in then in the positive direction around the $\mathbf{J}$-vector. Finally, $\mathbf{r}$ is the position vector of the pilot within the b-reference frame. This is given to be $\mathbf{r}^b = [10, 0, -5]^T$, but be careful! As I said, you need to consistently use the same reference system to express your vectors. For the previous vectors I used the E-frame every time: $d\mathbf{R}/dt$ and $\boldsymbol{\Omega}_{bE}$ are all expressed in that reference system. Thus, we need to express $\mathbf{r}$ in the E-frame, but currently it's expressed in the b-frame. Indeed, we need to apply our transformation matrix:

$$\mathbf{r}^E = \mathbb{T}_{Eb}\mathbf{r}^b = \mathbb{T}_{bE}^T\mathbf{r}^b = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} 10 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} \cos 10^\circ & 0 & \sin 10^\circ \\ 0 & 1 & 0 \\ -\sin 10^\circ & 0 & \cos 10^\circ \end{bmatrix}\begin{bmatrix} 10 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} 8.98 \\ 0 \\ -6.66 \end{bmatrix}$$

Thus, we obtain

$$\begin{aligned} \left.\frac{d\mathbf{R}_P}{dt}\right|_E^E &= \left.\frac{d\mathbf{R}}{dt}\right|_E^E + \left.\frac{d\mathbf{r}}{dt}\right|_b^E + \boldsymbol{\Omega}_{bE}^E \times \mathbf{r}^E \\ &= \begin{bmatrix} 200 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{5\pi}{180} \\ 0 \end{bmatrix} \times \begin{bmatrix} 8.98 \\ 0 \\ -6.66 \end{bmatrix} = \begin{bmatrix} 200 - 0.5812 \\ 0 \\ -0.78364 \end{bmatrix} \end{aligned}$$

I hope you agree it's not that hard in the end. Just a bit annoying notation. The superscripts E denote that each of those vectors is expressed in the E-frame.

3.3.1 Angular velocity vector

Now, how can you derive the angular velocity vector? In above example it was pretty easy, is it always this easy? Well, it's not *as* easy, but people who describe it as hard are the same people who'll probably end up failing this course.

Consider figure 3.13a, where in an infinitesimal time dt the reference system A is rotated an angle $d\alpha$ around the x_A-axis. The corresponding transformation matrix is thus

$$\mathbb{T}_{BA} = \mathbb{T}_x (d\alpha)\big|_A$$

and the corresponding angular velocity is

$$\boldsymbol{\Omega}_{BA} = \dot{\alpha}\mathbf{x}_A = \dot{\alpha}\mathbf{x}_B$$

since $\mathbf{x}_A = \mathbf{x}_B$. Now, what happens if we rotate around the Y_B-axis afterwards, for an angle $d\beta$, as shown in figure 3.13b? Well, the transformation matrix then becomes

$$\mathbb{T}_{CA} = \mathbb{T}_y (d\beta)\Big|_B \mathbb{T}_x (d\alpha)\big|_A$$

and the corresponding angular velocity vector simply becomes

$$\boldsymbol{\Omega}_{CA} = \dot{\beta}\mathbf{y}_B + \dot{\alpha}\mathbf{x}_A = \dot{\beta}\mathbf{y}_C + \dot{\alpha}\mathbf{x}_B$$

since $\mathbf{y}_C = \mathbf{y}_B$. See that we can simply sum them up, it's that easy. If you'd include an angular velocity over the Z_C-axis, you'd get

$$\mathbb{T}_{DA} = \mathbb{T}_z (d\gamma)\big|_C \mathbb{T}_y (d\beta)\Big|_B \mathbb{T}_x (d\alpha)\big|_A$$

and

$$\boldsymbol{\Omega}_{DA} = \dot{\gamma}\mathbf{z}_C + \dot{\beta}\mathbf{y}_B + \dot{\alpha}\mathbf{x}_A = \dot{\gamma}\mathbf{z}_D + \dot{\beta}\mathbf{y}_C + \dot{\alpha}\mathbf{x}_B$$

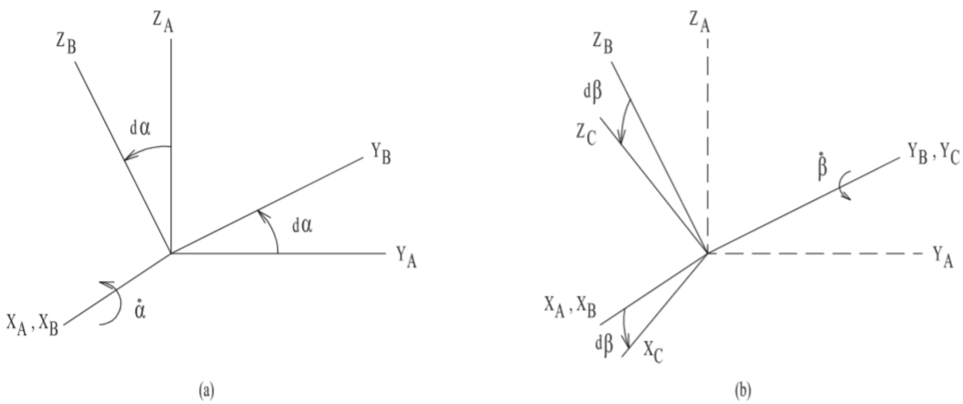


Figure 3.13: The transformation from frame A to frame B (a) and the one from frame B to frame C (b).

Now, there is one issue that remains still. As you remember from Dynamics, it's key to be consistent, and indeed, we need to express the vectors $\boldsymbol{\Omega}_{DA}$, $\mathbf{z}_D$, $\mathbf{y}_C$ and $\mathbf{x}_B$ all in the same coordinate system[1]. The easiest to choose is

[1]In case you don't get why: yes, these are all unit vectors, so you'd expect them to have two zero entries and one 1 in its vector. However, this is only true in the reference frame of which the unit vector originates. If you're a unit vector in $\mathbf{x}$-direction in frame A, you're likely not to be a unit vector perfectly aligned with the $\mathbf{x}$-direction in frame B if frame B is randomly rotated with respect to A. Indeed, you'd need to express the unit vector in the coordinate system of frame B by use of a transformation matrix.

simply reference frame D; $\mathbf{z}_D$ is already in this coordinate system, and the others simply follow as well:

$$\mathbf{y}^D = \mathbb{T}_z(d\gamma)\big|_C \mathbf{y}^C = \begin{bmatrix} \cos(d\gamma) & \sin(d\gamma) & 0 \\ -\sin(d\gamma) & \cos(d\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \sin(d\gamma) \\ \cos(d\gamma) \\ 0 \end{bmatrix}$$

$$\mathbf{x}^D = \mathbb{T}_z(d\gamma)\big|_C \mathbb{T}_y(d\gamma)\Big|_B \mathbf{x}^B = \begin{bmatrix} \cos(d\gamma) & \sin(d\gamma) & 0 \\ -\sin(d\gamma) & \cos(d\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(d\beta) & 0 & -\sin(d\beta) \\ 0 & 1 & 0 \\ \sin(d\beta) & 0 & \cos(d\beta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(d\gamma) & \sin(d\gamma) & 0 \\ -\sin(d\gamma) & \cos(d\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(d\beta) \\ 0 \\ \sin(d\beta) \end{bmatrix} = \begin{bmatrix} \cos(d\gamma)\cos(d\beta) \\ -\sin(d\gamma)\cos(d\beta) \\ \sin(d\beta) \end{bmatrix}$$

Thus, our angular velocity vector would become

$$\boldsymbol{\Omega}_{DA}\big|_D = \dot{\gamma}\mathbf{z}_D^D + \dot{\beta}\mathbf{y}_C^D + \dot{\alpha}\mathbf{x}_B^D = \dot{\gamma}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \dot{\beta}\begin{bmatrix} \sin(d\gamma) \\ \cos(d\gamma) \\ 0 \end{bmatrix} + \dot{\alpha}\begin{bmatrix} \cos(d\gamma)\cos(d\beta) \\ -\sin(d\gamma)\cos(d\beta) \\ \sin(d\beta) \end{bmatrix}$$

$$= \begin{bmatrix} \dot{\beta}\sin(d\gamma) + \dot{\alpha}\cos(d\gamma)\cos(d\beta) \\ \dot{\beta}\cos(d\gamma) - \dot{\alpha}\sin(d\gamma)\cos(d\beta) \\ \dot{\gamma} + \dot{\alpha}\sin(d\beta) \end{bmatrix}$$

3.4 Coordinate transformations

Now, I don't want to spoil the rest of this course for you, but what I already have to tell you is that the equations of motion that we'll derive are often solved for the velocity components V_N, V_E and V_D (which are the velocity components aligned with the F_E-frame, i.e. V_N points to the North, V_E to the East and V_D points downwards) and the **Euler angles** ϕ, θ and ψ.

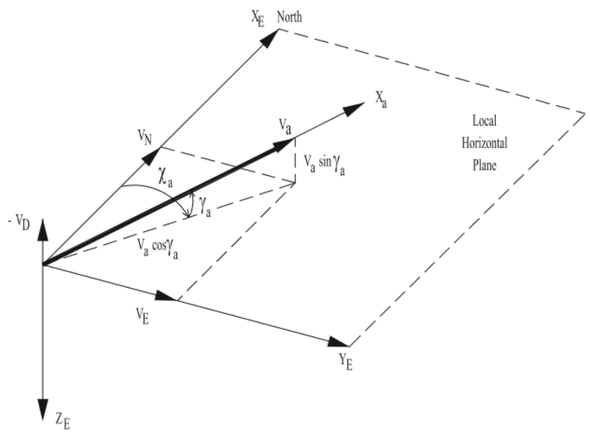


Figure 3.14: Relation between the spherical velocity components and Caretesian components in the V-frame.

Now, this is not always super useful information: maybe we want the velocity components in a different reference frame, or we want different angles, such as angle of attack, or flight path angle, etc. So, how do we transform between them? Well, for the velocity it is easy: you just apply the transformation matrix to whatever frame you want. For the angles, it's a bit more complicated. The aerodynamic heading angle ξ_a and flight path angle γ_a can be straightforwardly determined from figure 3.14: we simply have

RELATION BETWEEN ξ_a, γ_a AND ϕ, θ, ψ

The **aerodynamic heading angle** χ_a is given by

$$\chi_a = \arctan\left(\frac{V_E}{V_N}\right) \tag{3.24}$$

The **aerodynamic flight path angle** γ_a is given by

$$\gamma_a = -\arcsin\left(\frac{V_D}{V}\right) \tag{3.25}$$

where the minus sign comes from the fact that γ_a is positive 'upwards', but V_D is positive downwards.

For the angle of attack, angle of sideslip, and aerodynamic bank angle life is more complicated. The idea is as follows: $\mathbb{T}_{ab}$ is given by equation (3.19) and equals

$$\mathbb{T}_{ab} = \begin{bmatrix} \cos\beta_a & \sin\beta_a & 0 \\ -\sin\beta_a & \cos\beta_a & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\alpha_a & 0 & \sin\alpha_a \\ 0 & 1 & 0 \\ -\sin\alpha_a & 0 & \cos\alpha_a \end{bmatrix} = \begin{bmatrix} \cos\beta_a\cos\alpha_a & \sin\beta_a & \cos\beta_a\sin\alpha_a \\ -\sin\beta_a\cos\alpha_a & \cos\beta_a & -\sin\beta_a\sin\alpha_a \\ -\sin\alpha_a & 0 & \cos\alpha_a \end{bmatrix} \tag{3.26}$$

Now, we also know $\mathbb{T}_{Eb}$ (given by the transpose of equation (**??**)). Furthermore, we know the relation between the F_a- and F_E-frame, namely from equation (3.15) that

$$\mathbb{T}_{aE} = \mathbb{T}_x\left(-\mu_a\right)\Big|_{E''}\mathbb{T}_y\left(\gamma_a\right)\Big|_{E'}\mathbb{T}_z\left(\chi_a\right)\Big|_{E}$$

Now, obviously, we should have $\mathbb{T}_{aE}\mathbb{T}_{Eb} = \mathbb{T}_{ab}$, or in other words:

$$\mathbb{T}_{aE}\mathbb{T}_{Eb} = \mathbb{T}_{ab}$$
$$\mathbb{T}_x\left(-\mu_a\right)\mathbb{T}_y\left(\gamma_a\right)\mathbb{T}_z\left(\chi_a\right)\mathbb{T}_{Eb} = \mathbb{T}_{ab}$$

The $\mathbb{T}_x\left(-\mu_a\right)$ may be moved to the right-hand side by pre-multiplying with $\mathbb{T}_x^{-1}\left(-\mu_a\right) = \mathbb{T}_x\left(\mu_a\right)$, resulting in

$$\mathbb{T}_y\left(\gamma_a\right)\mathbb{T}_z\left(\chi_a\right)\mathbb{T}_{Eb} = \mathbb{T}_x\left(\mu_a\right)\mathbb{T}_{ab} \tag{3.27}$$

Call the left-hand side $\mathbb{T}_{Xb}$ (cause I'm not gonna do the matrix multiplication for that one), and the right-one can be expanded to

$$\mathbb{T}_x\left(\mu_a\right)\mathbb{T}_{ab} = \begin{bmatrix} \cos\alpha_a\cos\beta_a & \sin\beta_a & \sin\alpha_a\cos\beta_a \\ -\cos\alpha_a\sin\beta_a\cos\mu_a - \sin\alpha_a\sin\mu_a & \cos\beta_a\cos\mu_a & \cos\alpha_a\sin\mu_a - \sin\alpha_a\sin\beta_a\cos\mu_a \\ \cos\alpha_a\sin\beta_a\sin\mu_a - \sin\alpha_a\cos\mu_a & -\cos\beta_s\sin\mu_a & \cos\alpha_a\cos\mu_+\sin\alpha_a\sin\beta_a\sin\mu_a \end{bmatrix}$$

What's the point of all of this? Well, on the right-hand side of equation (3.27), we now have a matrix that consists solely of α_a, β_a and μ_a. On the left-hand side, we have a matrix $\mathbb{T}_{Xb}$ that solely consists of γ_a, χ_a, ϕ, θ and ψ. These matrices must be equal to each other (there's an equal sign between them, after all). Thus, the entries must be the same for each entry. This means that for example for β_a, we simply obtain from the entry in the first row, second column, that $\beta_a = \arcsin\left(\mathbb{T}_{Xb}(1,2)\right)$ where $\mathbb{T}_{Xb}(i,j)$ indicates the entry on the ith row and jth column. For α_a, we can do something similar: note that we must have

$$\frac{\sin\alpha_a\cos\beta_a}{\cos\alpha_a\cos\beta_a} = \frac{\left(\mathbb{T}_{Xb}(1,1)\right.}{\left(\mathbb{T}_{Xb}(1,3)\right.}$$
$$\alpha_a = \arctan\left(\frac{\left(\mathbb{T}_{Xb}(1,3)\right.}{\left(\mathbb{T}_{Xb}(1,1)\right.}\right)$$

and similarly for μ:

$$\frac{-\cos\beta_a\sin\mu_a}{\cos\beta_a\cos\mu_a} = \frac{\left(\mathbb{T}_{Xb}(3,2)\right.}{\left(\mathbb{T}_{Xb}(2,2)\right.}$$
$$\mu_a = -\arctan\left(\frac{\left(\mathbb{T}_{Xb}(3,2)\right.}{\left(\mathbb{T}_{Xb}(2,2)\right.}\right)$$

3.5 *Exam questions*

Exam April 2013: problem 1 (25p)

The attitude of an aircraft can be expressed by the Euler angles ϕ (roll), θ (pitch) and ψ (yaw), expressing the attitude of the body w.r.t. the inertial frame. For a flat, non-rotating Earth the vehicle-carried normal frame (the so-called E-frame) can be considered to be the inertial frame (the I-frame). The corresponding aircraft rotation is $\mathbf{\Omega}^b_{b/I} = (p, q, r)^T$. These angular-rate components describe the roll, pitch and yaw rate of the aircraft with respect to the inertial frame, expressed in body-frame component (index b).

a) Setup the transformation matrix from the b-frame to the I-frame, $\mathbb{T}_{bI}$, in terms of the correct sequence of unit-axis transformations.
b) Subsequently, from $\mathbb{T}_{bI}$ derive the angular rate of the b-frame w.r.t. the I-frame, by assuming that the angular displacements ϕ, θ and ψ take place in a time interval dt. Clearly indicate in which frames the individual components are expressed and which axes are involved.
c) Show that the attitude motion of the aircraft is given by

$$\begin{aligned}\dot{\phi} &= p + \sin\phi\tan\theta q + \cos\phi\tan\theta r\\ \dot{\theta} &= \cos\phi q - \sin\phi r\\ \dot{\psi} &= \frac{\sin\phi}{\cos\theta}q + \frac{\cos\phi}{\cos\theta}r\end{aligned}$$

Hints:
(1) Realise that the rotation vector from sub-question b) is equivalent to $\mathbf{\Omega}^b_{bI}$.
(2) Transform the individual components of rotation to one and the same frame (the b-frame).
(3) Solve the resulting system of equations.

For a), the Euler angles are first yaw ψ around the Z_E axis, then pitch θ around the $Y_{E'}$ axis, then roll ϕ around the $X_{E''}$ axis. Thus, we have

$$\mathbb{T}_{bI} = \mathbb{T}_{bE} = \mathbb{T}_x(\phi)\big|_{E''}\,\mathbb{T}_y(\theta)\big|_{E'}\,\mathbb{T}_z(\psi)\big|_E$$

For b), the corresponding rotation vector is simply

$$\mathbf{\Omega}_{b/I} = \dot{\phi}\mathbf{x}_b + \dot{\theta}\mathbf{y}_{E''} + \dot{\psi}\mathbf{z}_{E'}$$

For c), we need to transform all the unit vectors to a consistent frame; the b-frame is most convenient as p, q and r are expressed in that frame too. $\mathbf{x}_b$ is trivial to express in the b-frame as it's simply $[1,0,0]^T$, but $\mathbf{y}_{E''}$ and $\mathbf{z}_{E'}$ both need to be transformed:

$$\mathbf{y}^b_{E''} = \mathbb{T}_x(\phi)\,\mathbf{y}^{E''}_{E''} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\ \cos\phi\\ -\sin\phi\end{bmatrix}$$

$$\mathbf{z}^b_{E'} = \mathbb{T}_x(\phi)\,\mathbb{T}_y(\theta)\,\mathbf{z}^{E'}_{E'} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & \sin\phi\\ 0 & -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}-\sin\theta\\ 0\\ \cos\theta\end{bmatrix}$$
$$= \begin{bmatrix}-\sin\theta\\ \cos\theta\sin\phi\\ \cos\theta\cos\phi\end{bmatrix}$$

Thus,

$$\mathbf{\Omega}^b_{b/I} = \dot{\phi}\mathbf{x}^b_b + \dot{\theta}\mathbf{y}^b_{E''} + \dot{\psi}\mathbf{z}^b_{E'} = \dot{\phi}\begin{bmatrix}1\\0\\0\end{bmatrix} + \dot{\theta}\begin{bmatrix}0\\ \cos\phi\\ -\sin\phi\end{bmatrix} + \dot{\psi}\begin{bmatrix}-\sin\theta\\ \cos\theta\sin\phi\\ \cos\theta\cos\phi\end{bmatrix} = \begin{bmatrix}\dot{\phi} - \sin\theta\dot{\psi}\\ \cos\phi\dot{\theta} + \cos\theta\sin\phi\dot{\psi}\\ -\sin\phi\dot{\theta} + \cos\theta\cos\phi\dot{\psi}\end{bmatrix}$$

Thus, we have

$$\Omega_{bI}^{b} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\phi} - \sin\theta\dot{\psi} \\ \cos\phi\dot{\theta} + \cos\theta\sin\phi\dot{\psi} \\ -\sin\phi\dot{\theta} + \cos\theta\cos\phi\dot{\psi} \end{bmatrix}$$

Solving this equation for $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ can be done as follows (note: solving this matrix equation was worth only 2 points (of the 15 in total), so don't waste too much time thinking how did I manually solve matrix equations):

$$\begin{bmatrix} 1 & 0 & -\sin\theta & | & p \\ 0 & \cos\phi & \sin\phi\cos\theta & | & q \\ 0 & -\sin\phi & \cos\phi\cos\theta & | & r \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\sin\theta & | & p \\ 0 & \cos\phi & \sin\phi\cos\theta & | & q \\ 0 & 0 & \cos\phi\cos\theta + \frac{\sin\phi}{\cos\phi}\sin\phi\cos\theta & | & r + \frac{\sin\phi}{\cos\phi}q \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\sin\theta & | & p \\ 0 & \cos\phi & \sin\phi\cos\theta & | & q \\ 0 & 0 & \cos\theta\left(\cos\phi + \frac{\sin\phi}{\cos\phi}\sin\phi\right) & | & r + \tan\phi q \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\sin\theta & | & p \\ 0 & \cos\phi & \sin\phi\cos\theta & | & q \\ 0 & 0 & \cos\theta\left(\cos^2\phi + \sin^2\phi\right) & | & r\cos\phi + \sin\phi q \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\sin\theta & | & p \\ 0 & \cos\phi & \sin\phi\cos\theta & | & q \\ 0 & 0 & \cos\theta & | & r\cos\phi + \sin\phi q \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\sin\theta & | & p \\ 0 & \cos\phi & \sin\phi\cos\theta & | & q \\ 0 & 0 & 1 & | & r\frac{\cos\phi}{\cos\theta} + q\frac{\sin\phi}{\cos\theta} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & p + \sin\theta\left(r\frac{\cos\phi}{\cos\theta} + q\frac{\sin\phi}{\cos\theta}\right) \\ 0 & \cos\phi & 0 & | & q + \sin\phi\frac{\cos\theta}{\sin\theta}p \\ 0 & 0 & 1 & | & r\frac{\cos\phi}{\cos\theta} + q\frac{\sin\phi}{\cos\theta} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & p + \tan\theta\cos\phi r + q\tan\theta\sin\phi \\ 0 & 1 & 0 & | & \frac{q}{\cos\phi} + \frac{\tan\phi}{\tan\theta}p \\ 0 & 0 & 1 & | & r\frac{\cos\phi}{\cos\theta} + q\frac{\sin\phi}{\cos\theta} \end{bmatrix}$$

The first and third rows are correct, but idk how to get the 2nd one correct. But on the exam they already awarded the full points if a serious effort was shown.

Exam August 2012: problem 2 (23 points)

In general Newton's Laws hold for a non-rotating inertial frame. If one wants to express the equations of motion in a rotating frame, one has to compensate for this rotation. This process involves both static and dynamic transformations. In the following, these transformations are addressed.

a) State the three (static) unit-axis transformation matrices for a rotation α around the $X-$, $Y-$ and $Z-$axis.

b) In a dynamic rotation we assume that the angular displacements $d\alpha$ (the rotation) takes place in a certain time dt. Given the three frames A, B and C with corresponding rotations in figure 3.15, you are asked to:

   i) Set up the static transformation from frame A to C, $\mathbb{T}_{CA}$ in terms of the product of individual $\mathbb{T}$.

   ii) From the sequence of transformations, derive the angular rate of frame C with respect to frame A, $\mathbf{\Omega}_{CA}$. Make sure to express all components of $\mathbf{\Omega}_{CA}$ in components of frame C.

c) The position vector $\mathbf{R}$ is defined in the E-frame by $\mathbf{R} = -R\mathbf{z}_E$ (figure 3.16). The time derivative of $\mathbf{R}$ can be derived from this definition, and would include a component *relative* to the E-frame and one due to the *rotation* of the E-frame. You are asked to derive an expression for $d\mathbf{R}/dt$ expressed in components of the E-frame. Use can be made of the following:

$$\frac{d\mathbf{x}_E}{dt} = \boldsymbol{\Omega}_{EC}^{E} \times \mathbf{x}_E \qquad \frac{d\mathbf{y}_E}{dt} = \boldsymbol{\Omega}_{EC}^{E} \times \mathbf{y}_E \qquad \frac{d\mathbf{z}_E}{dt} = \boldsymbol{\Omega}_{EC}^{E} \times \mathbf{z}_E$$

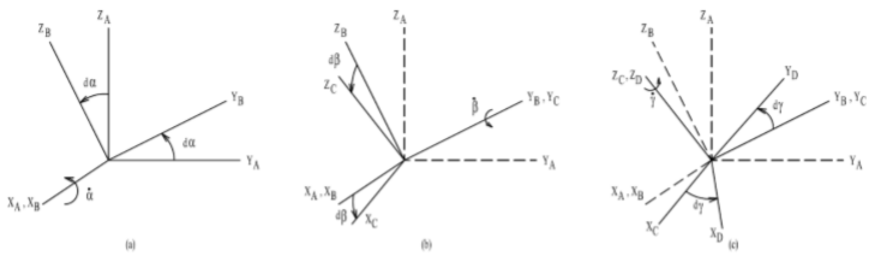


Figure 3.15: Transformation from A to C frame.

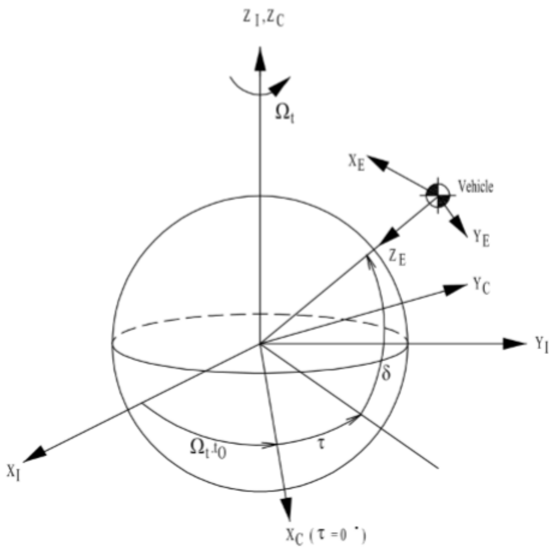


Figure 3.16: Relation between C and E frame.

For a), we simply have

$$\mathbb{T}_x(\theta_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

$$\mathbb{T}_y(\theta_2) = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

$$\mathbb{T}_z(\theta_3) = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For b), we have for i): we first rotate $d\alpha$ around X_B, then $d\beta$ around Y_C:

$$\mathbb{T}_{CA} = \mathbb{T}_y(d\beta)\Big|_B \mathbb{T}_x(d\alpha)\Big|_A = \begin{bmatrix} \cos d\beta & 0 & -\sin d\beta \\ 0 & 1 & 0 \\ \sin d\beta & 0 & \cos d\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos d\alpha & \sin d\alpha \\ 0 & -\sin d\alpha & \cos d\alpha \end{bmatrix}$$

For ii), the angular rate thus becomes

$$\mathbf{\Omega}_{CA} = \dot{\beta}\mathbf{y}_C + \dot{\alpha}\mathbf{x}_B$$

All components should be expressed in the same reference frame C: $\mathbf{y}_C^C$ is simply equal to $[0, 1, 0]^T$; $\mathbf{x}_B$ can be easily found as well:

$$\mathbf{x}_B^C = \mathbb{T}_y(d\beta)\Big|_B \mathbf{x}_B^B = \begin{bmatrix} \cos d\beta & 0 & -\sin d\beta \\ 0 & 1 & 0 \\ \sin d\beta & 0 & \cos d\beta \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos d\beta \\ 0 \\ \sin d\beta \end{bmatrix}$$

Thus,

$$\mathbf{\Omega}_{CA}^C = \dot{\beta}\mathbf{y}_C^C + \dot{\alpha}\mathbf{x}_B^C = \dot{\beta}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \dot{\alpha}\begin{bmatrix} \cos d\beta \\ 0 \\ \sin d\beta \end{bmatrix} = \begin{bmatrix} \dot{\alpha}\cos d\beta \\ \dot{\beta} \\ \dot{\alpha}\sin d\beta \end{bmatrix}$$

For c), we have

$$\frac{d\mathbf{R}}{dt} = -\frac{dR}{dt}\mathbf{z}_e - R\frac{d\mathbf{z}_e}{dt} = -\dot{R}\mathbf{z}_e - R\mathbf{\Omega}_{EC}^E \times \mathbf{z}_E$$

We thus need to find $\mathbf{\Omega}_{EC}^E$. Consider figure 3.16: we must first find $\mathbb{T}_{EC}$. We first rotate an angle τ about the Z_C-axis; τ is defined in positive direction. We first rotate an angle $-\delta - \pi/2$ over the Y_E-axis: δ itself is drawn in *negative* direction, so we get a minus sign. Thus,

$$\mathbb{T}_{EC} = \mathbb{T}_y\left(-\delta - \frac{\pi}{2}\right)\Big|_E \mathbb{T}_z(\tau)\big|_C$$

Thus,

$$\mathbf{\Omega}_{EC} = -\dot{\delta}\mathbf{y}_E + \dot{\tau}\mathbf{z}_E$$

We then transform $\mathbf{z}_{C'}$ to be in the correct reference frame:

$$\mathbf{z}_{C'}^E = \mathbb{T}_{EC'}\mathbf{z}_C^{C'} = \mathbb{T}_y\left(-\delta - \frac{\pi}{2}\right)\Big|_C \mathbf{z}_C^{C'} = \begin{bmatrix} \cos\left(-\delta - \frac{\pi}{2}\right) & 0 & -\sin\left(-\delta - \frac{\pi}{2}\right) \\ 0 & 1 & 0 \\ \sin\left(-\delta - \frac{\pi}{2}\right) & 0 & \cos\left(-\delta - \frac{\pi}{2}\right) \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} -\sin(\delta) & 0 & \cos(\delta) \\ 0 & 1 & 0 \\ -\cos(\delta) & 0 & -\sin(\delta) \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\delta) \\ 0 \\ -\sin(\delta) \end{bmatrix}$$

Thus,

$$\mathbf{\Omega}_{EC}^E = -\dot{\delta}\mathbf{y}_E^E + \dot{\tau}\mathbf{z}_E^E = -\dot{\delta}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \dot{\tau}\begin{bmatrix} \cos(\delta) \\ 0 \\ -\sin(\delta) \end{bmatrix} = \begin{bmatrix} \dot{\tau}\cos(\delta) \\ -\dot{\delta} \\ -\dot{\tau}\sin(\delta) \end{bmatrix}$$

leading to

$$\frac{d\mathbf{R}}{dt} = -\frac{dR}{dt}\mathbf{z}_e - R\frac{d\mathbf{z}_e}{dt} = -\dot{R}\mathbf{z}_e - R\mathbf{\Omega}_{EC}^E \times \mathbf{z}_E$$
$$\frac{d\mathbf{R}^E}{dt} = -\dot{R}\mathbf{z}_e^E - R\mathbf{\Omega}_{EC}^E \times \mathbf{z}_E^E = -\dot{R}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - R\begin{bmatrix} \dot{\tau}\cos(\delta) \\ -\dot{\delta} \\ -\dot{\tau}\sin(\delta) \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\dot{R} \end{bmatrix} - R\begin{pmatrix} \mathbf{x}_e & \mathbf{y}_e & \mathbf{z}_e \\ \dot{\tau}\cos(\delta) & -\dot{\delta} & -\dot{\tau}\sin(\delta) \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{bmatrix} -\dot{\delta} \\ -\dot{\tau}\cos(\delta) \\ 0 \end{bmatrix} = \begin{bmatrix} R\dot{\delta} \\ R\dot{\tau}\cos(\delta) \\ -\dot{R} \end{bmatrix}$$

Exam April 2011: question 6a

For the analysis of re-entry problems it is common to express the position and velocity in spherical components, whereas for aircraft studies usually Cartesian velocity components are used. In the accompanying figure the two definitions of the state variables are given.

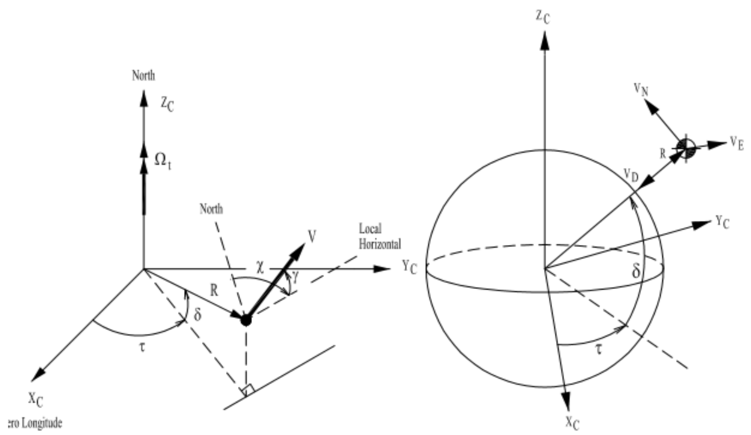


Figure 3.17: Definition of spherical position and velocity state variables (left) and definition of spherical position and Cartesian velocity (right).

a) Derive expressions for the spherical velocity components (velocity modulus V, flight-path angle γ and heading χ) as a function of the North, East and down velocity (V_N, V_E and V_D).

We simply have

$$V = \sqrt{V_N^2 + V_E^2 + V_D^2}$$

from Pythagoras. χ simply follows from

$$\chi = \arctan\left(\frac{V_E}{V_N}\right)$$

and finally, for γ we have

$$\gamma = -\arcsin\left(\frac{V_D}{V}\right)$$

4 Introduction to equations of motion

In this chapter, we'll make a general derivation of the equations of motion.

4.1 Relative acceleration

Remember the relation between velocity vectors in different reference frames, as described by equation (3.23):

RELATION BETWEEN VELOCITY VECTORS IN DIFFERENT REFERENCE FRAMES

Consider two reference frames A and B. Let $\mathbf{R}|_A$ denote the position vector of the origin of reference B with respect to the origin of reference frame A, and let $d\mathbf{R}/dt|_A$ denote the velocity vector of the origin of B in reference frame A. Let $\mathbf{r}$ denote the position vector of a point P in reference frame B, and $d\mathbf{r}/dt|_B$ the velocity vector of P within reference B. The velocity vector of this point in reference frame A is then given by

$$\left.\frac{d\mathbf{R}_P}{dt}\right|_A = \left.\frac{d\mathbf{R}}{dt}\right|_A + \left.\frac{d\mathbf{r}}{dt}\right|_B + \boldsymbol{\Omega}_{BA} \times \mathbf{r} \tag{3.23}$$

where $\boldsymbol{\Omega}_{BA}$ is the angular velocity vector of reference B *relative* to the angular velocity vector of reference A.

All vectors may be expressed in any reference system, even a reference system other than A or B. However, all terms should be expressed in the *same* reference system: be consistent!

Now, can we also find the derivative of this, so that we can find something about the acceleration? Yes we can unfortunately. The derivation is left as an exercise for the reader cause I can't be arsed to include it. However, believe me when I say that if you write equation (3.23) as

$$\left.\frac{d\mathbf{R}_P}{dt}\right|_A = \left.\frac{d\mathbf{R}}{dt}\right|_A + \left(\left.\frac{d\mathbf{r}}{dt}\right|_B + \boldsymbol{\Omega}_{BA} \times \mathbf{r}\right)$$

then the acceleration of point P in the A-frame is given by

$$\begin{aligned}
\left.\frac{d^2\mathbf{R}_P}{dt^2}\right|_A &= \left.\frac{d^2\mathbf{R}}{dt}\right|_A + \frac{d}{dt}\left(\left.\frac{d\mathbf{r}}{dt}\right|_B + \boldsymbol{\Omega}_{BA} \times \mathbf{r}\right) + \boldsymbol{\Omega}_{BA} \times \left(\left.\frac{d\mathbf{r}}{dt}\right|_B + \boldsymbol{\Omega}_{BA} \times \mathbf{r}\right) \\
&= \left.\frac{d^2\mathbf{R}}{dt}\right|_A + \left.\frac{d^2\mathbf{r}}{dt^2}\right|_B + \frac{d\boldsymbol{\Omega}_{BA}}{dt} \times \mathbf{r} + \boldsymbol{\Omega}_{BA} \times \left.\frac{d\mathbf{r}}{dt}\right|_B + \boldsymbol{\Omega}_{BA} \times \left.\frac{d\mathbf{r}}{dt}\right|_B + \boldsymbol{\Omega}_{BA} \times (\boldsymbol{\Omega}_{BA} \times \mathbf{r})
\end{aligned}$$

which reduces to

EQUATION FOR ABSOLUTE ACCELERATION WITH RESPECT TO THE INERTIAL FRAME

Consider two reference frames A and B, where A is an inertial frame of reference. Let $\mathbf{R}|_A$ denote the position vector of the origin of reference B with respect to the origin of reference frame A, let $d\mathbf{R}/dt|_A$ denote the velocity vector of the origin of B in reference frame A, and let $d^2\mathbf{R}/dt^2\big|_A$ denote the acceleration vector of the origin B in reference frame A. Let $\mathbf{r}$ denote the position vector of a point P in reference frame B, let $d\mathbf{r}/dt|_B$ the velocity vector of P within reference B and let $d^2\mathbf{r}/dt\big|_B$ denote the acceleration vector of a point P within reference frame B. The velocity vector of this point in reference frame A is then given by

$$\left.\frac{d^2\mathbf{R}_P}{dt^2}\right|_A = \left.\frac{d^2\mathbf{R}}{dt}\right|_A + \left.\frac{d^2\mathbf{r}}{dt^2}\right|_B + \frac{d\mathbf{\Omega}_{BA}}{dt}\times\mathbf{r} + 2\mathbf{\Omega}_{BA}\times\left.\frac{d\mathbf{r}}{dt}\right|_B + \mathbf{\Omega}_{BA}\times\left(\mathbf{\Omega}_{BA}\times\mathbf{r}\right) \tag{4.1}$$

where $\mathbf{\Omega}_{BA}$ is the angular velocity vector of reference B *relative* to the angular velocity vector of reference A.

All vectors may be expressed in any reference system, even a reference system other than A or B. However, all terms should be expressed in the *same* reference system: be consistent!

Note: frame A here *must* be an inertial frame of reference. This is in contrast to the one for velocity, where it was not required that one of them was inertial.

4.1.1 *Interpreting the relative motion equations*

Now, both equations may seem like total magic to you, especially the acceleration one, so let's discuss them in a bit more detail so that you understand what each term physically represents.

I think the velocity one makes quite a bit of sense: essentially in the first two terms, you just sum up the translation of frame B with respect to frame A and the translation of point P within frame B; the sum of $d\mathbf{R}/dt|_A + d\mathbf{r}/dt|_B$ gives you the translation of point P within frame A. Then you add $\mathbf{\Omega}_{BA}\times\mathbf{r}$ to this, so that you include the rotation of point P within the frame. Essentially it's just $\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega}\times\mathbf{r}_{B/A}$ which is something you've had back in the good old days of Dynamics.

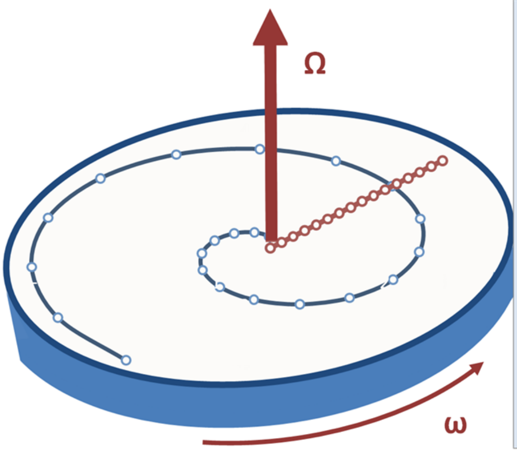
Now, the acceleration one is a bit more complicated. Let's consider a carousel where we sit on one of those horses on the outside. We choose an inertial frame of reference centered in the middle of the carousel (this reference frame does not even corotate with the carousel); we also choose a body-fixed reference frame, centered in our body's center of mass, with the X_b-axis pointing forward (in the same direction our eyes are pointing), Y_b-axis pointing outward, and the Z_b-axis pointing downward. The body-fixed reference frame moves along with us, so we're always at the origin of it, and it co-rotates with us as well.

In the body-fixed reference frame, our acceleration will be zero! After all, it moves along with us so we'll always be in the origin of it. Thus, from the point of view of the body-fixed reference frame, we're not moving or accelerating at all! Yes, the body-fixed reference frame itself is moving obviously, but within the body-fixed reference frame, you, on the horsey, are not moving as you are always at the origin of it.

Now, in the inertial reference frame, how does it work there? Our accelerations there won't be zero. After all, we're rotating around the center of the carousel, so at the very least there'll be normal acceleration. Well, the acceleration follows from an analysis of the terms in equation (4.1):

1. $d^2\mathbf{R}/dt\big|_A$ essentially represents the translational acceleration of frame B with respect to frame A. For example, suppose the horsey would start moving up and down as is often the case in the more advanced carousels: then frame B will have an acceleration in Z-direction with respect to frame A, and this term would take that into account.
2. $d^2\mathbf{r}/dt\big|_B$ essentially represents the translation acceleration of a point P with respect to frame B. For example, suppose you're not interested in the acceleration of your center of mass, but for example in the acceleration of your head. If you move your head forward, it'd get an acceleration in X_B-direction; the term $d^2\mathbf{r}/dt\big|_B$ would take that into account. Note that the sum of these first two terms, essentially represents the translational acceleration of point P with respect to frame A.

3. $d\mathbf{\Omega}_{BA}/dt \times \mathbf{r}$ represents the tangential acceleration of point P; after all, $d\mathbf{\Omega}_{BA}/dt = \boldsymbol{\alpha}_{BA}$, so this term becomes $\boldsymbol{\alpha}_{BA} \times \mathbf{r}$. For example, if the carousel guy decides to have fun and accelerate the carousel, you'd feel this tangential acceleration, obviously.
4. $\mathbf{\Omega}_{BA} \times (\mathbf{\Omega}_{BA} \times \mathbf{r})$ represents the apparent centripetal acceleration of P; after all, you can kinda recognize $\omega_{BA}^2\mathbf{r}$ in this, which you know very well from Dynamics to be the normal (centripetal) acceleration. You also feel this obviously: if you don't hold tight to your horsey, you may as well fall out of the carousel, which hurts.
5. $2\mathbf{\Omega}_{BA} \times d\mathbf{r}/dt|_B$ represents the Coriolis acceleration. What is this? Suppose you step off from the horsey, and for some reason, you stand in the middle of the carousel. You then walk in a *straight* line, i.e. in the instantaneous direction your eyes are pointing when you want to make a step[1]. The resultant path you walked is shown in figure 4.1: in a reference frame that corotates with the carousel, it seems as if you walked in a straight line. However, in the inertial reference frame, you have traced out a spiral: obviously this induces accelerations (not the least some additional normal acceleration), which is represented by the Coriolis acceleration.



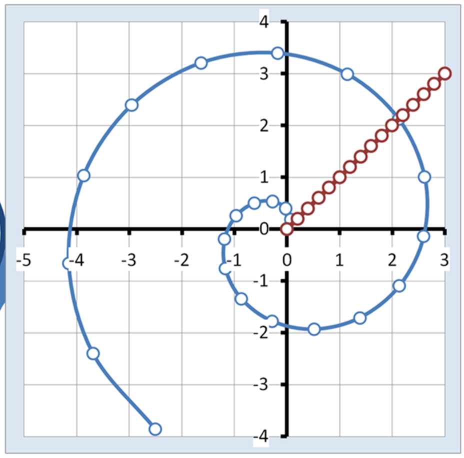


Figure 4.1: Crossing a rotating carousel walking at constant speed from the center of the carousel to its edge, a spiral is traced out in the inertial frame, while a simple straight radial path is seen in the frame of the carousel.

To close off this section, let me properly define the terms in the equation for absolute acceleration with respect to the inertial frame:

ACCELERATIONS

In equation (4.1), the following terms are defined:

- $d^2\mathbf{R}/dt^2$ is the **absolute acceleration of the moving w.r.t. the inertial frame**.
- $d\mathbf{\Omega}/dt \times \mathbf{r}$ is the **apparent (tangential) acceleration** due to angular acceleration of the moving frame.
- $\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r})$ is the **apparent (centripetal) acceleration** due to angular motion of the moving frame. Together with the two terms above, the resulting acceleration is called the **dragging acceleration**.
- $2\mathbf{\Omega} \times d\mathbf{r}/dt$ is the **Coriolis acceleration** due to the motion of p in the moving frame.
- $d^2\mathbf{r}/dt^2$ is the relative acceleration of p in the moving frame.

The absolute acceleration with respect to the inertial frame can be written as the summation of the dragging acceleration, Coriolis acceleration and relative acceleration.

Exam April 2015: problem 2acd (20p)

Consider the equations of translational motion in a rotating frame. It is common practice to express or transform the general formulation to the vehicle-carried normal Earth reference frame, also know as the E-frame, because the commonly used state variables like the Cartesian velocity V_N, V_E, V_D are usually given in this frame. The (spherical) position R, τ, δ gives the location of the aircraft (which is

[1] So if at $t = 0$ your eyes are pointing at that hot chick standing still at the outside of the carousel, then at $t = 0$ you set a step towards here, but for the next step, due to the rotation of the carousel, your eyes will be pointing to the left of her (or right depending on which direction it rotates), so then you're unfortunately no longer walking towards her.

the origin of the E-frame in the C-frame). Basically we want expressions for the derivatives of all state variables, both position and velocity. So starting with the general formulation:

$$m\frac{d\mathbf{V}_C^C}{dt} = \mathbf{F}_{ext}^C - 2m\boldsymbol{\Omega}_{CI}^C \times \mathbf{V}_C^C - m\boldsymbol{\Omega}_{CI}^C \times \left(\boldsymbol{\Omega}_{CI}^C \times \mathbf{r}_{cm}^C\right)$$

and

$$\frac{d\mathbf{r}_{cm}^C}{dt} = \mathbf{V}_C^C$$

with $\mathbf{r}_{cm}^C = (x, y, z)^T$ and $\mathbf{V}_C^C = (V_x, V_y, V_z)^T$.

(a) (**4 points**) We see that, to express the acceleration (applying Newton's $\mathbf{F} = m\mathbf{a}$ in a rotating frame), the external force needs two correction terms (the second and third right-hand terms of the first equation). Explain where these terms come from and what they represent. Note: it is not needed to derive these terms.

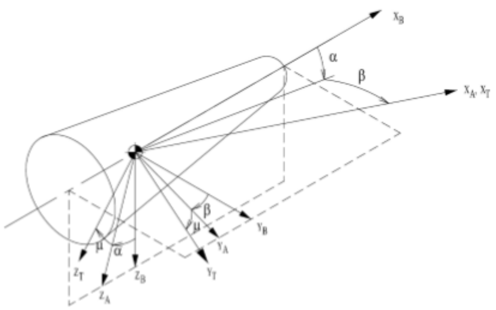
(c) (**8 points**) Given that $\mathbf{V}_C^E = (V_N, V_E, V_D)^T$, $\mathbf{r}_{cm}^E = (0, 0, -R)^T$, and $\boldsymbol{\Omega}_{CI}^C = (0, 0, \Omega_t)^T$, express the second right-hand term of the first equation in coordinates of the E-frame to get at

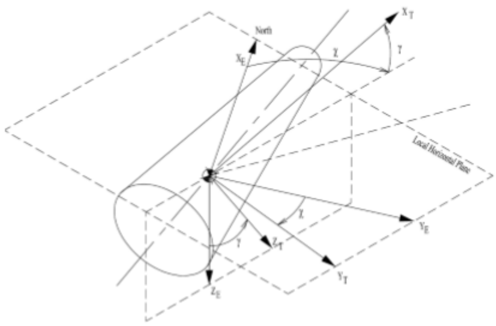
$$2m\Omega_t \left(V_E \sin\delta, -\left(V_D\cos\delta + V_E \sin\delta\right), V_E\cos\delta\right)^T$$

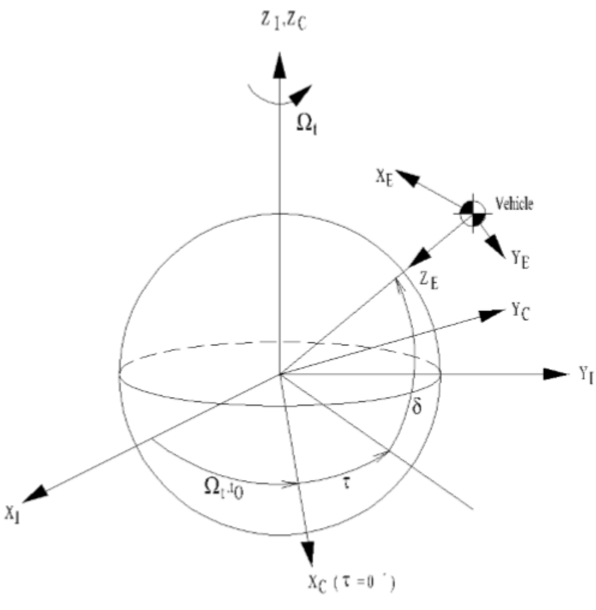
and do the same for the third right-hand term and check with the final answer

$$m\Omega_t^2 R\cos\delta\,(\sin\delta, 0, \cos\delta)^T$$

Make use of the given relations between the different frames involved as shown in the figures below:







(d) (**8 points.**) Also the external force needs to be transformed to the E-frame. Suppose the external force is the sum of the gravitational force (given in the E-frame $\mathcal{F}_E$), the aerodynamic forces (given in the aerodynamic frame $\mathcal{F}_a$), and the thrust force (given in the body frame $\mathcal{F}_b$). State the transformation matrices needed for all these forces (expressed in the E-frame) and subsequently define them as a sequence of unit axes rotations like $\mathbb{T}_{1n} = \mathbb{T}_x(\alpha_1)\,\mathbb{T}_y(\alpha_2)\cdots\mathbb{T}_?(\alpha_n)$. You do not have to actually multiply (or substitute) the matrices.

For a), see literally the definitions above:

- $2m\mathbf{\Omega}^C_{CI} \times \mathbf{V}^C_C$ is the force due to the coriolis acceleration, which occurs due to the motion of the object in the rotating frame.
- $m\mathbf{\Omega}^C_{CI} \times \left(\mathbf{\Omega}^C_{CI} \times \mathbf{r}^C_{cm}\right)$ is the apparent (centrifugal) force, due to the angular motion of the reference frame.

For c), we need to first evaluate $2m\mathbf{\Omega}^C_{CI} \times \mathbf{V}^C_C$, but in the E-frame, so we need to evaluate $2m\mathbf{\Omega}^E_{CI} \times \mathbf{V}^E_C$. Now, $\mathbf{V}^E_C$ is already given to us so that's nice, but $\mathbf{\Omega}^C_{CI}$ we need to first transform ourselves. We need to set up the transformation matrix $\mathbb{T}_{EC}$: we go from F_C to F_E by first rotating an angle τ around the Z_C-axis in positive direction (let's denote this intermediate reference system by C'); then by rotating an angle $\delta + \pi/2$ over the $Y_{C'}$-axis in *negative* direction, the transformation to the F_E is completed. Thus,

$$\mathbb{T}_{EC} = \mathbb{T}_y(-\delta - \pi/2)\,\mathbb{T}_z(\tau) = \begin{bmatrix} \cos\left(-\delta - \frac{\pi}{2}\right) & 0 & -\sin\left(-\delta - \frac{\pi}{2}\right) \\ 0 & 1 & 0 \\ \sin\left(-\delta - \frac{\pi}{2}\right) & 0 & \cos\left(-\delta - \frac{\pi}{2}\right) \end{bmatrix} \begin{bmatrix} \cos\tau & \sin\tau & 0 \\ -\sin\tau & \cos\tau & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} -\sin\delta\cos\tau & -\sin\delta\sin\tau & \cos\delta \\ -\sin\tau & \cos\tau & 0 \\ -\cos\delta\cos\tau & -\cos\delta\sin\tau & -\sin\delta \end{bmatrix}$$
$$\mathbf{\Omega}^E_{CI} = \mathbb{T}_{EC}\mathbf{\Omega}^C_{CI} = \begin{bmatrix} -\sin\delta\cos\tau & -\sin\delta\sin\tau & \cos\delta \\ -\sin\tau & \cos\tau & 0 \\ -\cos\delta\cos\tau & -\cos\delta\sin\tau & -\sin\delta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \Omega_t \end{bmatrix} = \Omega_t \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix}$$

Then simply performing the cross-product:

$$2m\mathbf{\Omega}^E_{CI} \times \mathbf{V}^E_C = 2m\Omega_t \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix} \times \begin{bmatrix} V_N \\ V_E \\ V_D \end{bmatrix} = 2m\Omega_t \begin{vmatrix} \mathbf{i}^E & \mathbf{j}^E & \mathbf{k}^E \\ \cos\delta & 0 & -\sin\delta \\ V_N & V_E & V_D \end{vmatrix} = 2m\Omega_t \begin{bmatrix} V_E\sin\delta \\ -\left(V_D\cos\delta + V_N\sin\delta\right) \\ V_E\cos\delta \end{bmatrix}$$

I know the question itself says it should be V_E rather than V_N, but I'm really, really sure that's a typo. V_E there doesn't make sense: in the case that $\delta = -90°$ and $\tau = 0°$, F_E and F_C are aligned in orientation (F_E is merely translated with respect to F_C). If you work out $[0, 0, \Omega_t]^T \times [V_N, V_E, V_D]^T$, you get $\Omega_t[-V_E, V_N, 0]^T$, which is what my solution gives but the one in the question itself doesn't give that. Nonetheless, we can do the same for the third right-hand term: we already have all the vectors, so we simply apply the cross-product twice:

$$m\mathbf{\Omega}^E_{CI} \times \left(\mathbf{\Omega}^E_{CI} \times \mathbf{r}^E_{cm}\right) = m\Omega_t \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix} \times \left(\Omega_t \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -R \end{bmatrix}\right)$$
$$= m\Omega_t^2 \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix} \times \begin{vmatrix} \mathbf{i}^E & \mathbf{j}^E & \mathbf{k}^E \\ \cos\delta & 0 & -\sin\delta \\ 0 & 0 & -R \end{vmatrix} = m\Omega_t^2 \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix} \times \begin{bmatrix} 0 \\ R\cos\delta \\ 0 \end{bmatrix}$$
$$= m\Omega_t^2 R \begin{vmatrix} \mathbf{i}^E & \mathbf{j}^E & \mathbf{k}^E \\ \cos\delta & 0 & -\sin\delta \\ 0 & R\cos\delta & 0 \end{vmatrix} = \begin{bmatrix} R\cos\delta\sin\delta \\ 0 \\ R\cos\delta\cos\delta \end{bmatrix} = m\Omega_t^2 R\cos\delta \begin{bmatrix} \sin\delta \\ 0 \\ \cos\delta \end{bmatrix}$$

For d), the transformation matrix for the gravitational force is simply the identity matrix as it's already in the desired reference frame.

For the aerodynamic force, we look carefully at the pictures. We see that we first have to transform to F_T, by rotating an angle μ around the X_a-axis, in positive direction, i.e. $\mathbb{T}_{Ta} = \mathbb{T}_x(\mu)\big|_a$. Then we need to rotate from F_T to F_E, which we do by looking at the top right figure. We see that we first must rotate in the direction opposite to how γ is drawn around the Y_T-axis, and γ itself is drawn in positive direction. Then we rotate in the opposite direction of χ around the Z_T'-axis, and χ is in positive direction. Thus, $\mathbb{T}_{ET} = \mathbb{T}_z(-\chi)\big|_{T'}\mathbb{T}_y(-\mu)\big|_{T}$, so the transformation matrix becomes

$$\mathbb{T}_{Ea} = \mathbb{T}_{ET}\mathbb{T}_{Ta} = \mathbb{T}_z(-\chi)\big|_{T'}\mathbb{T}_y(-\mu)\big|_{T}\mathbb{T}_x(\mu)\big|_a$$

For the thrust force, we need to transform from the body frame to the E-frame. We can write $\mathbb{T}_{Eb} = \mathbb{T}_{Ea}\mathbb{T}_{ab}$, where $\mathbb{T}_{Ea}$ was computed before. $\mathbb{T}_{ab}$ consists of a rotation α around the Y_b-axis, where α is drawn in negative direction. Afterwards, a rotation β follows around the Z_b'-axis, where β is drawn in positive direction. Thus, the transformation matrix becomes

$$\mathbb{T}_{ab} = \mathbb{T}_z(\beta)\big|_{b'}\mathbb{T}_y(-\alpha)\big|_b$$

so that the transformation matrix for the thrust force becomes

$$\mathbb{T}_{Eb} = \mathbb{T}_{Ea}\mathbb{T}_{ab} = \mathbb{T}_z(-\chi)\big|_{T'}\mathbb{T}_y(-\mu)\big|_{T}\mathbb{T}_x(\mu)\big|_a\mathbb{T}_z(\beta)\big|_{b'}\mathbb{T}_y(-\alpha)\big|_b$$

4.2 *General formulation of the equations of motion*

In this section, we will develop the general formulations of the equation of motion, akin to $F = ma$, but then more advanced.

4.2.1 *Translational motion*

Let us consider a vehicle with mass m, moving with a velocity $\mathbf{V}_C$ relative to the rotating Earth-Centered Earth-Fixed frame (index C) at a distance $\mathbf{r}_{cm}^C$ from the centre of the central body (see figure 4.2). The vehicle is subjected to an external force $\mathbf{F}_{ext}^C$ and has a rotation $\mathbf{\Omega}_{bI}^b$ w.r.t. the inertial reference frame. The C-frame is fixed to the central body with the origin in its center of mass and rotates with a constant angular velocity $\mathbf{\Omega}_{CI}^C = \left[0, 0, \Omega_t\right]^T$.

Now, Newton states that $\mathbf{F} = m\mathbf{a}$ *for an inertial frame of reference*, where $\mathbf{a}$ is the acceleration of the center of mass of the body. Thus, it is not as easy as taking $\mathbf{F}_{ext}^C = md^2\mathbf{R}/dt^2\big|_C^C$, where $d^2\mathbf{R}/dt^2\big|_C^C$ is the acceleration in the C-frame (subscript) expressed in the C-frame (superscript). Indeed, we need to express the acceleration in the I-frame, for which we need equation (4.1): expressing all vectors in the C-coordinate system (so each vector gets the superscript C) yields the equation of motion

$$\mathbf{F}_{ext}^C = m\left.\frac{d^2\mathbf{R}}{dt^2}\right|_I^C + m\left.\frac{d^2\mathbf{r}}{dt^2}\right|_C^C + m\frac{d\mathbf{\Omega}_{CI}^C}{dt}\times\mathbf{r}^C + 2m\mathbf{\Omega}_{CI}^C\times\left.\frac{d\mathbf{r}^C}{dt}\right|_C + \mathbf{\Omega}_{CI}^C\times\left(\mathbf{\Omega}_{CI}^C\times\mathbf{r}^C\right)$$

Now, you may wonder, why do I only take $\mathbf{\Omega}_{CI}$? What happens to $\mathbf{\Omega}_{bI}$? Well, $\mathbf{F} = m\mathbf{a}$ holds for the center of mass of the body. However, the body-fixed reference frame has its origin in the center of mass as well, so it's rotating about that point. You'd need the $\mathbf{\Omega}_{bI}$ to find velocities and accelerations on other points on the body, however, for the center of mass, it does not matter at all as the tangential velocity due to the angular velocity is zero there! It's similar to if you have a ball rolling on the ground and you hit it with a force acting through the center of mass: how it reacts to that is independent of the angular velocity it had. We'll soon see that $\mathbf{\Omega}_{bI}$ *is* important for the moment equation of motion.

Now, two terms drop out: we assume the Earth rotates at a constant angular velocity, so $d\mathbf{\Omega}_{CI}^C/dt = 0$. Furthermore, the Earth-Centered Earth-Fixed frame has its origin in the center of the Earth, coinciding with the origin of the Earth inertial frame. Thus, $d^2\mathbf{R}/dt^2 = 0$. Consequently, the equation reduces to

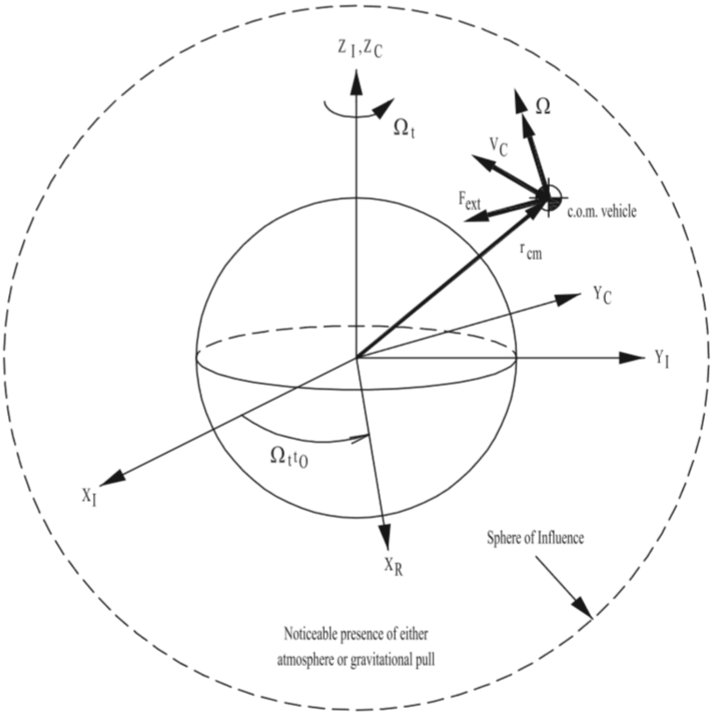


Figure 4.2: Definition of a vehicle moving in the sphere of influence of a celestial body. The inertial reference frame is indicated with index I. The Earth-centered Earth-fixed reference frame, with index C, is rigidly attached to the Earth and rotates with an angular velocity Ω_t, which is identical to the rotational rate of the Earth.

EQUATION OF MOTION FOR ROTATING REFERENCE FRAME

The equations of translational motion in the Earth-Centered Earth-Fixed frame are given by

$$\mathbf{F}_{ext}^{C} = m \left.\frac{d^2\mathbf{r}}{dt^2}\right|_C^C + 2m\mathbf{\Omega}_{CI}^{C} \times \left.\frac{d\mathbf{r}^C}{dt}\right|_C + \mathbf{\Omega}_{CI}^{C} \times \left(\mathbf{\Omega}_{CI}^{C} \times \mathbf{r}^C\right) \tag{4.2}$$

Yes it may seem hard at first but it's literally just $\mathbf{F} = m\mathbf{a}$, but then we need to transform the $\mathbf{a}$ to the inertial frame of reference, but this is trivial: we already derived equation (4.1) (okay we didn't but we know the formula for sure is correct), then plug replace A with I and B with C and that's it.

Note: you could just as well express this equation of motion in the E-frame: then you just transform each vector from the C-frame to the E-frame by multiplying with the transformation matrix $\mathbb{T}_{EC}$, and you'd get

$$\mathbf{F}_{ext}^{E} = m \left.\frac{d^2\mathbf{r}}{dt^2}\right|_C^E + 2m\mathbf{\Omega}_{CI}^{E} \times \left.\frac{d\mathbf{r}^E}{dt}\right|_C + \mathbf{\Omega}_{CI}^{E} \times \left(\mathbf{\Omega}_{CI}^{E} \times \mathbf{r}^E\right)$$

This may be done as the external forces are often given in the E-frame.

4.2.2 Rotational motion

Now, we also have moments etc. working on our aircraft, which means we have to analyse the aircraft as a rigid body. In all likelihood you sucked at rigid body motion in dynamics, but it's honestly not that difficult as long as you understand the concept. To derive the general formulation of the equations of motion in rotation, you have to realise where $\mathbf{F} = m\mathbf{a}$ comes from. Newton realised back in his days that the change in linear momentum is equal to the applied force, i.e.

$$F = \frac{d\,(mV)}{dt}$$

where $mV = P$ where P is the linear momentum. Unless you're dealing with relativity, m is constant so this becomes $F = ma$. However, for angular momentum we also have that the change in angular momentum is equal to the applied moment, i.e.

$$M = \frac{d\left(rmV\right)}{dt}$$

where $rmV = B$ where B is the angular momentum. In vector form, we have $\mathbf{B} = \mathbf{r} \times (m\mathbf{V})$. Now, again, we have to apply $\mathbf{M} = d\mathbf{B}/dt$ in an inertial frame of reference. For the moment equation, it is easiest to transform everything from the b-frame to the I-frame, rather than from the C-frame to the I-frame. Now, if you take moments around the center of mass of the aircraft, the result will be very similar to equation (4.2): each term is premultiplied with $\mathbf{r}^C\times$, each instance of m is replaced with an integral $\int_m dm$, to take into account the fact that each mass point has a different distance vector to the center of mass and the term $d\mathbf{\Omega}_{bI}^b/dt$ is nonzero now[2], resulting in

EQUATION OF MOTION ABOUT CENTER OF MASS

The equations of rotational motion about the center of mass are given by

$$\mathbf{M}_{c.o.m.}^b = \int_m \mathbf{r}^b \times \left.\frac{d^2\mathbf{r}}{dt^2}\right|_b^b dm + 2\int_m \mathbf{r}^b \times \mathbf{\Omega}_{bI}^b \times \left.\frac{d\mathbf{r}^b}{dt}\right|_b dm + \int_m \mathbf{r}^b \times \mathbf{\Omega}_{bI}^b \times \left(\mathbf{\Omega}_{bI}^b \times \mathbf{r}^b\right) dm + \int_m \mathbf{r}^b \times \frac{d\mathbf{\Omega}_{bI}^b}{dt} \times \mathbf{r}^b dm \tag{4.3}$$

Again, it may seem weird, but it's literally just basically applying equation (4.1), replacing As with I and Bs with b.

4.3 *External forces*

Let's focus on the equations of translational motion, and specifically, let's focus on the forces in (i.e. the left-hand side of the equation). Because we made it so general, it's actually laughably easy. We have three forces: gravity, aerodynamic forces, and propulsion forces.

4.3.1 *Gravity*

Using the assumption of a spherical Earth (so a 'perfect' sphere, with no irregularities in gravity field), the gravity vector in the E-frame becomes

GRAVITY FORCE VECTOR

The **gravity force vector** is expressed in the E-frame by

$$\mathbf{g}_G^E = \mathbf{g}_{r,G}^E = \begin{bmatrix} 0 \\ 0 \\ \frac{\mathcal{G}m_t}{(R_t+h)^2} \end{bmatrix} \tag{4.4}$$

where $\mathcal{G}$ is the constant of gravitation, m_t the Earth total mass, and R_t the mean Earth radius.

The weight is simply the mass times the gravity force vector. But honestly if there's anything in this subsection new to you, congrats cause I'd be impressed you managed to get to third year.

[2] If you're wondering now, but what happens to $\mathbf{\Omega}_{CI}$ then, isn't that important for something here? No, it isn't important at all. We're analysing the *rotation* around the center of mass of the vehicle. The fact that the Earth is rotating then doesn't matter at all. We're only looking at the vehicle. The Earth may well be spinning at 100 000 rad/s, but the vehicle doesn't care it's just flying around the Earth so it's not affected by it at all.

4.3.2 Aerodynamic forces

The aerodynamic forces are often given in the aerodynamic frame (gee, who'd have guessed). In that case, we have the following definition:

AERODYNAMIC FORCE VECTOR

The **aerodynamic force vector** is expressed in the a-frame by

$$\mathbf{F}_A^a = \begin{bmatrix} X^a \\ Y^a \\ Z^a \end{bmatrix} \tag{4.5}$$

Here, the lift L acts in *negative* Z^a direction, and the drag D acts in *negative* X^a direction. Note: the letter L will from now on forward refer to the moment around the X-axis.

Remember that X^a points forward and L points downward, which is why the lift and drag act in negative direction. This force vector may be transformed to the E-frame by multiplying with $\mathbb{T}_{Ea}$.

4.3.3 Propulsion forces

The propulsion forces are usually expressed in the body-axis reference frame, i.e. the b-frame. The decomposition is defined as

PROPULSION FORCE VECTOR

The **propulsion force vector** is expressed in the b-frame by

$$\mathbf{F}_P^b = \begin{bmatrix} F_{P,x}^b \\ F_{P,y}^b \\ F_{P,z}^b \end{bmatrix} \tag{4.6}$$

Honestly it's that easy.

4.4 External moments

Now let's focus on the left-hand side of the equations of rotational motion.

4.4.1 Gravity

Gravity acts through the center of gravity, so it will not cause a moment about it.

4.4.2 Aerodynamic forces

The aerodynamic forces create a moment about the center of gravity. We simply define this moment in the aerodynamic reference frame as

AERODYNAMIC MOMENT VECTOR

The **aerodynamic moment vector** is expressed in the a-frame by

$$M_{A,G}^a = \begin{bmatrix} L^a \\ M^a \\ N^a \end{bmatrix} \tag{4.7}$$

4.4.3 Propulsion forces

The propulsion forces also create a moment about the center of gravity, simply defined as

PROPULSION MOMENT VECTOR

The **propulsion moment vector** is expressed in the *b*-frame by

$$M^b_{P,G} = \begin{bmatrix} M^b_{P,x} \\ M^b_{P,y} \\ M^b_{P,z} \end{bmatrix} \tag{4.8}$$

4.5 A simple 2D example

In order to increase your understanding of how to derive equations of motions, and because it was in one of the old exams, we'll now do a relatively simple exam of 2D-translational motion, so that you get a flavour of how to derive stuff. We'll do the derivation in two ways: first the intuitive way, then the formal way using the definitions given above.

4.5.1 Intuitive way

First, consider figure 4.3, where an aircraft in a 2D translational motion is shown. We want to obtain the relevant equations to fully describe the 2D translational motion of this aircraft. Note that we have *three* unknowns: the flight path angle γ, the velocity V, and the distance to the center of the Earth R (you can also pick three other unknowns, but these are the most commonly used ones). Therefore, we need three equations: two equations of translational motion (since we are in 2D, we only have two equations of translational motion), and one kinematic insight[3].

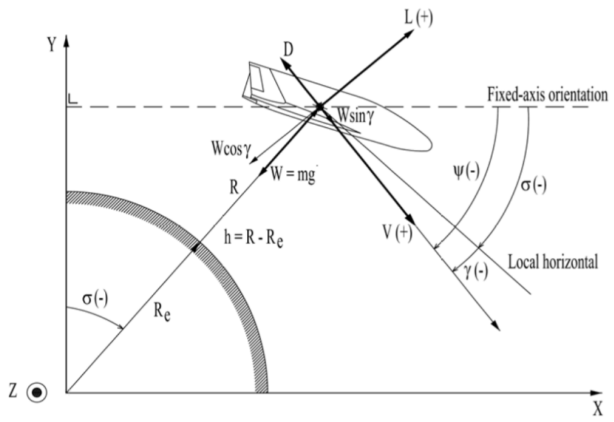


Figure 4.3: Free-body diagram for intuitive derivation. Note: the minus signs near the angles indicate that the drawn direction of those angles is the *negative* direction; thus, all angles are actually positive in *counter*clockwise direction.

Let's first derive the equations of translational motion. Rather intuitively, we have along the axis aligned with the velocity that

$$m\frac{dV}{dt} = -D - mg\sin\gamma \tag{4.9}$$

[3]If you're like, but why can't we use the equations of rotational motion? You could, but it'd just introduce an additional unknown, namely the pitch angle θ. So you're just making your own life more miserable.

If you're wondering, why is it $-mg\sin\gamma$ and not plus? Isn't $W\sin\gamma$ pointing in positive direction in the figure? Yes, but that's because they made a bit of a mess of the angles: the way γ is drawn, it's negative (e.g. in this picture $\gamma = -3°$); if γ, so that it would point in counterclockwise direction with respect to the local horizontal, W would have a component in *negative* direction of the velocity axis. Then, in the direction perpendicular to the velocity, we have

$$mV\frac{d\psi}{dt} = L - mg\cos\gamma$$

Now, is there a way to get rid of this ψ (as it'd be an additional unknown, meaning we'd need a second kinematic insight)? Yes, there is, fortunately. Note that we have $\psi = \sigma + \gamma$, so that

$$\frac{d\psi}{dt} = \frac{d\sigma}{dt} + \frac{d\gamma}{dt}$$

meaning we can write

$$mV\frac{d\gamma}{dt} = L - mg\cos\gamma - mV\frac{d\sigma}{dt}$$

Now, we see that σ is also the angle between the R_e and the Y-axis; as a result, we may write $Rd\sigma/dt = -V\cos\gamma$. Why? Remember that $R\dot{\theta}$ in general represents the tangential velocity component of a curvilinear motion (that's something you definitely should have remembered from Dynamics). $V\cos\gamma$ is the tangential component of the velocity, as it's the component perpendicular to $\mathbf{R}$. A minus sign needs to be added, as σ is defined to be positive in counterclockwise direction, but $V\cos\gamma$ points in clockwise direction. Thus, we may rewrite this equation of motion to

$$mV\frac{d\gamma}{dt} = L - mg\cos\gamma + m\frac{V^2}{R}\cos\gamma = L - mg\cos\gamma\left(1 - \frac{V^2}{gR}\right) = L - mg\cos\left(1 - \frac{V^2}{V_C^2}\right) \tag{4.10}$$

where $V_C = \sqrt{gR}$ is the circular velocity[4].

Now, finally, we see that we indeed have three unknowns: V, γ and R; the third equation follows from the kinematic insights that

$$\frac{dR}{dt} = \frac{dh}{dt} = V\sin\gamma \tag{4.11}$$

Equations (4.9)-(4.11) together make up a determined set of linear, first order differential equations, which may be rewritten in state space format rather straightforwardly. You can then solve them for $V(t)$, $\gamma(t)$ and $R(t)$, based on some initial conditions.

Note: the derivation in the slides is slightly different, but that's because theirs is wrong as it makes a complete mess of the signs.

4.5.2 Formal derivation

Now let's make a more formal derivation, based on figure 4.4. We want to set up the equations of motion in the T-frame, using V, γ and R; again, we'll need one kinematic insight to make it a determined system.

We need to solve the following equations of translational motion:

$$\mathbf{F}_{ext}^C = m\left.\frac{d^2\mathbf{r}}{dt^2}\right|_C^C + 2m\mathbf{\Omega}_{CI}^C \times \left.\frac{d\mathbf{r}^C}{dt}\right|_C + \mathbf{\Omega}_{CI}^C \times \left(\mathbf{\Omega}_{CI}^C \times \mathbf{r}^C\right) \tag{4.2}$$

Now, something beautiful happens: we are considering a two-dimensional motion. This means that the F_C and F_I frame are essentially coinciding: the F_C-frame rotates around the Z_I-axis, but we're not doing 3D right now, so the F_C and F_I-frame coincide (in other words, $Y_I = Y_C$, $X_I = X_C$ and $Z_I = Z_C$ in the figure). Thus, $\mathbf{\Omega}_{CI} = \mathbf{0}$, so we merely get

$$m\frac{d\mathbf{V}_C^C}{dt} = \mathbf{F}_{ext}^C$$

[4] The velocity of an object at distance R from the center of the Earth to remain in a circular orbit; think back to Noomen's lectures.

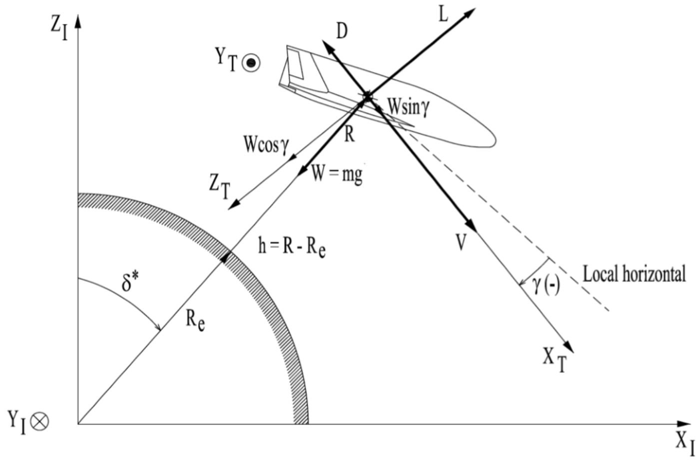


Figure 4.4: Free-body diagram for formal derivation. Note: Y_I points *into* the paper, Y_T points out of the paper. Consequentially, δ^* is positive clockwise (as it's drawn), and γ is positive counterclockwise (opposite to how it's drawn).

Furthermore, I can already tell you that the kinematic insight will be $d\mathbf{r}_{cm}^C/dt = \mathbf{V}_C^C$. Remember that the superscripts meant that that vector is expressed in that frame, meaning the equations are currently to be expressed all in the C-frame. In principle, this is fine, however, as we'll see, our lives become much, much easier if we just express everything in the T-frame, i.e.

$$m\frac{d\mathbf{V}_C^T}{dt} = \mathbf{F}_{ext}^T$$
$$\frac{d\mathbf{r}_{cm}^T}{dt} = \mathbf{V}_C^T$$

In case you forgot as well: $\mathbf{V}_C$ means the velocity of the aircraft within the F_C-frame, not the velocity of the F_C-frame itself, and $\mathbf{r}_{cm}^C$ is the position vector of the aircraft relative to the origin of the C-frame expressed within the F_C-frame.

We see that we need to come up with expressions for five things: $\mathbf{V}_C^T$, $\mathbf{r}_{cm}^T$ (we'll probably need this for $d\mathbf{r}_{cm}^T/dt$), $\mathbf{F}_{ext}^T$, $d\mathbf{V}_C^T/dt$ and $d\mathbf{r}_{cm}^T/dt$. This seems harder than it is really. Let's go through them one by one:

Velocity vector First, let's find an expression for $\mathbf{V}_C^T$. This is incredibly easy: the velocity expressed in the T-frame is simply $\mathbf{V}_C^T = V\mathbf{x}_T^T$.

Position vector Secondly, let's find an expression for $\mathbf{r}_{cm}^T$. Consider figure 4.4 once more; the vector $\mathbf{R}$ points from the origin of the I-frame to the aircraft. Evidently, $\mathbf{R}$ has a component along the *negative* Z_T-axis equal to $R\cos\gamma$, and a component along the positive X_T-axis of $\sin\gamma$ (note that if γ is negative, as drawn in the figure, it has a negative component along the X_T-axis). In other words, $\mathbf{r}_{cm}^T = R\sin\gamma\mathbf{x}_T^T - R\cos\gamma\mathbf{z}_T^T$.

Force vector For the force vector, we simply have

$$\mathbf{F}_{ext}^T = \begin{bmatrix} -D - mg\sin\gamma \\ 0 \\ -L + mg\cos\gamma \end{bmatrix}$$

where it should be noted that Z_T points *downwards*, which is why you get the minus sign for the lift L.

Velocity derivative Fourthly, for the velocity derivative $d\mathbf{V}_C^T/dt$, we have to use the previously found expression for $\mathbf{V}_C^T = V\mathbf{x}_T^T$. Differentiating, bearing the chain rule in mind, gives

$$\frac{d\mathbf{V}_C^T}{dt} = \frac{d}{dt}\left(V\mathbf{x}_T^T\right) = \frac{dV}{dt}\mathbf{x}_T^T + V\frac{d\mathbf{x}_T^T}{dt} \tag{4.12}$$

Here, $d\mathbf{x}_T/dt = \mathbf{\Omega}_{TC}^T \times \mathbf{x}_T$. Now, what is $\mathbf{\Omega}_{TC}^T$? The angular velocity vector of the T-frame, relative to the C-frame, expressed in the T-frame. We see merely a rotation around the Y_C-axis, of $\delta^* - \gamma$ in positive direction, i.e. $\mathbf{\Omega}_{TC}^C = \left(\dot{\delta}^* - \dot{\gamma}\right)\mathbf{y}_C^C$. Now, note that Y_C points in the opposite direction of Y_T; thus, we can simply multiply with -1 to obtain

$$\mathbf{\Omega}_{TC}^T = \begin{bmatrix} 0 \\ \dot{\gamma} - \dot{\delta}^* \\ 0 \end{bmatrix}$$

Then $d\mathbf{x}_T/dt$ follows straightforwardly:

$$\frac{d\mathbf{x}_T}{dt} = \mathbf{\Omega}_{TC}^T \times \mathbf{x}_T = \begin{bmatrix} 0 \\ \dot{\gamma} - \dot{\delta}^* \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \left(\dot{\delta}^* - \dot{\gamma}\right)\mathbf{z}_T \tag{4.13}$$

Thus, we obtain from equation (4.12)

$$\frac{d\mathbf{V}_C^T}{dt} = \frac{dV}{dt}\mathbf{x}_T^T + V\frac{d\mathbf{x}_T^T}{dt} = \dot{V}\mathbf{x}_T^T + V\left(\dot{\delta}^* - \dot{\gamma}\right)\mathbf{z}_T = \begin{bmatrix} \dot{V} \\ 0 \\ V\left(\dot{\delta}^* - \dot{\gamma}\right) \end{bmatrix} \tag{4.14}$$

Position derivative Finally, we can find the position derivative. This is done analogously to the velocity derivative, it's just more work, but the concept remains the same. We apply the chain rule to

$$\begin{aligned} \frac{d\mathbf{r}_{cm}^T}{dt} =& \frac{d}{dt}\left(R\sin\gamma\mathbf{x}_T^T - R\cos\gamma\mathbf{z}_T^T\right) \\ =& \dot{R}\sin\gamma\mathbf{x}_T^T + R\dot{\gamma}\cos\gamma\mathbf{x}_T^T + R\sin\gamma\frac{d\mathbf{x}_T^T}{dt} \\ &- \dot{R}\cos\gamma\mathbf{z}_T^T + R\dot{\gamma}\sin\gamma\mathbf{z}_T^T - R\cos\gamma\frac{d\mathbf{z}_T^T}{dt} \end{aligned} \tag{4.15}$$

Here, $d\mathbf{x}_T/dt$ was already found for the velocity derivative in equation (4.13) and equalled $d\mathbf{x}_T^T/dt = \left(\dot{\delta}^* - \dot{\gamma}\right)\mathbf{z}_T^T$. For $d\mathbf{z}_T^T/dt$, the same derivation can be done:

$$\frac{d\mathbf{x}_T}{dt} = \mathbf{\Omega}_{TC}^T \times \mathbf{x}_T = \begin{bmatrix} 0 \\ \dot{\gamma} - \dot{\delta}^* \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \left(\dot{\gamma} - \dot{\delta}^*\right)\mathbf{x}_T \tag{4.16}$$

Plugging these results into equation (4.15) results in

$$\begin{aligned} \frac{d\mathbf{r}_{cm}^T}{dt} =& \dot{R}\sin\gamma\mathbf{x}_T^T + R\dot{\gamma}\cos\gamma\mathbf{x}_T^T + R\sin\gamma\left(\dot{\delta}^* - \dot{\gamma}\right)\mathbf{z}_T^T \\ &- \dot{R}\cos\gamma\mathbf{z}_T^T + R\dot{\gamma}\sin\gamma\mathbf{z}_T^T - R\cos\gamma\left(\dot{\gamma} - \dot{\delta}^*\right)\mathbf{x}_T^T \\ =& \begin{bmatrix} \dot{R}\sin\gamma + R\dot{\delta}^*\cos\gamma \\ 0 \\ -\dot{R}\cos\gamma + R\dot{\delta}^*\sin\gamma \end{bmatrix} = \begin{bmatrix} V\cos^2\gamma + \dot{R}\sin\gamma \\ 0 \\ -\dot{R}\cos\gamma + V\cos\gamma\sin\gamma \end{bmatrix} \end{aligned} \tag{4.17}$$

where use was made of $R\dot{\delta}^* = V\cos\gamma$ (just like we used in the intuitive derivation).

Combining these results Now that we've got all these things, what was actually the point of all of it? Well, we were trying to solve the following system of equations:

$$m\frac{d\mathbf{V}_C^T}{dt} = \mathbf{F}_{ext}^T$$
$$\frac{d\mathbf{r}_{cm}^T}{dt} = \mathbf{V}_C^T$$

Let's focus on the second one first; from equation (4.17) and the fact that $\mathbf{V}_C^T = V\mathbf{x}_T^T$, we have

$$\frac{d\mathbf{r}_{cm}^T}{dt} = \mathbf{V}_C^T$$
$$\begin{bmatrix} V\cos^2\gamma + \dot{R}\sin\gamma \\ 0 \\ -\dot{R}\cos\gamma + V\cos\gamma\sin\gamma \end{bmatrix} = \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix}$$

All of these equations must be satisfied, and we can just pick one of them (except the second row obviously) to relate V and $\dot{R}$. Suppose we pick the first row:

$$V\cos^2\gamma + \dot{R}\sin\gamma = V$$
$$\dot{R}\sin\gamma = V\left(1-\cos^2\gamma\right) = V\sin^2\gamma$$
$$\dot{R} = V\sin\gamma \tag{4.18}$$

Note that if you plug this into the third row of the equation, the equation is still satisfied (indeed, all of the rows of the matrix are linearly dependent). This is the same kinematic insight that we found before in the intuitive derivation! Let's now focus on the equations of motion:

$$m\frac{d\mathbf{V}_C^T}{dt} = \mathbf{F}_{ext}^T$$
$$m\begin{bmatrix} \dot{V} \\ 0 \\ V\left(\dot{\delta}^* - \dot{\gamma}\right) \end{bmatrix} = \begin{bmatrix} -D - mg\sin\gamma \\ 0 \\ -L + mg\cos\gamma \end{bmatrix}$$

From the first row, we have

$$m\dot{V} = -D - mg\sin\gamma \tag{4.19}$$

From the third row, we have

$$mV\dot{\gamma} = L - mg\cos\gamma\left(1 - \frac{V^2}{gR}\right) \tag{4.20}$$

Note that equations (4.18)-(4.20) are identical to the equations found in the intuitive derivation!

Now, what's the use of this other than the fact that it was asked on an exam before (except there they didn't ask for the kinematic insight, so you didn't have to derive $\mathbf{r}_{cm}^T$ and $d\mathbf{r}_{cm}^T/dt$)? Well, we basically see the basic procedure:

- You identify the vectors you want to know.
- You find expressions for them: for regular components, this simply means either expressing them the correct frame at once, or expressing them in a different frame, and then transforming them by use of a transformation matrix. For time derivatives, you simply differentiate the regular components, by applying the chain rule! The derivatives of the unit vectors follow simply from the rotational rate of the frame; if you are interested in the velocity derivative $d\mathbf{V}_A^B/dt$, then the time-derivatives of the unit-vector are to be multiplied (by cross-product) with $\mathbf{\Omega}_{BA}^B$.

If you work very consistently, this is literally all you have to do. Really, everyone can do this, just be consistent and do the same thing over and over again.

Exam April 2015: problem 2b (5p)

Consider the equations of translational motion in a rotating frame. It is common practice to express or transform the general formulation to the vehicle-carried normal Earth reference frame, also know as the E-frame, because the commonly used state variables like the Cartesian velocity V_N, V_E, V_D are usually given in this frame. The (spherical) position R, τ, δ gives the location of the aircraft (which is the origin of the E-frame in the C-frame). Basically we want expressions for the derivatives of all state variables, both position and velocity. So starting with the general formulation:

$$m\frac{d\mathbf{V}_C^C}{dt} = \mathbf{F}_{ext}^C - 2m\mathbf{\Omega}_{CI}^C \times \mathbf{V}_C^C - m\mathbf{\Omega}_{CI}^C \times \left(\mathbf{\Omega}_{CI}^C \times \mathbf{r}_{cm}^C\right)$$

and

$$\frac{d\mathbf{r}_{cm}^C}{dt} = \mathbf{V}_C^C$$

with $\mathbf{r}_{cm}^C = (x, y, z)^T$ and $\mathbf{V}_C^C = (V_x, V_y, V_z)^T$.

(b) (**5 points**) In the two given equations distinguish and mention the 6 different components that need to be transformed to the E-frame, also mention per component in words what is needed for this transformation. Hint: make a distinction between regular components and components involving time derivatives.

The six components that need to be transformed to the E-frame are $\mathbf{F}_{ext}^C$, $\mathbf{\Omega}_{CI}^C$, $\mathbf{V}_C^C$, $\mathbf{r}_{cm}^C$, $d\mathbf{V}_C^C/dt$ and $d\mathbf{r}_{cm}^C/dt$. The first four of these can all be transformed by a simple transformation matrix. $d\mathbf{V}_C^C/dt$ and $d\mathbf{r}_{cm}^C/dt$ can be transformed by differentiating their regular components, applying the chain rule, and then finding expressions for the time derivatives of the unit vectors by use of $d\mathbf{x}_E/dt = \mathbf{\Omega}_{EC}^E \times \mathbf{x}_E$, $d\mathbf{y}_E/dt = \mathbf{\Omega}_{EC}^E \times \mathbf{y}_E$ and $d\mathbf{z}_E/dt = \mathbf{\Omega}_{EC}^E \times \mathbf{z}_E$.

5 Advanced equations of motion

Now we'll actually start deriving the equations of motion. Fortunately, you're basically guaranteed not to get an exam question about this chapter, so don't waste your time studying this chapter super well cause it's kinda useless. There's only one part that is asked sometimes on the exam but I'll clearly indicate that part. We'll first discuss the ones of translational motion, afterwards the ones of rotational motion. Both of them consist of two parts: the actual equations of motion, and the required kinematic insights. After we've done all of that, we'll simplify the equations of motion.

As there are a few cross-products we need to perform this chapter, and I don't feel like writing it out each and every time as it takes up quite a bit of space, let me repeat how you have to compute a cross product.

CROSS-PRODUCT

Consider two vectors,

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let the unit vectors along the axes be denoted by **x**, **y** and **z**; the **cross-product** is then defined by

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(a_2b_3 - a_3b_2\right)\mathbf{x} - \left(a_1b_3 - a_3b_1\right)\mathbf{y} + \left(a_1b_2 - a_2b_1\right)\mathbf{z} = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \tag{5.1}$$

Note that we are computing the determinant when we write $||$.

5.1 Equations of translational motion

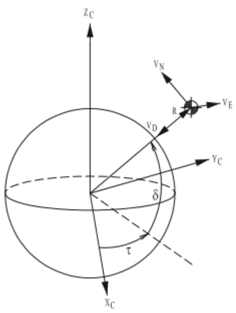


Figure 5.1: Euler angle singularity: aircraft in vertical climb.

First, we need to define what our states are, i.e. the stuff we want to solve for. They are shown in figure 5.1: the states are the velocity components V_N, V_E and V_D, and the spherical position components R (distance to center of the Earth, not the altitude), longitude τ and latitude δ. We thus have six unknowns: three equations originate from the equations of translational motion, given by equation (4.2), and three equations will originate from the kinematic insight $\mathbf{V}_C^C = d\mathbf{r}_{cm}^C/dt$. In other words, the governing equations will be

DYNAMIC AND KINEMATIC EQUATIONS FOR TRANSLATIONAL MOTION

For translational motion, the governing dynamic respectively kinematic equations are given by

$$m\frac{d\mathbf{V}_C^E}{dt} = \mathbf{F}_{ext}^E - 2m\mathbf{\Omega}_{CI}^E \times \mathbf{V}_C^E - m\mathbf{\Omega}_{CI}^E \times \left(\mathbf{\Omega}_{CI}^E \times \mathbf{r}_{cm}^E\right) \tag{5.2}$$

$$\frac{d\mathbf{r}_{cm}^E}{dt} = \mathbf{V}_C^E \tag{5.3}$$

where we'll derive everything in terms of the F_E-frame, as it's the easiest frame to do it in. We'll first focus on the dynamic equations given by equation (5.2), afterwards we'll derive the kinematic equations of equation (5.3).

5.1.1 Dynamic equations

For the dynamic equations, we see that we need to find expressions for $\mathbf{V}_C^E$, $\mathbf{r}_{cm}^E$, $\mathbf{\Omega}_{CI}^E$, $\mathbf{F}_{ext}^E$ and $d\mathbf{V}_C^E/dt$. Let's just find expressions for all of them.

Velocity vector For $\mathbf{V}_C^E$, we again just express everything in the F_E-frame, so that we simply have

$$\mathbf{V}_C^E = V_N\mathbf{x}_E + V_E\mathbf{y}_E + V_D\mathbf{z}_E = \begin{bmatrix} V_N \\ V_E \\ V_D \end{bmatrix} \tag{5.4}$$

Position vector For $\mathbf{r}_{cm}^E$, we very elegantly have

$$\mathbf{r}_{cm}^E = -R\mathbf{z}_E = \begin{bmatrix} 0 \\ 0 \\ -R \end{bmatrix} \tag{5.5}$$

Rotational velocity vector $\mathbf{\Omega}_{CI}^E$ is a bit more work. Remember that we simply have $\mathbf{\Omega}_{CI}^C = \Omega_t\mathbf{z}_C^C$, but we want to transform this to the F_E-frame. How do we do that? Well, remember that $\mathbb{T}_{EC}$ is given by (3.11), i.e.

$$\mathbb{T}_{EC} = \begin{bmatrix} -\sin\delta\cos\tau & -\sin\delta\sin\tau & \cos\delta \\ -\sin\tau & \cos\tau & 0 \\ -\cos\delta\cos\tau & -\cos\delta\sin\tau & -\sin\delta \end{bmatrix}$$

and thus we simply obtain

$$\mathbf{\Omega}_{CI}^E = \mathbb{T}_{EC}\mathbf{\Omega}_{CI}^C = \begin{bmatrix} -\sin\delta\cos\tau & -\sin\delta\sin\tau & \cos\delta \\ -\sin\tau & \cos\tau & 0 \\ -\cos\delta\cos\tau & -\cos\delta\sin\tau & -\sin\delta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \Omega_t \end{bmatrix} = \Omega_t \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix} \tag{5.6}$$

and that's it basically.

Force vector As already explained, we'll assume we have the external forces readily available in the F_E-frame, meaning we simply have

$$\mathbf{F}_{ext}^E = \mathbf{F}_A^E + \mathbf{F}_P^E + \mathbf{F}_G^E = \begin{bmatrix} F_x^E \\ F_y^E \\ F_z^E \end{bmatrix} \tag{5.7}$$

Velocity derivative Now let's find $d\mathbf{V}_C^E/dt$. It's very similar to how it was done in the 2D example of the previous chapter. We simply differentiate equation (5.4) to obtain

$$\frac{d\mathbf{V}_C^E}{dt} = \dot{V}_N \mathbf{x}_E^E + V_N \frac{d\mathbf{x}_E^E}{dt} + \dot{V}_E \mathbf{y}_E^E + V_E \frac{d\mathbf{y}_E^E}{dt} + \dot{V}_D \mathbf{z}_E^E + V_D \frac{d\mathbf{z}_E^E}{dt} \tag{5.8}$$

Here, we have $d\mathbf{x}_T/dt = \mathbf{\Omega}_{EC}^E \times \mathbf{x}_E$. Now, what is $\mathbf{\Omega}_{EC}^E$ again? It's the angular velocity of frame F_E relative to F_C, expressed in the F_E-frame. For this, we need the transformation matrix $\mathbb{T}_{EC} = \mathbb{T}_y\left(-\delta - \frac{\pi}{2} - \delta\right)\Big|_{C'} \mathbb{T}_z(\tau)|_C$, from which it is apparent that the angular velocity vector $\mathbf{\Omega}_{EC}$ is given by

$$\mathbf{\Omega}_{EC} = -\dot{\delta}\mathbf{y}_E + \dot{\tau}\mathbf{z}_C$$

Expressing this in the F_E-frame requires transforming $\mathbf{z}_C$ to the F_E-frame; this can be done by multiplying with $\mathbb{T}_{EC}$, given by equation (3.11) to be

$$\mathbb{T}_{EC} = \begin{bmatrix} -\sin\delta\cos\tau & -\sin\delta\sin\tau & \cos\delta \\ -\sin\tau & \cos\tau & 0 \\ -\cos\delta\cos\tau & -\cos\delta\sin\tau & -\sin\delta \end{bmatrix}$$

Performing the transformation results in

$$\mathbf{z}_C^E = \mathbb{T}_{EC}\mathbf{z}_C^C = \begin{bmatrix} -\sin\delta\cos\tau & -\sin\delta\sin\tau & \cos\delta \\ -\sin\tau & \cos\tau & 0 \\ -\cos\delta\cos\tau & -\cos\delta\sin\tau & -\sin\delta \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix}$$

In other words, we have

$$\mathbf{\Omega}_{EC}^E = \dot{\tau}\cos\delta\mathbf{x}_E - \dot{\delta}\mathbf{y}_E - \dot{\tau}\sin\delta\mathbf{z}_E = \begin{bmatrix} \dot{\tau}\cos\delta \\ -\dot{\delta} \\ -\dot{\tau}\sin\delta \end{bmatrix} \tag{5.9}$$

We can now find the derivatives $d\mathbf{x}_E^E/dt = \mathbf{\Omega}_{EC}^E \times \mathbf{x}_E^E$, $d\mathbf{y}_E/dt = \mathbf{\Omega}_{EC}^E \times \mathbf{y}_E$ and $d\mathbf{z}_E^E/dt = \mathbf{\Omega}_{EC}^E \times \mathbf{z}_E^E$:

$$\begin{aligned} \frac{d\mathbf{x}_E^E}{dt} &= \mathbf{\Omega}_{EC}^E \times \mathbf{x}_E^E = \begin{bmatrix} 0 \\ -\dot{\tau}\sin\delta \\ \dot{\delta} \end{bmatrix} \\ \frac{d\mathbf{y}_E^E}{dt} &= \mathbf{\Omega}_{EC}^E \times \mathbf{y}_E^E = \begin{bmatrix} \dot{\tau}\sin\delta \\ 0 \\ \dot{\tau}\cos\delta \end{bmatrix} \\ \frac{d\mathbf{z}_E^E}{dt} &= \mathbf{\Omega}_{EC}^E \times \mathbf{z}_E^E = \begin{bmatrix} -\dot{\delta} \\ -\dot{\tau}\cos\delta \\ 0 \end{bmatrix} \end{aligned} \tag{5.10}$$

Substituting equation (5.10) into (5.8) results in

$$\frac{d\mathbf{V}_C^E}{dt} = \begin{bmatrix} \dot{V}_N + V_E\dot{\tau}\sin\delta - V_D\dot{\delta} \\ \dot{V}_E - V_N\dot{\tau}\sin\delta - V_D\dot{\tau}\cos\delta \\ \dot{V}_D + V_N\dot{\delta} + V_E\dot{\tau}\cos\delta \end{bmatrix} \tag{5.11}$$

Combining everything Now again, why the fuck where we doing this again? Well we were analysing equation (5.2),

$$m\frac{d\mathbf{V}_C^E}{dt} = \mathbf{F}_{ext}^E - 2m\mathbf{\Omega}_{CI}^E \times \mathbf{V}_C^E - m\mathbf{\Omega}_{CI}^E \times \left(\mathbf{\Omega}_{CI}^E \times \mathbf{r}_{cm}^E\right) \tag{5.2}$$

We now have every term in this formula, we just need to work out the cross-products. We actually already did this in one of the examples of the previous chapter:

$$2m\mathbf{\Omega}_{CI}^E \times \mathbf{V}_C^E = 2m\Omega_t \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix} \times \begin{bmatrix} V_N \\ V_E \\ V_D \end{bmatrix} = 2m\Omega_t \begin{vmatrix} \mathbf{i}^E & \mathbf{j}^E & \mathbf{k}^E \\ \cos\delta & 0 & -\sin\delta \\ V_N & V_E & V_D \end{vmatrix} = 2m\Omega_t \begin{bmatrix} V_E\sin\delta \\ -\left(V_D\cos\delta + V_N\sin\delta\right) \\ V_E\cos\delta \end{bmatrix} \tag{5.12}$$

For the third term on the right:

$$
\begin{aligned}
m\mathbf{\Omega}_{CI}^{E} \times \left(\mathbf{\Omega}_{CI}^{E} \times \mathbf{r}_{cm}^{E}\right) &= m\Omega_t \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix} \times \left(\Omega_t \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -R \end{bmatrix}\right) \\
&= m\Omega_t^2 \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix} \times \begin{vmatrix} \mathbf{i}^E & \mathbf{j}^E & \mathbf{k}^E \\ \cos\delta & 0 & -\sin\delta \\ 0 & 0 & -R \end{vmatrix} = m\Omega_t^2 \begin{bmatrix} \cos\delta \\ 0 \\ -\sin\delta \end{bmatrix} \times \begin{bmatrix} 0 \\ R\cos\delta \\ 0 \end{bmatrix} \\
&= m\Omega_t^2 R \begin{vmatrix} \mathbf{i}^E & \mathbf{j}^E & \mathbf{k}^E \\ \cos\delta & 0 & -\sin\delta \\ 0 & R\cos\delta & 0 \end{vmatrix} = \begin{bmatrix} R\cos\delta\sin\delta \\ 0 \\ R\cos\delta\cos\delta \end{bmatrix} = m\Omega_t^2 R\cos\delta \begin{bmatrix} \sin\delta \\ 0 \\ \cos\delta \end{bmatrix}
\end{aligned}
\tag{5.13}
$$

Substituting equations (5.11), (5.7), (5.12) and (5.13) into equation (5.2) results in

$$
m\begin{bmatrix} \dot{V}_N + V_E\dot{\tau}\sin\delta - V_D\dot{\delta} \\ \dot{V}_E - V_N\dot{\tau}\sin\delta - V_D\dot{\tau}\cos\delta \\ \dot{V}_D + V_N\dot{\delta} + V_E\dot{\tau}\cos\delta \end{bmatrix} = \begin{bmatrix} F_x^E - 2m\Omega_t V_E\sin\delta - m\Omega_t^2 R\sin\delta\cos\delta \\ F_y^E + 2m\Omega_t\left(V_D\cos\delta + V_N\sin\delta\right) \\ F_z^E - 2m\Omega_t V_E\cos\delta - m\Omega_t^2 R\cos^2\delta \end{bmatrix}
$$

which may also be written as

$$
\begin{bmatrix} \dot{V}_N \\ \dot{V}_E \\ \dot{V}_D \end{bmatrix} = \begin{bmatrix} \frac{F_x^E}{m} - 2\Omega_t V_E\sin\delta - \Omega_t^2 R\sin\delta\cos\delta - V_E\dot{\tau}\sin\delta + V_D\dot{\delta} \\ \frac{F_y^E}{m} + 2\Omega_t\left(V_D\cos\delta + V_N\sin\delta\right) + V_N\dot{\tau}\sin\delta + V_D\dot{\tau}\cos\delta \\ \frac{F_z^E}{m} - 2\Omega_t V_E\cos\delta - \Omega_t^2 R\cos^2\delta - V_N\dot{\delta} - V_E\dot{\tau}\cos\delta \end{bmatrix}
\tag{5.14}
$$

These are our dynamic equations. We see all of our six state variables (and their derivatives) appearing, signifying the need for our kinematic insight.

5.1.2 Kinematic insight

Our kinematic insight is $d\mathbf{r}_{cm}^E/dt = \mathbf{V}_C^E$. Differentiating equation (5.5) yields

$$
\frac{d\mathbf{r}_{cm}^E}{dt} = -\frac{dR}{dt}\mathbf{z}_E - R\frac{d\mathbf{z}_E}{dt}
\tag{5.15}
$$

From equation (5.10), we have $d\mathbf{z}_E/dt = -\dot{\delta}\mathbf{x}_E - \dot{\tau}\cos\delta\mathbf{y}_E$, so that

$$
\frac{d\mathbf{r}_{cm}^E}{dt} = R\dot{\delta}\mathbf{x}_E + R\dot{\tau}\cos\delta\mathbf{y}_E - \dot{R}\mathbf{z}_E = \begin{bmatrix} R\dot{\delta} \\ R\dot{\tau}\cos\delta \\ -\dot{R} \end{bmatrix}
\tag{5.16}
$$

With $\mathbf{V}_C^E = V_N\mathbf{x}_E + V_E\mathbf{y}_E + V_D\mathbf{z}_E$, we obtain

$$
\begin{bmatrix} R\dot{\delta} \\ R\dot{\tau}\cos\delta \\ -\dot{R} \end{bmatrix} = \begin{bmatrix} V_N \\ V_E \\ V_D \end{bmatrix}
$$

meaning our kinematic insights yield us the additional set of equations

$$
\dot{\delta} = \frac{V_N}{R} \tag{5.17}
$$

$$
\dot{\tau} = \frac{V_E}{R\cos\delta} \tag{5.18}
$$

$$
\dot{R} = -V_D \tag{5.19}
$$

Equations (5.17) and (5.18) may be substituted in equation (5.14) to obtain the equations

$$\begin{bmatrix} \dot{V}_N \\ \dot{V}_E \\ \dot{V}_D \end{bmatrix} = \begin{bmatrix} \frac{F_x^E}{m} - 2\Omega_t V_E \sin\delta - \Omega_t^2 R \sin\delta\cos\delta - \frac{V_E^2\tan\delta - V_N V_D}{R} \\ \frac{F_y^E}{m} + 2\Omega_t\left(V_D\cos\delta + V_N\sin\delta\right) + \frac{V_E}{R}\left(V_N\tan\delta + V_D\right) \\ \frac{F_z^E}{m} - 2\Omega_t V_E\cos\delta - \Omega_t^2 R\cos^2\delta - \frac{V_E^2+V_N^2}{R} \end{bmatrix} \tag{5.20}$$

In overview, we can summarise the equations of translational motion as follows:

EQUATIONS OF TRANSLATIONAL MOTION

The **dynamic equations of translational motion** are given by

$$\dot{V}_N = \frac{F_x^E}{m} - 2\Omega_t V_E \sin\delta - \Omega_t^2 R\sin\delta\cos\delta - \frac{V_E^2\tan\delta - V_N V_D}{R} \tag{5.21}$$

$$\dot{V}_E = \frac{F_y^E}{m} + 2\Omega_t\left(V_D\cos\delta + V_N\sin\delta\right) + \frac{V_E}{R}\left(V_N\tan\delta + V_D\right) \tag{5.22}$$

$$\dot{V}_D = \frac{F_z^E}{m} - 2\Omega_t V_E\cos\delta - \Omega_t^2 R\cos^2\delta - \frac{V_E^2+V_N^2}{R} \tag{5.23}$$

The **kinematic equations of translational motion** are given by

$$\dot{\delta} = \frac{V_N}{R} \tag{5.24}$$

$$\dot{\tau} = \frac{V_E}{R\cos\delta} \tag{5.25}$$

$$\dot{R} = -V_D \tag{5.26}$$

This is a system of six non-linear differential equations that can be solved for V_N, V_E, V_D, δ, τ and R as a function of time, based on some initial condition. You can try to solve this with a computer, which admittedly will be very hard still because it's very non-linear so you'd have to come up with a good scheme to solve this properly.

5.2 *Equations of rotational motion*

Well that was fun, let's do it for rotational motion as well. Fortunately, strictly speaking, we only need to analyse the dynamic equations of rotational motion:

GOVERNING EQUATIONS OF ROTATIONAL MOTION

For rotational motion, the governing dynamic equations are given by

$$\mathbf{M}_{c.o.m.}^b = \int_m \mathbf{r}^b \times \left.\frac{d^2\mathbf{r}}{dt^2}\right|_b^b dm + 2\int_m \mathbf{r}^b \times \mathbf{\Omega}_{bI}^b \times \left.\frac{d\mathbf{r}^b}{dt}\right|_b dm + \int_m \mathbf{r}^b \times \mathbf{\Omega}_{bI}^b \times \left(\mathbf{\Omega}_{bI}^b \times \mathbf{r}^b\right) dm + \int_m \mathbf{r}^b \times \frac{d\mathbf{\Omega}_{bI}^b}{dt} \times \mathbf{r}^b dm \tag{5.27}$$

where $\mathbf{r}$ is the location of a point of the body with respect to the center of mass. Now, the time derivatives $d^2\mathbf{r}/dt^2$ and $d\mathbf{r}/dt$ are both equal to zero: they represent the movement of a point within the body: if we assume the body to be rigid, then all points stay at the same place within the body, so those time derivatives are zero. This means the equation reduces to

$$\mathbf{M}_{c.o.m.}^b = \int_m \mathbf{r}^b \times \mathbf{\Omega}_{bI}^b \times \left(\mathbf{\Omega}_{bI}^b \times \mathbf{r}^b\right) dm + \int_m \mathbf{r}^b \times \frac{d\mathbf{\Omega}_{bI}^b}{dt} \times \mathbf{r}^b dm$$

5.2.1 Dynamic equations

Now, actually, it's easier to just rederive this expression in a different way, so you can kinda forget about above expression. We start by analysing the angular momentum of a *rigid* body around its own center of mass. This is, by definition, equal to

$$\mathbf{B}^b_{c.o.m.} = \int\limits_m \mathbf{r}^b \times \left.\frac{d\mathbf{r}^b}{dt}\right|_I$$

where $\mathbf{r}^b$ is the position vector of an infinitesimal point on the body relative to the body's center of mass, expressed in the F_b-frame, and $d\mathbf{r}^b/dt|_I$ represents the time-derivative of this, as observed in the F_I-frame, expressed in the F_b-frame[1]. This time-derivative follows straightforwardly from equation (3.23); we simply get

$$\left.\frac{d\mathbf{r}^b}{dt}\right|_I = \left.\frac{d\mathbf{r}^b}{dt}\right|_b + \mathbf{\Omega}^b_{bI} \times \mathbf{r}^b$$

where $d\mathbf{r}^b/dt|_b$ is obviously zero, as explained before (they represent the movement of points within the body, but if we assume it's rigid, there's no such movement). Consequentially, we simply obtain

$$\mathbf{B}^b_{c.o.m.} = \int\limits_m \mathbf{r}^b \times \left(\mathbf{\Omega}^b_{bI} \times \mathbf{r}^b\right) dm$$

We can actually evaluate this integral relatively simply. Let $\mathbf{r}^b = [x, y, z]^T$ and $\mathbf{\Omega}^b_{bI} = [p, q, r]^T$, and with the triple product

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}\,(\mathbf{b} \cdot \mathbf{a})$$

we can write

$$\begin{aligned}
\mathbf{B}^b_{c.o.m.} &= \int\limits_m \mathbf{r}^b \times \left(\mathbf{\Omega}^b_{bI} \times \mathbf{r}^b\right) dm = \int\limits_m \left[\mathbf{\Omega}^b_{bI}\left(\mathbf{r}^b \cdot \mathbf{r}^b\right) - \mathbf{r}^b\left(\mathbf{\Omega}^b_{bI} \cdot \mathbf{r}^b\right)\right] dm \\
&= \int\limits_m \left\{ \begin{bmatrix} p \\ q \\ r \end{bmatrix} \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) - \begin{bmatrix} x \\ y \\ z \end{bmatrix} \left(\begin{bmatrix} p \\ q \\ r \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) \right\} dm = \int\limits_m \left\{ \begin{bmatrix} p \\ q \\ r \end{bmatrix} \left(x^2 + y^2 + z^2\right) - \begin{bmatrix} x \\ y \\ z \end{bmatrix} (px + qy + rz) \right\} \\
&= \int\limits_m \begin{bmatrix} px^2 + py^2 + pz^2 - px^2 - qxy - rxz \\ qx^2 + qy^2 + qz^2 - pxy - qy^2 - ryz \\ rx^2 + ry^2 + rz^2 - pxz - qyz - rz^2 \end{bmatrix} dm = \int\limits_m \begin{bmatrix} p\left(y^2 + z^2\right) - qxy - rxz \\ q\left(x^2 + z^2\right) - pxy - ryz \\ r\left(x^2 + y^2\right) - pxz - qyz \end{bmatrix} dm
\end{aligned}$$

Then, let us define the mass moments of inertia

$$I_{xx} = \int\limits_m \left(y^2 + z^2\right) dm$$

$$I_{yy} = \int\limits_m \left(x^2 + z^2\right) dm$$

$$I_{zz} = \int\limits_m \left(x^2 + y^2\right) dm$$

[1] What the fuck does that mean again? Well, F_b is a translating and rotating frame, as it rotates along with the body. Within this reference frame, a point may not be moving at all (as is in fact the case for rigid bodies). However, as the frame itself is rotating with respect to inertial frame, it may actually be moving with respect to F_I, which is why we need to take the time-derivative as observed in the F_I-frame. We can express this time-derivative using the unit vectors of the F_b-frame, though.

and the product mass moments of inertia

$$I_{xy} = \int\limits_m xy dm$$
$$I_{xz} = \int\limits_m xz dm$$
$$I_{yz} = \int\limits_m yz dm$$

then we may write it as

$$\mathbf{B}^b_{c.o.m.} = \begin{bmatrix} pI_{xx} - qI_{xy} - rI_{xz} \\ -pI_{xy} + qI_{yy} - rI_{yz} \\ -pI_{xz} - qI_{yz} + rI_{zz} \end{bmatrix}$$

Or, in matrix notation:

ANGULAR MOMENTUM AROUND OWN CENTER OF MASS

The angular momentum around a rigid body's own center of mass is given by

$$\mathbf{B}^b_{c.o.m.} = \mathbf{I}^b \cdot \mathbf{\Omega}^b_{bI} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} \tag{5.28}$$

where $\mathbf{I}^b$ is called the **inertia tensor**.

Now, by Newton's laws, we must have that the applied external moment is equal to the time rate of change of the angular momentum. However, you need to be careful here: we only computed the angular moment around its own center of mass. In reality, however, our body as a whole is rotating around the Earth as well. Indeed, a correctional factor needs to be applied, very similar to the correction to the velocity: we need to add a term $\mathbf{\Omega}^b_{bI} \times \mathbf{B}^b_{cm}$, such that

$$\mathbf{M}_{cm} = \frac{d\mathbf{B}^b_{cm}}{dt} + \mathbf{\Omega}^b_{bI} \times \mathbf{B}^b_{cm}$$

With $\mathbf{B}^b_{cm}/dt = \mathbf{I}^b \cdot \mathbf{\Omega}^b_{bI}$, applying the chain rule to this leads to

$$\frac{d\mathbf{B}^b_{cm}}{dt} = \frac{d}{dt}\left(\mathbf{I}^b \cdot \mathbf{\Omega}^b_{bI}\right) = \frac{d\mathbf{I}}{dt} \cdot \mathbf{\Omega}^b_{bI} + \mathbf{I}^b \cdot \frac{d\mathbf{\Omega}^b_{bI}}{dt} = \mathbf{I}\dot{\mathbf{\Omega}}^b_{bI}$$

as $d\mathbf{I}^b/dt = \mathbf{0}$ for rigid bodies. Thus, we obtain

GOVERNING DYNAMIC EQUATIONS FOR ROTATIONAL MOTION

The governing dynamic equations for rotational motion are

$$\mathbf{M}_{cm} = \frac{d\mathbf{B}^b_{cm}}{dt} + \mathbf{\Omega}^b_{bI} \times \mathbf{B}^b_{cm} = \mathbf{I}^b \cdot \dot{\mathbf{\Omega}}^b_{bI} + \mathbf{\Omega}^b_{bI} \times \mathbf{I}^b \cdot \mathbf{\Omega}^b_{bI} \tag{5.29}$$

Written out in scalar components, we obtain

$$M_x = I_{xx}\dot{p} + \left(I_{zz} - I_{yy}\right) qr - I_{xz}\left(\dot{r} + pq\right) \tag{5.30}$$
$$M_y = I_{yy}\dot{q} + \left(I_{xx} - I_{zz}\right) rp + I_{xz}\left(p^2 - r^2\right) \tag{5.31}$$
$$M_z = I_{zz}\dot{r} + \left(I_{yy} - I_{xx}\right) pq - I_{xz}\left(\dot{p} - rq\right) \tag{5.32}$$

This may be solved for $\dot{p}$, $\dot{q}$ and $\dot{r}$. We can write equation (5.29) as an explicit expression for $\dot{\mathbf{\Omega}}^b_{bI}$:

$$\dot{\mathbf{\Omega}}^b_{bI} = \mathbf{I}^b\left(\mathbf{M}^b_{cm} - \mathbf{\Omega}^b_{bI} \times \left(\mathbf{I} \cdot \mathbf{\Omega}^b_{bI}\right)\right)$$

This is an explicit expression for $\dot{p}$, $\dot{q}$ and $\dot{r}$, and in principle we're done. However, inverting $\mathbf{I}^b$ analytically is very difficult. Therefore, let's make a simplification, and assume the $X_B Z_B$-plane is a plane of symmetry, so that $I_{xy} = I_{yz} = 0$. Then equation (5.29) reduces to

$$\begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} - \begin{bmatrix} p \\ q \\ r \end{bmatrix} \times \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

Solving for $\dot{p}$, $\dot{r}$ and $\dot{q}$ yields

$$\begin{aligned} \dot{p} &= \frac{I_{zz}}{I_{xx}I_{zz} - I_{xz}^2} M_x + \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2} M_z + \frac{\left(I_{xx} - I_{yy} + I_{zz}\right) I_{xz}}{I_{xx}I_{zz} - I_{xz}^2} pq + \frac{\left(I_{yy} - I_{zz}\right) I_{zz} - I_{xz}^2}{I_{xx}I_{zz} - I_{xz}^2} qr \\ \dot{q} &= \frac{M_y}{I_{yy}} + \frac{I_{xz}}{I_{yy}} \left(r^2 - p^2\right) + \frac{I_{zz} - I_{xx}}{I_{yy}} pr \\ \dot{r} &= \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2} M_z + \frac{I_{xx}}{I_{xx}I_{zz} - I_{xz}^2} M_z + \frac{\left(I_{xx} - I_{yy}\right) I_{xx} + I_{xz}^2}{I_{xx}I_{zz} - I_{xz}^2} pq + \frac{\left(-I_{xx} + I_{yy} - I_{zz}\right) I_{xz}}{I_{xx}I_{zz} - I_{xz}^2} qr \end{aligned}$$

We can simplify this further if we assume there are two planes of symmetry, cause then all $I_{xy} = I_{xz} = I_{yz} = 0$, simplifying these expressions to

$$\begin{aligned} \dot{p} &= \frac{M_x}{I_{xx}} + \frac{I_{yy} - I_{zz}}{I_{xx}} qr \\ \dot{q} &= \frac{M_y}{I_{yy}} + \frac{I_{zz} - I_{xx}}{I_{yy}} pr \\ \dot{r} &= \frac{M_z}{I_{zz}} + \frac{I_{xx} - I_{yy}}{I_{zz}} pq \end{aligned}$$

5.2.2 Kinematic equations

Now, above equations are very nice, and they are a determined set of non-linear differential equations that you can solve for $p(t)$, $q(t)$ and $r(t)$ based on some initial conditions. However, we are actually far more interested in the pitch angle θ, yaw angle ψ and roll angle ϕ. So, what is exactly the difference between p, q and r, and $\dot{\theta}$, $\dot{\psi}$ and $\dot{\phi}$? Well, there are two difference, which are apparent from how these quantities are defined. First, p, q and r represent the rotational rate of the body with respect to the inertial frame, i.e.

$$\mathbf{\Omega}_{bI} = p\mathbf{x}_b + q\mathbf{y}_b + r\mathbf{z}_b \tag{5.33}$$

where $\dot{\theta}$, $\dot{\psi}$ and $\dot{\phi}$ represent the rotational rate of the body with respect to the local horizontal , i.e. the vehicle carried Earth normal frame F_E (we'll get to this formula in a bit more detail in a bit):

$$\mathbf{\Omega}_{bE} = \dot{\phi}\mathbf{x}_b + \dot{\theta}\mathbf{y}_{E''} + \dot{\psi}\mathbf{z}_E \tag{5.34}$$

There are two important differences between the two:

- p, q and r are the rotational rates with respect to the *inertial* frame, whereas $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ are with respect to F_E. Consequently, p, q and r essentially include corrections for the rotational rate of the Earth, whereas $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ don't include those.
- For $\mathbf{\Omega}_{bI}$ we nicely have $\mathbf{x}_b$, $\mathbf{y}_b$ and $\mathbf{z}_b$. However, for the $\mathbf{\Omega}_{bE}$, we have those weird subscripts. So why are those subscripts there again? Well, for a very simple reason. Remember that the transformation matrix from the E-frame to the b-frame is given by Equation (3.12),

$$\mathbb{T}_{bE} = \mathbb{T}_{bE''}\mathbb{T}_{E''E'}\mathbb{T}_{E'E} = \mathbb{T}_x(\phi)\big|_{E''}\ \mathbb{T}_y(\theta)\Big|_{E'}\ \mathbb{T}_z(\psi)\big|_E \tag{??}$$

  This meant that the rotational rate would be equal to

$$\mathbf{\Omega}_{bE} = \mathbf{\Omega}_{bE''} + \mathbf{\Omega}_{E''E'} + \mathbf{\Omega}_{E'E} = \dot{\phi}\mathbf{x}_{E''} + \dot{\theta}\mathbf{y}_{E'} + \dot{\psi}\mathbf{z}_E = \dot{\phi}\mathbf{x}_b + \dot{\theta}\mathbf{y}_{E''} + \dot{\psi}\mathbf{z}_{E'} \tag{5.35}$$

since $\mathbf{x}_b = \mathbf{x}_{E''}$, $\mathbf{y}_{E''} = \mathbf{y}_{E'}$ and $\mathbf{z}_{E'} = \mathbf{z}_E$, i.e. we can also write $\mathbf{\Omega}_{bE} = \dot{\phi}\mathbf{x}_b + \dot{\theta}\mathbf{y}_{E''} + \dot{\psi}\mathbf{z}_E$. Now, the subscripts mean we first have to transform those unit vectors to be $\mathbf{x}_b$, $\mathbf{y}_b$ and $\mathbf{z}_b$ (although we don't to do anything for $\mathbf{x}_b$). Physically, this means that even if we would assume a non-rotating Earth, p, q and r wouldn't be exactly the same as $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$, as they are defined about different axes.

Let's deal with these problems as follows: we'll rewrite $\mathbf{\Omega}_{bE}$ to be consistently expressed in the F_b-frame, thus with $\mathbf{x}_b$, $\mathbf{y}_b$ and $\mathbf{z}_b$ rather than $\mathbf{x}_b$, $\mathbf{y}_{E''}$ and $\mathbf{z}_E$. Furthermore, we'll *subtract* the effect of the rotation of the Earth from $\mathbf{\Omega}^b_{bI}$ (the one containing p, q and r), such that we are left with some quantities $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$, which would represent the rotational rates with the rotation of the Earth removed from p, q and r (as those currently take the rotation of the Earth into account, whereas $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ aren't (so we need to make them consistent: we could also have added the effect of rotation to $\mathbf{\Omega}^b_{bE}$, but that's just a matter of own preference).

Mathematically, we can write what we want to do as follows:

$$\mathbf{\Omega}^b_{bI} - \mathbf{\Omega}^b_{EI} = \mathbf{\Omega}^b_{bE} \tag{5.36}$$

The term on the right-hand side is $\mathbf{\Omega}^b_{bE}$ consistently expressed in the F_b-frame. The left-hand side represents the rotational rate of the Earth taken out of the $\mathbf{\Omega}^b_{bI}$.[2]. The left-hand side we'll abbreviate to

$$\mathbf{\Omega}^b_{bI} - \mathbf{\Omega}^b_{EI} = \begin{bmatrix} \tilde{p} \\ \tilde{q} \\ \tilde{r} \end{bmatrix}$$

So we'll first find expressions for $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$, and then relate them to the right-hand side vector $\mathbf{\Omega}^b_{bE}$, containing $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$. I hope the general idea is clear now.

Finding $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ Let's first focus on the left-hand side of the equation. From equations (3.8)-(3.11), we have

$$\mathbb{T}_{EI} = \mathbb{T}_{EC}\mathbb{T}_{CI} = \mathbb{T}_y(-\delta - \pi/2)\Big|_{C'} \mathbb{T}_z\left(\tau + \Omega_t \cdot t_O\right)\Big|$$

and consequentially, we have

$$\mathbf{\Omega}_{EI} = -\dot{\delta}\mathbf{y}_E + \left(\dot{\tau} + \Omega_t\right)\mathbf{z}_I$$

We need to transform this to the b-frame however, i.e. we need to compute

$$\mathbf{y}^b_E = \mathbb{T}_{bE}\mathbf{y}^E_E$$
$$\mathbf{z}^b_I = \mathbb{T}_{bI}\mathbf{z}^I_I$$

Sparing yuo the detailed computations (it should be obvious how to compute them, $\mathbf{y}^E_E = [0,1,0]^T$ and $\mathbf{z}^I_I = [0,0,1]$ (after all they're the unit vectors and they are expressed in the frame they are defined in), it's just a matter of having time and the will to compute them), the results are

$$\mathbf{y}^b_E = \begin{bmatrix} \cos\theta\sin\psi \\ \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi \\ \sin\psi\cos\phi\sin\theta - \cos\psi\sin\phi \end{bmatrix}$$
$$\mathbf{z}^b_I = \begin{bmatrix} \cos\delta\cos\psi\cos\theta + \sin\delta\sin\theta \\ \cos\delta\left(\cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi\right) - \sin\delta\cos\theta\sin\phi \\ \cos\delta\left(\sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi\right) - \sin\delta\cos\theta\cos\phi \end{bmatrix}$$

which is absolutely delightful. It means that the left-hand side of equation (5.36) becomes

$$\begin{bmatrix} \tilde{p} \\ \tilde{q} \\ \tilde{r} \end{bmatrix} = \mathbf{\Omega}^b_{bI} - \mathbf{\Omega}^b_{EI} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} - \left(-\dot{\delta}\mathbf{y}^b_E + \left(\dot{\tau} + \mathbf{\Omega}_t\right)\mathbf{z}^b_I\right) \tag{5.37}$$

$$= \begin{bmatrix} p + \cos\theta\sin\psi\dot{\delta} - \left[\cos\delta\cos\psi\cos\theta + \sin\delta\sin\theta\right]\left(\dot{\tau} + \Omega_t\right) \\ q + \left(\sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi\right)\dot{\delta} - \left[\cos\delta\left(\cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi\right) - \sin\delta\cos\theta\sin\phi\right]\left(\dot{\tau} + \Omega_t\right) \\ r + \left(\sin\psi\cos\phi\sin\theta - \cos\psi\sin\phi\right)\dot{\delta} - \left[\cos\delta\left(\sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi\right) - \sin\delta\cos\theta\cos\phi\right]\left(\dot{\tau} + \Omega_t\right) \end{bmatrix} \tag{5.38}$$

which is absolutely beautiful once more.

[2]In case you're wondering, why the subscripts like this? Remember that normally speaking, $\mathbf{\Omega}_{ab} + \mathbf{\Omega}_{bc} = \mathbf{\Omega}_{ac}$, just like $\mathbb{T}_{ab}\mathbb{T}_{bc} = \mathbb{T}_{ac}$. In other words, $\mathbf{\Omega}_{ab} = \mathbf{\Omega}_{ac} - \mathbf{\Omega}_{bc}$, which explains the subscripts.

Finding $\boldsymbol{\Omega}_{bE}^{b}$ Let's now find on the right-hand side of equation (5.36). This is literally the only thing that ever gets asked in the exams (together with solving the equation for $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$. We had

$$\boldsymbol{\Omega}_{bE} = \dot{\phi}\mathbf{x}_b + \dot{\theta}\mathbf{y}_{E''} + \dot{\psi}\mathbf{z}_E$$

but we need to express this in terms of $\mathbf{x}_b$, $\mathbf{y}_b$ and $\mathbf{z}_b$ only. We can do this by just transforming $\mathbf{y}_{E''}$ and $\mathbf{z}_E$ to the F_b-frame. Leaving out the computations, these are simply[3]

$$\mathbf{y}_{E''}^{b} = \mathbb{T}_{bE''}\mathbf{y}_{E''}^{E''} = \begin{bmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{bmatrix}$$
$$\mathbf{z}_{E}^{b} = \mathbb{T}_{bE}\mathbf{z}_{E}^{E} = \begin{bmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix}$$

In other words, we have

$$\Omega_{bI}^{b} = \dot{\phi}\mathbf{x}_b + \dot{\theta}\mathbf{y}_{E''} + \dot{\psi}\mathbf{z}_E = \dot{\phi}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \dot{\theta}\begin{bmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{bmatrix} + \dot{\psi}\begin{bmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}$$

This means that equation (5.36) has become

$$\boldsymbol{\Omega}_{bI}^{b} - \boldsymbol{\Omega}_{EI}^{b} = \boldsymbol{\Omega}_{bE}^{b} \tag{5.36}$$
$$\begin{bmatrix} \tilde{p} \\ \tilde{q} \\ \tilde{r} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix}\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \tag{5.39}$$

with $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ following from equation (5.38). Solving above equation yields finally the kinematic attitude equations:

$$\dot{\phi} = \tilde{p} + \sin\phi\tan\theta\tilde{q} + \cos\phi\tan\theta\tilde{r}$$
$$\dot{\theta} = \cos\phi\tilde{q} - \sin\phi\tilde{r}$$
$$\dot{\psi} = \frac{\sin\phi}{\cos\theta}\tilde{q} + \frac{\cos\phi}{\cos\theta}\tilde{r}$$

Summarising this section:

[3] I mean, at the end of the day it's just a matrix multiplication. Ask a first year in a few months from now to do it for you and they'll do it, although probably they mess everything up because they're stupid, but it's the idea that counts. The transformation matrix itself is also just a bunch of matrix multiplications after looking carefully at the drawings of chapter 3, but yeah that's also not too hard, just time-consuming if you want to do it again by hand.

EQUATIONS OF ROTATIONAL MOTION

The **dynamic equations of rotational motion**, for bodies for which the $X_B Z_B$-plane is a symmetry plane, are given by

$$M_x = I_{xx}\dot{p} + \left(I_{zz} - I_{yy}\right) qr - I_{xz}\left(\dot{r} + pq\right) \tag{5.40}$$

$$M_y = I_{yy}\dot{q} + \left(I_{xx} - I_{zz}\right) rp + I_{xz}\left(p^2 - r^2\right) \tag{5.41}$$

$$M_z = I_{zz}\dot{r} + \left(I_{yy} - I_{xx}\right) pq - I_{xz}\left(\dot{p} - rq\right) \tag{5.42}$$

for which an explicit solution for $\dot{p}$, $\dot{q}$ and $\dot{r}$ is given by

$$\dot{p} = \frac{I_{zz}}{I_{xx}I_{zz} - I_{xz}^2} M_x + \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2} M_z + \frac{\left(I_{xx} - I_{yy} + I_{zz}\right) I_{xz}}{I_{xx}I_{zz} - I_{xz}^2} pq + \frac{\left(I_{yy} - I_{zz}\right) I_{zz} - I_{xz}^2}{I_{xx}I_{zz} - I_{xz}^2} qr \tag{5.43}$$

$$\dot{q} = \frac{M_y}{I_{yy}} + \frac{I_{xz}}{I_{yy}}\left(r^2 - p^2\right) + \frac{I_{zz} - I_{xx}}{I_{yy}} pr \tag{5.44}$$

$$\dot{r} = \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2} M_z + \frac{I_{xx}}{I_{xx}I_{zz} - I_{xz}^2} M_z + \frac{\left(I_{xx} - I_{yy}\right) I_{xx} + I_{xz}^2}{I_{xx}I_{zz} - I_{xz}^2} pq + \frac{\left(-I_{xx} + I_{yy} - I_{zz}\right) I_{xz}}{I_{xx}I_{zz} - I_{xz}^2} qr \tag{5.45}$$

The **kinematic equations of rotational motion** are given by

$$\dot{\phi} = \tilde{p} + \sin\phi\tan\theta\tilde{q} + \cos\phi\tan\theta\tilde{r} \tag{5.46}$$

$$\dot{\theta} = \cos\phi\tilde{q} - \sin\phi\tilde{r} \tag{5.47}$$

$$\dot{\psi} = \frac{\sin\phi}{\cos\theta}\tilde{q} + \frac{\cos\phi}{\cos\theta}\tilde{r} \tag{5.48}$$

where

$$\tilde{p} = p + \cos\theta\sin\psi\dot{\delta} - \left[\cos\delta\cos\psi\cos\theta + \sin\delta\sin\theta\right]\left(\dot{\tau} + \Omega_t\right) \tag{5.49}$$

$$\tilde{q} = q + \left(\sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi\right)\dot{\delta} - \left[\cos\delta\left(\cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi\right) - \sin\delta\cos\theta\sin\phi\right]\left(\dot{\tau} + \Omega_t\right) \tag{5.50}$$

$$\tilde{r} = r + \left(\sin\psi\cos\phi\sin\theta - \cos\psi\sin\phi\right)\dot{\delta} - \left[\cos\delta\left(\sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi\right) - \sin\delta\cos\theta\cos\phi\right]\left(\dot{\tau} + \Omega_t\right) \tag{5.51}$$

I'll add question 2 of the 2017 exam later to this section.

5.2.3 *Euler angle singularity*

Now, you may look at equation (5.48) and think, but what happens at $\theta = 90°$? Does the whole world crash? Fortunately, no one dies then: it's simply a result of how we defined the Euler angles. It's not a physical phenomenon, obviously. It arises from the following: consider figure 5.2. An aircraft in vertical climb is oriented as shown in that figure. How did we get from the F_E-frame to this specific F_b-frame? Well, there are tons of answers to this question: one way would be to rotate $-90°$ about the Z_E-axis, then $90°$ about the $Y_{E'}$-axis. Or you could rotate $90°$ about the Y_E-axis to begin with, and then $90°$ about the $X_{E'}$-axis. There are infinitely many possibilities, which is this singularity.

Of course, for aircraft, a pitch angle of $90°$ will basically never happens, and if it happens, you're gonna die anyway so why even bother. However, in rocketry, it's obviously a bit more common; in that case, it may be helpful to define your equations of motion in a different reference frame (such as the aerodynamic one). However, it should be noted that the singularity will always exist, if it's not about the Y-axis then about a different one.

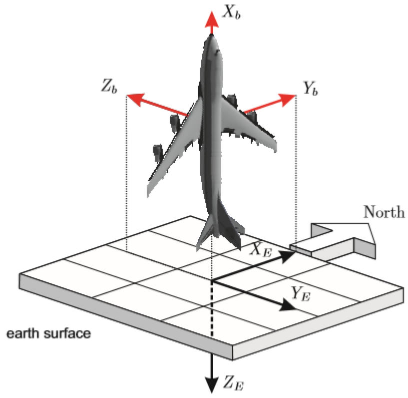


Figure 5.2: Euler angle singularity: aircraft in vertical climb.

5.3 Simplifications of the equations of motion

Now that we've seen the full set of equations of motion without any assumptions (except a few). We'll make two major assumptions:

- We assume a non-rotating Earth. This means that Ω_t reduces to zero. This will slightly simplify the dynamic equations of translational motion, as the apparent Coriolis and centrifugal forces drop out. But other than that it doesn't simplify a whole lot.
- We assume a flat Earth. This means that $R \to \infty$ (as the radius of curvature becomes infinity); consequently, from equation (5.24) and (5.25), $\dot{\delta} = \dot{\tau} = 0$. This hugely simplifies the kinematic equations of rotational motion; equations (5.49)-(5.51) reduce to simply $\tilde{p} = p$, $\tilde{q} = q$ and $\tilde{r} = r$. Beautiful stuff. The dynamic equations of rotational motion do not change, unfortunately.

5.3.1 Simplified equations of motion in the body frame

In the next chapter (which you can read about in part IV of this beautiful summary), we'll use the simplified equations of motion in the body frame. So, let's derive them here. Let's start with the equations of rotational motion for a change, because those are easy to transform as they are already in the body frame. Then, moving on to the equations of translational motion. For the dynamic equations of translational motion, it is easier to just rederive the whole thing to be honest. Since we have a non-rotating Earth, we simply have

$$\left.\frac{d\mathbf{V}_G}{dt}\right|_E^b = \mathbf{F}_{ext}^b \tag{5.52}$$

where the left-hand side is the time rate of change (derivative) of the velocity ($\mathbf{V}$) of the center of gravity (subscript G) in the F_E-frame (subscript E), expressed in the F_b-frame (subscript b). Let's first focus on the right-hand side of this equation. Let's assume the aerodynamic and propulsion force are combined into one single force with components X^b, Y^b and Z^b. Then the external force is given by

$$\mathbf{F}_{ext}^b = \mathbb{T}_{bE}\begin{bmatrix} 0 \\ 0 \\ mg_{r,0} \end{bmatrix} + \begin{bmatrix} X^b \\ Y^b \\ Z^b \end{bmatrix} = mg_{r,0}\begin{bmatrix} -\sin\theta \\ \sin\phi\cos\theta \\ \cos\phi\cos\theta \end{bmatrix} + \begin{bmatrix} X^b \\ Y^b \\ Z^b \end{bmatrix}$$

The left-hand side of the equation may be obtained as follows: the velocity component along the X_b-axis is denoted by u; the velocity component along the Y_b-axis is denoted by v and the velocity component along the Z_b-axis is denoted by w. This means that

$$\left.\frac{d\mathbf{V}_G}{dt}\right|_b^b = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

However, this is the rotating reference frame[4], an we want it with respect to the non-rotating vehicle-carried normal Earth frame, the F_E-frame[5]. We simply obtain

$$\left.\frac{d\mathbf{V}_G}{dt}\right|_E^b = \left.\frac{d\mathbf{V}_G}{dt}\right|_b^b + \mathbf{\Omega}_{bE}^b \times \mathbf{V}_G^b = \begin{bmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{bmatrix} + \begin{bmatrix}p\\ q\\ r\end{bmatrix} \times \begin{bmatrix}u\\ v\\ w\end{bmatrix} = \begin{bmatrix}\dot{u} + qw - rv\\ \dot{v} + ru - pw\\ \dot{w} + pv - qu\end{bmatrix}$$

Thus, our simplified dynamic equations of translational equation become

$$m\begin{bmatrix}\dot{u} + qw - rv\\ \dot{v} + ru - pw\\ \dot{w} + pv - qu\end{bmatrix} = mg_{r,0}\begin{bmatrix}-\sin\theta\\ \sin\phi\cos\theta\\ \cos\phi\cos\theta\end{bmatrix} + \begin{bmatrix}X\\ Y\\ Z\end{bmatrix}$$

The kinematic insights will now relate V_N, V_E and V_D to u, v and w, which will 'simply' be

$$\begin{aligned}\begin{bmatrix}V_N\\ V_E\\ V_D\end{bmatrix} &= \mathbb{T}_{Eb}\begin{bmatrix}u\\ v\\ w\end{bmatrix}\\ &= \begin{bmatrix}[u\cos\theta + (v\sin\phi + w\cos\phi)\sin\theta]\cos\psi - (v\cos\phi - w\sin\phi)\sin\psi\\ [u\cos\theta + (v\sin\phi + w\cos\phi)\sin\theta]\sin\psi + (v\cos\phi - w\sin\phi)\cos\psi\\ u\sin\theta - (v\sin\phi + w\cos\phi)\cos\theta\end{bmatrix}\end{aligned}$$

The equations of rotational motion do not change at all, as I already said. This means that we can finally close off this beautiful chapter with the simplified equations of motion:

[4] If you're like, but dude we just assumed non-rotating and flat Earth what the hell is still rotating now, it's the aircraft itself obviously, once it starts pitching, rolling and yawing.

[5] If we assume a flat, non-rotating Earth, this frame is merely translating over time, but it'll never rotate, as it's X_E-axis is defined to point North (so it can't rotate about Z_E or Y_E) and it's Z_B-axis always points down, but if the Earth is flat, this means it always points in the same direction, so it can't rotate at all, but only translate.

SIMPLIFIED EQUATIONS OF MOTION

The **simplified equations of motion** are given by the following sets of equations: the **simplified dynamic equations of translational motion** are given by

$$m(\dot{u} + qw - rv) = -mg_{r,0}\sin\theta + X \tag{5.53}$$
$$m(\dot{v} + ru - pw) = mg\sin\phi\cos\theta + Y \tag{5.54}$$
$$m(\dot{w} + pv - qu) = mg\sin\phi\sin\theta + Z \tag{5.55}$$

The **simplified kinematic equations of translational motion** are given by

$$V_N = [u\cos\theta + (v\sin\phi + w\cos\phi)\sin\theta]\cos\psi - (v\cos\phi - w\sin\phi)\sin\psi \tag{5.56}$$
$$V_E = [u\cos\theta + (v\sin\phi + w\cos\phi)\sin\theta]\sin\psi + (v\cos\phi - w\sin\phi)\cos\psi \tag{5.57}$$
$$V_D = u\sin\theta - (v\sin\phi + w\cos\phi)\cos\theta \tag{5.58}$$

The **dynamic equations of rotational motion**, for bodies for which the X_BZ_B-plane is a symmetry plane, are given by

$$M_x = I_{xx}\dot{p} + \left(I_{zz} - I_{yy}\right)qr - I_{xz}(\dot{r} + pq) \tag{5.40}$$
$$M_y = I_{yy}\dot{q} + \left(I_{xx} - I_{zz}\right)rp + I_{xz}\left(p^2 - r^2\right) \tag{5.41}$$
$$M_z = I_{zz}\dot{r} + \left(I_{yy} - I_{xx}\right)pq - I_{xz}(\dot{p} - rq) \tag{5.42}$$

for which an explicit solution for $\dot{p}$, $\dot{q}$ and $\dot{r}$ is given by

$$\dot{p} = \frac{I_{zz}}{I_{xx}I_{zz} - I_{xz}^2}M_x + \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2}M_z + \frac{\left(I_{xx} - I_{yy} + I_{zz}\right)I_{xz}}{I_{xx}I_{zz} - I_{xz}^2}pq + \frac{\left(I_{yy} - I_{zz}\right)I_{zz} - I_{xz}^2}{I_{xx}I_{zz} - I_{xz}^2}qr \tag{5.43}$$

$$\dot{q} = \frac{M_y}{I_{yy}} + \frac{I_{xz}}{I_{yy}}\left(r^2 - p^2\right) + \frac{I_{zz} - I_{xx}}{I_{yy}}pr \tag{5.44}$$

$$\dot{r} = \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2}M_z + \frac{I_{xx}}{I_{xx}I_{zz} - I_{xz}^2}M_z + \frac{\left(I_{xx} - I_{yy}\right)I_{xx} + I_{xz}^2}{I_{xx}I_{zz} - I_{xz}^2}pq + \frac{\left(-I_{xx} + I_{yy} - I_{zz}\right)I_{xz}}{I_{xx}I_{zz} - I_{xz}^2}qr \tag{5.45}$$

The **kinematic equations of rotational motion** are given by

$$\dot{\phi} = p + \sin\phi\tan\theta q + \cos\phi\tan\theta r \tag{5.46}$$
$$\dot{\theta} = \cos\phi q - \sin\phi r \tag{5.47}$$
$$\dot{\psi} = \frac{\sin\phi}{\cos\theta}q + \frac{\cos\phi}{\cos\theta}r \tag{5.48}$$

5.3.2 Overview of assumptions made

To close off this chapter, let's have an overview of the assumptions that were either implicitly or explicitly made:

ASSUMPTIONS MADE FOR THE SIMPLIFIED EQUATIONS OF MOTION

1. Spherical Earth: for the gravity force, the Earth is assumed to be a perfect sphere, such that gravity perturbations do not need to be taken into account.
2. Flat Earth: the Earth's curvature is neglected, i.e. $R \to \infty$, so that $\dot{\delta} \to 0$ and $\dot{\tau} \to 0$.
3. Non-rotating Earth: the Coriolis and centrifugal accelerations are neglected.
4. Constant mass: to describe the motion at a specific instant of time, the mass is assumed to be assumed constant over that very short period of time. Otherwise you'd have to take the impulse of the propellant into account.
5. Rigid body: no parts of the body are moving, i.e. the flaps are not extending or retracting.

6. No rotating masses: turbines, propellers, etc. are ignored, as these would add gyroscopic forces.
7. Symmetric aircraft: no exotic configurations are chosen such as unsymmetric aircraft.
8. Constant wind (if at all): this means we exclude wind shear and turbulence from our velocities u, v and w.
9. Constant gravity: we neglect altitude variations of graivty in our analysis.
10. We choose F_b such that the $X_B Z_B$-plane is a symmetry plane, such that $I_{xy} = I_{yz} = 0$. This means we neglect mass asymmetries.

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