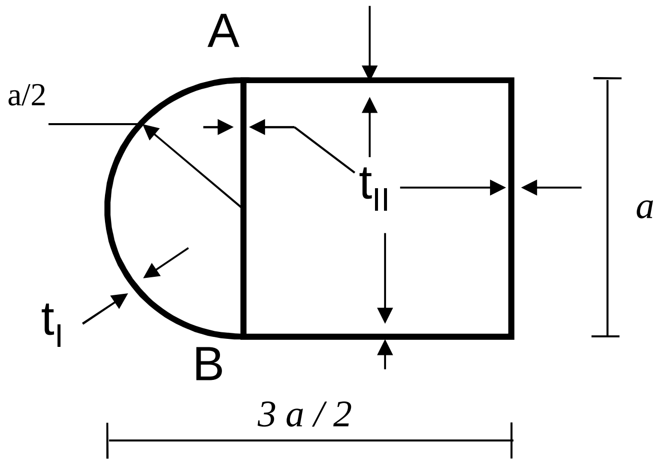
Torsion of multi-cell cross-sections - Example



For the 2-cell cross-section shown under torque T, with thickness t_I for the curved portion and t_{II} everywhere else, find the relation between t_I and t_{II} so that she shear flow in the front straight web AB is zero

- first determine the shear flows q_I and q_{II} (the curved cell is I and the square cell is II)
- to determine the shear flows we use:
  - torque equivalence
  - equality of rates of twist

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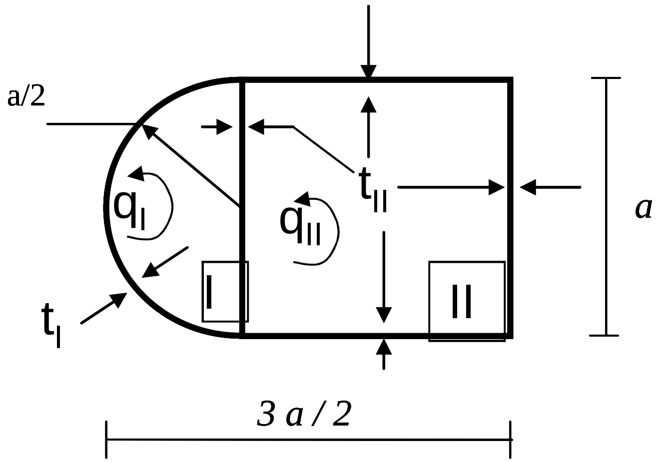

- the areas of cells I and II are:

$$A_I = \pi \frac{a^2}{8} \qquad A_{II} = a^2$$

- the torque in each cell is given by:

$$T_I = 2A_I q_I \qquad T_{II} = 2A_{II} q_{II}$$

- the rate of twist in cell I is (note both cells have the same material so G=const)

$$\left(\frac{d\theta}{dz}\right)_I = \frac{1}{2A_I}\oint \frac{q\,ds}{tG} = \frac{1}{2A_I G}\left[\int_0^{\pi} \frac{q_I \frac{a}{2} d\theta}{t_I} + \int_0^{a} \frac{(q_I - q_{II})\,ds}{t_{II}}\right] = \frac{1}{2A_I G}\left(\frac{q_I \pi a}{2t_I} + \frac{(q_I - q_{II})\,a}{t_{II}}\right)$$

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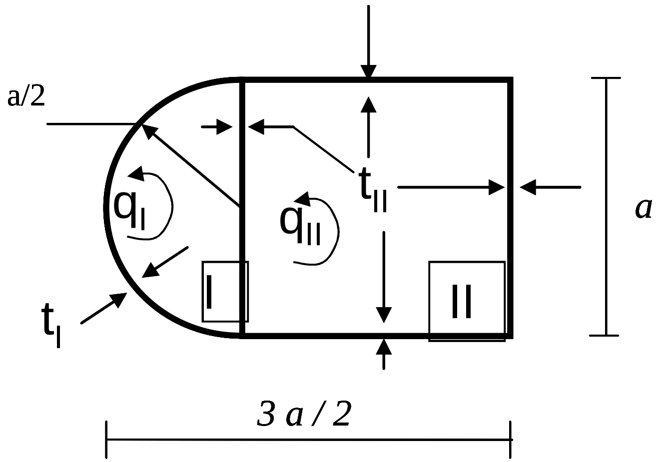

$$\left(\frac{d\theta}{dz}\right)_I = \frac{1}{2A_I G}\left(\frac{q_I \pi a}{2t_I} + \frac{(q_I - q_{II})a}{t_{II}}\right)$$

- the rate of twist in cell II is

$$\left(\frac{d\theta}{dz}\right)_{II} = \frac{1}{2A_{II}}\oint\frac{qds}{tG} = \frac{1}{2A_{II}G}\left[\int_0^{3a}\frac{q_{II}ds}{t_{II}} + \int_0^{a}\frac{(q_{II}-q_I)ds}{t_{II}}\right] = \frac{1}{2A_{II}G}\left(\frac{q_{II}3a}{t_{II}} + \frac{(q_{II}-q_I)a}{t_{II}}\right)$$

- since the rates of twist are equal

$$\left(\frac{d\theta}{dz}\right)_I = \left(\frac{d\theta}{dz}\right)_{II} = \frac{d\theta}{dz} \Rightarrow \frac{1}{2A_I G}\left(\frac{q_I \pi a}{2t_I} + \frac{(q_I - q_{II})a}{t_{II}}\right) = \frac{1}{2A_{II}G}\left(\frac{q_{II}3a}{t_{II}} + \frac{(q_{II}-q_I)a}{t_{II}}\right)$$

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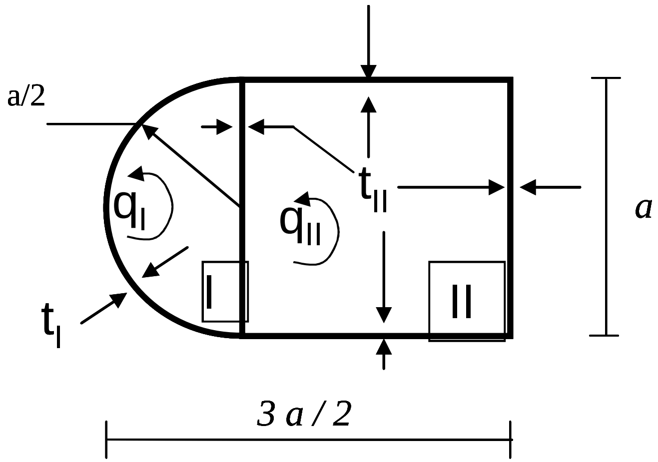


$$T_I = 2A_I q_I \qquad T_{II} = 2A_{II} q_{II}$$

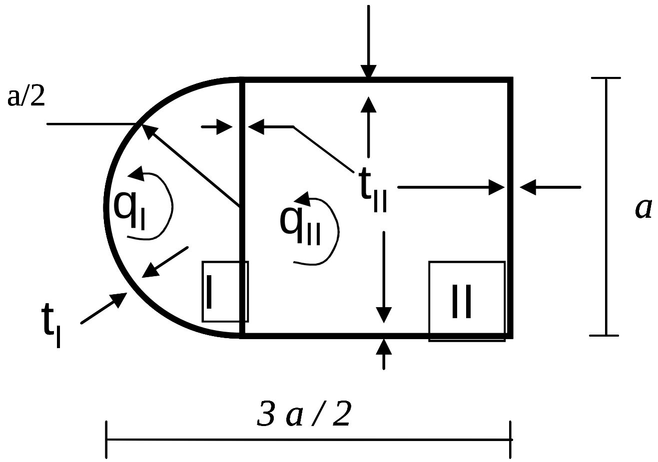
$$\frac{1}{2A_I G}\left(\frac{q_I \pi a}{2t_I} + \frac{(q_I - q_{II})a}{t_{II}}\right) = \frac{1}{2A_{II} G}\left(\frac{q_{II} 3a}{t_{II}} + \frac{(q_{II} - q_I)a}{t_{II}}\right)$$

• now torque equivalence requires that the torques from the two cells add up (or are equivalent to) the total torque T:

$$T = T_I + T_{II} \Rightarrow T = 2A_I q_I + 2A_{II} q_{II}$$

two equations in the two unknowns q_I and q_{II}

Torsion of multi-cell cross-sections - Example



$$\frac{1}{2A_I G}\left(\frac{q_I \pi a}{2t_I}+\frac{(q_I-q_{II})a}{t_{II}}\right)=\frac{1}{2A_{II}G}\left(\frac{q_{II}3a}{t_{II}}+\frac{(q_{II}-q_I)a}{t_{II}}\right)$$

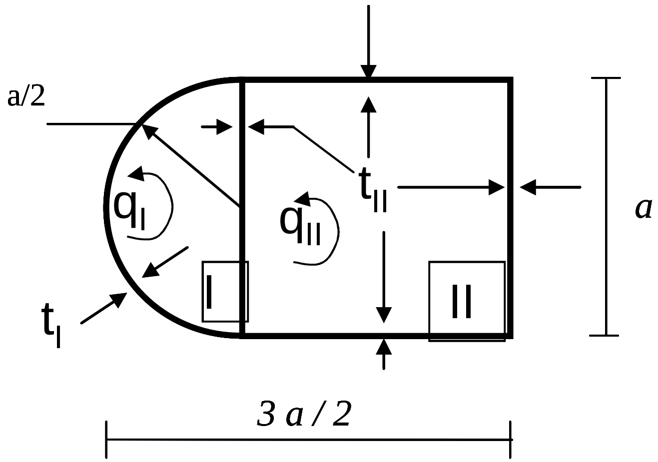
$$T = T_I + T_{II} \Rightarrow T = 2A_I q_I + 2A_{II} q_{II}$$

- to solve, simplify the first equation:

$$\frac{1}{A_I}\left(\frac{q_I \pi}{2t_I}+\frac{(q_I-q_{II})}{t_{II}}\right)=\frac{1}{A_{II}}\left(\frac{3q_{II}}{t_{II}}+\frac{(q_{II}-q_I)}{t_{II}}\right)\Rightarrow$$

$$q_I\left(\frac{1}{A_I}\left(\frac{\pi}{2t_I}+\frac{1}{t_{II}}\right)+\frac{1}{A_{II}t_{II}}\right)=q_{II}\left(\frac{1}{A_I t_{II}}+\frac{4}{A_{II}t_{II}}\right)$$

Torsion of multi-cell cross-sections - Example



$$q_I\left(\frac{1}{A_I}\left(\frac{\pi}{2t_I}+\frac{1}{t_{II}}\right)+\frac{1}{A_{II}t_{II}}\right)=q_{II}\left(\frac{1}{A_I t_{II}}+\frac{4}{A_{II}t_{II}}\right)$$

$$T=2A_I q_I+2A_{II}q_{II}$$

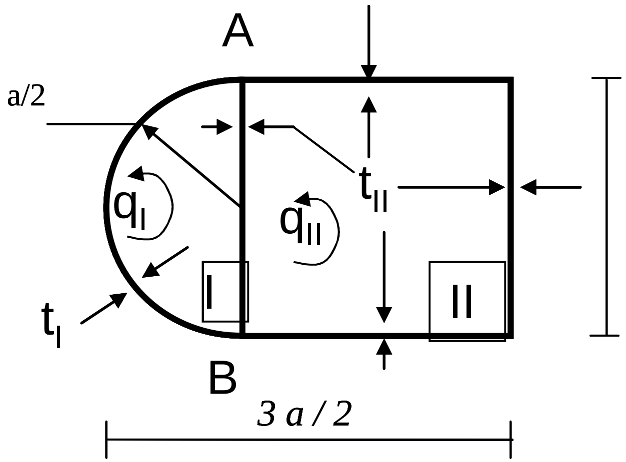
use the first to solve for q_{II} as a function of q_I and substitute in the second and solve for q_I

- this leads to: $$q_I=\frac{T}{2}\frac{4A_I+A_{II}}{\left(A_I+A_{II}\right)^2+3A_I{}^2+A_{II}{}^2\frac{\pi}{2}\frac{t_{II}}{t_I}}$$

- and substituting in the first to solve for q_{II}:

$$q_{II}=\frac{T}{2}\frac{A_I+A_{II}\left(\frac{\pi}{2}\frac{t_{II}}{t_I}+1\right)}{\left(A_I+A_{II}\right)^2+3A_I{}^2+A_{II}{}^2\frac{\pi}{2}\frac{t_{II}}{t_I}}$$

Torsion of multi-cell cross-sections - Example



$$q_I = \frac{T}{2} \frac{4A_I + A_{II}}{\left(A_I + A_{II}\right)^2 + 3A_I{}^2 + A_{II}{}^2 \frac{\pi}{2} \frac{t_{II}}{t_I}}$$

$$q_{II} = \frac{T}{2} \frac{A_I + A_{II}\left(\frac{\pi}{2} \frac{t_{II}}{t_I} + 1\right)}{\left(A_I + A_{II}\right)^2 + 3A_I{}^2 + A_{II}{}^2 \frac{\pi}{2} \frac{t_{II}}{t_I}}$$

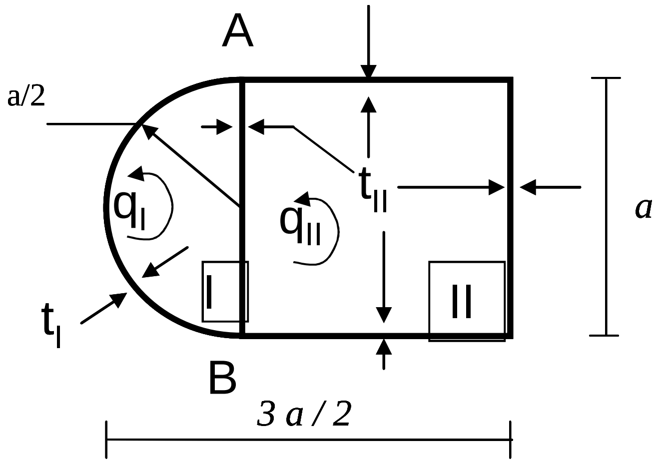
- now that the shear flows are determined, making the shear flow in AB equal to zero means q_I-q_{II}=0; this gives

$$4A_I + A_{II} = A_I + A_{II}\left(\frac{\pi}{2} \frac{t_{II}}{t_I} + 1\right)$$

- which simplifies to:

$$3A_I = A_{II} \frac{\pi}{2} \frac{t_{II}}{t_I}$$

Torsion of multi-cell cross-sections - Example



$$3A_I = A_{II}\,\frac{\pi}{2}\frac{t_{II}}{t_I}$$

$$A_I = \pi\frac{a^2}{8} \qquad A_{II} = a^2$$

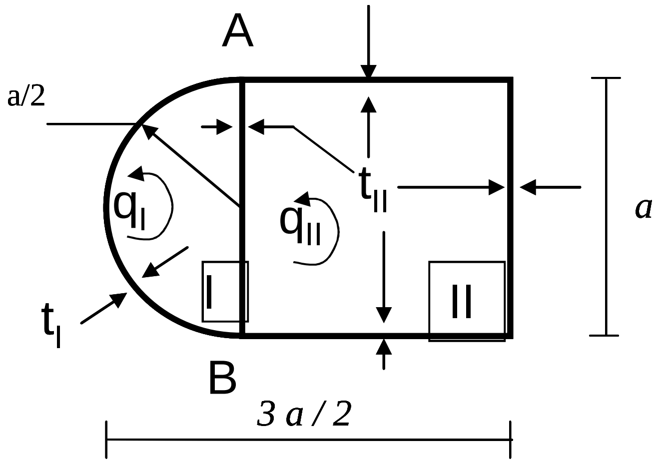
- using the expressions for A_I and A_{II} we can solve for t_{II} as a function of t_I:

$$3\frac{\pi}{8}a^2 = a^2\,\frac{\pi}{2}\frac{t_{II}}{t_I} \Rightarrow t_{II} = \frac{3}{4}t_I$$

interestingly, this relation is independent of a !!

which guarantees that $q_{AB}=0$

Torsion of multi-cell cross-sections - Example



$$3\frac{\pi}{8}a^2 = a^2\frac{\pi}{2}\frac{t_{II}}{t_I} \Rightarrow t_{II} = \frac{3}{4}t_I$$

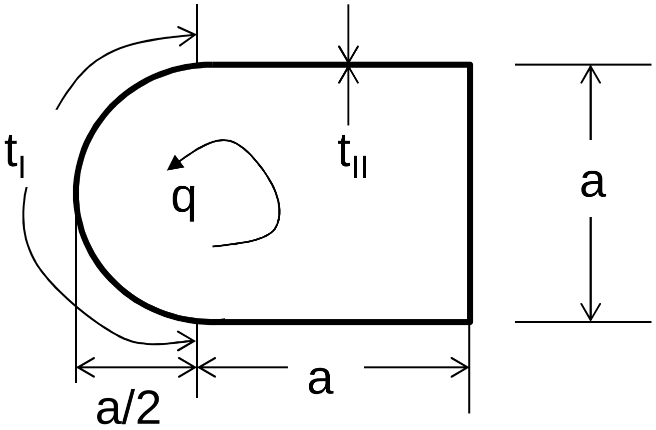
$$q_I = q_{II} = q = \frac{T}{2a^2}\frac{\left(\frac{\pi}{2}+1\right)}{\frac{\pi^2}{16}+1+\frac{5\pi}{8}} = \underbrace{\frac{T}{2a^2}\frac{1}{\frac{\pi}{8}+1}}$$

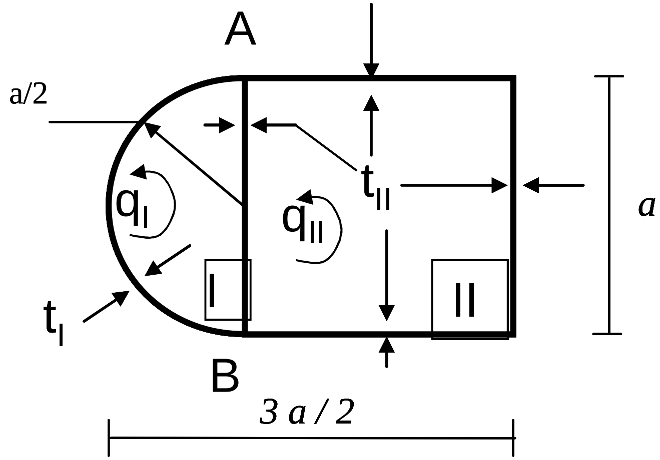
this is exactly the same answer we would have gotten from q=T/(2A) if AB were not present

- with q_{AB}=0, the shear flows q_I and q_{II} are equal and the two cells act as if they were a single cell (consisting of the outer skins) with constant shear flow; this means the vertical web AB can be removed; but is this saving us weight?

Torsion of multi-cell cross-sections - Example

- compare the weight of the following cases:





$$t_{II} = \frac{3}{4} t_I$$

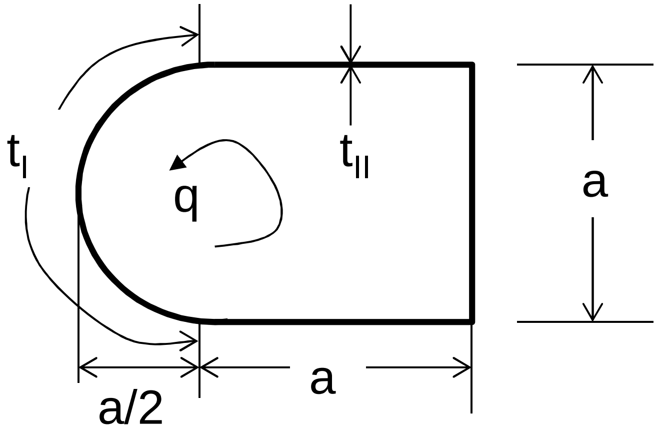
$$q = \frac{T}{2a^2} \frac{1}{\frac{\pi}{8} + 1}$$

$$q_I = \frac{T}{2} \frac{4A_I + A_{II}}{\left(A_I + A_{II}\right)^2 + 3A_I{}^2 + A_{II}{}^2 \frac{\pi}{2} \frac{t_{II}}{t_I}}$$

$$q_{II} = \frac{T}{2} \frac{A_I + A_{II}\left(\frac{\pi}{2} \frac{t_{II}}{t_I} + 1\right)}{\left(A_I + A_{II}\right)^2 + 3A_I{}^2 + A_{II}{}^2 \frac{\pi}{2} \frac{t_{II}}{t_I}}$$

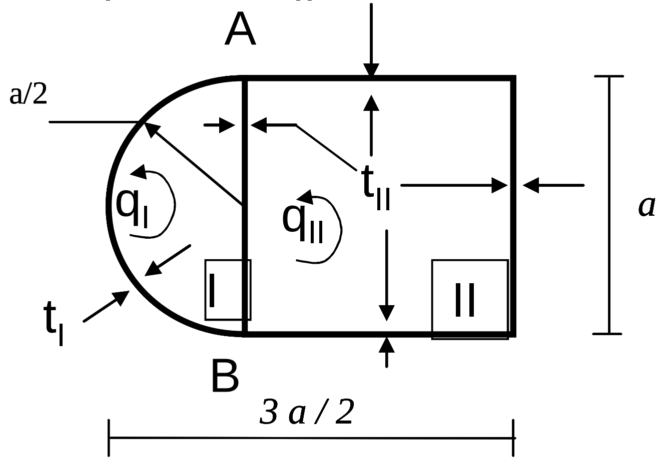
Torsion of multi-cell cross-sections - Example

- or, if we substitute for A_I and A_{II} in the q_I and q_{II} expressions



$$t_{II} = \frac{3}{4} t_I$$

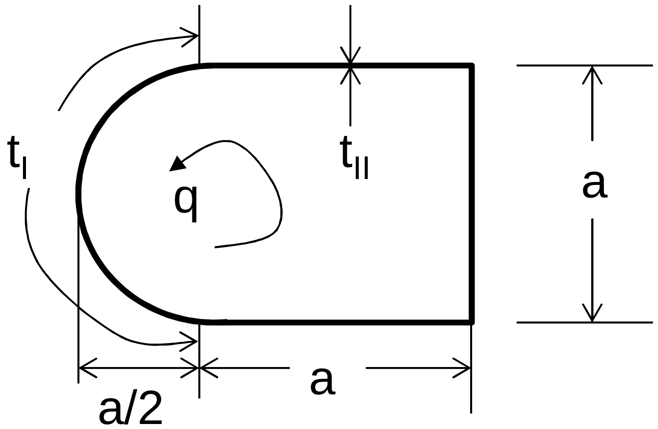
$$q = \frac{T}{2a^2} \frac{1}{\frac{\pi}{8} + 1}$$

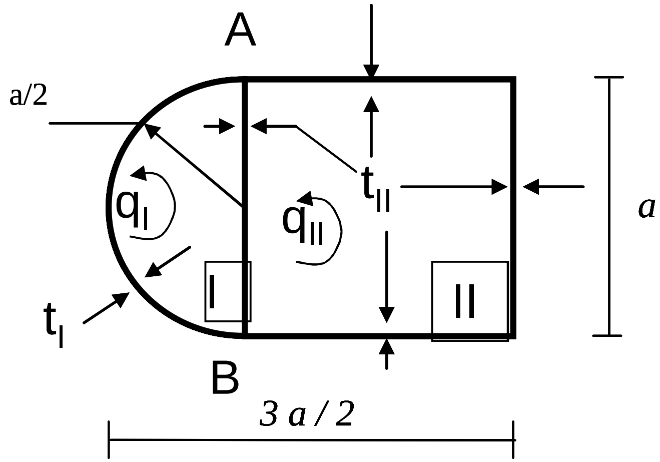


$$q_I = \frac{T}{2a^2} \frac{\frac{\pi}{2} + 1}{\frac{\pi^2}{16} + \frac{\pi}{4} + 1 + \frac{\pi}{2}\frac{t_{II}}{t_I}}$$

$$q_{II} = \frac{T}{2a^2} \frac{\frac{\pi}{8} + \frac{\pi}{2}\frac{t_{II}}{t_I} + 1}{\frac{\pi^2}{16} + \frac{\pi}{4} + 1 + \frac{\pi}{2}\frac{t_{II}}{t_I}}$$

Torsion of multi-cell cross-sections - Example

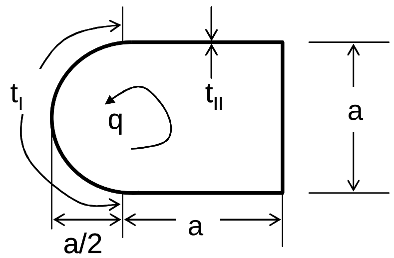


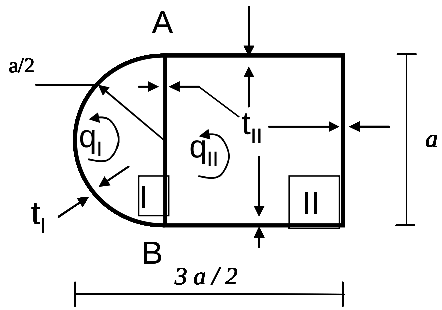


• note that in the first case (without the front web) we know the ratio of the thicknesses t_{II}/t_I and the shear flow q

• in the second case (with the front web) we only know the two shear flows q_I and q_{II} but know nothing about the thicknesses

Torsion of multi-cell cross-sections - Example

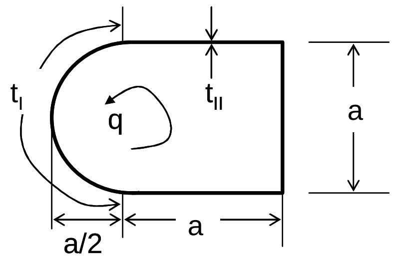


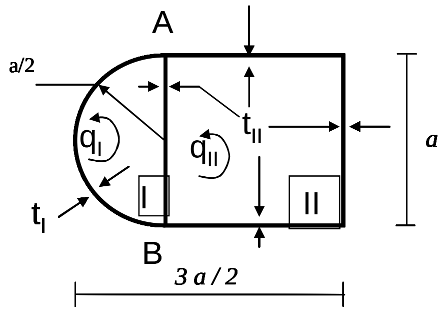


- in order to determine the best weight for each option we have to design it so it does not fail under the applied torque T

- the design means that we pick a material (the same for all skins and webs and for both options) and then determine the thicknesses everywhere such that there is no failure and the weight is as low as possible

Torsion of multi-cell cross-sections – Example: Min weight design





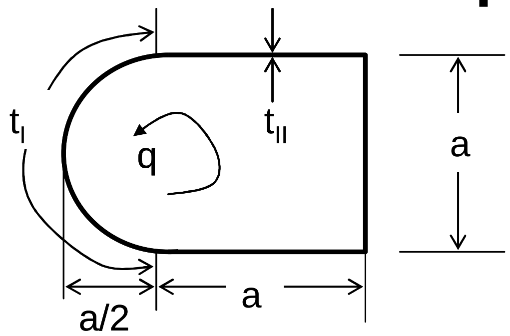
• the weight is proportional to the area of each cross-section (not the enclosed area)

• to determine the cross-sectional area we need the thickness of each skin or web segment

• the most efficient design (from a min weight point of view) is the one that just fails when the applied torque T is reached

• this means that the **highest** shear stress in any skin or web equals the shear yield stress of the material τ_y

Torsion of multi-cell cross-sections – Example: Min weight design



• option 1: no fwd web

$$q = \frac{T}{2a^2}\frac{1}{\frac{\pi}{8}+1}$$

• the applied shear stress equals the shear flow q divided by the thickness of the segment

• since the shear flow is constant, the part that will fail first is the one with the lowest thickness (because it has highest shear stress); thus, the shear stress in the aft square part is:

$$\tau_{II} = \frac{q}{t_{II}} \Rightarrow \tau_{II} = \frac{1}{t_{II}}\frac{T}{2a^2}\frac{1}{\frac{\pi}{8}+1}$$

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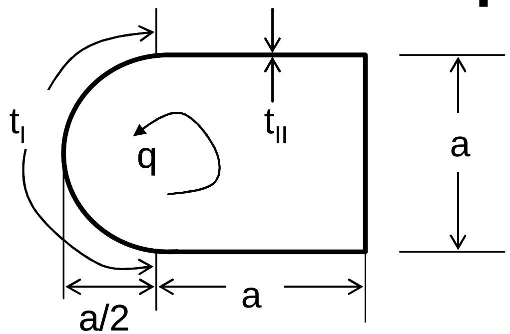

$$\tau_{II} = \frac{q}{t_{II}} \Rightarrow \tau_{II} = \frac{1}{t_{II}} \frac{T}{2a^2} \frac{1}{\frac{\pi}{8}+1}$$

$$t_{II} = \frac{3}{4} t_I$$

- at failure, $\tau_{II} = \tau_y$ so:

$$\tau_y = \frac{1}{t_{II}} \frac{T}{2a^2} \frac{1}{\frac{\pi}{8}+1}$$

- solving for t_{II}:

$$t_{II} = \frac{T}{2a^2 \tau_y} \frac{1}{\frac{\pi}{8}+1}$$

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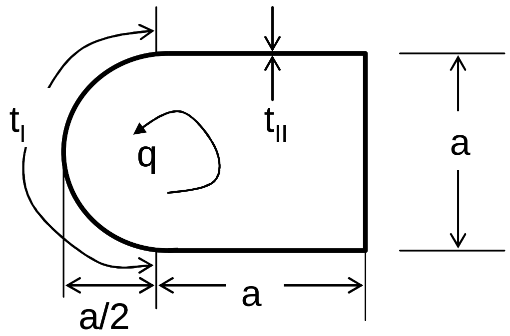


$$t_{II} = \frac{T}{2a^2\tau_y}\frac{1}{\frac{\pi}{8}+1} \qquad t_{II} = \frac{3}{4}t_I$$

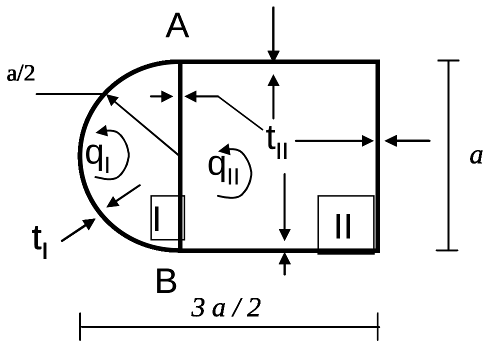
- the weight is then given by (for beam length L)

$$W_1 = \rho L\left[\frac{\pi a}{2}t_I + 3at_{II}\right] = \rho La\left[\frac{\pi}{2}\frac{4}{3}\frac{T}{2a^2\tau_y}\frac{1}{\frac{\pi}{8}+1} + 3\frac{T}{2a^2\tau_y}\frac{1}{\frac{\pi}{8}+1}\right] = \rho La\frac{T}{2a^2\tau_y}\frac{1}{\frac{\pi}{8}+1}\left[\frac{2\pi}{3}+3\right] \Rightarrow$$

$$W_1 = \rho L\frac{T}{6a\tau_y}\frac{(2\pi+9)}{\frac{\pi}{8}+1}$$

- or substituting values: $W_1 = (1.83)\rho L\dfrac{T}{a\tau_y}$

Torsion of multi-cell cross-sections – Example: Min weight design

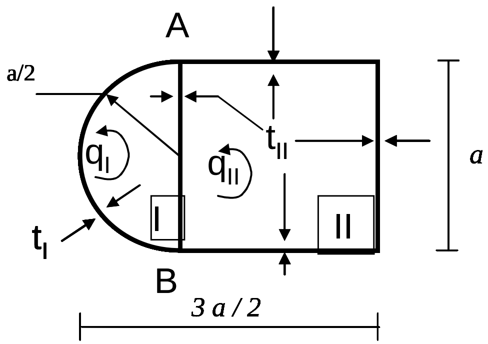


- Option 2: with fwd web

$$q_I = \frac{T}{2a^2} \frac{\frac{\pi}{2}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}} \qquad q_{II} = \frac{T}{2a^2} \frac{\frac{\pi}{8}+\frac{\pi}{2}\frac{t_{II}}{t_I}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}}$$

- we have three different portions to consider:
  - circular portion (leading edge) with q_I and thickness t_I
  - aft portion (except for the front web) with q_{II} and thickness t_{II}
  - fwd web with q_I-q_{II} and thickness t_{II}

Torsion of multi-cell cross-sections – Example: Min weight design



$$q_I = \frac{T}{2a^2}\frac{\frac{\pi}{2}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}} \qquad q_{II} = \frac{T}{2a^2}\frac{\frac{\pi}{8}+\frac{\pi}{2}\frac{t_{II}}{t_I}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}}$$

- for the front circular portion, equating the shear stress to the material yield shear stress:

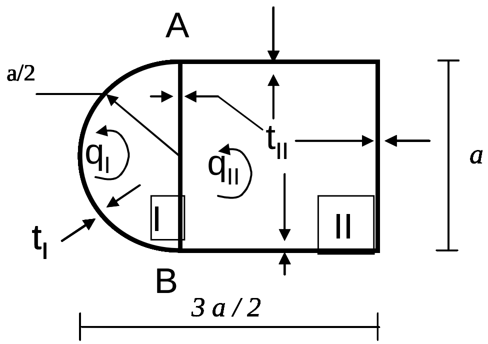
$$\frac{q_I}{t_I} = \tau_y \Rightarrow \frac{1}{t_I}\frac{T}{2a^2}\frac{\frac{\pi}{2}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}} = \tau_y$$

- and solving for t_I

$$t_I = \frac{T}{2a^2\tau_y}\frac{\frac{\pi}{2}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}}$$

note that t_I appears on both sides of the equation!

Torsion of multi-cell cross-sections – Example: Min weight design



$$q_I = \frac{T}{2a^2}\frac{\frac{\pi}{2}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}} \qquad q_{II} = \frac{T}{2a^2}\frac{\frac{\pi}{8}+\frac{\pi}{2}\frac{t_{II}}{t_I}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}}$$

- for the aft square section (without the front web), the failure condition gives:

$$\frac{q_{II}}{t_{II}} = \tau_y \Rightarrow \frac{1}{t_{II}}\frac{T}{2a^2}\frac{\frac{\pi}{8}+\frac{\pi}{2}\frac{t_{II}}{t_I}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}} = \tau_y$$

- and solving for t_{II} (in terms of t_{II}/t_I),

$$t_{II} = \frac{T}{2a^2\tau_y}\frac{\frac{\pi}{8}+\frac{\pi}{2}\frac{t_{II}}{t_I}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}}$$

note that t_{II} appears on both sides of the equation!

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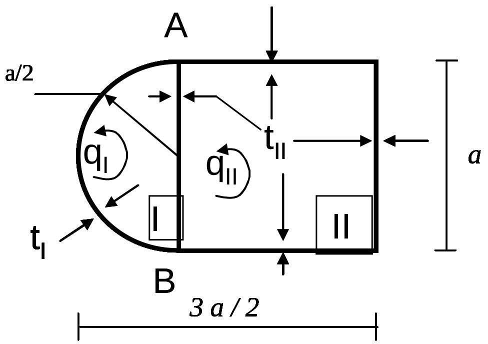


$$t_I = \frac{T}{2a^2\tau_y}\frac{\frac{\pi}{2}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}} \qquad t_{II} = \frac{T}{2a^2\tau_y}\frac{\frac{\pi}{8}+\frac{\pi}{2}\frac{t_{II}}{t_I}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}}$$

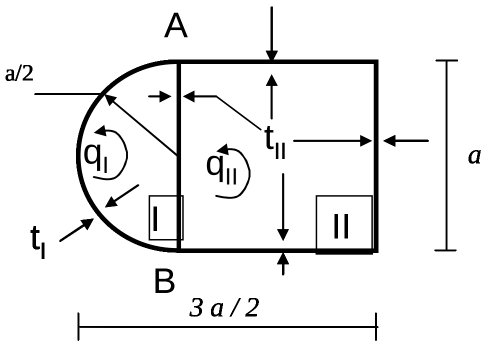
- if we divide the second equation by the first we get an equation with only one unknown, t_{II}/t_I

$$\frac{t_{II}}{t_I} = \frac{\frac{\pi}{8}+\frac{\pi}{2}\frac{t_{II}}{t_I}+1}{\frac{\pi}{2}+1}$$

- which is solved for t_{II}/t_I

$$\frac{t_{II}}{t_I} = \frac{\pi}{8}+1$$

Torsion of multi-cell cross-sections – Example: Min weight design



$$t_I = \frac{T}{2a^2\tau_y}\,\frac{\frac{\pi}{2}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}} \qquad t_{II} = \frac{T}{2a^2\tau_y}\,\frac{\frac{\pi}{8}+\frac{\pi}{2}\frac{t_{II}}{t_I}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}}$$

$$\frac{t_{II}}{t_I} = \frac{\pi}{8}+1$$

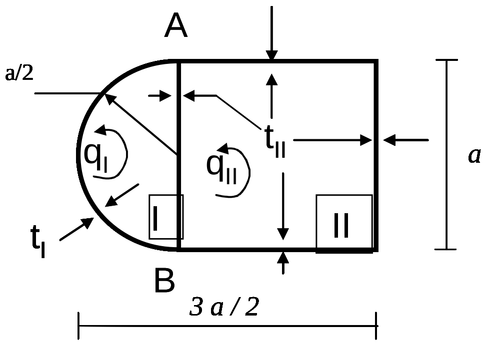
• substituting in the expressions for t_I and t_{II} and performing the calculations:

$$t_I = 0.28\frac{T}{a^2\tau_y} \qquad t_{II} = 0.39\frac{T}{a^2\tau_y}$$

both leading edge and aft square (minus fwd spar) fail simultaneously

• before calculating the weight for this case we must make sure that the fwd web with shear flow q_I-q_{II} does not fail under T when its thickness is the t_{II} value just calculated

Torsion of multi-cell cross-sections – Example: Min weight design



$$q_I = \frac{T}{2a^2}\frac{\frac{\pi}{2}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}} \qquad q_{II} = \frac{T}{2a^2}\frac{\frac{\pi}{8}+\frac{\pi}{2}\frac{t_{II}}{t_I}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}}$$

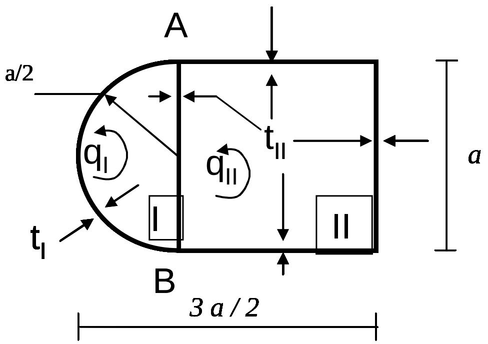
$$\frac{t_{II}}{t_I} = \frac{\pi}{8}+1$$

- examine the quantity q_{II}-q_I:

$$q_{II}-q_I = \frac{T}{2a^2}\frac{\frac{\pi}{8}+\frac{\pi}{2}\frac{t_{II}}{t_I}+1-\left(\frac{\pi}{2}+1\right)}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}} = \frac{T}{2a^2}\frac{\frac{\pi}{8}\left(\frac{\pi}{2}+1\right)}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}}$$

- this quantity is smaller than q_{II} !

Torsion of multi-cell cross-sections – Example: Min weight design



$$q_I = \frac{T}{2a^2} \frac{\frac{\pi}{2}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}} \qquad q_{II} = \frac{T}{2a^2} \frac{\frac{\pi}{8}+\frac{\pi}{2}\frac{t_{II}}{t_I}+1}{\frac{\pi^2}{16}+\frac{\pi}{4}+1+\frac{\pi}{2}\frac{t_{II}}{t_I}}$$

$$\frac{t_{II}}{t_I} = \frac{\pi}{8}+1 \qquad t_I = 0.28\frac{T}{a^2\tau_y} \qquad t_{II} = 0.39\frac{T}{a^2\tau_y}$$

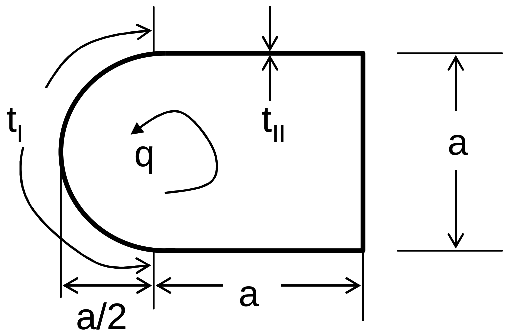
- since q_I-q_{II}<q_{II} (in absolute value), the fwd web with thickness t_{II} does not fail under q_I-q_{II} (because the rest of the aft square with thickness t_{II} under q_{II}, which is greater than q_I-q_{II}, does not fail either)

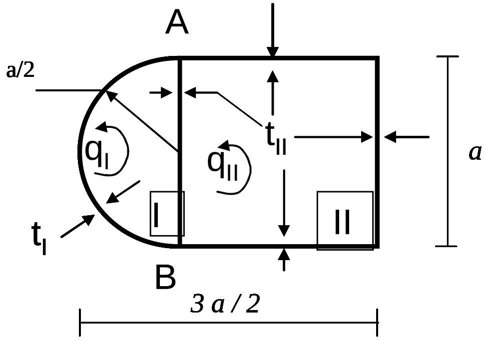
- substituting in the weight expression:

$$W_2 = \rho L\left[\frac{\pi a}{2}t_I + 4at_{II}\right] \Rightarrow W_2 = \rho L\frac{T}{a^2\tau_y}\left[\frac{\pi a}{2}(0.28)+4a(0.39)\right] \Rightarrow W_2 = 2\rho L\frac{T}{a\tau_y}$$

note this is 4 instead of 3 it was before because the front web is present!

Torsion of multi-cell cross-sections – Example: Min weight design



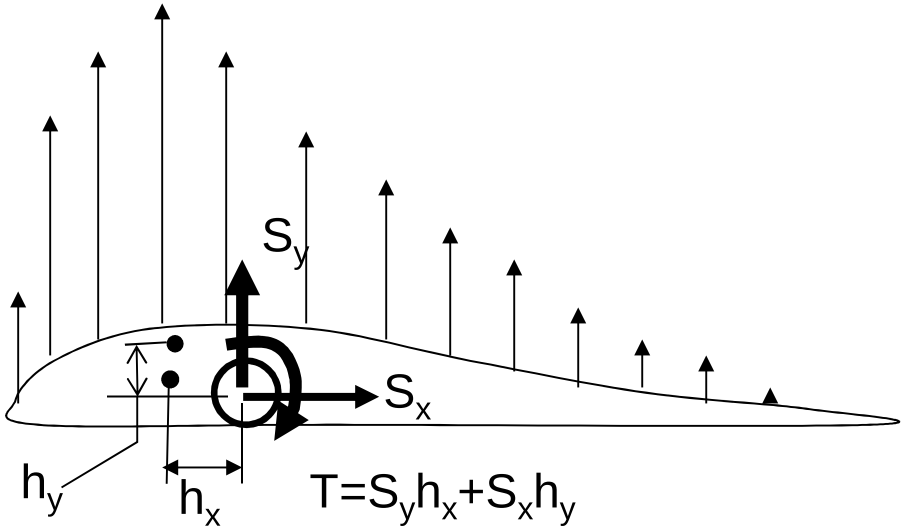


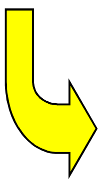
$$W_1 = (1.83)\rho L \frac{T}{a\tau_y}$$

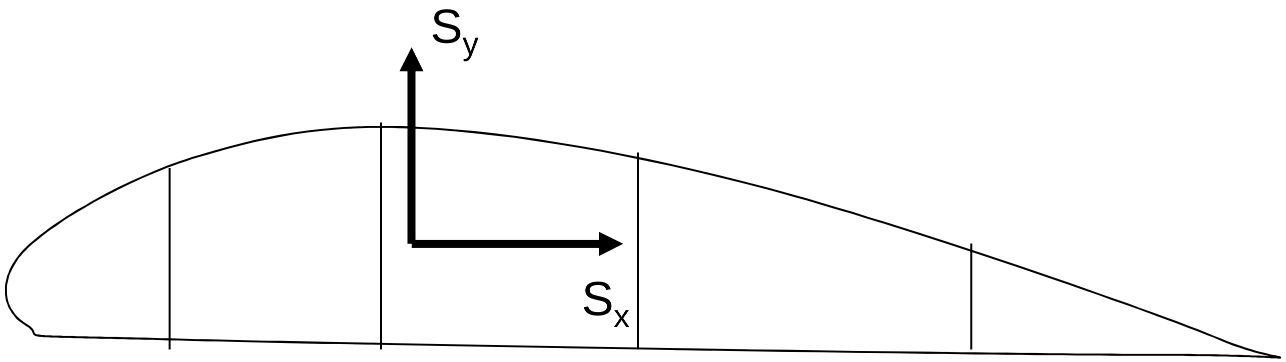
$$W_2 = 2\rho L \frac{T}{a\tau_y}$$

- therefore, removing the front web decreases the weight (for the case of pure torsion) by 8.5%
- conclusion: for pure torsion cases it pays to have a single cell

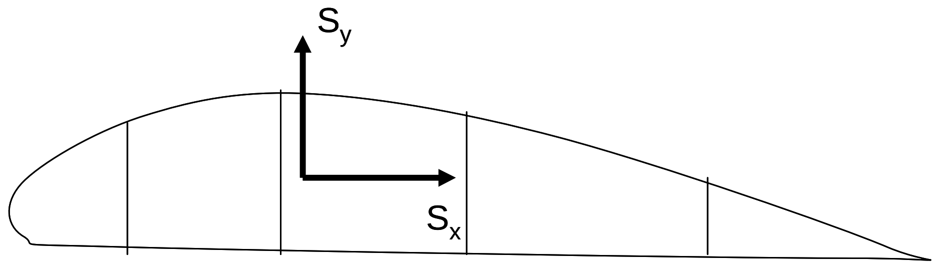
Shear of multi-cell beam





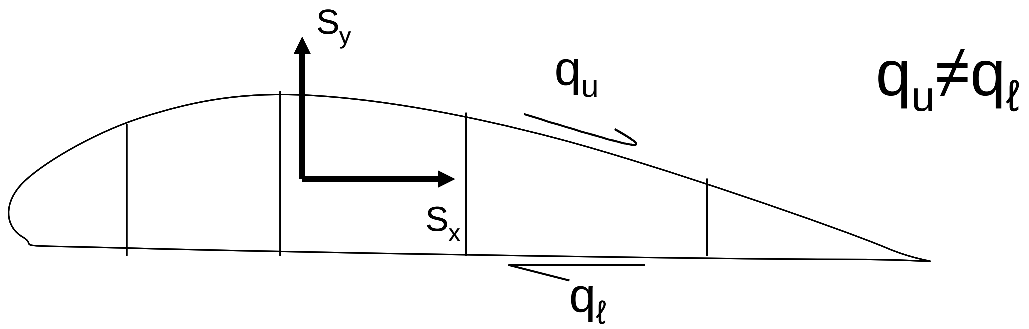


Shear of multi-cell beam



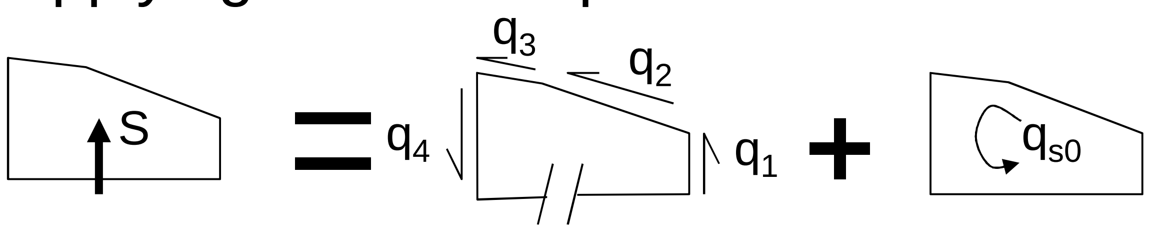
- the two shear forces S_y and S_x represent net lift (Lift-Weight) and drag respectively
- S_y and S_x do not necessarily pass through the shear center so they also cause a torque
- the torque portion of the problem was just examined before (this and previous lecture)

Shear of multi-cell beam

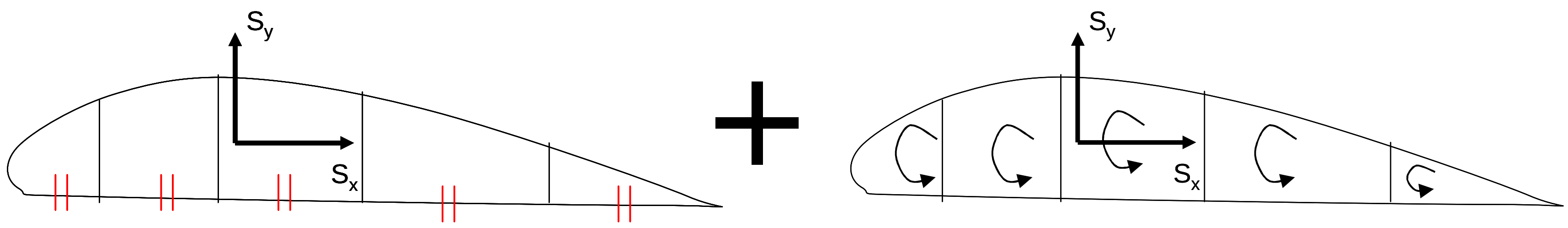


• note that, unlike the case of pure torque where the upper and lower shear flows in each cell were the same, here they are, in general, different

• for a single cell, the shear flows were determined by cutting at a convenient place solving for the shear flows, then closing the cut and adding a constant shear flow and applying moment equivalence



Shear of multi-cell beam



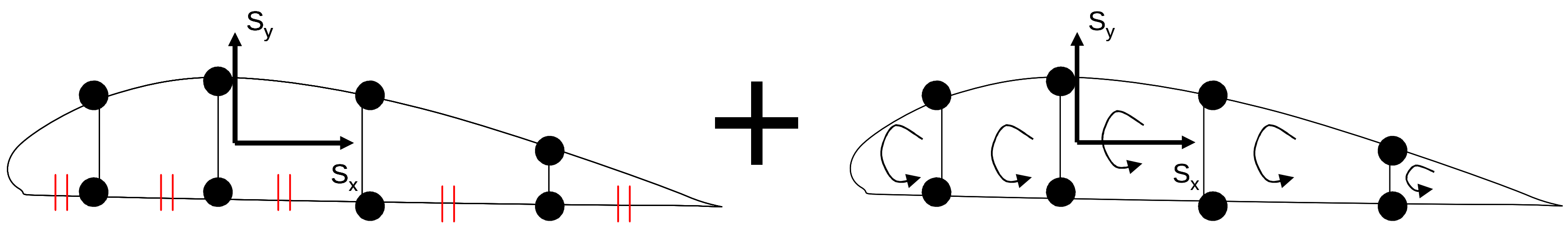
- in an analogous fashion, each cell is cut at a convenient point and the procedure is repeated for all cells

- the cells are then closed and constant shear flows (different for each cell) are added

- the procedure leads to a system of equations

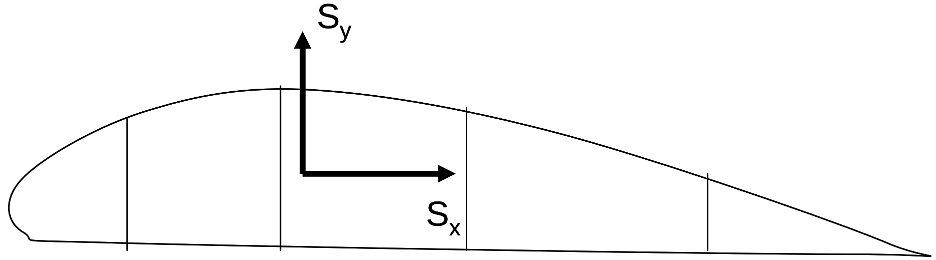
- to simplify things, the cuts should be made near the mid-point of the upper or lower skins; this makes the q_{s0} shear flows (in each cell) small and avoids numerical problems; otherwise we end up subtracting large numbers which requires carrying a lot of significant digits

Shear of multi-cell beam



• even in idealized structure with booms where the shear flows are constant between booms, it pays to cut at the middle of upper (or lower) skins **when only a vertical shear is applied**: then the shear flow at the cut skin is zero and, from horizontal equilibrium, the shear flow at the opposite skin is also zero => less unknowns to solve for

Shear of multi-cell beam



• for each cut cell, the shear flows are determined by (see lecture 7)

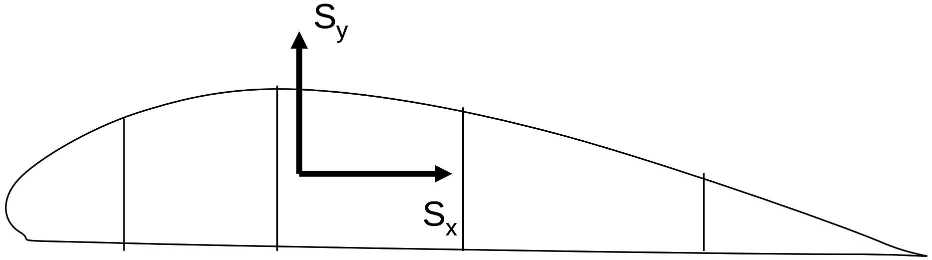
$$q_s = -\frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s txds - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2}\int_0^s tyds \qquad (5.6)$$

if the skins can carry bending loads or, (see lecture 9)

$$q_s = -\frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2}\left[\int_0^s t_D xds + \sum_{r=1}^{n} B_r x_r\right] - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2}\left[\int_0^s t_D yds + \sum_{r=1}^{n} B_r y_r\right] \qquad (7.12)$$

if the skins are idealized and booms are present (t_D=0)

Shear of multi-cell beam



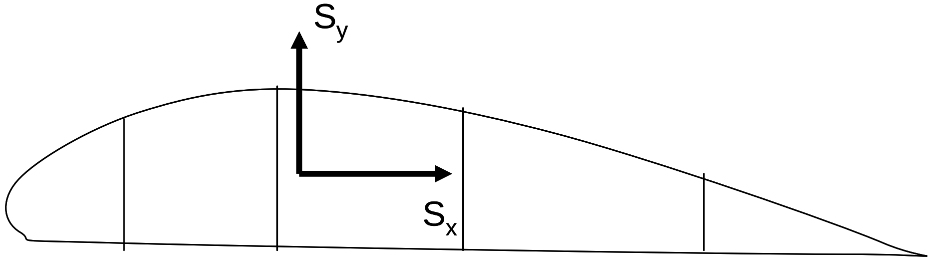
• for each closed cell, we have one additional unknown, the constant shear flow q_{s0}

• in addition, since the shear forces do not necessarily act through the shear center, there is a rate of twist $d\theta/dz$ common to all cells

• so for n cells, there are n unknown q_{s0} values **plus** $d\theta/dz$ or n+1 unknowns

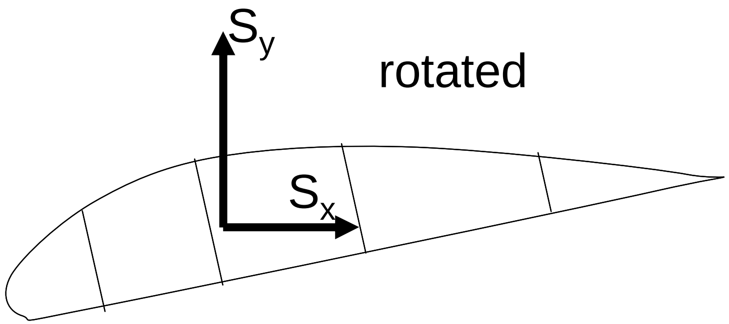
Note that Megson uses the symbol q_b for the shear flow q_s of the cut cross-section; we will use q_b for consistency=>q_b is the shear flow distribution in each cell (changes from cell to cell) obtained when the cell is cut

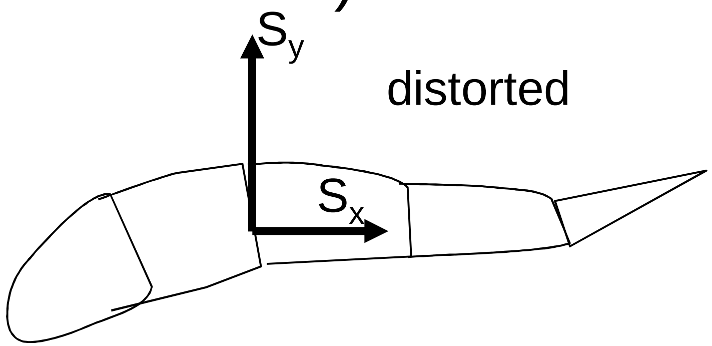
Shear of multi-cell beam



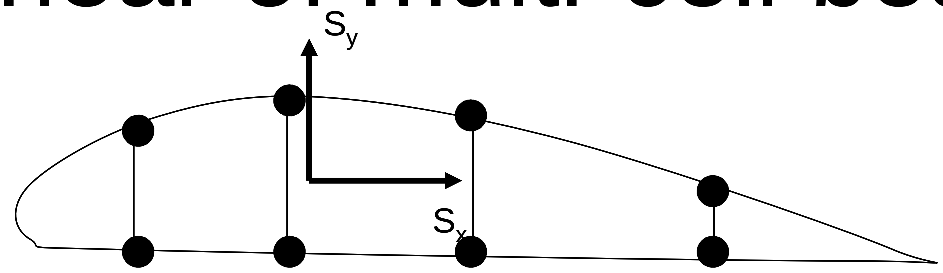
• we are looking for n+1 equations to determine the n+1 unknowns: q_{s0} for n cells and dθ/dz

• we use the fact that under the applied shear loads S_x and S_y, each cell, even if the entire cross-section rotates, will have the same rate of twist dθ/dz as every other cell (i.e. there may be rotation but no distortion)





Shear of multi-cell beam



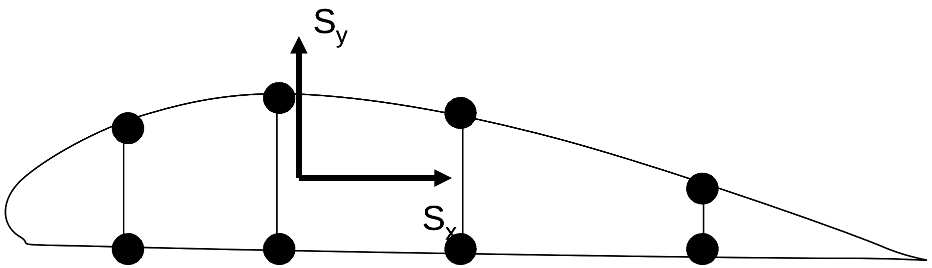
• for simplicity, consider an idealized cross-section where the skins carry only (constant) shear flows and the bending loads are taken by the booms

• the equation for the rate of twist dθ/dz was found (lecture 7) to be:

$$\frac{d\theta}{dz} = \frac{1}{2A}\oint \frac{q_s}{tG} ds \qquad (5.12)$$

• apply this equation for each cell

Shear of multi-cell beam



$$\frac{d\theta}{dz} = \frac{1}{2A}\oint \frac{q_s}{tG} ds \qquad (5.12)$$

• in eq (5.12), q_s was the total shear flow at each skin or web; to use it, we must use the total shear flow for our case which is $q_b + q_{s0}$

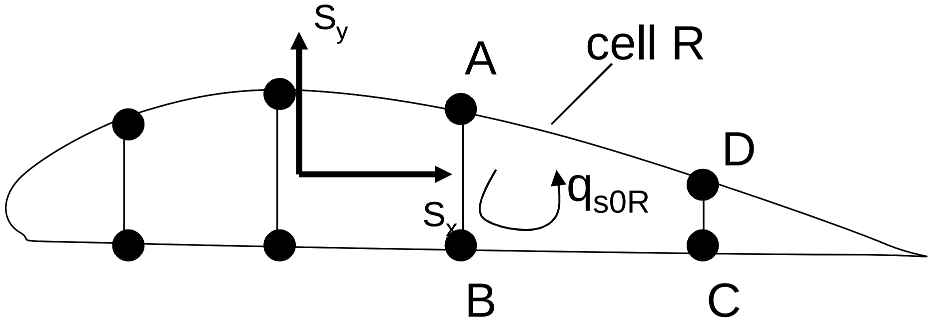
when cell is cut

constant shear flow when cell is closed

• so for the Rth cell, of enclosed area A_R (and assuming G is the same everywhere):

$$\frac{d\theta}{dz} = \frac{1}{2A_R G}\oint_R \frac{(q_{bR} + q_{s0R})}{t} ds \qquad (9.1)$$

Shear of multi-cell beam



$$\frac{d\theta}{dz}=\frac{1}{2A_RG}\oint_R\frac{\left(q_{bR}+q_{s0R}\right)}{t}ds \qquad (9.1)$$

- we then have

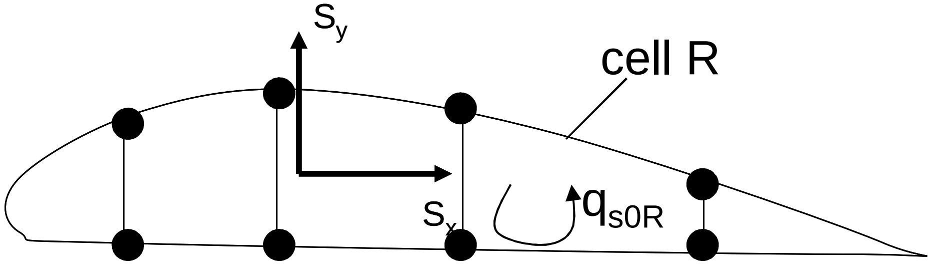
$$\oint_R\frac{q_{s0R}}{t}ds=\frac{q_{AB}L_{AB}}{t_{AB}}+\frac{q_{BC}L_{BC}}{t_{BC}}+\frac{q_{CD}L_{CD}}{t_{CD}}+\frac{q_{DA}L_{DA}}{t_{DA}}=\frac{\left(q_{s0R}-q_{s0R-1}\right)L_{AB}}{t_{AB}}+\frac{q_{s0R}L_{BC}}{t_{BC}}+\frac{\left(q_{s0R}-q_{s0R+1}\right)L_{CD}}{t_{CD}}+\frac{q_{s0R}L_{DA}}{t_{DA}} \qquad (9.2)$$

- substituting in eq (9.1):

$$\frac{d\theta}{dz}=\frac{1}{2A_RG}\left[\frac{\left(q_{s0R}-q_{s0R-1}\right)L_{AB}}{t_{AB}}+\frac{q_{s0R}L_{BC}}{t_{BC}}+\frac{\left(q_{s0R}-q_{s0R+1}\right)L_{CD}}{t_{CD}}+\frac{q_{s0R}L_{DA}}{t_{DA}}+\oint_R\frac{q_{bR}ds}{t}\right] \qquad (9.3)$$

- note that q_{bR}, the shear flow**s** (more than one) in cell R when it is cut are known from the standard shear flow equation

Shear of multi-cell beam

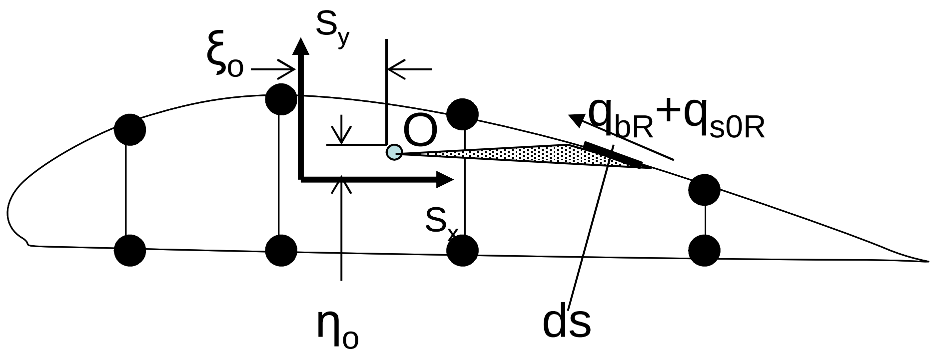


$$\frac{d\theta}{dz}=\frac{1}{2A_R G}\left[\frac{\left(q_{s0R}-q_{s0R-1}\right)L_{AB}}{t_{AB}}+\frac{q_{s0R}L_{BC}}{t_{BC}}+\frac{\left(q_{s0R}-q_{s0R+1}\right)L_{CD}}{t_{CD}}+\frac{q_{s0R}L_{DA}}{t_{DA}}+\oint_R\frac{q_{bR}ds}{t}\right] \quad (9.3)$$

- eq. (9.3) can be written for each of the n cells; so we get n equations in the n+1 unknowns, q_{s0R} and $d\theta/dz$

- we need one more equation; this is obtained by **moment equivalence**: the moment caused by the applied loads S_x and S_y about any convenient point must equal the moment caused by the resulting shear flows about the same point

Shear of multi-cell beam

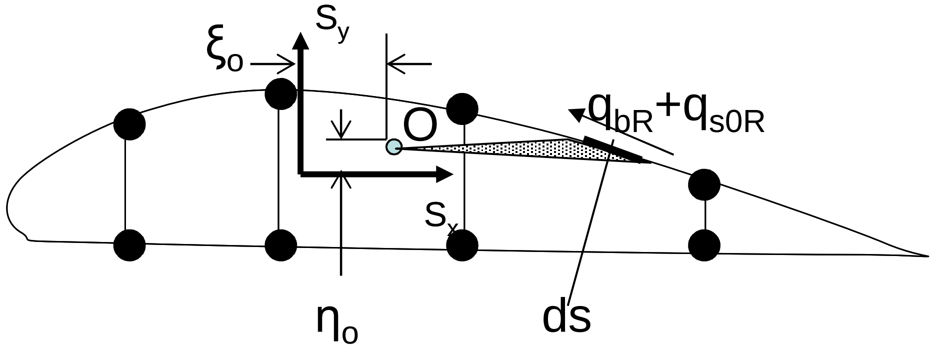


• the contribution to the moments about point O from $q_{bR}+q_{s0R}$ over an element of length ds is $(q_{bR}+q_{s0R})pds$ where p is the vertical distance between O and the axis of ds

• the complete moment equivalence equation is the same as eq (5.9) applied to all cells (5.9 was for a single cell)

$$S_x\eta_o - S_y\xi_o = \oint_s pq_b ds + 2Aq_{s0} \qquad (5.9)$$

Shear of multi-cell beam



$$S_x\eta_o - S_y\xi_o = \oint_s pq_b ds + 2Aq_{s0} \qquad (5.9)$$

- applying eq (5.9) over all cells:

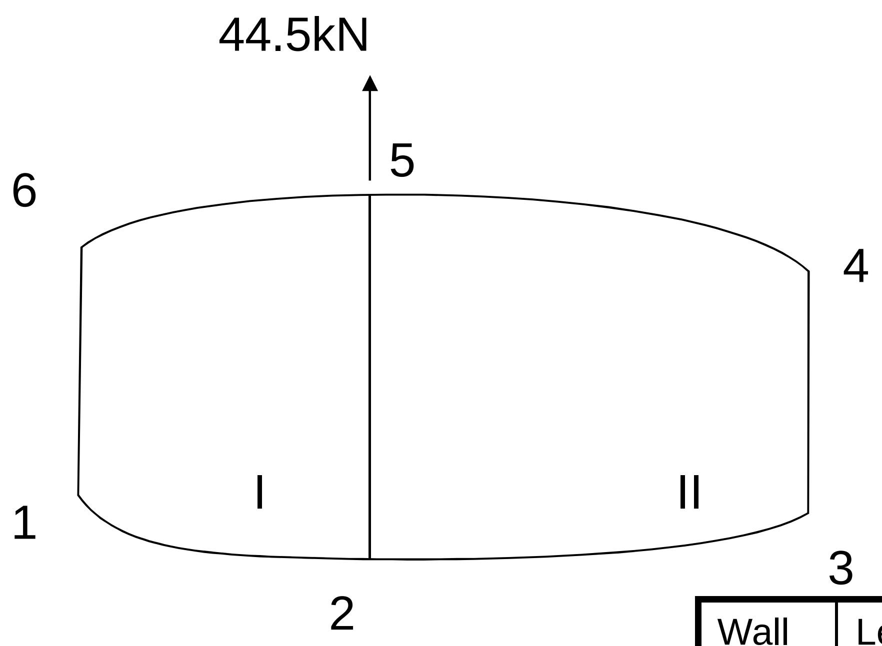
$$S_x\eta_o - S_y\xi_o = \sum_{R=1}^{n}\oint_s pq_{bR}ds + \sum_{R=1}^{n} 2A_R q_{s0R} \qquad (9.4)$$

- if point O, about which moments are taken, coincides with the point of intersection of the lines of action of S_x and S_y then $\eta_o = \xi_o = 0$ and eq. (9.4) reduces to:

$$0 = \sum_{R=1}^{n}\oint_s pq_{bR}ds + \sum_{R=1}^{n} 2A_R q_{s0R} \qquad (9.5)$$

- eq (9.4) or (9.5) is the last equation needed to solve for our n+1 unknowns

Example: Determine shear flows for a two-cell beam



Determine shear flows when S_y=44.5kN; skins are idealized and G is the same everywhere

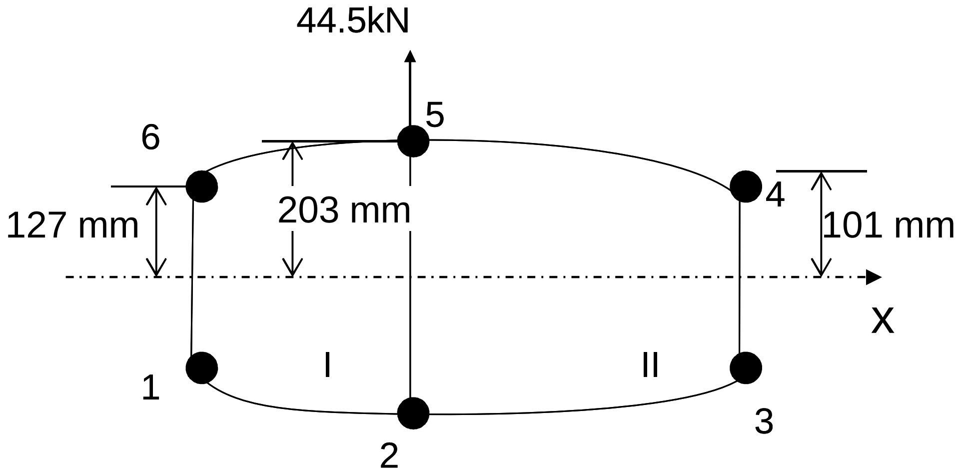
Also given the following:

Enclosed Area for Cell I, A_I=232000mm^2

Enclosed Area for Cell II, A_{II}=258000mm^2

Wall	Length (mm)	Thickness (mm)	Boom	Area (mm^2)
16	254	1.625	1,6	1290
25	406	2.032	2,5	1936
34	202	1.220	3,4	645
12,56	647	0.915		
23,45	775	0.559		

Example: Determine shear flows for a two-cell beam



the boom areas are symmetric wrt x axis ($B_1=B_6$, $B_2=B_5$, $B_3=B_4$); then, $I_{xy}=0$

since booms are present, the pertinent equation is (7.12):

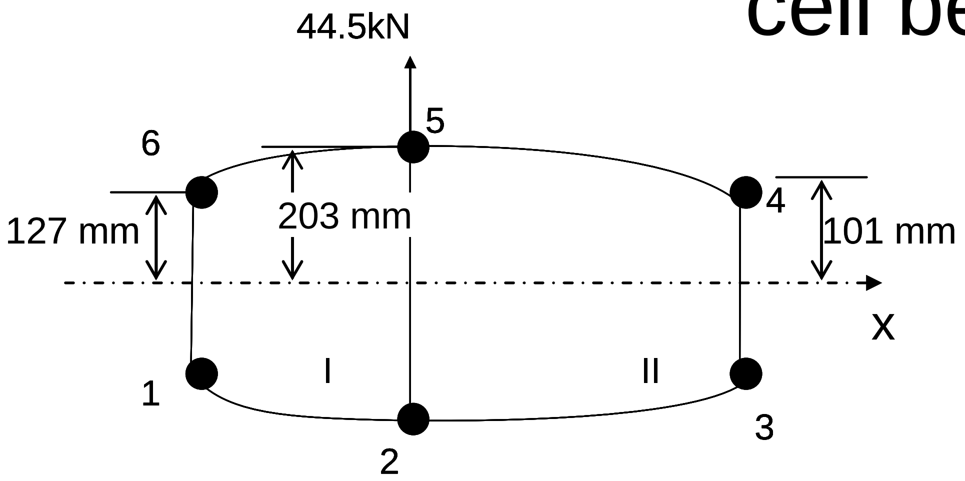
$$q_s = -\frac{I_{xx}S_x - I_{xy}S_y}{I_{xx}I_{yy} - I_{xy}^2}\left[\int_0^s t_D x\,ds + \sum_{r=1}^{n} B_r x_r\right] - \frac{I_{yy}S_y - I_{xy}S_x}{I_{xx}I_{yy} - I_{xy}^2}\left[\int_0^s t_D y\,ds + \sum_{r=1}^{n} B_r y_r\right] \qquad (7.12)$$

- which, for skin carrying only shear ($t_D=0$) and $I_{xy}=S_x=0$ reduces to:

$$q_b = -\frac{S_y}{I_{xx}}\sum_{r=1}^{n} B_r y_r \qquad (9.6)$$

(with q_s replaced by q_b to stay consistent with Megson's notation in this chapter)

Example: Determine shear flows for a two-cell beam



$$q_b = -\frac{S_y}{I_{xx}} \sum_{r=1}^{n} B_r y_r \qquad (9.6)$$

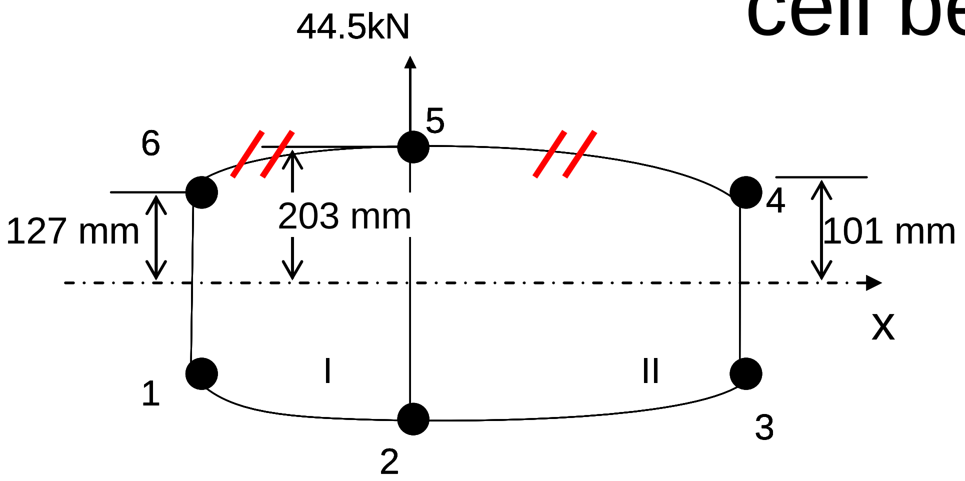
the moment of inertia I_{xx} is:

$I_{xx} = 2B_1(127)^2 + 2A_2(203)^2 + 2A_3(101)^2 = 214.3x10^6$ mm^4

- using this value and the value of S_y to substitute in (9.6):

$$q_b = -2.08x10^{-4} \sum_{r=1}^{n} B_r y_r \qquad (9.6a)$$

Example: Determine shear flows for a two-cell beam



$$q_b = -2.08x10^{-4}\sum_{r=1}^{n} B_r y_r \qquad (9.6a)$$

- cut the upper skins 65 and 54 (anywhere since shear flows are constant in each skin); for each cell, start from the cut and go around (assume shear flows positive counterclockwise):

$$q_{b65} = 0$$

$$q_{b61} = -\frac{S_y}{I_{xx}} B_1 y_1 = -2.08x10^{-4} x1290x127 = -34.01N/mm$$

$$q_{b12} = q_{b65} = 0$$ (symmetry)

$$q_{b25} = -\frac{S_y}{I_{xx}} B_2 y_2 = -2.08x10^{-4} x1936x(-203) = 81.60N/mm$$

cell 1

$$q_{b45} = q_{b23} = 0$$

$$q_{b34} = -2.08x10^{-4} B_3 y_3 =$$

$$= -2.08x10^{-4} x645x(-101) = 13.52N/mm$$

cell 2