Flight Dynamics summary IV: 2019-2020 edition

Based on *Lecture Notes Flight Dynamics* by J.A. Mulder et al.

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Contents

Preface

The final part of this summary, and the very last summary of the full bachelor. It shall go without saying that under no circumstance you'll forward this summary to anyone else.

1 Linearised equations of motion

In the last part, we finished with the beautiful equations

SIMPLIFIED EQUATIONS OF MOTION

The **simplified equations of motion** are given by the following sets of equations: the **simplified dynamic equations of translational motion** are given by

$$m(\dot{u} + qw - rv) = -mg_{r,0}\sin\theta + X \tag{1.1}$$
$$m(\dot{v} + ru - pw) = mg\sin\phi\cos\theta + Y \tag{1.2}$$
$$m(\dot{w} + pv - qu) = mg\sin\phi\sin\theta + Z \tag{1.3}$$

The **simplified kinematic equations of translational motion** are given by

$$V_N = [u\cos\theta + (v\sin\phi + w\cos\phi)\sin\theta]\cos\psi - (v\cos\phi - w\sin\phi)\sin\psi \tag{1.4}$$
$$V_E = [u\cos\theta + (v\sin\phi + w\cos\phi)\sin\theta]\sin\psi + (v\cos\phi - w\sin\phi)\cos\psi \tag{1.5}$$
$$V_D = u\sin\theta - (v\sin\phi + w\cos\phi)\cos\theta \tag{1.6}$$

The **dynamic equations of rotational motion**, for bodies for which the $X_B Z_B$-plane is a symmetry plane, are given by

$$M_x = I_{xx}\dot{p} + \left(I_{zz} - I_{yy}\right)qr - I_{xz}(\dot{r} + pq) \tag{1.7}$$
$$M_y = I_{yy}\dot{q} + \left(I_{xx} - I_{zz}\right)rp + I_{xz}\left(p^2 - r^2\right) \tag{1.8}$$
$$M_z = I_{zz}\dot{r} + \left(I_{yy} - I_{xx}\right)pq - I_{xz}(\dot{p} - rq) \tag{1.9}$$

for which an explicit solution for $\dot{p}$, $\dot{q}$ and $\dot{r}$ is given by

$$\dot{p} = \frac{I_{zz}}{I_{xx}I_{zz} - I_{xz}^2}M_x + \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2}M_z + \frac{\left(I_{xx} - I_{yy} + I_{zz}\right)I_{xz}}{I_{xx}I_{zz} - I_{xz}^2}pq + \frac{\left(I_{yy} - I_{zz}\right)I_{zz} - I_{xz}^2}{I_{xx}I_{zz} - I_{xz}^2}qr \tag{1.10}$$
$$\dot{q} = \frac{M_y}{I_{yy}} + \frac{I_{xz}}{I_{yy}}\left(r^2 - p^2\right) + \frac{I_{zz} - I_{xx}}{I_{yy}}pr \tag{1.11}$$
$$\dot{r} = \frac{I_{xz}}{I_{xx}I_{zz} - I_{xz}^2}M_z + \frac{I_{xx}}{I_{xx}I_{zz} - I_{xz}^2}M_z + \frac{\left(I_{xx} - I_{yy}\right)I_{xx} + I_{xz}^2}{I_{xx}I_{zz} - I_{xz}^2}pq + \frac{\left(-I_{xx} + I_{yy} - I_{zz}\right)I_{xz}}{I_{xx}I_{zz} - I_{xz}^2}qr \tag{1.12}$$

The **kinematic equations of rotational motion** are given by

$$\dot{\phi} = p + \sin\phi\tan\theta q + \cos\phi\tan\theta r \tag{1.13}$$
$$\dot{\theta} = \cos\phi q - \sin\phi r \tag{1.14}$$
$$\dot{\psi} = \frac{\sin\phi}{\cos\theta}q + \frac{\cos\phi}{\cos\theta}r \tag{1.15}$$

Now, these equations are still a non-linear system of differential equations, severely complicating solving them. Indeed, we will, as the title of this chapter gave away, linearise them, such that we can write them as matrices, so that we can easily solve them with a computer.

1.1 Linearisation in general

1.1.1 Single-variable linearisation

First, let's start with a recap on linearisation. Linearisation is based on Taylor series. In general, for single-variable functions, we have that

TAYLOR SERIES

If a function $f(x)$ is infinitely times differentiable on the interval $[x_0, x]$, then the Taylor expansion of $f(x)$ about x_0 is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \tag{1.16}$$

where $f^{(n)}$ is the nth derivative of f (where $f^{(0)}$ is the undifferentiated function).

In practical terms, we have

$$f(x) = \frac{f(x_0)}{0!}(x-x_0)^0 + \frac{f'(x_0)}{1!}(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2 + ...$$

In linearisation, we'll ignore all terms that are second order or higher.

Example 1

Linearise the function $f(x) = e^{\cos(x)}$ around $x_0 = 1$.

We have

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0)$$

$f(x_0)$ is simply $e^{\cos(x_0)} = e^{\cos(1)}$. Furthermore, the derivative is simply

$$f'(x) = -e^{\cos(x)}\sin(x)$$
$$f'(x_0) = -e^{\cos(x_0)}\sin(x_0) = -e^{\cos(1)}\sin(1)$$

Thus, our linearisation becomes

$$f(x) \approx e^{\cos(1)} - e^{\cos(1)}\sin(1)(x-1)$$

1.1.2 Multi-variable linearisation

If we have a function f of multiple variables, where the variables are contained in a vector $\mathbf{X}$, the linearisation is simple[1]

LINEARISATION OF MULTI-VARIABLE FUNCTION

A function $f(\mathbf{X})$, where $\mathbf{X}$ is a vector containing n variables $x_1, ..., x_n$, may be linearised around a point $\mathbf{X}_0$ as

$$f(\mathbf{X}) = f(\mathbf{X}_0) + f_{x_1}(\mathbf{X}_0)\left(x_1 - x_{1_0}\right) + ... + f_{x_n}(\mathbf{X}_0)\left(x_n - x_{n_0}\right) \tag{1.17}$$

Now, the book writes Δx_1 etc. instead of $x_1 - x_{1_0}$, which is a minor change that we can adapt to. Let's do an example.

[1] Just to clarify something in case you ever got confused a multi-variable function is different from a vector-valued function! A vector-valued function returns a vector as an output, whereas the functions we're talking about here are scalar-valued functions: although they may depend on multiple inputs, the output is a simple scalar. A vector-valued function is typically denoted by use of $\mathbf{f}$ rather than f. Note that multi-variable vector-valued functions are common, i.e. $\mathbf{f}(\mathbf{X})$ is a common occurrence in real life.

Example 2

Linearise the function $f(x,y) = xy + x^2 \sin(e^y)$ around $\mathbf{X}_0 = [x_0, y_0]$.

We very simply have

$$f\left(\mathbf{X}_0\right) = f\left(\left[x_0, y_0\right]\right) = x_0 y_0 + x_0^2 \sin\left(e^{y_0}\right)$$

Nothing more than that. For the derivatives, we have

$$\begin{aligned} f_x\left(\mathbf{X}\right) &= y + 2x \sin\left(e^y\right) \\ f_x\left(\mathbf{X}_0\right) &= y_0 + 2x_0 \sin\left(e^{y_0}\right) \\ f_y\left(\mathbf{X}\right) &= x + x^2 \cos\left(e^y\right) e^y \\ f_y\left(\mathbf{X}_0\right) &= x_0 + x_0^2 \cos\left(e^{y_0}\right) e^{y_0} \end{aligned}$$

Thus,

$$\begin{aligned} f\left(\mathbf{X}\right) \approx & f\left(\mathbf{X}_0\right) + f_x\left(\mathbf{X}_0\right)\Delta x + f_y\left(\mathbf{X}_0\right)\Delta y = x_0 y_0 + x_0^2 \sin\left(e^{y_0}\right) \\ & + \left(y_0 + 2x_0 \sin\left(e^{y_0}\right)\right)\Delta x + \left(x_0 + x_0^2 \cos\left(e^{y_0}\right) e^{y_0}\right)\Delta y \end{aligned}$$

It may seem a fair bit of work at first: the linearised equation is much longer than the original function. However, if you work consistently, it's all just a matter of differentiating the function. Really, first years can do that.

1.2 *Linearisation of the equations of motion*

In this section, we'll purely focus on the dynamic equations of translational and rotational motion, i.e. equations (1.1)-(1.3) and (1.7)-(1.9):

DYNAMIC EQUATIONS OF MOTION

The **simplified dynamic equations of translational motion** are given by

$$\begin{aligned} m\left(\dot{u} + qw - rv\right) &= -mg_{r,0}\sin\theta + X && (1.1) \\ m\left(\dot{v} + ru - pw\right) &= mg\sin\phi\cos\theta + Y && (1.2) \\ m\left(\dot{w} + pv - qu\right) &= mg\sin\phi\sin\theta + Z && (1.3) \end{aligned}$$

The **dynamic equations of rotational motion**, for bodies for which the $X_B Z_B$-plane is a symmetry plane, are given by

$$\begin{aligned} I_{xx}\dot{p} + \left(I_{zz} - I_{yy}\right)qr - I_{xz}\left(\dot{r} + pq\right) &= M_x && (1.7) \\ I_{yy}\dot{q} + \left(I_{xx} - I_{zz}\right)rp + I_{xz}\left(p^2 - r^2\right) &= M_y && (1.8) \\ I_{zz}\dot{r} + \left(I_{yy} - I_{xx}\right)pq - I_{xz}\left(\dot{p} - rq\right) &= M_z && (1.9) \end{aligned}$$

We will first focus on the left-hand sides of these equations.

1.2.1 *Linearisation of states*

Let's start with the translational motion equations. Take for example equation (1.1): this is a function with variables $\dot{u}$, q, w, r and v; the mass m is obviously not a variable as we assume it's constant. This means that $f\left(\mathbf{X}_0\right) = m\left(\dot{u}_0 + q_0 w_0 - r_0 v_0\right)$. Then, we need to find $f_{\dot{u}}$, f_q, f_w, f_r and f_v. These are very simple: $f_{\dot{u}} = m$, $f_q = mw$, $f_w = mq$, $f_r = -mv$ and $f_v = -mr$: they are all just linear functions so differentiating them is trivial.

Thus, we obtain

$$\begin{aligned} f^{(x)}(\mathbf{X}) \approx& f\left(\mathbf{X}_0\right)+f_{\dot{u}}\left(\mathbf{X}_0\right)\Delta\dot{u}+f_q\left(\mathbf{X}_0\right)\Delta q+f_w\left(\mathbf{X}_0\right)\Delta w+f_r\left(\mathbf{X}_0\right)\Delta r+f_v\left(\mathbf{X}_0\right)\Delta v \\ =& m\left(\dot{u}_0+q_0w_0-r_0v_0\right)+m\Delta\dot{u}+mw_0\Delta q+mq_0\Delta w-mv_0\Delta r-mr_0\Delta v \\ =& m\left(\dot{u}_0+q_0w_0-r_0v_0\right)+m\left(\Delta\dot{u}+w_0\Delta q+q_0\Delta w-v_0\Delta r-r_0\Delta v\right) \end{aligned} \tag{1.18}$$

For the other two translational equations we also simply obtain

$$\begin{aligned} f^{(y)}(\mathbf{X}) \approx& f\left(\mathbf{X}_0\right)+f_{\dot{v}}\left(\mathbf{X}_0\right)\Delta\dot{v}+f_r\left(\mathbf{X}_0\right)\Delta r+f_u\left(\mathbf{X}_0\right)\Delta u+f_p\left(\mathbf{X}_0\right)\Delta p+f_w\left(\mathbf{X}_0\right)\Delta w \\ =& m\left(\dot{v}_0+r_0u_0-p_0w_0\right)+m\left(\Delta\dot{v}+u_0\Delta r+r_0\Delta u-w_0\Delta p-p_0\Delta w\right) \end{aligned} \tag{1.19}$$

$$\begin{aligned} f^{(z)}(\mathbf{X}) \approx& f\left(\mathbf{X}_0\right)+f_{\dot{w}}\left(\mathbf{X}_0\right)\Delta\dot{w}+f_p\left(\mathbf{X}_0\right)\Delta p+f_v\left(\mathbf{X}_0\right)\Delta v+f_q\left(\mathbf{X}_0\right)\Delta q+f_u\left(\mathbf{X}_0\right)\Delta u \\ =& m\left(\dot{w}_0+p_0v_0-q_0u_0\right)+m\left(\Delta\dot{w}+v_0\Delta p+p_0\Delta v-u_0\Delta q-q_0\Delta u\right) \end{aligned} \tag{1.20}$$

It may seem overwhelming at first, but literally all you're doing is just differentiating $m(\dot{u}+qw-rv)$ and the other equations a bunch of times.

Similarly, we can do the same for the left-hand side of the dynamic equations of rotational motion. Consider again the equation for the x-axis in a bit more detail. We have

$$f^{(x)}(\mathbf{X}) = I_{xx}\dot{p}+\left(I_{zz}-I_{yy}\right)qr-I_{xz}\left(\dot{r}+pq\right)$$

which is a function dependent on $\dot{p}$, q, r, $\dot{r}$ and p. Thus, we simply have $f^{(x)}\left(\mathbf{X}_0\right)=I_{xx}\dot{p}_0+\left(I_{zz}-I_{yy}\right)q_0r_0-I_{xz}\left(\dot{r}_0+p_0q_0\right)$. Additionally, $f_{\dot{p}}=I_{xx}$, $f_q=\left(I_{zz}-I_{yy}\right)r-I_{xz}p$, $f_r=\left(I_{zz}-I_{yy}\right)q$, $f_{\dot{r}}=-I_{xz}$ and $f_p=-I_{xz}q$. Thus, we obtain

$$\begin{aligned} f^{(x)}(\mathbf{X}) \approx& f\left(\mathbf{X}_0\right)+f_{\dot{p}}\left(\mathbf{X}_0\right)\Delta\dot{p}+f_q\left(\mathbf{X}_0\right)\Delta q+f_r\left(\mathbf{X}_0\right)\Delta r+f_{\dot{r}}\left(\mathbf{X}_0\right)\Delta\dot{r}+f_p\left(\mathbf{X}_0\right)\Delta p \\ =& I_{xx}\dot{p}_0+\left(I_{zz}-I_{yy}\right)q_0r_0-I_{xz}\left(\dot{r}_0+p_0q_0\right)+I_{xx}\Delta\dot{p}+\left[\left(I_{zz}-I_{yy}\right)r_0-I_{xz}p_0\right]\Delta q \\ &+\left(I_{zz}-I_{yy}\right)q_0\Delta r-I_{xz}\Delta\dot{r}-I_{xz}q_0\Delta p \end{aligned} \tag{1.21}$$

Similarly, for the other equations, we obtain

$$\begin{aligned} f^{(y)}(\mathbf{X}) \approx& f\left(\mathbf{X}_0\right)+f_{\dot{q}}\left(\mathbf{X}_0\right)\Delta\dot{q}+f_r\left(\mathbf{X}_0\right)\Delta r+f_p\left(\mathbf{X}_0\right)\Delta p \\ =& I_{yy}\dot{q}_0+\left(I_{xx}-I_{zz}\right)p_0r_0+I_{xz}\left(p_0^2-r_0^2\right)+I_{yy}\Delta\dot{q}+\left[\left(I_{xx}-I_{zz}\right)p_0+I_{xz}r_0\right]\Delta r+\left[\left(I_{xx}-I_{zz}\right)r_0+I_{xz}p_0\right]\Delta p \end{aligned} \tag{1.22}$$

$$\begin{aligned} f^{(z)}(\mathbf{X}) \approx& f\left(\mathbf{X}_0\right)+f_{\dot{r}}\left(\mathbf{X}_0\right)\Delta\dot{r}+f_p\left(\mathbf{X}_0\right)\Delta p+f_q\left(\mathbf{X}_0\right)\Delta q+f_{\dot{p}}\left(\mathbf{X}_0\right)\Delta\dot{p}+f_r\left(\mathbf{X}_0\right)\Delta r \\ =& I_{zz}\dot{r}_0+\left(I_{yy}-I_{xx}\right)p_0q_0-I_{xz}\left(\dot{p}_0-q_0r_0\right)+I_{zz}\Delta\dot{r}+\left(I_{yy}-I_{xx}\right)q_0\Delta p \\ &+\left[\left(I_{yy}-I_{xx}\right)p_0+I_{xz}r_0\right]\Delta q-I_{xz}\Delta\dot{p}+I_{xz}q_0\Delta r \end{aligned} \tag{1.23}$$

LINEARISED LEFT-HAND SIDES OF EQUATIONS OF MOTION

The linearised left-hand sides for the translational equations of motion are given by

$$F_x = m\left(\dot{u}_0+q_0w_0-r_0v_0\right)+m\left(\Delta\dot{u}+w_0\Delta q+q_0\Delta w-v_0\Delta r-r_0\Delta v\right) \tag{1.24}$$

$$F_y = m\left(\dot{v}_0+r_0u_0-p_0w_0\right)+m\left(\Delta\dot{v}+u_0\Delta r+r_0\Delta u-w_0\Delta p-p_0\Delta w\right) \tag{1.25}$$

$$F_z = m\left(\dot{w}_0+p_0v_0-q_0u_0\right)+m\left(\Delta\dot{w}+v_0\Delta p+p_0\Delta v-u_0\Delta q-q_0\Delta u\right) \tag{1.26}$$

The linearised left-hand side for the rotational equations of motion are given by

$$\begin{aligned} M_x =& I_{xx}\dot{p}_0+\left(I_{zz}-I_{yy}\right)q_0r_0-I_{xz}\left(\dot{r}_0+p_0q_0\right)+I_{xx}\Delta\dot{p}+\left[\left(I_{zz}-I_{yy}\right)r_0-I_{xz}p_0\right]\Delta q \\ &+\left(I_{zz}-I_{yy}\right)q_0\Delta r-I_{xz}\Delta\dot{r}-I_{xz}q_0\Delta p \end{aligned} \tag{1.27}$$

$$M_y = I_{yy}\dot{q}_0+\left(I_{xx}-I_{zz}\right)p_0r_0+I_{xz}\left(p_0^2-r_0^2\right)+I_{yy}\Delta\dot{q}+\left[\left(I_{xx}-I_{zz}\right)p_0+I_{xz}r_0\right]\Delta r+\left[\left(I_{xx}-I_{zz}\right)r_0+I_{xz}p_0\right]\Delta p \tag{1.28}$$

$$\begin{aligned} M_z =& I_{zz}\dot{r}_0+\left(I_{yy}-I_{xx}\right)p_0q_0-I_{xz}\left(\dot{p}_0-q_0r_0\right)+I_{zz}\Delta\dot{r}+\left(I_{yy}-I_{xx}\right)q_0\Delta p \\ &+\left[\left(I_{yy}-I_{xx}\right)p_0+I_{xz}r_0\right]\Delta q-I_{xz}\Delta\dot{p}+I_{xz}q_0\Delta r \end{aligned} \tag{1.29}$$

1.2.2 Linearisation of forces and moments

Now that we've discussed the left-hand sides, let's evaluate the right-hand sides of the equations. Again, first let's do the equations of translational motion. If we assume that our altitude is more or less constant so that g is constant, and define $mg = W$ (which shall be constant as well), we see that for example our first equation only depends on θ and X (the force acting on the aircraft in X-direction). Now, there's a distinct difference between θ and X: θ itself is a state on its own. However, X is not: by itself X depends on many actual states and inputs: it is a function of the three velocities u, v and w, the angular rates p, q and r, the elevator deflection δ_e, trim tab deflection δ_t, etc. In other words, X on its own depends on a lot of factors and is not a state on itself.

Indeed, this means that you're not allowed to simply include the term $f_X\left(\mathbf{x}_0\right)\Delta X$ in your linearisation: ΔX would not be known and thus it would be rather useless. Rather, we must differentiate it with respect to all of those states that X depends upon, i.e. we must include terms $f_u\left(\mathbf{x}_0\right)\Delta u$, for all of the relevant states that affect X.

In case you're a bit confused right now with what I exactly mean with states:

STATES AND INPUTS

States are the *smallest set* of parameters that describe your state. In the analysis of aircraft, the usual chosen states are the translational velocity components u, v and w, the angles θ, ϕ and ψ, and the rotational rates p, q and r. Note that more parameters may describe the state, e.g. X in principle also describes the state of the system. However, this is not part of the *smallest* set of parameters that can describe the state, as those are already filled by the previously listed states.

The **inputs** are controllable inputs you yourself are able to feed to the aircraft. They are the elevator deflection δ_e, trim tab deflection δ_t, aileron deflection δ_a and rudder deflection δ_r.

At all times, it holds that one is only allowed to differentiate with respect to these states, or their time derivatives. It is *not* allowed to differentiate with respect to stuff that aren't states.

So, basically we have to differentiate X with respect to all the states, and with their time-derivatives! So fuck how many time derivatives are there that we need to differentiate with respect to? Aren't there infinitely many time derivatives? Yess, so let's analyse for a moment those time-derivatives, and while doing so, let's also take into account their effects on Y, Z, M_x, M_y and M_z. It turns out that actually almost all time-derivatives are insignificant. Thus, in general, X, Y, Z, M_x, M_y and M_z are all only functions of u, v, w, p, q, r, δ_e, δ_t, δ_a and δ_r. However, there are four exceptions: the time-derivative $\dot{v}$ is important for both F_y and M_z, and $\dot{w}$ is important for both F_z and M_y. Additionally, the trim tab deflection δ_t is generally negligible for F_y, M_x and M_z. Thus, we can say that F_x, F_y, F_z, M_x, M_y and M_z (the right-hand sides of the equations of motion I mean) are functions of

$$
\begin{aligned}
F_x &= f\left(\theta, u, v, w, p, q, r, \delta_e, \delta_t, \delta_a, \delta_r\right)\\
F_y &= f\left(\theta, \phi, u, v, w, \dot{v}, p, q, r, \delta_e, \delta_a, \delta_r\right)\\
F_z &= f\left(\theta, \phi, u, v, w, \dot{w}, p, q, r, \delta_e, \delta_t, \delta_a, \delta_r\right)\\
M_x &= f\left(u, v, w, p, q, r, \delta_e, \delta_a, \delta_r\right)\\
M_y &= f\left(u, v, w, \dot{w}, p, q, r, \delta_e, \delta_t, \delta_a, \delta_r\right)\\
M_z &= f\left(u, v, w, \dot{v}, p, q, r, \delta_e, \delta_a, \delta_r\right)
\end{aligned}
$$

We can make an additional simplification, however: we can assume that no aerodynamic coupling exists between the symmetric and the asymmetric degrees of freedom, as long as the deviations and disturbances remain small. What does this mean? This means that e.g the symmetrical velocity u (this velocity is located in the symmetry-plane after all) does not affect the asymmetric forces and moments (i.e. F_y (which points out of the symmetry plane), M_x and M_z are not affected by this[2]. In general, the symmetrical states θ, u, w, $\dot{w}$ and q, and symmetrical inputs δ_e and δ_t, do not affect the asymmetric forces and moments F_y, M_x and M_z, and the

[2] M_x and M_z are not affected by u as the X_bZ_b-plane is assumed to be a plane of symmetry; thus, moments around these axes correspond to asymmetric moments, as motions around these axes would move the aircraft out of the symmetry plane. Only a moment around the Y_b-axis would lead to the symmetry plane to be preserved.

asymmetrical states ϕ, v, $\dot{v}$, p and r, and asymmetrical inputs δ_a and δ_r, do not affect the symmetric forces and moments F_x, F_z and M_y. This reduces the dependencies to

$$F_x = f\left(\theta, u, w, q, \delta_e, \delta_t\right) \tag{1.30}$$
$$F_y = f\left(\phi, v, \dot{v}, p, r, \delta_a, \delta_r\right) \tag{1.31}$$
$$F_z = f\left(\theta, u, w, \dot{w}, q, \delta_e, \delta_t\right) \tag{1.32}$$
$$M_x = f\left(v, p, r, \delta_a, \delta_r\right) \tag{1.33}$$
$$M_y = f\left(u, w, \dot{w}, q, \delta_e, \delta_t\right) \tag{1.34}$$
$$M_z = f\left(v, \dot{v}, p, r, \delta_a, \delta_r\right) \tag{1.35}$$

Okay why the fuck did I dwell off so seemingly much? Well, let us consider once more the right-hand-sides of the equations of motion:

DYNAMIC EQUATIONS OF MOTION

The **simplified dynamic equations of translational motion** are given by

$$m\left(\dot{u} + qw - rv\right) = -mg_{r,0}\sin\theta + X \tag{1.1}$$
$$m\left(\dot{v} + ru - pw\right) = mg\sin\phi\cos\theta + Y \tag{1.2}$$
$$m\left(\dot{w} + pv - qu\right) = mg\sin\phi\sin\theta + Z \tag{1.3}$$

The **dynamic equations of rotational motion**, for bodies for which the X_BZ_B-plane is a symmetry plane, are given by

$$I_{xx}\dot{p} + \left(I_{zz} - I_{yy}\right)qr - I_{xz}\left(\dot{r} + pq\right) = M_x \tag{1.7}$$
$$I_{yy}\dot{q} + \left(I_{xx} - I_{zz}\right)rp + I_{xz}\left(p^2 - r^2\right) = M_y \tag{1.8}$$
$$I_{zz}\dot{r} + \left(I_{yy} - I_{xx}\right)pq - I_{xz}\left(\dot{p} - rq\right) = M_z \tag{1.9}$$

Now, we see that for example the right-hand side of equation (1.1), we need to differentiate only with respect to θ, u, w, q, δ_e and δ_t, according to equation (1.30). Thus, we simply get

$$F_x\left(\mathbf{X}\right) \approx F_x\left(\mathbf{X}_0\right) + \frac{\partial F_x}{\partial\theta}\Delta\theta + \frac{\partial F_x}{\partial u}\Delta u + \frac{\partial F_x}{\partial w}\Delta w + \frac{\partial F_x}{\partial q}\Delta q + \frac{\partial F_x}{\partial\delta_e}\Delta\delta_e + \frac{\partial F_x}{\partial\delta_t}\Delta\delta_t$$

Now, $F_x\left(\mathbf{X}_0\right) = -W\sin\theta_0 + X_0$, nothing more, nothing less. The derivative with respect to θ is simply equal to $-W\cos\theta$. The other derivatives are all very elegant: $\frac{\partial F_x}{\partial u} = X_u$, $\frac{\partial F_x}{\partial w} = X_w$, etc. etc. Thus, we simply obtain

$$F_x\left(\mathbf{X}\right) \approx -W\sin\theta_0 + X_0 - W\cos\theta_0\Delta\theta + X_u\Delta u + X_w\Delta w + X_q\Delta q + X_{\delta_e}\Delta\delta_e + X_{\delta_t}\Delta\delta_t \tag{1.36}$$

Again, looks intimidating at first, but if you look at it for a second, it's not all that bad. Similarly, if we analyse F_y, we realise that although we see θ appearing, we do not need to differentiate with respect to θ! After all, we said that θ, a symmetric angle, wouldn't influence an asymmetrical force (F_y), thus we do not need to include that derivative anymore[3]. Thus, we only obtain

$$F_y\left(\mathbf{X}\right) \approx F_y\left(\mathbf{X}_0\right) + \frac{\partial F_y}{\partial\phi}\Delta\phi + \frac{\partial F_y}{\partial v}\Delta v + \frac{\partial F_y}{\partial\dot{v}}\Delta\dot{v} + \frac{\partial F_y}{\partial p}\Delta p + \frac{\partial F_y}{\partial r}\Delta r + \frac{\partial F_y}{\partial\delta_a}\Delta\delta_a + \frac{\partial F_y}{\partial\delta_r}\Delta\delta_r$$

Here, $F_y\left(\mathbf{X}_0\right)$ is simply $W\sin\phi_0\cos\theta_0 + Y_0$, nothing more, nothing less. $\partial F_y/\partial\phi$ is $W\cos\phi\cos\theta$, that's a differentiation even 16-year-olds can do. The other derivatives are as elegant as before, so we obtain

$$F_y\left(\mathbf{X}\right) \approx W\sin\phi_0\cos\theta_0 + Y_0 + W\cos\phi_0\cos\theta_0\Delta\phi + Y_v\Delta v + Y_{\dot{v}}\Delta\dot{v} + Y_p\Delta p + Y_r\Delta r + Y_{\delta_a}\Delta\delta_a + Y_{\delta_r}\Delta\delta_r \tag{1.37}$$

[3] If θ doesn't influence F_y, $dF_y/d\theta = 0$. So why include it at all.

Similarly, we obtain

$$F_z(\mathbf{X}) \approx W\cos\phi_0\cos\theta_0 + Z_0 - W\cos\phi_0\sin\theta_0\Delta\theta + Z_u\Delta u + Z_w\Delta w + Z_{\dot{w}}\Delta\dot{w} + Z_q\Delta q + Z_{\delta_e}\Delta\delta_e + Z_{\delta_t}\Delta\delta_t \tag{1.38}$$

$$M_x(\mathbf{X}) \approx L_0 + L_v\Delta v + L_p\Delta p + L_r\Delta r + L_{\delta_a}\Delta\delta_a + L_{\delta_r}\Delta\delta_r \tag{1.39}$$

$$M_y(\mathbf{X}) \approx M_0 + M_u\Delta u + M_w\Delta w + M_{\dot{w}}\Delta\dot{w} + M_q\Delta q + M_{\delta_e}\Delta\delta_e + M_{\delta_t}\Delta\delta_t \tag{1.40}$$

$$M_z(\mathbf{X}) \approx N_0 + N_v\Delta v + N_{\dot{v}}\Delta\dot{v} + N_p\Delta p + N_r\Delta r + N_{\delta_a}\Delta\delta_a + N_{\delta_r}\Delta\delta_r \tag{1.41}$$

LINEARISED RIGHT-HAND SIDES OF EQUATIONS OF MOTION

The linearised right-hand sides for the translational equations of motion are given by

$$F_x(\mathbf{X}) \approx -W\sin\theta_0 + X_0 - W\cos\theta_0\Delta\theta + X_u\Delta u + X_w\Delta w + X_q\Delta q + X_{\delta_e}\Delta\delta_e + X_{\delta_t}\Delta\delta_t \tag{1.42}$$

$$F_y(\mathbf{X}) \approx W\sin\phi_0\cos\theta_0 + Y_0 + W\cos\phi_0\cos\theta_0\Delta\phi + Y_v\Delta v + Y_{\dot{v}}\Delta\dot{v} + Y_p\Delta p + Y_r\Delta r + Y_{\delta_a}\Delta\delta_a + Y_{\delta_r}\Delta\delta_r \tag{1.43}$$

$$F_z(\mathbf{X}) \approx W\cos\phi_0\cos\theta_0 + Z_0 - W\cos\phi_0\sin\theta_0\Delta\theta + Z_u\Delta u + Z_w\Delta w + Z_{\dot{w}}\Delta\dot{w} + Z_q\Delta q + Z_{\delta_e}\Delta\delta_e + Z_{\delta_t}\Delta\delta_t \tag{1.44}$$

The linearised right-hand sides for the rotational equations of motion are given by

$$M_x(\mathbf{X}) \approx L_0 + L_v\Delta v + L_p\Delta p + L_r\Delta r + L_{\delta_a}\Delta\delta_a + L_{\delta_r}\Delta\delta_r \tag{1.45}$$

$$M_y(\mathbf{X}) \approx M_0 + M_u\Delta u + M_w\Delta w + M_{\dot{w}}\Delta\dot{w} + M_q\Delta q + M_{\delta_e}\Delta\delta_e + M_{\delta_t}\Delta\delta_t \tag{1.46}$$

$$M_z(\mathbf{X}) \approx N_0 + N_v\Delta v + N_{\dot{v}}\Delta\dot{v} + N_p\Delta p + N_r\Delta r + N_{\delta_a}\Delta\delta_a + N_{\delta_r}\Delta\delta_r \tag{1.47}$$

Now, we can set these equal to equations (1.24)-(1.29) and obtain six linearised equations of motion; as the equations are pretty long I won't write out the full equations (also cause we'll be setting some stuff equal to zero in section 1.3, so let's wait for that). However, because you're super smart, you must be thinking, but wait this would be six equations of motion, but we've used eight states so far: θ, ϕ, u, v, w, p, q and r. So, we need to generate at least two extra equations. Well, I listed a lot of equations at the beginning of the chapter, probably for a reason. Indeed, we can use the kinematic insights from the rotational motion!

1.2.3 *Linearisation of kinematic relations*

We have the kinematic insights

KINEMATIC EQUATIONS OF ROTATIONAL MOTION

The **kinematic equations of rotational motion** are given by

$$\dot{\phi} = p + \sin\phi\tan\theta q + \cos\phi\tan\theta r \tag{1.13}$$

$$\dot{\theta} = \cos\phi q - \sin\phi r \tag{1.14}$$

$$\dot{\psi} = \frac{\sin\phi}{\cos\theta}q + \frac{\cos\phi}{\cos\theta}r \tag{1.15}$$

These equations relate (the derivatives of) the already included states to each other; only ψ is introduced as state here, but that makes nine equations and nine unknowns so we're happy 😀 (I'm pretty proud of getting this emoji into latex don't hate on it). However, they're non-linear equations, so we must linearise them. Let's first consider equation (1.13). We have $\dot{\phi}(\mathbf{X}_0) = p_0 + q_0\sin\phi_0\tan\theta_0 + r_0\cos\phi_0\tan\theta_0$. We need to differentiate with respect p, ϕ, θ, q and r, so we'll end up at

$$\dot{\phi}(\mathbf{X}) \approx \dot{\phi}(\mathbf{X}_0) + \frac{\partial\dot{\phi}}{\partial p}\Delta p + \frac{\partial\dot{\phi}}{\partial\phi}\Delta\phi + \frac{\partial\dot{\phi}}{\partial\theta}\Delta\theta + \frac{\partial\dot{\phi}}{\partial q}\Delta q + \frac{\partial\dot{\phi}}{\partial r}\Delta r$$

Now, $\partial\dot{\phi}/\partial p = 1$, $\partial\dot{\phi}/\partial\phi = q\cos\phi\tan\theta$, $\partial\dot{\phi}/\partial\theta = (q\sin\phi + r\cos\phi)/\cos^2\theta$, where it is helpful to remember that

$$\frac{d\tan\theta}{d\theta} = \frac{d\frac{\sin\theta}{\cos\theta}}{d\theta} = \frac{\cos\theta\cos\theta + \sin\theta\sin\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

where use was made of the quotient rule; additionally, $\partial\dot{\phi}/\partial q = \sin\phi\tan\theta$ and $\partial\dot{\phi}/\partial r = \cos\phi\tan\theta$. Therefore,

$$\begin{aligned}\dot{\phi}(\mathbf{X}) \approx\ & p_0 + q_0\sin\phi_0\tan\theta_0 + r_0\cos\phi_0\tan\theta_0 + \Delta p + q_0\cos\phi_0\tan\theta_0\Delta\phi + \frac{q\sin\phi + r\cos\phi}{\cos^2\theta}\Delta\theta \\ & + \sin\phi\tan\theta\Delta q + \cos\phi\tan\theta\Delta r \end{aligned} \tag{1.48}$$

Similarly, we can find the linearisation of $\dot{\theta}$. We'll have

$$\dot{\theta}(\mathbf{X}) \approx \dot{\theta}\left(\mathbf{X}_0\right) + \frac{\partial\dot{\theta}}{\partial\phi}\Delta\phi + \frac{\partial\dot{\theta}}{\partial q}\Delta q + \frac{\partial\dot{\theta}}{\partial r}\Delta r$$

Here, $\dot{\phi}\left(\mathbf{X}_0\right) = q_0\cos\phi_0 - r_0\sin\phi_0$. Furthermore, $\partial\dot{\theta}/\partial\phi = -\sin\phi q - \cos\phi r$; $\partial\dot{\theta}/\partial q = \cos\phi$ and $\partial\dot{\theta}/\partial r = -\sin\phi$. Thus,

$$\dot{\theta}(\Delta\mathbf{X}) \approx \dot{\theta}\left(\mathbf{X}_0\right) + \left(-q_0\cos\phi_0 - r_0\sin\phi_0\right)\Delta\phi_0 + \cos\phi_0\Delta q - \sin\phi_0\Delta r \tag{1.49}$$

Finally, $\dot{\psi}$ may be linearised as well:

$$\dot{\psi}(\mathbf{X}) \approx \dot{\psi}\left(\mathbf{X}_0\right) + \frac{\partial\dot{\psi}}{\partial\phi}\Delta\phi + \frac{\partial\dot{\psi}}{\partial\theta}\Delta\theta + \frac{\partial\dot{\psi}}{\partial q}\Delta q + \frac{\partial\dot{\psi}}{\partial r}\Delta r$$

Here, $\dot{\psi}\left(\mathbf{X}_0\right) = q_0\sin\phi_0/\cos\theta_0 + r_0\cos\phi_0/\cos\theta_0$. Additionally, $\partial\dot{\theta}/\partial\phi = q\cos\phi/\cos\theta - r\sin\phi/\cos\theta$; $\partial\dot{\theta}/\partial\theta = (q\sin\phi + r\cos\phi)\tan\theta/\cos\theta$, where it is helpful to remember that

$$\frac{d\frac{1}{\cos\theta}}{d\theta} = \frac{d(\cos\theta)-1}{d\theta} = -(\cos\theta)^{-2}\cdot -\sin\theta = \frac{\sin\theta}{\cos^2\theta} = \frac{\tan\theta}{\cos\theta}$$

Furthermore, $\partial\dot{\psi}/\partial q = \sin\phi/\cos\theta$ and $\partial\dot{\psi}/\partial r = \cos\phi/\cos\theta$. Thus, we obtain

$$\dot{\psi}(\mathbf{X}) \approx q_0\frac{\sin\phi_0}{\cos\theta_0} + r_0\frac{\cos\phi_0}{\cos\theta_0} + \left(q_0\frac{\cos\phi_0}{\cos\theta_0} - r_0\frac{\sin\phi_0}{\cos\theta_0}\right)\Delta\phi + \frac{\left(q\sin\phi + r_0\cos\phi_0\right)\tan\theta_0}{\cos\theta_0}\Delta\theta + \frac{\sin\phi_0}{\cos\theta_0}\Delta q + \frac{\cos\phi_0}{\cos\theta_0}\Delta r \tag{1.50}$$

We now have a system of 9 by 9 linear equations; it should be obvious that if desired, this can be straightforwardly solved by a computer. If desired, the kinematic equations of translational motion may also be computed, i.e. equations (1.4)-(1.6); these equations do *not* need to be linearised, as all of the required parameters can be solved by the previously set up nine equations[4]. In short, the linearised kinematic insights are:

LINEARISED KINEMATIC RELATIONS

The linearised kinematic equations of rotational motion are given by

$$\begin{aligned}\dot{\phi}(\mathbf{X}) \approx\ & p_0 + q_0\sin\phi_0\tan\theta_0 + r_0\cos\phi_0\tan\theta_0 + \Delta p + q_0\cos\phi_0\tan\theta_0\Delta\phi + \frac{q\sin\phi + r\cos\phi}{\cos^2\theta}\Delta\theta \\ & + \sin\phi\tan\theta\Delta q + \cos\phi\tan\theta\Delta r \end{aligned} \tag{1.51}$$

$$\dot{\theta}(\Delta\mathbf{X}) \approx \dot{\theta}\left(\mathbf{X}_0\right) + \left(-q_0\cos\phi_0 - r_0\sin\phi_0\right)\Delta\phi_0 + \cos\phi_0\Delta q - \sin\phi_0\Delta r \tag{1.52}$$

$$\begin{aligned}\dot{\psi}(\mathbf{X}) \approx\ & q_0\frac{\sin\phi_0}{\cos\theta_0} + r_0\frac{\cos\phi_0}{\cos\theta_0} + \left(q_0\frac{\cos\phi_0}{\cos\theta_0} - r_0\frac{\sin\phi_0}{\cos\theta_0}\right)\Delta\phi \\ & + \frac{\left(q\sin\phi + r_0\cos\phi_0\right)\tan\theta_0}{\cos\theta_0}\Delta\theta + \frac{\sin\phi_0}{\cos\theta_0}\Delta q + \frac{\cos\phi_0}{\cos\theta_0}\Delta r \end{aligned} \tag{1.53}$$

The kinematic relations of translational motion do not need to be linearised and still are given by

$$V_N = [u\cos\theta + (v\sin\phi + w\cos\phi)\sin\theta]\cos\psi - (v\cos\phi - w\sin\phi)\sin\psi \tag{1.4}$$

$$V_E = [u\cos\theta + (v\sin\phi + w\cos\phi)\sin\theta]\sin\psi + (v\cos\phi - w\sin\phi)\cos\psi \tag{1.5}$$

$$V_D = u\sin\theta - (v\sin\phi + w\cos\phi)\cos\theta \tag{1.6}$$

[4] So if you're smart, you just compute those first, and afterwards you can find V_N, V_E and V_D directly from that. There's no need to linearise the equations, as you're not solving a system of equations for them.

1.3 Linearisation about steady, symmetric, straight flight

The above 9 equations (excluding the kinematic relations of translational motion) can be written in state-space format:

$$\dot{\mathbf{x}} + \mathbf{\Delta}\dot{\mathbf{x}} = \mathbf{A}_0\mathbf{x}_0 + \mathbf{A}\mathbf{\Delta}\mathbf{x} + \mathbf{B}_0\mathbf{u}_0 + \mathbf{B}\mathbf{\Delta}\mathbf{u}$$

where $\mathbf{A}_0$, $\mathbf{A}$, $\mathbf{B}_0$ and $\mathbf{B}$ are all constant matrices, and $\dot{\mathbf{x}}_0$ is also a constant. The simulation is initialized at the initial state of linearisation $\mathbf{X}_0$. If this is a steady state, then $\dot{\mathbf{x}}_0$ is zero, and consequentially, as we also require $\dot{\mathbf{x}}_0 = \mathbf{A}_0\mathbf{x}_0 + \mathbf{B}_0\mathbf{u}_0$, the terms $\mathbf{A}_0\mathbf{x}_0$ and $\mathbf{B}_0\mathbf{u}_0$ also drop out. Thus, for a steady state initial state, we simply have the very familiar

$$\mathbf{\Delta}\dot{\mathbf{x}} = A\mathbf{\Delta}\mathbf{x} + \mathbf{B}\mathbf{\Delta}\mathbf{u} \tag{1.54}$$

The matrix $\mathbf{A}$ contains the stability derivatives and $\mathbf{B}$ the control derivatives.

Now, this is all very nice, but you may wonder now, what state did we actually linearise about? Truth is, we didn't specify a state that we linearised about at all. We will now simply choose linearisation about steady, symmetric, straight flight. This allows for a significant simplification of the equations of motion. First, this means that almost all translational and angular velocities are zero initially; the only nonzero initial terms are u, w, θ, X and Z. All other initial states are zero, meaning in a lot of the linearised equations, the first few terms drop out.

Furthermore, you may remember some stuff about the body axis frame (denoted by subscript b) and how it wasn't fully fixed cause you were free to pick the direction of X_b within the $X_b - Z_b$-symmetry plane. Indeed, that's something that we'll use here: we'll use the F_S reference frame, the stability reference frame, from here on forward. The X_S-axis of the stability reference frame pointed in the direction of the projection of the freestream velocity on the symmetry plane; this means that $u_0 = V$ and $w_0 = 0$, and $\alpha_0 = 0$ and $\theta_0 = \gamma_0$ (and for straight flight, $\gamma_0 = 0$).

If you now look at the red boxes throughout the previous sections, there is soooo much stuff that drops out. Almost all initial states except for u_0 (which equals V now), X_0 and Z_0 are zero; this means that rather than including a Δ everytime we'll remove the Δ as well to speed up the writing; we just need to remember that u does not represent the true velocity along the X_S-axis, but one should add the initial condition u_0 to obtain the true value. For the symmetrical equations of motion, i.e. translational equations of motion in X and Z-direction (equating equations (1.24) and (1.26) to (1.42) and (1.44)), rotational equation of motion in Y-direction (equating equations (1.28) to (1.46)) and the kinematic relation (1.52), resulting in

SYMMETRICAL MOTION

The symmetrical motion for steady, straight, symmetric flight is given by

$$m\dot{u} = -W\cos\theta_0\theta + X_u u + X_w w + X_q q + X_{\delta_e}\delta_e + X_{\delta_t}\delta_t \tag{1.55}$$

$$m(\dot{w} - qV) = -W\sin\theta_0\theta + Z_u u + Z_w w + Z_{\dot{w}}\dot{w} + Z_q q + Z_{\delta_e}\delta_e + Z_{\delta_t}\delta_t \tag{1.56}$$

$$I_{yy}\dot{q} = M_u u + M_{\dot{w}}\dot{w} + M_q q + M_{\delta_e}\delta_e + M_{\delta_t}\delta_t \tag{1.57}$$

$$\dot{\theta} = q \tag{1.58}$$

The asymmetrical equations of motion, i.e. translational equation of motion in Z-direction (equating equations (1.25) to (1.43)), rotational equations of motion in X- and Z-direction (equating equations (1.27) and (1.29) to equations (1.45) and (1.47)), and rotational kinematic relations (1.51) and (1.53) are given by

ASYMMETRICAL MOTION

The asymmetrical motion for steady, straight, symmetric flight is given by

$$m(\dot{v} + rV) = W\cos\theta_0\phi + Y_v v + Y_{\dot{v}}\dot{v} + Y_p p + Y_r r + Y_{\delta_a}\delta_a + Y_{\delta_r}\delta_r \tag{1.59}$$

$$I_{xx}\dot{p} - I_{xz}\dot{r} = L_v v + L_p p + L_r r + L_{\delta_a}\delta_a + L_{\delta_r}\delta_r \tag{1.60}$$

$$I_{zz}\dot{r} - I_{xz}\dot{p} = N_v v + N_{\dot{v}}\dot{v} + N_p p + N_r r + N_{\delta_a}\delta_a + N_{\delta_r}\delta_r \tag{1.61}$$

$$\dot{\psi} = \frac{r}{\cos\theta_0} \tag{1.62}$$

$$\dot{\phi} = p + r\tan\theta_0 \tag{1.63}$$

1.3.1 *Examples*

Exam October 2011: problem 3e (9p)

For the linearisation of the re-entry equations of motion, it is common to express the position and velocity in spherical components. The kinematic equations are in that case given by:

$$\dot{R} = V \sin\gamma$$
$$\dot{\tau} = \frac{V\cos\gamma\sin\chi}{R\cos\delta}$$
$$\dot{\delta} = \frac{V}{R}\cos\gamma\cos\chi$$

Linearise the above kinematic equations, writing the result as $\Delta\dot{\mathbf{x}} = \mathbf{A}\Delta\mathbf{x}$. Note: one can NOT assume that the nominal flight-path angle is small; only perturbations are considered to be small.

The first step is to identify the stuff that remains constant: in this case, nothing can be assumed constant (sometimes for example the wing span b would appear in the equation; this is obviously a constant and thus you do not need to differentiate with respect to this). The states are R, τ, δ, V, γ and χ, and thus we need to differentiate with respect to those variables. The first equation is found easily; we have (differentiating in the order of the states I listed)

$$\Delta\dot{R} = \frac{\partial\dot{R}}{\partial V}\Delta V + \frac{\partial\dot{R}}{\partial\gamma}\Delta\gamma = \sin\gamma_0\Delta V + V_0\cos\gamma_0\Delta\gamma$$

The second equation may be found similarly, just a bit more work to write it all out. Once again, we differentiate in the order of the states I listed before, noting it doesn't depend on τ itself:

$$\begin{aligned}\Delta\dot{\tau} &= \frac{\partial\dot{\tau}}{\partial R}\Delta R + \frac{\partial\dot{\tau}}{\partial\delta}\Delta\delta + \frac{\partial\dot{\tau}}{\partial V}\Delta V + \frac{\partial\dot{\tau}}{\partial\gamma}\Delta\gamma + \frac{\partial\dot{\tau}}{\partial\chi}\Delta\chi \\ &= -\frac{V_0\cos\gamma_0\sin\xi_0}{R_0^2\cos\delta_0}\Delta R - \frac{V_0\cos\gamma_0\sin\chi_0\sin\delta_0}{R_0\cos^2\delta_0}\Delta\delta + \frac{\cos\gamma_0\sin\chi_0}{R_0\cos\delta_0}\Delta V - \frac{V_0\sin\gamma_0\sin\chi_0}{R_0\cos\delta_0}\Delta\gamma + \frac{V_0\cos\gamma_0\cos\chi_0}{R_0\cos\delta_0}\Delta\chi\end{aligned}$$

The last equation can also be linearised relatively easily, noting it doesn't depend on τ or δ:

$$\begin{aligned}\Delta\dot{\delta} &= \frac{\partial\dot{\delta}}{\partial R}\Delta R + \frac{\partial\dot{\delta}}{\partial V}\Delta V + \frac{\partial\dot{\delta}}{\partial\gamma}\Delta\gamma + \frac{\partial\dot{\delta}}{\partial\chi}\Delta\chi \\ &= -\frac{V_0}{R_0^2}\cos\gamma_0\cos\chi_0\Delta R + \frac{1}{R_0}\cos\gamma_0\cos\chi_0\Delta V - \frac{V_0}{R_0}\sin\gamma_0\cos\chi_0\Delta\gamma - \frac{V_0}{R_0}\cos\gamma_0\sin\chi_0\Delta\chi\end{aligned}$$

In matrix format, the first three rows of the state-space system could thus be written as

$$\begin{bmatrix}\Delta\dot{R}\\ \Delta\dot{\tau}\\ \Delta\dot{\delta}\\ \Delta\dot{V}\\ \Delta\dot{\gamma}\\ \Delta\dot{\chi}\end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \sin\gamma_0 & V_0\cos\gamma_0 & 0 \\ \frac{V_0\cos\gamma_0\sin\xi_0}{R_0^2\cos\delta_0} & 0 & -\frac{V_0\cos\gamma_0\sin\chi_0\sin\delta_0}{R_0\cos^2\delta_0} & \frac{\cos\gamma_0\sin\chi_0}{R_0\cos\delta_0} & -\frac{V_0\sin\gamma_0\sin\chi_0}{R_0\cos\delta_0} & \frac{V_0\cos\gamma_0\cos\chi_0}{R_0\cos\delta_0} \\ -\frac{V_0}{R_0^2}\cos\gamma_0\cos\chi_0 & 0 & 0 & \frac{1}{R_0}\cos\gamma_0\chi_0 & -\frac{V_0}{R_0}\sin\gamma_0\cos\chi_0 & -\frac{V_0}{R_0}\cos\gamma_0\sin\chi_0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}\begin{bmatrix}\Delta R\\ \Delta\tau\\ \Delta\delta\\ \Delta V\\ \Delta\gamma\\ \Delta\chi\end{bmatrix}$$

Exam April 2017: question 2 (adapted) (20 points)

a) Given the formulation of the state-space form $\mathbf{\Delta\dot{x}} = \mathbf{A\Delta x} + \mathbf{B\Delta u}$, explain in mathematical terms how you obtain the matrices $\mathbf{A}$ and $\mathbf{B}$ from a set of n non-linear state equations $\mathbf{f}$, with n state variables and m control variables. Provide the expressions for $\mathbf{A}$ and $\mathbf{B}$.

b) Consider the equation

$$m\frac{d\mathbf{V}_C^C}{dt} = \mathbf{F}_{ext}^C - 2m\mathbf{\Omega}_{CI}^C \times \mathbf{V}_C^C - m\mathbf{\Omega}_{CI}^C\left(\mathbf{\Omega}_{CI}^C \times \mathbf{r}_{cm}^C\right) \tag{1.64}$$

with $\mathbf{r}_{cm}^C = [x, y, z]^T$ and $\mathbf{V}_C^C = [V_x, V_y, V_z]^T$. Additionally, $\mathbf{\Omega}_{CI}^C = [0, 0, \Omega_t]^T$. Consider only the second and third right-hand term of the equation:

i) write the vector components of these three terms as three scalar functions and add them together

ii) identify the state variables

c) Linearise each of the three scalar functions separately.

For a), the partial derivatives of $\mathbf{f}$ with respect to all state *and* control variables need to be computed, evaluated at x_0 and u_0. Then the matrix $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\mathbf{A} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x=x_0}$$

$$\mathbf{B} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\right|_{u=u_0}$$

Please note that you should not leave out the statement that you evaluate the derivatives at $x = x_0$ and $u = u_0$, else you'll be subtracted points.

For b), this question seems like a total shithole, but it's actually not that hard if you just compute the cross-products. After all, $\mathbf{\Omega}_t$ has only one nonzero entry, so it's really easy to compute the cross-product of it. We have

$$\mathbf{\Omega}_{CI}^C \times \mathbf{V}_C^C = \begin{bmatrix} 0 \\ 0 \\ \Omega_t \end{bmatrix} \times \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \Omega_t \\ V_x & V_y & V_z \end{pmatrix} = -\Omega_t V_y \mathbf{i} + \Omega_t V_x \mathbf{j} + 0\mathbf{k} = \begin{bmatrix} -\Omega_t V_y \\ \Omega_t V_x \\ 0 \end{bmatrix}$$

$$\mathbf{\Omega}_{CI}^C \times \mathbf{r}_{cm}^C = \begin{bmatrix} 0 \\ 0 \\ \Omega_t \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \Omega_t \\ x & y & z \end{pmatrix} = -\Omega_t y \mathbf{i} + \Omega_t x \mathbf{j} + 0\mathbf{k} = \begin{bmatrix} -\Omega_t y \\ \Omega_t x \\ 0 \end{bmatrix}$$

Thus,

$$\mathbf{\Omega}_{CI}^C \times \left(\mathbf{\Omega}_{CI}^C \times \mathbf{r}_{cm}^C\right) = \begin{bmatrix} 0 \\ 0 \\ \Omega_t \end{bmatrix} \times \begin{bmatrix} -\Omega_t y \\ \Omega_t x \\ 0 \end{bmatrix} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \Omega_t \\ -\Omega_t y & \Omega_t x & 0 \end{pmatrix} = -\Omega_t^2 x \mathbf{i} - \Omega_t^2 y \mathbf{j} + 0\mathbf{k} = \begin{bmatrix} -\Omega_t^2 x \\ -\Omega_t^2 y \\ 0 \end{bmatrix}$$

We can thus write is as three scalar functions:

$$\mathbf{f} = -2m\mathbf{\Omega}_{CI}^C \times \mathbf{V}_C^C - m\mathbf{\Omega}_{CI}^C\left(\mathbf{\Omega}_{CI}^C \times \mathbf{r}_{cm}^C\right)$$

$$\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = -2m\begin{bmatrix} -\Omega_t V_y \\ \Omega_t V_x \\ 0 \end{bmatrix} - m\begin{bmatrix} -\Omega_t^2 x \\ -\Omega_t^2 y \\ 0 \end{bmatrix}$$

$$f_x = 2m\Omega_t V_y + m\Omega_t^2 x$$

$$f_y = -2m\Omega_t V_x + m\Omega_t^2 y$$

$$f_z = 0$$

The state variables are x, y, V_x and V_y (z and V_z would normally also be state variables, but they do not appear in the equations and thus are not state variables for this specific question). m and Ω_t are not states, but are constants.

Okay maybe I totally fucked up before lol but this is incredibly easy now. We simply get

$$\begin{aligned} f_x &= f_x\big|_{x_0} + \left.\frac{\partial f_x}{\partial x}\right|_{x_0} \Delta x + \left.\frac{\partial f_x}{\partial V_y}\right|_{x_0} \Delta V_y = 2m\Omega_t V_{y_0} + m\Omega_t^2 x_0 + 2m\Omega_t \Delta V_y + m\Omega_t^2 \Delta x \\ f_y &= f_y\big|_{x_0} + \left.\frac{\partial f_y}{\partial y}\right|_{x_0} \Delta y + \left.\frac{\partial f_y}{\partial V_x}\right|_{x_0} \Delta V_x = 2m\Omega_t V_{x_0} + m\Omega_t^2 y_0 + 2m\Omega_t \Delta V_x + m\Omega_t^2 \Delta y \\ f_z &= 0 \end{aligned}$$

Honestly how do you get 12 points for linearising a function that's already linear lol.

In hindsight, perhaps they meant you were to do this question by making use of an example discussed in part III of the summary: this question followed that question, and there you had to prove that

$$\begin{aligned} 2m\mathbf{\Omega}_{CI}^C &= 2m\Omega_t \begin{bmatrix} V_E \sin\delta \\ -\left(V_D \cos\delta + V_E \sin\delta\right) \\ V_E \cos\delta \end{bmatrix} \\ m\mathbf{\Omega}_{CI}^C \times \left(\mathbf{\Omega}_{CI}^C \times \mathbf{r}_{cm}^C\right) &= m\Omega_t^2 R \cos\delta \begin{bmatrix} \sin\delta \\ 0 \\ \cos\delta \end{bmatrix} \end{aligned}$$

Doing questions b) and c) again now yields the following:

For b), the scalar components are

$$\begin{aligned} f_x &= -2m\Omega_t V_E \sin\delta - m\Omega_t^2 R \cos\delta \sin\delta \\ f_y &= 2m\Omega_t \left(V_D \cos\delta + V_N \sin\delta\right) \\ f_z &= -2m\Omega_t V_E \cos\delta - m\Omega_t^2 R \cos^2\delta \end{aligned}$$

and the states would be V_N, V_E, V_D, R and δ. Thus, we can linearise everything. First, we have for f_x:

$$\Delta f_x = \left.\frac{\partial f_x}{\partial V_E}\right|_0 \Delta V_E + \left.\frac{\partial f_x}{\partial R}\right|_0 \Delta R + \left.\frac{\partial f_x}{\partial \delta}\right|_0 \Delta\delta$$

We simply have

$$\begin{aligned} \left.\frac{\partial f_x}{\partial V_E}\right|_0 &= -2m\Omega_t \sin\delta_0 \\ \left.\frac{\partial f_x}{\partial R}\right|_0 &= -m\Omega_t^2 \cos\delta_0 \sin\delta_0 \\ \frac{\partial f_x}{\partial \delta} &= -2m\Omega_t V_E \cos\delta + m\Omega_t^2 R \sin^2\delta - m\Omega_t^2 R \cos^2\delta \\ \left.\frac{\partial f_x}{\partial \delta}\right|_0 &= -2m\Omega_t V_{E_0} \cos\delta_0 + m\Omega_t^2 R_0 \left(\sin^2\delta_0 - \cos^2\delta_0\right) \end{aligned}$$

Thus,

$$\Delta f_x = -2m\Omega_t \sin\delta_0 \Delta V_E - m\Omega_t^2 \cos\delta_0 \sin\delta_0 \Delta R + \left[-2m\Omega_t V_{E_0} \cos\delta_0 + m\Omega_t^2 R_0 \left(\sin^2\delta_0 - \cos^2\delta_0\right)\right] \Delta\delta$$

For f_y, we have

$$\Delta f_y = \left.\frac{\partial f_y}{\partial V_N}\right|_0 \Delta V_N + \left.\frac{\partial f_y}{\partial V_D}\right|_0 \Delta V_D + \left.\frac{\partial f_y}{\partial \delta}\right|_0 \Delta\delta$$

All derivatives are straightforward:

$$\left.\frac{\partial f_y}{\partial V_N}\right|_0 = 2m\Omega_t \sin\delta_0$$
$$\left.\frac{\partial f_y}{\partial V_D}\right|_0 = 2m\Omega_t \cos\delta_0$$
$$\left.\frac{\partial f_y}{\partial \delta}\right|_0 = 2m\Omega_t \left(-V_{D_0}\sin\delta_0 + V_{N_0}\cos\delta_0\right)$$

Thus,

$$\Delta f_y = 2m\Omega_t \sin\delta_0 \Delta V_N + 2m\Omega_t \cos\delta_0 \Delta V_D + 2m\Omega_t \left(-V_{D_0}\sin\delta_0 + V_{N_0}\cos\delta_0\right)\Delta\delta$$

Finally, for f_z,

$$\Delta f_z = \left.\frac{\partial f_z}{\partial V_E}\right|_0 \Delta V_E + \left.\frac{\partial f_z}{\partial R}\right|_0 \Delta R + \left.\frac{\partial f_z}{\partial \delta}\right|_0 \Delta\delta$$

We simply have

$$\left.\frac{\partial f_z}{\partial V_E}\right|_0 = -2m\Omega_t \cos\delta_0$$
$$\left.\frac{\partial f_z}{\partial R}\right|_0 = -m\Omega_t^2 \cos^2\delta_0$$
$$\frac{\partial f_z}{\partial \delta} = 2m\Omega_t V_E \sin\delta + m\Omega_t^2 R \sin\delta\cos\delta$$
$$\left.\frac{\partial f_z}{\partial \delta}\right|_0 = 2m\Omega_t V_{E_0} \sin\delta_0 + m\Omega_t^2 R_0 \sin\delta_0\cos\delta_0$$

Thus,

$$\Delta f_z = -2m\Omega_t \cos\delta_0 \Delta V_E + -m\Omega_t^2 \cos^2\delta_0 \Delta R + 2m\Omega_t V_{E_0}\sin\delta_0 + m\Omega_t^2 R_0 \sin\delta_0\cos\delta_0\Delta\delta$$

Exam August 2012: question 3a (16 points)

The open-loop flight behaviour of an entry capsule is characterised by very strong oscillations around all three axes. Individual components of motion are hard to distinguish, because of a strong dynamic coupling, partially the result of an offset in the location of the center of mass in Z-direction. This offset gives a product of inertia I_{xz} that is too large to be ignored.

a) The following set of equations describes the pitch motion of the entry capsule:

$$\dot{q} = \frac{M_y}{I_{yy}} + \frac{I_{xz}}{I_{yy}}\left(r^2 - p^2\right) + \frac{I_{zz} - I_{xx}}{I_{yy}} pr$$
$$\dot{\alpha} \approx q - \frac{L}{mV}$$

The external moment M_y contains an aerodynamic moment $\mathcal{M}$ and a reaction-control moment $M_{T,y}$. Assume the following (READ THIS CAREFULLY):

- Both pitch moment $\mathcal{M}$ and lift L are a function of the angle of attack α, and Mach number, M.
- The nominal state is *trimmed* condition. Realise what this means for the nominal pitch moment.
- For the rotational motion considered, the atmospheric properties are constant.
- Due to the oscillatory nature of the rotational motion, the nominal angular rates p_0, q_0 and r_0 cannot be considered small.
- The Mach number is *not* a state and has to be properly linearised.

- Translational and rotational motion are *not* decoupled.
- Consider all state and control variables (even though remaining state equations are not shown here).

You are asked to linearise the above equation and put it in state-space form. To facilitate you, follow the questions below. Clearly explain what you are doing.

i) Given the formulation for state-space form, $\mathbf{\Delta \dot{x}} = \mathbf{A\Delta x} + \mathbf{B\Delta u}$, explain how you obtain the matrices $\mathbf{A}$ and $\mathbf{B}$ from a set of n state equations $\mathbf{f}$.

ii) Indicate the state variables $\mathbf{\Delta x}$ and control variables $\mathbf{\Delta u}$.

iii) As mentioned, the pitch-moment is, amongst others, a function of Mach number, but also of course dynamic pressure. Derive the partial derivatives for the Mach number and dynamic pressure w.r.t. the related state variables (again: the atmospheric properties are constant). Formulate your answer such that the atmospheric properties do *not* appear directly in your answer.

iv) Linearise both equations and write your answer in the form $\mathbf{\Delta \dot{x}} = \mathbf{A\Delta x} + \mathbf{B\Delta u}$ as two scalar equations.

It takes like an hour to just read the question, and the question in itself is impossible to do on your first attempt if you don't know what to do. So, just follow my lead for all of them. First, for i), just remember by heart, the partial derivatives of $\mathbf{f}$ with respect to all state *and* control variables need to be computed, evaluated at x_0 and u_0. Then the matrix $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\mathbf{A} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x=x_0}$$

$$\mathbf{B} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \dots & \frac{\partial f_1}{\partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \dots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}\right|_{u=u_0}$$

Please note that you should not leave out the statement that you evaluate the derivatives at $x = x_0$ and $u = u_0$, else you'll be subtracted points.

For ii), the state variables are $\mathbf{\Delta x} = [\Delta V, \Delta p, \Delta q, \Delta r, \Delta \alpha]^T$ and the control variables are $\mathbf{\Delta u} = [M_{T,y}]$. Note that q represents pitch rate; $\bar{q}$ represents dynamic pressure.

For iii), let's first focus on the Mach number. We have $M = V/a$, where as is the speed of sound. As the atmospheric properties are assumed to be constant, a is constant as well. Thus,

$$\frac{\partial M}{\partial V} = \frac{1}{a} = \frac{V}{V}\frac{1}{a} = \frac{V}{a}\frac{1}{V} = \frac{M}{V}$$

Furthermore, we have for the dynamic pressure $\bar{q}$, that

$$\frac{\partial \bar{q}}{\partial V} = \frac{\partial \left(0.5\rho V^2\right)}{\partial V} = \frac{\rho V dV}{dV} = \rho V = \frac{0.5\rho V^2}{0.5V} = \frac{\bar{q}}{0.5V} = \frac{2\bar{q}}{V}$$

For iv), let's first focus on the first equation. We have

$$\Delta \dot{q} = \left.\frac{\partial \dot{q}}{\partial V}\right|_0 \Delta V + \left.\frac{\partial \dot{q}}{\partial p}\right|_0 \Delta p + \left.\frac{\partial \dot{q}}{\partial r}\right|_0 \Delta r + \left.\frac{\partial \dot{q}}{\partial \alpha}\right|_0 \Delta \alpha + \left.\frac{\partial \dot{q}}{\partial M_{T,y}}\right|_0 \Delta M_{T,y}$$

Let's do the derivatives one-by-one. Only M_y depends on the velocity, so all other terms in the equation for $\dot{q}$ do not need to be considered at all. Writing $M_y = C_m q_0 S_{ref} c_{ref}$ yields

$$\frac{\partial M_y}{\partial V} = \frac{\partial C_m}{\partial V}\bar{q} S_{ref} q_{ref} + C_m \frac{\partial \bar{q}}{\partial V} S_{ref} q_{ref} = \frac{\partial C_m}{\partial M}\frac{\partial M}{\partial V}\bar{q} S_{ref} q_{ref} + C_m \frac{\partial \bar{q}}{\partial V} S_{ref} c_{ref}$$

Here, as we evaluate the derivative at nominal conditions, $C_m = 0$ (the nominal conditions are trim, after all). Furthermore, as derived before, $\partial M/\partial V = M/V$. Thus,

$$\left.\frac{\partial M_y}{\partial V}\right|_0 = \frac{\partial C_m}{\partial M}\frac{M_0}{V_0}\bar{q}_0 S_{ref} c_{ref}$$

and thus we obtain

$$\left.\frac{\partial \dot{q}}{\partial V}\right|_0 = \frac{1}{I_{yy}}\frac{\partial C_m}{\partial M}\frac{M_0}{V_0} q_0 S_{ref} c_{ref}$$

Then, for $\partial \dot{q}/\partial p$ and $\partial \dot{q}/\partial r$, we obtain simply

$$\begin{aligned}\left.\frac{\partial \dot{q}}{\partial p}\right|_0 &= -2\frac{I_{xz}}{I_{yy}} p_0 + \frac{I_{zz} - I_{xx}}{I_{yy}} r_0 \\ \left.\frac{\partial \dot{q}}{\partial r}\right|_0 &= 2\frac{I_{xz}}{I_{yy}} r_0 + \frac{I_{zz} - I_{xx}}{I_{yy}} p_0\end{aligned}$$

Furthermore, $\partial \dot{q}/\partial \alpha$ may be obtained from realising that only M_y depends on α, so again writing $M_y = C_m \bar{q} S_{ref} c_{ref}$, we obtain

$$\frac{\partial M_y}{\partial \alpha} = C_{m_\alpha}\bar{q} S_{ref} c_{ref}$$

and thus

$$\left.\frac{\partial \dot{q}}{\partial \alpha}\right|_0 = \frac{1}{I_{yy}}\frac{\partial C_m}{\partial \alpha}\bar{q} S_{ref} c_{ref}$$

Finally, we simply have

$$\left.\frac{\partial \dot{q}}{\partial M_{T,y}}\right|_0 = \frac{1}{I_{yy}}$$

Thus,

$$\begin{aligned}\Delta \dot{q} = \frac{1}{I_{yy}}\frac{\partial C_m}{\partial M}\frac{M_0}{V_0} q_0 S_{ref} c_{ref}\Delta V - 2\frac{I_{xz}}{I_{yy}} p_0 + \frac{I_{zz} - I_{xx}}{I_{yy}} r_0 \Delta p \\ + 2\frac{I_{xz}}{I_{yy}} r_0 + \frac{I_{zz} - I_{xx}}{I_{yy}} p_0 \Delta r + \frac{1}{I_{yy}}\frac{\partial C_m}{\partial \alpha}\bar{q} S_{ref} c_{ref}\Delta \alpha + \frac{1}{I_{yy}}\Delta M_{T,y}\end{aligned}$$

We can do similar stuff for $\Delta\dot{\alpha}$:

$$\Delta \dot{\alpha} = \left.\frac{\partial \dot{\alpha}}{\partial V}\right|_0 \Delta V + \left.\frac{\partial \dot{\alpha}}{\partial q}\right|_0 \Delta q + \left.\frac{\partial \dot{\alpha}}{\partial \alpha}\right|_0 \Delta \alpha$$

differentiating with respect to V is again the hardest one. q doesn't depend on V so that's nice, but for L we must substitute $C_L 0.5\rho V^2 S_{ref}$, leading to

$$\begin{aligned}\dot{\alpha} &\approx q - \frac{C_L 0.5\rho V^2 S_{ref}}{mV} = q - \frac{C_L 0.5\rho V S_{ref}}{m} \\ \frac{\partial \dot{\alpha}}{\partial V} &= -\frac{\partial C_L}{\partial V}\frac{0.5\rho V S_{ref}}{m} - \frac{C_L 0.5\rho S_{ref}}{m} = -\left(\frac{\partial C_L}{\partial M}\frac{\partial M}{\partial V} V - C_L\right)\frac{0.5\rho S_{ref}}{m}\end{aligned}$$

Here, $\partial M/\partial V = M/V$ as derived before; thus,

$$\frac{\partial \dot{\alpha}}{\partial V} = -\left(\frac{\partial C_L}{\partial M}\frac{M}{V}V - C_L\right)\frac{0.5\rho S_{ref}}{m} = -\left(\frac{\partial C_L}{\partial M}M - C_L\right)\frac{0.5\rho V^2 S_{ref}}{mV^2}$$
$$\frac{\partial}{\partial \dot{\alpha}}\bigg|_0 = -\left(\frac{\partial C_L}{\partial M}M_0 - C_L\right)\frac{\bar{q}_0 S_{ref}}{mV_0^2}$$

$\partial\dot{\alpha}/\partial q$ is trivial to compute; it's equal to

$$\frac{\partial \dot{\alpha}}{\partial q}\bigg|_0 = 1$$

Finally, we have

$$\dot{\alpha} \approx q - \frac{C_L 0.5\rho V^2 S_{ref}}{mV} = q - \frac{C_L 0.5\rho V S_{ref}}{m}$$
$$\frac{\partial \dot{\alpha}}{\partial \alpha} = -\frac{\partial C_L}{\partial \alpha}\frac{0.5\rho V S_{ref}}{m} = -\frac{1}{V}\frac{\partial C_L}{\partial \alpha}\frac{0.5\rho V^2 S_{ref}}{m}$$
$$\frac{\partial \dot{\alpha}}{\partial \alpha}\bigg|_0 = -\frac{1}{mV_0}\frac{\partial C_L}{\partial \alpha}\bar{q}S_{ref}$$

and thus we have

$$\Delta\dot{\alpha} = -\left(\frac{\partial C_L}{\partial M}\frac{\partial M}{\partial V}V - C_L\right)\frac{0.5\rho S_{ref}}{m}\Delta V + \Delta q - \frac{1}{mV_0}\frac{\partial C_L}{\partial \alpha}\bar{q}S_{ref}\Delta\alpha$$

Yes it's an absolutely awful derivation, but the stuff with the Mach number and dynamic pressure not being a state appears very frequently on exams so it's recommended you understand them well.

1.4 Normalising the equations of motion

Remember we had the equations of motion, given below:

SYMMETRICAL MOTION

The symmetrical motion for steady, straight, symmetric flight is given by

$$m\dot{u} = -W\cos\theta_0\theta + X_u u + X_w w + X_q q + X_{\delta_e}\delta_e + X_{\delta_t}\delta_t \tag{1.65}$$
$$m(\dot{w} - qV) = -W\sin\theta_0\theta + Z_u u + Z_w w + Z_{\dot{w}}\dot{w} + Z_q q + Z_{\delta_e}\delta_e + Z_{\delta_t}\delta_t \tag{1.66}$$
$$I_{yy}\dot{q} = M_u u + M_{\dot{w}}\dot{w} + M_q q + M_{\delta_e}\delta_e + M_{\delta_t}\delta_t \tag{1.67}$$
$$\dot{\theta} = q \tag{1.68}$$

ASYMMETRICAL MOTION

The asymmetrical motion for steady, straight, symmetric flight is given by

$$m(\dot{v} + rV) = W\cos\theta_0\phi + Y_v v + Y_{\dot{v}}\dot{v} + Y_p p + Y_r r + Y_{\delta_a}\delta_a + Y_{\delta_r}\delta_r \tag{1.69}$$
$$I_{xx}\dot{p} - I_{xz}\dot{r} = L_v v + L_p p + L_r r + L_{\delta_a}\delta_a + L_{\delta_r}\delta_r \tag{1.70}$$
$$I_{zz}\dot{r} - I_{xz}\dot{p} = N_v v + N_{\dot{v}}\dot{v} + N_p p + N_r r + N_{\delta_a}\delta_a + N_{\delta_r}\delta_r \tag{1.71}$$
$$\dot{\psi} = \frac{r}{\cos\theta_0} \tag{1.72}$$
$$\dot{\phi} = p + r\tan\theta_0 \tag{1.73}$$

Flight Dynamics summary: part IV

Based on *Lecture Notes Flight Dynamics* by J.A. Mulder et al.

Sam van Elsloo

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We can make these non-dimensional[5], so that we can compare aircraft independent of aircraft dimensions and weight. We do this by dividing the symmetric ones by $1/2\rho V^2 S\bar{c}$, and the asymmetric ones by $1/2\rho V^2 Sb$. This division isn't important at all for the exam, nor is it for the SVV project, so I'll omit most of it, except the general things that may clarify why some stuff looks different from before.

1.4.1 Symmetric equations of motion in non-dimensional form

For the symmetric equations of motion, the following 'tricks' are applied:

- We define the forces $W\sin\theta_0 = X_0$ and $W\cos\theta_0 = -Z_0$. These represent the force due to the weight of the aircraft in the negative X_S-direction and Z_S-direction.
- The normalised time derivative is $D_c = \bar{c}/V \cdot d/dt$.
- We define $\hat{u} = u/V$, $\alpha = w/V$ and $\dot{\alpha} = \dot{w}/V$.
- For forces, the force derivatives with respect to regular components (e.g. X_u) is normalised with respect to $1/2\rho V S$, i.e. $C_{X_u} = X_u/(1/2\rho V S)$; note that V does not appear squared. For force derivatives with respect to time-derivative components (e.g. $Z_{\dot{w}}$. Note that q is already a time-derivative), they are normalised with respect to $1/2\rho S\bar{c}$, i.e. $C_{Z_{\dot{w}}} = Z_{\dot{w}}/(1/2\rho S\bar{c})$. This is because the time-derivative means that a factor $\bar{c}/V$ goes into the normalising of the time-derivative; see bullet 2.
- For moments, the moment derivatives with respect to regular components (e.g. M_u) is normalised with respect to $1/2\rho V S\bar{c}$, i.e. $C_{M_u} = M_u/(1/2\rho V S\bar{c})$. For moment derivatives with respect to time derivatives (e.g. $C_{m_{\dot{\alpha}}}$), they are normalised with respect to $1/2\rho S\bar{c}^2$, i.e. $C_{m_{\dot{\alpha}}} = M_{\dot{\alpha}}/(1/2\rho S\bar{c}^2)$.
- We define non-dimensional mass and products of inertia $\mu_c = m/(\rho S\bar{c})$ and $K_Y^2 = I_{yy}/(m\bar{c}^2)$.

If you take all these things into account, it'll be absolutely no surprise to anyone that it straightforwardly (no jk it's absolutely awful to write down, there are no difficult computations, but it's just so much work, that's why I can't be arsed to nicely put it into LaTeX) leads to

$$
\begin{aligned}
\left(C_{X_u} - 2\mu_c D_c\right)\hat{u} + C_{X_\alpha}\alpha + C_{Z_0}\theta + C_{X_q}\frac{q\bar{c}}{V} + C_{X_{\delta_e}} + C_{X_{\delta_t}} &= 0 \\
C_{Z_u}\hat{u} + \left[C_{Z_\alpha} + \left(C_{Z_{\dot{\alpha}}} + 2\mu_c\right)D_c\right]\alpha - C_{X_0}\theta + \left(C_{Z_q} + 2\mu_c\right)\frac{q\bar{c}}{V} + C_{Z_{\delta_e}}\delta_e + C_{Z_{\delta_t}}\delta_t &= 0 \\
-D_c\theta + \frac{q\bar{c}}{V} &= 0 \\
C_{m_u}\hat{u} + \left(C_{m_\alpha} + C_{m_{\dot{\alpha}}}D_c\right)\alpha + \left(C_{m_q} - 2\mu_c K_Y^2 D_c\right)\frac{q\bar{c}}{V} + C_{m_{\delta_e}}\delta_e + C_{m_{\delta_t}}\delta_t &= 0
\end{aligned}
$$

In matrix notation, this becomes

LINEARISED EQUATIONS OF SYMMETRIC MOTION IN MATRIX FORMAT

The linearised symmetric equations of motion in matrix format are given by

$$
\begin{bmatrix}
C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & C_{X_q} \\
C_{Z_u} & C_{Z_\alpha} + \left(C_{Z_{\dot{\alpha}}} - 2\mu_c\right)D_c & -C_{X_0} & C_{Z_q} + 2\mu_c \\
0 & 0 & -D_c & 1 \\
C_{m_u} & C_{m_\alpha} + C_{m_{\dot{\alpha}}}D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_C
\end{bmatrix}
\begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix}
=
\begin{bmatrix}
-C_{X_{\delta_e}} & -C_{X_{\delta_t}} \\
-C_{Z_{\delta_e}} & -C_{Z_{\delta_t}} \\
0 & 0 \\
-C_{m_{\delta_e}} & -C_{m_{\delta_t}}
\end{bmatrix}
\begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix}
\tag{1.74}
$$

Note that the Cessna 550 aircraft used in the flight test has such a small trim tab that it's not included in the equations of motion for that one in the assignment. This may be written as an equation with $\dot{\mathbf{x}}$ on one side and $\mathbf{x}_s$ on the other side. This can be obtained by splitting the time-derivatives from the remainder of the equations

[5] I included the front page again cause some of you fuckers don't print the front page, so hopefully you won't notice this until printing it.

of motion and straightforwardly leads to (with $D_c = \bar{c}/V \cdot d/dt$)

$$P_s \dot{\mathbf{x}}_s = Q_s \mathbf{x}_s + R_s \mathbf{u}_s$$

$$\begin{bmatrix} -2\mu_c \frac{\bar{c}}{V} & 0 & 0 & 0 \\ 0 & \left(C_{Z_{\dot{\alpha}}} - 2\mu_c\right)\frac{\bar{c}}{V} & 0 & 0 \\ 0 & 0 & -\frac{\bar{c}}{V} & 0 \\ 0 & C_{m_{\dot{\alpha}}}\frac{\bar{c}}{V} & 0 & -2\mu_c K_Y^2 \frac{\bar{c}}{V} \end{bmatrix} \begin{bmatrix} \dot{\hat{u}} \\ \dot{\alpha} \\ \dot{\theta} \\ \frac{\dot{q}\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} -C_{X_u} & -C_{X_\alpha} & -C_{Z_0} & 0 \\ -C_{Z_u} & -C_{Z_\alpha} & C_{X_0} & -\left(C_{Z_q} + 2\mu_c\right) \\ 0 & 0 & 0 & -1 \\ -C_{m_u} & -C_{m_\alpha} & 0 & -C_{m_q} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} + \begin{bmatrix} -C_{X_{\delta_e}} \\ -C_{Z_{\delta_e}} \\ 0 \\ -C_{m_{\delta_e}} \end{bmatrix} \begin{bmatrix} \delta_e \end{bmatrix}$$

An explicit expression for $\dot{\mathbf{x}}_s$ may be obtained by premultiplying both sides of the equation by the inverse of P_s.

1.4.2 Asymmetric equations of motion in non-dimensional form

For the asymmetric equations of motions, the following ‘tricks’ are applied:

- We let $C_L = W \cos\theta_0 / (1/2\rho V^2 S)$.
- The normalised time derivative is $D_b = b/V \cdot d/dt$.
- We define $\beta = v/V$ and $\dot{\beta} = \dot{v}/V$.
- For the force and moment derivatives with respect to (time-derivatives of) the sideslip, the same rules apply as before, except all $\bar{c}$ are replaced with b.
- For the force and moment derivatives with respect to the roll rate p and yaw rate r (note that these are time-derivatives by themselves), the same rules apply as before, except they are now multiplied with an additional factor 2, e.g. $C_{l_p} = 2L_p/(1/2\rho V S b^2)$.
- We define non-dimensional mass and products of inertia $\mu_b = m/(\rho S \bar{b})$, $K_X^2 = I_{xx}/(mb^2)$, $K_Z = I_{zz}/(mb^2)$ and $J_{XZ} = I_{xz}/(mb^2)$.
- The kinematic equation relating $\dot{\psi}$ to $r/\cos\theta_0$ will be ignored, as it is not necessary to obtain a determined system of equations.
- In the kinematic equation relating $\dot{\phi}$ to $p + r\tan\theta_0$, θ_0 will be replaced with γ_0, which equals zero as it is level flight we are analysing.

If you take all these things into account, it’ll be absolutely no surprise to anyone that it straightforwardly (no jk it’s absolutely awful to write down, there are no difficult computations, but it’s just so much work, that’s why I can’t be arsed to nicely put it into LaTeX) leads to

$$\begin{aligned} \left[C_{Y_\beta} + \left(C_{Y_{\dot{\beta}}} - 2\mu_b\right) D_b\right]\beta + C_L\phi + C_{Y_p}\frac{pb}{2V} + \left(C_{Y_r} - 4\mu_b\right)\frac{rb}{2V} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r &= 0 \\ -\frac{1}{2}D_b\phi + \frac{pb}{2V} &= 0 \\ C_{l_\beta}\beta + \left(C_{l_p} - 4\mu_b K_X^2 D_b\right)\frac{pb}{2V} + \left(C_{l_r} + 4\mu_b K_{xxZ} D_b\right)\frac{rb}{2V} + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r &= 0 \\ \left(C_{n_\beta} + C_{n_{\dot{\beta}}} D_b\right)\beta + \left(C_{n_p} + 4\mu_b K_{XZ} D_b\right)\frac{pb}{2V} + \left(C_{n_r} - 4\mu_b K_Z^2 D_b\right)\frac{rb}{2V} + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r &= 0 \end{aligned}$$

In matrix notation, this becomes

LINEARISED EQUATIONS OF ASYMMETRIC MOTION IN MATRIX FORMAT

The linearised asymmetric equations of motion in matrix format are given by

$$\begin{bmatrix} C_{Y_\beta} + \left(C_{Y_{\dot{\beta}}} - 2\mu_b\right) D_b & C_L & C_{Y_p} & C_{Y_r} - 4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} - 4\mu_b K_X^2 D_b & C_{l_r} + 4\mu_b K_{XZ} D_b \\ C_{n_\beta} + C_{n_{\dot{\beta}}} & 0 & C_{n_p} + 4\mu_b K_{XZ} D_b & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} = \begin{bmatrix} -C_{Y_{\delta_a}} & -C_{Y_{\delta_r}} \\ 0 & 0 \\ -C_{l_{\delta_a}} & -C_{l_{\delta_r}} \\ -C_{n_{\delta_a}} & -C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \tag{1.75}$$

This may be written as an equation with $\dot{\mathbf{x}}$ on one side and $\mathbf{x}_s$ on the other side. This can be obtained by splitting the time-derivatives from the remainder of the equations of motion and straightforwardly leads to (with

$D_b = \bar{b}/V \cdot d/dt)$

$$P_a\dot{\mathbf{x}}_a = Q_a\mathbf{x}_a + R_a\mathbf{u}_a$$

$$\begin{bmatrix} \left(C_{Y_{\dot{\beta}}} - 2\mu_b\right)\frac{b}{V} & 0 & 0 & 0 \\ 0 & -\frac{1}{2}\frac{b}{V} & 0 & 0 \\ 0 & 0 & -4\mu_b K_X^2 \frac{b}{V} & 4\mu_b K_{XZ}\frac{b}{V} \\ C_{n_{\dot{\beta}}}\frac{b}{V} & 0 & 4\mu_b K_{XZ}\frac{b}{V} & -4\mu_b K_Z^2\frac{b}{V} \end{bmatrix} \begin{bmatrix} \dot{\beta} \\ \dot{\phi} \\ \frac{\dot{p}b}{2V} \\ \frac{\dot{q}b}{2V} \end{bmatrix} = \begin{bmatrix} -C_{Y_\beta} & -C_L & -C_{Y_p} & -\left(C_{Y_r} - 4\mu_b\right) \\ 0 & 0 & -1 & 0 \\ -C_{l_\beta} & 0 & -C_{l_p} & -C_{l_r} \\ -C_{n_\beta} & 0 & -C_{n_p} & -C_{n_r} \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} + \begin{bmatrix} -C_{Y_{\delta_a}} & -C_{Y_{\delta_r}} \\ 0 & 0 \\ -C_{l_{\delta_a}} & -C_{l_{\delta_r}} \\ -C_{n_{\delta_a}} & -C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix}$$

An explicit expression for $\dot{\mathbf{x}}_a$ is obtained by premultiplying both sides of the equation with the inverse of P_a.

2 Analysis of symmetric equations of motion

2.1 Derivation of eigenvalues

Consider again the symmetric equations of motion, as shown in Equation (1.74).

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & C_{X_q} \\ C_{Z_u} & C_{Z_\alpha} + \left(C_{Z_{\dot\alpha}} - 2\mu_c\right) D_c & -C_{X_0} & C_{Z_q} + 2\mu_c \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot\alpha}} D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_C \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} -C_{X_{\delta_e}} & -C_{X_{\delta_t}} \\ -C_{Z_{\delta_e}} & -C_{Z_{\delta_t}} \\ 0 & 0 \\ -C_{m_{\delta_e}} & -C_{m_{\delta_t}} \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix} \tag{1.74}$$

The dynamic stability of an aircraft can be established by analysing the response of an aircraft to a disturbance from its equilibrium condition, i.e. the independent variables, δ_e and δ_t, are assumed to be zero. As you may well know, the eigenvalues tell a great deal about the stability of the solution

$$\mathbf{x} = \sum \boldsymbol{\xi}_i e^{\lambda_{c,i} s_c} \tag{2.1}$$

may be assumed, where $\boldsymbol{\xi}_i$ are the eigenvectors and λ_i are the eigenvalues. The eigenvalues may be obtained straightforwardly; applying the differential operator D_c to equation (2.1)

$$D_c \mathbf{x} = \frac{\bar{c}}{V}\frac{d}{dt} \sum \boldsymbol{\xi}_i e^{\lambda_{c,i} s_c} = \sum \lambda_{c,i} \boldsymbol{\xi}_i e^{\lambda_{c,i} s_c} \tag{2.2}$$

Here, s_c is the dimensionless time-parameter, $s_c = (V/\bar{c})t$. Evidently, for the components x_j of $\mathbf{x}$, it holds that $D_c x_j = \lambda_{c,j} x_j$. Substituting this expression into equation (1.74), together with the fact that the input can be ignored, leads to

$$\begin{bmatrix} C_{X_u} - 2\mu_c \lambda_c & C_{X_\alpha} & C_{Z_0} & C_{X_q} \\ C_{Z_u} & C_{Z_\alpha} + \left(C_{Z_{\dot\alpha}} - 2\mu_c\right) \lambda_c & -C_{X_0} & C_{Z_q} + 2\mu_c \\ 0 & 0 & -\lambda & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot\alpha}} \lambda_c & 0 & C_{m_q} - 2\mu_c K_Y^2 \lambda_c \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \mathbf{0} \tag{2.3}$$

The eigenvalues can then be found by finding the determinant of the matrix and setting it equal to zero. This leads to a **characteristic equation** of the form

$$A\lambda_c^4 + B\lambda_c^3 + C\lambda_c^2 + D\lambda_c + E = 0 \tag{2.4}$$

Solving this equation leads to for eigenvalues λ_c. Note that these eigenvalues are non-dimensional! You need to multiply with $V/\bar{c}$ to get the dimensional ones (the ones you obtain when you don't normalise the equations of motion), and if you want to obtain stuff like period, time to damp half the amplitude, you need to multiply by $V/\bar{c}$ as well (of course you don't need to do this for stuff like the damping ratio, which are dimensionless anyway). I've mentioned a few characteristics now already, so let's dive deeper into them.

2.1.1 Characteristics of real eigenvalues

For real eigenvalues, two important characteristics exist. However, it should first be mentioned that obviously only if the eigenvalue is negative, the corresponding eigenmotion is zero; else $e^{\lambda_c s_c}$ blows up to infinity.

Time to damp to half the amplitude

The time to damp to half the amplitude is derived as follows, taking $s_c = (V/\bar{c})t$: the time to damp to half the amplitude, $T_{\frac{1}{2}}$ represents

$$\begin{aligned}
x\left(t + T_{\frac{1}{2}}\right) &= \frac{1}{2}x(t) \\
e^{\lambda_c \frac{V}{\bar{c}}\left(t + T_{\frac{1}{2}}\right)} &= \frac{1}{2}e^{\lambda_c \frac{V}{c}t} \\
\lambda_c \frac{V}{\bar{c}}\left(t + T_{\frac{1}{2}}\right) &= \ln\left(\frac{1}{2}\right) + \lambda_c \frac{V}{\bar{c}}t \\
\lambda_c \frac{V}{\bar{c}}T_{\frac{1}{2}} &= \ln\left(\frac{1}{2}\right) \\
T_{\frac{1}{2}} &= \frac{\ln\left(\frac{1}{2}\right)}{\lambda_c}\frac{\bar{c}}{V}
\end{aligned}$$

Time constant

The time constant τ is the time interval in which the exponent decreases by 1 and the exponential function itself decreases by the factor $1/e$. Then,

$$\begin{aligned}
x(t+\tau) &= \frac{1}{e}x(t) \\
e^{\lambda_c \frac{V}{\bar{c}}(t+\tau)} &= \frac{1}{e}e^{\lambda_c \frac{V}{\bar{c}}t} \\
\lambda_c \frac{V}{\bar{c}}(t+\tau) &= -1 + \lambda_c \frac{V}{\bar{c}}t \\
\lambda_c \frac{V}{\bar{c}}\tau &= -1 \\
\tau &= -\frac{1}{\lambda_c}\frac{\bar{c}}{V}
\end{aligned}$$

Summarising:

CHARACTERISTICS OF REAL EIGENVALUES

The time to damp to half the amplitude $T_{\frac{1}{2}}$ is given by

$$T_{\frac{1}{2}} = \frac{\ln\left(\frac{1}{2}\right)}{\lambda_c}\frac{\bar{c}}{V} \tag{2.5}$$

The time constant τ is given by

$$\tau = -\frac{1}{\lambda_c}\frac{\bar{c}}{V} \tag{2.6}$$

It should be noted that when the eigenmotion is unstable, i.e. when λ_c is positive, then both $T_{\frac{1}{2}}$ and τ denote the time to *double* the amplitude T_2 and the time to *increase* the exponent by 1, respectively.

2.1.2 *Characteristics of complex eigenvalues*

For complex eigenvalues, written as

$$\lambda_c = \xi_c + j\eta_c \tag{2.7}$$

it should be noted that for an eigenmotion with eigenvalues that are complex, the eigenvalues are conjugate, i.e. $\lambda_{c_{1,2}} = \xi_c \pm j_c$. We can now once again find some interesting parameters.

Period

First, the period P of oscillation is the time it takes for the argument of the harmonic function (due to the imaginary component) to increase by 2π; i.e.

$$\eta_c \frac{V}{\bar{c}} P = 2\pi$$
$$P = \frac{2\pi}{\eta_c} \frac{\bar{c}}{V}$$

Time to damp to half amplitude

The time to damp to half amplitude $T_{\frac{1}{2}}$ is once more given by

$$T_{\frac{1}{2}} = \frac{\ln\left(\frac{1}{2}\right)}{\lambda_c} \frac{\bar{c}}{V}$$

Number of periods in which the amplitude decreases to half its original value

Okay lol I don't there's a sexier name for this tbh. This parameter $C_{\frac{1}{2}}$ is simply

$$C_{\frac{1}{2}} = \frac{T_{\frac{1}{2}}}{P}$$

Logarithmic decrement

The logarithmic decrement δ is the ratio of the oscillation's amplitude in two successive maxima,

$$\delta = \ln \frac{e^{\xi_c \frac{V}{\bar{c}}(t+P)}}{e^{\xi_c \frac{V}{\bar{c}} t}} = \xi_c \frac{V}{\bar{c}} P$$

Damping ratio

The damping ratio ζ is given by

$$\zeta = -\frac{\xi_c}{\sqrt{\xi_c^2 + \eta_c^2}}$$

CHARACTERISTICS OF COMPLEX EIGENVALUES

The period P is given by

$$P = \frac{2\pi}{\eta_c}\frac{\bar{c}}{V} \tag{2.8}$$

The time to damp to half amplitude $T_{\frac{1}{2}}$ is given by

$$T_{\frac{1}{2}} = \frac{\ln\left(\frac{1}{2}\right)}{\lambda_c}\frac{\bar{c}}{V} \tag{2.9}$$

The number of periods in which the amplitude decreases to half its original value $C_{\frac{1}{2}}$ is given by

$$C_{\frac{1}{2}} = \frac{T_{\frac{1}{2}}}{P} \tag{2.10}$$

The logarithmic decrement δ is given by

$$\delta = \xi_c \frac{V}{\bar{c}} P \tag{2.11}$$

The damping ratio is given by

$$\zeta = -\frac{\xi_c}{\sqrt{\xi_c^2 + \eta_c^2}} \tag{2.12}$$

Note: these formulas are not given on the formula sheet; you'll have to know them by heart, unfortunately (as sometimes you do need them).

2.2 *Stability criteria*

We require the real (parts of the complex) eigenvalues to be negative. Consider the following equation once more:

$$A\lambda_c^4 + B\lambda_c^3 + C\lambda_c^2 + D\lambda_c + E = 0 \tag{2.4}$$

Two people, Routh and Hurwitz, did some analysis, and they found that if $A > 0$, then if

$$A > 0, \qquad B > 0, \qquad C > 0, \qquad D > 0, \qquad E > 0$$

and if

$$BDC - AD^2 - B^2E > 0$$

then all eigenvalues are negative. These stability criteria are called the **Routh-Hurwitz Stability Criteria**; the latter expression,

$$R = BCD - AD^2 - B^2E > 0 \tag{2.13}$$

is often called **Routh's discriminant**.

In case $A < 0$, them the signs of B, C, D, E and R must also be negative to obtain negative real parts of the eigenvalues.

2.3 *Approximate solutions*

Exact solutions to the full equations of motion are possible to compute but are extremely laborious; their results are included the reader but I won't bother with them as it's not important at all for the exam. Instead, we'll make some simplifications, reducing the size of the matrix to a more reasonable size, so that we can determine the eigenvalues from there. Remember one more time the matrix equation,

LINEARISED EQUATIONS OF SYMMETRIC MOTION IN MATRIX FORMAT

The linearised symmetric equations of motion in matrix format are given by

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & C_{X_q} \\ C_{Z_u} & C_{Z_\alpha} + \left(C_{Z_{\dot{\alpha}}} - 2\mu_c\right) D_c & -C_{X_0} & C_{Z_q} + 2\mu_c \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & 0 & C_{m_q} - 2\mu_c K_Y^2 D_C \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} -C_{X_{\delta_e}} & -C_{X_{\delta_t}} \\ -C_{Z_{\delta_e}} & -C_{Z_{\delta_t}} \\ 0 & 0 \\ -C_{m_{\delta_e}} & -C_{m_{\delta_t}} \end{bmatrix} \begin{bmatrix} \delta_e \\ \delta_t \end{bmatrix} \tag{1.74}$$

Please note: you do not need to know the final results by heart: it is important that you understand the method to get there.

2.3.1 *Short-period motion*

The short-period motion is characterized by its heavily damped oscillation and its motion of rapid pitching about the aircraft center of gravity. This oscillation period is so short that the airspeed can be assumed to remain constant, and that it is essentially purely an angle of attack variation. This implies that $\hat{u} = 0$ and the first column of the matrix in (1.74) will be neglected. This also implies that the forces in X_B direction should remain in equilibrium, and therefore the X_B equation (first row in the matrix) can be entirely neglected. Furthermore, the condition of initial steady flight is assumed to be level; therefore $\gamma_0 = 0$ and $C_{X_0} = 0$. Finally, the trivial kinematic relation $-D_c\theta + \frac{q\bar{c}}{V} = 0$ can be omitted as well. This simplification for short period motion reduces the (1.74) to the final expression as

$$\begin{bmatrix} C_{Z_\alpha} + \left(C_{Z_{\dot{\alpha}}} - 2\mu_c\right) D_c & C_{Z_q} + 2\mu_c \\ C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & C_{m_q} - 2\mu_c K_Y^2 D_C \end{bmatrix} \begin{bmatrix} \alpha \\ \frac{q\bar{c}}{V} \end{bmatrix} = \mathbf{0}$$

This equation can be used in order to obtain the characteristic equation, and finding the eigenvalues by equating the characteristic determinant to zero. This leads to two expressions of eigenvalues $\lambda_{c_{1,2}}$ of the following.

$$\lambda_{c_{1,2}} = \frac{-B \pm j\sqrt{4AC - B^2}}{2A}$$

Where the coefficients are given by

$$A = 2\mu_c {K_Y}^2 \left(2\mu_c - C_{Z_{\dot{\alpha}}}\right)$$
$$B = -2\mu_c {K_Y}^2 C_{Z_\alpha} - \left(2\mu_c + C_{z_q}\right) C_{m_{\dot{\alpha}}} - \left(2\mu_c - C_{Z_{\dot{\alpha}}}\right) C_{m_q}$$
$$C = C_{z_\alpha} C_{m_q} - (2\mu_c + C_{z_q}) C_{m_\alpha}$$

2.3.2 *Phugoid*

The phugoid is characterized by a long-period oscillation around the equilibrium condition. Due to the long period, it may be assumed that accelerations are negligible; consequently, the time rate of change of the pitch rate q may be assumed to be zero, meaning that in Equation (1.74), all contributions due to $D_c(q\bar{c}/V)$ are ignored. Additionally, the time rate of change of the angle of attack may be neglected, as the angle of attack remains nearly constant during the motion. Consequentially, the contributions due to $D_c\alpha$ are neglected. Furthermore, in horizontal flight, $\gamma = 0$ and thus $C_{X_0} = 0$. Finally, C_{z_q} is neglected as it is relatively small compared to $2\mu_c$. Implementing these simplifications in Equation (1.74) leads to

$$\begin{bmatrix} C_{X_u} - 2\mu_c D_c & C_{X_\alpha} & C_{Z_0} & 0 \\ C_{Z_u} & C_{Z_\alpha} & 0 & 2\mu_c \\ 0 & 0 & -D_c & 1 \\ C_{m_u} & C_{m_\alpha} & 0 & C_{m_q} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \alpha \\ \theta \\ \frac{q\bar{c}}{V} \end{bmatrix} = \mathbf{0}$$

The corresponding eigenvalues are given by

$$\lambda_{c_{3,4}} = \frac{-B \pm j\sqrt{4AC - B^2}}{2A}$$

with the coefficients given by

$$\begin{aligned}
A &= 2\mu_c \left(C_{Z_\alpha} C_{m_q} - 2\mu_c C_{m_\alpha} \right) \\
B &= 2\mu_c \left(C_{X_u} C_{m_\alpha} - C_{m_u} C_{X_\alpha} \right) + C_{m_q} \left(C_{Z_u} C_{X_\alpha} - C_{X_u} C_{Z_\alpha} \right) \\
C &= C_{Z_0} \left(C_{m_u} C_{Z_\alpha} - C_{Z_u} C_{m_\alpha} \right)
\end{aligned}$$

3 Analysis of asymmetric equations of motion

For the characteristics of the eigenvalues, literally nothing changes, except for the fact you should use b everywhere where we used $\bar{c}$. Thus I won't repeat them here. The stability criteria are also still exactly the same.

3.1 Spiral and Dutch roll mode, the lateral stability diagram

The spiral motion and Dutch roll eigenmotions may both be unstable, depending on the Routh-Hurwitz Stability Criteria. To be specific, if $E < 0$, then the spiral in unstable; if $R < 0$, then the Dutch roll is unstable. Now, the magnitudes of E and R are influenced to a large extent by the effective dihedral C_{l_β} and the static directional stability C_{n_β}. The former is usually negative; the latter is usually positive. We can plot the lines $E = 0$ and $R = 0$ as functions of these two parameters as shown in figure 3.1: in the region to the left of the $E = 0$ curve the spiral is unstable. In the region to the right of the $R = 0$ curve, the Dutch roll is unstable. Only in between the two curves our aircraft is stable in both eigenmotions, which is nice to know.

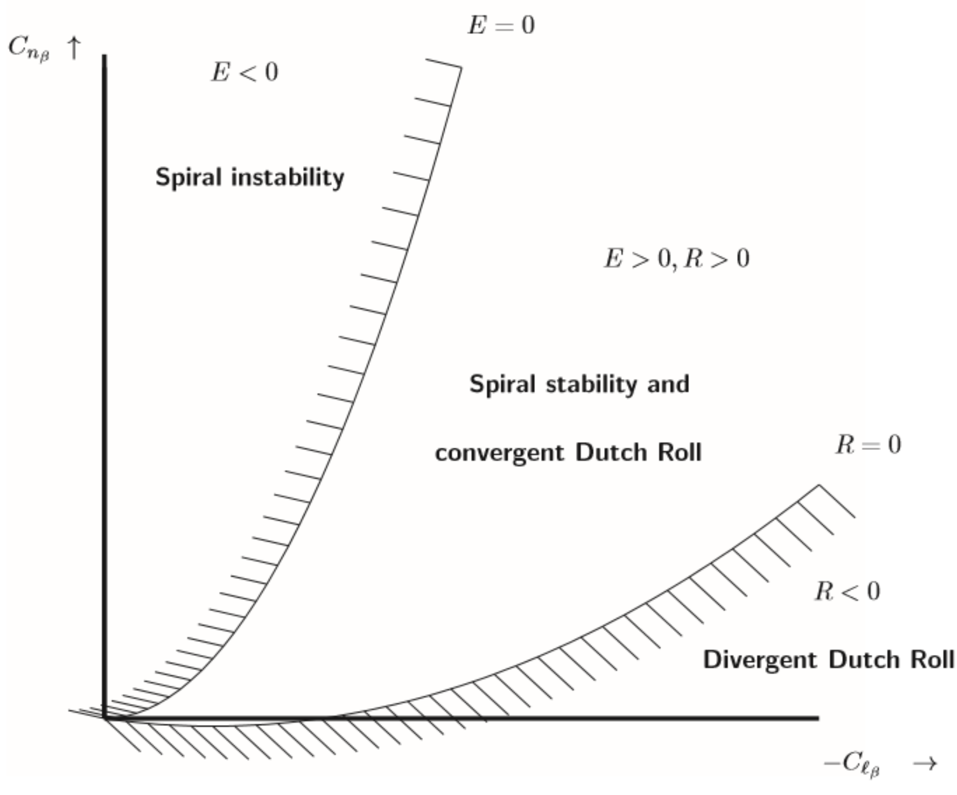

Figure 3.1: Lateral stability diagram.

Additionally, it can be shown that for spiral stability, we must specifically have

$$C_{l_\beta} C_{n_r} - C_{n_\beta} C_{l_r} > 0 \tag{3.1}$$

which is also nice to know I suppose. Now, something actually important is, suppose we have designed our aircraft, and we find out that we have either spiral or Dutch roll is unstable. What do we do? Well, we can change the values of C_{n_β} and C_{l_β}; this will cause us to shift in the lateral stability diagram to the safe zone. C_{l_β}

can be made more negative by increasing the dihedral of the wing (or increasing the sweep of it). C_{n_β} can be made more positive by increasing the vertical tail surface area S_v or increasing the tail length l_v.

Finally, it should be noted that spiral instability is generally considered less bad than Dutch roll instability as the spiral motion takes much longer than the Dutch roll.

3.2 Approximate solutions

Once again, we can come up with approximate solutions by considering the asymmetric equations of motion once more:

LINEARISED EQUATIONS OF ASYMMETRIC MOTION IN MATRIX FORMAT

The linearised asymmetric equations of motion in matrix format are given by

$$\begin{bmatrix} C_{Y_\beta} + \left(C_{Y_{\dot\beta}} - 2\mu_b\right) D_b & C_L & C_{Y_p} & C_{Y_r} - 4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} - 4\mu_b K_X^2 D_b & C_{l_r} + 4\mu_b K_{XZ} D_b \\ C_{n_\beta} + C_{n_{\dot\beta}} & 0 & C_{n_p} + 4\mu_b K_{XZ} D_b & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} = \begin{bmatrix} -C_{Y_{\delta_a}} & -C_{Y_{\delta_r}} \\ 0 & 0 \\ -C_{l_{\delta_a}} & -C_{l_{\delta_r}} \\ -C_{n_{\delta_a}} & -C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \delta_a \\ \delta_r \end{bmatrix} \tag{1.75}$$

3.2.1 Aperiodic Roll

The aperiodic roll motion is the motion that occurs when rolling the aircraft at such a roll rate that the roll damping moment counteracts the rolling moment induced by the ailerons, such that no resultant rolling moment acts on the aircraft, leading to a constant roll rate. By ignoring the yawing moment, the columns corresponding to the angle of sideslip β and the non-dimensional yaw rate $rb/(2V)$ disappear. The equations for the lateral force and the yawing moment may be omitted as well, such that only the second and third row in equation (1.75) remain. As the roll angle ϕ does not appear in the third row, the kinematic relation $-1/2D_b\phi + pb/(2V)$ may be removed as well. Thus, equation (1.75) reduces to merely

$$\left(C_{l_p} - 4\mu_b K_X^2 D_b\right) \frac{pb}{2V} = 0$$

so that the real eigenvalue is given by

$$\lambda_{b,1} = \frac{C_{l_p}}{4\mu_b K_X^2}$$

3.2.2 Dutch Roll

Dutch roll is an eigenmotion in which the aircraft rolls and yaws in an oscillatory fashion, with the yaw lagging the roll by a quarter period. To obtain an approximation for the eigenvalues, it may be assumed that the rolling component is discarded. Consequentially, the columns in equation (1.75) corresponding to ϕ and $pb/(2V)$ are ignored. The row corresponding to the kinematic relation $-1/2D_b\phi + pb/(2V)$ and the row corresponding to the rolling moment may also be ignored with this assumption. Neglecting $C_{Y_{\dot\beta}}$, $C_{n_{\dot\beta}}$ and C_{Y_r} results in the simplified system

$$\begin{bmatrix} C_{Y_\beta} - 2\mu_b D_b & -4\mu_b \\ C_{n_\beta} & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \frac{rb}{2V} \end{bmatrix} = \mathbf{0}$$

The eigenvalues are then given by

$$\lambda_{b_{2,3}} = \frac{2\left(C_{n_r} + 2K_Z^2 C_{Y_\beta}\right) \pm \sqrt{64K_Z^2\left(4\mu_b C_{n_\beta} + C_{Y_\beta}C_{n_r}\right) - 4\left(C_{n_r} + 2K_Z^2 C_{Y_\beta}\right)^2}}{16\mu_b K_Z^2}$$

3.2.3 Spiral Motion

Usually, the spiral motion is a very slow and possibly unstable eigenmotion, where the aircraft sidedslips, yaws, and rolls. Hence, it is possible to assume all linear and angular accelerations to be negligible. This implies that $D_b\beta = D_b\frac{pb}{2V} = D_b\frac{rb}{2V} = 0$. In addition, C_{Y_r} and C_{Y_P} can be neglected. These assumptions simplifies to the following matrix:

$$\begin{bmatrix} C_{Y_\beta} & C_L & 0 & -4\mu_b \\ 0 & -\frac{1}{2}D_b & 1 & 0 \\ C_{l_\beta} & 0 & C_{l_p} & C_{l_r} \\ C_{n_\beta} & 0 & C_{n_p} & C_{n_r} \end{bmatrix} \begin{bmatrix} \beta \\ \phi \\ \frac{pb}{2V} \\ \frac{rb}{2V} \end{bmatrix} = \mathbf{0}$$

Due to the many zeros, the characteristic equation reduces to first order, of which the solution is given by

$$\lambda_{b_4} = \frac{2C_L(C_{l_\beta}C_{n_r} - C_{n_\beta}C_{l_r})}{C_{l_p}(C_{Y_\beta}C_{n_r} + 4\mu_b C_{n_\beta}) - C_{n_p}(C_{Y_\beta}C_{l_r} + 4\mu_b C_{l_\beta})}$$

4 Exam questions

Exam August 2011: problem 4 (20 points)

a) Draw the lateral stability diagram for a conventional aircraft. Clearly indicate which parameters are on the axes. Also indicate the regions where the Dutch roll and spiral motion are stable and where they are unstable.
b) Imagine an aircraft that has a convergent spiral motion while the Dutch roll is unstable. If you could change the geometric properties of just one part of the aircraft (wings, tail, fuselage, vertical stabiliser or horizontal stabiliser) what alteration is required to make the Dutch roll stable while keeping a convergent spiral motion? Present two possible alterations.
c) Clearly indicate in the figure of question (4a) in which the region the aircraft of question (4b) falls before the alteration (mark the location with a circle). Indicate by means of an arrow which path is followed through the lateral stability diagram when performing each alteration presented in question (4b). Indicate which arrow belongs to which alteration.
d) What is the sign of the aerodynamic coefficient C_{n_p} for a conventional aircraft which a positive wing sweep? Explain what the contribution of the wing is for this aerodynamic coefficient. Clearly mention how the aerodynamic forces which define this contribution are generated.

See figure 4.1 (ignore the crosses and arrows for now).

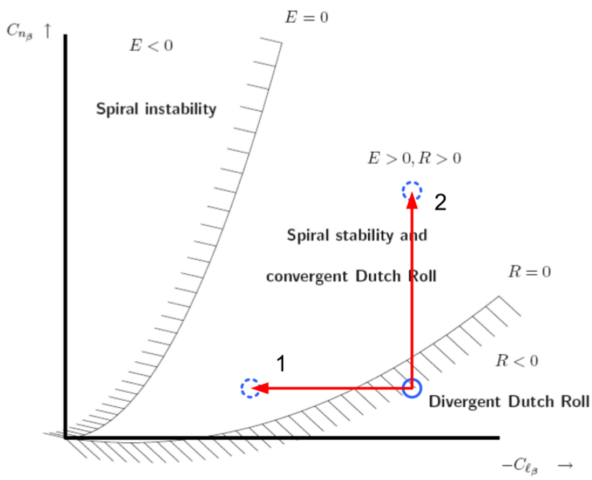

Figure 4.1: Lateral stability diagram.

For b), apparently the C_{l_β} is too negative, and C_{n_β} not positive enough. C_{n_β} can be made more positive by increasing the vertical tail surface area: the tail will then produce more force in negative Y_b-direction when under influence of sideslip, creating a more positive yawing moment. C_{l_β} can be made less negative by decreasing the dihedral of the wing. In general, dihedral makes the right wing produce more lift under sideslip as the angle of attack is increased, whereas for the left wing less will produces as the angle of attack is decreased for that wing. Thus, a negative rolling moment is induced; decreasing dihedral will make this less effective, decreasing the magnitude of C_{l_β}.

For c), see figure 4.1. Arrow 1 corresponds to the decrease in dihedral; arrow 2 corresponds to the increase in vertical tail surface area.

For d), C_{n_p} is generally negative: under influence of a positive rolling velocity p, the right wing will go down and the left wing will go up. This means that the angle of attack of the down-going wing is increased, whereas it will decrease for the up-going wing. In other words, the velocity vector is inclined downward for the down-going wing, meaning the lift (perpendicular to the velocity vector) will have a component in positive X_b-direction, pushing the right-wing forward. The up-going wing will have a velocity vector that's inclined upwards, so the lift will have a component in negative X_b-direction, pulling it aft. Thus, a negative yawing moment is created.

Exam August 2014: problem 3bcd (13 points)

b) To analytically study the open-loop pitch motion, we make some simplifications. In that case, the state-space model is given by:

$$\Delta\dot{q} = \frac{1}{I_{yy}} C_{m_\alpha} \bar{q}_0 S_{ref} c_{ref} \Delta\alpha + \frac{\Delta M_{t,y}}{I_{yy}}$$
$$\Delta\dot{\alpha} = \Delta q - \frac{1}{mV_0} C_{L_\alpha} \bar{q}_0 S_{ref} \Delta\alpha$$

You are asked to derive an analytical expression for the eigenvalues related to the longitudinal oscillation.

c) A numerical representation of the eigenvalues found under (b) is $\lambda_{1,2} = -1.9744 \cdot 10^{-2} \pm 1.5534 \cdot 10^{1} j$. You are asked to calculate the *dimensional* values of the damping factor, ζ, period, P, and amplitude half (or double) time, $T_{\frac{1}{2}}$ or T_2, depending on the nature of the eigenmotion. First state the used equation, then your numerical value.

d) Is the mode under (c) stable or unstable? Explain why.

For b), in matrix notation we have

$$\begin{bmatrix} \Delta\dot{q} \\ \Delta\dot{\alpha} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{I_{yy}} C_{m_\alpha} \bar{q}_0 S_{ref} c_{ref} \\ 1 & -\frac{1}{mV_0} C_{L_\alpha} \bar{q}_0 S_{ref} \end{bmatrix} \begin{bmatrix} \Delta q \\ \Delta\alpha \end{bmatrix} + \begin{bmatrix} \frac{1}{I_{yy}} \\ 0 \end{bmatrix} \begin{bmatrix} \Delta M_{T,y} \end{bmatrix}$$

The eigenvalues of the first matrix ought to be determined. This can be done straightforwardly, as the determinant of $\mathbf{A} - \lambda\mathbf{I}$ is required to be zero:

$$\begin{vmatrix} -\lambda & \frac{1}{I_{yy}} C_{m_\alpha} \bar{q}_0 S_{ref} c_{ref} \\ 1 & -\frac{1}{mV_0} C_{L_\alpha} \bar{q}_0 S_{ref} - \lambda \end{vmatrix} = -\lambda \cdot \left(-\frac{1}{mV_0} C_{L_\alpha} \bar{q}_0 S_{ref} - \lambda \right) - \frac{1}{I_{yy}} C_{m_\alpha} \bar{q}_0 S_{ref} c_{ref} = 0$$

$$\lambda^2 + \lambda \cdot \frac{1}{mV_0} C_{L_\alpha} \bar{q}_0 S_{ref} - \frac{1}{I_{yy}} C_{m_\alpha} \bar{q}_0 S_{ref} c_{ref} = 0$$

From the quadratic formula, it follows that

$$\lambda = \frac{-\frac{1}{mV_0} C_{L_\alpha} \bar{q}_0 S_{ref} \pm \sqrt{\left(\frac{1}{mV_0} C_{L_\alpha} \bar{q}_0 S_{ref} \right)^2 - 4 \cdot -\frac{1}{I_{yy}} C_{m_\alpha} \bar{q}_0 S_{ref} c_{ref}}}{2}$$

Yes it's ugly but it's literally just the quadratic formula.

For the damping factor, we have

$$\zeta = -\frac{\xi}{\sqrt{\xi^2+\eta^2}} = -\frac{-1.9744\cdot 10^{-2}}{\sqrt{\left(-1.9744\cdot 10^{-2}\right)^2+\left(1.5334\cdot 10^{1}\right)^2}} = 0.001288$$

$$P = \frac{2\pi}{\eta} = \frac{2\pi}{1.5334\cdot 10^{1}} = 0.4098\,\mathrm{s}$$

$$T_{\frac{1}{2}} = \frac{\ln\left(\frac{1}{2}\right)}{\xi} = \frac{\ln\left(\frac{1}{2}\right)}{-1.9744\cdot 10^{-2}} = 35.11\,\mathrm{s}$$

For d), the mode is stable: the real eigenvalue is negative, thus the exponential is monotonously decreasing.

Exam April 2013: problem 3 (25 points)

Figure 4.2: Northrop B-2 Spirit.

A flying wing is an aircraft without a tailplane, see figure 4.2. As a result, the characteristic modes of the flying wing differ significantly from those of a conventional aircraft. In this question the dynamics of the short period mode of a flying wing will be examined.

a) From the full linearised longitudinal equations of motion, derive a simplified form describing the short period motion. Assume that $\hat{u}$ and that $\theta = 0$. Additionally, the stability derivatives $C_{m_{\dot{\alpha}}}$, $C_{Z_{\dot{\alpha}}}$, C_{m_q} and C_{Z_q} are assumed to be zero. Note: γ is allowed to vary and *not* assumed to be constant.

b) Derive the characteristic equation of the simplified short period motion and determine the non-dimensional eigenvalues. Does the flying wing have stable short period motion? Use the numerical data from table 4.3.

c) In order to improve the flying qualities of the flying wing, a simple pitch-rate feedback controller is integrated with the flight control system. This feedback controller has the following definition: $\delta_e = k_q \frac{q\bar{c}}{V}$. Derive the new characteristic equation for the simplified short period motion from part (b).

d) For what range of the feedback gain is the short period both stable and periodic. Use the numerical data from table 4.3.

$V = 180[ms^{-1}]$	$\mu_c = 205$	$K_Y^2 = 0.98$	$C_{X_0} = 0$	$C_{X_u} = -0.22$	$C_{X_\alpha} = 0.465$
$C_{Z_0} = -1.14$	$C_{Z_u} = -2.72$	$C_{Z_\alpha} = 0.5$	$C_{Z_q} = 0$	$C_{Z_{\dot{\alpha}}} = 0$	$C_{Z_{\delta_e}} = -0.44$
$C_{m_\alpha} = -0.45$	$C_{m_u} = 0$	$C_{m_{\dot{\alpha}}} = 0$	$C_{m_{\delta_e}} = -1.28$	$C_{m_q} = 0$	

Figure 4.3: Aircraft data.

For a), first, note that the equations of motion in matrix format appear on the formula sheet. Then, $\theta = 0$ and $\hat{u} = 0$ mean that the first and third column may be discarded from the matrix equation. The third row may then also be discarded as it is a trivial kinematic relation. The first row may also be omitted: this equation represents force equilibrium in X_s-direction; however, it is assumed that $\hat{u} = 0$ so the forces should be in equilibrium at all times. Furthermore, setting the specified derivatives equal to zero leads to

$$\begin{bmatrix} C_{Z_\alpha} - 2\mu_c D_c & 2\mu_c \\ C_{m_\alpha} & -2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \alpha \\ \frac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} -C_{Z_{\delta_e}} \\ -C_{m_{\delta_e}} \end{bmatrix} \delta_e$$

For b), we substitute D_c with the non-dimensional eigenvalues λ_c, and set the determinant equal to zero:

$$\begin{vmatrix} C_{Z_\alpha} - 2\mu_c \lambda_c & 2\mu_c \\ C_{m_\alpha} & -2\mu_c K_Y^2 \lambda_c \end{vmatrix} = \left(C_{Z_\alpha} - 2\mu_c \lambda\right)\left(-2\mu_c K_Y^2 \lambda_c\right) - 2\mu_c C_{m_\alpha} = 0$$

This can be solved for λ_c:

$$\begin{aligned} 4\mu_c^2 K_Y^2 \lambda_c^2 - 2\mu_c C_{Z_\alpha} K_Y^2 \lambda_c - 2\mu_c C_{m_\alpha} &= 0 \\ 164738\lambda_c^2 - 200.9\lambda_c + 184.5 &= 0 \end{aligned}$$

$$\lambda = \frac{200.9 \pm \sqrt{(-200.9)^2 - 4 \cdot 164738 \cdot 184.5}}{2 \cdot 164738} = 0.0006 \pm 0.0335i$$

As the real part of this complex eigenvalue is positive, it is unstable.

For c), we now obtain for our simplified short period motion

$$\begin{bmatrix} C_{Z_\alpha} - 2\mu_c D_c & 2\mu_c \\ C_{m_\alpha} & -2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \alpha \\ \frac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} -C_{Z_{\delta_e}} \\ -C_{m_{\delta_e}} \end{bmatrix} k_q \frac{q\bar{c}}{V}$$

Moving the right-hand side term to the left-side leads to

$$\begin{bmatrix} C_{Z_\alpha} - 2\mu_c D_c & 2\mu_c + C_{Z_{\delta_e}} k_q \\ C_{m_\alpha} & -2\mu_c K_Y^2 D_c + C_{m_{\delta_e}} k_q \end{bmatrix}$$

Once again substituting λ_c and computing the determinant leads to

$$\begin{aligned} \begin{vmatrix} C_{Z_\alpha} - 2\mu_c \lambda_c & 2\mu_c + C_{Z_{\delta_e}} k_q \\ C_{m_\alpha} & -2\mu_c K_Y^2 \lambda_c + C_{m_{\delta_e}} k_q \end{vmatrix} =& 4\mu_c^2 K_y^2 + \left(-2\mu_c C_{Z_\alpha} K_Y^2 - 2\mu_c C_{m_{\delta_e}} k)q\right) \lambda_c \\ & - 2\mu_c C_{m_\alpha} + \left(-C_{m_\alpha} C_{Z_{\delta_e}} + C_{Z_\alpha} C_{m_{\delta_e}}\right) k_q = 0 \end{aligned}$$

Writing this is $A\lambda^2 + B\lambda + C = 0$, we obtain

$$A = 4\mu_c^2 K_Y^2 = 164738$$
$$B = -2\mu_c C_{Z_\alpha} K_Y^2 - 2\mu_c C_{m_{\delta_e}} k_q = -200.9 + 542.8k_q$$
$$C = -2\mu_c C_{m_\alpha} + \left(C_{Z_\alpha} - C_{m_{\delta_e}} - C_{m_\alpha} C_{Z_{\delta_e}}\right) k_q = 184.5 - 0.838k_q$$

For d), we require for stability that $-B/2A < 0$, and for periodicity we require $B^2 - 4AC < 0$. First focussing on the first requirement; A is positive and not affected by k_q; thus we require $-B < 0$, or $B > 0$. Thus,

$$\begin{aligned} -200.9 + 524.8k_q &> 0 \\ k_q &> 0.3828 \end{aligned}$$

For $B^2 - 4AC < 0$, we obtain

$$\begin{aligned} \left(-200.9 + 524.8k_q\right)^2 - 4 \cdot 164738 \cdot \left(184.5 - 0.838k_q\right) &< 0 \\ 40360.81 - 218097.04k_q + 275415.05k_q^2 - 121576644 + 552201.78k_q &< 0 \\ 275415.05k_q^2 + 334104.74k_q - 121436283 &< 0 \end{aligned}$$

This has solutions

$$\begin{aligned} k_q &= \frac{-334104.74 \pm \sqrt{334104.74^2 - 4 \cdot 275415.05 \cdot -121436283}}{2 \cdot 275415.05} \\ k_{q_{max}} &= 20.04 \\ k_{q_{min}} &= -22.05 \end{aligned}$$

$B^2 - 4AC$ is smaller than zero for values of k_q between these two limits (as the parabola is convex). Combined with the previous requirement, we thus have

$$0.3828 < k_q < 20.04$$

Exam October 2014: problem 4 (20 points)

a) How many eigenmotions does a conventional aircraft have and what are they called?
b) What is the name of the lateral eigenmotion which is characterised by an oscillation? Draw the time response of the eigenmotion during three periods in a figure with p and r on the axes taking into account the following assumptions:
   - The eigenmotion is stable
   - Only this eigenmotion is excited and that the initalisation phase (i.e. the time interval when the eigenmotion is introduced) has passed.
   - The time to half amplitude is equal to the period.
   - The maximal values of p and r is 0.08 rad/s during the considering time interval.

   Clearly indicate the direction of time by using arrows.

Consider the aircraft data considered in table 4.4.

c) Derive the simplified EOM for the short period mode assuming that the initial steady flight condition is level, i.e. $\gamma_0 = 0$, and the airspeed is constant (note that the flight path angle is allowed to vary!).
d) Compute the dimensionless eigenvalues and the dimensional time to half amplitude, period, damped natural frequency, and the damping ratio for this eigenmotion.

V	=	51.82 m/sec	m	=	5897 kg	$\bar{c}$	=	2.134 m
S	=	21.37 m^2				μ_c	=	105.56
K_Y^2	=	0.8979	x_{cg}	=	0.32 $\bar{c}$			
C_{X_0}	=	0	C_{Z_0}	=	-1.640			
C_{X_u}	=	0	C_{Z_u}	=	-3.72	C_{m_u}	=	-0.004
C_{X_α}	=	0.580	C_{Z_α}	=	-5.5296	C_{m_α}	=	-0.660
$C_{X_{\dot{\alpha}}}$	=	0	$C_{Z_{\dot{\alpha}}}$	=	-0.80	$C_{m_{\dot{\alpha}}}$	=	-2.50
C_{X_q}	=	0	C_{Z_q}	=	-2.050	C_{m_q}	=	-6.75
$C_{X_{\delta_e}}$	=	0	$C_{Z_{\delta_e}}$	=	-0.400	$C_{m_{\delta_e}}$	=	-0.980

Figure 4.4: Aircraft data.

For a), we have

- Short period motion
- Phugoid
- Dutch Roll
- Aperiodic roll
- Spiral motion

For b), the eigenmotion is called Dutch Roll, and see figure 4.5: however, please note that the the plot should be mirrored with repsect to the horizontal axis! For aircraft, C_{l_r} is positive: a positive yaw rate produces a positive roll rate, not the other way around that's shown in the plot.

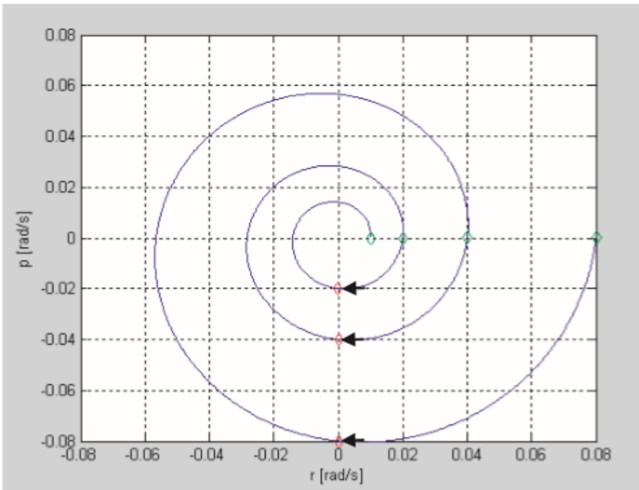

Figure 4.5: Aircraft data.

For c), assuming that $\gamma_0 = 0$ means that C_{X_0} reduces to zero. Additionally, $\hat{u}$ is assumed to be zero as airspeed is assumed constant, thus, the entire first column can be omitted. That the velocity is constant also implies force equilibrium in X, thus the full first row can be neglected. Finally, the now trivial kinematic equation of the third row may be omitted, resulting in

$$\begin{bmatrix} C_{Z_\alpha} + \left(C_{Z_{\dot{\alpha}}} - 2\mu_c\right) D_c & 2\mu_c + C_{Z_q} \\ C_{m_\alpha} + C_{m_{\dot{\alpha}}} D_c & C_{m_q} - 2\mu_c K_Y^2 D_c \end{bmatrix} \begin{bmatrix} \alpha \\ \frac{q\bar{c}}{V} \end{bmatrix} = \begin{bmatrix} -C_{Z_{\delta_e}} \\ -C_{m_{\delta_e}} \end{bmatrix} \delta_e$$

For d), we first obtain the dimensionless eigenvalues, by substituting D_c with λ_c, computing the characteristic equation and setting it equal to zero:

$$\begin{vmatrix} C_{Z_\alpha} + \left(C_{Z_{\dot\alpha}} - 2\mu_c\right)\lambda_c & 2\mu_c + C_{Z_q} \\ C_{m_\alpha} + C_{m_{\dot\alpha}}\lambda_c & C_{m_q} - 2\mu_c K_Y^2 \lambda_c \end{vmatrix} =$$

$$\left(C_{Z_\alpha} + \left(C_{Z_{\dot\alpha}} - 2\mu_c\right)\lambda_c\right) \cdot \left(C_{m_q} - 2\mu_c K_Y^2 \lambda_c\right) - \left(C_{Z_q} + 2\mu_c\right) \cdot \left(C_{m_\alpha} + C_{m_{\dot\alpha}}\lambda_c\right) = 0$$

$$2\mu_c K_Y^2 \left(2\mu_c - C_{Z_{\dot\alpha}}\right)\lambda_c^2 + \left[-2\mu_c K_Y^2 C_{Z_\alpha} - \left(2\mu_c + C_{Z_q}\right)C_{m_{\dot\alpha}} - \left(2\mu_c - C_{Z_{\dot\alpha}}\right)C_{m_q}\right]\lambda_c$$
$$+C_{Z_\alpha} - \left(2\mu_c + C_{Z_q}\right)C_{m_\alpha} = 0$$

Writing it as $A\lambda^2 + B\lambda + C$ means we have

$$A = 2\mu_c K_Y^2 \left(2\mu_c - C_{Z_{\dot\alpha}}\right) = 40172.5$$
$$B = -2\mu_c K_Y^2 C_{Z_\alpha} - \left(2\mu_c + C_{Z_q}\right)C_{m_{\dot\alpha}} - \left(2\mu_c - C_{Z_{\dot\alpha}}\right)C_{m_q} = 3001.4$$
$$C = C_{Z_\alpha} - \left(2\mu_c + C_{Z_q}\right)C_{m_\alpha} = 175.31$$

Thus, the eigenvalues are

$$\lambda_c = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-3001.4 \pm \sqrt{3001.4^2 - 4 \cdot 40172.5 \cdot 175.31}}{2 \cdot 40172.5} = -0.0374 \pm 0.0545i$$

The requested characteristics are then easily computed:

$$P = \frac{2\pi}{\eta_c}\frac{\bar{c}}{V} = \frac{2\pi}{0.0545}\frac{2.134}{51.82} = 4.75\,\text{s}$$
$$T_{\frac{1}{2}} = \frac{\ln\frac{1}{2}}{\xi_c}\frac{\bar{c}}{V} = 0.763\,\text{s}$$
$$\omega_n = \frac{2\pi}{P} = \frac{2\pi}{4.75} = 1.323\,\text{rad/s}$$
$$\zeta = \frac{-\xi_c}{\sqrt{\xi_c^2 + \eta_c^2}} = -\frac{-0.0374}{\sqrt{(-0.0374)^2 + (0.0545)^2}} = 0.566$$
$$\omega_0 = \frac{\omega_n}{\sqrt{1-\zeta^2}} = \frac{1.323}{\sqrt{1 - 0.566^2}} = 1.323\,\text{rad/s}$$

Exam April 2014: problem 5 (22 points)

$V = 125[ms^{-1}]$	$b = 13.36[m]$	$\mu_b = 15.5$	$K_X^2 = 0.012$	$K_{XZ} = 0.002$
$K_Z^2 = 0.037$	$C_L = 1.136$	$C_{Y_\beta} = -0.9896$	$C_{Y_p} = -0.087$	$C_{Y_r} = 0.43$
$C_{Y_{\dot\beta}} = 0$	$C_{l_\beta} = -0.0772$	$C_{l_p} = -0.3444$	$C_{l_r} = 0.28$	$C_{n_\beta} = 0.1638$
$C_{n_p} = -0.0108$	$C_{n_r} = -0.1930$	$C_{n_{\dot\beta}}$ =-0.15		

Figure 4.6: Aircraft data.

The linearised equations of motion can be significantly simplified without loosing too much of their accuracy for predicting the characteristic modes. In this question, the linearised asymmetric equations of motion will be simplified for the Dutch roll motion.

a) Starting with the full, linearised asymmetric equations of motion, derive the simplified set of equations describing the Dutch roll motion assuming that the roll angle ϕ and roll rate r can be discarded. Also assume that $C_{Y_{\dot{\beta}}}$ can be neglected and that C_{Y_r} is insignificant relative to $4\mu_b$. Note that in this case $C_{n_{\dot{\beta}}}$ will *not* be neglected.

b) Using the numerical data in table 4.6, calculate the eigenvalues corresponding with the simplified set of equations of motion from part a).

c) Calculate the range of values of C_{n_β} for which the Dutch roll motion becomes non-periodic.

d) Is the range of values for C_{n_β} you found in part c) beneficial for static lateral stability.

For a), the fact that ϕ and p are discarded means that the second and third column can be omitted completely; hence, the second row may also be omitted. Additionally, as $p = 0$ implies moment equilibrium about the X-axis, the third row may also be omitted. Neglecting the contributions of C_{Y_r} and $C_{\dot{\beta}}$, we obtain

$$\begin{bmatrix} C_{Y_\beta} - 2\mu_b D_b & -4\mu_b \\ C_{n_\beta} + C_{n_{\dot{\beta}}} D_b & C_{n_r} - 4\mu_b K_Z^2 D_b \end{bmatrix} \begin{bmatrix} \beta \\ \frac{rb}{2V} \end{bmatrix} = 0$$

For b), we obtain the eigenvalues by substituting $\lambda_c = D_c$ and determining the characteristic equation by setting the determinant equal to 0:

$$\begin{vmatrix} C_{Y_\beta} - 2\mu_b \lambda_b & -4\mu_b \\ C_{n_\beta} + C_{n_{\dot{\beta}}} \lambda_b & C_{n_r} - 4\mu_b K_Z^2 \lambda_b \end{vmatrix} = 0$$

$$\left(C_{Y_\beta} - 2\mu_b \lambda_b\right)\left(C_{n_r} - 4\mu_b K_Z^2 \lambda_b\right) - \left(-4\mu_b\right)\left(C_{n_\beta} + C_{n_{\dot{\beta}}} \lambda_b\right) = 0$$

Writing this as $A\lambda_c^2 + B\lambda_c + C = 0$ and collecting terms leads to

$$\begin{aligned} A &= 8\mu_b^2 K_Z^2 = 71.114 \\ B &= -2\mu_b \left(C_{n_r} + 2K_Z^2 C_{Y_\beta} - 2C_{n_{\dot{\beta}}}\right) = -1.04685 \\ C &= 4\mu_b C_{n_\beta} + C_{Y_\beta} C_{n_r} = 10.346 \end{aligned}$$

Solving for $\lambda_{c_{1,2}}$ yields

$$\begin{aligned} \lambda_{c_{1,2}} &= \frac{-B + \sqrt{B^2 - 4AC}}{2A} = \frac{1.04685 \pm \sqrt{(-1.04685)^2 - 4 \cdot 71.114 \cdot 10.346}}{2 \cdot 71.114} \\ &= 7.3604 \cdot 10^{-3} \pm 0.3814i \end{aligned}$$

For c), the Dutch roll becomes non-periodic for $4AC < B^2$, i.e. when

$$4 \cdot 71.114 \cdot \left[4\mu_b C_{n_\beta} + C_{Y_\beta} C_{n_r}\right] < (-1.04685)^2$$

$$C_{n_\beta} < \frac{1}{4\mu_b}\left[\frac{(-1.04685)^2}{4 \cdot 71.114} - (-0.9896) \cdot (-0.193)\right] = -3.0134 \cdot 10^{-3}$$

So, for $C_{n_\beta} < -3.0134 \cdot 10^{-3}$ the Dutch roll is a-periodic.

For d), we apparently require a negative C_{n_β}, which is is not desired for lateral stability: C_{n_β} needs to be negative to have weathervane stability, as then a positive sideslip will result in a counteracting yawing moment. So, this is not beneficial for lateral stability.

Figure 4.7: Congratulations, you have reached the end of the final summary of the Bachelor!

Index