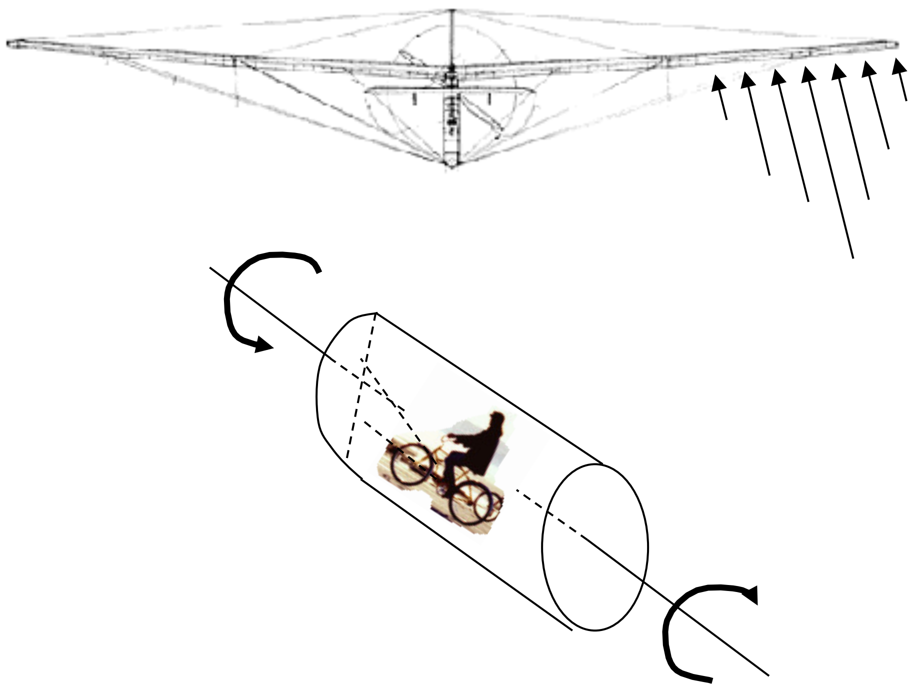
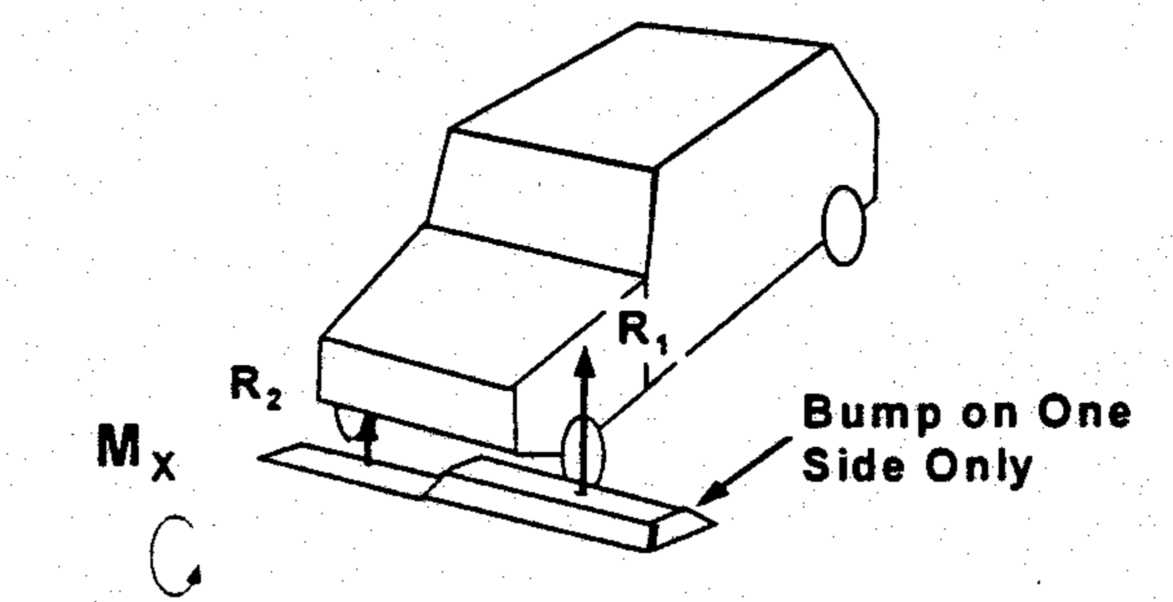
Megson Ch 3.1, 1.5, 18.1, 17.1

Torsion



e.g. wind gust acting on one wing tip twists the fuselage and puts it under torsion

or in cars…



Hello guys and gals I'm brand new to the forum and have question, here goes;

I've done a torsion test on a car chassis with one end secured and the other end attached to a device which has a bar connected perpendicular to the chassis which rests on an arced section of steel on the floor. Now the arced section was placed centrally to the chassis and a load applied on the end of the bar. There was 6 DTI's set up at various points of the chassis (3 either side) and the deflection was taken. Now this was repeated when the shear plates were removed and only the top one in place then only the bottom. So the data that I've got is the load applied, angle of twist, radius of twist, distance from fixed point to load applied and deflection. Now what I was wondering is, does anyone know how to calculate the torque of the chassis per degree, I know how to calculate it for simple round bars/tubes but am confused on how to do it for a chassis. Any help would be very much appreciated.

Torque applied= load*radius

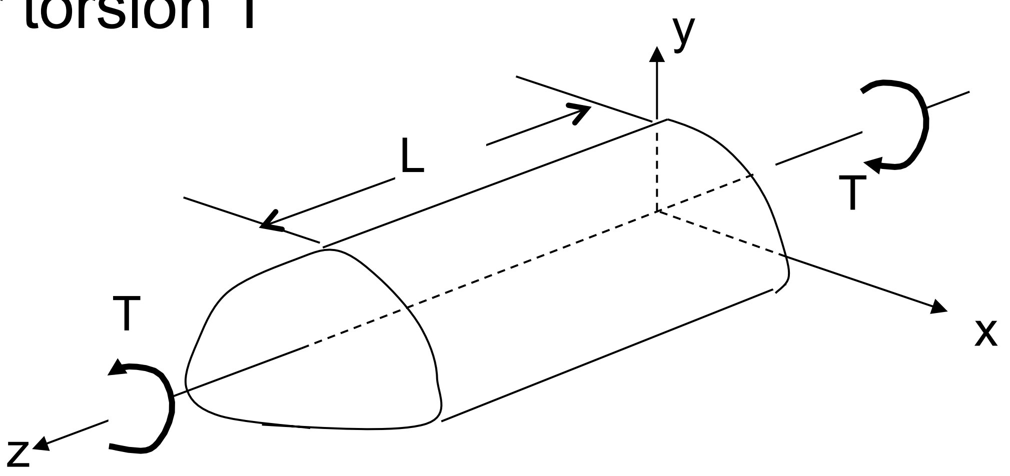
Doesn't get much easier than that does it?

Your degrees of twist are just the angular motion of your rocker system.
Cheers

I don't think that is right for what I need as this would give me the same torsional stiffness for the chassis when it had the shear panels on and off? Or are you saying to

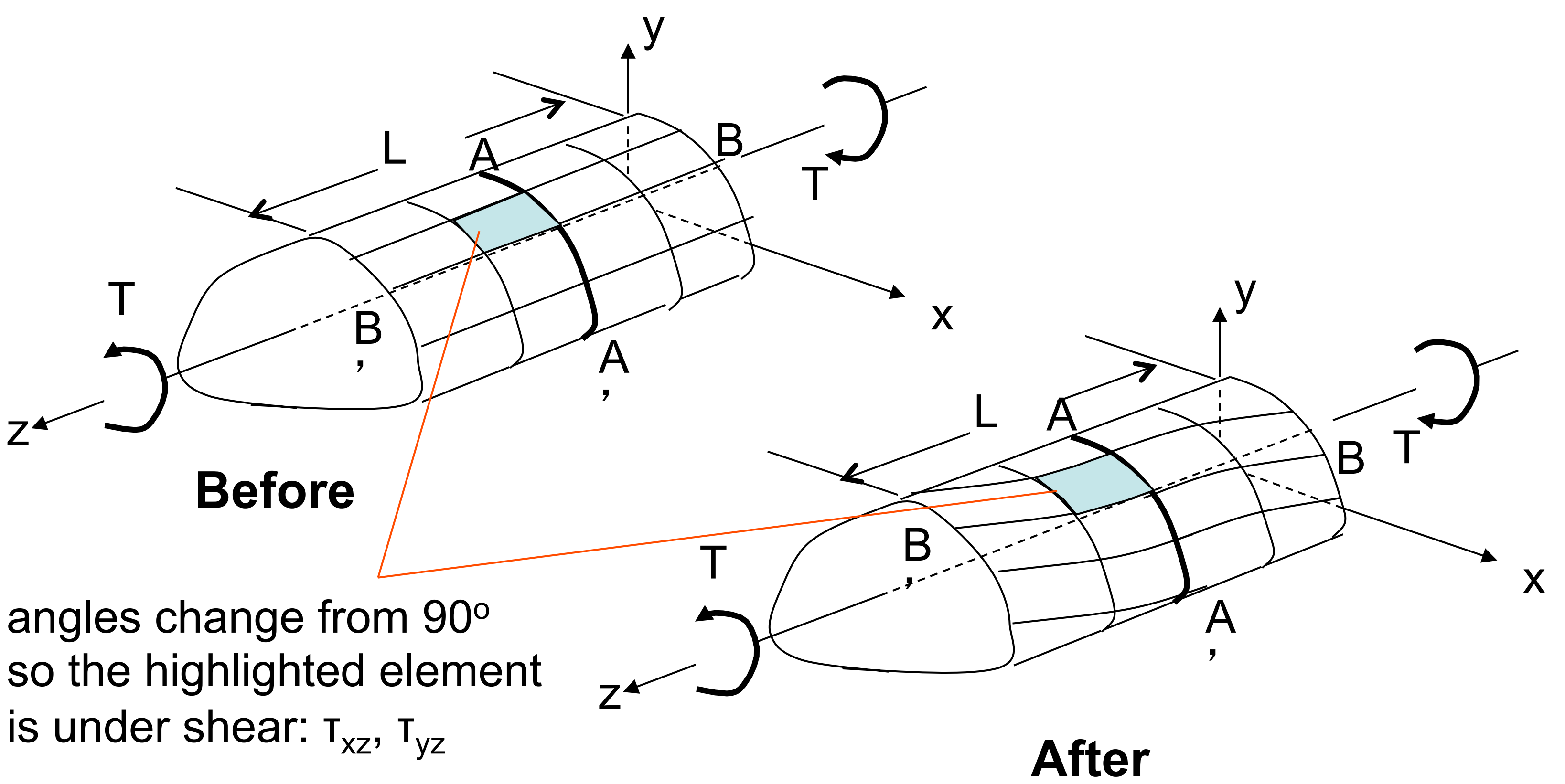
Torsion of a solid bar

- determine the stresses in a **solid** bar of any shape under torsion T



- Assumptions:
  - direct stresses are zero: $\sigma_x = \sigma_y = \sigma_z = 0$
  - shear stress $\tau_{xy} = 0$
  - no restraint at the ends (sections perpendicular to z axis are free to warp out of plane)

Torsion of a solid bar



- Lines AA' parallel to the end cross-sections remain parallel to the end cross-sections
- Lines BB' parallel to the axis of the bar become curved

Torsion of solid bar – Determination of stresses

- recall, from lecture 1, eqs 1.7-1.9 (equilibrium eqns):

$$\cancel{\frac{\partial \sigma_x}{\partial x}} + \cancel{\frac{\partial \tau_{xy}}{\partial y}} + \frac{\partial \tau_{xz}}{\partial z} + \cancel{X} = 0$$

$$\cancel{\frac{\partial \tau_{xy}}{\partial x}} + \cancel{\frac{\partial \sigma_y}{\partial y}} + \frac{\partial \tau_{yz}}{\partial z} + \cancel{Y} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \cancel{\frac{\partial \sigma_z}{\partial z}} + \cancel{Z} = 0$$

- neglecting body forces, X, Y, Z, and applying the assumptions stated earlier on the stresses:

$$\frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\frac{\partial \tau_{yz}}{\partial z} = 0 \qquad (3.1\text{-}3.3)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$$

Torsion of solid bar – Determination of stresses

$$\frac{\partial \tau_{xz}}{\partial z} = 0$$

$$\frac{\partial \tau_{yz}}{\partial z} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0$$

• first two equations state that the shear stresses (and thus the strains also) are not a function of z (depend only on x and y)

• third equation provides a relation between the two unknown shear stresses

• how do we solve the third equation?

Torsion of solid bar – Prandtl's stress function formulation

• Prandtl came up with a way of solving the third equation. He defined the stress function φ:

$$\tau_{xz} = \frac{\partial \varphi}{\partial y} \qquad (3.4)$$

$$\tau_{yz} = -\frac{\partial \varphi}{\partial x} \qquad (3.5)$$

• Eqs (3.4) and (3.5) identically satisfy eq. (3.3) irrespective of what φ is

• By selecting an appropriate φ (which is a function of the cross-sectional shape of the bar), different problems can be solved

• Note that, so far, only the equilibrium equations are satisfied; φ must be such that the compatibility eqs (1.1-1.6 after u,v, and w are eliminated) and BC's are satisfied

Torsion of solid bar – Prandtl's stress function formulation

- strain-displacement equations 1.1-1.6:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

- are simplified if we use the fact that: $\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$

and

$$\varepsilon_x = \frac{1}{E}\left[\cancel{\sigma_x} - \nu\left(\cancel{\sigma_y} + \cancel{\sigma_z}\right)\right] \qquad \gamma_{yz} = \frac{1}{G}\tau_{yz}$$

$$\varepsilon_y = \frac{1}{E}\left[\cancel{\sigma_y} - \nu\left(\cancel{\sigma_x} + \cancel{\sigma_z}\right)\right] \qquad \gamma_{xz} = \frac{1}{G}\tau_{xz} \qquad (3.6\text{-}3.11)$$

$$\varepsilon_z = \frac{1}{E}\left[\cancel{\sigma_z} - \nu\left(\cancel{\sigma_x} + \cancel{\sigma_y}\right)\right] \qquad \gamma_{xy} = \frac{1}{G}\cancel{\tau_{xy}}$$

inverted eqs
(1.10-1.15) !!

Torsion of solid bar – Prandtl's stress function formulation

- from $\varepsilon_x=0$ we get $\partial u/\partial x=0$ => $u= f(y,z)$ only
- from $\varepsilon_z=0$ we get $\partial w/\partial z=0$ => $w=f(x,y)$ only
- from $\partial \tau_{xz} /\partial z=0$ (see 3 pages ago) we get $\frac{\partial \gamma_{xz}}{\partial z} = 0 \Rightarrow \frac{\partial^2 u}{\partial z^2} + \cancel{\frac{\partial^2 w}{\partial x \partial z}} = 0$
- from which: $u=zf_1(y)+f_2(y)$
- but $u=0$ at $z=0$ so $f_2(y)=0$ and $u=zf_1(y)$
- in an analogous fashion, can show that $v=zg_1(x)$
- then, from $\tau_{xy}=0$ => $\gamma_{xy}=0$ => $\partial f_1/dy =C_1$ and $\partial g_1/\partial x=-C_1$
- therefore: $u=kzy$ and $v=-kzx$ (3.10a,b)

leading to

$$\gamma_{yz} = \frac{\partial w}{\partial y} - kx$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + ky$$

(3.12-3.13)

Torsion of solid bar – Prandtl's stress function formulation

- from (3.12) we can write:

$$\frac{\partial^3 w}{\partial x^2 \partial y} = \frac{\partial^2 \gamma_{yz}}{\partial x^2} \quad \text{and} \quad \frac{\partial^3 w}{\partial x \partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial x \partial y}$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} - kx$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + ky \qquad \text{(3.12-3.13)}$$

- similarly from (3.13) we can write:

$$\frac{\partial^3 w}{\partial x^2 \partial y} = \frac{\partial^2 \gamma_{xz}}{\partial x \partial y} \quad \text{and} \quad \frac{\partial^3 w}{\partial x \partial y^2} = \frac{\partial^2 \gamma_{xz}}{\partial y^2}$$

- from the first two we obtain:

$$\frac{\partial}{\partial x}\left[\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y}\right] = 0 \qquad \text{(3.14)}$$

- and from the second pair we obtain:

$$\frac{\partial}{\partial y}\left[\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y}\right] = 0 \qquad \text{(3.15)}$$

Torsion of solid bar – Prandtl's stress function formulation

$$\frac{\partial}{\partial x}\left[\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y}\right] = 0 \qquad (3.14)$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz} \qquad (3.9)$$

$$\gamma_{xz} = \frac{1}{G}\tau_{xz} \qquad (3.10)$$

$$\tau_{xz} = \frac{\partial \varphi}{\partial y} \qquad (3.4)$$

$$\tau_{yz} = -\frac{\partial \varphi}{\partial x} \qquad (3.5)$$

$$\frac{\partial \gamma_{yz}}{\partial x} = -\frac{1}{G}\frac{\partial^2 \phi}{\partial x^2}$$

$$-\frac{\partial \gamma_{xz}}{\partial y} = -\frac{1}{G}\frac{\partial^2 \phi}{\partial y^2}$$

$$-\frac{\partial}{\partial x}\left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right] = 0 \qquad (3.16)$$

Torsion of solid bar – Prandtl's stress function formulation: Governing equation

• use eqs (3.9), (3.10), (3.4) and (3.5) to substitute in eqs (3.14) and (3.15) to get:

$$-\frac{\partial}{\partial x}\left[\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right]=0 \qquad (3.16)$$

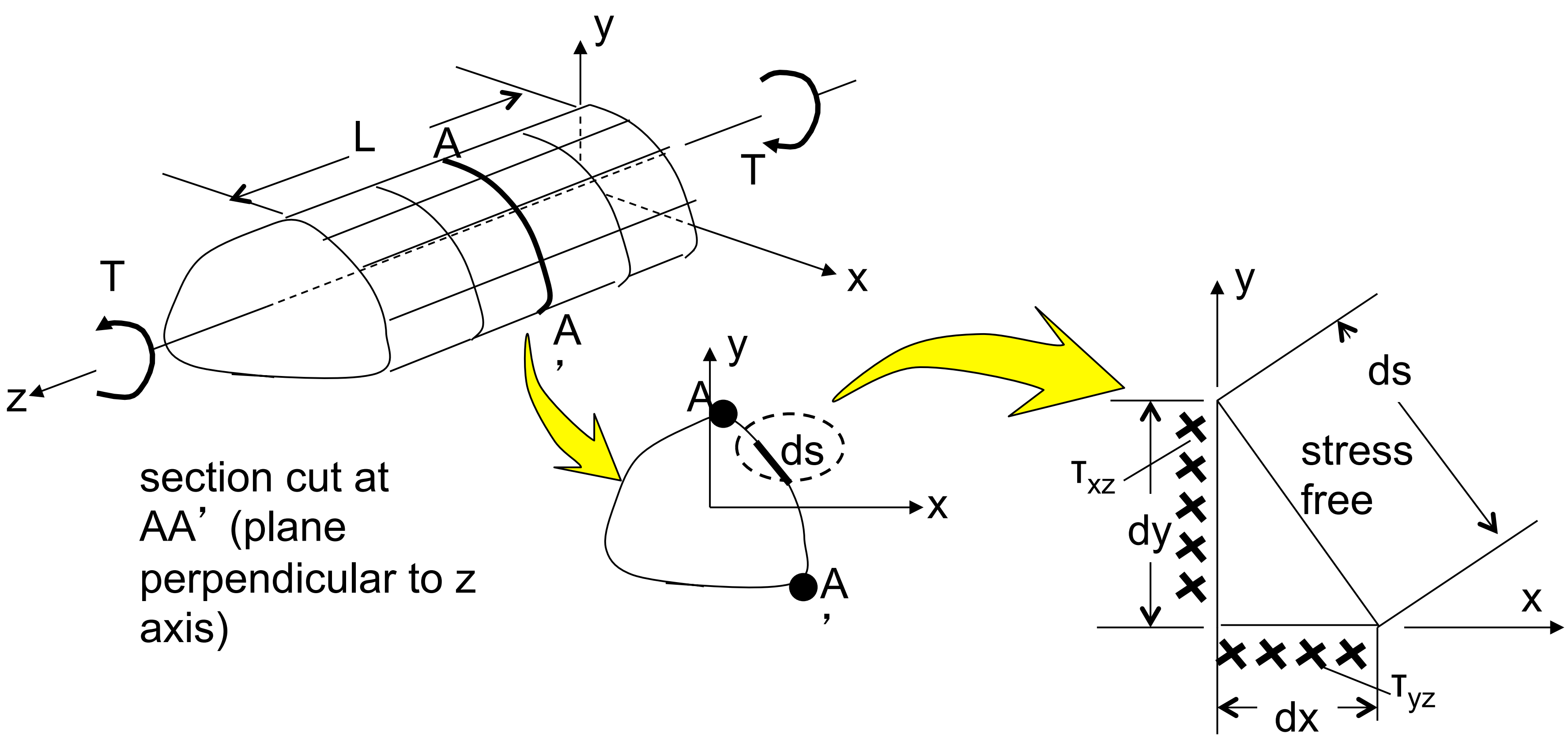
$$-\frac{\partial}{\partial y}\left[\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right]=0 \qquad (3.17)$$

• the only way equations (3.16) and (3.17) are compatible with each other is if:

$$\boxed{\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=\nabla^2\varphi=F \quad \text{(a constant)}} \qquad (3.18)$$

governing equation for torsion of solid bar

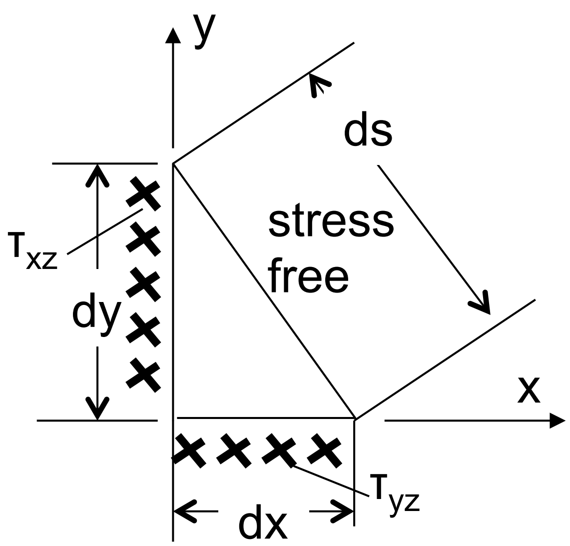
Torsion of solid bar – Prandtl's stress function formulation: Boundary Conditions



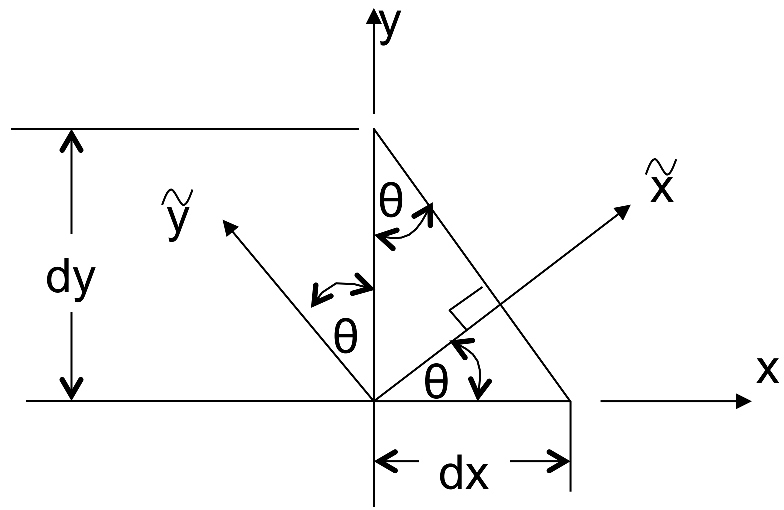
section cut at AA' (plane perpendicular to z axis)

- isolate an element dx dy ds of length dz and determine the stresses on the surface ds dz

Torsion of solid bar – Prandtl's stress function formulation: Boundary Conditions



The surface ds dz is stress free. The stresses on that surface, $\sigma_{\tilde{x}\tilde{x}}$, $\tau_{\tilde{x}y}$, $\tau_{\tilde{x}z}$ are zero:

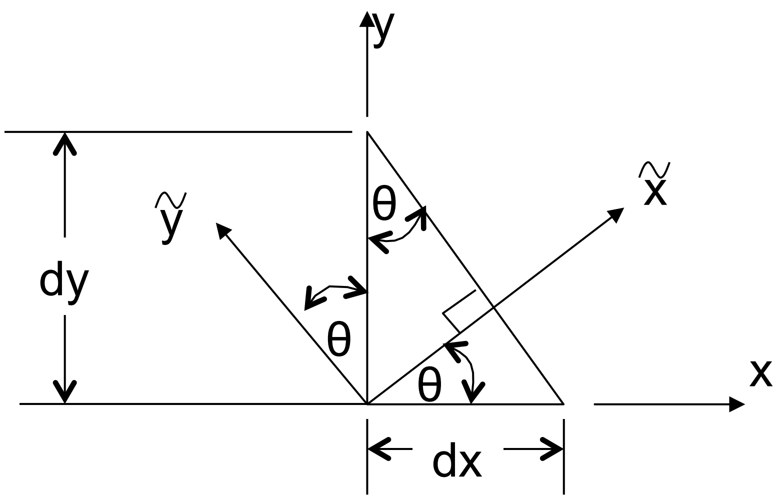


- use eq. (7a) fm lecture 1 to write the stresses in the rotated coordinate system (3-D version!)

- use the fact that σ_{xx}, σ_{yy}, σ_{zz}, and τ_{xy} are zero in the xy system

Note that Megson (section 3.1) does it slightly differently; this is equivalent

Torsion of solid bar – Prandtl's stress function formulation: Boundary Conditions



Eq. (7a) rewritten:

$$\sigma_{\tilde{m}\tilde{n}} = l_{\tilde{m}p}\, l_{\tilde{n}q}\, \sigma_{pq} \tag{3.19}$$

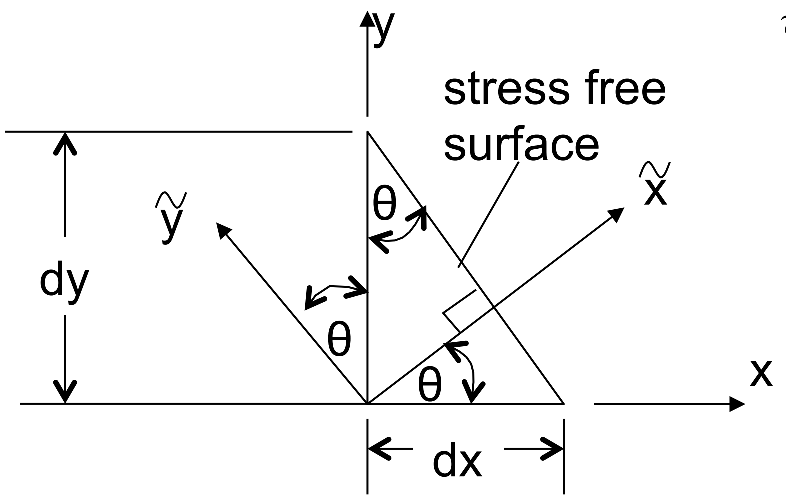
- it is easy to show, using eq. (3.19), that $\sigma_{\tilde{x}\tilde{x}} = \tau_{\tilde{x}\tilde{y}} = 0$
- the only one left is $\tau_{\tilde{x}z}$. Applying (3.19):

$$\tau_{\tilde{x}z} = l_{\tilde{1}1} l_{\tilde{3}1} \cancel{\sigma_{xx}} + l_{\tilde{1}2} l_{\tilde{3}2} \cancel{\sigma_{yy}} + l_{\tilde{1}3} l_{\tilde{3}3} \cancel{\sigma_{zz}} + l_{\tilde{1}1} l_{\tilde{3}2} \cancel{\tau_{xy}} + l_{\tilde{1}2} l_{\tilde{3}1} \cancel{\tau_{yx}} +$$

$$l_{\tilde{1}2} l_{\tilde{3}3} \tau_{yz} + l_{\tilde{1}3} l_{\tilde{3}2} \tau_{zy} + l_{\tilde{1}1} l_{\tilde{3}3} \tau_{xz} + l_{\tilde{1}3} l_{\tilde{3}1} \tau_{zx}$$

recall that $\sigma_{ij}=\sigma_{ji}$ and $\tau_{ij}=\tau_{ji}$

Torsion of solid bar – Prandtl's stress function formulation: Boundary Conditions



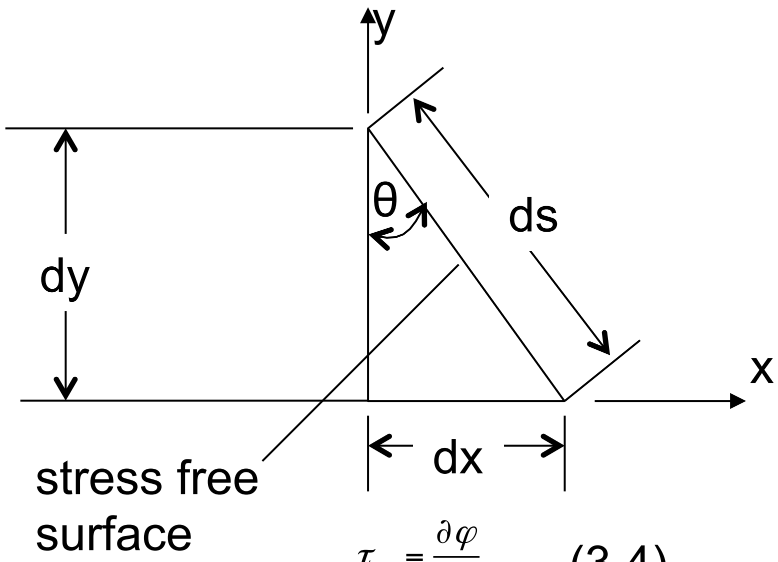
$$\tau_{\tilde{x}\tilde{z}} = \ell_{\tilde{1}1}\ell_{\tilde{3}1}\cancel{\sigma_{xx}} + \ell_{\tilde{1}2}\ell_{\tilde{3}2}\cancel{\sigma_{yy}} + \ell_{\tilde{1}3}\ell_{\tilde{3}3}\cancel{\sigma_{zz}} + \ell_{\tilde{1}1}\ell_{\tilde{3}2}\cancel{\tau_{xy}} + \ell_{\tilde{1}2}\ell_{\tilde{3}1}\cancel{\tau_{yx}} +$$
$$\ell_{\tilde{1}2}\ell_{\tilde{3}3}\tau_{yz} + \ell_{\tilde{1}3}\ell_{\tilde{3}2}\tau_{zy} + \ell_{\tilde{1}1}\ell_{\tilde{3}3}\tau_{xz} + \ell_{\tilde{1}3}\ell_{\tilde{3}1}\tau_{zx}$$

(axes 1, 2, 3 coincide with x, y, z; 3 is out of the page)

Note that:
$$\begin{cases} l_{\tilde{1}2} = \cos(90-\theta) = \sin\theta \\ l_{\tilde{3}3} = \cos(0) = 1 \\ l_{\tilde{1}3} = \cos(90) = 0 \\ l_{\tilde{3}2} = \cos(90) = 0 \\ l_{\tilde{1}1} = \cos\theta \\ l_{\tilde{3}1} = \cos(90) = 0 \end{cases}$$

- substituting: $\tau_{\tilde{x}\tilde{z}} = \sin\theta\tau_{yz} + \cos\theta\tau_{xz}$
- and since $\tau_{\tilde{x}\tilde{z}}=0$ on the outer surface: $\sin\theta\tau_{yz} + \cos\theta\tau_{xz} = 0 \Rightarrow \tan\theta\tau_{yz} + \tau_{xz} = 0$
- but from the above sketch: tanθ=-dx/dy (because as ds increases x decreases) So: $-dx(\tau_{yz}) + dy(\tau_{xz}) = 0$

Torsion of solid bar – Prandtl's stress function formulation: Boundary Conditions



$$\tau_{xz} = \frac{\partial \varphi}{\partial y} \qquad (3.4)$$

$$\tau_{yz} = -\frac{\partial \varphi}{\partial x} \qquad (3.5)$$

$$-dx(\tau_{yz}) + dy(\tau_{xz}) = 0$$

- divide through by ds:

$$-\frac{dx}{ds}\tau_{yz} + \frac{dy}{ds}\tau_{xz} = 0$$

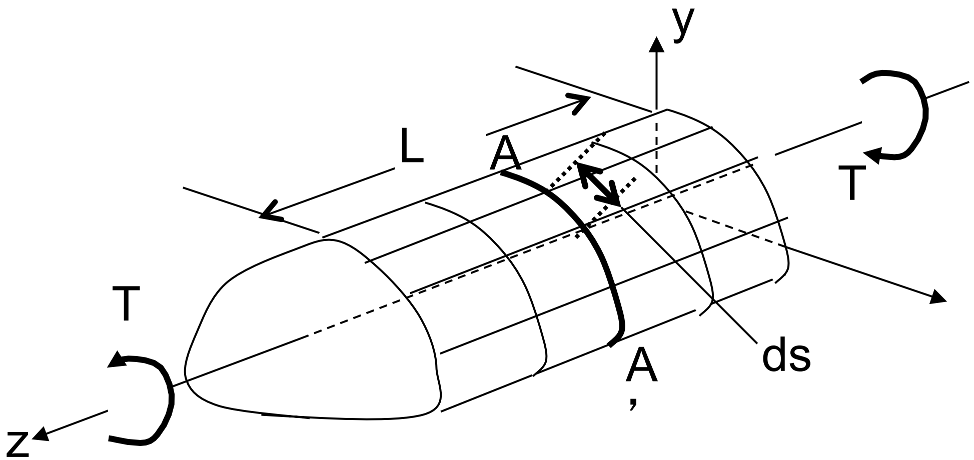
- and use (3.4) and (3.5)

$$\frac{dx}{ds}\frac{\partial \varphi}{\partial x} + \frac{dy}{ds}\frac{\partial \varphi}{\partial y} = 0 \qquad (3.20)$$

- but from basic calculus: $\frac{dx}{ds}\frac{\partial \varphi}{\partial x} + \frac{dy}{ds}\frac{\partial \varphi}{\partial y} = \frac{\partial \varphi}{\partial s}$

- combining with (3.20): $\boxed{\frac{\partial \varphi}{\partial s} = 0}$ (3.21)

Torsion of solid bar – Prandtl's stress function formulation: Boundary Conditions



$$\frac{\partial \varphi}{\partial s} = 0 \qquad (3.21)$$

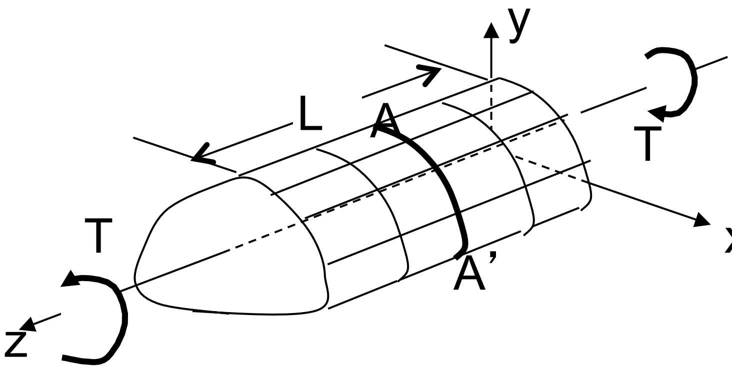
• eq. (3.21) implies that φ is constant on the surface of the bar

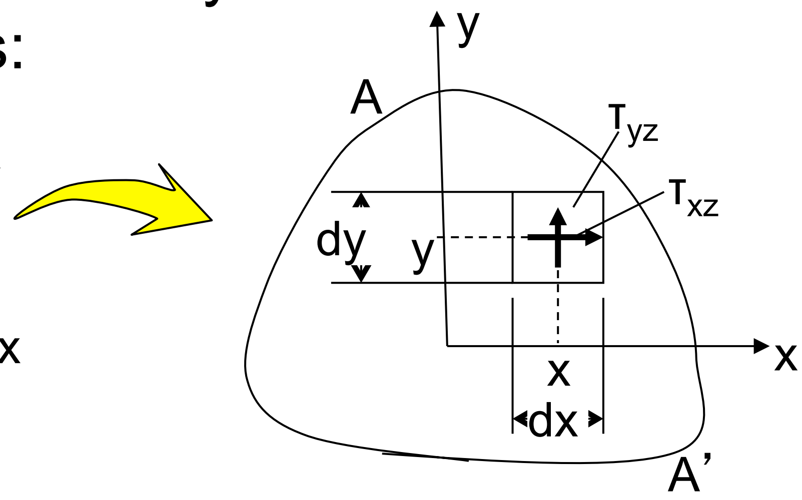
• however, the value of the constant does not affect the stresses because they are defined as derivatives of φ (see eqs (3.4) and (3.5)); therefore, we can set

$\boxed{\varphi = 0}$ on the bar boundary BC for governing eq. (3.18) (3.22)

Torsion of solid bar – Implications for φ

• consider now the stresses at any cross-section perpendicular to the z axis:





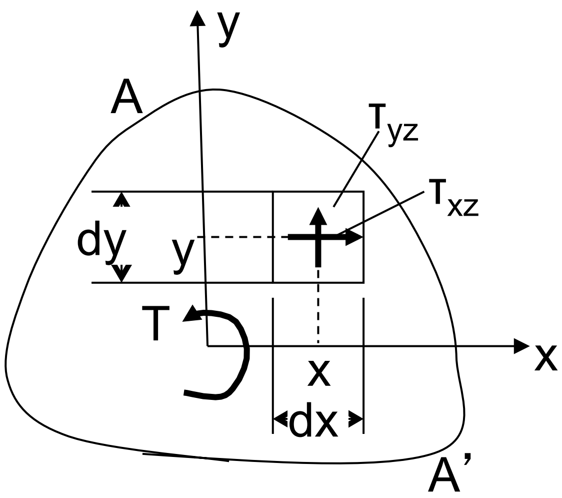
• at point (x,y) the stresses acting are τ_{xz}, τ_{yz} as shown

• the corresponding forces over an element of area dxdy are:

$$\tau_{xz} dxdy$$

$$\tau_{yz} dydx$$

Torsion of solid bar – Implications for φ



the two shear stresses τ_{xz}, τ_{yz} combine to "create" the torque T (i.e. the stresses are equivalent to the applied torque)

$$\tau_{xz} = \frac{\partial \varphi}{\partial y} \qquad (3.4)$$

$$\tau_{yz} = -\frac{\partial \varphi}{\partial x} \qquad (3.5)$$

- if T is positive as shown (counter-clockwise),

$$T = \iint \left(\tau_{yz} x - \tau_{xz} y \right) dxdy \qquad (3.23)$$

moment arms

- and using (3.4) and (3.5):

$$T = -\iint \left(x \frac{\partial \varphi}{\partial x} + y \frac{\partial \varphi}{\partial y} \right) dxdy \qquad (3.24)$$

Torsion of solid bar – Implications for φ

$$T = -\iint \Bigg(\underbrace{x\frac{\partial \varphi}{\partial x}}_{\text{integrate by parts w.r.t x}} + \underbrace{y\frac{\partial \varphi}{\partial y}}_{\text{integrate by parts w.r.t y}} \Bigg) dxdy \qquad (3.24)$$

$$\int x\frac{\partial \varphi}{\partial x}dx = x\varphi - \int \varphi dx \qquad \int y\frac{\partial \varphi}{\partial y}dy = x\varphi - \int \varphi dy$$

Also note that φ=0 on the boundary from (3.22) so these two terms are zero

- then,

$$T = -\left\{\int dy\left[-\int \varphi dx\right] + \int dx\left[-\int \varphi dy\right]\right\} \Rightarrow \boxed{T = 2\iint \varphi dxdy} \qquad (3.25)$$

- therefore, solving the torsion problem of a solid bar amounts to determining the solution to (3.18) provided that (3.22) and (3.25) are satisfied

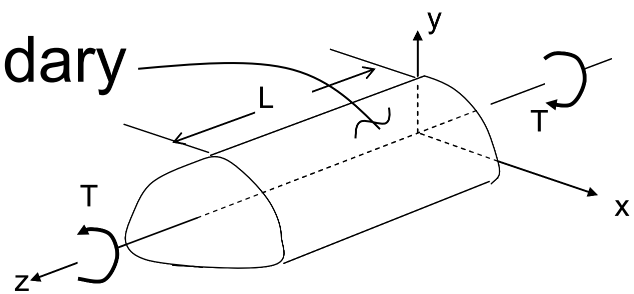
Torsion of solid bar – Implications for φ

- solve the equation:

$$\boxed{\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \nabla^2 \varphi = F \quad \text{(a constant)}} \qquad (3.18)$$

- subject to the conditions

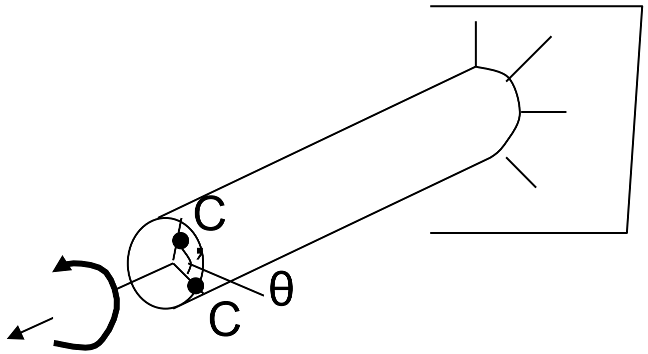
$\boxed{\varphi = 0}$ on the bar boundary (3.22)



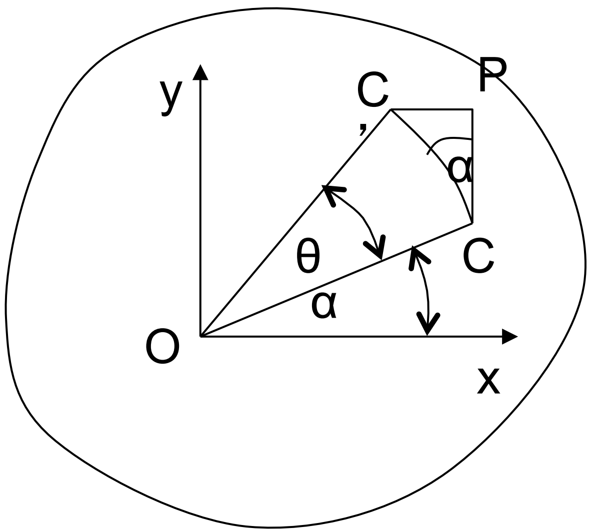
$$\boxed{T = 2\iint \varphi \, dx\, dy} \qquad (3.25)$$

Determination of angle of twist

- the two ends of a bar under torsion will rotate with respect to each other over an angle of twist θ



if one end is considered stationary, point C at the other end moves to C’ over an angle θ



OC=OC’=r

For **small** angle θ, CC’ = (θ) OC

Angle C’CP=α (sides are perp to COx)

C’P=-u (+T causes –u)

CP=v

Fm right triangle CPC’:

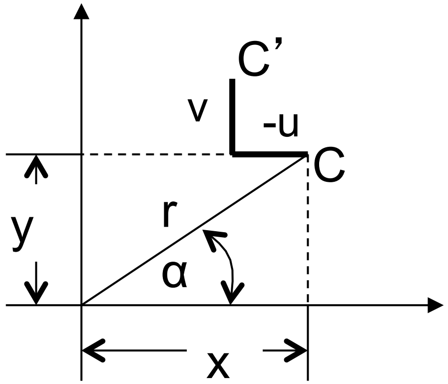
CP=CC’ cosα

C’P=CC’ sinα

$$u = -r\theta \sin\alpha$$

$$v = r\theta \cos\alpha$$

Determination of angle of twist



$$u = -r\theta \sin\alpha$$
$$v = r\theta \cos\alpha \qquad (3.26a,b)$$

- but

$$x = r\cos\alpha$$
$$y = r\sin\alpha$$

- and, therefore,

$$u = -y\theta$$
$$v = x\theta \qquad (3.27a,b)$$

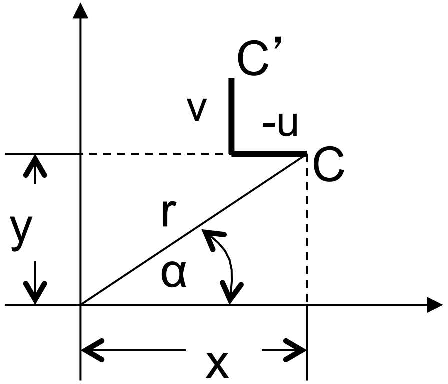
from before
$$\begin{cases} \gamma_{yz} = \dfrac{\partial w}{\partial y} - kx \\ \gamma_{xz} = \dfrac{\partial w}{\partial x} + ky \end{cases} \qquad (3.12\text{-}3.13)$$

- now use (3.12) and (3.13) to obtain,

$$\frac{\partial w}{\partial y} = \gamma_{yz} + kx$$
$$\frac{\partial w}{\partial x} = \gamma_{xz} - ky$$

- and γ_{yz} and γ_{xz} can be replaced using (1.13) and (1.14)

Determination of angle of twist



u=kzy and v=-kzx (3.10a,b)

$$u = -y\theta$$
$$v = x\theta$$ (3.27a,b)

$$\frac{\partial w}{\partial y} = \frac{\tau_{yz}}{G} + kx$$
$$\frac{\partial w}{\partial x} = \frac{\tau_{xz}}{G} - ky$$

(where $\tau_{ij} = G\gamma_{ij}$ was used)

- comparing now (3.10a,b) with (3.27a,b):

$$\theta = -kz \Rightarrow k = -\frac{d\theta}{dz}$$

- which, substituted in the two equations above, gives:

$$\frac{\partial w}{\partial y} = \frac{\tau_{yz}}{G} - x\frac{d\theta}{dz}$$
$$\frac{\partial w}{\partial x} = \frac{\tau_{xz}}{G} + y\frac{d\theta}{dz} \qquad (3.28)$$

- if we differentiate (3.28a) w.r.t x and (3.28b) w.r.t y, the left hand sides will be equal; so subtracting the right hand sides (after differentiation) gives

Determination of angle of twist and torsion constant

$$\frac{1}{G}\frac{\partial \tau_{yz}}{\partial x} - \frac{d\theta}{dz} = \frac{1}{G}\frac{\partial \tau_{xz}}{\partial y} + \frac{d\theta}{dz} \Rightarrow \frac{1}{G}\left(\frac{\partial \tau_{xz}}{\partial y} - \frac{\partial \tau_{yz}}{\partial x}\right) + 2\frac{d\theta}{dz} = 0$$

- using eqs (3.4) and (3.5) to substitute for τ_{xz} and τ_{yz} gives

$$\left(\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial x^2}\right) = -2G\frac{d\theta}{dz} \qquad (3.29)$$

- and using (3.18):

$$\boxed{-2G\frac{d\theta}{dz} = F} \quad \text{(a constant)} \qquad (3.30)$$

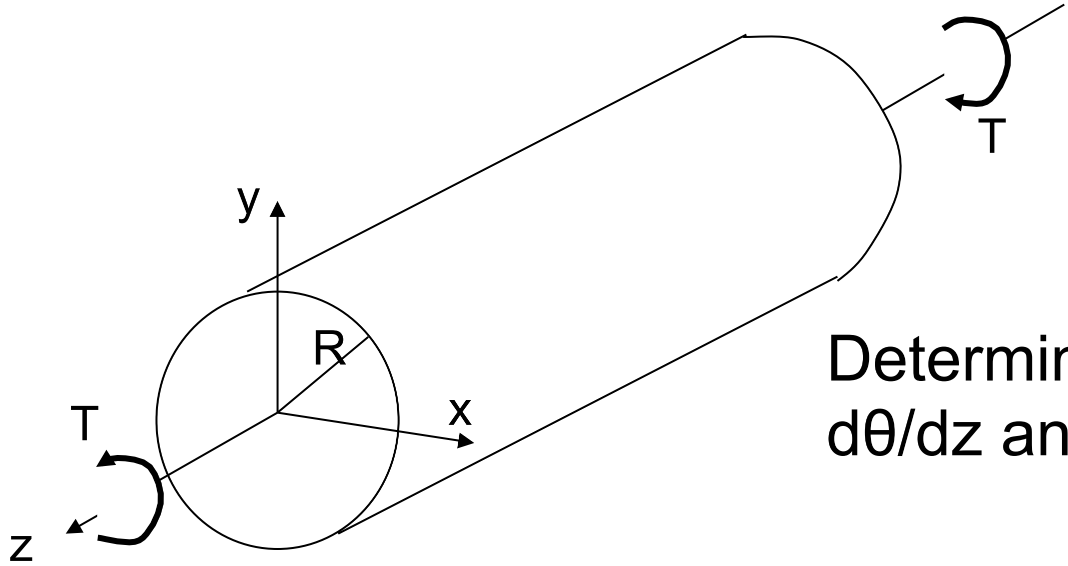
- also define the torsion constant J such that

$$\boxed{T = GJ\frac{d\theta}{dz}} \qquad (3.31)$$

note similarity/analogy between this equation and the axial Force-displacement equation: F=EA(du/dx) or the bending moment-curvature equation: $M=-EI(d^2w/dx^2)$

- GJ is also called the torsional rigidity (the same way EA is the membrane stiffness and EI is the bending stiffness)

Application: Cylindrical bar under torsion



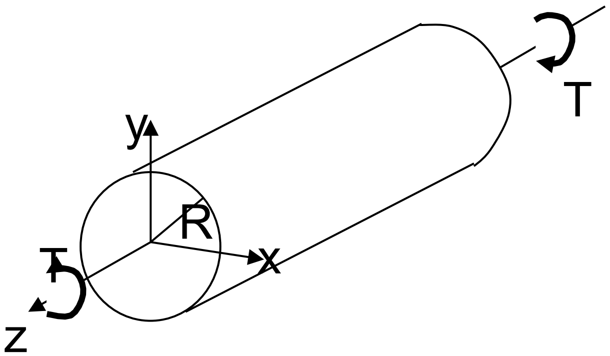
Determine the rate of twist $d\theta/dz$ and stress distribution

need to solve eq. (3.18): $$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = \nabla^2 \varphi = F \quad \text{(a constant)}$$

subject to BC (3.22): φ = 0 on the bar boundary

and knowing from (3.30): $$-2G\frac{d\theta}{dz} = F \qquad (d\theta/dz=\text{constant})$$

Application: Cylindrical bar under torsion



We are looking for a function φ that is zero on the boundary of the bar (from eq 3.22); The easiest way to do this is to start from the equation that defines points on the boundary (equation of a circle):

$$x^2 + y^2 = R^2 \Rightarrow$$

$$\underbrace{x^2 + y^2 - R^2} = 0$$

this expression on the LHS is zero on the bar boundary

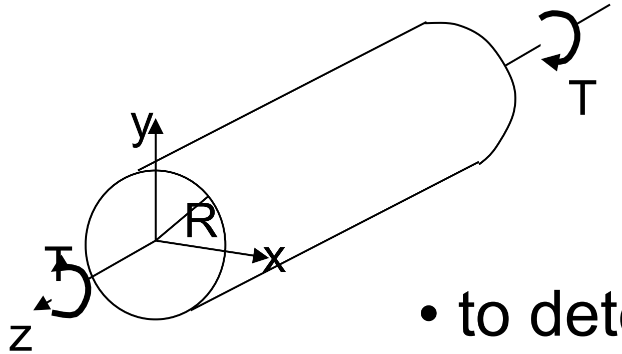
- so try $\varphi = C(x^2 + y^2 - R^2)$ (3.32)

  with C an unknown constant

- substitute in the governing equation (3.18) modified by (3.30) (which is the same as 3.29):

$$\left(\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial x^2}\right) = -2G\frac{d\theta}{dz}$$

Application: Cylindrical bar under torsion



$$\left.\begin{aligned}&\left(\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial x^2}\right)=-2G\frac{d\theta}{dz}\\ &\varphi=C(x^2+y^2-R^2)\end{aligned}\right\} \quad 4C=-2G\frac{d\theta}{dz}\Rightarrow C=-\frac{G}{2}\frac{d\theta}{dz}$$

- to determine dθ/dz we use eq (3.25):

$$T=2\iint\varphi\,dxdy \Rightarrow T=2\left(-\frac{G}{2}\frac{d\theta}{dz}\right)\iint\left(x^2+y^2-R^2\right)dxdy \qquad (3.25a)$$

But: $\iint x^2dxdy=\int_0^R 4\int_0^{\sqrt{R^2-y^2}}x^2dxdy=\int_0^R 4\left[\frac{x^3}{3}\right]_0^{\sqrt{R^2-y^2}}dy=\int_0^R\frac{4}{3}\left(R^2-y^2\right)^{3/2}dy$

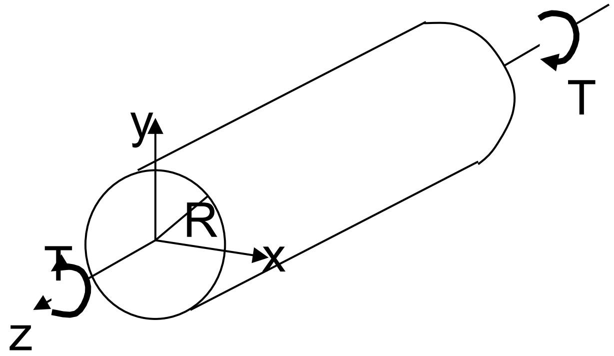
and: $\int(r^2-x^2)^{3/2}\,dx=\frac{1}{8}\left(x\left(5r^2-2x^2\right)\sqrt{r^2-x^2}+3r^4\tan^{-1}\left(\frac{x}{\sqrt{r^2-x^2}}\right)\right)$ from: http://integrals.wolfram.com

therefore: $\int x^2dx=\frac{\pi R^4}{4}$ and by cyclic symmetry: $\int y^2dx=\frac{\pi R^4}{4}$

finally: $\iint R^2dxdy=R^2\iint dxdy=R^2(Area)=R^2\pi R^2$

combining all in (3.25a): $T=-G\frac{d\theta}{dz}\left(\frac{\pi R^4}{4}+\frac{\pi R^4}{4}-\pi R^4\right)\Rightarrow T=G\frac{\pi R^4}{2}\frac{d\theta}{dz}$

Application: Cylindrical bar under torsion: rate of twist and polar moment of inertia



$$T = G\frac{\pi R^4}{2}\frac{d\theta}{dz} \tag{3.33}$$

- from (3.33),

$$\frac{d\theta}{dz} = \frac{2T}{G\pi R^4} \tag{3.34}$$

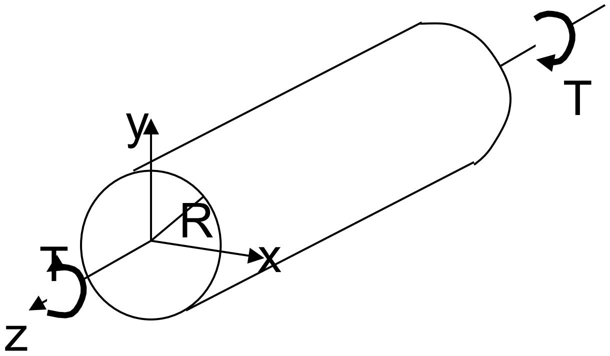
- compare also (3.33) with (3.31), $T = GJ\dfrac{d\theta}{dz}$, to obtain:

$$J = \frac{\pi R^4}{2} \tag{3.35}$$

- using (3.35) and (3.34) to substitute for C in φ gives:

$$\varphi = -\frac{T}{\pi R^4}\left(x^2 + y^2 - R^2\right) = -\frac{T}{2J}\left(x^2 + y^2 - R^2\right) \tag{3.36}$$

Application: Cylindrical bar under torsion: determination of stresses

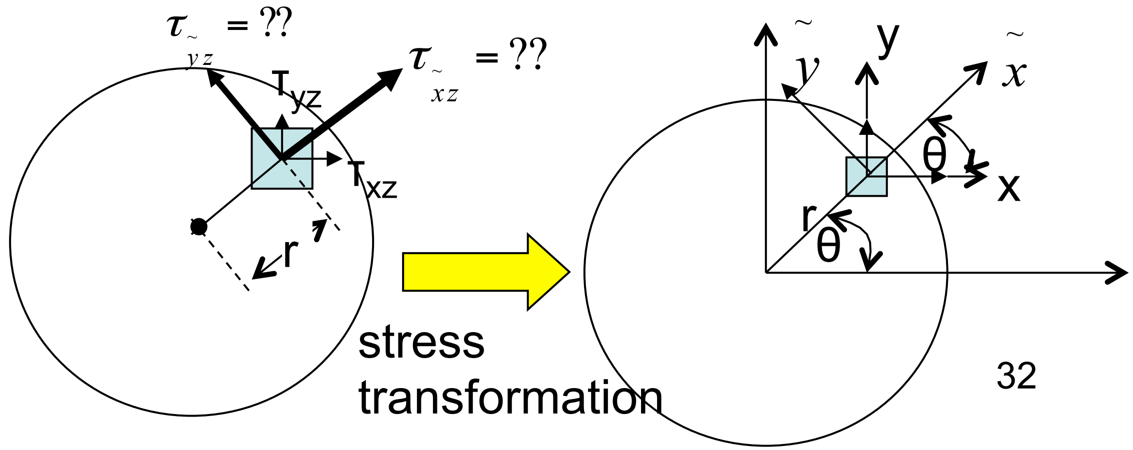


with φ known from (3.36), use (3.4) and (3.5) to determine stresses

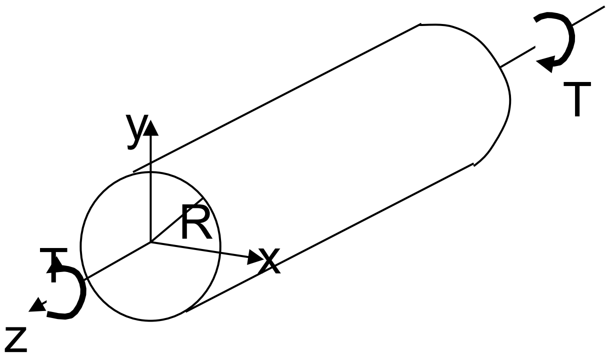
$$\left.\begin{aligned}\varphi &= -\frac{T}{2J}\left(x^2+y^2-R^2\right)\\ \tau_{xz} &= \frac{\partial\varphi}{\partial y}\\ \tau_{yz} &= -\frac{\partial\varphi}{\partial x}\end{aligned}\right\}\qquad \begin{aligned}\tau_{xz} &= -\frac{Ty}{J} \qquad (3.37)\\ \tau_{yz} &= \frac{Tx}{J} \qquad (3.38)\end{aligned}$$

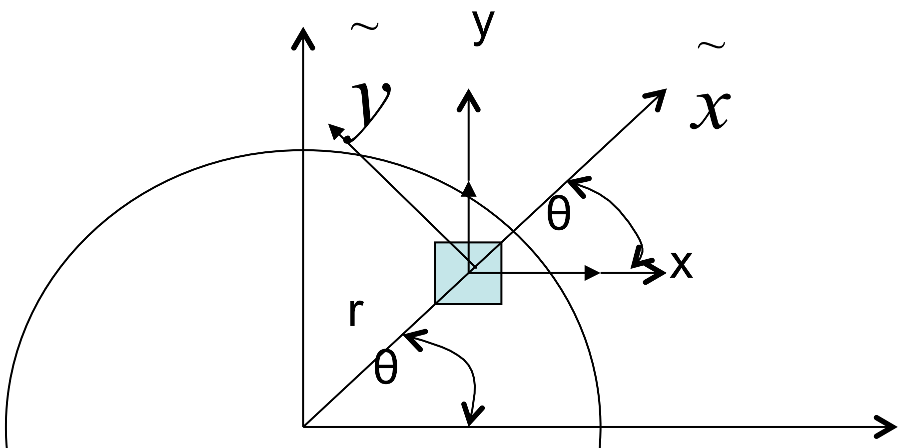
(recall that, except for these two, all other stresses in the bar are zero)

Determine now the shear stresses in the radial and tangential direction a distance r from the origin (center of rotation)



Application: Cylindrical bar under torsion: determination of stresses





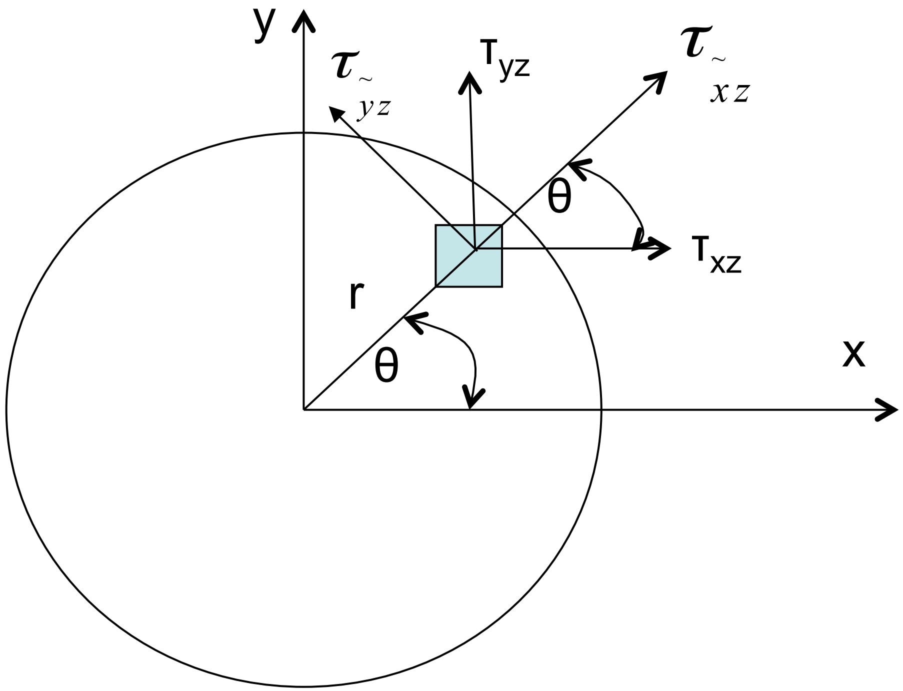
- recall the stress transformation when we calculated the BC's of the problem

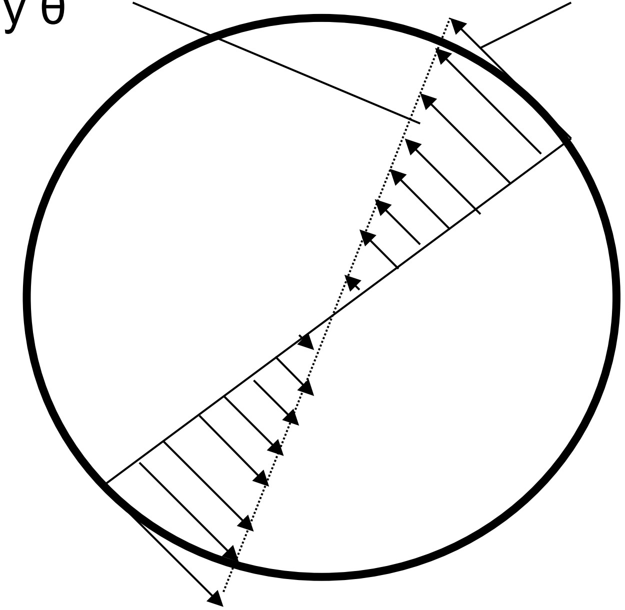
$$\tau_{\tilde{x}\tilde{z}} = \ell_{\tilde{1}1}\ell_{\tilde{3}1}\cancel{\sigma_{xx}} + \ell_{\tilde{1}2}\ell_{\tilde{3}2}\cancel{\sigma_{yy}} + \ell_{\tilde{1}3}\ell_{\tilde{3}3}\cancel{\sigma_{zz}} + \ell_{\tilde{1}1}\ell_{\tilde{3}2}\cancel{\tau_{xy}} + \ell_{\tilde{1}2}\ell_{\tilde{3}1}\cancel{\tau_{yx}} +$$
$$\ell_{\tilde{1}2}\ell_{\tilde{3}3}\tau_{yz} + \ell_{\tilde{1}3}\ell_{\tilde{3}2}\tau_{zy} + \ell_{\tilde{1}1}\ell_{\tilde{3}3}\tau_{xz} + \ell_{\tilde{1}3}\ell_{\tilde{3}1}\tau_{zx}$$

which led to $\tau_{\tilde{x}\tilde{z}} = \sin\theta\tau_{yz} + \cos\theta\tau_{xz}$

- but $x = r\cos\theta \Rightarrow \cos\theta = \frac{x}{r}$, $y = r\sin\theta \Rightarrow \sin\theta = \frac{y}{r}$ and using (3.37), (3.38): $\tau_{\tilde{x}\tilde{z}} = \frac{y}{r}\frac{Tx}{J} - \frac{x}{r}\frac{Ty}{J} = 0$
- in an analogous fashion: $\tau_{\tilde{y}\tilde{z}} = -\sin\theta\tau_{xz} + \cos\theta\tau_{yz} = -\frac{y}{r}\left(-\frac{Ty}{J}\right) + \frac{x}{r}\frac{Tx}{J} = \frac{x^2 + y^2}{rJ}T$

Application: Cylindrical bar under torsion: determination of stresses





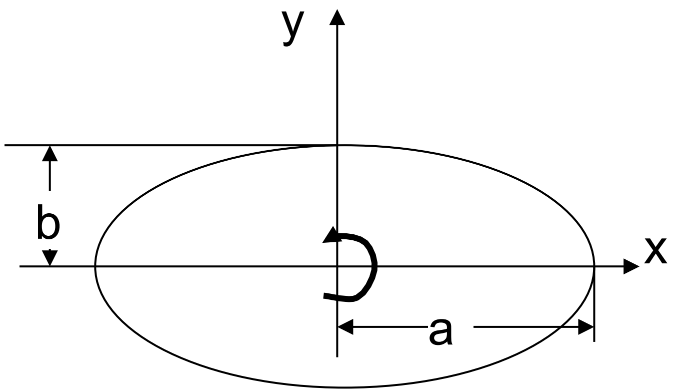
$$\boxed{\begin{aligned} \tau_{\tilde{x}z} &= 0 \\ \tau_{\tilde{y}z} &= \tau = \frac{Tr}{J} \end{aligned}}$$

(3.39)

(3.40)

the shear stress in the radial direction is zero; the shear stress τ in the tangential direction varies linearly with r from 0 at the center to maximum value at the edge and is independent of θ

Application: Elliptical bar under torsion



Since φ=0 on the boundary, determine a function that is zero there. The equation of an ellipse is

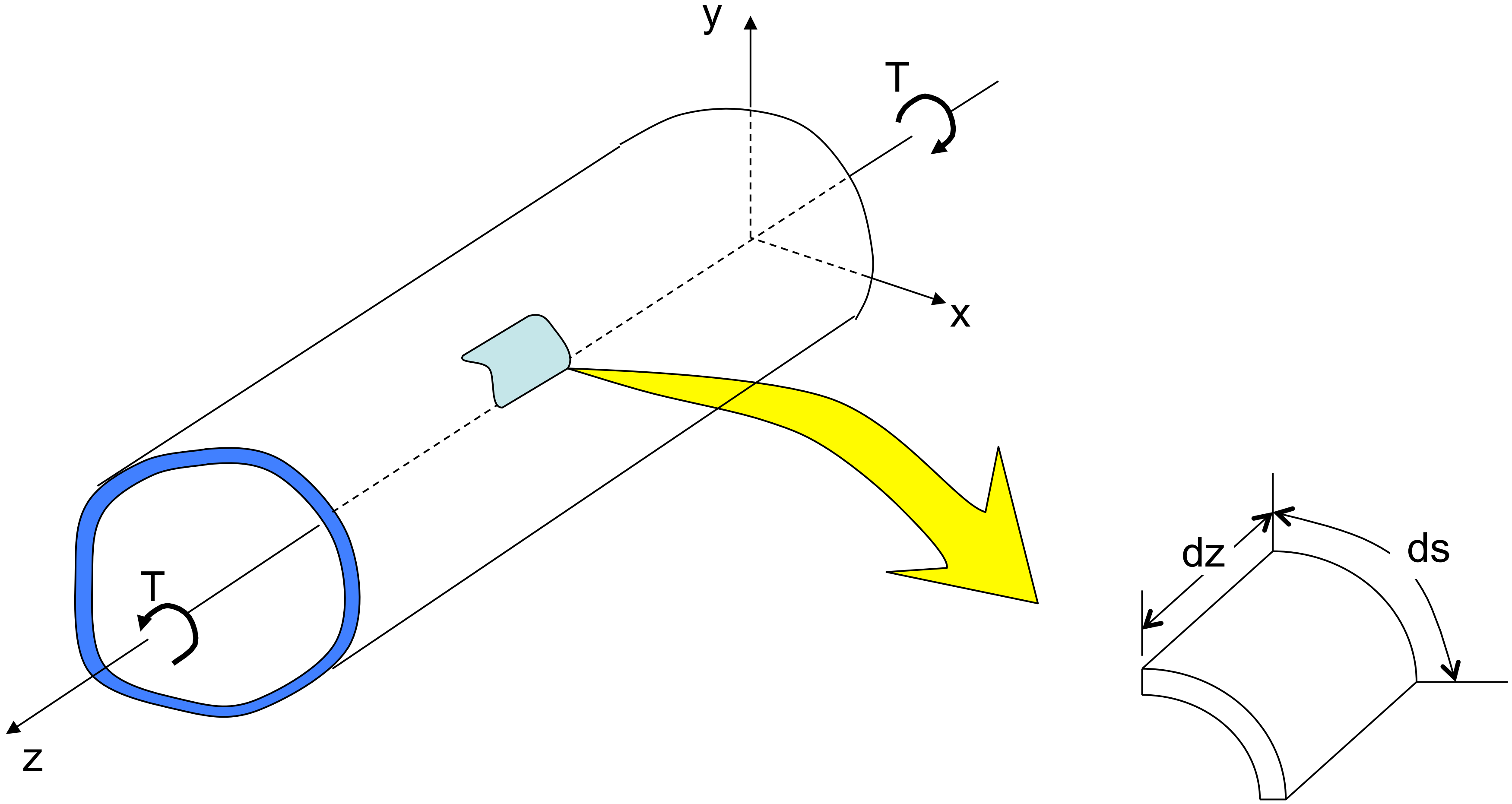
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- try $\varphi = C\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

- and follow the same procedure as before

- the τ_{xz} and τ_{yz} stresses will again be linear in y and x respectively

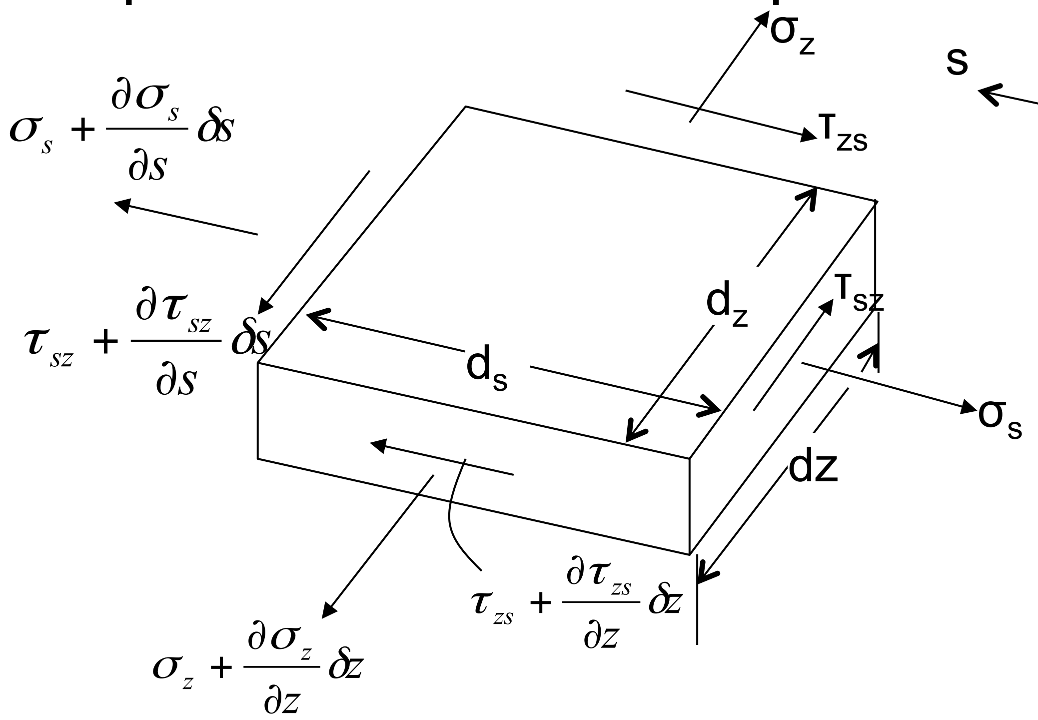
Torsion of closed section beams



- isolate an element ds by dz

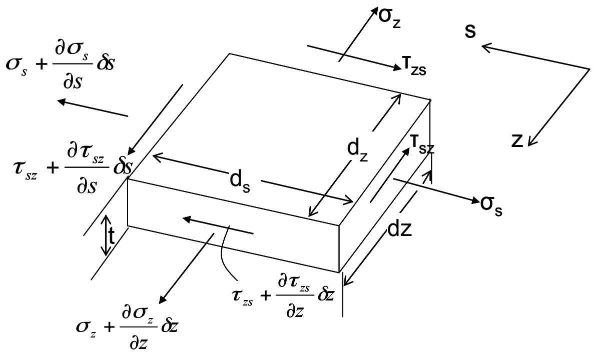
Torsion of closed section beams

- place the element in equilibrium:



- force equilibrium in z direction: $\left(\frac{\partial \sigma_z}{\partial z} dz\right) tds + \left(\frac{\partial \tau_{sz}}{\partial s} ds\right) tdz = 0$
- force equilibrium in s direction: $\left(\frac{\partial \sigma_s}{\partial s} ds\right) tdz + \left(\frac{\partial \tau_{zs}}{\partial z} dz\right) tds = 0$

Torsion of closed section beams



$$\left(\frac{\partial \sigma_z}{\partial z} dz\right) t ds + \left(\frac{\partial \tau_{sz}}{\partial s} ds\right) t dz = 0 \qquad (3.41)$$

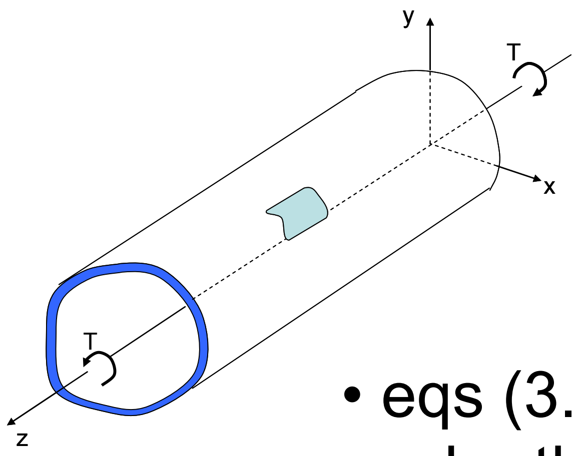
$$\left(\frac{\partial \sigma_s}{\partial s} ds\right) t dz + \left(\frac{\partial \tau_{zs}}{\partial z} dz\right) t ds = 0 \qquad (3.42)$$

- for pure torsion, there are no direct stresses=> $\sigma_s = \sigma_z = 0$
- also use the fact that $\tau_{zs} = \tau_{sz} = \tau$ which is the same as the tangential stress examined before
- define shear flow q: $q = t\tau$ (3.43)
- then eqs (3.41) and (3.42) become

$$\frac{\partial q}{\partial s} = 0 \qquad (3.41a)$$

$$\frac{\partial q}{\partial z} = 0 \qquad (3.42a)$$

Torsion of closed section beams – Shear flow



$$\frac{\partial q}{\partial s} = 0 \qquad (3.41a)$$

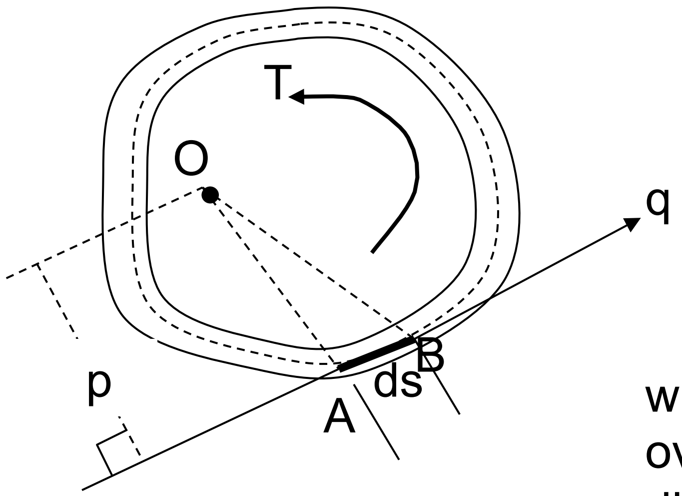
$$\frac{\partial q}{\partial z} = 0 \qquad (3.42a)$$

- eqs (3.41a) and (3.42a) are compatible with each other ONLY if q=const

- so pure torsion results in constant shear flow q in the wall of closed section beam

- note that this does not mean the shear stress τ is constant; if the thickness changes, τ changes (see eq. 3.43) even though q stays constant

Torsion of closed section beams – Relation of shear flow to applied torque



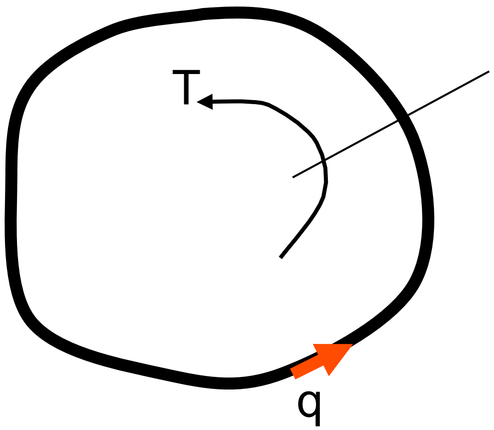
The shear flow q causes the torque T. So integrating the elemental force caused by q all around the beam wall would give the applied torque:

$$T = \oint_{wall} p(df)$$

where f is the force caused by shear flow q acting over length ds and p is the moment arm (vertical distance from origin O to the line of action of q)

- substituting f=q ds : $T = \oint_{wall} pqds$
- since q = const: $T = q\oint_{wall} pds$
- but pds is twice the area of triangle OAB; so the complete line integral is twice the enclosed area of the beam A; therefore: $\boxed{T = 2Aq}$ (3.44)

Torsion of closed section beams – Relation of shear flow to applied torque



enclosed area A

$$T = 2Aq \tag{3.44}$$

For **any** closed beam section, the product of the shear flow times twice the enclosed area gives the applied torque

• note that the origin O can be anywhere inside or outside the beam walls; the total moment does not change; one may have to account for negative swept areas when evaluating the line integral (there is a sign convention, see Megson p. 528); in the end, eq. (3.44) is always valid