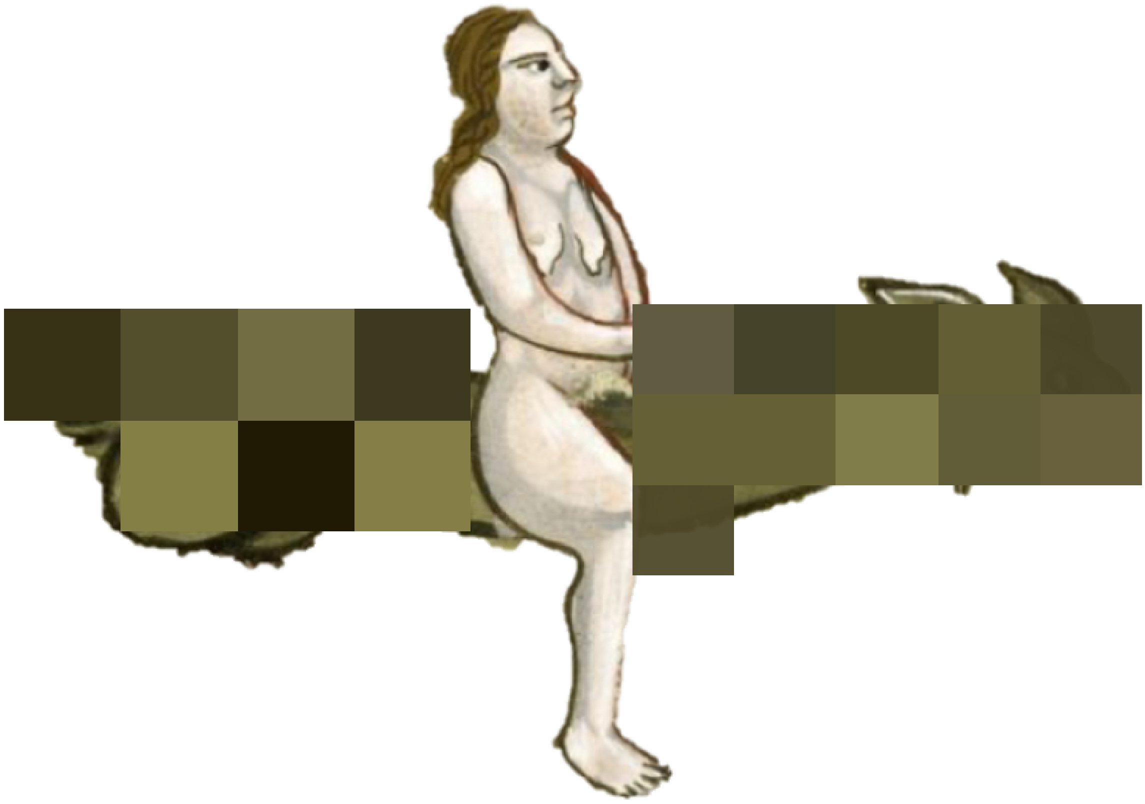
Flight Dynamics summary I: 2019-2020 edition

Based on *Lecture Notes Flight Dynamics* by J.A. Mulder et al.



Sam van Elsloo

February - April 2018

Version 1.0

©Sam van Elsloo

Contents

©Sam van Elsloo

1 What is flight dynamics?

The day that you know that would come: the final summary of your bachelor. Fortunately for you, it's a tetralogy of summaries so you have plenty of stuff still to read.

Summary structure

Furthermore, there's quite a bit to remark about the structure of the summary, and I'd like to make it as clear as I can how I divided everything up since it can be a bit confusing if you're following the lectures. First, let me list which chapters each part of the summary covers:

- Part I of this summary, which is what you have in front of you right now, covers chapter 9 and 10 of the reader. It corresponds to longitudinal stability and control.
- Part II covers chapter 8 and 11 of the reader. It corresponds to lateral stability and control.
- Part III covers chapter 2, 3 and 4 of the reader. It's about deriving equations of motion.
- Part IV covers chapter 5 and 6 of the reader. It's about eigenmotions.

Now, let me explain why I choose to divide it like this:

- First, the summary as a whole is rather long; personally I don't like having parts that are longer than approximately 75 pages due to the compilation time it takes. Furthermore, it gets annoying to staple so that's another reason for splitting it up in parts.
- Secondly, the order of the regular lectures is super messed up. The reason for this is because the fact the SVV flight dynamics assignment, for which they need some of the corresponding material. However, personally I like to use summaries to study for the exam, and so it makes more sense to me to put it in an order that makes sense to study by for the exam. Therefore, topics that relate to each other are put together.

Now, before you're like, but how can I use the summary then to study in the meantime and if I want to keep up with the lectures etc.? Here's your answer:

- For part I, nothing changes. You start with it, and it's approximately the first six lectures or so.
- For part II, the lectures start to deviate from the summary. To be specific, the lectures stop somewhere halfway through chapter 8, as they need to start with deriving equations of motion (part III of the summary). In my opinion, this is quite unideal: personally, I found it much nicer to keep on studying chapter 8 because it's honestly not that much (you can do it in a few hours easily, it's just a qualitative analysis of why some stuff is positive and some stuff is negative, but there's no real mathematics involved), and it's still rather useful for the flight-test itself because you know a bit more what all the control derivatives mean. Also, they only get back to the second part of chapter 8 towards the end of the course (in lecture 17), so by then you'll probably have forgotten everything already. That's why I think it's nicer the way I put it in the summary: you can just study everything in one go, and it's not that much extra work to finish it. I also included chapter 11 in part II; for this you should read the note I included at the beginning of the corresponding chapter in the summary. But basically: if you have time, it's best to just study it already when you're there (even though the lectures skip it until the end of the course), cause else you'll have forgotten about everything already, even though you already had the knowledge after chapter 8 to do it.
- For part III, this is basically where the lectures pick up after lecture 10 or so. The first chapter of this part is important for the flight-test, as I give a brief introduction to eigenmotions so that you know what everyone talks when they're talking about Dutch-roll etc. Once again, it's beneficial if you've studied chapter 8 completely, as it helps me explain stuff (so in there, I assume you already studied it). I should note that the lecture about the introduction to eigenmotions was given directly after part I has finished, but personally it made more sense to me to put it in this part of the summary.
- For part IV, it's nothing really special I think.

So in short, for as much as you have time for, please just stick to the structure that I use in the summary rather than the order used in the lectures[1]. It'll save you a lot of time in the long-run as it doesn't jump from one topic to another.

Finally, I've put together a table containing what chapters in which parts of the summary correspond to which chapters in the book, and which lectures (using the ones on Collegerama as reference), as shown in table 1.1.

Table 1.1: Reference list between order of summary, reader, and lectures

Part	Chapter in summary	Chapter in reader	Lecture
Part I	Chapter 1	-	-
	Chapter 2	-	1
	Chapter 3	Chapter 9.1	1 - 2
	Chapter 4	Chapter 9.2.1	2
	Chapter 5	Chapter 9.2.2-9.2.4	2 - 3
	Chapter 6	Chapter 10.1	4 - 5
	Chapter 7	Chapter 10.2	5 - 6
	Chapter 8	-	-
Part II	Chapter 1.1-1.2	8.1-8.2	8
	Chapter 1.3-1.4	8.3-8.4	17
	Chapter 2	8.5-8.8	17
	Chapter 3	11.1-11.5	18
Part III	Chapter 1	-	7, 9
	Chapter 2	Chapter 2.1	10
	Chapter 3	Chapter 2.2-2.3	11 - 12
	Chapter 4-5	Chapter 3	13 - 14
Part IV	Chapter 1	Chapter 4.1-4.3	15-16
	Chapter 2	Chapter 4.4	19
	Chapter 3	Chapter 4.5	19
	Chapter 4	-	-

Most notably, this summary does not cover everything that is in the reader. The simple reason for this is that not everything in the reader is part of the study material. I've only included stuff that's covered in the lectures, so if you decide to be stubborn and study the reader instead, don't be surprised when you found out you've been wasting your time studying stuff that's not part of the exam.

Examples

Regarding the examples I've used in this summary: they are all taken from old exams. On old exams, you'll find both multiple-choice questions, and open questions:

- Multiple-choice questions are included whenever the treatment of a subject has been finished. As they are all fairly easy, they serve as a good way to check whether you've understood the material (if you get any of them wrong, I'd strongly advise you to take a good look at the previously discussed material to make sure you understand it). As a bonus, the multiple-choice questions seem to be taken from a fixed pool of multiple-choice questions, so I've included all of the ones I could find, so it's good practice for the exam as well (multiple choice questions appear in about 80% of the exams, and when they do appear, they're usually worth 10 points in total (and you'll be asked 5-10 multiple choice questions), for about 10 points, out of 100 points in total (so it's not much, but it's something).
- Open questions are included as much as possible, but do note that it takes quite a while before you're able to do one of the open questions, so just wait patiently. I've indicated which exam question it is (year + problem number), and how many points it is worth (out of 100 points).

[1] Honestly if it weren't for the flight-test I'm sure they'd have used the same order.

©Sam van Elsloo

Part specific comments

Unfortunately, for this part of the summary, there were some applicable exam questions about chapter 2, but after that there were no suitable questions until you've finished chapter 6. That's why there are unfortunately almost no examples in chapters 3-6 and instead I've put all of the examples relating to these chapters in a separate chapter 7. This was simply necessary because chapters 3 and 4 are mostly just background to chapters 5 and 6, and questions about chapters 5 and 6 are usually combined.

Furthermore, you'll find a *lot* of derivations and formulas. It's absolutely pointless to learn the formulas by heart (you have a formula sheet after all) and regarding the derivations: they never ask for the most complicated ones. However, for the easier ones, you should have an understanding of how to derive it[2].

Also, if you're wondering about the front page: I listened to the two most frequently critiques of the previous front pages:

- "It's too color intensive, which sucks when printing, because I'm stupid for aerospace engineering and I didn't realize that TU Delft doesn't care how much color there is on a page, it's the same prize for each page.": it's almost completely white now.
- "There should be an aerospace related picture on the front page.": well there's something flying now.

(also yes this chapter should have been called "Preface", but you infidels don't print prefaces)

[2]For example, you'll see a derivation similar to what was treated in intro I regarding stability. Of course, you should be able to do such a derivation still.

©Sam van Elsloo

2 *Basic definitions of control theory*

First off some very basic stuff: this course primarily focusses around stability of aircraft. It should be noted that stability of aircraft is semantically wrong: an aircraft can be in equilibrium; this equilibrium can be stable or not:

- If an equilibrium unstable, then after a disturbance, it does not return to the equilibrium condition, but diverges away from it.
- If an equilibrium is neutrally stable, then after a disturbance, the disturbance stays present, but does not increase either.
- If an equilibrium is stable, then after a disturbance, it returns back to original equilibrium.

All of this is much clearer shown in figure 2.1.

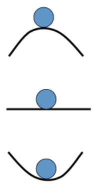
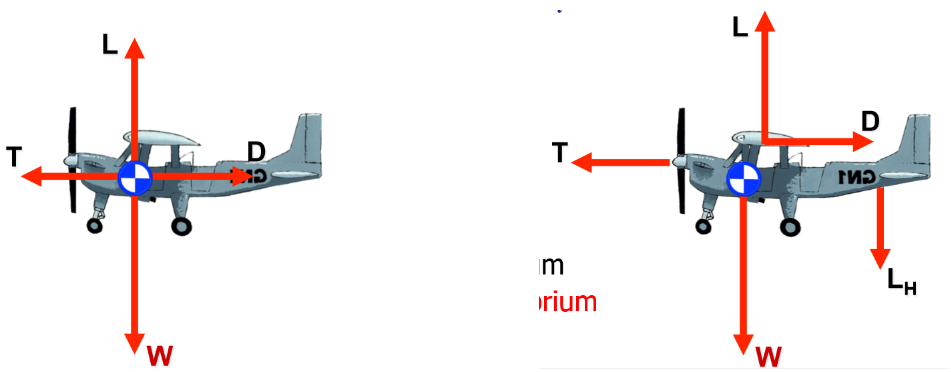


Figure 2.1: Stability: unstable equilibrium (top figure), neutrally stable equilibrium (middle figure), stable equilibrium (bottom figure).

Furthermore, in the design of control systems for aircraft, it's usually a trade-off between agility (ability to realize attitude changes quickly) and stability.

To close this chapter off, it is important to keep in mind what this course is all about: you may think, it's just flight mechanics but more intense, but actually it's not. Compare figure 2.2a with 2.2b. Flight Mechanics focussed on the *force* equilibria (figure 2.2a): we checked whether the forces in vertical add up to zero, and do the same for horizontal direction. We didn't really care about the points of application of the forces. This is a fine approach if you're looking at macro-level, i.e. when you plan the route for an aircraft etc.

On the other hand, Flight Dynamics focusses on the *moment* equilibria (figure 2.2b). We'll look at whether the moments around a certain point are in equilibrium. These moment equilibria are important on the micro-level, i.e. when you're looking at how exactly an aircraft rolls in the air etc.



(a) Simple free-body diagram of an aircraft.

(b) More detailed free-body diagram of an aircraft.

Figure 2.2: Free-body diagrams of aircraft.

So, it's worthwhile to keep in mind that in this course, we'll not really consider the force-equilibria but mainly focus on the moment equilibria.

©Sam van Elsloo

3 Aerodynamic center

In this chapter, we'll review some basic concepts such as aerodynamic center and center of pressure. Although we've had those things before, we'll discuss them more formally this time. Furthermore, although I'll repeat it later, please note that we're actually looking at 3D wings, not at 2D airfoils (in case it's unclear).

3.1 Resultant force on wing

You'll hopefully recall how lift and drag is generated by a wing. The air flows over the wing, at varying speed, creating lower pressure on the upper side and higher pressure on the upper side. This pressure distribution is shown in figure 3.1. This pressure distribution can rather straightforwardly be replaced by a single resultant force vector, **R**.

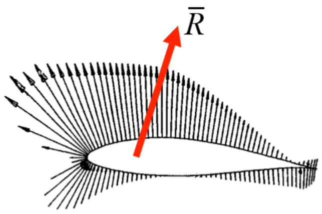


Figure 3.1: Pressure distribution.

This force can be decomposed as shown in figure 3.2 (please ignore C_m for now, we'll discuss it in the next section):

COMPONENTS OF DIMENSIONLESS FORCE VECTOR

$\mathbf{C}_R$ is the **dimensionless total aerodynamic force vector**; it is given by

$$\mathbf{C}_r = \frac{\mathbf{R}}{\frac{1}{2}\rho V^2 S} \tag{3.1}$$

C_N and C_T are the components of $\mathbf{C}_R$ normal respectively tangential to the body axis (the chordline).

C_L and C_D are the components of $\mathbf{C}_R$ normal respectively tangential to the freestream airflow.

Note that C_N, C_T etc. are also all normalized; e.g.

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 S}$$

C_N, C_T, C_L and C_D are related as follows:

RELATION BETWEEN COEFFICIENTS W.R.T. BODY AXIS AND FREESTREAM AIRFLOW

$$C_N = C_L \cos\alpha + C_D \sin\alpha \tag{3.2}$$
$$C_T = C_D \cos\alpha - C_L \sin\alpha \tag{3.3}$$

Now, normally speaking, $C_N \approx C_L$, but C_T does not approximate C_D. Why is this? Well, let's use the small-angle approximation:

$$\begin{aligned} C_N &= C_L + C_D\alpha \\ C_T &= C_D - C_L\alpha \end{aligned}$$

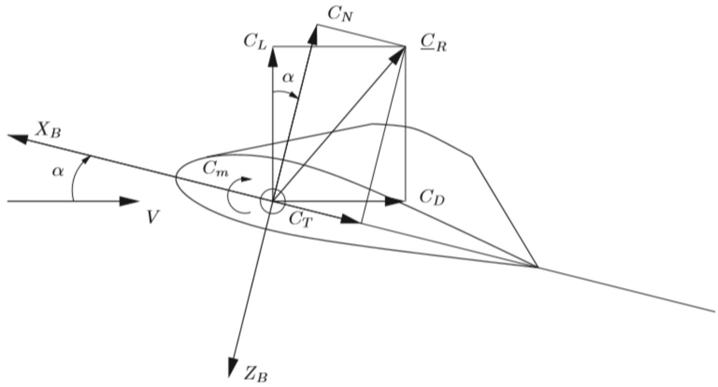
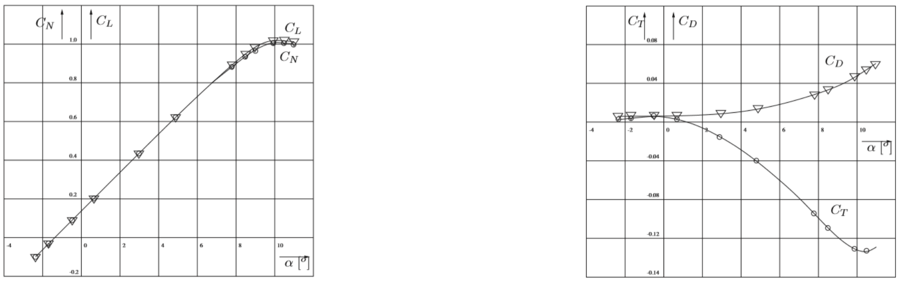


Figure 3.2: The aerodynamic forces and the moment acting on a wing in symmetric flight.

For C_N, C_D is typically small compared to C_L, and α is by definition small as well (we use the small-angle approximation after all). Thus, $C_N \approx C_L$. On the other hand, for C_T, again C_D and α are small compared to C_L: this means that at the end of the day, the term $C_L\alpha$ is not negligible at all, so C_T is not approximately the same as C_D. If you don't believe me, please see figures 3.3a and 3.3b: in figure 3.3a, C_N and C_L are largely the same; in figure 3.3b, we clearly see that C_T and C_D differ a lot once α starts to increase a bit.



(a) C_N and C_L as functions of α for some aircraft.

(b) C_T and C_D as functions of α for some aircraft.

Figure 3.3: C_N, C_L and C_T, C_D as functions of α for some aircraft.

Now, note that C_T actually becomes negative for $\alpha > 1°$, which would mean that the wing is actually pulled *forward* in chord-wise direction rather than dragged backwards. How can this? Well, look at figure 3.1: it's not that hard to imagine that if you increase the angle of attack, the low pressure zone slightly above the leading edge (where there are big arrows pointing towards the left-top corner) will actually suck the wing forward. This means that C_T becomes negative. The drag will still point backwards, as C_N will be more inclined backwards, meaning you still have drag; see figure 3.4.

Multiple-choice questions, part 1

Which of the following statements is/are true?

a) Over a large angle-of-attack ($-5° < \alpha < 10°$) range it holds that $C_N \approx C_L$ and $C_D \approx C_T$.
b) For $\alpha < 5°$ it holds that $C_N \approx C_L$.
c) For $\alpha < 5°$ it holds that $C_N \approx C_L$ ánd $C_D \approx C_T$.
d) Only when the Mach influence can be neglected, it always holds that $C_N \approx C_L$ and $C_D \approx C_T$.

The (only) correct answer B. $C_D \approx C_T$ is simply never true.

©Sam van Elsloo

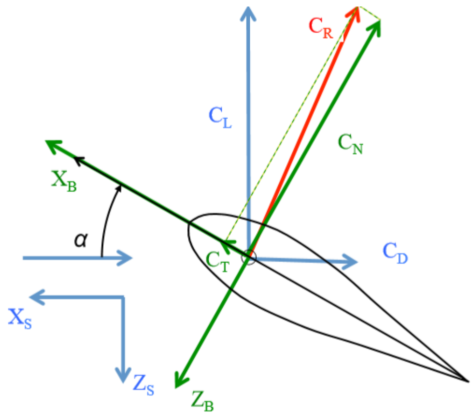


Figure 3.4: The aerodynamic forces and the moment acting on a wing in symmetric flight at a large angle of attack.

3.2 Aerodynamic moment

Now, I've said before that you should ignore C_m as we'd discuss it later; the time to discuss C_m has come now. Note that

MOMENT COEFFICIENT OF WING

The **moment coefficient** of a wing is given by

$$C_m = \frac{M}{\frac{1}{2}\rho V^2 S\bar{c}}$$

where $\bar{c}$ is the **mean aerodynamic chord**, i.e. the average chord length of the wing, abbreviated by **mac**.

Naturally, the aerodynamic force generates a moment. However, this moment is a little bit ambiguous as it heavily depends on where you calculate the moment about. Consider figure 3.5. You're in a wind tunnel doing experiments with a wing. Once we measure C_N, C_T and C_m at location 1; afterwards we measure it at location 2. Obviously, the measured C_N and C_T will be the same. However, the measured moment coefficients will be different: just think what happens when you hold a wing by its leading edge vs. when you hold it by its trailing edge. If you hold it by its leading edge, the back of the wing will pull up, i.e. you get a negative pitching moment (as nose upwards = positive, by convention). On the other hand, if you hold it by the trailing edge, the nose of the wing will tend upwards, meaning you get a positive pitching moment. Thus, clearly, $C_{m_1} \neq C_{m_2}$. We can actually relate them pretty easily though. If we know the value of C_N and C_T, and C_m at location 1, then the C_m at location 2 is (clockwise (nose upward) positive)

MOMENT COEFFICIENT

$$C_{m_{(x_2,z_2)}} = C_{m_{(x_1,z_1)}} + C_N\frac{x_2 - x_1}{\bar{c}} - C_T\frac{z_2 - z_1}{\bar{c}} \tag{3.4}$$

where $(x_2 - x_1)/\bar{c}$ and $(z_2 - z_1)/\bar{c}$ are the normalized distances[1].

3.2.1 Center of pressure

Now, you may have looked at figure 3.1 and immediately thought, wait a sec how do I know where to draw the resultant vector? Well, there is a logical point to draw it. It makes sense that there will be a point on the mean aerodynamic chord (mac) about which, when you measure the moment, the moment is zero (we'll see

[1]If you don't immediately see where this formula comes from: we're just calculating the moment around point 2. C_{m_1} is positive itself, C_N contributes in clockwise direction looking at the figure, and C_T contributes negatively according to the figure. We have to take normalized distances as otherwise you'd multiply the dimensionless C_N (or C_T) with meters, whilst C_m is dimensionless.

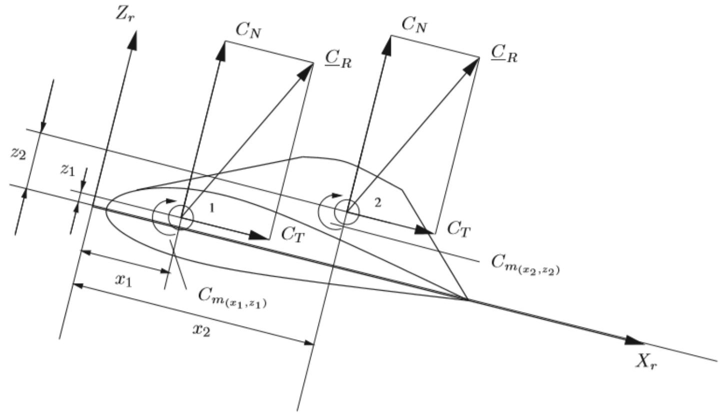


Figure 3.5: The variation of the moment with changes in the reference point.

in a bit how to calculate where this location is). Thus, it's advantageous to draw our resultant vector through this point (called the **center of pressure**): then you don't have an inherent moment coefficient you have to deal with[2].

Now, the question becomes, where should we draw the resultant vector such that it indeed does not also produce a couple C_m? For this, we can use equation (3.4); we'll consider the mac (mean aerodynamic chord) drawn in figure 3.6. This mac may look a little bit strange to you (the bolded part doesn't cover the entire chord: this is intended), but you have to keep the following in mind: we are considering 3D wings in this chapter! The outline of the airfoil is simply the wing profile at one location along the span, for example the root. The mac is projected on top of this, essentially.

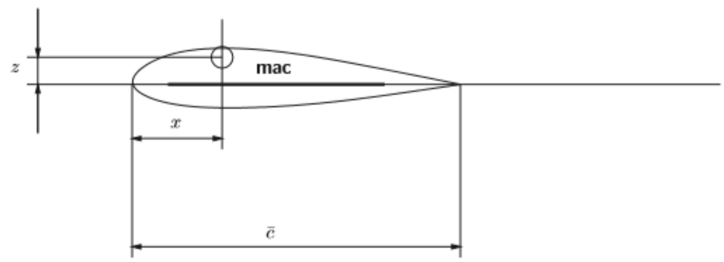


Figure 3.6: Location of point (x, z) with respect to the position of the mean aerodynamic chord.

Now, to use equation (3.4), let point 1 be the leading edge of the mac, i.e. (x_0, z_0), with moment coefficient C_{m_0}. Let point 2 be the center of pressure, located at the mac, i.e. (x_d, z_0). Note that both points have the same z-coordinate. Furthermore, the center of pressure has zero moment coefficient, so we get from equation (3.4):

$$\begin{aligned} C_{m_{(x_2,z_2)}} &= C_{m_{(x_1,z_1)}} + C_N \frac{x_2 - x_1}{\bar{c}} - C_T \frac{z_2 - z_1}{\bar{c}} \\ 0 &= C_{m_{(x_0,z_0)}} + C_N \frac{x_d - x_0}{\bar{c}} - C_T \frac{z_0 - z_0}{\bar{c}} = C_{m_{(x_0,z_0)}} + C_N \frac{x_d - x_0}{\bar{c}} \end{aligned}$$

The center of pressure is thus as follows:

[2]What I mean is: suppose we know that this center of pressure lays at 0.75c (i.e. at 3/4 of the mac), then calculating the moment coefficient about the leading edge would be very easy: it's simply $C_{m_2} = 0 + C_N \cdot -0.75$, where C_N is easily known. On the other hand, if we'd had arbitrarily drawn the force vector somewhere else, e.g. at the trailing edge, then it'd have become $C_{m_2} = C_{m_1} + C_N \cdot -1$: this is more fucked up, because now we'd also have to know C_{m_1}: the moment coefficient at the trailing edge. We may not know the moment coefficient at every single point on the wing, so we'd rather draw the force vector somewhere where we know it to be zero and then use that as reference point.

LOCATION OF CENTER OF PRESSURE

$$\frac{x_d - x_0}{\bar{c}} = \frac{e}{\bar{c}} = -\frac{C_{m_{(x_0,z_0)}}}{C_N} \tag{3.5}$$

We can rewrite this a bit. Looking at figure 3.7, we realize that the moment coefficient (shown at various location along the MAC) varies pretty much linearly with angle of attack for $\alpha < 10°$. Thus, we can state that

$$C_{m_{(x_0,z_0)}} = C_{m_0} + \frac{dC_{m_{(x_0,z_0)}}}{dC_N} C_N$$

where C_{m_0} is the moment coefficient around the leading edge of the mac (lemac) when $C_N = 0$. This allows us to rewrite the location of the center of pressure (CoP) to

$$\frac{e}{\bar{c}} = \frac{e}{\bar{c}} = -\frac{C_{m_{(x_0,z_0)}}}{C_N} = \frac{e}{\bar{c}} = -\frac{C_{m_0} + \frac{dC_{m_{(x_0,z_0)}}}{dC_N} C_N}{C_N}$$

or:

LOCATION OF CENTER OF PRESSURE

$$\frac{e}{\bar{c}} = -\frac{C_{m_0}}{C_N} - \frac{dC_{m_{(x_0,z_0)}}}{dC_N} \tag{3.6}$$

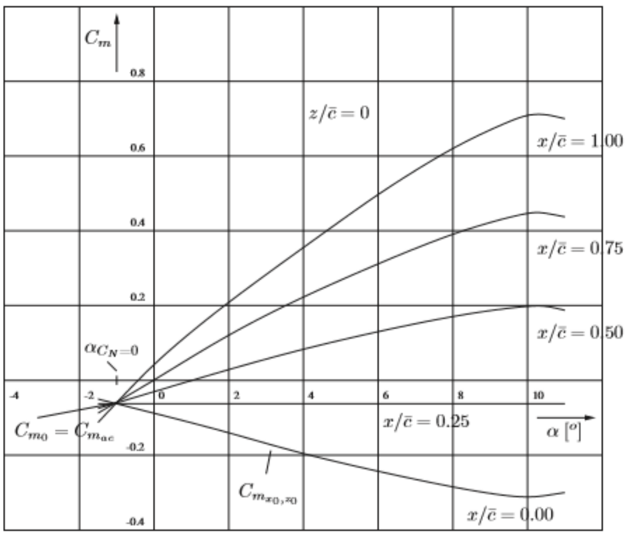


Figure 3.7: Moment curves for various positions of the reference point along the mac. $x/\bar{c} = 0.00$ corresponds to the leading edge.

Looking at figure 3.7, we realize that C_{m_0} is negative (remember it's a constant value: it's the moment coefficient about the leading edge when $C_N = 0$), and the slope $dC_{m_{(x_0,z_0)}}/dC_N$ is negative as well when looking at the leading edge (the graph associated with $x/\bar{c} = 0.00$); this means that $e/\bar{c}$ will be positive and thus that the center of pressure is located behind the leading edge (except when you try very negative values for α but why would you wanna do that?). In fact, note that C_{m_0} and $dC_{m_{(x_0,z_0)}}/dC_N$ are both constants, and C_N is the only thing that varies with α: this means that for very small C_N, the center of pressure will be located very far behind the leading edge; as C_N increases, it'll move closer and closer to it. This movement of the center of pressure at various angles of attack is shown in figure 3.8. Note that both the magnitude and direction of $\mathbf{C}_R$ change as C_N and C_T vary with angle of attack as well.

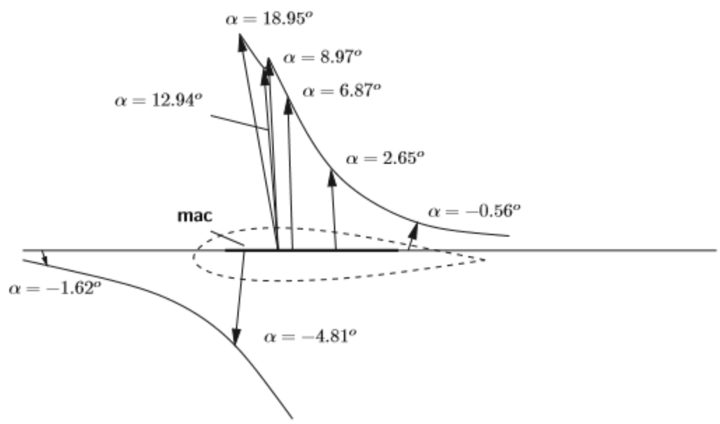


Figure 3.8: The magnitude of $\mathbf{C}_R$ and the position of the line of action of $\mathbf{C}_R$ as a function of the angle of attack.

3.3 Finding the aerodynamic center

Now, so far, we've basically have seen two ways of characterizing the aerodynamic moment: the $C_m - \alpha$ curves shown in figure 3.7, and locating the center of pressure. However, both methods have their disadvantages: the $C_m - \alpha$ curves depend on what reference point you take, and the location of the center of pressure varies with angle of attack. Therefore, let's get rid of these problems by considering the aerodynamic center:

AERODYNAMIC CENTER

> The **aerodynamic center** (ac) is the moment reference point about which the moment does not vary with angle of attack (i.e. the moment is constant).

Note that this circumvents both of the problems associated with the previous methods: it's a fixed, single reference point, and the location of it does not vary with angle of attack.

Now, last year during aero I, you proved that in theory, the aerodynamic center was located at the quarter-chord point for thin airfoils. However, in theory, communism works too, but in practice, millions die, so it's a good idea to consider a more practical approach. Unfortunately, the practical approach is a rather long, boring road, so try to keep your attention with me for a bit.

3.3.1 The first metacenter and the neutral line

We have two important points to discuss: the neutral point and the aerodynamic center. We'll first discuss the neutral point (don't worry yet about what it means), the aerodynamic center comes in the next subsection.

We start with analysing the wing at two angles of attack: once at α, and once at $\alpha + \Delta\alpha$. Remembering figure 3.8, we realize that when we increment α with $\Delta\alpha$, two things change: the location of the center of pressure along the mac, and how $\mathbf{C}_R$ looks like (e.g. compare $\alpha = 2.65°$ and $\alpha = -0.56°$ in figure 3.8). We can draw this again in figure 3.9. If you're intimidated by this figure due to all the new information, let me explain: it's just comparing two adjacent $\mathbf{C}_R$s in figure 3.8 with each other[3] however, we draw the resultant vector lines through this time. We make the lines of action of $\mathbf{C}_R$ as long as is required to make the lines intersect[4] this intersection is called the **first metacenter**, although this term is only used in naval architecture and I don't often see boats flying around so I don't know why the book mentions this, but anyway. To clarify, this first metacenter is not a fixed location for all α; its location varies with α (looking at figure 3.8, this makes sense: it's clear that the intersection point of two adjacent $\mathbf{C}_R$ won't necessarily be in the same location). Now, what's so special about this first metacenter? Note that for both α and $\alpha + \Delta\alpha$, $\mathbf{C}_R$ is acting through the first metacenter. Thus,

[3] The vector associated with $\alpha + \Delta\alpha$ can be written as $\mathbf{C}_R + \Delta\mathbf{C}_R$.

[4] So, in figure 3.8, you'd draw the lines of $\alpha = -0.56°$ and $\alpha = 2.65°$ a bit longer so that they intersect (you wouldn't change the position of the vectors though.

©Sam van Elsloo

the following holds:

$$\begin{aligned} C_m &= 0 \\ \frac{dC_m}{d\alpha} &= 0 \end{aligned}$$

After all, $\mathbf{C}_R$ passes through the first metacenter and thus C_m is zero; furthermore, it stays zero when we go from α to $\alpha + \Delta\alpha$, thus the derivative is also zero.

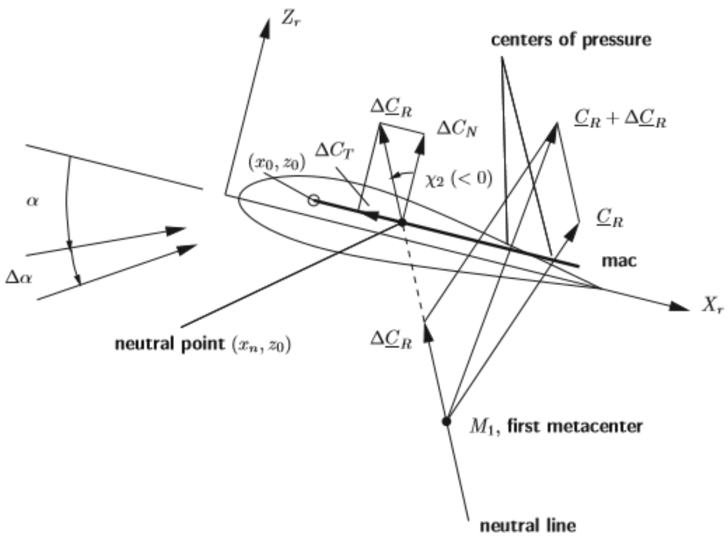


Figure 3.9: The line of action of the difference between aerodynamic force vectors at adjacent values of the angle of attack.

Now, in figure 3.9, you also see $\Delta\mathbf{C}_R$ drawn: this one should be pretty self-explanatory, it's simply the difference between the two resultant force vectors for α and $\alpha + \Delta\alpha$. Note that when you look at a fixed reference point along the line of action of $\Delta\mathbf{C}_R$, the change of aerodynamic moment will be zero if you change α[5]. Taking the limit:

NEUTRAL LINE

For all points on the line of action of $d\mathbf{C}_R$,

$$\frac{dC_m}{d\alpha} \tag{3.7}$$

The line of action of $d\mathbf{C}_R$ is called the **neutral line**. The intersection of the neutral line with the mac is called the **neutral point**.

Now, note that much like the first metacenter, the neutral line also varies with α (i.e. at different α, the first metacenter and the neutral line are oriented differently). However, the neutral *point* is actually pretty much fixed in location, as we can prove as follows: remember that in general, we have

$$C_{m_{(x_2,z_2)}} = C_{m_{(x_1,z_1)}} + C_N \frac{x_2 - x_1}{\bar{c}} - C_T \frac{z_2 - z_1}{\bar{z}}$$

Let the first location be the leading edge of the mac, i.e. $x_1 = x_0$, $z_1 = z_0$ and $C_{m_{(x_1,z_1)}} = C_{m_{(x_0,z_0)}}$. Let the second location be the neutral points with location $x_2 = x_n$ and $z_2 = z_n$. Now, if we differentiate with respect to α, we naturally get

$$\frac{dC_{m_{(x_n,z_n)}}}{d\alpha} = \frac{dC_{m_{(x_0,z_0)}}}{d\alpha} + \frac{dC_N}{d\alpha}\frac{x - x_0}{\bar{c}} - \frac{dC_T}{d\alpha}\frac{z - z_0}{\bar{c}} = 0$$

[5] After all, remember that the vector associated with $\alpha + \Delta\alpha$ can be written as $\mathbf{C}_R + \Delta\mathbf{C}_R$. The resultant moment about an arbitrary point is naturally $\left(\mathbf{C}_R + \Delta\mathbf{C}_R\right) \cdot \mathbf{r}$, with $\mathbf{r}$ being the position vector. This can be rewritten to $\mathbf{C}_R \cdot \mathbf{r} + \Delta\mathbf{C}_R \cdot \mathbf{r}$; this first term is equivalent to the moment that the first vector generates around the same point. Thus, the change in moment is merely given by $\Delta\mathbf{C}_R \cdot \mathbf{r}$ when comparing the moments about the same point for different angles of attack. If you take the moment around an arbitrary point on the line of action of $\Delta\mathbf{C}_R$, then $\mathbf{r} = \mathbf{0}$ so the change of moment will be zero.

However, $dC_{m_{(x_n,z_n)}}/d\alpha$ will be zero, as the neutral point lays on the neutral line, where $dC_M/d\alpha = 0$. Furthermore, $z_n = z_0$ as we are on the mac, so in fact we get

$$0 = \frac{dC_{m_{(x_0,z_0)}}}{d\alpha} + \frac{dC_N}{d\alpha}\frac{x_n - x_0}{\bar{c}}$$

Multiply with $d\alpha$ and divide by dC_N to get

LOCATION OF NEUTRAL POINT

The location of the neutral point follows from

$$\frac{x_n - x_0}{\bar{c}} = -\frac{dC_{m_{(x_0,z_0)}}}{dC_N} \tag{3.8}$$

Since everything is constant (as long as we are in the linear region of the $C_m - \alpha$ curve[6]), the location of x_n is constant as well. In case you were interested, the angle χ_2 in figure 3.9 is given by[7]

$$\chi_2 = \arctan\frac{dC_T}{dC_N}$$

Note that dC_T/dC_N will always be negative: when α increases, C_N also increases, but C_T decreases (remember figure 3.3).

3.3.2 *The second metacenter and the aerodynamic center*

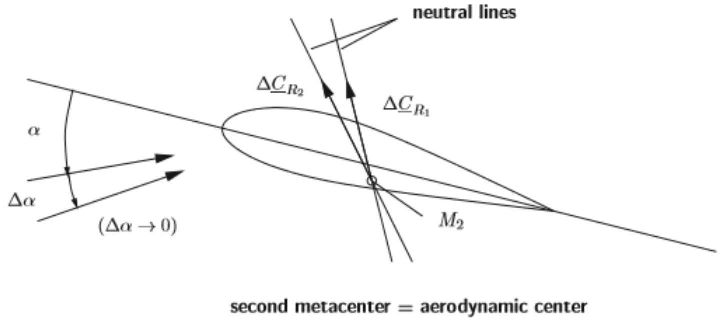


Figure 3.10: The position of the aerodynamic center.

Now, as I said before, the neutral line varies with angle of attack. Therefore, let us consider $\alpha_1 = \alpha$ and $\alpha_2 = \alpha + \Delta\alpha$ and draw their associated neutral lines, as done in figure 3.10[8]. At the point where these two neutral lines intersect, the following conditions are satisfied:

$$\left(\frac{dC_m}{d\alpha}\right)_{\alpha_1} = 0$$
$$\left(\frac{dC_m}{d\alpha}\right)_{\alpha_2} = 0$$

In other words, for both angles of attack, the moment coefficient about the intersection point is zero if you increase the angle of attack. This means that between α and $\alpha + \Delta\alpha$, the derivatives are constant (equal to zero). This means that the second derivative is zero as well: $d^2C_M/d\alpha^2 = 0$ at the intersection point as well. This intersection point has a familiar name:

[6]Since then both C_m and C_N vary linearly with α, so dC_m/dC_N is a constant.

[7]In case you're wondering, wait where did ξ_1 go, looking at figure 3.5, χ_2 is the angle between C_N and $\mathbf{C}_R$, given by $\chi_1 = \arctan C_T/C_N$.

[8]I agree that it's a bit badly drawn, as the neutral point (where the neutral line intersects with the mac), should not move. Note that this should mean that the neutral lines intersect at the mac and not somewhere else. We'll soon see that they actually indeed intersect at the mac, so it's just a poor drawing.

©Sam van Elsloo

SECOND METACENTER AND AERODYNAMIC CENTER

The point of intersection of two neutral lines belonging to two adjacent angles of attack is called the **second metacenter**, M_2 in naval architecture. It is also known as the **aerodynamic center**.

Finding the location of the aerodynamic center is done as follows: for the aerodynamic center, it holds that

$$\begin{aligned}\frac{dC_m}{d\alpha} &= 0\\ \frac{d^2C_m}{d\alpha^2} &= 0\end{aligned}$$

Now, we apply tricks very similar to finding the neutral point. The first point will be the leading edge of the mac; the second point will be the aerodynamic center with $x = x_{ac}$ and $z = z_{ac}$. We then get the following equations:

$$\begin{aligned}\frac{dC_{m_{ac}}}{d\alpha} &= \frac{dC_{m_{(x_0,z_0)}}}{d\alpha} + \frac{dC_N}{d\alpha}\frac{x_{ac}-x_0}{\bar{c}} - \frac{dC_T}{d\alpha}\frac{z_{ac}-z_0}{\bar{c}} = 0\\ \frac{d^2C_{m_{ac}}}{d\alpha^2} &= \frac{d^2C_{m_{(x_0,z_0)}}}{d\alpha^2} + \frac{d^2C_N}{d\alpha^2}\frac{x_{ac}-x_0}{\bar{c}} - \frac{d^2C_T}{d\alpha^2}\frac{z_{ac}-z_0}{\bar{c}} = 0\end{aligned}$$

As x_{ac} and z_{ac} are the only unknowns in these two equations, it can easily be solved to find x_{ac} and z_{ac}. The moment coefficient is then given by

$$C_{m_{ac}} = C_{m_{(x_0,z_0)}} + C_N\frac{x_{ac}-x_0}{\bar{c}} - C_T\frac{z_{ac}-z_0}{\bar{c}}$$

3.3.3 Relation between aerodynamic center and neutral point

Now, I said before that figure 3.10 is pretty fucked up: the neutral lines all need to go through the same point on the mac. However, that means that I'm implying the aerodynamic center has the same location as the neutral point: we can prove that this is true (for small α, but that assumption holds for basically everything we're gonna be doing in this course). Look at equation (3.9): note that the moment coefficient and normal coefficient vary almost perfectly linearly with α; this means that the second derivatives are zero. This means that equation (3.9) reduces to

$$\frac{d^2C_T}{d\alpha^2}\frac{z_{ac}-z_0}{\bar{c}} = 0$$

leading to $z_{ac} = 0$ (remember figure 3.3b: the second derivative of C_T is definitely not zero). Thus, the aerodynamic center is also located at the mac, and thus the neutral point and aerodynamic center coincide.

3.3.4 Use of aerodynamic center

The aerodynamic center makes the model of aerodynamic forces and moment acting on a wing much nicer; as by definition the moment $C_{m_{ac}}$ is constant, we only have a constant $C_{m_{ac}}$ about the ac of the wing and the total aerodynamic force $\mathbf{C}_R$ through the ac for all angles of attack; see figure 3.11.

Furthermore, if $\frac{x_{ac}}{\bar{c}}$ and $C_{m_{ac}}$ are known, the aerodynamic coefficient about an arbitrary reference poin ton the mac is given by the following very elegant expression:

$$C_m(x) = C_{m_{ac}} + C_N\frac{x - x_{ac}}{\bar{c}}$$

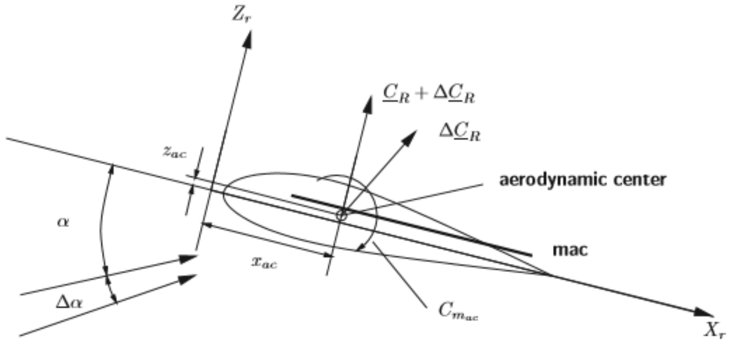


Figure 3.11: The aerodynamic forces and moment, using the ac as the moment reference point.

Multiple choice questions, part 2

Decide for every statement whether it is True or False. No explanation is required!

1. For a wing, the neutral point is the point where the line of action of the resultant aerodynamic force C_R crosses the mean aerodynamic chord.
2. There is only one unique $C_m - \alpha$ curve for a wing in a flow.
3. In the aerodynamic center the change in aerodynamic moment due to a change in angle of attack is zero. Moreover, the second order derivative of C_m with respect to α is zero.
4. The aerodynamic center is the point on the mean aerodynamic chord about which the longitudinal aerodynamic moment is zero.
5. The aerodynamic center is defined as the point where two neutral lines belonging to two adjacent angles of attack intersect. This point is also called the second metacenter.
6. The neutral point is also called the second metacenter.
7. The aerodynamic center is always a fixed point for the entire flight envelope, i.e. the full range of angle of attack.

The correct answers are:

1. False: the neutral point is the point where the line of action of $d\mathbf{C}_R$ crosses the mean aerodynamic chord, not of $\mathbf{C}_R$.
2. False: there are infinitely many $C_m - \alpha$ curve for a wing in a flow: the shape of the $C_m - \alpha$ curve depends on what reference point you take, after all.
3. True: the first sentence is the definition of aerodynamic center. The second sentence is correct since when a value is constant, its second order derivative is zero.
4. False: this point is called the *center of pressure*. For the aerodynamic center, the *change* in (longitudinal) aerodynamic moment is zero.
5. True: this is how we located the aerodynamic center at the beginning of this section.
6. False: only if we make certain assumptions, the neutral point *coincides* with the aerodynamic center (also known as second metacenter). Even then: it only coincides, it is not *called* the second metacenter (I hope you get the semantic difference). Note: the official answer model says that the neutral point is called the *first* metacenter, but that's quite a lot of bullshit, I mean just look at figure 3.9.
7. False: it's only for small angles of attack; in reality, it varies with angle of attack (and angle of sideslip, but we haven't discussed that yet).

3.4 *Aerodynamic center and stability*

Throwback time to the nostalgic times of designing the flying wing for our first project (and take a moment of silence for your group mates who didn't make it past the first year). How were we able to make our flying wing fly without it crashing into the ground? The key was stability, and specifically, the location of the aerodynamic center compared to the center of gravity.

Let us test a flying wing in a wind tunnel, suspending it in the suspension point. We'll consider three cases:

- The suspension point lays ahead of the aerodynamic center ($(x - x_{ac})/\bar{c} < 0$). Looking at figure 3.12a, we see that equilibrium is possible, as long as $C_{m_{ac}} > 0$ (remember that clockwise is positive for moments). Now, is this equilibrium stable? Yes, it is: if we increase the angle of attack slightly as done in figure 3.12b (and thus C_N is increased slightly as well), we see that a counterclockwise moment is generated around the suspension point due to the increased C_N. This pushes the leading edge down again, meaning the disturbance is removed. Thus, when the suspension point is ahead of the aerodynamic center, if $C_{m_{ac}}$ is positive and dC_m/dC_N is negative[9], the wing is stable. Graphically, we can depict it as done in figure 3.12c: α_1 is the equilibrium state: C_m (taken about the suspension point) equals zero at α_1 (after all, it's in equilibrium); it is required that $C_{m_{ac}} > 0$ and the derivative of C_m is negative, i.e. $dC_m/dC_N < 0$. In case you don't fully get the graph:
  - $C_{m_{ac}}$ is indicated as follows: remember that

$$C_m = C_{m_{ac}} + C_N \frac{x - x_{ac}}{\bar{c}}$$

    So, when $C_N = 0$, $C_m = C_{m_{ac}}$ and that is how $C_{m_{ac}}$ can be read off from the graph (by just looking at C_m when $C_N = 0$).
  - The line going up is simply the $C_N - \alpha$ line (we assume it's fully linear).
  - The line going down is the $C_m - \alpha$ line. We require the slope of this line to be negative, as we require $dC_m/dC_N < 0$ (and $dC_N/d\alpha > 0$, so we actually require $dC_m/d\alpha < 0$).
- The suspension point lays exactly at the aerodynamic center. Looking at figure 3.13a, we see that equilibrium is possible, as long as $C_{m_{ac}} = 0$. However, this equilibrium is merely *neutrally stable*: when we increase the angle of attack as done in figure 3.13b, we don't generate a countermoment, so the disturbance is kept in place (but it does not increase either). Thus, $dC_m/dC_N = 0$. Graphically, we can depict this as done in figure 3.13c: $C_{m_{ac}} = 0$, and the $C_m - \alpha$ curve is just a straight line.
- The suspension point lays behind the aerodynamic center. Looking at figure 3.14a, we see that equilibrium is possible, as long as $C_{m_{ac}} < 0$ (it needs to be counterclockwise after all). However, if we then increase the angle of attack in figure 3.14b, we see that the equilibrium is unstable: the increased C_N will only cause the leading edge to pitch up even more, causing an even bigger increase in C_N, pitching the wing even up more, etc.: it's unstable. In figure 3.14c, this is graphically depicted: $C_{m_{ac}}$ is negative, and dC_m/dC_N is positive.

In short, we can say:

NOTES REGARDING STABILITY OF A SINGLE WING

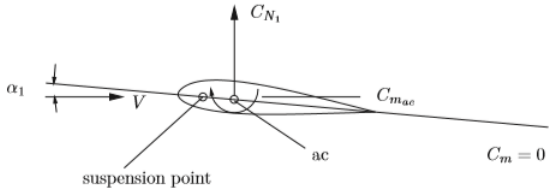
- When the suspension point is in front of the ac, there is a stable equilibrium, on the condition that $C_{m_{ac}} < 0$. dC_m/dC_N will be negative[a].
- When the suspension point is exactly at the ac, there is a neutrally stable equilibrium, on the condition that $C_{m_{ac}} = 0$. $dC_m/dC_N = 0$.
- When the suspension point is behind the ac, there is an unstable equilibrium[b].

[a] To be clear: this is not a condition for stability; it follows inherently from the fact that the suspension point is in front of the ac.

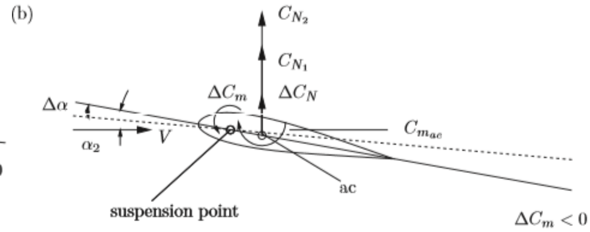
[b] Note that when $C_{m_{ac}} < 0$, there is still the possibility for an equilibrium, but this will be achieved for $C_N < 0$ (i.e. when the wing is producing downforce instead of lift).

Now, in reality, your flying wing wasn't suspended in a windtunnel or something, but flew through the sports hall. In that case, the wing does not rotate around the suspension point, but about the center of gravity. In all of the analysis above, you can read center of gravity everywhere where you see suspension point.

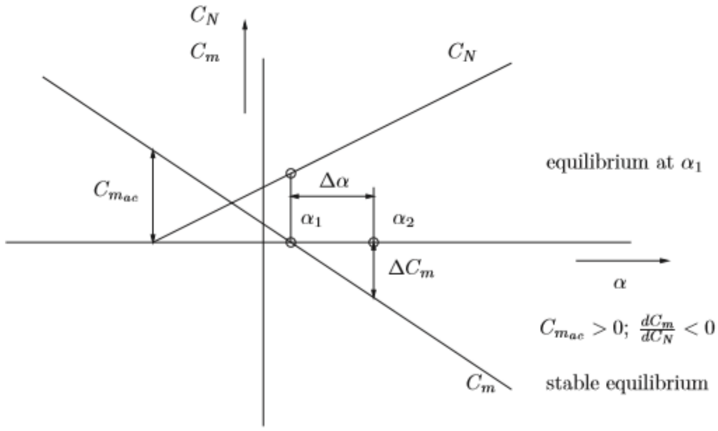
[9] After all, we saw that C_m became negative when C_N increased.



(a) Moments and forces at α_1.

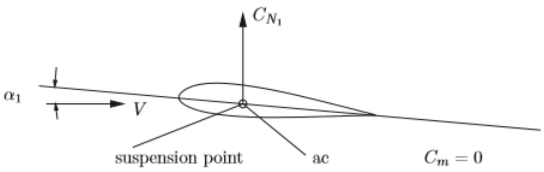


(b) Moments and forces at α_2.

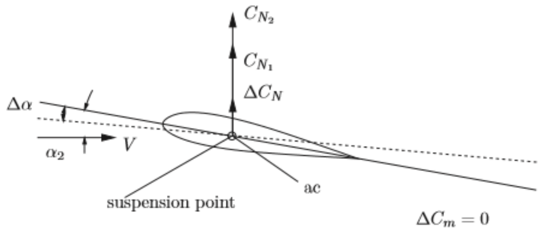


(c) Associated $C_N - \alpha$ and $C_m - \alpha$ graphs.

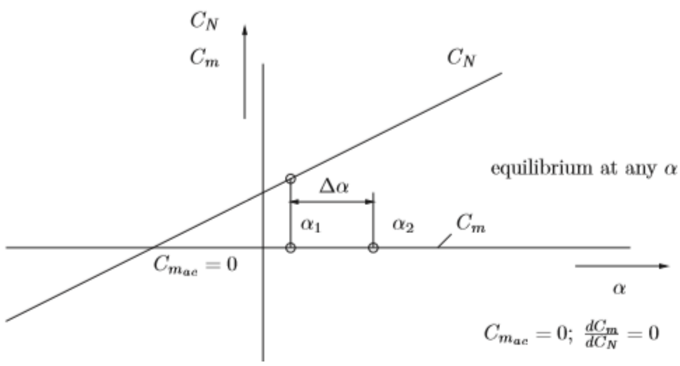
Figure 3.12: Stability analysis of wing with suspension point in front of the ac.



(a) Moments and forces at α_1.

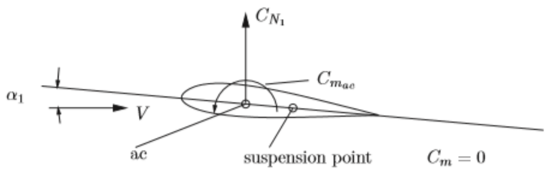


(b) Moments and forces at α_2.

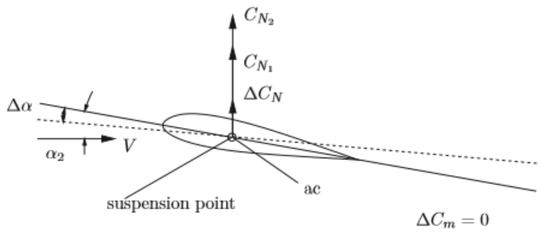


(c) Associated $C_N - \alpha$ and $C_m - \alpha$ graphs.

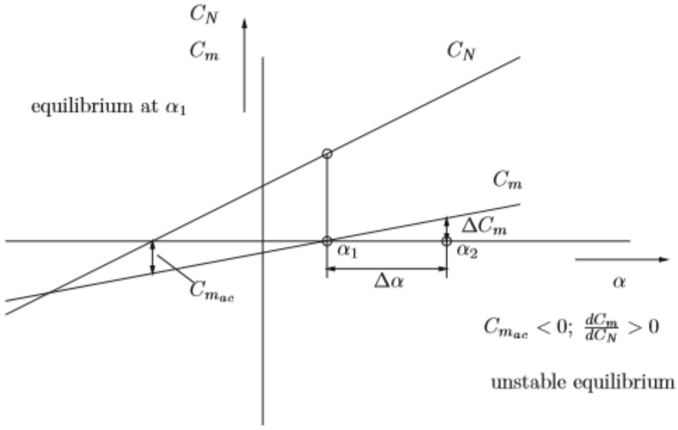
Figure 3.13: Stability analysis of wing with suspension point exactly at the ac.



(a) Moments and forces at α_1.



(b) Moments and forces at α_2.



(c) Associated $C_N - \alpha$ and $C_m - \alpha$ graphs.

Figure 3.14: Stability analysis of wing with suspension point behind the ac.

Multiple-choice questions, part (3)

A model of a wing is positioned in a windtunnel and can rotate freely along a hinge-line. The hinge-line coincides with the Y-axis (lateral-axis) of the model, and the origin of the hinge-line is located in the center of gravity of the wing. The hinge-axis is located at a distance x_s behind the leading edge of the mean aerodynamic chord (m.a.c.). The aerodynamic center is located at a distance $x_{a.c.}$ behind the leading edge of the m.a.c. The windtunnel is activated (the dynamic pressure increases in the windtunnel). Which of the following statements is (are) correct?

a) If $C_{m_{a.c.}} > 0$ and $x_s < x_{a.c.}$ then the model will assume a position with $C_N = 0$ and the equilibrium is stable.
b) If $C_{m_{a.c.}} = 0$ and $x_s < x_{a.c.}$ then the model will assume a position with $C_N = 0$ and the equilibrium is stable.
c) If $C_{m_{a.c.}} > 0$ and $x_s < x_{a.c.}$ then the model will assume a position with $C_N > 0$ and the equilibrium is stable.
d) If $C_{m_{a.c.}} < 0$ and $x_s > x_{a.c.}$ then the model will assume a position with $C_N < 0$ and the equilibrium is stable.
e) If $C_{m_{a.c.}} < 0$ and $x_s < x_{a.c.}$ then the model will assume a position with $C_N > 0$ and the equilibrium is stable.
f) If $C_{m_{a.c.}} < 0$ and $x_s < x_{a.c.}$ then the model will assume a position with $C_N < 0$ and the equilibrium is stable.

Correct are b), c) and f). Unfortunately, you'll have to verify each statement independently from each other, by drawing a sketch (unless you're really smart and can imagine it in your head, but personally I wouldn't take that risk as the signs are pretty confusing at times) akin to figures 3.12a and 3.12b etc. We must then first verify that C_m is zero in the mentioned position (if it's not, it's not even an equilibrium at all), and if it is, whether it is stable.

a) This sketch is pretty much shown in figure 3.12a, except that $C_N = 0$ now. If you sketch this, it'll be obvious that the model will *not* assume a position with $C_N = 0$: then C_m about the suspension point would be equal to $C_{m_{a.c.}}$, so it wouldn't be in equilibrium. We don't even have to wonder whether the equilibrium is stable or not, as there is no equilibrium. This statement is **false**.

b) Again, relatively similar to figure 3.12a, except that now both $C_{m_{a.c.}}$ and C_N are zero. Then around the suspension point we'll indeed have $C_m = 0$, so it is an equilibrium. It's stable as well: if we we increase the angle of attack (see figure 3.12b), C_N will increase, producing a negative C_m which helps pitch the leading edge down again. Thus, the equilibrium is stable. This statement is **true**.

c) This one is exactly figure 3.12a: $C_{m_{a.c.}}$ is positive, so there is a $C_N > 0$ possible such that there is an equilibrium. This equilibrium is also stable. This statement is **true**.

d) This is slightly similar to figure 3.14a: the suspension point now lays behind the aerodynamic center. However, given that $C_{m_{a.c.}} < 0$ (as is done in figure 3.14a, to have equilibrium, you should have $C_N > 0$, not $C_N < 0$. Thus, there is no equilibrium with $C_N < 0$ and thus this statement is **false**.

e) This is similar to figure 3.12a again: however, this time, since $C_N > 0$ and $C_{m_{a.c.}} < 0$, there *will* be a nonzero C_m, thus there is no equilibrium for $C_N > 0$. There could have been an equilibrium if $C_N < 0$ (then it'd be possible to have $C_m = 0$ around the suspension point), but for $C_N > 0$, that's not possible. This statement is **false**.

f) Pretty similar to f), although now $C_N < 0$, so an equilibrium is possible this time (note that this means that you're probably flying at a negative angle of attack, but doesn't matter regarding stability). However, is it stable? Yes: if you'd increase the angle of attack (i.e. make it less negative), C_N will become less negative, causing the leading edge of the wing to go down again, decreasing the angle of attack again. Thus, this equilibrium is stable, and this statement is **true**.

I'll admit that above question was a bit tough, but as long as you draw sketches everything and understand how equilibrium and stability are derived from those, you should be fine. Just be careful with the signs.

Multiple-choice questions, part (4)

Which of the following statements are true and which are false:

1. Increasing the wing sweep for a conventional aircraft without any tapering of the wings will increase longitudinal static stability, i.e. C_{m_α} will become more negative.

Correct are:

1. True: it's seems a bit out of place question, but I didn't know where to place it otherwise. Note that if you look at figure 3.12, if you increase the sweep, the aerodynamic center will move backwards. This means that the distance to the point of rotation will increasing, meaning an increase in C_N will cause a larger change in C_m. Thus, dC_m/dC_N becomes more negative, and so does C_{m_α}. In case you're thinking, but doesn't the center of gravity also move backwards? Yes, it does, but since it's a full aircraft, it won't move backwards as fast as the aerodynamic center (the aerodynamic center only depends on the geometry of the wing; the center of gravity also takes into account the fuselage etc., so it won't move as fast backwards).

3.5 *Characteristics of wing-fuselage-nacelle*

Now, if you remember correctly, an aircraft does not consists of *only* a wing, also of a fuselage and some engines, which are encapsulated by nacelles. Let's first consider the influence of the fuselage.

In figure 3.15, the flow around a fuselage is simulated. A fuselage is not designed to produce lift: if you look at figure 3.15, you see that a bit of lift is generated at the front of the fuselage, but at the back of the fuselage, it generates a bit of downforce, so the net result is basically no lift (a fuselage is designed to produce minimal drag, after all). However, an upward pointing force at the front and a downward pointing force at the back means that a (positive) moment is generated by the fuselage, causing the aircraft to pitch up (we have $dC_m/d\alpha > 0$ for the fuselage: with increasing angle of attack, the moment caused by the fuselage becomes stronger).

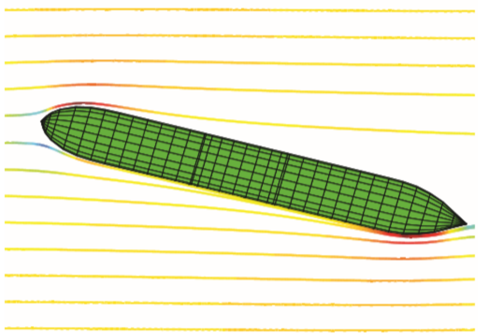


Figure 3.15: Numerical simulation of the pressure distribution around a fuselage.

To make it a bit easier, we'll consider this effect of the fuselage and just include it in the aerodynamic center of the wing, i.e. we get

$$C_{m_{w+f}} = C_{m_{ac_{w+f}}} + C_n \frac{x - x_{ac_{w+f}}}{\bar{c}}$$

However, we can also find $C_{m_{w+f}}$ differently: you could place the fuselage in a wind tunnel, then place the wing in a wind tunnel, and measure the C_m twice, and then simply add the results to each other. However, this is not entirely true: the wing influences the performance of the fuselage, and vice versa. So, in fact, we can write it as

$$C_{m_{w+f}} = C_{m_w} + C_{m_f} + \Delta C_m$$

with

$$\Delta C_m = C_{m_{f.i.}} + C_{m_{w.i.}}$$

where $C_{m_{f.i.}}$ is the wing interference on fuselage and $C_{m_{w.i.}}$ the fuselage interference on the wing.

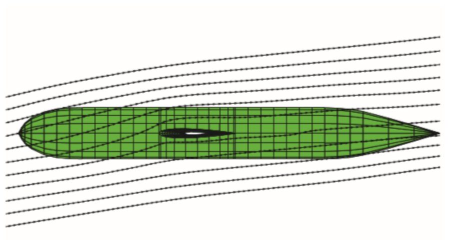
3.5.1 Influence wing on fuselage

Consider figure 3.16a: if you place a fuselage separately in a wind tunnel, the streamlines will remain straight and parallel. However, due to the presence of the wings, there is a bit of upwash just in front of the wing (the local angle of attack there is higher), and downwash behind the wing (the local angle of attack there is lower). This means that the front of the aircraft will produce slightly more lift, and the back of the fuselage slightly less lift, as shown in figure 3.16b. This means that the moment produced by the wing is increased even more.

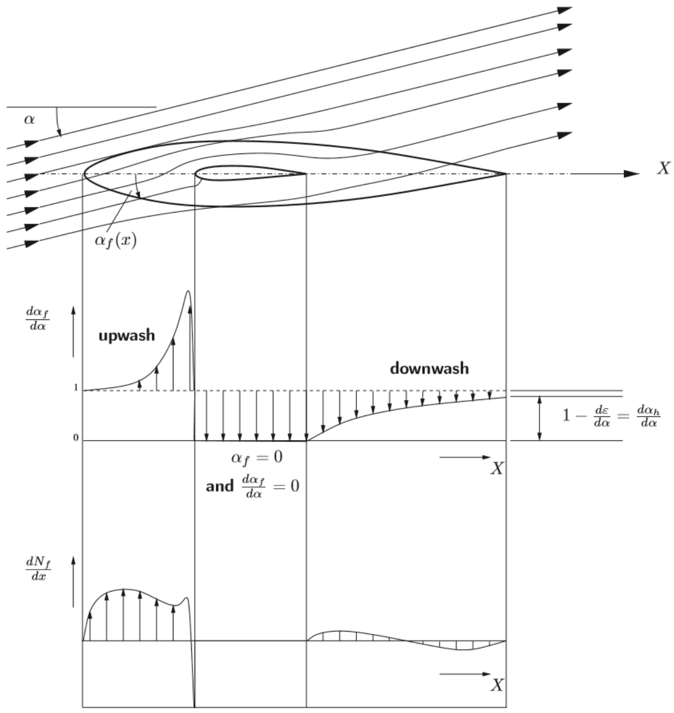
3.5.2 Influence fuselage on wing

The influence of the fuselage on the wing is pretty straightforward: remember that a flow velocity can be decomposed in a vertical component and a horizontal component, and the ratio between those makes up the angle of attack (if the horizontal component is very large in comparison to the vertical component, α will be very small as well). Now, let's take an aircraft from behind, as done in figure 3.17a. We have sketched the *vertical* velocity components of the flow. We see that the flow will have a higher vertical velocity around the fuselage (where it is bended): the air is compressed a bit there, so it has to move faster to allow for the same amount of air to move through[10]. This means that the angle of attack is higher near the fuselage, as shown in figure 3.17b. This obviously has an effect on the moment and aerodynamic center, although in reality, it can be pretty much neglected.

[10]Yes this is really basic aerodynamics, but I don't know how much you have forgotten in half a year.

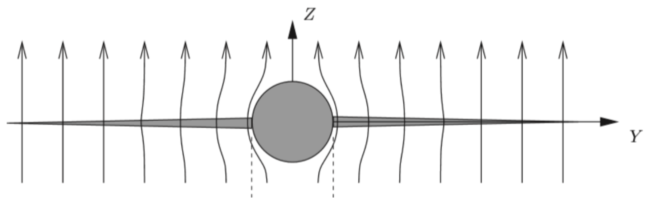


(a) Numerical simulation of the velocity field of a wing-fuselage configuration, $\alpha = 10°$.

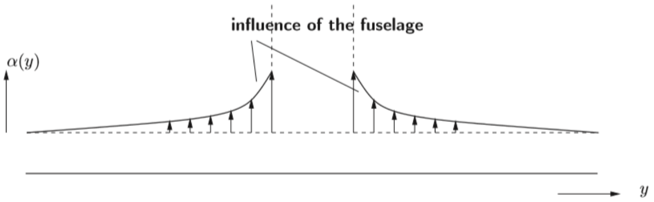


(b) The variation of the angle of attack and the normal force along the fuselage axis in a wing induced flow field.

Figure 3.16: Influence wing on fuselage.



(a) Flow around the fuselage.



(b) The variation of the angle of attack along the wingspan.

Figure 3.17: Fuselage induced variations of the local angle of attack along the wing span.

4 Equilibrium in steady, straight, symmetric flight

4.1 Quick recap on angle of attack, flight path angle, pitch angle, and lift vs. normal force

We start by analysing a very simple case: equilibrium in steady, straight and symmetric flight. However, I think it is important to recap on what these things mean, since you've probably forgotten about them since last year. First, look at figure 4.1:

BASIC DEFINITIONS OF SOME THINGS

Some angles:
- The **angle of attack** α is the angle between the velocity and the **body axis** of the aircraft (the axis that goes through the nose of the aircraft and through the center of gravity of it).
- The **flight path angle** γ is the angle between the horizontal and the velocity vector.
- The **pitch angle** θ is the sum of α and γ (i.e. $\theta = \alpha + \gamma$).

Some force vectors:
- The **lift** vector L is defined to act perpendicular to the velocity.
- The **drag** vector D is defined to act parallel to the velocity.
- The **normal force** vector N is defined to act perpendicular to the body axis
- The **tangential force** vector T acts tangential to the body axis.

Some flight conditions:
- **Steady flight** means that the velocity V is constant.
- **Straight flight** means that the flight path angle γ is constant.
- **Symmetric flight** means that there is no roll angle and the turn rate is zero.

Yes this is really basic stuff so if you managed to remember all of this correctly, good on you, but it's good to have an overview before we start on this chapter.

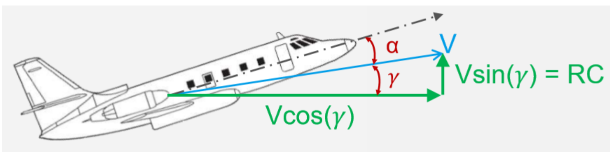


Figure 4.1: Angle of attack α, flight path angle γ and pitch angle $\theta = \alpha + \gamma$.

4.2 Basic force equilibrium

From the course on flight mechanics, you probably remember a figure similar to figure 4.2: we have an aircraft, and somewhere there is the weight $\mathbf{W}$ acting downward and the resultant aerodynamic force $\mathbf{R}$. However, do note, in this course (and elsewhere), we consider the thrust to be part of the the aerodynamic force, so we don't have a separate $\mathbf{T}$ vector[1]. The aerodynamic force can be decomposed into a component Z perpendicular to the body axis, and a component X in the direction of the body axis.

Then, we take force equilibria in X_b and Z_b directions (and moment equilibrium about the centre of grav-

[1] In fact, we consider every thing that is not part of the inertia of the aircraft (i.e. the weight) to be part of the aerodynamic force vector.

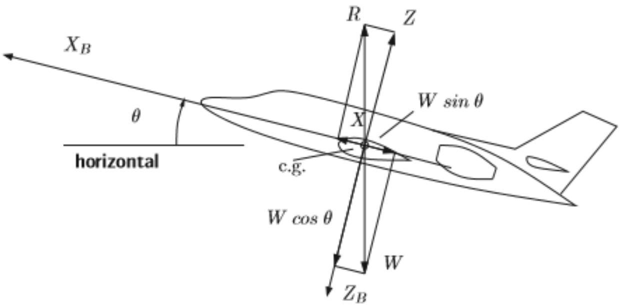


Figure 4.2: The equilibrium in steady, symmetric flight.

ity)[2]

$$\begin{aligned} -W\sin\theta + X &= 0 \\ W\cos\theta - Z &= 0 \\ M &= 0 \end{aligned}$$

This shouldn't be hard at all. Now, naturally, we may also replace Z with the normal force N (as Z *is* the normal force) and X with T (as X *is* the tangential force), so we get

$$\begin{aligned} W\sin\theta &= T \\ W\cos\theta &= N \\ M &= 0 \end{aligned}$$

This is almost exactly the same as last year during flight mechanics: in addition to that we lump thrust and 'true' aerodynamic force together, there is one small difference: during flight mechanics, we took the force equilibria in the with respect to the *velocity* instead of with respect to the body-axis as we do now: this was the reason why last year we got L and D in our equations. However, in this course, N and T are more useful, which is why we take the force equilibria with respect to the body axis.

4.3 More advanced force equilibrium

Now, even your own mother won't be impressed with the free body diagram of figure 4.2 (and if she would be, then she doesn't have too high expectations of you so idk if you should be happy with that). So, we can complicate our lives a bit and take a look at figure 4.3. Yes, it looks intimidating as hell right now, but we're gonna neglect basically everything but one thing, so just keep calm.

First off, we have three main parts of the aircraft:

- The main wing, which produces a normal force N_w (normal to the body-axis) and a tangential force T_w (tangential to the body-axis). It also produces a moment, M_{ac_w}. Furthermore, the chord may be inclined an angle i_w with respect to the body-axis. However, N_w and T_w still act normal respectively tangential to the *body*-axis, not to the chord-line. So, i_w may be whatever you want, no one cares about it.
- The horizontal tailplane, which produces a normal force N_h (normal to the body-axis) and a tangential force T_h (tangential to the body-axis). It also produces a moment, M_{ac_h}. Again, the chord of the tail may be inclined an angle i_h but no one cares as the chord doesn't matter.
- The propulsion unit, which produces thrust inclined an angle i_p with respect to the body-axis (so now i_p is important), and an upward force depending on the angle of attack of the plane (it may produce a tiny bit of lift).

Note that the subscript w is used to refer to the main wing (although often it is also omitted); h to refer to the horizontal tail and p to refer to the propulsion unit. Of course, there's also the weight W at an angle θ with respect to the body axis.

[2] Of course, you can also take force equilibria in the moving Earth or local horizon system (the coordinate system where Z_e points directly downwards (in the same direction as **W** and X_e points horizontally); however, this is not the convention as it complicates everything a lot.

©Sam van Elsloo

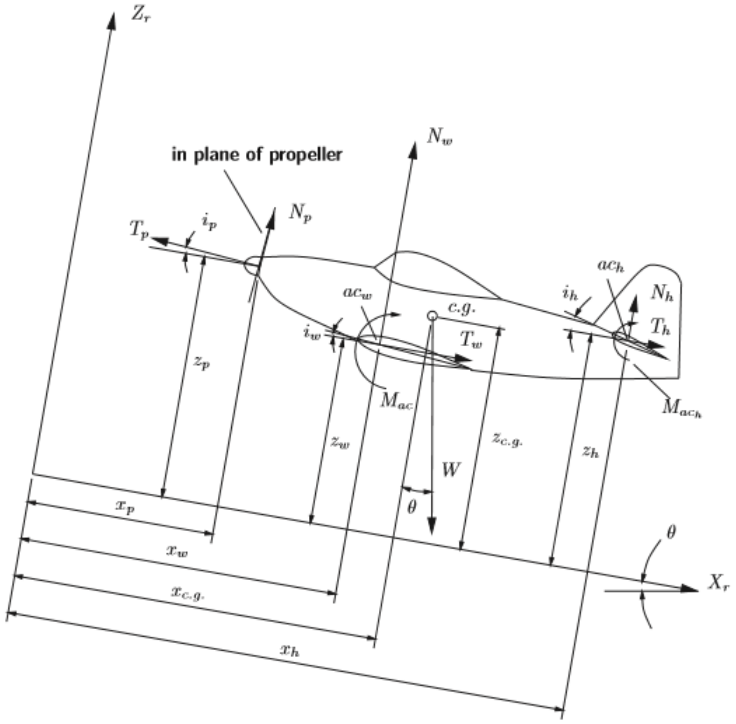


Figure 4.3: The equilibrium in steady, symmetric flight, on a more advanced level.

Then, we can again take force equilibria (but like I said, we're gonna ignore a lot of the terms appearing, so don't bother making notes of these equations as a lot of stuff will disappear; only the end-result is important): first, along the X_b-axis (to the right is positive):

$$\overset{+}{\rightarrow} \sum F_{X_b} : \quad T_w + T_h - T_p \cos i_p + N_p \sin i_p + W \sin\theta = 0 \tag{4.1}$$

Along the Z_b-axis (upwards is positive):

$$+\uparrow \sum F_{X_z} : \quad N_w + N_h + N_p \cos i_p + T_p \sin i_p - W \cos\theta = 0 \tag{4.2}$$

Finally, taking moments around the Y_b-axis (i.e. around the c.g. in clockwise direction):

$$\begin{aligned} \circlearrowright + \sum M : \quad & M_{ac_w} + N_w \left(x_{c.g.} - x_w\right) - T_w \left(z_{c.g.} - z_w\right) \\ & + M_{ac_h} + N_h \left(x_{c.g.} - x_h\right) - T_h \left(z_{c.g.} - z_h\right) \\ & + \left(N_p \cos i_p + T_p \sin i_p\right) \cdot \left(x_{c.g.} - x_p\right) \\ & + \left(T_p \cos i_p - N_p \sin i_p\right) \left(z_{c.g.} - z_p\right) = 0 \end{aligned} \tag{4.3}$$

Now we make everything dimensionless, i.e. divide by equations (4.1) and (4.2) by $\left(\rho V^2 S\right)/2$ and equation (4.3) by $\left(\rho V^2 S \bar{c}\right)/2$, where S refers to the *wing* surface area (not including the tail). For this, let us write out several of the forces and moments that do not correspond to the wing:

$$\begin{aligned} N_h &= C_{N_h} \frac{1}{2} \rho V_h^2 S_h \\ T_h &= C_{T_h} \frac{1}{2} \rho V_h^2 S_h \\ M_{ac_h} &= C_{m_{ac_h}} \frac{1}{2} \rho V_h^2 S_h \bar{c}_h \\ T_p &= T_c \rho V^2 D^2 \\ N_p &= C_{N_p} \frac{1}{2} \rho V^2 S_p \end{aligned}$$

Most symbols speak for themselves; note that for the horizontal tailplane we use the velocity over the tailplane rather than the velocity over the wing (and similarly we use the surface area of the tailplane there). Furthermore, T_c is the thrust coefficient (indeed it's not a typo, don't write C_T or something, for obvious reasons); D is the diameter of the propeller and S_p the area of the propeller, $S_p = \pi D^2/4$. Furthermore, $T_p \cos i_p - N_p \sin i_p \approx T_p$ and $\cos i_p \approx 1$, and thus we obtain the non-dimensional equilibrium equations

$$\overset{+}{\rightarrow} \sum F_{X_b} : C_{T_w} + C_{T_h} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} - T_c \frac{2D^2}{S} + \frac{W}{\frac{1}{2}\rho V^2 S} \sin\theta = 0 \quad (4.4)$$

$$+\uparrow \sum F_{Z_b} : C_{N_w} + C_{N_h} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} + T_c \frac{2D^2}{S} \sin i_p + C_{N_p} \frac{S_p}{S} - \frac{W}{\frac{1}{2}\rho V^2 S} \cos\theta = 0 \quad (4.5)$$

$$\begin{aligned} \circlearrowright + \sum M : & C_{m_{ac_w}} + C_{N_w} \frac{x_{c.g.} - x_w}{\bar{c}} - C_{T_w} \frac{z_{c.g.} - z_w}{\bar{z}} \\ & + C_{m_{ac_h}} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \frac{\bar{c}_h}{\bar{c}} + C_{N_h} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \frac{x_{c.g.} - x_h}{\bar{c}} - C_{T_h} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \frac{z_{c.g.} - z_h}{\bar{c}} \\ & + \left(T_c \frac{2D^2}{S} \sin i_p + C_{N_p} \frac{S_p}{S}\right) \frac{x_{c.g.} - x_p}{\bar{c}} + T_c \frac{2D^2}{S} \frac{z_{c.g.} - z_p}{\bar{c}} = 0 \quad (4.6) \end{aligned}$$

These are still ugly, so let's simply a bit more:

1. Let's neglect propulsion effects, i.e. terms involving T_c and C_{N_p}.
2. Let's neglect the contribution of the tangential force of the tail wing, C_{T_h}, to equations (4.4) and (4.6). After all, this force will nearly always be small.
3. Let's neglect the contribution of the tangential force of the wing, C_{T_w}, to the moment equation (4.6). After all, this force is small in comparison with C_{N_w}, and the vertical distance is usually small. If one of these assumptions is not satisfied, this simplification of course can't be made.
4. Let's neglect the contribution of the moment of the tailplanes, $C_{m_{ac_h}}$ as this is often small, even 0 when a symmetric airfoil is chosen[3]. This means that $C_{m_{ac}}$ can be used to refer to $C_{m_{ac_w}}$ now.

Applying these simplifications lead to some logical equations:

SIMPLIFIED EQUILIBRIUM EQUATIONS ALONG BODY AXIS

The equilibrium equations of a plane along a body-axis reference system are given by

$$\overset{+}{\rightarrow} \sum F_{X_b} : C_{T_w} + \frac{W}{\frac{1}{2}\rho V^2 S} \sin\theta = 0 \quad (4.7)$$

$$+\uparrow \sum F_{Z_b} : C_{N_w} + C_{N_h} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} - \frac{W}{\frac{1}{2}\rho V^2 S} \cos\theta = 0 \quad (4.8)$$

$$\circlearrowright + \sum M : C_{m_{ac_w}} + C_{N_w} \frac{x_{c.g.} - x_w}{\bar{c}} + C_{N_h} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \frac{x_{c.g.} - x_h}{\bar{c}} = 0 \quad (4.9)$$

The corresponding free-body diagram is shown in figure 4.4.

Now, let's focus on the moment equation. Let us define the following:

TAIL LENGTH

The **tail length** of an aircraft is the distance between the aerodynamic center of the horizontal tailplane and the aerodynamic center of the wing with fuselage and nacelles, i.e.

$$l_h = x_h - x_w \quad (4.10)$$

For conventional aircraft, $l_h = x_h - x_w \approx x_h - x_{c.g.}$.

[3] For tailplanes this is much more common than for wings.

©Sam van Elsloo

This means that Equation (4.9) may be written as

$$C_m = C_{m_w} + C_{m_h} = C_{m_{ac}} + C_{N_w}\frac{x_{c.g.} - x_w}{\bar{c}} - C_{N_h}\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} = 0 \tag{4.11}$$

Here, $S_h l_h/(S\bar{c})$ is known as the **tailplane volume**. Evidently, the tailplane is present to provide a counteracting moment C_{m_h} to oppose the moment caused by the wing, C_{m_w}. The value of C_{m_h} can be controlled by an appropriate deflection of the elevator or changing the incidence angle of the whole horizontal tailplane.

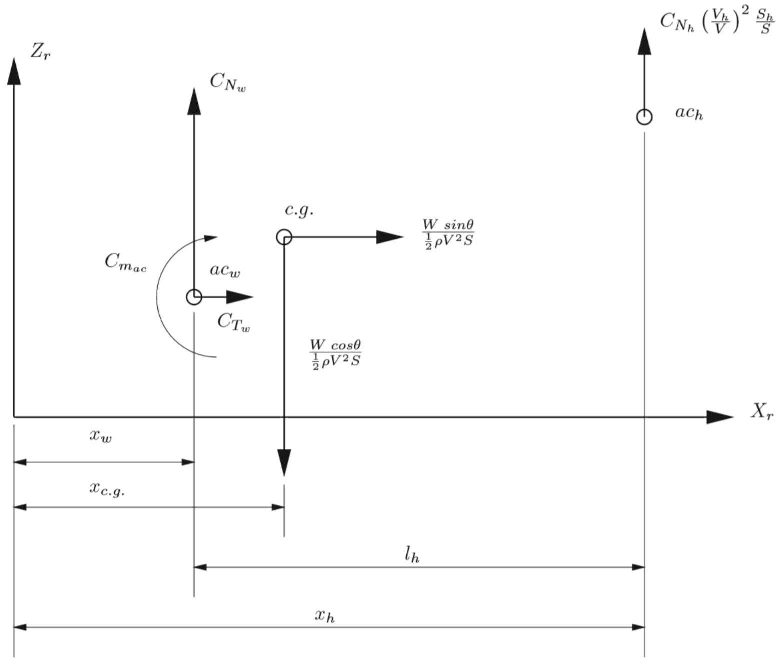


Figure 4.4: The equilibrium in steady, symmetric flight, on a more advanced level, simplified.

©Sam van Elsloo

5 Horizontal tailplane

The previous chapter finished with the equation

$$C_m = C_{m_w} + C_{m_h} = C_{m_{ac}} + C_{N_w}\frac{x_{c.g.} - x_w}{\bar{c}} - C_{N_h}\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} = 0 \tag{5.1}$$

This chapter focusses on the parameter C_N and other stuff related to the horizontal tailplane. For this, we consider the geometry of the horizontal tailplane, as visualized in Figure 5.1. Our horizontal tailplane consists of three components:

- The horizontal tailplane itself which is at an angle α_h relative to the flow.
- The elevator, which is at an angle δ_e relative to the horizontal tailplane.
- The trim tab, which is at an angle δ_t relative to the elevator.

Please bear in mind that all angles are relative to the previous element. Furthermore, for all of them, clockwise is positive (as usual).

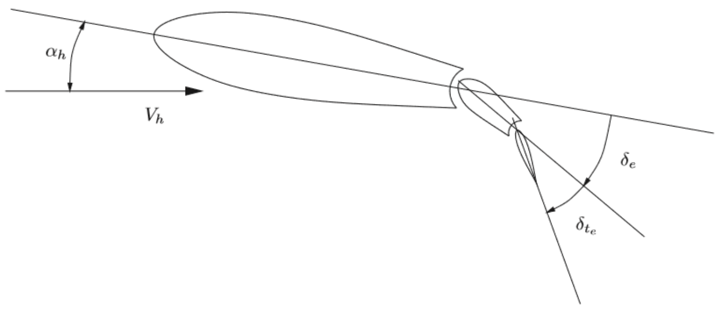


Figure 5.1: Geometry of a horizontal tailplane.

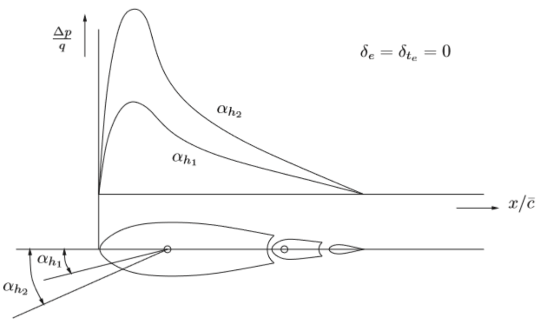
5.1 Normal force on horizontal tailplane

Let's first consider the normal force on the tailplane, i.e. C_{N_h} itself. We do this by considering the pressure distribution over the wing and how it is affected by α_h, δ_e and δ_t. Let's first consider figure 5.2a: here we have plotted the pressure distribution for two different values of α_h, whilst we keep the deflection of the elevator and tab 0. The pressure distribution should then look kinda familiar; we see that most lift is being generated at around the quarter-chord of the chord (remember: the total area below the graph is equal to C_{N_h}).

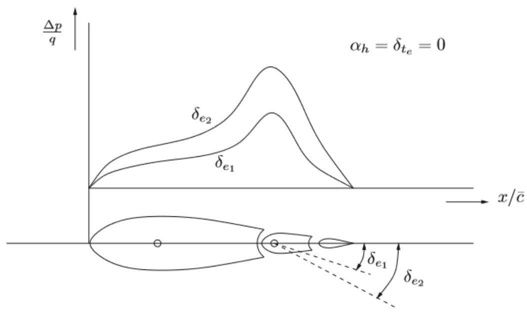
Let's then consider the case where the angle of attack is 0, but we deflect the elevator (but keep the tab angle 0, so that the tab is aligned with the elevator), as shown in figure 5.2b. We see that when we deflect the elevator, there is an increased lift production around the quarter-chord of the elevator. Note that the pressure distribution in front of the elevator is also affected; this is due to the subsonic nature of the flow[1]. The lift production is a bit smaller than in figure 5.2a, but is still quite significantly affected by the elevator deflection.

Finally, let's consider the case where the angle of attack and the elevator deflection are 0, but the tab is deflected, as shown in figure 5.2c. We see that we obtain a minor pressure peak at around the quarter-chord of the tab, but nothing really significant at all if we're really honest.

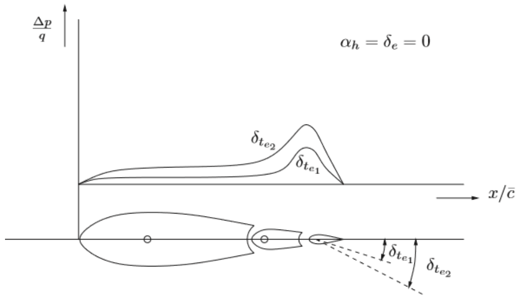
[1]The precise reasoning is something you were taught aero I but I won't repeat that here since no one cares.



(a) Pressure distribution due to α_h.



(b) Pressure distribution due to δ_e.



(c) Pressure distribution due to δ_{t_e}.

Figure 5.2: Pressure distribution due to α_h, δ_e and δ_{t_e}.

Now, it turns out (as you'd expect) that the normal force of the horizontal tailplane, for small deflections, varies linearly with the deflections. In other words,

$$C_{N_h} = C_{N_h}\left(\alpha_h, \delta_e, \delta_{t_e}\right) = C_{N_{h_0}} + \frac{\partial C_{N_h}}{\partial \alpha_h}\alpha_h + \frac{\partial C_{N_h}}{\partial \delta_e}\delta_e + \frac{\partial C_{N_h}}{\partial \delta_{t_e}}\delta_{t_e} \tag{5.2}$$

where $C_{N_{h_0}}$ is the offset, i.e. the C_{N_h} when all angles are 0. Furthermore, remember that for example C_{L_α} represented $\partial C_L/\partial\alpha$; in a very similar fashion, equation (5.2) may be written as

$$C_{N_h} = C_{N_{h_0}} + C_{N_{h_\alpha}}\alpha_h + C_{N_{h_\delta}}\delta_e + C_{N_{h_{\delta_t}}}\delta_{t_e}$$

Now, for symmetric airfoils (and for horizontal tailplanes, we often use symmetric airfoils), $C_{N_{h_0}} = 0$; furthermore, C_{N_h} changes very little with δ_{t_e}, thus $C_{N_{h_{\delta_t}}} \approx 0$. Therefore, we simply have

NORMAL FORCE COEFFICIENT OF HORIZONTAL TAILPLANE

$$C_{N_h} = C_{N_{h_\alpha}}\alpha_h + C_{N_{h_\delta}}\delta_e \tag{5.3}$$

where $C_{N_{h_\alpha}} > 0$ and $C_{N_{h_\delta}} > 0$.

The slopes are obviously positive as an increase in angle will cause an increase in normal force and not a decrease.

5.2 *Hinge moment of the elevator*

Now, when you have an elevator, you must have a hinge where it can rotate around, as depicted in figure 5.3. There is a hinge line close to the front of the elevator, around which the elevator can rotate. Now, if this was a simple hinge, the elevator would be *free* to rotate and would just oscillate randomly; obviously this is not desired which is why we must provide a counteracting torque around this hinge line so that the elevator stays in the

©Sam van Elsloo

desired orientation. For that, it is of course interesting to know what the moment created by the aerodynamic forces around the hinge line is. For this, we introduce the hinge moment coefficient C_{h_e}:

$$C_{h_e} = \frac{H_e}{\frac{1}{2}\rho V_h^2 S_e \bar{c}_e}$$

where H_e is the moment around the hinge line of the elevator (hence the subscript e), and S_e and $\bar{c}_e$ are defined in figure 5.3.

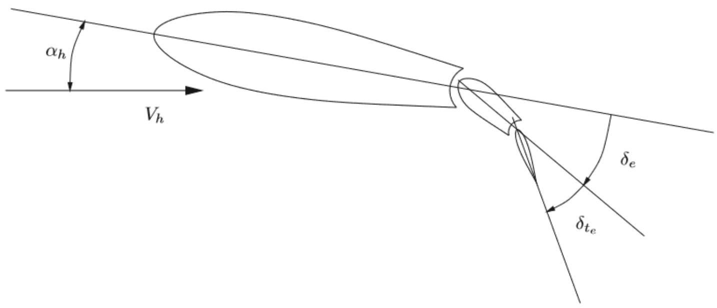


Figure 5.3: Geometry of a horizontal tailplane.

Now, it is natural to assume that C_{h_e} will be a function of α_h, δ_e and δ_{t_e}:

$$C_{h_e} = C_{h_e}\left(\alpha_h, \delta_e, \delta_{t_e}\right)$$

Now, where exactly do the moments come from? Well, let's first just consider the effect of deflecting the elevator itself (again, clockwise is positive, i.e. nose up and tail down). Suppose our hinge line is at the very front of elevator: then the you are essentially pushing the tail of the elevator into the flow, which pushes it back upward, causing a counterclockwise (negative) moment. This is desired: if the elevator experiences a disturbance in the flow, it'll automatically return to the correct position.

However, there is something undesirable about this as well: suppose you want to have a certain elevator deflection: then the flow is continuously pushing it upward again, but you want to keep it deflected. This means that you either have to use manpower (fly-by-wire system) or hydraulics to keep it in place, both of which are not desired. Fortunately for us, we can alleviate our moment: consider what happens when we deflect the trim tab upward relative to the elevator (so in *negative* direction). This means that the tab will start producing *downforce* rather than lift, and will therefore cause a *positive* moment around the hinge line. This compensates for the negative moment caused by the deflection of the elevator[2]. The effect of the combination of changing δ_e and δ_{e_t} is shown in figure 5.4.

Finally, what happens if we alter α_h and keep the other deflections constant? Not much actually. The pressure distribution over the entire elevator is increased by almost the same factor (as shown in figure 5.2a); if we take moments around a hinge-line close to the front, this means that a nose down moment is created by the increased pressure distribution, meaning that for an increase in α_h, a negative moment is created.

In any case, if we only consider small deflections, we may once more simply write

$$C_{h_e} = C_{h_0} + C_{h_\alpha}\alpha_h + C_{h_\delta}\delta_e + C_{h_{\delta_t}}\delta_{t_e} \tag{5.4}$$

Again, $C_{h_0} = 0$ as it is a symmetric airfoil; however, none of the derivatives may be neglected this time (as the trim tab *was* important this time, due to the moment arm), and thus we obtain

HINGE MOMENT COEFFICIENT OF ELEVATOR

$$\boxed{C_{h_e} = C_{h_\alpha}\alpha_h + C_{h_\delta}\delta_e + C_{h_{\delta_t}}\delta_{t_e}} \tag{5.5}$$

If the hinge-line is towards the front of the elevator, $C_{h_\alpha} < 0$, $C_{h_\delta} < 0$ and $C_{h_{\delta_t}} < 0$, as explained previously[3].

[2]This also answers your question: why did we bother to include a trim tab at all, if it does not influence C_{N_h}? Well, for the moment around the hinge-line it is important, due to the distance between the hinge-line and the trim tab.

[3]Note that α and δ_e are typically speaking positive, once more confirming that δ_t needs to be negative to get the value of C_{h_e} close to 0.

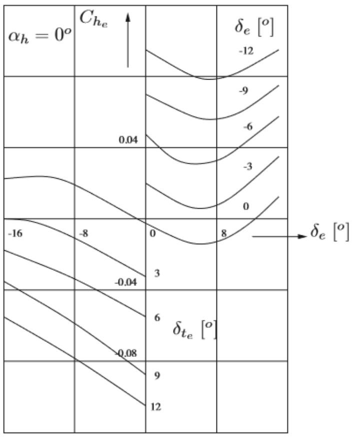


Figure 5.4: Hinge moment coefficient C_{h_e} as a function of α_h, δ_e and δ_{t_e} for a tailplane of the Fokker F-27. Note: in the top-right corner it should read δ_{e_t} rather than δ_e.

We'll now discuss what happens when the hinge-line is more aft.

5.2.1 *Effect of moving hinge line backward*

Above discussion assumed that the hinge line was towards the front of the elevator; however, are there any advantages to placing it more aft? Well, consider figure 5.5. In figure 5.5a, we have plotted the increased pressure distribution over the elevator due to increased α_h. If we put the hinge-line very far aft (the hinge-line is the small circle drawn in the elevator), we see that more of the lift is being produced in front of the hinge-line, meaning that a nose-up attitude is created (a positive moment). Thus, by placing the hinge-line more aft, you can make $C_{h_\alpha} > 0$. This may be slightly preferred over having it negative; after all, suppose the aircraft as a whole experiences a small disturbance causing a slightly higher angle of attack: the elevator will then experience a nose-up attitude, increasing its angle of attack, increasing lift at the tail and thus providing a nose-down attitude for the aircraft as a whole, cancelling out the disturbance. So, it helps a little in making the aircraft more stable (not a whole lot though, so it's not essential).

Furthermore, consider figure 5.5b, where we plotted the increased pressure distribution over the elevator due to increased δ_e. Once more, if we put the hinge line farther back, more lift will be produced in front of the hinge line, meaning that a nose-up (positive) attitude is created if δ_e is increased. This is very, very bad: if your elevator experiences a small disturbance due to some reason, it'll automatically be forced to deflect more and more until the aircraft crashes and everyone dies. So, we absolutely never want $C_{h_\delta} > 0$.

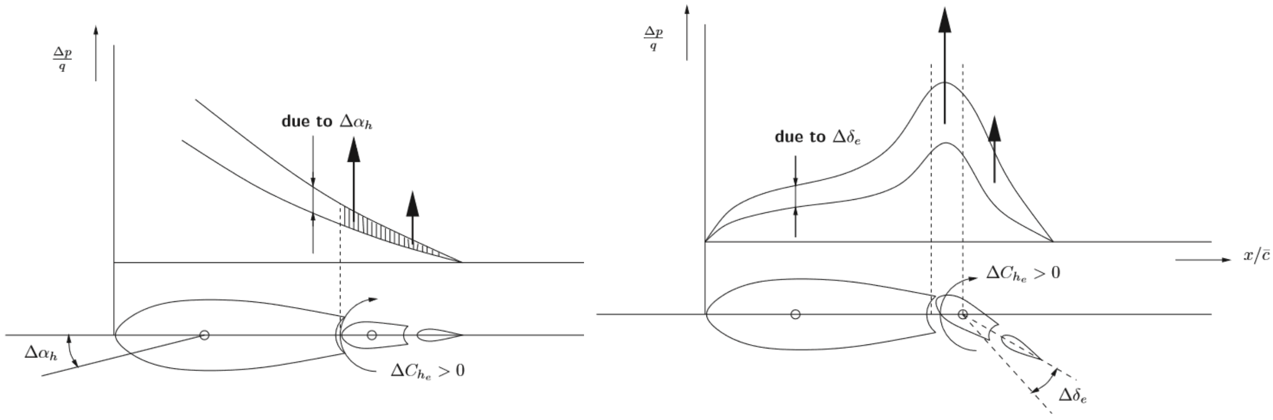
So, in conclusion, although it may be preferable (although it's only a small preference) to have C_{h_α} be positive by putting the hinge line backward, we absolutely do not want C_{h_δ} to be positive.

If a control surface has both C_{h_α} and C_{h_δ} negative, it is called **aerodynamically underbalanced**. If it has both as positive, it is called **aerodynamically overbalanced**.

Multiple-choice questions, part (1)

For some aircraft longitudinal control is performed by an all flying (or moving) horizontal tail instead of an elevator. In this case i_h and δ_e are equal. If the horizontal tailplane is located in front of the horizontal tailplane's aerodynamic center, which of the following statements is/are true?

(a) $C_{N_{h_\alpha}} > C_{N_{h_\delta}}$, C_{h_α} and $C_{h_\delta} < 0$.
(b) $C_{N_{h_\alpha}} = C_{N_{h_\delta}}$, C_{h_α} and C_{h_δ} are both negative.
(c) $C_{N_{h_\alpha}} = C_{N_{h_\delta}}$, C_{h_α} and C_{h_δ} are both positive.
(d) $C_{N_{h_\alpha}} = C_{N_{h_\delta}}$, $C_{h_\alpha} > 0$ and $C_{h_\delta} < 0$.
(e) $C_{N_{h_\alpha}} \neq C_{N_{h_\delta}}$, C_{h_α} equals C_{h_δ} and both equal to zero.



(a) Pressure distribution over elevator due to change in α_h.

(b) Pressure distribution over elevator due to change in δ_e.

Figure 5.5: Change in pressure distribution over elevator.

The only correct statement is b). If i_h and δ_e are equal, then essentially the elevator is the tailplane, so $C_{N_\alpha} = C_{N_{h_\delta}}$, and C_{h_α} equals C_{h_δ}. They are both negative, because when axis of rotation is in front of the aerodynamic center, most of the lift is created behind the axis of rotation, meaning a negative moment is created.

5.3 *Flow direction at tailplane*

Till now you've seen α_h frequently, but how exactly do α and α_h relate? Well, consider figure 5.6: first, before the wing, the freestream velocity has an angle of attack α. However, due to the wing, a downwash ϵ is introduced[4]; this causes the angle of the flow to decrease. Furthermore, the horizontal tailplane may be at an inclination i_h compared to the wing, increasing the angle again. In other words,

$$\alpha_h = \alpha - \epsilon + i_h \tag{5.6}$$

The downwash is then given by

DOWNWASH ANGLE

$$\epsilon = (\alpha - \alpha_0)\frac{d\epsilon}{d\alpha} \tag{5.7}$$

Here, $d\epsilon/d\alpha$ can be roughly estimated by $d\epsilon/d\alpha = 4/(A+2)$ (for aspect ratios A larger than 5). This is beyond the scope of the course however, but it's just to let you know that it depends three-dimensionality of the wing (the downwash is created by the wing-tip vortices after all). Substituting equation (5.9) into equation (5.6) leads to

$$\alpha_h = \alpha - (\alpha - \alpha_0)\frac{d\epsilon}{d\alpha} + i_h = \alpha - \alpha_0 - (\alpha - \alpha_0)\frac{d\epsilon}{d\alpha} + (\alpha_0 + i_h) \tag{5.8}$$

such that

ANGLE OF ATTACK OF HORIZONTAL TAILPLANE

$$\alpha_h = (\alpha - \alpha_0)\left(1 - \frac{d\epsilon}{d\alpha}\right) + (\alpha_0 + i_h) \tag{5.9}$$

$d\epsilon/d\alpha$ is pretty much independent of α (see formula I listed); α_0 and i_h are both constants as well, thus differentiating with respect to α yields

[4]After all, the wing is pushed up, so the air is pushed down.

DERIVATIVE OF ANGLE OF ATTACK OF HORIZONTAL TAILPLANE

$$\frac{d\alpha_h}{d\alpha} = \left(1 - \frac{d\epsilon}{d\alpha}\right) \tag{5.10}$$

Note that this derivative is always smaller than 1, meaning that the positive contribution of the horizontal tailplane to the static stability of conventional aircraft with rear mounted tail planes. Furthermore, it's worth bearing in mind that the ratio V_h/V is also always smaller than one: the main wing of the aircraft slows the flow down (due to the drag), so this ratio is smaller than 1.

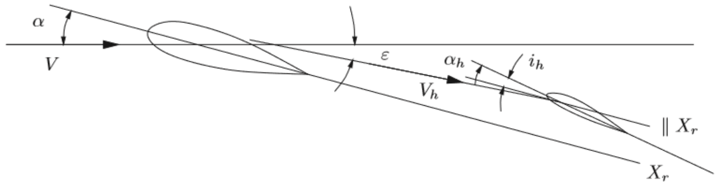


Figure 5.6: Angle of attack α_h of the horizontal tailplane.

Multiple-choice questions, part (1)

Decide for every statement whether it is True or False. No explanation is required!

1. The main wing of a conventional aircraft reduces the local dynamic pressure at the horizontal tailplane.
2. The vortex plane behind the main wing of a conventional aircraft causes a decrease in angle of attack of the horizontal stabilizer.

The correct answer are:

1. See literally what I wrote above. This statement is **true**.
2. This is known as downwash. This statement is **true**.

5.4 *Effect of airspeed and center of gravity on tail load*

Now, multiply equation (4.11) with $\rho V^2 S\bar{c}/2$ to make it dimensional again, then we obtain

$$M = C_{m_{ac}}\frac{1}{2}\rho V^2 S\bar{c} + N_w\left(x_{c.g.} - x_w\right) - N_h l_h = 0 \tag{5.11}$$

which can be rewritten to

$$N_h = \frac{1}{l_h}\left[C_{m_{ac}}\frac{1}{2}\rho V^2 S\bar{c} + N_w\left(x_{c.g.} - x_w\right)\right] \tag{5.12}$$

For small pitch angles and for $N_h << N_w$, $N_w \approx W$ so that

NORMAL FORCE OF HORIZONTAL TAIL PLANE

In order to have horizontal and rotational equilibrium in steady, straight, symmetric flight, the **tail load** equals

$$N_h \approx \frac{1}{l_h}\left[C_{m_{ac}}\frac{1}{2}\rho V^2 S\bar{c} + W\left(x_{c.g.} - x_w\right)\right] \tag{5.13}$$

So this is the value that N_h must have, otherwise there's no rotational or vertical equilibrium and we die. If you're wondering, but aren't we now first considering N_h to be zero (since $N_w \approx W$), so shouldn't this then equal zero as well? No, when we say that $N_w + N_h = W$ basically means $N_w \approx W$, we assume that N_h is small *in comparison to* N_w. However, N_h in itself is not small if you don't compare it with anything else, which is why we still get a value of N_h here, even if we assumed it to be 0 before.

©Sam van Elsloo

Let's analyse equation (5.13) a bit further. We see that the tail load N_h compensates the moment caused by the wing, $C_{m_{ac}}$; it also compensates the moment of N_w (approximately equal to W) around the center of gravity:

- The term containing the moment coefficient around the aerodynamic center $C_{m_{ac}}$ depends quadratically on the airspeed. Note that the value of $C_{m_{ac}}$ itself is constant; this was basically the entire point of chapter 2 so if you've already forgotten that, good job. Note that $C_{m_{ac}}$ may be either negative, zero or positive. If it is negative, then N_h will decrease quadratically with increasing V; if it is 0 it will remain constant with increasing V; if it is positive it will increase with increasing V.
- The term containing the center of gravity varies linearly with $x_{c.g.}$: if $x_{c.g.}$ increases, N_h will consequently also increase. Note that it does not matter whether the center of gravity will be in front of the aerodynamic center x_w or not. If it is in front, it'd mean that at $V = 0$, you'd have negative N_h, but this is not something that breaks the aircraft as you simply deflect your elevator in the opposite direction.

Equation (5.13) can be plotted as done in figure 5.7: there we see that the parabolic variation in C_N. We have plotted 6 graphs: they are taken for two different positions of the center of gravity; once in front of the ac, and once aft of it. Note how the initial vertical coordinate is purely determined by $W(x_{c.g.} - x_w)/l_h$ (which can also be negative). For both positions, we have drawn the graph for a certain positive $C_{m_{ac}}$, for a certain negative $C_{m_{ac}}$ and for $C_{m_{ac}} = 0$.

Also, just to point out, why is the tail load even important at all? Well it's really important for the people in the structural department obviously, as it tells them the load on the tail (which the name tail load kinda told you).

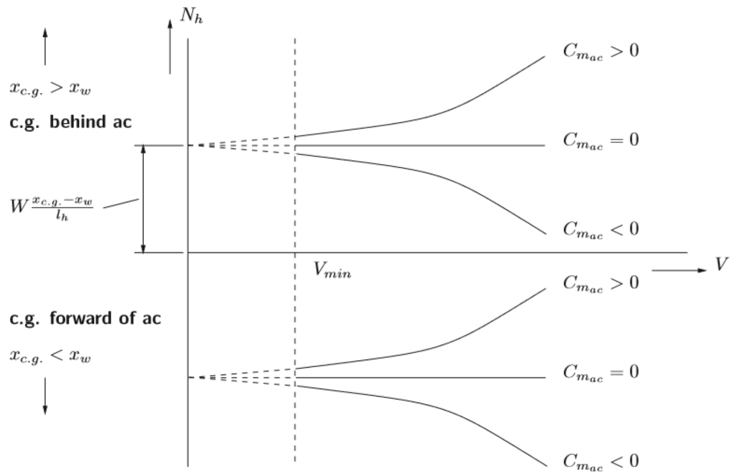


Figure 5.7: The variation of the tail load with airspeed at different center of gravity positions and values of $C_{m_{ac}}$.

5.5 *Elevator deflection required for moment equilibrium*

Now, we can put the pieces of the previous sections together and compute the required deflection angle of the deflection to have equilibrium for a certain angle of attack α. For this, consider Equation (5.13) once more:

$$C_m = C_{m_{ac}} + C_{N_w}\frac{x_{c.g.} - x_w}{\bar{c}} - C_{N_h}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}} = 0 \tag{4.11}$$

Here, C_{N_w} and C_{N_h} may both be written as

$$C_{N_w} = C_{N_{w_\alpha}}\left(\alpha - \alpha_0\right) \tag{5.14}$$

$$C_{N_h} = C_{N_{h_\alpha}}\alpha_h + C_{N_{h_\delta}}\delta_e \tag{5.15}$$

where equation (5.15) is equation (5.3). Furthermore, α_h in equation (5.15) may be written as (from equation (5.9))

$$C_{N_h} = C_{N_{h_\alpha}} \left[\left(\alpha - \alpha_0\right) \left(1 - \frac{d\epsilon}{d\alpha}\right) + \left(\alpha_0 + i_h\right) \right] + C_{N_{h_\delta}} \delta_e \tag{5.16}$$

Combining equations (5.14) and (5.16) into equation (4.11) leads to

$$\begin{aligned} C_m =& C_{m_{ac}} + C_{N_{w_\alpha}} \left(\alpha - \alpha_0\right) \frac{x_{c.g.} - x_w}{\bar{c}} \\ &- \left\{ C_{N_{h_\alpha}} \left[\left(\alpha - \alpha_0\right) \left(1 - \frac{d\epsilon}{d\alpha}\right) + \left(\alpha_0 + i_h\right) \right] + C_{N_{h_\delta}} \delta_e \right\} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} = 0 \end{aligned} \tag{5.17}$$

Now, let us aim to rewrite this equation a bit, such that there is only one constant term, one term purely dependent on α and one term purely term dependent on δ_e (as those are the only variables here); in other words,

$$C_m = C_{m_0} + C_{m_\alpha} \left(\alpha - \alpha_0\right) + C_{m_{\delta_e}} \delta_e = 0 \tag{5.18}$$

where we define

MOMENT DEFINITIONS

We define:

- A constant C_{m_0}:

$$C_{m_0} = C_{m_{ac}} - C_{N_{h_\alpha}} \left(\alpha_0 + i_h\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{5.19}$$

- The **static longitudinal stability, stick fixed** C_{m_α},

$$C_{m_\alpha} = C_{N_{w_\alpha}} \frac{x_{c.g.} - x_w}{\bar{x}} - C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{5.20}$$

- The **elevator efficiency** $C_{m_{\delta_e}}$,

$$C_{m_{\delta_e}} = -C_{N_{h_\delta}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{5.21}$$

With these definitions in mind, solving equation (5.18) becomes trivial. For moment equilibrium, $C_m = 0$, and thus we simply obtain

REQUIRED ELEVATOR DEFLECTION ANGLE

For steady, straight, symmetric flight, the elevator deflection angle δ_e is given by

$$\delta_e = -\frac{1}{C_{m_{\delta_e}}} \left[C_{m_0} + C_{m_\alpha} \left(\alpha - \alpha_0\right) \right] \tag{5.22}$$

with $C_{m_{\delta_e}}$, C_{m_0} and C_{m_α} given by equations (5.21), (5.19) and (5.20) respectively.

Some final remarks on this section:

- $C_{m_{ac}}$, $C_{N_{w_\alpha}}$ and α_0 are fixed by the aerodynamic design of the wing, fuselage and nacelles.
- The required tail volume $S_h l_h/(S\bar{c})$ follows primarily from the requirement that the plane shall be statically stable ($C_{m_\alpha} < 0$) at the *rear*most permissible center of gravity; $C_{N_{h_\alpha}}$ follows from the geometry of the horizontal tailplane.
- i_h (if it can't be adjusted during flight) is often chosen such that it minimises parasite tailplane drag when the elevator angle is 0.
- Both $d\epsilon/d\alpha$ and $(V_h/V)^2$ follow from characteristics of wing and location of horizontal tailplane relative to wing and fuselage.
- For tail planes with fixed i_h, $(C_m)_{\delta_e=0}$, i.e. the moment coefficient when the elevator deflection is 0, depends on the angle of attack and center of gravity position, and has to be compensated for by the term $C_{m_{\delta_e}} \delta_e$, as equation (5.18) may be written as

$$C_m = \left(C_m\right)_{\delta_e=0} + C_{m_{\delta_e}} \delta_e \tag{5.23}$$

©Sam van Elsloo

- For most aircraft the maximum positive value of $C_{m_{\delta_e}} \cdot \delta_e$ (elevator up) is larger than the minimum negative value.
- The largest positive value of $C_{m_{\delta_e}} \delta_e$ will be needed to compensate for the largest $(C_m)_{\delta_e} = 0$. The latter occurs at the most forward c.g. position and the maximum angle of attack.

©Sam van Elsloo

6 Stick fixed static longitudinal stability and control

Remember the fundamental equilibrium equations of steady, straight, symmetric flight:

$$\overset{+}{\rightarrow}\sum F_{X_b}: C_{T_w} + \frac{W}{\frac{1}{2}\rho V^2 S}\sin\theta = 0 \tag{4.7}$$

$$+\uparrow\sum F_{Z_b}: C_{N_w} + C_{N_h}\left(\frac{V_h}{V}\right)^2\frac{S_h}{S} - \frac{W}{\frac{1}{2}\rho V^2 S}\cos\theta = 0 \tag{4.8}$$

$$\circlearrowright + \sum M: C_{m_{ac_w}} + C_{N_w}\frac{x_{c.g.} - x_w}{\bar{c}} + C_{N_h}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}} = 0 \tag{4.11}$$

with $l_h = x_h - x_w \approx x_h - x_{c.g.}$.

The previous chapters concerned itself with equilibrium: when do we have equilibrium? This chapter will be about *stability*: for small disturbances, the aircraft should go back to its equilibrium state, rather than having the disturbance increasing. It'll be a rather long chapter but it's not too difficult fortunately. We will first discuss stick *fixed* stability, afterwards we'll discuss stick *free* stability, afterwards we'll discuss some aircraft manoeuvres. We'll get to what those terms mean once we get to it.

6.1 Stick fixed static longitudinal stability

We saw in Chapter 2, that for a wing to be stable, we required

$$C_{m_\alpha} < 0, \quad \text{at } C_m = 0 \tag{6.1}$$

This also holds for aircraft[1]: if $C_{m_\alpha} > 0$ when $C_m = 0$, your aircraft is unstable (if it's zero, it's neutrally stable)[2].

Remember equation (5.17):

$$\begin{aligned} C_m = &C_{m_{ac}} + C_{N_{w_\alpha}}\left(\alpha - \alpha_0\right)\frac{x_{c.g.} - x_w}{\bar{c}} \\ &- \left\{C_{N_{h_\alpha}}\left[\left(\alpha - \alpha_0\right)\left(1 - \frac{d\epsilon}{d\alpha}\right) + \left(\alpha_0 + i_h\right)\right] + C_{N_{h_\delta}}\delta_e\right\}\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}} = 0 \end{aligned} \tag{5.17}$$

Now, what does stick fixed mean? Well, your elevator is connected to the stick of the pilot: if he deflects the stick forward or backward, the elevator will go up or down. Stick fixed then means that he keeps the position of the stick fixed, so that the deflection angle of elevator, δ_e, remains constant. So, for this section, we'll assume that δ_e is constant. In the next section, about stick free static longitudinal stability, we'll see what happens if we release it.

Now, let's rewrite equation (5.17) to a sum of moment caused by wing and fuselage, and a moment caused by the horizontal tailplane, i.e. write it as

[1] After all: if our aircraft pitches up for some reason (i.e. $\Delta\alpha$ is positive), we want to create a *negative* moment ΔC_m, so that it automatically pitches down again.

[2] Do bear in mind: it's not a day-and-night difference between a slightly positive and slightly negative value. It's much more gradual: if you have a very small positive C_{m_α} it's not as if your aircraft is completely uncontrollable. If you have a very small negative C_{m_α} (i.e. close to 0), yes your disturbance dampens out over time, but it takes ages to do so.

MOMENT COEFFICIENT OF AIRCRAFT COMPONENTS

The moment coefficient of the aircraft C_m is given by

$$C_m = C_{m_w} + C_{m_h} \tag{6.2}$$

with the **wing moment coefficient** given by

$$C_{m_w} = C_{m_{ac}} + C_{N_{w_\alpha}} \left(\alpha - \alpha_0\right) \frac{x_{c.g.} - x_w}{\bar{c}} \tag{6.3}$$

and the **horizontal tailplane moment coefficient** given by

$$C_{m_h} = -\underbrace{\left\{ C_{N_{h_\alpha}} \underbrace{\left[\left(\alpha - \alpha_0\right)\left(1 - \frac{d\epsilon}{d\alpha}\right) + \left(\alpha_0 + i_h\right)\right]}_{\alpha_h} + C_{N_{h_\delta}} \delta_e \right\}}_{C_{N_h}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{6.4}$$

where α_h is the angle of attack of the horizontal tailplane and C_{N_h} the normal force coefficient of the horizontal tailplane.

Now, if we differentiate equation (6.2), we simply get $C_{m_\alpha} = C_{m_{\alpha_w}} + C_{m_{\alpha_h}}$, where we obtain by differentiating equations (6.3) and (6.4):

$$C_{m_{\alpha_w}} = C_{N_{w_\alpha}} \frac{x_{c.g.} - x_w}{\bar{c}} \tag{6.5}$$

$$C_{m_{\alpha_h}} = -C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{6.6}$$

Thus, we see that $C_{m_{\alpha_w}}$ will always be positive, whereas $C_{m_{\alpha_h}}$ will always be negative. Indeed, if we sum these together, then the sum must be smaller than 0, so in other words: the absolute value of $C_{m_{\alpha_h}}$ must be larger than the absolute value of $C_{m_{\alpha_w}}$. Graphically, we can depict this as shown in figure 6.1: the wing-fuselage provide an ever-increasing C_{m_w} for increasing α; the tailplane produces an ever-decreasing C_{m_h} for increasing α. Superimposing them must have a negative slope at the α where $C_m = 0$. Now, if you're wondering, but wait, doesn't this fix our value of α, i.e. is the α we fly determined by moment equilibrium rather than being free to fly at optimum α for maximum L/D? No it doesn't. By changing i_h and δ_e, you can alter the α at which C_m is zero (just look at equation (5.17), so that you can fly at whichever α you wanna fly.

6.2 *Neutral point, stick fixed*

Now, I said before that we require

$$\left|C_{m_{\alpha_h}}\right| \geq \left|C_{m_{\alpha_w}}\right|$$

Let us reconsider equations (6.5) and (6.6) and write them as

$$C_{m_{\alpha_w}} = C_{N_{w_\alpha}} \frac{x_{c.g.} - x_w}{\bar{c}} \tag{6.5}$$

$$C_{m_{\alpha_h}} = C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \frac{x_{c.g.} - x_h}{\bar{c}} \tag{6.7}$$

where we rewrite equation (6.6) using $l_h = x_h - x_w \approx x_h - x_{c.g}$; by switching the order of x_h and $x_{c.g.}$ we get rid of the minus sign at the front. If we put the center of gravity increasingly far aft, i.e. $x_{c.g.}$ increases, $C_{m_{\alpha_h}}$ becomes increasingly positive, whereas $C_{m_{\alpha_h}}$ becomes less negative (as the difference $x_{c.g.} - x_h$ becomes smaller). Thus, there will be a point at which

$$\left|C_{m_{\alpha_h}}\right| = \left|C_{m_{\alpha_w}}\right|$$

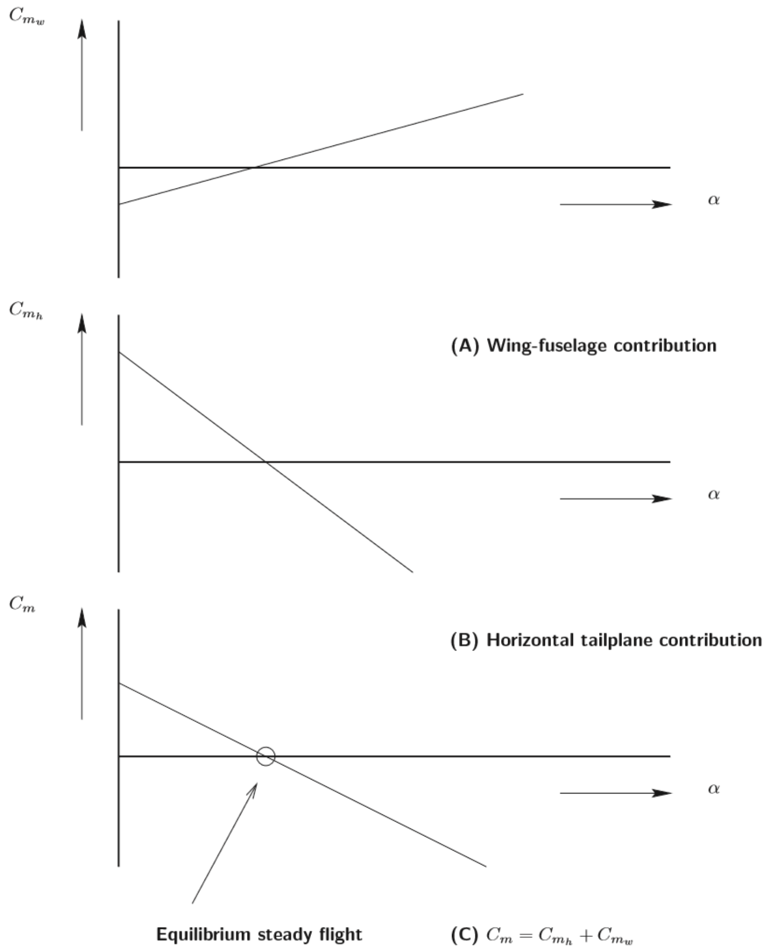


Figure 6.1: The contribution of various parts of the aircrtaft to the moment curve $C_m - \alpha$.

and this is a very interesting point, as at this point $C_{m_\alpha} = 0$ so that the aircraft becomes *neutrally stable*. Consequentially, it has a very fancy name, namely

NEUTRAL POINT

The **neutral point** of an aircraft is the location of the center of gravity at which

$$C_{m_\alpha} = 0 \tag{6.8}$$

The corresponding location of the center of gravity is denoted by x_n.

As we are discussing stick fixed, we write $x_{n_{fix}}$. Now, what is the value of $x_{n_{fix}}$? This is simply the value of $x_{c.g.}$ such that $C_{m_\alpha} = 0$. We simply have

$$C_{m_\alpha} = C_{m_{\alpha_w}} + C_{m_{\alpha_h}} = C_{N_{w_\alpha}} \frac{x_{n_{fix}} - x_w}{\bar{c}} + C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \frac{x_{n_{fix}} - x_h}{\bar{c}} = 0 \tag{6.9}$$

Now, we could solve this for $x_{n_{fix}}$, but we like to get rid of $C_{N_{w_\alpha}}$. For this, note that $C_N = C_{N_w} + C_{N_h}$, where C_{N_h} is indicated in equation (6.4), so that

$$C_N = C_{N_w} + C_{N_{h_\alpha}} \left[(\alpha - \alpha_0) \left(1 - \frac{d\epsilon}{d\alpha}\right) + (\alpha_0 + i_h) \right] + C_{N_{h_\delta}} \delta_e$$

Differentiate with respect to α, then multiply by $(x_{n_{fix}} - x_w)/\bar{c}$:

$$C_{N_\alpha} = C_{N_{w_\alpha}} + C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \tag{6.10}$$

$$C_{N_\alpha} \frac{x_{n_{fix}} - x_w}{\bar{c}} = C_{N_{w_\alpha}} \frac{x_{n_{fix}} - x_w}{\bar{c}} + C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \frac{x_{n_{fix}} - x_w}{\bar{c}} \tag{6.11}$$

Now, subtract equation (6.9) from (6.11), and one obtains

$$C_{N_\alpha} \frac{x_{n_{fix}} - x_w}{\bar{c}} = C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \left(\frac{x_{n_{fix}} - x_w}{\bar{c}} - \frac{x_{n_{fix}} - x_h}{\bar{c}} \right) \tag{6.12}$$

$$= C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \frac{x_h - x_w}{\bar{c}} = C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{6.13}$$

where we used $x_h - x_w = l_h$. In other words,

LOCATION OF NEUTRAL POINT STICK FIXED

The location of the neutral point for stick fixed is given by

$$\frac{x_{n_{fix}} - x_w}{\bar{c}} = \frac{C_{N_{h_\alpha}}}{C_{N_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{6.14}$$

Now, we can also subtract equation (6.9) from (6.7). We then obtain

$$\begin{aligned} C_{m_\alpha} &= C_{N_{w_\alpha}} \left(\frac{x_{c.g.} - x_w}{\bar{c}} - \frac{x_{n_{fix}} - x_w}{\bar{c}} \right) + C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \left(\frac{x_{c.g.} - x_w}{\bar{c}} - \frac{x_{n_{fix}} - x_w}{\bar{c}} \right) \\ &= \left[C_{N_{w_\alpha}} + C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S} \right] \frac{x_{c.g.} - x_{n_{fix}}}{\bar{c}} \end{aligned}$$

Now, from equation (6.10), this may be simplified to

RELATION BETWEEN MOMENT DERIVATIVE AND NORMAL FORCE DERIVATIVE

C_{m_α} relates to C_{N_α} via

$$C_{m_\alpha} = C_{N_\alpha} \frac{x_{c.g.} - x_{n_{fix}}}{\bar{c}} \tag{6.15}$$

or taking the differentials, $dC_m = dC_N(x_{c.g.} - x_{n_{fix}})/\bar{c}$. In other words, the countermoment generated by a disturbance that generates an increased normal force dC_N, is equal to dC_N times the normalised distance between the center of gravity and the neutral point. This can be graphically represented as done in figure 6.2: the combined change of dC_{N_w} and $dC_{N_h}(V_h/V)^2 S_h/S$ is located in the neutral point; evidently it creates a moment equal to $dC_N(x_{c.g.} - x_{n_{fix}})/\bar{c}$. This is an additional way of interpreting the neutral point (we also see that when the c.g. coincides with the neutral point, no countermoment is generated, and thus the aircraft is neutrally stable. When the c.g. is behind the neutral point, a positive moment is created, meaning we have unstable equilibrium. For your information, the non-dimensional distance $(x_{c.g.} - x_{n_{fix}})/\bar{c}$ is called the **stability margin**, stick fixed.

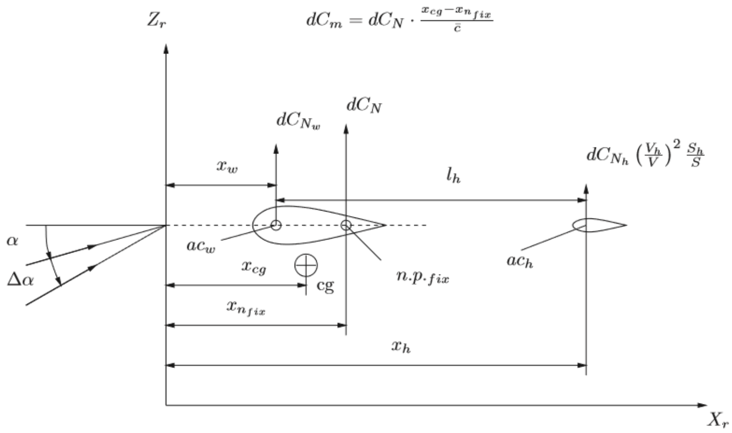


Figure 6.2: Change in moment dC_m duie to change in the angle of attack $\Delta\alpha$.

Multiple choice questions, part (1)

Which of the following statements are true:

1. The center of gravity position, at which the equilibrium of the moment is neutrally stable, stick fixed, is called the neutral point, stick fixed.
2. Physically, the stick-fixed neutral point is the point where the change in the normal forced (dC_N) acts.
3. The neutral point of a statically stable aircraft is positioned ahead of the center of gravity.

The correct answers are:

1. This is literally the definition. This is **true**.
2. Yeah, see literally above picture. This is **true**.
3. No, it's *behind* the center of gravity. This is **false**.

6.3 *Elevator trim curve*

Remember that from equation (5.22)

ELEVATOR DEFLECTION ANGLE AS FUNCTION OF ANGLE OF ATTACK

$$\delta_e = -\frac{1}{C_{m_{\delta_e}}}\left[C_{m_0} + C_{m_\alpha}\left(\alpha - \alpha_0\right)\right] \tag{5.22}$$

Remember that $C_{m_{\delta_e}}$ should always be negative! We have plotted this equation in the left of figure 6.3, which I'll explain if you don't fully understand what's being drawn. First, it's important to know that it's common practice to plot negative deflection angles of elevators up[3]. This is why for $\alpha = \alpha_0$, δ_e will be on the lower part

[3]This is because although elevator up is a negative δ_e, it causes pitch up (which is positive).

of the graph, even though $-1/C_{m_{\delta_e}} \cdot C_{m_0}$ will always be positive. Then, if $C_{m_\alpha} > 0$, δ_e will become increasingly positive for increasing α, which is why the graph for $C_{m_\alpha} > 0$ goes *down*. For $C_{m_\alpha} = 0$, δ_e stays constant and thus a straight line follows. For $C_{m_\alpha} < 0$, δ_e will become smaller and smaller and become negative, meaning the graph goes *up*.

What is the use of the left of figure 6.3? Well, as mentioned before, we really want $C_{m_\alpha} < 0$ for our aircraft. Now, measuring C_{m_α} is very difficult, as it'd require putting the full aircraft in a windtunnel, which usually won't fit. What we can do based on the left of figure 6.3 however, is that we measure the required deflection angle for several values of α; if it becomes increasingly negative for increasing α, we apparently had $C_{m_\alpha} < 0$ (according to the left of figure 6.3).

Now, what's plotted on the right of figure 6.3? Note that equation (5.22) may be rewritten according to

$$\delta_e = -\frac{1}{C_{m_{\delta_e}}}\left[C_{m_0} + \frac{C_{m_\alpha}}{C_{N_\alpha}} C_{N_\alpha}\left(\alpha - \alpha_0\right)\right]$$

Here,

$$C_{N_\alpha}\left(\alpha - \alpha_0\right) \approx C_N \approx \frac{W}{\frac{1}{2}\rho V^2 S}$$

and thus

ELEVATOR DEFLECTION ANGLE AS FUNCTION OF VELOCITY

$$\delta_e = -\frac{1}{C_{m_{\delta_e}}}\left(C_{m_0} + \frac{C_{m_\alpha}}{C_{N_\alpha}} \frac{W}{\frac{1}{2}\rho V^2 S}\right) \tag{6.16}$$

Everything is known a priori for the aircraft; only the velocity can be varied throughout flight. Once more, we can use this to experimentally determine the sign of C_{m_α}, as shown in the right of figure 6.3: we fly at different airspeeds, adjust the angle of attack as necessary to preserve vertical force equilibrium, then measure what the required deflection angle is for $C_m = 0$. Based on the shape of the resultant graph, we can compare it with the right of figure 6.3 and determine the sign of C_{m_α}.

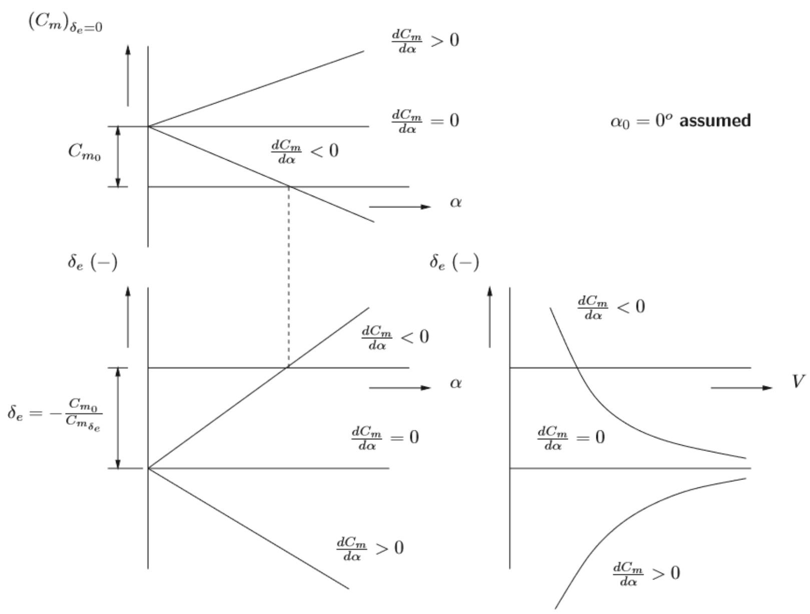


Figure 6.3: Moment curves and corresponding trim curves.

©Sam van Elsloo

6.4 *Elevator stick position stability*

I'm not entirely sure what the point of this section is but anyway. Note that from equation (5.22), we have

$$\frac{d\delta_e}{d\alpha} = -\frac{C_{m_\alpha}}{C_{m_{\delta_e}}} \tag{6.17}$$

Note that for a stable aircraft, both C_{m_α} and $C_{m_{\delta_e}}$ are negative; thus, it always holds that $d\delta_e/d\alpha < 0$. Similarly, from equation (6.16), we obtain

$$\frac{d\delta_e}{dV} = \frac{4W}{\rho V^3 S}\frac{1}{C_{m_{\delta_e}}}\frac{C_{m_\alpha}}{C_{N_\alpha}} \tag{6.18}$$

Now, C_{m_α} and $C_{m_{\delta_e}}$ are both always negative for stable aircraft; thus, it always holds that $d\delta_e/dV > 0$. Furthermore, we have the following definition:

TRIM STABILITY

The **trim stability** is defined as the slope $d\delta_e/dV$ of the elevator trim curve $\delta_e - V$. The higher, the more stable.

Why does anyone care about this? The pilot slightly cares: he controls the elevator with a system similar to what's depicted in figure 6.4. If he pushes the stick in a positive displacement s_e, it will move the elevator in a positive (downward) deflection δ_e. Clearly, there's a linear relationship between s_e and δ_e.

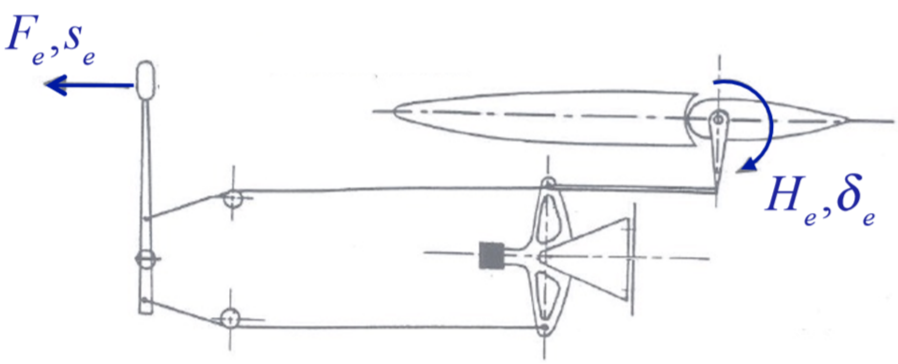


Figure 6.4: Stick connected to elevator.

Now, suppose you are flying at a certain velocity V and a certain elevator deflection angle δ_e, and you want to lower your airspeed a bit. Consequentially, you need to increase α, by increasing the pitch angle of the aircraft, which is achieved by *decreasing* δ_e a bit[4]. Now, you want to do this a bit quickly, so the pilot pulls back the stick strongly, so that the initial change in deflection angle is $\Delta\delta_{e_i}$. He then releases it slightly, so that now the ultimate change in deflection angle, *compared to the original* δ_e, is merely $\Delta\delta_{e_u}$.[5]

Well, what you want as a pilot of course is that the displacement initial Δs_{e_i} has the same sign as the ultimate displacement Δs_{e_u}: if this is not the case, this means that you first have to pull the stick backward, and then actually *push* it forward to remain in equilibrium. You much rather want that the ultimate displacement is reasonably close to the initial displacement, otherwise you have to move your arm so much each time. If you don't really follow what I'm saying, consider figure 6.5: what we want is shown on the left: we want $\Delta\delta_{e_u}$ (the ultimate change in deflection angle, compared to the *original* δ_e) to be reasonably close to $\Delta\delta_{e_i}$: this way, if we first pull it back a displacement Δs_{e_i}, we don't have to move our arm much to reach the new position Δs_{e_u}. What we don't want is shown on the right: there we have to first pull back the stick, but for some reason (the aircraft is unstable, as we'll find out), for the new equilibrium position, in the end we'll have to push it *forward* (in the middle, it is shown for when the ultimate displacement will just equal 0 again).

[4]Then the tail produces a little bit more downforce after all, causing the pitch to increase.

[5]Of course, you can also do it more gradually, but then it'll take ages to reach your new pitch angle, so we look for a way to do it quicker.

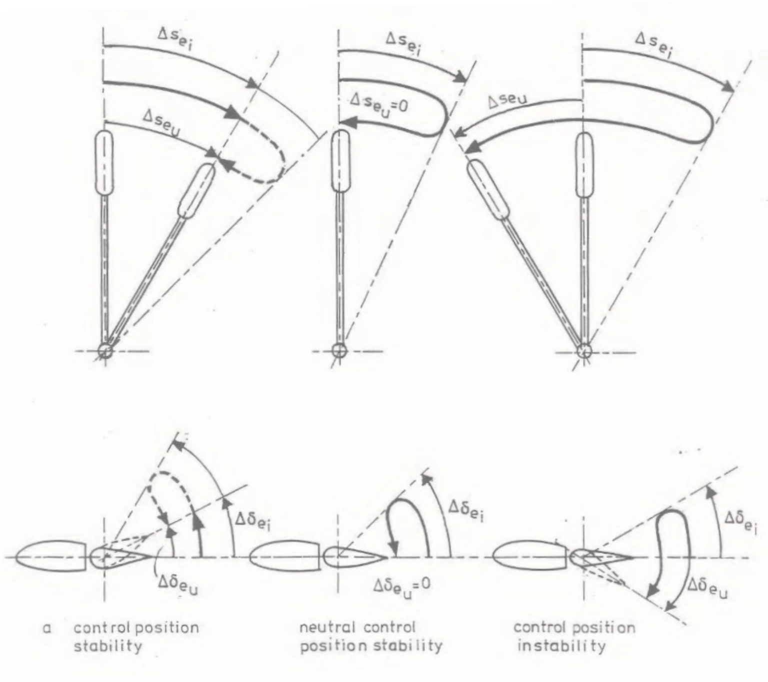


Figure 6.5: Initial and ultimate control displacement and control surface deflection for the transition to lower airspeed.

So, we require that Δs_{e_u} and Δs_{e_i} are of the same sign; due to the linear relationship between s_e and δ_e, this means that $\Delta\delta_{e_i}$ and $\Delta\delta_{e_u}$ are also of the same sign. Mathematically, this means

$$\frac{\Delta s_{e_u}}{\Delta s_{e_i}} > 0, \quad \text{and} \quad \frac{\Delta \delta_{e_u}}{\Delta \delta_{e_i}} > 0$$

This is automatically the case for stable aircraft: we can write

$$\frac{\Delta \delta_{e_u}}{\Delta \delta_{e_i}} = \frac{\frac{\Delta \delta_{e_u}}{\Delta V}}{\frac{\Delta \delta_{e_i}}{\Delta V}}$$

However, the derivative of δ_e with respect to V was always positive, and thus the result is indeed always positive for a stable aircraft. Thus, for stable aircraft, we also have stable control position stability.

For unstable aircraft, we cannot safely determine the sign, and thus these may lead to unstable control positions. Consider figure 6.6: the top part corresponds to a *stable* aircraft, as $d\delta_e/dV > 0$ for that curve (remember that negative values of δ_e appear on the top of the graph). We see that to go from an initial flight situation (1) to an ultimate flight situation (2), we first give it some negative change in deflection $\Delta\delta_{e_i}$ and ultimately some smaller negative change in deflection (compared to original deflection at (1), not to the new deflection after the initial change) $\Delta\delta_{e_u}$. These have the same sign and differ not too much in magnitude.

However, for the unstable aircraft, plotted in the bottom, the story is a little different: there we must first decrease δ_e by some $\Delta\delta_{e_i}$, but ultimately we'll have to *in*crease δ_e by some $\Delta\delta_{e_u}$. This is very tiring for the pilot.

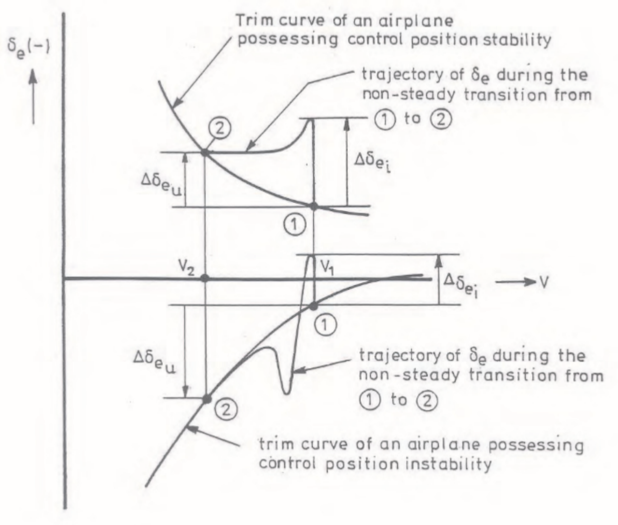


Figure 6.6: Aircraft response to a control deflection.

6.5 Influence of center of gravity position on elevator trim curve

Consider once more equations (5.22) and (6.16):

$$\delta_e = -\frac{1}{C_{m_{\delta_e}}}\left[C_{m_0} + C_{m_\alpha}\left(\alpha - \alpha_0\right)\right] \tag{5.22}$$

$$\delta_e = -\frac{1}{C_{m_{\delta_e}}}\left(C_{m_0} + \frac{C_{m_\alpha}}{C_{N_\alpha}}\frac{W}{\frac{1}{2}\rho V^2 S}\right) \tag{6.16}$$

Furthermore, remember equation (6.15), $C_{m_\alpha} = C_{N_\alpha}\left(x_{c.g.} - x_{n_{fix}}\right)/\bar{c}$. Substituting this into above equations results in

$$\delta_e = -\frac{1}{C_{m_{\delta_e}}}\left[C_{m_0} + C_{N_\alpha}\left(\alpha - \alpha_0\right)\frac{x_{c.g.} - x_{n_{fix}}}{\bar{c}}\right] \tag{6.19}$$

$$\delta_e = -\frac{1}{C_{m_{\delta_e}}}\left(C_{m_0} + \frac{W}{\frac{1}{2}\rho V^2 S}\right)\frac{x_{c.g.} - x_{n_{fix}}}{\bar{c}} \tag{6.20}$$

We can now investigate the effect of shifting the c.g. aft. Plotting the $\delta_e - \alpha$ and $\delta_e - V$ graphs for various values of $x_{c.g.}$ is done in figure 6.7; we see that the aircraft becomes less and less stable as $d\delta_e/dV$, the trim stability, decreases, thus the elevator control position stability logically also decreases.

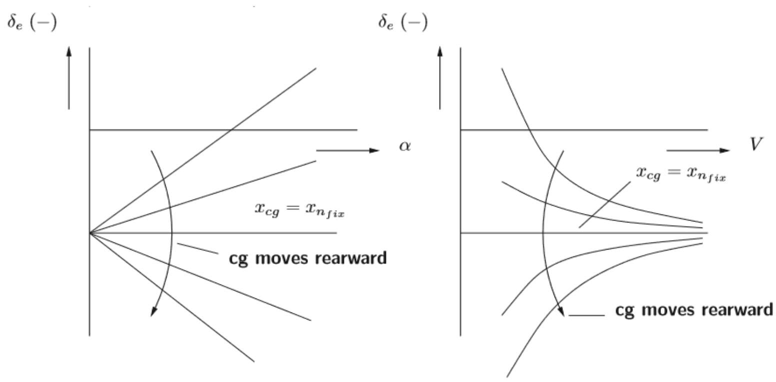


Figure 6.7: Influence of c.g. position on trim curve.

7 Stick free static longitudinal stability and control

Now, previously, we assumed δ_e to be fixed, but what happens if we let go of the stick and let it rotate freely due to the aerodynamic forces? This is actually the nicest situation for a pilot after all, cause then he can just fly hands-free and just do other stuff with his hands. Indeed, the flight condition at which the pilot does not have to exert a force on the control stick is called the **trimmed** flight condition.

Once again, we start with

MOMENT COEFFICIENT OF AIRCRAFT COMPONENTS

The moment coefficient of the aircraft C_m is given by

$$C_m = C_{m_w} + C_{m_h} \tag{6.2}$$

with the **wing moment coefficient** given by

$$C_{m_w} = C_{m_{ac}} + C_{N_{w_\alpha}} \left(\alpha - \alpha_0\right) \frac{x_{c.g.} - x_w}{\bar{c}} \tag{6.3}$$

and the **horizontal tailplane moment coefficient** given by

$$C_{m_h} = -\underbrace{\left\{ C_{N_{h_\alpha}} \underbrace{\left[\left(\alpha - \alpha_0\right)\left(1 - \frac{d\epsilon}{d\alpha}\right) + \left(\alpha_0 + i_h\right)\right]}_{\alpha_h} + C_{N_{h_\delta}} \delta_e \right\}}_{C_{N_h}} \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{6.4}$$

where α_h is the angle of attack of the horizontal tailplane and C_{N_h} the normal force coefficient of the horizontal tailplane.

Once more, we are interested $C_{m_\alpha} = C_{m_{\alpha_w}} + C_{m_{\alpha_h}}$, but now we obtain by differentiating equations (6.3) and (6.4):

$$\begin{aligned} C_{m_{\alpha_w}} &= C_{N_{w_\alpha}} \frac{x_{c.g.} - x_w}{\bar{c}} \\ C_{m_{\alpha_h}} &= -\left[C_{N_{h_\alpha}}\left(1 - \frac{d\epsilon}{d\alpha}\right) + C_{N_{h_\delta}} \frac{d\delta_e}{d\alpha}\right]\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \end{aligned} \tag{7.1}$$

$C_{m_{\alpha_w}}$ is the same as before, but comparing equation (7.1) to (6.6), we see the appearance of a term $C_{N_{h_\delta}} d\delta_e/d\alpha$. Now, $C_{N_{h_\delta}}$ is known: we already saw it in equation (5.3): it is the derivative of the normal force of the horizontal tailplane with respect to the elevator deflection angle, i.e. how much C_{N_h} changes with a change in δ_e. Therefore, let's find the derivative $d\delta_e/d\alpha$, and for that, let's think about what actually happens when we have our stick free. Consider figure 7.1: the moment around the hingeline is given by equation (5.5)[1]:

$$C_{h_e} = C_{h_\alpha}\alpha_h + C_{h_\delta}\delta_e + C_{h_{\delta_t}}\delta_{t_e} \tag{5.5}$$

If our stick is free, we are not applying a reaction torque to the hingeline, thus, to have equilibrium, we *must* have $C_{h_e} = 0$. In other words, suppose you are flying at a certain equilibrium state, and then change α_h or δ_{t_e} a bit, then the elevator deflection angle will, due to the aerodynamic forces, automatically converge to a new δ_e such that the result $C_{h_{e_{free}}} = 0$. Of course, this won't happen instantaneously, but it will happen soon enough.

[1]Remember: C_h represented *moment* coefficients around the hingeline. C_{h_α} is then the derivative of this with respect to α_h, etc.

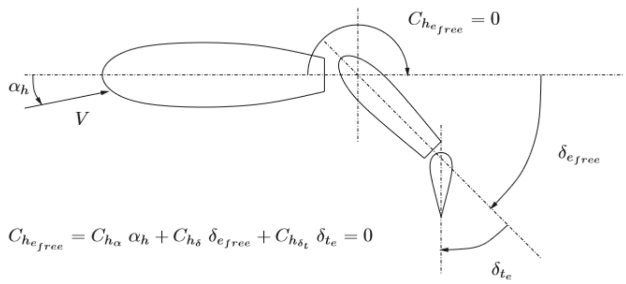


Figure 7.1: The equilibrium of the free elevator.

In other words, we have

$$C_{h_{e_{free}}} = C_{h_\alpha}\alpha_h + C_{h_\delta}\delta_{e_{free}} + C_{h_{\delta_t}}\delta_{t_e} = 0$$

which means that

ELEVATOR DEFLECTION ANGLE STICK FREE

The stick free elevator deflection angle is given by

$$\delta_{e_{free}} = -\frac{C_{h_\alpha}}{C_{h_\delta}}\alpha_h - \frac{C_{h_{\delta_t}}}{C_{h_\delta}}\delta_{t_e} \tag{7.2}$$

This means that

$$\left(\frac{d\delta_e}{d\alpha}\right)_{free} = \frac{d\delta_{e_{free}}}{d\alpha_h}\frac{d\alpha_h}{d\alpha} = -\frac{C_{h_\alpha}}{C_{h_\delta}}\left(1 - \frac{d\epsilon}{d\alpha}\right) \tag{7.3}$$

as $d\alpha_h/d\alpha = 1 - d\epsilon/d\alpha$ (equation (5.10)).

Substituting equation (7.3) into equation (7.1), we obtain

$$\begin{aligned} C_{m_{\alpha_h}} &= -\left[C_{N_{h_\alpha}}\left(1 - \frac{d\epsilon}{d\alpha}\right) - C_{N_{h_\delta}}\frac{C_{h_\alpha}}{C_{h_\delta}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\right]\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}} \\ &= -\left(C_{N_{h_\alpha}} - C_{N_{h_\delta}}\frac{C_{h_\alpha}}{C_{h_\delta}}\right)\left(1 - \frac{d\epsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}} \end{aligned} \tag{7.4}$$

This equation is very similar to equation (6.6),

$$C_{m_{\alpha_h}} = -C_{N_{h_\alpha}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}} \tag{6.6}$$

Indeed, let us define

NORMAL FORCE GRADIENT OF TAILPLANE STICK FREE

The **normal force gradient of the horizontal tailplane in the stick free situation** is defined as

$$C_{N_{h_{\alpha_{free}}}} = C_{N_{h_\alpha}} - C_{N_{h_\delta}}\frac{C_{h_\alpha}}{C_{h_\delta}} \tag{7.5}$$

so that equation (7.4) becomes

$$C_{m_{\alpha_h}} = -C_{N_{h_{\alpha_{free}}}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2\frac{S_h l_h}{S\bar{c}} \tag{7.6}$$

and now equations (7.6) and (6.6) are essentially the same except for the subscript of $C_{N_{h_\alpha}}$. Indeed, looking back at equation (7.5), we see that literally the only difference is the inclusion of the term $C_{N_{h_\delta}}C_{h_\alpha}/C_{h_\delta}$. In this term, $C_{N_{h_\delta}}$ will naturally be positive, C_{h_δ} should always be negative[2] as extensively discussed before in section

[2]Note that it is now also clear that $C_{N_{h_\delta}}$ should also not be too small, otherwise the fraction tends to infinity.

©Sam van Elsloo

5.2.1. We'll see in the next section what the effect is of the sign of C_{h_α}.

7.1 Neutral point, stick free

To literally no one's surprise, we get a very similar expression for our neutral point of the stick free situation, compared to the location of the neutral point of the stick fixed situation, equation (6.14):

$$\frac{x_{n_{fix}} - x_w}{\bar{c}} = \frac{C_{N_{h_\alpha}}}{C_{N_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{6.14}$$

For the stick free situation, it becomes

LOCATION OF NEUTRAL POINT STICK FREE

The location of the neutral point for stick free is given by

$$\frac{x_{n_{free}} - x_w}{\bar{c}} = \frac{C_{N_{h_{\alpha_{free}}}}}{C_{N_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha}\right) \left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{7.7}$$

And similar to equation (6.15),

$$C_{m_\alpha} = C_{N_\alpha} \frac{x_{c.g.} - x_{n_{fix}}}{\bar{c}} \tag{6.15}$$

we obtain

RELATION BETWEEN MOMENT DERIVATIVE AND NORMAL FORCE DERIVATIVE

$C_{m_{\alpha_{free}}}$ relates to $C_{N_{\alpha_{free}}}$ by

$$C_{m_{\alpha_{free}}} = C_{N_{\alpha_{free}}} \frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} \tag{7.8}$$

or $dC_{m_{\alpha_{free}}} = dC_{N_{\alpha_{free}}}(x_{c.g.} - x_{n_{free}})/\bar{c}$, which can once more be visualised as done in figure 7.2: the change in normal force acts through the neutral point, creating a countermoment.

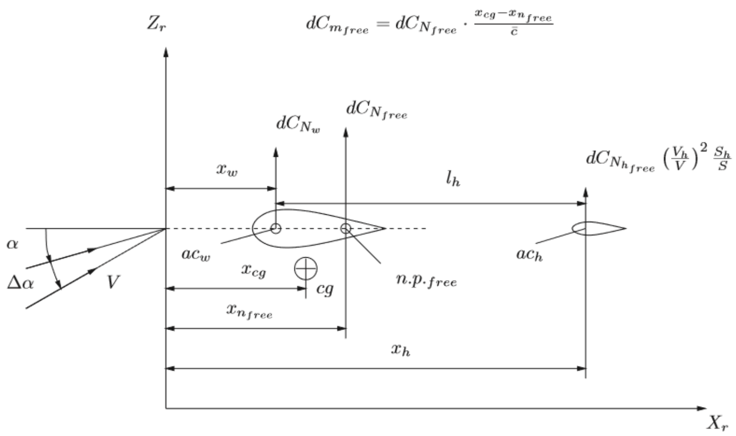


Figure 7.2: The change in the moment $dC_{m_{free}}$ due to a change in angle of attack $\Delta\alpha$.

This is all very similar to before, so how exactly do the positions of the neutral points, stick fixed and stick free, relate to each other? Simply subtract equation (6.14) from equation (7.7), so that we obtain

$$\frac{x_{n_{free}} - x_w}{\bar{c}} - \frac{x_{n_{fix}} - x_w}{\bar{c}} = \left(\frac{C_{N_{h_{\alpha free}}}}{C_{N_\alpha}} - \frac{C_{N_{h_\alpha}}}{C_{N_\alpha}}\right)\left(1 - \frac{d\epsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}$$

$$\frac{x_{n_{free}} - x_{n_{fix}}}{\bar{c}} = \frac{C_{N_{h_{\alpha free}}} - C_{N_{h_{\alpha fix}}}}{C_{N_\alpha}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{7.9}$$

Now, remember equation (7.5),

$$C_{N_{h_{\alpha free}}} = C_{N_{h_\alpha}} - C_{N_{h_\delta}} \frac{C_{h_\alpha}}{C_{h_\delta}} \tag{7.5}$$

This means that equation (7.9) may be rewritten to

$$\frac{x_{n_{free}} - x_{n_{fix}}}{\bar{c}} = -\frac{C_{N_{h_\delta}}}{C_{N_\alpha}} \frac{C_{h_\alpha}}{C_{h_\delta}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}}$$

Making use of equation (5.21),

$$C_{m_\delta} = -C_{N_{h_\delta}}\left(\frac{V_h}{V}\right)^2 \frac{S_h l_h}{S\bar{c}} \tag{5.21}$$

RELATIVE POSITION OF THE NEUTRAL POINTS

The relative position of the neutral points is given by

$$\frac{x_{n_{free}} - x_{n_{fix}}}{\bar{c}} = \frac{C_{m_\delta}}{C_{N_\alpha}} \frac{C_{h_\alpha}}{C_{h_\delta}}\left(1 - \frac{d\epsilon}{d\alpha}\right) \tag{7.10}$$

What's the point of this? Well, realise that C_{h_δ} should always be negative; C_{N_α} will always be positive; C_{m_δ} will always be negative[3]. Thus,whether the neutral point stick free lays behind of the neutral point stick fixed or not depends on the sign of C_{h_α}: if $C_{h_\alpha} > 0$, then the right side will be positive, thus $x_{n_{free}} > x_{n_{fix}}$ and the neutral point stick free lies behind the neutral point stick fixed. This means that you have a larger stability margin in the stick free case! Why does this make sense? Well, consider equation (5.5) once more:

$$C_{h_e} = C_{h_\alpha}\alpha_h + C_{h_\delta}\delta_e + C_{h_{\delta_t}}\delta_{t_e} \tag{5.5}$$

If C_{h_α} is positive, then if the aircraft increases a nose-up attitude (i.e. slightly increased α), then since C_{h_δ} will be negative, δ_e will be positive to make C_{h_e} 0 again (as required for stick free). This means that the elevator will automatically go down, increasing lift at the horizontal tailplane, and thus providing an extra countermoment to counteract the disturbed nose-up attitude. Thus, a positive C_{h_α} improves stability. If C_{h_α} is negative, then δ_e will be negative as well if α is increased; this means that the elevator goes up, decreasing lift at the horizontal tailplane. The aircraft may well still be stable, but it will be less stable than if C_{h_α} had been positive. So, if C_{h_α} is negative, the stability margin will be smaller for the stick free situation compared to fixed stick, meaning that the neutral point will be less far aft. If C_{h_α} is positive, the stability margin will be larger for the stick free situation. What happens to the elevator for various values of C_{h_α} also has been depicted in figure 7.3.

Multiple choice questions, part 1

Which of the following statements are true:

1. For conventional aircraft the neutral point, stick free lies aft of the neutral point, stick fixed.
2. If the hinge-moment coefficients C_{h_α} and C_{h_δ} are both negative, the free-stick static stability ($C_{m_{\alpha_{free}}}$) is smaller (in absolute value) than the fixed-stick static stability ($C_{m_{\alpha_{fix}}} = C_{m_\alpha}$).

The correct answers are:

[3]Just look at equation (5.21): $C_{N_{h_\delta}}$ is always positive and all the other terms will also be positive.

©Sam van Elsloo

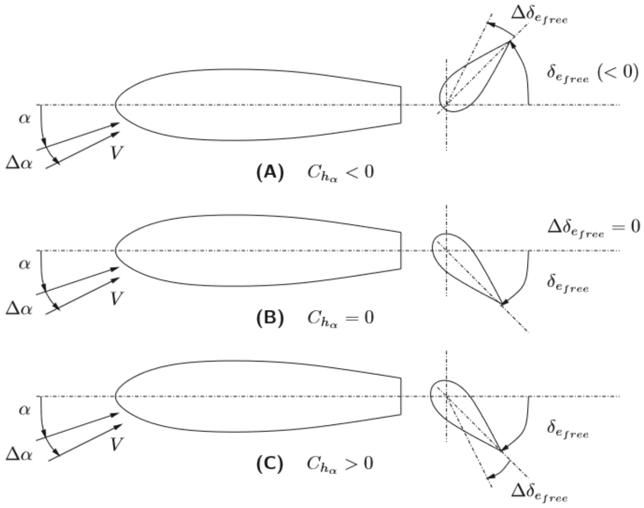


Figure 7.3: The behaviour of the free elevator after a change of the angle of attack.

1. This is not generally true, and actually most of the time the stick free will lay ahead of the stick fixed one. This is **false**.
2. This is true, see literally what I wrote above. This is **true**.

7.2 *The control force curve*

This section contains some absolutely awful derivations if you want to derive it perfectly. To derive everything perfectly, it'd take me multiple pages and it wouldn't really help you understand the theory better, and since the front page still bears the name summary, I'm not gonna include it. The gist is that we first consider figure 6.4 once more: for equilibrium we must have

$$F_e \Delta s_e + H_e \Delta \delta_e = 0$$

Taking infinitesimally small displacements, we obtain

$$F_e = -\frac{\partial \delta_e}{\partial s_e} H_e = -\frac{\partial \delta_e}{\partial s_e}\frac{1}{2}\rho V_h^2 S_e \bar{c}_e C_{h_e} = -\frac{\partial \delta_e}{\partial s_e}\frac{1}{2}\rho V_h^2 S_e \bar{c}_e \left(C_{h_\alpha}\alpha_h + C_{h_\delta}\delta_e + C_{h_{\delta_t}}\delta_{t_e}\right)$$

Now, we have already seen the equations

$$\alpha_h = \left(\alpha - \alpha_0\right)\left(1 - \frac{d\epsilon}{d\alpha}\right) + \left(\alpha_0 + i_h\right) \tag{5.9}$$

$$\delta_e = -\frac{1}{C_{m_{\delta_e}}}\left[C_{m_0} + C_{m_{\alpha_{fix}}}\left(\alpha - \alpha_0\right)\right] \tag{5.22}$$

so that

$$\begin{aligned} F_e =& -\frac{\partial \delta_e}{\partial s_e}\frac{1}{2}\rho V_h^2 S_e \bar{c}_e \left(C_{h_\alpha}\left[\left(\alpha - \alpha_0\right)\left(1 - \frac{d\epsilon}{d\alpha}\right) + \left(\alpha_0 + i_h\right)\right] - \frac{C_{h_\delta}}{C_{m_{\delta_e}}}\left[C_{m_0} + C_{m_{fix}}\left(\alpha - \alpha_0\right)\right] + C_{h_{\delta_t}}\delta_{t_e}\right) \\ =& -\frac{\partial \delta_e}{\partial s_e}\frac{1}{2}\rho V_h^2 S_e \bar{c}_e \left(\left[C_{h_\alpha}\left(\alpha_0 + i_h\right) - \frac{C_{h_\delta}}{C_{m_{\delta_e}}} C_{m_0} + C_{h_{\delta_t}}\delta_{t_e}\right]\right. \\ &+ \left.\left[C_{h_\alpha}\left(1 - \frac{d\epsilon}{d\alpha}\right) - \frac{C_{h_\delta}}{C_{m_{\delta_e}}} C_{m_{\alpha_{fix}}}\right]\left(\alpha - \alpha_0\right)\right) \end{aligned}$$

So, we have a term that's independent of α and a term that's dependent of α. Therefore, we can write it as

$$F_e = -\frac{d\delta_e}{ds_e}\frac{1}{2}\rho V_h^2 S_e \bar{c}_e \left[C'_{h_0} + C'_{h_\alpha}\left(\alpha - \alpha_0\right)\right] \tag{7.11}$$

Here,

THESE THINGS DON'T HAVE A PROPER NAME I THINK

$$C'_{h_0} = -\frac{C_{h_\delta}}{C_{m_\delta}} C_{m_{ac}} - \frac{C_{h_\delta}}{C_{N_{h_\delta}}} C_{N_{h_{\alpha_{free}}}} \left(\alpha_0 + i_h\right) + C_{H_{\delta_t}} \delta_{t_e} \tag{7.12}$$

$$C'_{h_\alpha} = C_{h_\alpha}\left(1 - \frac{d\epsilon}{d\alpha}\right) - \frac{C_{h_\delta}}{C_{m_\delta}} C_{m_{\alpha_{fix}}} \tag{7.13}$$

$$= -\frac{C_{h_\delta}}{C_{m_\delta}} C_{m_{\alpha_{free}}} \tag{7.14}$$

$$= -\frac{C_{h_\delta}}{C_{m_\delta}} C_{N_\alpha} \frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} \tag{7.15}$$

Just trust me that these expressions are correct. Note that we have multiple equivalent expressions for C'_{h_α}. Now, note that

$$\left(\alpha - \alpha_0\right) = \frac{W}{\frac{1}{2}\rho V^2 S}\frac{1}{C_{L_\alpha}} \approx \frac{W}{\frac{1}{2}\rho V^2 S}\frac{1}{C_{N_\alpha}}$$

as C_{L_α} is simply the lift-curve slope. Consequentially, we may write equation (7.11) as

$$F_e = -\frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \left[C'_{h_0}\frac{1}{2}\rho V_h^2 + C'_{h_\alpha}\frac{W}{S}\frac{1}{C_{N_\alpha}}\right] \tag{7.16}$$

We'll make two more substitutions. First, note that from equation (7.15)

$$\frac{C_{h_\alpha}}{C_{N_\alpha}} = -\frac{C_{h_\delta}}{C_{m_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} \tag{7.17}$$

Secondly, from equation (7.12) it is apparent that C'_{h_0} depends on δ_{t_e}, and indeed, there is a $\delta_{t_{e_0}}$ for which $C'_{h_0} = 0$. This value of $\delta_{t_{e_0}}$ can be found straightforwardly by solving (7.12);

$$\delta_{t_{e_0}} = \frac{\frac{C_{h_\delta}}{C_{m_\delta}} C_{m_{ac}} + \frac{C_{h_\delta}}{C_{N_{h_\delta}}} C_{N+h_{\alpha_{free}}} \left(\alpha_0 + i_h\right)}{C_{h_{\delta_t}}} \tag{7.18}$$

By doing so, we may write $C'_{h_0} = C_{h_{\delta_t}}\left(\delta_{t_e} - \delta_{t_{e_0}}\right)$. Substituting this and equation (7.17) into equation (7.16) leads to

ELEVATOR CONTROL FORCE

The force required to keep the elevator in the correct position for moment equilibrium is

$$F = \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \left[\underbrace{\frac{W}{S}\frac{C_{h_\delta}}{C_{m_{\delta_e}}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}}}_{\text{independent of } V} - \underbrace{\frac{1}{2}\rho V^2 C_{h_{\delta_t}}\left(\delta_{t_e} - \delta_{t_{e_0}}\right)}_{\text{dependent on } V}\right] \tag{7.19}$$

And now you may very well wonder, what was the point of all of this? Well, note that for *stick free* flight, we have $F = 0$: after all, we don't set any force on the stick. Thus, if we know the value of $\delta_{t_{e_0}}$ (which can

©Sam van Elsloo

be straightforwardly computed from equation (7.18)), perhaps we can find out what the required δ_{t_e} is such that $F = 0$: the pilot then knows what the deflection of the trim tab should be, so he can simply select the corresponding setting and for the rest just relax.

Now, note that in equation (7.19), we have a part that is independent of airspeed and a part that is dependent. Furthermore, remember that all of C_{h_δ}, C_{m_δ} and $C_{h_{\delta_t}}$ are negative[4], and that for stable aircraft, $x_{c.g.} < x_{n_{free}}$. This means that for $V = 0$, the value of F_e is completely determined by the first term, which will be *negative*. Then, if we increase V, the value of F_e will change: suppose first we take some trim tab deflection angle $\delta_{t_e} < \delta_{t_{e0}}$: then $(\delta_{t_e} - \delta_{t_{e0}})$ would be negative, and $C_{h_{\delta_t}}$ was already negative, so those minus signs cancel out. This means that F_e becomes increasingly negative! However, want it to go towards 0; evidently, we should pick $\delta_{t_e} > \delta_{t_{e0}}$: then F_e will become less negative, and eventually go through $F_e = 0$. This is exactly what we want! We want stick free, after all.

Graphically this is depicted in figure 7.4: again, *negative* values of F_e are plotted upward. This is because negative F_e means you are pulling the stick back, which for pilots seems more positive. However, we see that indeed for $\delta_{t_e} > \delta_{t_{e0}}$ the plot goes down and that it'd eventually cross the horizontal axis. By constructing many such graphs, for various values of δ_{t_e}, you can look up what the required trim tab deflection is to fly at a certain velocity. The speed at which the control force is zero has a special name:

TRIM SPEED

> The **trim speed** is the airspeed at which the control force is zero, indicated by V_{tr}.

If the elevator is let free at this airspeed, the control position will not change.

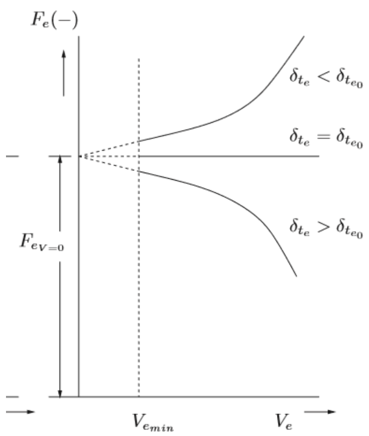


Figure 7.4: Schematic form of the elvator control force curve, F_e as a function of equilvant airspeed V_e.

7.3 *Elevator stick force stability*

Now we'll do a similar analysis as before in how the control stick should move. For a pilot and basically for any human being, it is natural to have the feeling that if you pull the stick backwards, that the stick is trying to push you back in the opposite direction. You do not want it to happen that if you pull the stick backwards, that it feels as if you have to pull it back even more; this would be very unnatural. Indeed, we want a positive feel, which requires, looking at figure 6.4:

$$\frac{dF_e}{ds_e} > 0$$

[4] C_{h_δ} and C_{m_δ} are negative as otherwise you have instability; $C_{h_{\delta_t}}$ is negative because when you increase the deflection angle of the trim tab, the lift over the trim tab will increase, causing a negative moment about the hingeline of the elevator.

In other words: if we want to move it backwards, we need to give it a backwards force as well[5]. Now, once more, $d\delta_e/ds_e > 0$, and thus we actually have

$$\frac{dF_e}{d\delta_e} = \frac{dF_e/dV}{d\delta_e/dV} > 0$$

Now, as in section 6.4, $d\delta_e/dV > 0$, and thus we merely require

REQUIREMENT FOR ELEVATOR CONTROL FORCE STABILITY

If the elevator control force curve at the trim speed satisfies the condition

$$\left(\frac{dF_e}{dV}\right)_{F_e=0} > 0 \tag{7.20}$$

the aircraft is said to have **elevator control force stability** in that particular flight condition. The higher this value, the more stable.

So, we must evaluate the control force derivative with respect to velocity at the trim conditions. Let's just differentiate equation (7.19):

$$\frac{dF_e}{dV} = -\frac{\partial\delta_e}{\partial s_e} S_e\bar{c}_e \left(\frac{V_h}{V}\right)^2 \rho V C_{h_{\delta_t}} \left(\delta_{t_e} - \delta_{t_{e_0}}\right) \tag{7.21}$$

Now, trim conditions are obtained by solving equation (7.19) for $F = 0$ (as the trim conditions occur when $F = 0$). That yields the equation

$$\frac{W}{S}\frac{C_{h_\delta}}{C_{m_{\delta_e}}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} = \frac{1}{2}\rho V^2 C_{h_{\delta_t}} \left(\delta_{t_e} - \delta_{t_{e_0}}\right) \tag{7.22}$$

which may be written as

$$\rho V C_{h_{\delta_t}} \left(\delta_{t_e} - \delta_{t_{e_0}}\right) = 2\frac{W}{S}\frac{C_{h_\delta}}{C_{m_{\delta_e}}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}}\frac{1}{V_{tr}}$$

Substituting this into equation (7.21) leads to

CONTROL FORCE DERIVATIVE AT TRIM SPEED

The control force derivative at trim speed is given by

$$\left(\frac{dF_e}{dV}\right)_{F_e=0} = -2\frac{d\delta_e}{ds_e} S_e\bar{c}_e \left(\frac{V_h}{V}\right)^2 \frac{W}{S}\frac{C_{h_\delta}}{C_{m_{\delta_e}}}\frac{x_{c.g.} - x_{free}}{\bar{c}}\frac{1}{V_{tr}} \tag{7.23}$$

Now, $C_{m_{\delta_e}}$ and C_{h_δ} are both negative. For *stable* aircraft, for which the center of gravity is in front of the neutral point, $x_{c.g.} - x_{n_{free}}$ is negative, so then the derivative is indeed positive. So, statically stable, stick free aircraft are automatically control force stable and vice versa. Note that the derivative can easily be measured in flight: we just measure these derivatives and if it's positive, it's stable.

7.4 *Influence of center of gravity position*

If the center of gravity moves forward, the entire curve shown in figure 7.4 will move upward. This means that for the same δ_{e_t}, you need to fly at a higher velocity to have $F_e = 0$, or you need a larger δ_{e_t} to fly at the same speed. If the c.g. moves backward, the curve will move downward.

[5]This may seem like super weird, but think about the following: suppose you have a very long stick standing up straight (vertically). Somehow, it's in equilibrium. However, you decide to pull it slightly towards you: due to gravity, it will then automatically want to fall farther towards you, meaning you actually have to *push* it away to keep it in the same spot. This is rather counterintuitive and makes it an unstable equilibrium (and thus undesired). That's what I'm trying to convey here. You much rather have something with a torsional spring so that if you pull it towards you, you have to *keep* pulling as it wants to go back to its equilibrium position.

©Sam van Elsloo

Multiple-choice questions, part (2)

Which of the following statements are true and which are false:

1. Shifting the position of the center of gravity forward will decrease the magnitude of the required control forces exerted by the pilot during normal operations.
2. Most pilots will find an aircraft unpleasant to fly if the elevator control force curve has a negative slope: $(dF_e/dV)_{F_e=0} < 0$.

The correct answers are:

1. This statement is false, but I'm not sure whether I agree with the reasoning of the official solutions. They say that the force does not change, but in my opinion, it will *increase*. To quote the book: "If the c.g. is shifted in X-direction this constant term in equation (7.19) varies and, at a constant trim tab angle, the entire control force curve will shift up or down, parallel to itself. If the c.g. is moved forward the control force wil shift upwards (a larger pull force will be required) at constant δ_{t_e} and i_h." That clearly seems to say that the control forces increase, making this statement **false**.
2. Yeah they hate it, this statement is **true**.

7.5 *Influence of trim tab angle*

We can also solve equation (7.22) for δ_e:

$$\frac{W}{S}\frac{C_{h_\delta}}{C_{m_{\delta_e}}}\frac{x_{c.g.}-x_{n_{free}}}{\bar{c}} = \frac{1}{2}\rho V^2 C_{h_{\delta_t}}\left(\delta_{t_e}-\delta_{t_{e_0}}\right) \tag{7.22}$$

$$\delta_{t_e} = \delta_{t_{e_0}} + \frac{W}{\frac{1}{2}\rho V_{tr}^2 S}\frac{1}{C_{h_{\delta_t}}}\frac{C_{h_\delta}}{C_{m_{\delta_e}}}\frac{x_{c.g.}-x_{n_{free}}}{\bar{c}} \tag{7.24}$$

So, if you want to fly at a different trim speed, you simply change δ_{t_e} accordingly, which should have become clear from what I said before. Furthermore, note that we have

$$\frac{d\delta_{t_e}}{dV_{tr}} = \frac{-4W}{\rho V_{tr}^3 S}\frac{1}{C_{h_{\delta_t}}}\frac{C_{h_\delta}}{C_{m_{\delta_e}}}\frac{x_{c.g.}-x_{n_{free}}}{\bar{c}}$$

Here, $C_{h_{\delta_t}}$, C_{h_δ} and $C_{m_{\delta_e}}$ are all negative. This means that for a stable aircraft, for which $x_{c.g.}-x_{n_{free}}$ is negative, $d\delta_{t_e}/dV_{tr}$ is negative! Vice versa, if it turns out that $d\delta_{t_e}/dV_{tr}$ is positive, it means that the aircraft is unstable (both statically unstable and control force unstable).

©Sam van Elsloo

8 Exam questions

Here are a few exam questions to practice with. The first ones are a bit easy, after that they get progressively more involved. If you freak out because you don't know a single answer to them: just bear in mind that the exams are quite repetitive, so by just doing enough old exams the exam is still passable, even if you don't truly know all of the derivations.

Exam April 2011: problem 2 (15 points)

(a) Sketch the elevator trim curve ($\delta_e - V$) if the following conditions hold:
- Statically stable, stick fixed
- $C_{m_0} > 0$

Clearly indicate the negative and positive side of the axes!

(b) Sketch the elevator control force curve ($F_e - V$) if the following conditions hold:
- Statically **un**stable, stick free
- $\delta_{t_e} < \delta_{t_0}$

Clearly indicate the negative and positive side of the axes!

For a), you may either remember figure 6.3 by heart, or you just look at your formula sheet. δ_e is given by

$$\delta_e = -\frac{1}{C_{m_{\delta_e}}}\left(C_{m_0} + \frac{W}{\frac{1}{2}\rho V^2 S}\frac{x_{c.g.} - x_{n_{fixed}}}{\bar{c}}\right)$$

Key to these questions is to analyse what happens for $V \to 0$ and for $V \to \infty$. For $V \to \infty$, the right term diminishes to 0 and C_{m_0} dominates. Remember that $C_{m_{\delta_e}} < 0$, and thus if $C_{m_0} > 0$, this means that δ_e will tend towards a positive value for $V \to \infty$. Then, for $V \to 0$, the right term will dominate. For a stable aircraft, $x_{c.g.} < x_{n_{fixed}}$, so this term will be negative as $x_{c.g.} - x_{n_{fixed}} < 0$. This means that when this term is multiplied with $-1/C_{m_{\delta_e}}$, the result will be negative, as $C_{m_{\delta_e}}$ is negative as well. Thus, for $V \to 0$, δ_e starts from a negative value. Then, drawing a hyperbolic curve starting at negative values and ending somewhere positive, taking into account that negative δ_e are plotted upwards, yields part (A) of figure 8.1.

For b), we do a similar analysis, but now we look at our formula sheet to see

$$F_e = \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \left[\frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} - \frac{1}{2}\rho V^2 C_{h_{\delta_t}}\left(\delta_{t_e} - \delta_{t_{e0}}\right)\right]$$

Once more, we analyse $V \to 0$ and $V \to \infty$. For $V \to 0$, the left term dominates. If the aircraft is unstable, then $x_{c.g.} > x_{n_{free}}$, so $x_{c.g.} - x_{n_{free}} > 0$. C_{h_δ} and C_{m_δ} are both negative, so those cancel out. The terms in front are also all positive, thus the result is positive. Thus, for $V \to 0$, F_e is positive. For $V \to \infty$, the right term dominates. $C_{h_{\delta_t}} < 0$, and if $\delta_{t_e} < \delta_{t_{e0}}$ the term between brackets is also negative, meaning those minus signs cancel out, meaning that the minus in front of it stands. As a result, for $V \to \infty$, F_e becomes negative. Drawing it parabolically (and remembering that negative δ_e are plotted upwards), we obtain part (B) of figure 8.1.

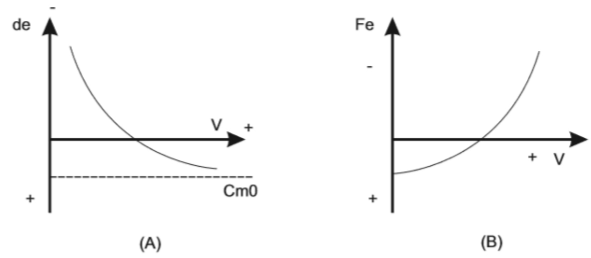


Figure 8.1: Elevator trim curve and elevator control force curve.

Exam August 2011: problem 3 (15 points)

Consider the aircraft in figure 8.2 which is the Rutan Voyager. The Rutan Voyager was the first aircraft to fly around the world non-stop. In order to maximize its range, the Voyager had 17 fuel tanks; 8 in each wing and nacelle and 1 in the fuselage. In total, the fuel tanks contained over 3000 kg of fuel, while its empty weight was around 1000 kg.

(a) Explain where the center of gravity is located: aft of the main wing, between the main wing and the canard or in front of the canard.
(b) This aircraft is longitudinally stable (i.e. $C_{m_\alpha} < 0$). Explain, using only words, why it is longitudinally stable.
(c) In the case of (b), what is the advantage of having a canard instead of a horizontal stabilizer?
(d) The pilots of the Voyager had to constantly trim the aircraft. Why was this the case, and how do you suggest they should perform this task? Clearly explain your answer in terms of the static longitudinal stability!
(e) Describe the effect on the stick-position stability if the canard is removed from the aircraft and a conventional horizontal stabilizer is added. Assume that the shift of the center of gravity after the reconfiguration is negligible.

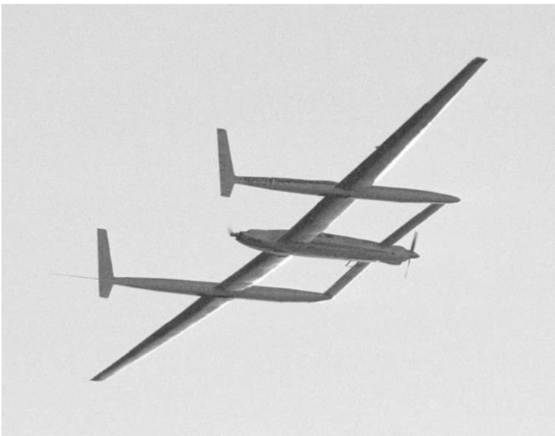


Figure 8.2: Rutan Voyager.

For a), the center of gravity must naturally be placed between the main wing and the canard. This way, both can generate positive lift (which is desirable) and equilibrium can still be attained.

For b), the question is maybe a bit weirdly posed, but it's simply because the neutral point is aft of the center of gravity.

For c), advantages of canards are:

- They can produce positive lift, whereas horizontal tailplanes have to generate negative lift most

of the time.
- Due to a higher efficiency of producing lift (due to the reason above), the overall drag of the aircraft is less.
- A canard operates in clean air, meaning the control surfaces are more efficient.

For d), as the plane contains so much fuel (75% of total weight), the center of gravity constantly changes position as fuel is burned. Static stability is lost if the c.g. moves aft of the neutral point. This can be prevented by the pilots by pumping fuel from the tanks in the wing to the tank in the canard, to ensure that the c.g. stays in front of the neutral point.

For e), using a conventional horizontal stabilizer will shift the neutral point backward. Thus, the stick-position stability is increased.

These two were pretty easy, so let's try a slightly harder one.

Exam April 2012: problem 4 (25 points)

The force and moment equilibrium for an aircraft with a canard configuration differs significantly from that of an aircraft with a conventional horizontal tailplane.

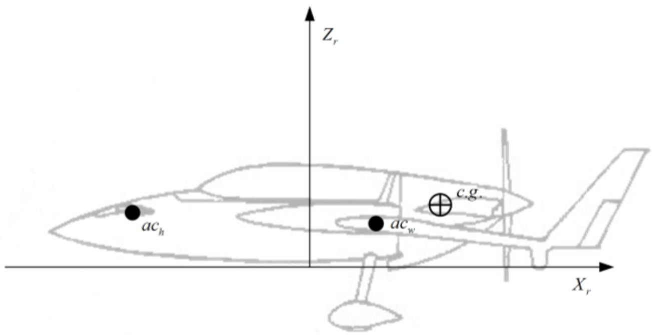
(a) In the figure below, draw the non-dimensional forces and moments necessary for longitudinal equilibrium for an aircraft with a canard configuration and the given center of gravity. Make sure that the forces and moments have the correct direction! Assume that for the canard, C_{T_h} and $C_{m_{ac_h}}$ are negligible. For the main wing, assume that the contribution of C_{T_w} to the aerodynamic moment is negligible.

(b) Formulate the equilibrium equation for the pitching moment coefficient C_m. Note that for a canard configuration it is safe to assume that $(V_h/V)^2 = 1$, while the factor S_h/S must not be neglected.

(c) Derive from the moment equation the expression for the static stability C_{m_α}.

(d) Derive an expression for the x-coordinate of the neutral point stick-fixed $x_{n_{fix}}$.

(e) The situation in the figure below is not very efficient not statically stable. Suggest a configuration change in the figure resulting in a more efficient, and stable design. Mark the changed forced, moments, lengths and points with an asterisk.



For a), see figure 8.3. Note that the normal force of the canard must point downward to be able to creat equilibrium around the c.g. Furthermore, M_{ac} is drawn clockwise as we'll assume it's positive for now.

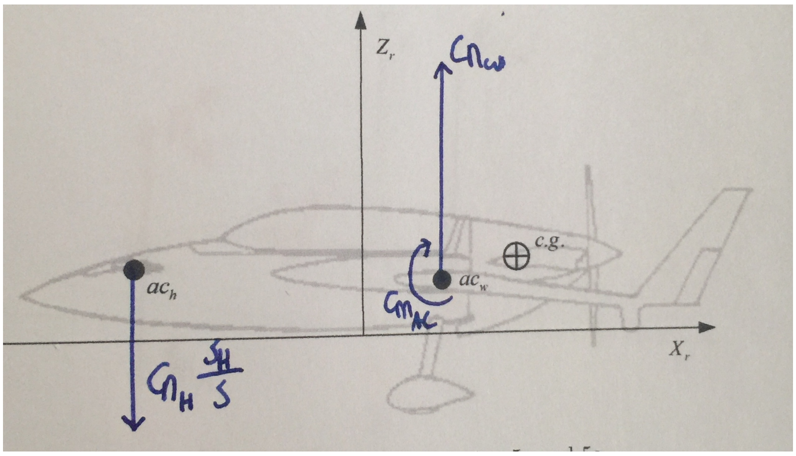


Figure 8.3: Answer to a).

For b), we have

$$\circlearrowright + \sum M \; : \; M_{ac} + N_w \left(x_{c.g.} - x_w \right) - N_h \left(x_{c.g.} - x_h \right) = 0$$

Normalising by dividing by $1/2 \rho V^2 S \bar{c}$ leads to

$$C_m = C_{m_{ac}} + C_{N_w} \frac{x_{c.g.} - x_w}{\bar{c}} - C_{N_h} \frac{S_h}{S} \frac{x_{c.g.} - x_h}{\bar{c}} = 0$$

For c), we simply differentiate with respect to α: $C_{m_{ac}}$ is constant, as are the distances, so we simply obtain

$$C_{m_\alpha} = C_{N_{w_\alpha}} \frac{x_{c.g.} - x_w}{\bar{c}} - \frac{dC_{N_h}}{d\alpha} \frac{S_h}{S} \frac{x_{c.g.} - x_h}{\bar{c}}$$

Here, $dC_{N_h}/d\alpha$ may be obtained by

$$\frac{dC_{N_h}}{d\alpha} = \frac{dC_{N_h}}{d\alpha_h} \frac{d\alpha_h}{d\alpha} = C_{N_{h_\alpha}} \frac{d\alpha_h}{d\alpha}$$

We use the following expression for $d\alpha_h/d\alpha$:

$$\alpha_h = \alpha - \epsilon + i_h$$

where ϵ is the downwash and i_h the incidence angle of the horizontal stabiliser. We then obtain

$$\frac{d\alpha_h}{d\alpha} = 1 - \frac{d\epsilon}{d\alpha}$$

so that

$$C_{m_\alpha} = C_{N_{w_\alpha}} \frac{x_{c.g.} - x_w}{\bar{c}} - \frac{dC_{N_h}}{d\alpha} \frac{S_h}{S} \frac{x_{c.g.} - x_h}{\bar{c}} = C_{N_{w_\alpha}} \frac{x_{c.g.} - x_w}{\bar{c}} - C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha} \right) \frac{S_h}{S} \frac{x_{c.g.} - x_h}{\bar{c}}$$

For d), the neutral point is the value of $x_{c.g.}$ such that $C_{m_\alpha} = 0$. In other words, we require

$$0 = C_{N_{w_\alpha}} \frac{x_{n_{fix}} - x_w}{\bar{c}} - C_{N_{h_\alpha}} \left(1 - \frac{d\epsilon}{d\alpha} \right) \frac{S_h}{S} \frac{x_{n_{fix}} - x_h}{\bar{c}}$$

This can be obtained by realising that we also have

$$N = N_w - N_h = W$$
$$C_N = C_{N_w} - C_{N_h}\frac{S_h}{S} = \frac{W}{\frac{1}{2}\rho V^2 S}$$

Differentiate with respect to α:

$$C_{N_\alpha} = C_{N_{w_\alpha}} - C_{N_{h_\alpha}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{S_h}{S} = 0$$

Multiplying with $(x_{n_{fix}} - x_w)/\bar{c}$ yields

$$C_{N_\alpha}\frac{x_{n_{fix}} - x_w}{\bar{c}} = C_{N_{w_\alpha}}\frac{x_{n_{fix}} - x_w}{\bar{c}} - C_{N_{h_\alpha}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{S_h}{S}\frac{x_{n_{fix}} - x_w}{\bar{c}}$$

Subtract first equation from this, to obtain

$$\begin{aligned} C_{N_\alpha}\frac{x_{n_{fix}} - x_w}{\bar{c}} =& C_{N_{w_\alpha}}\frac{x_{n_{fix}} - x_w}{\bar{c}} - C_{N_{w_\alpha}}\frac{x_{n_{fix}} - x_w}{\bar{c}} \\ & - C_{N_{h_\alpha}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{S_h}{S}\left(\frac{x_{n_{fix}} - x_w}{\bar{c}} - \frac{x_{n_{fix}} - x_h}{\bar{c}}\right) \\ =& - C_{N_{h_\alpha}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{S_h}{S}\frac{x_h - x_w}{\bar{c}} = C_{N_{h_\alpha}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{S_h}{S}\frac{x_w - x_h}{\bar{c}} \end{aligned}$$

Letting $l_h = x_w - x_h$, we obtain

$$\frac{x_{n_{fix}} - x_w}{\bar{c}} = C_{N_{h_\alpha}}\left(1 - \frac{d\epsilon}{d\alpha}\right)\frac{S_h l_h}{S\bar{c}}$$

For e), by placing the center of gravity in front of the neutral point, the aircraft becomes stable, and the canard is also required to produce, resulting in a more efficient design. See figure 8.4

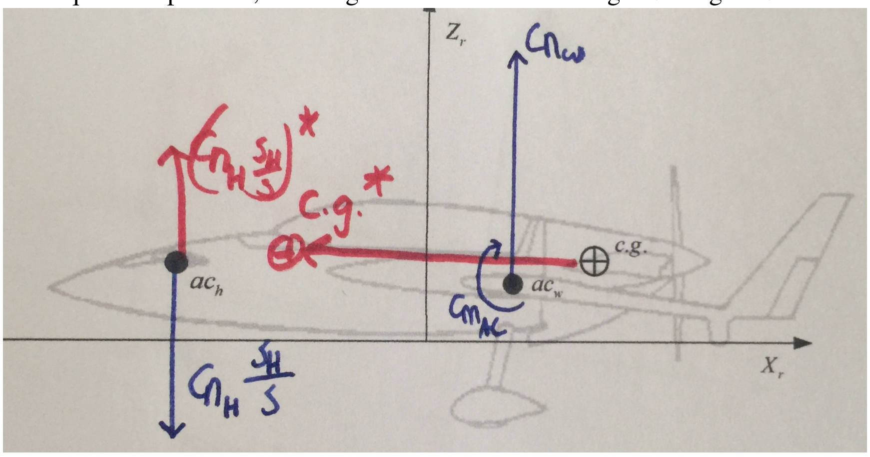


Figure 8.4: Answer to e).

Maybe you found the above question already a bit hard due to the derivations, unfortunately you just gotta live with it. This kind of simple derivation you are required to know and be able to produce yourself. Also because literally first years are able to do part b) and c)[1]. In any case, here are the main steps you have to remember to derive the location of the neutral point:

[1]Okay maybe not *all* of the first years as based on their free-body diagrams for dynamics, some of them seem to think that gravity acts upwards so idk how badly they'd manage to fuck up these questions, but in principle, they had question b) and c) in intro I.

DERIVING THE NEUTRAL POINT OF A STICK FIXED AIRCRAFT

1. Draw the free-body diagram of the aircraft.
2. Set up the equilibrium equation for the moments.
3. Differentiate the moment equation with respect to α.
4. Set up the equilibrium equation for the vertical forces.
5. Differentiate the normal force equation with respect to α.
6. Subtract the derivative of the moment equation from this equation and rewrite as desired.

Exam April 2014: problem 4 (25 points)

$W = 155000[N[$	$\rho = 0.65[kgm^{-3}]$	$V = 100[ms^{-1}]$	$g = 9.81[Nkg^{-1}]$
$S = 65[m^2]$	$S_e = 3[m^2]$	$\left(\frac{V_h}{V}\right)^2 = 1$	$\frac{d\delta_e}{ds_e} = 2.0[radm^{-1}]$
$\bar{c} = 2.5[m]$	$\bar{c}_e = 0.4[m]$	$\frac{x_{cg}}{\bar{c}} = 0.33$	$\frac{x_{n_{free}}}{\bar{c}} = 0.62$
$C_{m_\delta} = -2.0[rad^{-1}]$	$C_{h_\delta} = -0.3[rad^{-1}]$	$C_{h_{\delta_t}} = -0.12[rad^{-1}]$	

Table 8.1: Aircraft parameters.

Consider a conventional aircraft during trimmed horizontal flight ($F_e = 0$), and for which the parameters in table 8.1 hold.

(a) **(5 points)** Calculate both $(\delta_{t_e} - \delta_{t_{e0}})$ and the control force stability $(dF_e/dV)_{F_e=0}$ (answer must be given with an accuracy of 3 significant decimals behind the decimal point).

(b) **(8 points)** At a certain altitude a mass of 1500 kg is being released out of the aircraft, causing a relocation of the center of gravity at $x_{cg-new} = 0.25\bar{c}$. Calculate the magnitude as well as the sign of the control force, which is necessary to keep flying at the original airspeed V. Assume that the trim tab is not being adjusted. Should the pilot exert a pulling force or a pushing force at the control column? (If you are unable to answer question (a) then use: $(\delta_{t_e} - \delta_{t_{e0}}) = 0.3$).

(c) **(6 points)** Calculate the change in trim tab angle if the control force is being reduced to zero, and assuming that the airspeed does not change. Should the trim tab be rotated upward or downward?

(d) **(6 points)** Does the aircraft possess more control force stability or less control force stability after the weight release? Support your conclusion by computing the control force stability for the aircraft after the weight release.

May seem overwhelming at first, but just look at your formula sheet a few times and you'll be fine. First, we use the elevator control force to compute $(\delta_{t_e} - \delta_{t_{e0}})$:

$$F_e = \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \left[\frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} - \frac{1}{2}\rho V^2 C_{h_{\delta_t}}\left(\delta_{t_e} - \delta_{t_{e0}}\right)\right]$$

We require $F_e = 0$, which is achieved by setting

$$\frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} = \frac{1}{2}\rho V^2 C_{h_{\delta_t}}\left(\delta_{t_e} - \delta_{t_{e0}}\right)$$

$$\frac{2W}{\rho S V^2}\frac{C_{h_\delta}}{C_{m_\delta} C_{h_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} = \left(\delta_{t_e} - \delta_{t_{e0}}\right)$$

and this is literally it, just plug in the numbers now:

$$\left(\delta_{t_e} - \delta_{t_{e0}}\right) = \frac{2 \cdot 155000}{0.65 \cdot 65 \cdot 100^2}\frac{-0.3}{-2.0 \cdot -0.12} \cdot (0.33 - 0.62) = 0.266\,\text{rad}$$

©Sam van Elsloo

To obtain $(dF_e/dV)_{F_e=0}$, we differentiate F_e with respect to V, to obtain

$$\frac{dF_e}{dV} = \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \cdot -\rho V C_{h_{\delta_t}} \left(\delta_{t_e} - \delta_{t_{e0}}\right)$$

We simply substitute in our previously computed $\left(\delta_{t_e} - \delta_{t_{e0}}\right)$ (after all, this was calculated to set $F_e = 0$) to obtain

$$\left(\frac{dF_e}{dV}\right)_{F_e=0} = -2 \cdot 3 \cdot 0.4 \cdot 1 \cdot 0.65 \cdot 100 \cdot -0.12 \cdot 0.266 = 4.979\,\mathrm{Ns/m}$$

For b), we simply use our expression for F_e, and simply only change W and $x_{c.g.}/\bar{c}$ accordingly. The new weight becomes

$$W = 155000 - 9.81 \cdot 1500 = 140\,285\,\mathrm{N}$$

Then we plug this in so that

$$\begin{aligned} F_e &= \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \left[\frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} - \frac{1}{2}\rho V^2 C_{h_{\delta_t}} \left(\delta_{t_e} - \delta_{t_{e0}}\right)\right] \\ &= 2 \cdot 3 \cdot 0.4 \cdot 1 \cdot \left[\frac{140285}{65} \cdot \frac{-0.3}{-2.0} \cdot (0.25 - 0.62) - \frac{1}{2} \cdot 0.65 \cdot 100^2 \cdot -0.12 \cdot 0.266\right] = -38.5\,\mathrm{N} \end{aligned}$$

As this force is negative, he has to *pull* on the stick.

For c), we just redo the first part of a) again, just with different W and $x_{c.g.}/\bar{c}$:

$$\begin{aligned} \left(\delta_{t_e} - \delta_{t_{e0}}\right) &= \frac{2W}{\rho S V^2} \frac{C_{h_\delta}}{C_{m_\delta} C_{h_\delta}} \frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} \\ &= \frac{2 \cdot 140285}{0.65 \cdot 65 \cdot 100^2} \frac{-0.3}{-2.0 \cdot -0.12} \cdot (0.25 - 0.62) = 0.307\,\mathrm{rad} \end{aligned}$$

So the change in trim tab angle is $0.307 - 0.266 = 0.0411$ rad. As this angle is positive, the tab should be rotated *down*ward (after all, positive deflections are downward for the elevator and trim tab).

For d), we just redo the second part of a) again, just with different numbers:

$$\begin{aligned} \frac{dF_e}{dV} &= \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \cdot -\rho V C_{h_{\delta_t}} \left(\delta_{t_e} - \delta_{t_{e0}}\right) \\ &= -2 \cdot 3 \cdot 0.4 \cdot 1 \cdot 0.65 \cdot 100 \cdot -0.12 \cdot 0.307 = 5.750\,\mathrm{Nm/s} \end{aligned}$$

Since this value is large than before, the aircraft now possesses more control force stability.

Honestly, I think the hardest part of this question was just realising that it was really this easy. Just look at your formula sheet, really.

Exam March 2007: problem 3 (5 points)

Consider an aircraft with properties provided in table 8.2.

W = 150000 N	g = 9.81 m/s^2	$\left(\frac{V_h}{V}\right)^2$ = 1
S = 60 m^2	S_h = 15 m^2	S_e = 3 m^2
$\bar{c}$ = 3.0 m	l_h = 12 m	$\bar{c}_e$ = 0.4 m
$\frac{d\delta_e}{ds_e}$ = 2.25 rad m^{-1}	C_{m_α} = -1.5 rad^{-1}	C_{m_q} = -14 rad^{-1}
x_{cg} = 0.25$\bar{c}$	C_{N_α} = 6 rad^{-1}	$C_{N_{h_\delta}}$ = 2.0 rad^{-1}
ρ = 1.225 kg/m^3	C_{h_α} = -0.1	C_{h_δ} = -0.4 rad^{-1}
$\frac{d\epsilon}{d\alpha}$ = 0.8	V = 125 m/s	$C_{h_{\delta_t}}$ = -0.12 rad
	$C_{m_{\delta_e}}$ = −2.0 rad^{-1}	

Table 8.2: Aircraft parameters.

(a) Calculate the position of the neutral point, stick free expressed in terms of the mean aerodynamic cord $\bar{c}$ for straight flight. It is assumed that the contribution of the horizontal tailplane to C_{N_α} is negligible.

Once more, simply looking at your formula sheet does miracles. The relative position of the neutral point, stick free and stick fixed is given by

$$\frac{x_{n_{free}} - x_{n_{fix}}}{\bar{c}} = \frac{C_{m_\delta}}{C_{N_\alpha}} \frac{C_{h_\alpha}}{C_{h_\delta}} \left(1 - \frac{d\epsilon}{d\alpha}\right)$$

Now, every value except $x_{n_{fix}}/\bar{c}$ is known. However, that can be straightforwardly computed from

$$\frac{x_{c.g.} - x_{n_{fix}}}{\bar{c}} = \frac{C_{m_\alpha}}{C_{N_\alpha}}$$

$$\frac{x_{n_{fix}}}{\bar{c}} = \frac{x_{c.g.}}{\bar{c}} - \frac{C_{m_\alpha}}{C_{N_\alpha}} = 0.25 - \frac{-1.5}{6} = 0.5$$

and thus

$$\frac{x_{n_{free}}}{\bar{c}} - 0.5 = \frac{-2}{6} \frac{-0.1}{-0.4} (1 - 0.8)$$

$$x_{n_{free}} = 0.483\bar{c}$$

As long as you weren't a dumbass like me and spend hours thinking how to do this question before looking at your formula sheet, I think this question wasn't too hard.

Exam January 2005: problem 7 (10 points)

Consider a Fokker F27 from which the flight conditions and some numerical aircraft details are given in table 8.3 and 8.4:

$H_p = 10000$ m ISA	$V = 120$ m/s	$\rho = 0.4127$ kg/m^3
$C_{m_\alpha} = -0.983$	$\frac{x_{cg}}{\bar{c}} = 0.346$	$S = 70\ m^2$
$\bar{c} = 2.58\ \ m$		

Table 8.3: Aircraft parameters.

W = 142000 N	$\rho = 0.4127$ kg/m^3	$\frac{d\delta_e}{ds_e} = 2.06$ rad m^{-1}
$\left(\frac{V_h}{V}\right)^2 = 0.95$	$S_e = 3.17$ m^2	$\bar{c}_e = 0.38$ m
$C_{h_\alpha} = 0$	$C_{N_\alpha} = 6.59$ rad^{-1}	$C_{m_\delta} =$ -1.67 rad^{-1}
$C_{h_\delta} =$ -0.372 rad^{-1}	$C_{h_{\delta_t}} =$ -0.125 rad^{-1}	$C_{m_q} =$ -16.50 [rad/s]$^{-1}$
$\delta_{t_e} = 0$ rad	$\delta_{t_{e_0}} = 0$ rad	

Table 8.4: Aircraft parameters.

Consider three different control mechanisms,

- Mechanism 1: a control mechanism without augmentation.
- Mechanism 2: a control mechanism that contains a spring which pulls the column forward with an approximately constant control force of 50 N, see figure 8.5.

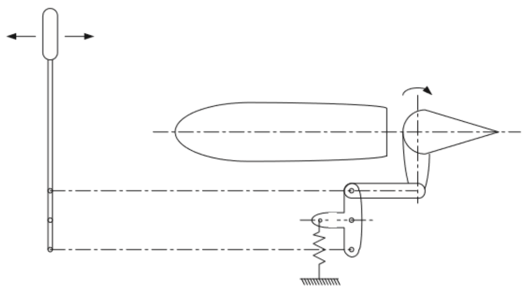


Table 8.5: Mechanism 2.

For both control mechanisms, compute:

(a) The neutral point, stick free $x_{n_{free}}$
(b) The stick force stability dF_e/dV

Furthermore, compare the values and explain which one you think is best.

Once more, more a test of how well you can read formula sheets than anything else. The neutral point for mechanism 1 is especially easy. Similar to the previous example, we use the relation between $x_{n_{free}} - x_{n_{fix}}$:

$$\frac{x_{n_{free}} - x_{n_{fix}}}{\bar{c}} = \frac{C_{m_\delta}}{C_{N_\alpha}} \frac{C_{h_\alpha}}{C_{h_\delta}} \left(1 - \frac{d\epsilon}{d\alpha}\right)$$

Note that C_{h_α} is given to be 0, so this entire thing reduces to zero, so $x_{n_{free}} = x_{n_{fix}}$. We can then compute $x_{n_{fix}}$ from the equation

$$\frac{x_{c.g.} - x_{n_{fix}}}{\bar{c}} = \frac{C_{m_\alpha}}{C_{N_\alpha}}$$

$$\frac{x_{n_{fix}}}{\bar{c}} = \frac{x_{c.g.}}{\bar{c}} - \frac{C_{m_\alpha}}{C_{N_\alpha}} = 0.346 - \frac{-0.983}{6.59} = 0.495$$

so $x_{n_{free}}/\bar{c} = 0.495$. For mechanism 2, the computation is slightly more involved. We now have to use

$$F_e = \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \left[\frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} - \frac{1}{2}\rho V^2 C_{h_{\delta_t}}\left(\delta_{t_e} - \delta_{t_{e0}}\right)\right]$$

How do we include the spring force then? Well, we simply add 50 N to this equation:

$$F_e = \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \left[\frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} - \frac{1}{2}\rho V^2 C_{h_{\delta_t}}\left(\delta_{t_e} - \delta_{t_{e0}}\right)\right] + F_{e_{aug}}$$

However, $\delta_{t_e} = \delta_{t_{e0}} = 0$, so this reduces to

$$F_e = \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} + F_{e_{aug}}$$

$$\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} = \frac{F_e - F_{e_{aug}}}{\frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}}$$

$$\frac{x_{n_{free}}}{\bar{c}} = \frac{x_{c.g.}}{\bar{c}} - \frac{F_e}{\frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}} + \frac{F_{e_{aug}}}{\frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}}$$

Now, the question is, what is F_e? Well, note that

$$\frac{F_e}{\frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}}$$

is equal to $(x_{c.g.} - x_{n_{free}})\bar{c}$ had there *not* been a spring force. So, we simply can write

$$\begin{aligned}
\frac{x_{n_{free}}}{\bar{c}} &= \frac{x_{c.g.}}{\bar{c}} - \frac{x_{c.g.} - x_{n_{free_{old}}}}{\bar{c}} + \frac{F_{e_{aug}}}{\frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}} \\
&= \frac{x_{n_{free_{old}}}}{\bar{c}} + \frac{F_{e_{aug}}}{\frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}} \\
&= 0.495 + \frac{50}{2.06 \cdot 3.17 \cdot 0.38 \cdot 0.95 \cdot \frac{142000}{70} \cdot \frac{-0.372}{-1.67}} = 0.5421
\end{aligned}$$

I'll agree that this one was a bit tough, unfortunately I also don't have any tips for it in general as it's a rather specific question.

For b), it may be very tempting to differentiate F_e to obtain

$$\frac{dF_e}{dV} = \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \cdot -\rho V C_{h_{\delta_t}}\left(\delta_{t_e} - \delta_{t_{e0}}\right)$$

and say lol $\delta_{t_e} = \delta_{t_{e0}} = 0$, so this entire thing is zero. This would be a fundamental mistake: the stick force stability is the value of this derivative *at* $F_e = 0$. This is a very important distinction, and it means that we must first compute at which δ_{t_e} it holds that $F_e = 0$. We do this by simply solving the equation for F_e for $F_e = 0$. For mechanism 1, this means solving

$$F_e = \frac{d\delta_e}{ds_e} S_e \bar{c}_e \left(\frac{V_h}{V}\right)^2 \left[\frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} - \frac{1}{2}\rho V^2 C_{h_{\delta_t}}\left(\delta_{t_e} - \delta_{t_{e0}}\right)\right] = 0$$

This is satisfied when

$$\frac{W}{S}\frac{C_{h_\delta}}{C_{m_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} - \frac{1}{2}\rho V^2 C_{h_{\delta_t}}\left(\delta_{t_e} - \delta_{t_{e0}}\right) = 0$$

Note that $\delta_{t_{e0}}$ is still zero, so we have

$$\begin{aligned}\delta_{t_e} &= \frac{W}{\frac{1}{2}\rho V^2 S}\frac{1}{C_{h_{\delta_t}}}\frac{C_{h_\delta}}{C_{m_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} = \frac{142000}{\frac{1}{2}\cdot 0.4127 \cdot 120^2 \cdot 70}\cdot\frac{1}{-0.125}\cdot\frac{-0.372}{-1.67}(0.346 - 0.495)\\ &= 0.181\,27\,\text{rad}\end{aligned}$$

Plugging this into the derivative yields

$$\frac{dF_e}{dV} = 2.06 \cdot 3.17 \cdot 0.38 \cdot 0.95 \cdot -0.4127 \cdot 120 \cdot -0.125\,(0.18127 - 0) = 2.6483\,\text{Ns/m}$$

For mechanism 2, the trimmed situation is when $F_e = 50\,\text{N}$, so trimmed only means that it should be such that the elevator itself does not require a force (yeah idk how you're supposed to know that either). THis means that we simply have to find δ_{t_e} again, by plugging in

$$\begin{aligned}\delta_{t_e} &= \frac{W}{\frac{1}{2}\rho V^2 S}\frac{1}{C_{h_{\delta_t}}}\frac{C_{h_\delta}}{C_{m_\delta}}\frac{x_{c.g.} - x_{n_{free}}}{\bar{c}} = \frac{142000}{\frac{1}{2}\cdot 0.4127 \cdot 120^2 \cdot 70}\cdot\frac{1}{-0.125}\cdot\frac{-0.372}{-1.67}(0.346 - 0.5421)\\ &= 0.2384\,\text{rad}\end{aligned}$$

Thus, we obtain

$$\frac{dF_e}{dV} = 2.06 \cdot 3.17 \cdot 0.38 \cdot 0.95 \cdot -0.4127 \cdot 120 \cdot -0.125\,(0.2384 - 0) = 3.479\,\text{Ns/m}$$

For c), the spring system is more stable: the neutral point lies farther aft, and the elevator control stability is better. It should be noted that the spring system requires larger trim tab deflections, which would result in larger drag, so from an aerodynamic point of view, the normal system is better.

©Sam van Elsloo

Index