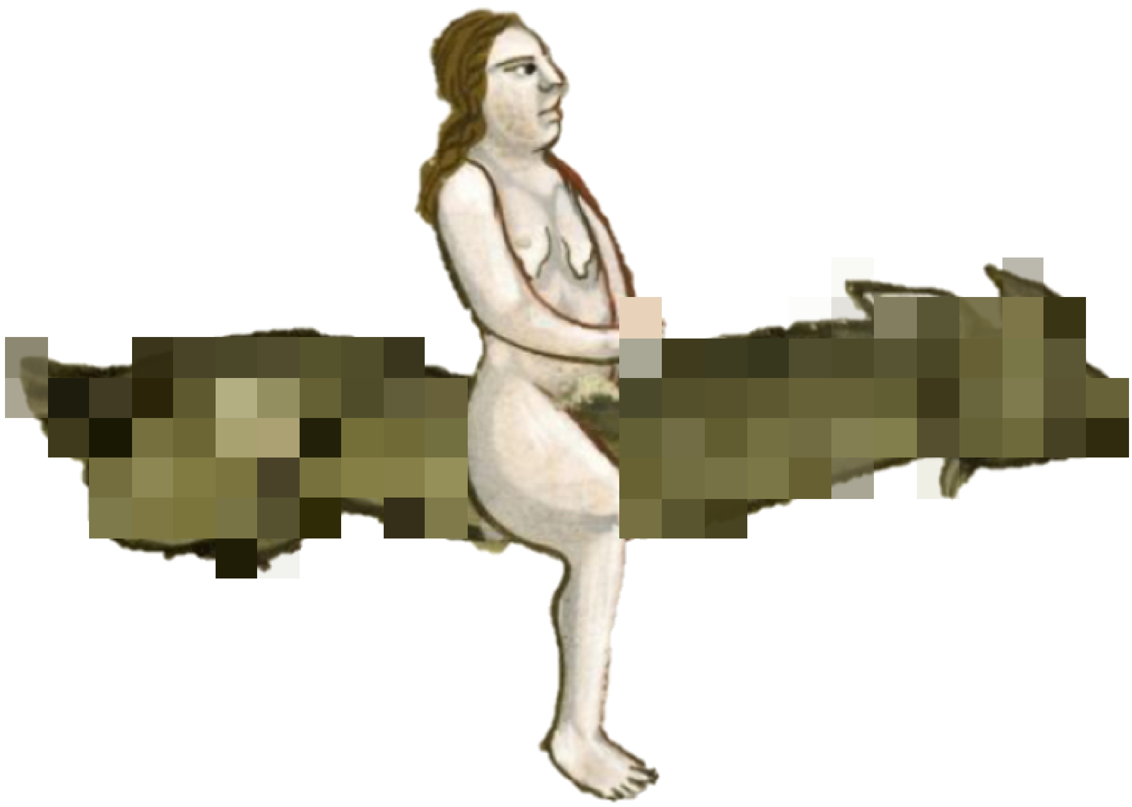
Flight Dynamics summary II: 2019-2020 edition

Based on *Lecture Notes Flight Dynamics* by J.A. Mulder et al.



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Preface

Okay so this part is pretty short (less than 35 pages if you discount the exam questions I included). The first two chapters are purely remembering stuff by heart. Personally I don't really like those chapters cause memorizing stuff is not something I really enjoy, but maybe you think it's nice. However, even though I sometimes include equations in the first two chapters, you absolutely don't need to remember them. Just memorize the blue boxes I put at the end of each section, that's literally all you need to know from those chapters. Chapter 3 is a lot more fun imo: it's just a bunch of rather straightforward calculations with matrices (which becomes very easy with your graphical calculator). However, all in all this part isn't terribly difficult.

1 Lateral stability and control derivatives

Part I of this summary was about longitudinal, symmetric stability. This part forcusses on *lateral*, asymmetric stability, i.e. when there's roll or yaw involved. Let me first just repeat once more what exactly symmetric/asymmetric, straight etc. mean, cause it's always good to have a clear understanding of that to avoid confusion:

BASIC DEFINITIONS OF FLIGHT CONDITIONS

- **Steady flight** means that the velocity V is constant.
- **Straight flight** means that the flight path angle γ is constant.
- **Symmetric flight** means that there is no roll angle and the turn rate is zero.

Part I was mostly about symmetric flight, where pitch rate is important. For asymmetric flight, pitch rate is unimportant and we deal mostly with roll and yaw[1].

1.1 Aerodynamic force and moments due to side slipping, rolling and yawing

First, it's important to define our coordinate system. We use the coordinate system shown in figure 1.1. The axes are defined as follows:

DIRECTIONS OF BODY AXIS SYSTEM

For a **body axis system**, we define the following axes (all axes originate from the center of gravity):

- The X_b**-axis** is aligned with the centerline of the fuselage, and points towards the nose. The velocity component along this axis is denoted by u.
- The Y_b**-axis** points towards the right wing, when looking at the aircraft from behind. The velocity component along this axis is denoted by v.
- The Z_b**-axis** points down, perpendicular to the fuselage centerline. The velocity component along this axis is denoted by w.

Figure 1.2 may be helpful in understanding the directions of the axes.

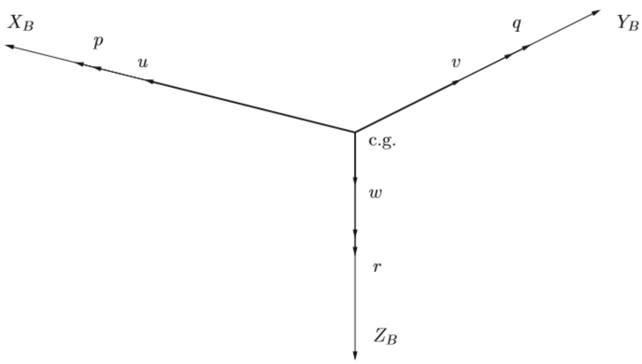


Figure 1.1: Six degrees of freedom of a rigid aircraft.

Figure 1.1 also defines the positive rotations:

[1] In fact, we'll see that stuff that was important in symmetric flight is unimportant now, and stuff that was unimportant in symmetric flight is important now.

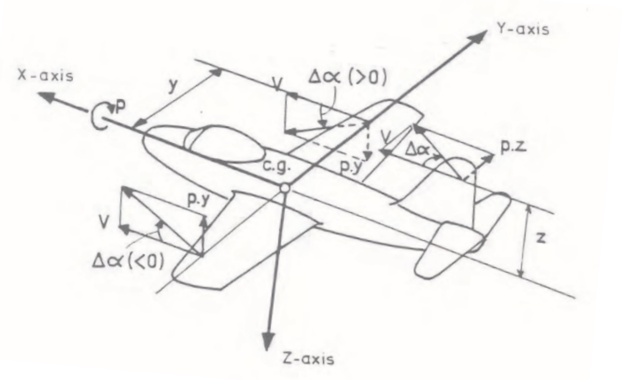


Figure 1.2: An airplane with inscribed body axis system.

ANGULAR VELOCITY COMPONENTS OF BODY AXIS SYSTEM

For a body axis system, we define the following rotations:
- Rotation around the X_b-axis is denoted by p and corresponds to the **rolling velocity**. Often the non-dimensional parameter $\frac{pb}{2V}$ is used.
- Rotation around the Y_b-axis is denoted by q and corresponds to the **pitching velocity**.
- Rotation around the Z_b-axis is denoted by r and corresponds to the **yawing velocity**. Often the non-dimensional parameter $\frac{rb}{2V}$ is used.

Positive directions follow from use of the right-hand rule along the corresponding axis.

Note that for asymmetric flight, the pitching velocity is unimportant (as that one corresponds to symmetric flight). Don't ask me why we non-dimensionalise like $pb/(2V)$, it's just how it is. Note that it is indeed dimensionless; we multiply radians per unit of time with a unit of distance (so we get distance over time), and then divide by velocity so it becomes unitless.

Furthermore, we define the following forces:

FORCES OF BODY AXIS SYSTEM

For a body axis system, we define the following forces:
- The force along the X_b-axis is denoted by X. It'll mostly deal with thrust and drag.
- The force along the Y_b-axis is denoted by Y. This force is generated by asymmetric motion.
- The force along the Z_b-axis is denoted by Z. It'll mostly deal with lift and gravity.

Positive directions are the directions of the corresponding axis.

Note that the X- and Z-forces will be unimportant to us most of the time: these correspond to symmetric motion rather than asymmetric motion. Only the Y-component is important.

Finally, we define the following moments:

FORCES OF BODY AXIS SYSTEM

For a body axis system, we define the following forces:
- The force along the X_b-axis is denoted by L. This moment corresponds to a **rolling moment**.
- The force along the Y_b-axis is denoted by M. This moment corresponds to a **pitching moment**.
- The force along the Z_b-axis is denoted by N. This moment corresponds to a **yawing moment**.

Positive directions follow from the right-hand rule of the corresponding axis.

Again, note that the M-moment will be unimportant to us as the pitching moment was already discussed extensively in the previous part of the summary. Only the L- and N-components are important.

So, in short, we only have three forces/moments we are interested in: the force Y, and the moments L and N. Non-dimensionalising those:

NON-DIMENSIONAL ASYMMETRIC FORCES AND MOMENTS

The **non-dimensional asymmetric forces and moments** are given by

$$C_Y = \frac{Y}{\frac{1}{2}\rho V^2 S} \tag{1.1}$$

$$C_l = \frac{L}{\frac{1}{2}\rho V^2 S b} \tag{1.2}$$

$$C_n = \frac{N}{\frac{1}{2}\rho V^2 S b} \tag{1.3}$$

Note that the moments are normalised by dividing by the wingspan b; opposed to the pitching moment which was normalised by dividing by $\bar{c}$. Just a matter of convention, really. Furthermore, please note that here L is the *rolling moment*, not the lift. Usually it's clear from context whether L is lift or the rolling moment, but if necessary I'll indicate which L I'm referring to. Same for C_l, although it's pretty rare to be confused whether something is the rolling moment coefficient of a wing or the lift coefficient of an airfoil (similarly N and C_n don't refer to the normal force but to the yawing moment).

We also define the following angles:

ANGLES OF BODY AXIS SYSTEM

For a body axis system, we define the following angles:

- The angle between the velocity vector and the $X_B - Y_B$-plane is defined as the **angle of attack** α. It is equal to

$$\alpha = \arcsin\frac{w}{V} \approx \frac{w}{V} \tag{1.4}$$

  for small angles.
- The angle between the velocity vector and the $X_B - Z_B$-plane is defined as the **sideslip angle** β. It is equal to

$$\beta = \arcsin\frac{v}{V} \approx \frac{v}{V} \tag{1.5}$$

  for small angles.

These angles, along with the asymmetric forces and moments, are shown in figure 1.3.

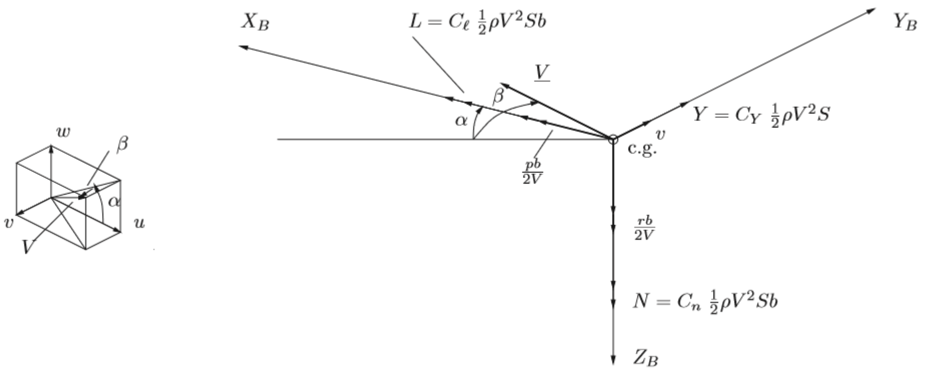


Figure 1.3: Asymmetric force and moments.

I agree that it's a lot of definitions, but try to draw figure 1.3 yourself; it's honestly just a matter of remembering how all the forces are defined more than anything. That's all there is to it. And if you're reading the following sections and get confused by the signs, just go back to these pages to see how everything is defined.

1.1.1 Rewriting the non-dimensional forces and moments

Now we want to come up with some expressions for C_Y, C_l and C_n. They'll be very basic expressions, similar to $C_L = C_{L_\alpha}\alpha$. Indeed, let's define the following derivatives:

$$C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta} \qquad C_{l_\beta} = \frac{\partial C_l}{\partial \beta} \qquad C_{n_\beta} = \frac{\partial C_n}{\partial \beta} \tag{1.6}$$

$$C_{Y_p} = \frac{\partial C_Y}{\partial \frac{pb}{2V}} \qquad C_{l_p} = \frac{\partial C_l}{\partial \frac{pb}{2V}} \qquad C_{n_p} = \frac{\partial C_n}{\partial \frac{pb}{2V}} \tag{1.7}$$

$$C_{Y_r} = \frac{\partial C_Y}{\partial \frac{rb}{2V}} \qquad C_{l_r} = \frac{\partial C_l}{\partial \frac{rb}{2V}} \qquad C_{n_r} = \frac{\partial C_n}{\partial \frac{rb}{2V}} \tag{1.8}$$

$$\tag{1.9}$$

Note that although we differentiate with respect to $\frac{pb}{2V}$ and $\frac{rb}{2V}$ respectively, we still only include the p and r int he subscripts. With these derivatives in mind, we are allowed to write, for small angles:

NON-DIMENSIONAL ASYMMETRIC FORCES AND MOMENTS

The non-dimensional asymmetric aerodynamic forces and moments caused by the lateral motions are

$$C_Y = C_{Y_\beta}\beta + C_{Y_p}\frac{pb}{2V} + C_{Y_r}\frac{rb}{2V} \tag{1.10}$$

$$C_l = C_{l_\beta}\beta + C_{l_p}\frac{pb}{2V} + C_{l_r}\frac{rb}{2V} \tag{1.11}$$

$$C_n = C_{n_\beta}\beta + C_{n_{\dot\beta}}\frac{\dot\beta b}{V} + C_{n_p}\frac{pb}{2V} + C_{n_r}\frac{rb}{2V} \tag{1.12}$$

In case you're freaking out right now: it's just writing it similar to

$$C_m = C_{m_0} + C_{m_\alpha}\alpha + C_{m_\delta}\delta$$

which you more or less saw in the previous part of the summary. We just linearise the shit out of everything, meaning we can write it in these forms. Nothing magic about it, honestly. It's similar to $C_L = C_{L_\alpha}\alpha$ but then with more partial derivatives involved. Nothing special at all. Two things though:

- We multiply the partial derivatives with subscript p and r by $pb/(2V)$ and $rb/(2V)$. After all, those derivatives were differentiated with respect to $pb/(2V)$ and $rb/(2V)$, so that's why you now have to multiply with them again (the inclusion of just the p and r in the subscripts is a bit deceitful; it should have included $pb/(2V)$ and $rb/(2V)$ but that'd have been more work to write down in the subscripts).
- The term $C_{n_{\dot\beta}}\frac{\dot\beta b}{V}$ is a bit weird, but fortunately it's influence can be neglected so we won't discuss it in this chapter.

All we're gonna do in this chapter is make a qualitative analysis of equations (1.10)-(1.12): we'll discuss the magnitude and signs of each of the derivatives listed in equations (1.6)-(1.8), but nothing more than that. No big-ass derivations like we did in part I. There are some equations in there but those are generally important. The only thing you really need to remember is summarised in blue boxes at the end of each section; the main text is just to provide explanations why that is. So just read through the main text once, and then remember the stuff in the blue boxes.

1.2 Derivatives with respect to sideslip angle β

In this section we'll discuss the derivatives in the first 'column' of equations (1.10)-(1.12), the ones that are with respect to β.

1.2.1 Stability derivative C_{Y_β}

If you have a side-slipping plane, as shown in figure 1.4, the air will push it to the left, which is the negative Y_B-direction. Thus, it is no surprise that C_{Y_β} is negative: for increasingly β, C_Y becomes increasingly negative (as the resultant force points in negative Y_B direction), so there's a negative relation between the two.

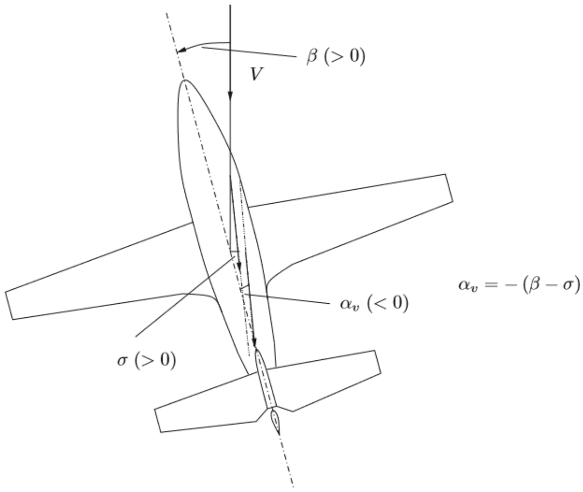


Figure 1.4: Relation between the sideslip angle β, the sidewash angle σ and the angle of attack α_v at the vertical tailplane.

The single biggest contributor to C_{Y_β} is the vertical tailplane. The reason for this is depicted in figure 1.4. The vertical tailplane is essentially a vertical airfoil. As a result of the positive sideslip, it'll experience a *negative* angle of attack, creating a 'lift' force that points in negative Y_B-direction[2]. Note that similar to downwash ϵ for the horizontal taiplane, we have sidewash β for the vertical tailplane. Indeed, the angle of attack at the vertical tailplane is given by

$$\alpha_v = -(\beta - \sigma) \tag{1.13}$$

After all, for increasingly positive β, α_v becomes increasingly negative, and for increasingly positive σ, α_v becomes less negative (so there's a positive relation between σ and α_h). Just as for the normal force of the horizontal tailplane, where we had

$$\left(C_{N_\alpha}\right)_h = C_{N_{h_\alpha}} \frac{d\alpha_h}{d\alpha} \left(\frac{V_h}{V}\right)^2 \frac{S_h}{S}$$

we now have

$$\left(C_{Y_\beta}\right)_v = C_{Y_{v_\alpha}} \frac{d\alpha_v}{d\beta} \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S}$$

Using equation (1.13), we obtain

$$\frac{d\alpha_v}{d\beta} = -\left(1 - \frac{d\sigma}{d\beta}\right)$$

so that

$$\left(C_{Y_\beta}\right)_v = -C_{Y_{v_\alpha}} \left(1 - \frac{d\sigma}{d\beta}\right) \left(\frac{V_v}{V}\right)^2 \frac{S_v}{S} \tag{1.14}$$

Later on, we'll discuss $d\sigma/d\beta$ in more detail. However, it'll never be larger than 1, so we see that indeed $(C_{Y_\beta})_v < 0$.

So, to summarize, C_{Y_β} is mostly determined by the vertical tailplane, and is negative in sign.

[2] After all, a positive angle of attack is defined such that it produces a lift force in positive Y_B-direction.

1.2.2 Stability derivative C_{l_β}

Now, what happens to our rolling moment when we increase the sideslip angle? Well, there are three main factors in play: the dihedral of the wing (how much the wing is curved upwards when looking from the front), the sweep of the wing, and whether it is a low-wing or high-wing configuration.

Let's first think, would we want C_{l_β} to be negative or positive? Well, consider this: suppose, due to some disturbance, our aircraft gets an angle of roll to the right (so in the positive direction denoted in figure 1.1). As a result, the Y_b-axis now points slightly downward, meaning that gravity will have a component along this axis, towards the right: as a result, it starts sideslipping to the right, so β will automatically increase. If we'd then have C_{l_β} be negative, this means that the rolling moment would become negative! This would automatically counteract the original positive rolling moment, without the pilot having to do anything. So, we prefer C_{l_β} to be negative.

Let's now first analyse the effect of the wing dihedral, as this one is the hardest of the three mentioned factors to analyse. Suppose β increases suddenly; this causes an increase in velocity component v (the one aligned with the Y_B-axis); see part a) of figure 1.5. The horizontal component u, aligned with the X_B-axis, will remain the same however. Now, look at figure part c) of figure 1.5: as we'll see, due to the dihedral, the change in v will cause a change in velocity components normal to the wings, V_{n_l} for the left wing and V_{n_r} for the right wing, respectively. As a result, the angles of attack will change, causing a difference in lift produced on both sides, and as a result, a rolling motion.

Now, what is the change in normal components on the wings? Well, for this, consider part b) of figure 1.5: we look at the wing in the $Y_B - Z_B$ plane (and we take the plane from behind); the w- and v-components of the velocity are drawn. The u-component points out of the page, so that one does not matter to determine the velocity component normal to the wing's surfaces. Now, let's analyse the right-wing: the velocity component of w that's perpendicular to the wing surface and aligned with V_{n_r} there is equal to $w \cos\Gamma$; the component of v that's perpendicular to the wing surface and aligned with V_{n_r} is equal to $v \sin\Gamma$. In other words,

$$V_{n_r} = w\cos\Gamma + v\sin\Gamma = w + v\Gamma \tag{1.15}$$

for small dihedral angles. Similarly, for the left wing, the component of w that's aligned with V_{n_l} is equal to $w\cos\Gamma$; however, the component of v that's aligned with V_{n_l} is equal to $-v\sin\Gamma$: after all, this components points in opposite direction of V_{n_l}. Thus, we obtain

$$V_{n_l} = w\cos\Gamma - v\sin\Gamma = w - v\Gamma \tag{1.16}$$

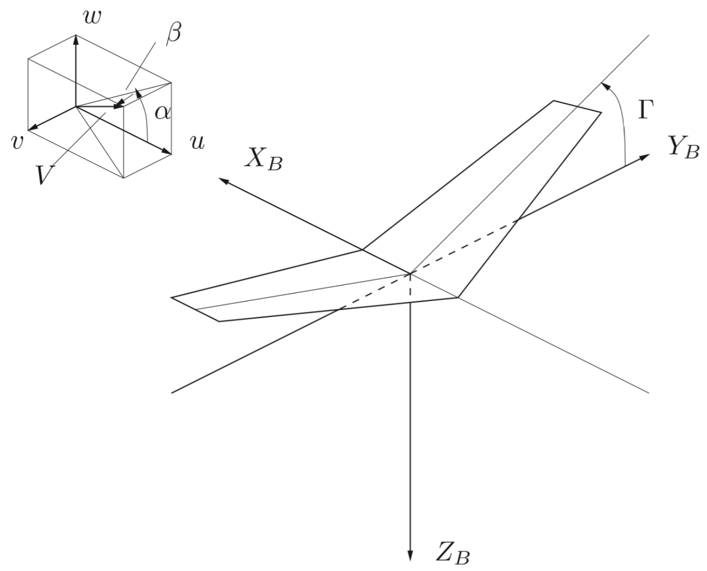
Furthermore, $w = V\alpha$ and $v = V\beta$, so we obtain from equations (1.15) and (1.16):

$$\begin{aligned} V_{n_r} &= V\alpha + V\beta\Gamma = V(\alpha + \beta\Gamma) \\ V_{n_l} &= V\alpha - V\beta\Gamma = V(\alpha - \beta\Gamma) \end{aligned}$$

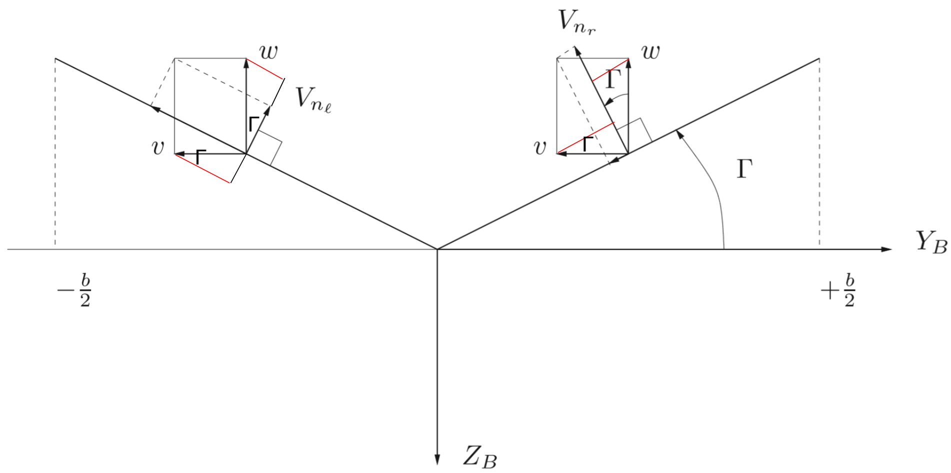
Thus, relatively, the left wing experiences a reduces angle of attack due to the positive sideslip, whereas the right-wing experiences an increased angle of attack (as the normal component of the velocity is higher, see part c) of figure 1.5). Indeed, the right wing will produce more lift and the left wing less lift: as a result, a *negative* rolling moment is created, as this is in opposite direction of positive rotation (just use your right-hand rule for axis X_B in part a) of figure 1.5). In other words, for positive dihedral, C_{l_β} is *negative*: the more positive β, the more negative C_l becomes.

Note that if you'd have negative dihedral, C_{l_β} would be positive.

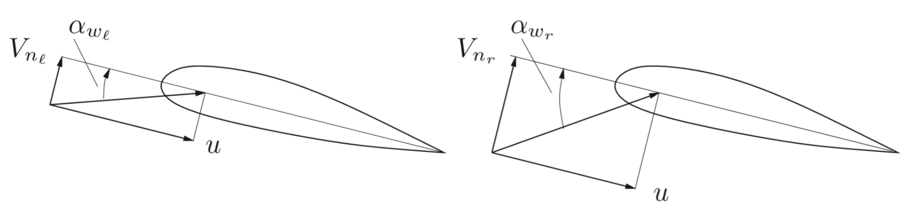
Now, let's go onto sweep. A similar story holds there: consider figure 1.6: if there is some sweep Λ and some positive sideslip β, then the right wing will merely experience an effective sweep of $\Lambda - \beta$, whereas the left wing will experience an effective sweep of $\Lambda + \beta$. As a result, the lift over the right wing increases, but decreases over the left. The difference in lift will equal (okay I don't know why we're doing this derivation you may as



(A) Wing with dihedral in sideslipping flight



(B) The normal velocities V_{n_ℓ} and V_{n_r} at two sides of the wing



(C) The angles of attack at the left and right wing

Figure 1.5: The origin of the difference in angles of attack at the left and right wing for a wing with dihedral in sideslipping flight.

well skip it):

$$\begin{aligned}
\Delta L &= C_L \frac{1}{2}\rho \left(V\cos\left(\Lambda-\beta\right)\right)^2 \frac{S}{2} - C_L \frac{1}{2}\rho \left(V\cos\left(\Lambda+\beta\right)\right)^2 \frac{S}{2} \\
&= C_L \frac{1}{2}\rho V^2 \frac{S}{2}\left[\cos^2\left(\Lambda-\beta\right)-\cos^2\left(\Lambda+\beta\right)\right] = C_L \frac{1}{2}\rho V^2 \frac{S}{2}\left\{\frac{1}{2}+\cos\left[2\left(\Lambda-\beta\right)\right]-\frac{1}{2}-\frac{1}{2}\cos\left[2\left(\Lambda+\beta\right)\right]\right\} \\
&= C_L \frac{1}{2}\rho V^2 \frac{S}{2}\left[\frac{1}{2}\cos\left(2\Lambda\right)\cos\left(2\beta\right)-\frac{1}{2}\sin\left(2\Lambda\right)\sin\left(2\beta\right)-\frac{1}{2}\cos\left(2\Lambda\right)\cos\left(2\beta\right)+\frac{1}{2}\sin\left(2\Lambda\right)\sin\left(2\beta\right)\right] \\
&= C_L \frac{1}{2}\rho V^2 \frac{S}{2}\left[\sin\left(2\Lambda\right)\sin\left(2\beta\right)\right] = C_L \frac{1}{2}\rho V^2 S \sin\left(2\Lambda\right)\beta
\end{aligned}$$

As the right wing generates more lift than the left, a negative rolling motion is generated, meaning that for positive sweep, C_{l_β} is negative.

The effects of dihedral and sweep are first order accurate additive (that means, as a first order approximation, you can simply add them and you don't have to do analyse the interference between them). As a result, we often speak about **effective dihedral** of a wing, which combines the two.

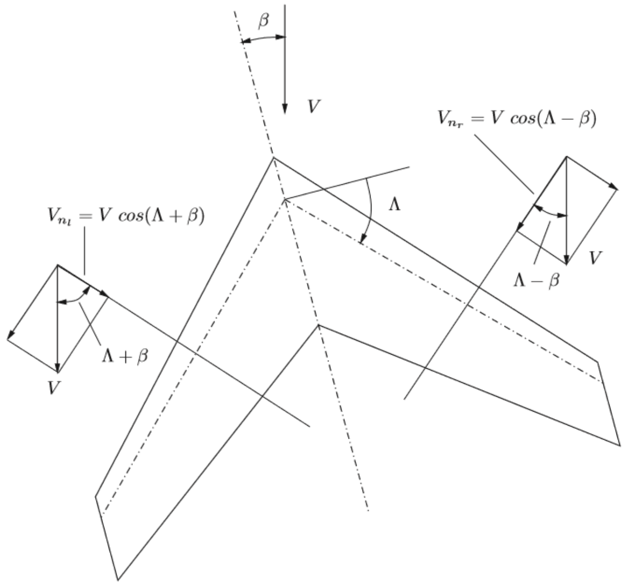


Figure 1.6: Origin of a rolling moment caused by wing-fuselage interactions in sideslipping flight.

Finally, the interaction between the fuselage and wing also has an effect on the rolling motion under influence of sideslip. If there is some positive sideslip β, we get a velocity profile over the fuselage as shown in figure 1.7: on top a high-wing configuration is depicted, below a low-wing configuration is shown.

We see that for the high-wing configuration, for positive β, the flow is pushed up underneath right wing, and goes down underneath the left wing. As a result, the angle of attack increases on the right wing (especially near the fuselage), and decreases for the left wing (especially near the fuselage), as shown in the c_l-distribution over the wing on the right. As a result, a negative rolling moment is created, meaning that C_{l_β} is negative as there exists a negative relation between β and C_l.

On the other hand for the low-wing configuration, the right wing will experience a lower angle of attack whereas the left wing will experience a larger angle of attack. As a result, a positive rolling moment is generated: C_{l_β} is positive.

In short, this section can be summarised as:

1. A negative C_{l_β} is desired.
2. Positive dihedral causes a negative C_{l_β}.
3. Positive sweep causes a negative C_{l_β}.
4. High-wing configuration causes negative C_{l_β}; low-wing configuration causes positive C_{l_β}.

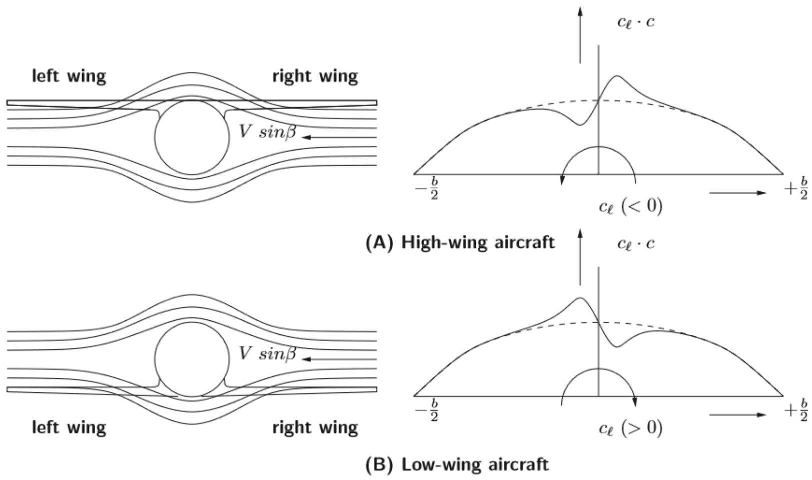


Figure 1.7: Origin of a rolling moment caused by wing-fuselage interactions in sideslipping flight. Note that we are taking the aircraft from behind.

1.2.3 Stability derivative C_{n_β}

Now, what happens to the yawing rate if β is increased a bit? Well, first and foremost, the vertical tailplane plays an important role here. Remember that as I discussed for C_{Y_β}, the vertical tailplane will produce a force in negative Y_B direction. This causes a yawing moment around the center of gravity, in positive direction (just apply your right-hand rule to the Z_B axis shown in figure 1.1). Indeed, C_{n_β} is positive: for positive β, a positive yawing moment is created. This is a desired result: it means that the aircraft will automatically turn to whatever direction the velocity was coming from, eliminating the sideslip. It's essentially like a weathervane then; C_{l_β} is fittingly called the **weathervane stability**. If your aircraft doesn't have positive C_{n_β}, it becomes very hard to fly.

It should be noted that the moment arm between the force created by the vertical tail is not super straightforward. To see this, consider figure 1.8b, which is a schematic sketch of figure 1.8a. The *horizontal* distance (the one aligned with $X_{B,s}$) is the distance we're interested in. We see that for large angles of attack, this is not simply $x_v - x_{c.g.}$ any more: rather, we must use the geometry of figure 1.8b to find the moment arm; this equals from simple geometry

$$r = \left(x_v - x_{c.g.}\right)\cos\alpha + \left(z_v - z_{c.g.}\right)\sin\alpha$$

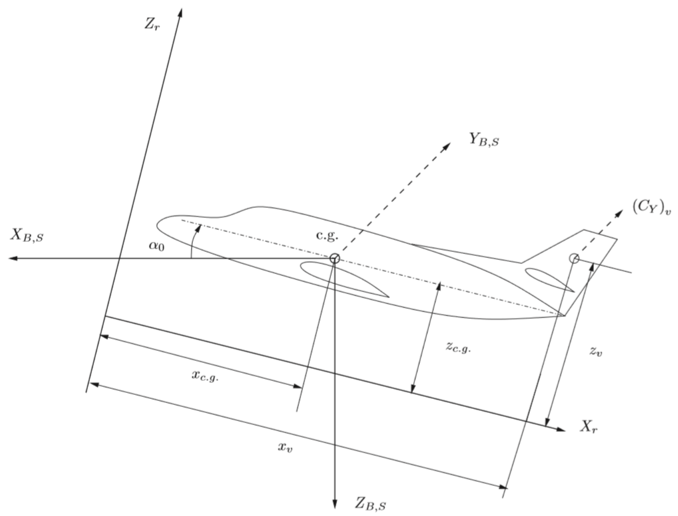
However, for small angles of attack we'll simply say $r \approx x_v - x_{c.g.} = l_v$. The moment created by C_{Y_β} is thus simply (using equation (1.14))

$$\left(C_{n_\beta}\right)_v = C_{Y_{v_\alpha}}\left(1 - \frac{d\sigma}{d\beta}\right)\left(\frac{V_v}{V}\right)^2 \frac{S_v l_v}{Sb}$$

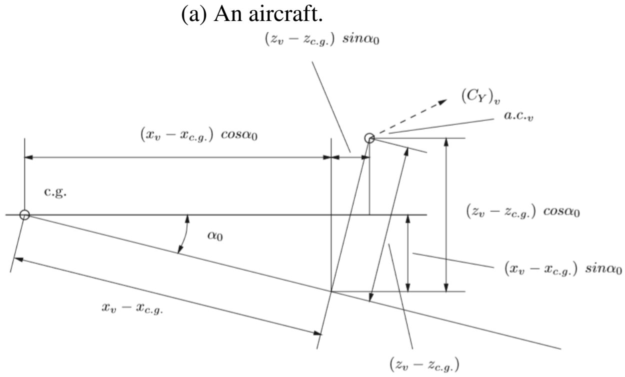
which is very similar to the equation for $(C_{m_\alpha})_h$ you saw in part I of the summary.

Now, let's focus on $d\sigma/d\beta$. Generally speaking this is a parameter that's very hard to determine. However, we can at least investigate the sign of it. Suppose we have a low-wing aircraft. Suppose there's some sideslip β. As depicted in figure 1.7, this causes more lift to be produced on the left wing than on the right wing. This means that the pressure is lower on the left side of the fuselage than on the right side of it. This causes circulation over the fuselage; for the part above the wing this will be in counterclockwise direction. As a result, the velocity component normal to the vertical tail increases (see figure 1.9), increasing the angle of attack there, having a stabilising effect as the normal force produced by the vertical tail will obviously increase. Thus, $d\sigma/d\beta$ must evidently be negative.

For high-wing aircraft the story is the other way around: there the circulation is the other way around, thus having a destabilising effect. $d\sigma/d\beta$ is then positive.



(a) An aircraft.



(b) Geometry of the aircraft.

Figure 1.8: The position of action of $(C_{Y_v})_v$ relative to the X- and Z-axis in the stability reference frame.

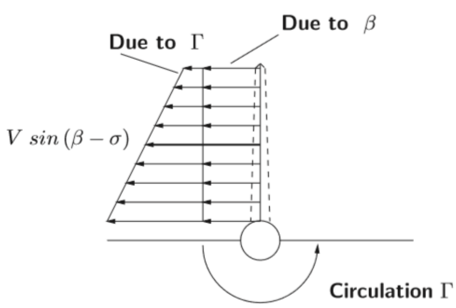


Figure 1.9: The change in sidewash at the vertical tailplane due to wing-fuselage interactions in a low-wing aircraft in sideslipping flight.

Furthermore, the fuselage has some influence as well: remember that the aircraft pivots around its center of gravity. Now, looking at figure 1.4, you can imagine that if the center of gravity is very far back (as is the case when the engines are in the back), there's a large part of the fuselage in front of it (say 70% of the fuselage). As a result, the air pushes the fuselage to rotate around its center of gravity, in *negative* direction, as β becomes larger and larger[3]. Thus, if your center of gravity is too far aft, you get a *negative* C_{n_β} from your fuselage (which is undesirable).

Finally, if you increase your wing sweep, you place more of the aerodynamic forces, including the lateral ones that are present due to sweep, behind the center of gravity. Consequentially, it produces a positive yawing moment, making C_{n_β} more positive.

In short, this section can be summarised as:

1. A positive C_{n_β} is desirable.
2. The vertical tailplane is responsible for a negative C_{n_β} due to the moment arm of the normal force on the vertical tailplane.
3. If it's a low-wing configuration, $d\sigma/d\beta$ is negative; if it's high-wing, $d\sigma/d\beta$ is positive.
4. The fuselage itself has a negative C_{n_β}, especially when the center of gravity is in the back.
5. Increasing wing sweep makes C_{n_β} more positive, as it puts the lateral forces behind the center of gravity, introducing a positive yawing moment.

1.2.4 Section summary

In short, you have to remember the following:

SIDE-SLIP STABILITY DERIVATIVES

- C_{Y_β} is mostly determined by the vertical tailplane, and is negative in sign.
- With regards to C_{l_β}:
  - A negative C_{l_β} is desired.
  - Positive dihedral causes a negative C_{l_β}.
  - Positive sweep causes a negative C_{l_β}.
  - High-wing configuration causes negative C_{l_β}; low-wing configuration causes positive C_{l_β}.
- With regards to C_{n_β}:
  - A positive C_{n_β} is desirable.
  - The vertical tailplane is responsible for a negative C_{n_β} due to the moment arm of the normal force on the vertical tailplane.
  - If it's a low-wing configuration, $d\sigma/d\beta$ is negative; if it's high-wing, $d\sigma/d\beta$ is positive.
  - The fuselage itself has a negative C_{n_β}, especially when the center of gravity is in the back.
  - Increasing wing sweep makes C_{n_β} more positive, as it puts the lateral forces behind the center of gravity, introducing a positive yawing moment.

Multiple-choice questions: part (2)

Decide for every statement whether it is True or False. No explanation is required!

1. The sign of C_{Y_β} for a conventional aircraft is negative and the dominant contribution to this stability derivative is made by the main wing.
2. The contribution of the fuselage to C_{n_β} is destabilizing.
3. Both wing-sweep angle and dihedral affect the stability derivative C_{l_β}.
4. The forward pointing half wing of a high wing aircraft experiences a decrease in angle of attack due to sideslip.

Correct answers are:

[3] If you don't get what I mean: just imagine the center of gravity would be all the way at the back. Then the flow will push the aircraft to rotate around this point in counterclockwise direction. It's literally pushing the aircraft to rotate that way as there's partial frontal impact (that increases as β becomes larger and larger).

1. This is false; it's mainly caused by the vertical tailplane. This statement is **false**.
2. The fuselage causes a positive C_{n_β}, but a negative value is desired. This statement is **true**.
3. Yeah, this statement is **true**.
4. The forward-pointing half wing is the side from which the sideslip velocity component originates from. As seen in figure 1.7, this means that the angle of attack is increased here, not decreased. This statement is **false**.

1.3 Stability derivatives with respect to roll rate

Now let's consider the derivatives in equations (1.8)-(1.14) in the second column, the one with respect to p. For this, consider figure 1.10 once more.

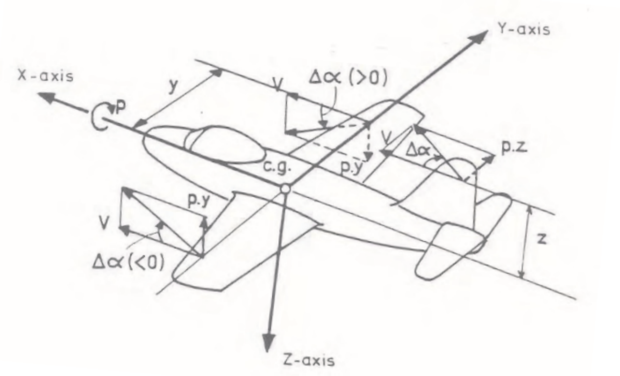


Figure 1.10: An airplane with inscribed body axis system.

1.3.1 Stability derivative C_{Y_p}

First, I should note that C_{Y_p} is generally speaking very small and thus usually ignored. The only contributing factor to it is the vertical tailplane. If an aircraft starts rolling with a rolling velocity $pb/(2V)$, the tail of the aircraft is pushed into the flow. As a result, it experiences an angle of attack; it's as if the vertical tail experiences some sideslip. As a result, a normal force in *negative* Y-direction is produced for positive $pb/(2V)$, thus C_{Y_p} is negative.

1.3.2 Stability derivative C_{l_p}

This one is more important. However, you've already seen it in ADSEE II (the 2016-2017 edition at least), for that amazing aircraft assignment, where you had to design an aileron which was really a lot of fun. Note that if your aircraft is rolling, then the angle of attack is increased at parts where the wing is going down (as it moves into the flow), and decreased where the wing is going up (as it moves out of the flow); see figure 1.11. In fact, at a position y along the wing, the change in geometric angle of attack, when under a roll rate p is given by[4]

$$\Delta\alpha = \frac{py}{V}$$

Maybe now it's also clear why we normalise p by multiplying by $b/(2V)$: we like to write above equation as

$$\Delta\alpha = \frac{pb}{2V}\frac{y}{b/2}$$

[4] After all, if the radial velocity is p, then the vertical velocity component will be $w = py$. The angle of attack is simply $\arctan(w/V) \approx w/V$.

because y-varies from $-b/2$ to $b/2$ (it's similar to how we prefer to write $x/\bar{c}$ for longitudinal stability). We see the term $pb/2V$ appearing now, which is dimensionless.

In any case, the side going up experiences a decrease in angle of attack, whereas the side going down experiences an increase in angle of attack. As a result, the left side in figure 1.11 experiences a smaller lift, and the right side a larger lift: this creates a rolling motion in opposite direction of p. Thus, C_{l_p} is negative: this is called **roll damping** the rolling itself damps itself.

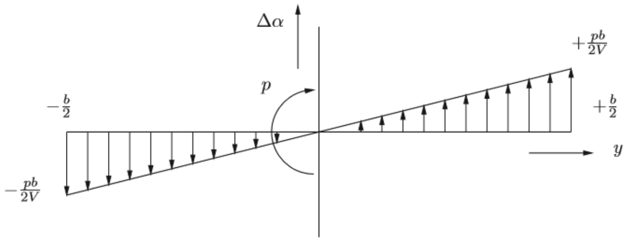


Figure 1.11: The variation of the local geometric angle of attack alopng the span of a rolling wing.

This isn't too hard imo, although the concept of roll damping is important obviously (but this is literally all there is to it: one side experiences a larger α, the other a smaller).

There is one additional thing influencing C_{l_p}: remember that the vertical tailplane produced a force in negative Y_b direction (that was literally the point of the previous subsection). Looking at figure 1.11, we see that this induces a further negative rolling moment, thus this C_{l_p} is negative as well.

1.3.3 Stability derivative C_{n_p}

This one is quite logical at first, but then gets completely counter-intuitive. We want to know what happens to the yawing moment if we're subject to some rolling rate. Well, just like before, we have that the downgoing will experience a larger angle of attack, and thus produce more lift, and also more drag! Similarly, the upgoing wing will produce less lift and drag. Now, you may look at figure 1.10 and think, do the right wing will produce more drag, so that one will move backward, causing *positive* moment around the Z_b-axis (just use your right-hand rule to determine that it's positive). However, it's really deceptive: the down-going wing will actually move *for*ward. Why? Well, the definition of lift and drag is that they are perpendicular and parallel to the *velocity-vector*, respectively. If there's an increased angle of attack, the velocity vector starts to point slightly downward, as shown in figure 1.10[5]. The lift vector will act perpendicular to this, and thus actually have a component in positive x-direction! Yes, the drag points mostly in negative x-direction, but the lift-component more than makes up for this. On the other hand, for the down-going wing, the velocity vector will point a bit upwards. As a result, the lift will point slightly to the right, as it's perpendicular to this. As a result, the up-going wing experiences a larger force backwards. As a result, you get a *negative* moment around the Z_b-axis. Thus, C_{n_p} is negative: for positive roll rates, you get positive yawing moments.

There's another way of explaining the seemingly counterintuitiveness of the sign of C_{n_p}: remember the very second chapter of part I: amongst others, I showed there that the tangential force actually becomes negative with increasing α, as shown in figure 1.12: evidently, the up-going wing will experience a positive tangential force, pulling it forward and creating a *negative* moment around the Z_b-axis.

Furthermore, sweep also plays a role in C_{n_p}. After all, consider figure 1.13. Both sides of the wing will have a tangential force component, one that is perpendicular to the quarter-chord sweep. As I explained, the down-going wing, with its larger angle of attack, will have a tangential force component pointing forward (as follows from the negative tangential force component for high α depicted in figure 1.12). The up-going wing will still have a tangential force component forcing backward. As a result, the ΔC_Y of both wings will be in the same direction. Depending on the location of the c.g., this will cause another yawing moment around the c.g.

[5] If you're like, shouldn't velocity point in the opposite direction though, as if it'd comes from the front? Note that the velocity we drew is the velocity of the aircraft with the air standing still (whereas normally draw it with the aircraft standing still and the air moving relative to the aircraft). But whether you draw it in positive X_b-direction or negative, it doesn't matter for this concept.

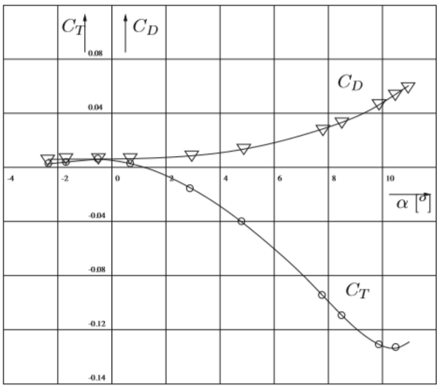


Figure 1.12: C_T and C_D as functions of α for some aircraft.

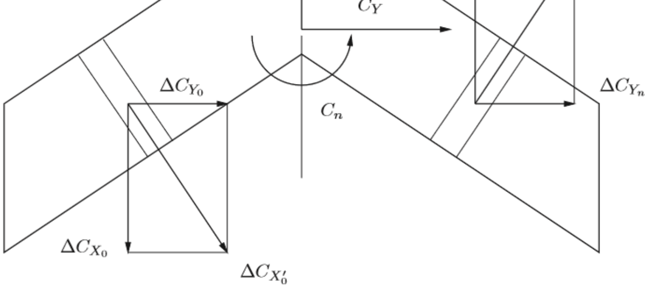


Figure 1.13: The side force and yawing moment on a rolling, swept back wing.

Finally, the vertical tailplane produces a positive contribution to C_{l_p}. Due to the lateral force it creates in negative Y_B-direction, it creates a positive yawing moment around the c.g. (just use your righthand-rule around the Z_B-axis).

One thing that may be important to realise: the fact that $C_{n_p} < 0$ (in combination with the discussion for C_{l_p}) is the reason for Dutch roll: when the aircraft starts rolling, the down-going side will move forward as well, and vice versa.

1.3.4 Section summary

In short, you have to remember the following:

ROLLING RATE STABILITY DERIVATIVES

- C_{Y_p} is very small, primarily caused by the vertical tailplane and negative.
- With regards to C_{l_p}:
  - C_{l_p} is mainly caused by the main wing, which causes a negative C_{l_p} (roll damping).
  - The vertical tailplane induces an additional negative C_{l_p}.
- With regards to C_{n_p}:
  - C_{n_p} is negative, as the down-going wing will experience more lift and drag, and since the velocity vector is inclined downward, the lift will now have a component in positive x-direction which offsets the increased drag. As a result, a negative yawing moment is created.
  - Depending on the sweep and the location of the center of gravity, the sweep of the wing may also contribute to C_{n_p}.
  - The vertical tail produces a positive contribution C_{n_p}.

Multiple-choice questions: part (3)

Decide for every statement whether it is True or False. No explanation is required!

1. The wing of a conventional aircraft has a negative contribution to C_{l_p}.
2. The stability derivative C_{n_p} is determined mainly by the contribution of the wing
3. Positive rolling motion ($p > 0$) causes an increase of the effective angle of attack on the right wing half.

Correct answers are:

1. Yeah this is roll damping, this statement is **true**.
2. Yeah, this statement is **true**.
3. Yeap, this is also **true**.

1.4 Stability derivatives with respect to yaw rate

Now, finally, the derivatives of equations (1.10)-(1.12) with respect to the yaw rate r. For this, we'll introduce the r-motion. This is nothing more than an idealised turn with a turn radius R, being performed with a constant angular velocity r (the yaw rate). See also figure 1.14; it's literally nothing more than that, but we put a fancy name on it just because we can. Note that the angular velocity of the airplane around the midpoint of the circle is the same as the angular velocity (i.e. the roll rate) of the airplane around its own Z_B-axis: after all, when the airplane will have traversed a full circle around the midpoint, it'll also have rotated a full 360° around its Z_B-axis.

There are two important implications from this r-motion. First, note that there is slideslip over the aircraft, but that it changes over the distance of the fuselage: at the center of gravity, there is no sideslip at all as the velocity is aligned with the fuselage. However, at the front of the aircraft, there is a bit of positive sideslip, and near the tail there is negative sideslip. Secondly, the velocity varies along the wing, as shown in figure 1.15: farther away from the origin the tangential velocity is higher.

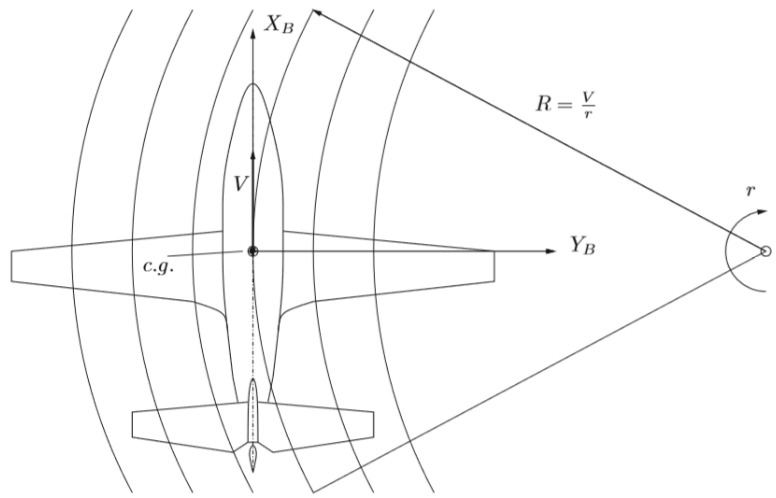


Figure 1.14: The ideal r-motion.

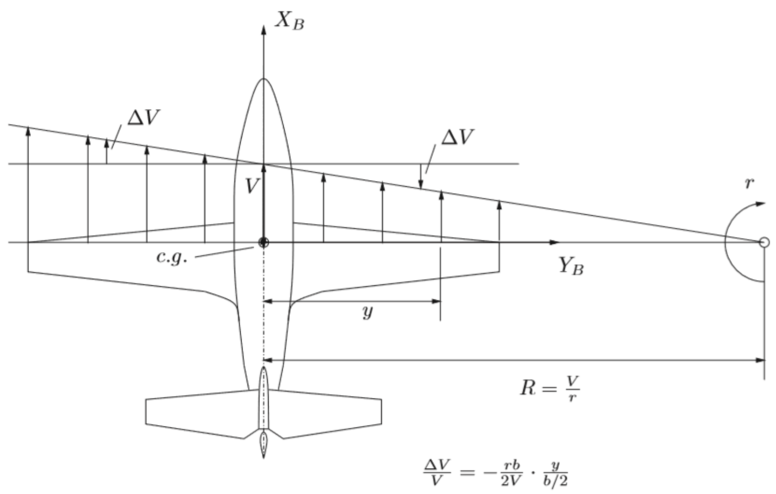


Figure 1.15: The variation in airspeed in spanwise direction due to an r-motion.

These things are the only two things that matter basically.

1.4.1 Stability derivative C_{Y_r}

Again, this derivative is usually negligibly small so we don't care a lot about it. However, if you do care, it's positive for the following reason: the sidewash at the tail causes a positive angle of attack for the vertical tail, causing a positive force in Y_B-direction, as shown in figure 1.16. So, C_{Y_r} is positive.

1.4.2 Stability derivative C_{l_r}

Now, how is the roll l affected by the yaw rate r? Well, it's pretty simple. The left wing in figure 1.15 will experience a higher velocity, thus produce more lift, thus it'll start create a positive rolling moment around the X_B axis (just use your right-hand rule to determine it's positive). Additionally, although it's less significant, the vertical tailplane also causes a positive rolling moment: remember it creates a force that points in positive

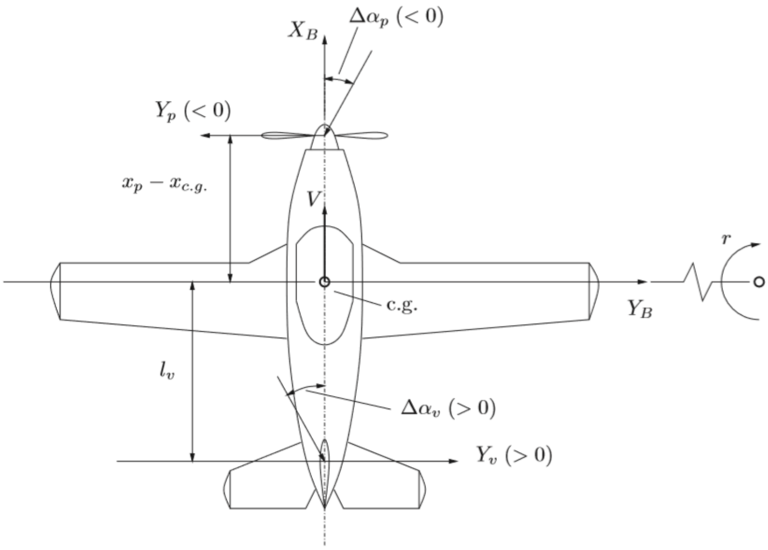


Figure 1.16: The side forces on the vertical tailplane and the propeller due to an r-motion.

Y_B direction; usually it's located a small distance above the center of gravity of the fuselage. Consequently, a positive rolling moment is created. So, in short, C_{l_r} is positive.

1.4.3 Stability derivative C_{n_r}

Once more, the force that the vertical tailplane produces in positive Y_B direction has a very large arm (equal to $x_v - x_{c.g.}$) and thus produces a negative yawing moment around the c.g. (remember that Z_B points into the paper, and then use your right-hand rule). The fuselage also adds negatively to C_{n_r}.

1.4.4 Section summary

Well this was short, let's summarise anyway cause coloured boxes look nice.

YAW RATE STABILITY DERIVATIVES

- C_{Y_r} is very small, primarily caused by the vertical tailplane and positive.
- With regards to C_{l_r}:
  - C_{l_r} is mainly caused by the main wing, which causes a positive C_{l_r} due to velocity differential over the wing span.
  - The vertical tailplane induces an additional positive C_{l_r}.
- C_{n_r} is mainly caused by the vertical tailplane, which cause a negative C_{n_r}. The fuselage also adds negatively to C_{n_r}, though.

Multiple-choice questions: part (4)

Decide for every statement whether it is True or False. No explanation is required!

1. Increasing the surface of the vertical tail S_v increases the magnitude of C_{n_r}.

Correct answers are:

1. Yeah it does, this statement is **true**.

2 Forces and moments due to aileron and rudder deflections

In this chapter we'll see the forces and moments created by ailerons and rudders. Ailerons are the things attached to the wing tips that can go up and down to primarily induce roll; rudders are the thing at the end of the vertical tail to induce yaw. They are graphically depicted, along with their positive directions, in figure 2.1. Note that for the ailerons, you always deflect one of them downward and the other upwards; this is why a downward deflection of the *right* aileron is defined as being positive. The deflection of the aileron is denoted by δ_a and is the *difference* between the deflection angles of the left and right ailerons (thus, if you deflect both of them, say, 30°, the deflection of the aileron is $\delta_a = 60°$). A deflection of the rudder is denoted by δ_r.

Now, deflecting the aileron will cause changes in the force Y, and the rolling moment L and yawing moment N. Consequently, just like equations (1.6)-(1.8), we have the partial derivatives

$$C_{Y_{\delta_a}} = \frac{\partial C_Y}{\partial \delta_a} \qquad C_{l_{\delta_a}} = \frac{\partial C_l}{\partial \delta_a} \qquad C_{n_{\delta_a}} = \frac{\partial C_n}{\partial \delta_a}$$

Similarly, for the rudder we have the partial derivatives

$$C_{Y_{\delta_r}} = \frac{\partial C_Y}{\partial \delta_r} \qquad C_{l_{\delta_r}} = \frac{\partial C_l}{\partial \delta_r} \qquad C_{n_{\delta_r}} = \frac{\partial C_n}{\partial \delta_r}$$

In the next two sections, we'll again analyse the signs of these derivatives without going too in-depth. The blue boxes are all you have to remember, really.

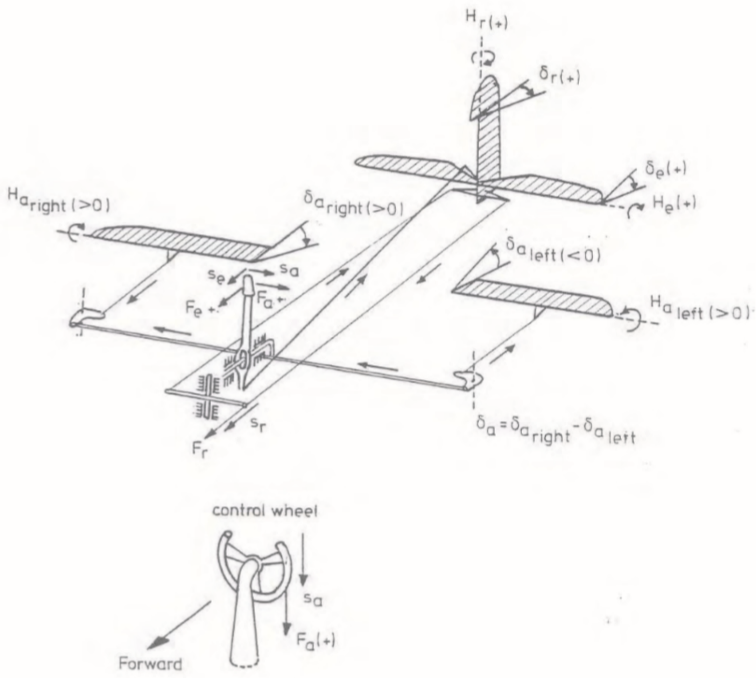


Figure 2.1: The positive direction of control deflections, contyrol forces, control surface deflections and hinge moments.

2.1 Aileron control derivatives

2.1.1 Control derivative $C_{Y_{\delta_a}}$

If you deflect your ailerons, this will not cause a lateral force. So, $C_{Y_{\delta_a}} = 0$. NEXT.

2.1.2 Control derivative $C_{l_{\delta_a}}$

What happens to your roll moment if you deflect your ailerons? Well, if you deflect your right aileron downward and the left upward, the right wing will produce more lift and the left wing less. As a result, you get a *negative* rolling moment (just use your right-hand rule about the X_B axis), which points in forward direction when looking at figure 2.1. The derivative $C_{l_{\delta_a}}$ (which is thus negative) is called the **aileron effectiveness**. NEXT.

2.1.3 Control derivative $C_{n_{\delta_a}}$

What happens then to your yawing moment if you deflect your ailerons? Well, as the lift over the right wing will increase, drag will increase there (and similarly, lift over left wing decreases so draw decreases there). Consequently, the right wing is pulled back relative to the left wing, thus a *positive* yawing moment is created (just use your right-hand rule for the Z_B-axis, which points downward)[1]. Thus, $C_{n_{\delta_a}}$ is positive. Note that this weird: you deflect your ailerons in positive direction to make a left-turn, but your yaw moment will initially point your nose to the right. This is therefore called **adverse yaw**. NEXT.

2.1.4 Section summary

AILERON CONTROL DERIVATIVES

- $C_{Y_{\delta_a}} = 0$.
- $C_{l_{\delta_a}}$ is negative; this derivative is called the aileron effectiveness.
- $C_{n_{\delta_a}}$ is positive; this is called adverse yaw.

2.2 Rudder control derivatives

2.2.1 Control derivative $C_{Y_{\delta_r}}$

If you deflect your rudder in positive direction, it'll experience an angle of attack and thus create a lateral force in positive Y_B-direction, as shown in figure 2.2. NEXT.

2.2.2 Control derivative $C_{l_{\delta_r}}$

Once more, the lateral force of the rudder will create a rolling moment due its vertical distance (in Z_B-direction) with the center of gravity. This rolling moment is positive (just use your right-hand rule along the X_B-axis), meaning that $C_{l_{\delta_r}}$ is positive. NEXT.

[1] In case you're wondering, but why is it then not the other way around, like it was for C_{n_p}? Well, there the change in lift was because of the induced vertical velocity, meaning that the velocity vector became inclined which was the reason why it was so counter-intuitive. Here that is not the case: the velocity stays the same direction; the lift merely increases/decreases due to increased/decreased camber.

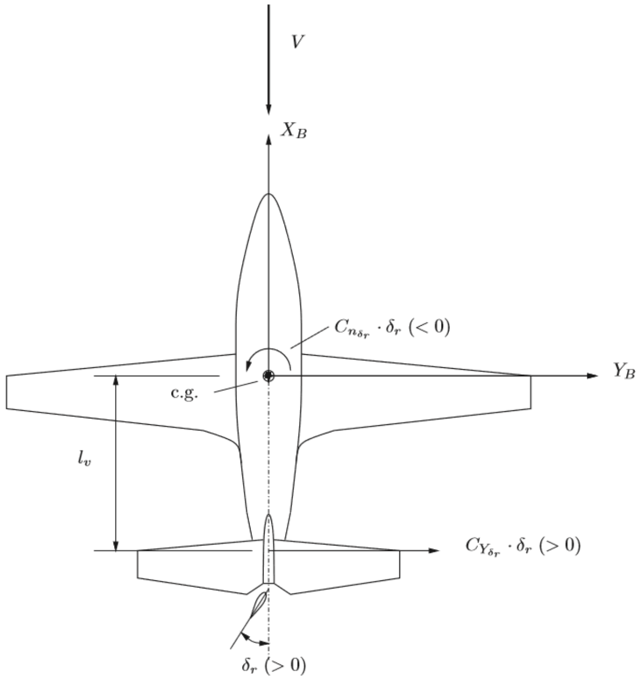


Figure 2.2: The side force and yawing moment due to a rudder deflection.

2.2.3 *Control derivative $C_{n_{\delta_r}}$*

The lateral force $C_{Y_{\delta_r}} \cdot \delta_r$, due to its moment arm l_v depicted in figure 2.2, creates a *negative* yawing moment around the c.g. (just use your right-hand rule along the Z_B-axis, which points into the paper). Thus, $C_{n_{\delta_r}}$ is negative. And we're finished with this chapter, that was quick.

2.2.4 *Section summary*

RUDDER CONTROL DERIVATIVES

- $C_{Y_{\delta_r}}$ is positive.
- $C_{l_{\delta_r}}$ is positive.
- $C_{n_{\delta_r}}$ is negative.

2.3 *Exam questions*

Exam August 2009: problem 3 (20p)

Consider a statically and dynamically stable aircraft with a straight, low wing configuration during sideslipping flight, ($\beta > 0$). The c.g. of the aircraft is located slightly behind the main wing. Provide your answers in the following table form:

Coefficient	Sign	Case 1	Case 2	Case 3

(a) What is the conventional sign of the following lateral stability and control derivatives: C_{l_β}, C_{n_β}, C_{l_p}, C_{n_p}, C_{l_r}, C_{n_r}, $C_{l_{\delta_a}}$, $C_{n_{\delta_a}}$? Give your answer using the following notation: > 0, < 0, $= 0$.

(b) What happens to these stability and control derivatives in the following cases:

1) The diameter of the fuselage is doubled. All other aircraft parameters are unaffected. Neglect

the extra drag on the fuselage.
2) The area of the vertical stabilizer is halved. All other aircraft parameters are unaffected.
3) The main wing is given a large positive sweep angle Λ. Assume also that the c.g. is located slightly in front of the main wing. All other aircraft parameters are unaffected.

Give your answer in terms of the following statements: more/less negative/positive or negligible (≈ 0).

(c) Briefly explain which of the above parameters you think is the most important to the flying qualities of the aircraft from case 2 during sideslipping flight.

(d) Briefly discuss 1 structural change that can be made to the aircraft from case 2 (other than increasing the surface of the vertical tail) to improve its flying qualities during sideslipping flight. Which of the above parameters are affected by your proposed structural change?

For a), it's important to just remember what each derivative represented and what its causes were. Let's just go through each of them:

- C_{l_β} is resultant rolling moment due to sideslip. This was mainly caused due to sweep/dihedral: for positive dihedral, the right-wing would generate more lift as it experienced an increased angle of attack, and for sweep something similar holds. Consequently, a *negative* rolling moment around the X_B-axis was created; thus, $C_{l_\beta} < 0$. Since the aircraft is low-wing, the fuselage causes C_{l_β} to be less negative due to wing-fuselage interaction.
- C_{n_β} is the resultant yawing moment due to sideslip. The vertical tailplane induces a positive C_{n_β} due to the moment arm of its normal force, so $C_{n_\beta} > 0$. The fuselage has a disturbing influence however (the fuselage itself has a negative C_{n_β}).
- C_{l_p} is the resultant rolling moment due to a rolling velocity. You should really know that this is negative, as the rolling velocity causes an increasing angle of attack for the down-going wing, causing an increase in lift over there, opposing the rolling motion. Furthermore, the vertical tail also produces a rolling moment in the opposite direction of the motion due to the vertical distance (in Z_B-direction) to the center of gravity, thus $C_{l_p} < 0$.
- C_{n_p} is the resultant yawing moment due to a rolling velocity. As the down-going wing experiences a larger angle of attack, it'll experience more drag; however, due to the fact that the velocity vector becomes inclined, the increased lift actually causes the down-going wing to be pulled *forward*. As a result, a negative yaw-moment is produced, meaning that C_{n_p} is negative. It should be noted that this is offset by the vertical tail, which, due to the horizontal distance (in X_B-direction) to the center of gravity, will cause C_{n_p} to be more positive.
- C_{l_r} is the resultant rolling moment due to yawing. When looking at r-motion, the left wing will experience a higher V than the right-wing, so the lift over the left wing will be higher. This causes a *positive* rolling moment, so C_{l_r} is positive. This is further amplified by the vertical tailplane, which creates a force in positive Y_B-direction due to the resulting sideslip from yawing, and due to the vertical distance between the center of gravity, the vertical tail produces a positive rolling moment.
- C_{n_r} is the resultant yawing moment due to yawing. The vertical tailplane produces a force in positive Y_B-direction, which creates a negative yawing moment around the c.g. The fuselage also contributes negatively to C_{n_r}.
- $C_{l_{\delta_a}}$ is the resultant rolling moment due to a positive aileron deflection. For a positive aileron deflection, you deflect the right one downward, and the left one upward. As a result, the right wing produces more lift than the left wing, and you generate a *negative* rolling moment. Thus, $C_{l_\delta} < 0$.
- $C_{n_{\delta_a}}$ is the resulting yawing moment due to a positive aileron deflection. If the right wing produces more lift than the left wing, the right wing will produce more drag and is thus being pulled back. A positive yawing moment results, thus $C_{n_{\delta_a}} > 0$.

This leads to the results summarised below.

Coefficient	Sign	Case 1	Case 2	Case 3
C_{l_β}	< 0			
C_{n_β}	> 0			
C_{l_p}	< 0			
C_{n_p}	< 0			
C_{l_r}	> 0			
C_{n_r}	< 0			
$C_{l_{\delta_a}}$	< 0			
$C_{n_{\delta_a}}$	> 0			

For b), it's mostly just a case of looking what I wrote down in part a). First, for case 1): We see that the fuselage has a disturbing influence on C_{l_β} and C_{n_β}, so if we double the diameter, these will be less negative and less positive, respectively. Other than that, a larger fuselage results in a more negative C_{n_r}.

For case 2), we see that the vertical tail induces a positive C_{n_β}, so halving it makes it less positive. Similarly, C_{l_p} will be less negative, and C_{n_p} will be more negative as the positive contribution of the tail becomes smaller. C_{l_r} will also become less positive, and C_{n_r} less negative.

For case 3), a larger wing sweep makes C_{l_β} more negative, and C_{n_β} more positive as it puts more of the wing behind the center of gravity, meaning the lateral forces of the wing generate a more positive yawing moment. C_{l_p} becomes less negative, as the wing sweep decreases lift across the wing, so the roll is dampened less. Similarly, $C_{l_{\delta_a}}$ goes down, as the ailerons become less effective as the velocity component perpendicular to the aileron becomes smaller.

This can be summarised as shown below:

Coefficient	Sign	Case 1	Case 2	Case 3
C_{l_β}	< 0	less neg	≈ 0	more neg
C_{n_β}	> 0	less pos	less pos	more pos
C_{l_p}	< 0	≈ 0	less neg	less neg
C_{n_p}	< 0	≈ 0	more neg	≈ 0
C_{l_r}	> 0	≈ 0	less pos	≈ 0
C_{n_r}	< 0	more neg	less neg	≈ 0
$C_{l_{\delta_a}}$	< 0	≈ 0	≈ 0	less neg
$C_{n_{\delta_a}}$	> 0	≈ 0	≈ 0	≈ 0

For c), the most important parameter is C_{n_β}: it is the ability of the aircraft to, when under some sideslip, return to the position where its velocity is in the direction of the velocity (so without sideslip). It's also called the weathercock stability.

For d), just look at question b) honestly. We could do something similar to case 1, but rather decrease the fuselage diameter this time: this would make C_{l_β} *more* negative (which is desired), and C_{n_β} more positive (which is desired). Furthermore, C_{n_r} will become less negative, so there'd be less yaw damping; however, C_{n_β} (and C_{l_β} are much more important for stability. Alternatively, you could introduce wing sweep: this would make C_{l_β} more negative and C_{n_β} more positive. C_{l_p} would become less negative so there'd be less roll damping. It should be noted that it make $C_{l_{\delta_a}}$ less negative though, making the ailerons less effective.

Exam August 2006: problem 5 (5p)

The lateral stability and control derivatives C_{Y_β}, C_{l_β}, C_{n_β}, C_{Y_p}, C_{l_p}, C_{n_p}, C_{Y_r}, C_{l_r}, C_{n_r}, $C_{Y_{\delta_r}}$, $C_{l_{\delta_r}}$, $C_{n_{\delta_r}}$.

(a) What is the sign of these stability and control derivatives?

(b) Assume that the surface of the vertical tailplane is *doubled*. What happens to the *magnitude* of the derivatives considered in question (a)? Note that all other parameters remain unchanged (so l_h is at its original value).

For a), regarding the signs, we can make the table shown below.

Coefficient	Sign
C_{Y_β}	-
C_{l_β}	-
C_{n_β}	+
C_{Y_p}	-
C_{l_p}	-
C_{n_p}	-
C_{Y_r}	+
C_{l_r}	+
C_{n_r}	-
$C_{Y_{\delta_r}}$	+
$C_{l_{\delta_r}}$	+
$C_{n_{\delta_r}}$	-

For b), we can append the table to what's shown below (+ is increase, - is decrease in magnitude). Personally I think they all make sense: in almost all cases, the contribution of the vertical tail was of the same sign as the sign of the coefficient (so if the coefficient was negative, then the vertical tail also contributed negatively to that). The only exception is C_{n_p}, the resulting yawing moment due to roll. The vertical tail has a positive effect on this, as you get a force in negative Y_B-direction due to the roll; a resultant positive yawing moment is created around the Z_B-axis. C_{n_p} is negative due to the large contribution of the main wing however, as explained earlier. So, it'll now becomes less negative, so its magnitude will decrease.

Coefficient	Sign	Change
C_{Y_β}	-	+
C_{l_β}	-	+
C_{n_β}	+	+
C_{Y_p}	-	+
C_{l_p}	-	+
C_{n_p}	-	-
C_{Y_r}	+	+
C_{l_r}	+	+
C_{n_r}	-	+
$C_{Y_{\delta_r}}$	+	+
$C_{l_{\delta_r}}$	+	+
$C_{n_{\delta_r}}$	-	+

Exam April 2012: problem 5 (10p)

Consider a conventionally statically stable aircraft (low wing configuration) at cruising flight, i.e. α is small. Please provide your answers in the following table form:

Coefficient	Sign	Case 1	Case 2	Case 3

(a) What is the conventional sign of the following lateral stability and control derivatives: C_{n_β}, C_{l_r}, C_{n_p}, $C_{l_{\delta_a}}$?

(b) What happens to these stability and control derivatives in the following cases:
  1) Increase of dihedral Γ. All other aircraft parameters remain unchanged.
  2) Increase of wing sweep angle Λ. All other aircraft parameters remain unchanged.
  3) The vertical tail length l_v is doubled. All other aircraft parameters remain unchanged.

Give your answer in terms of the following statements: more/less negative/positive or unchanged (0).

For a), C_{n_β} is positive as the vertical tail will experience a force in negative Y_B-direction, resulting in a positive yawing moment. C_{l_r} is positive: if the aircraft yaws in positive direction, the left wing will experience a higher velocity, thus produce more lift, inducing a positive rolling moment (also the vertical tail experiences a negative angle of attack, causing a force in positive Y_B-direction, resulting in a positive rolling moment as the tail is located above the c.g.). C_{n_p} is negative: this is the roll damping. $C_{l_{\delta_a}}$ is negative as well: the right wing would produce more lift than the left wing, so it creates a negative rolling moment. This leads to the table shown below:

Coefficient	Sign	Case 1	Case 2	Case 3
C_{n_β}	+			
C_{l_r}	+			
C_{n_p}	-			
$C_{l_{\delta_a}}$	-			

For b), let's analyse them separately:

- Increasing the dihedral has no effect on any of the coefficients. Only the coefficients C_{l_β} is affected by dihedral, but that's not here. The coefficients all remain the same.
- Increasing the sweep has little effect as well. It does make C_{n_β} more positive, as the lateral forces of the wing become more and more behind the center of gravity of the plane due to the sweep. However, the other ones do not change at all.
- Increasing the vertical tail length will result in a more positive C_{n_β}. For the rest, nothing changes. The official solutions say that C_{l_r} would become more positive as well, however, I'm not sure whether I can agree with this: yes, a positive rolling moment is caused by the vertical tail, but that's due to the moment arm in *vertical* direction between the positive Y_B-force generated by the vertical tail and the c.g. Increasing the tail length has no effect on this, I think, at least not for small α.

In short, we get the table below.

Coefficient	Sign	Case 1	Case 2	Case 3
C_{n_β}	+	0	more pos	more pos
C_{l_r}	+	0	0	0
C_{n_p}	-	0	0	0
$C_{l_{\delta_a}}$	-	0	0	0

3 Lateral stability and control in steady flight

Okay so if you gave the preface a thorough read, you'll know that the structure of the course is a bit messed up, and of the most complicating affairs for me is where to put this chapter. For your information, this corresponds to chapter 11 of the book, and lecture 18 of the lecture series, so it's pretty much the last thing that's treated in the lectures. However, there are multiple reasons to put it here:

- It uses exactly the stuff of the previous chapters. You basically already know everything you need for this chapter, and you're probably nicely in a flow studying about lateral stability, so it's nice to continue that flow I think, rather than abruptly stopping it, doing something completely else for half of the course, only to get back at it at the very end of the course.
- The chapter itself is rather short, so it shouldn't take you too long anyway.
- Exam questions about this chapter are quite common so it's a relatively important chapter.
- At long last, we'll finally do some actual calculations and exam questions. Personally I think this is a nice change as this part largely focussed on just reading and remembering stuff and nothing more than that.

So, in short, you can already do this chapter, it's short and it makes more sense to do it now.

3.1 Equilibrium equations

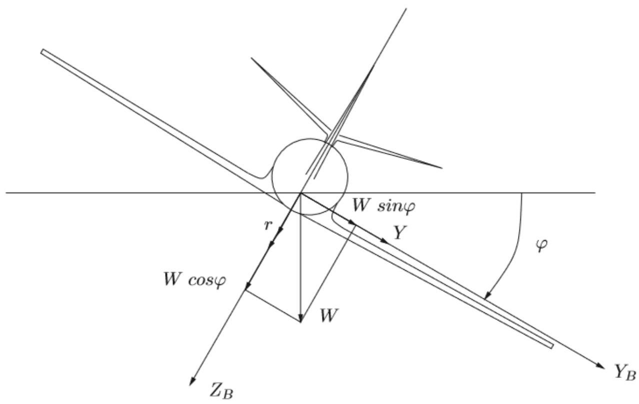


Figure 3.1: The forces along the Y_B-axis of an aircraft in steady, horizontal asymmetric flight.

This chapter focusses on horizontal, steady, asymmetric flight. The aircraft sideslips and yaws at constant β and $rb/(2V)$, respectively. It has a rolling angle ϕ, but no rolling velocity. The corresponding free-body diagram is shown in figure 3.1, and the corresponding equations of motion are

DIMENSIONAL EQUATIONS OF MOTION HORIZONTAL, STEADY, ASYMMETRIC FLIGHT

The equations of motion in straight, symmetric flight are given by

$$\sum F_{Y_B} : W \sin\phi + Y = mVr \tag{3.1}$$

$$\sum M_X : L = 0 \tag{3.2}$$

$$\sum M_Z : N = 0 \tag{3.3}$$

To clarify a few things:

- We take moments around the X_B-axis for equation (3.2), which points into the page. Just look at figure 3.1: rather evidently, there's no moment around this axis.

- We take moments around the Z_B-axis for equation (3.3). Again, evidently we don't have a moment around this axis.
- For equation (3.1), the term Vr is the normal acceleration in the direction of Y_B. Now, you may wonder, but isn't the normal acceleration given by V^2/R? Yes it is, so it is quintessential for you to remember that r is the (dimensional) *yaw-rate*, not the radius R of the circle. This is the same as the angular velocity ω around the midpoint of the circle the aircraft is rotating around. With $\omega = V/R$ (with R now the radius again), we see that indeed $Vr = V\omega = V^2/R$, so indeed the normal acceleration is V^2/R, but we write it as Vr with r the dimensional yaw-rate (so not normalised yet).

Now, we can make these equations of motion dimensionless. First, for equation (3.1): for small ϕ, $W \approx L$, so making it dimensionless by dividing by $(1/2)\rho V^2 S$ results in (for small ϕ)

$$\begin{aligned} W\sin\phi + Y &= mVr \\ C_L\phi + C_Y &= \frac{mVr}{\frac{1}{2}\rho V^2 S} = \frac{2mr}{\rho V S} \end{aligned} \tag{3.4}$$

Now, let's focus on the right-hand side of this equation. Let me rewrite it as

$$\frac{2mr}{\rho V S} = 4\frac{m}{\rho S b}\frac{rb}{2V}$$

We now see that we have the non-dimensional velocity in the right-fraction. Furthermore, we define

NON-DIMENSIONAL MASS PARAMETER

The **non-dimensional mass parameter** μ_b is defined as

$$\mu_b = \frac{m}{\rho S b} \tag{3.5}$$

so that we can actually write equation (3.4) as

$$C_L\phi - 4\mu_b\frac{rb}{2V} + C_Y = 0$$

Equations (3.2) and (3.3) are trivial to normalise by just dividing by $1/2\rho V^2 Sb$, meaning that the non-dimensional equations of motion are given by

NON-DIMENSIONAL EQUATIONS OF MOTION HORIZONTAL, STEADY, ASYMMETRIC FLIGHT

The equations of motion in straight, symmetric flight are given by

$$\sum F_{Y_B} : C_L\phi - 4\mu_b\frac{rb}{2V} + C_Y = 0 \tag{3.6}$$

$$\sum M_X : C_l = 0 \tag{3.7}$$

$$\sum M_Z : C_n = 0 \tag{3.8}$$

These may be expanded, using equations (1.10)-(1.12):

$$C_Y = C_{Y_\beta}\beta + C_{Y_p}\frac{pb}{2V} + C_{Y_r}\frac{rb}{2V} \tag{1.10}$$

$$C_l = C_{l_\beta}\beta + C_{l_p}\frac{pb}{2V} + C_{l_r}\frac{rb}{2V} \tag{1.11}$$

$$C_n = C_{n_\beta}\beta + C_{n_p}\frac{pb}{2V} + C_{n_r}\frac{rb}{2V} \tag{1.12}$$

By also including the effects of the aileron and rudder deflections, expanding like this leads to

$$\begin{aligned} C_L\phi + C_{Y_\beta}\beta + \left(C_{Y_r} - 4\mu_b\right)\frac{rb}{2V} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r &= 0 \\ C_{l_\beta}\beta + C_{l_r}\frac{rb}{2V} + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r &= 0 \\ C_{n_\beta}\beta + C_{n_r}\frac{rb}{2V} + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r &= 0 \end{aligned}$$

which looks ugly but is actually just a system of three equations with five unknowns (ϕ, β, $rb/(2V)$, δ_a and δ_r). We'll make a few further simplifications before analysing however; $C_{Y_{\delta_a}}$ and $C_{l_{\delta_r}}$ will be neglected, and C_{Y_r}, $C_{n_{\delta_a}}$, $C_{Y_{\delta_r}}$ will also be neglected, due to their small sizes. Thus, the simplified equations of motion become

SIMPLIFIED NON-DIMENSIONAL EQUATIONS OF MOTION HORIZONTAL, STEADY, ASYMMETRIC FLIGHT

The simplified equations of motion in straight, symmetric flight are given by are given by:

$$C_L\phi + C_{Y_\beta}\beta - 4\mu_b\frac{rb}{2V} = 0 \tag{3.9}$$

$$C_{l_\beta}\beta + C_{l_r}\frac{rb}{2V} + C_{l_{\delta_a}}\delta_a = 0 \tag{3.10}$$

$$C_{n_\beta}\beta + C_{n_r}\frac{rb}{2V} + C_{n_{\delta_r}}\delta_r = 0 \tag{3.11}$$

In matrix format, these can be written as

$$\begin{bmatrix} C_L & C_{Y_\beta} & -4\mu_b & 0 & 0 \\ 0 & C_{l_\beta} & C_{l_r} & C_{l_{\delta_a}} & 0 \\ 0 & C_{n_\beta} & C_{n_r} & 0 & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \phi \\ \beta \\ \frac{rb}{2V} \\ \delta_a \\ \delta_r \end{bmatrix} = \mathbf{0} \tag{3.12}$$

Now, what was the point of all this? Well, we have a system of 3 equations and 5 unknowns. We thus have two free variables. We can thus arbitrarily set one of the unknowns, and then see what the required ϕ, β etc. are to attain a certain turn rate r. That's what we'll do in the next section: sequentially, we'll set $\phi = 0$, $\beta = 0$, $\delta_a = 0$ etc., and see what can be said about the other variables. To be honest I'm not quite sure what the point of the next section is since it in itself is never asked on exams, but rather they come up with new situations[1]. So, just read through it to get a flavour of what to do, but don't bother studying it too intensively (but focus on the exam questions at the end of the chapter).

3.2 Steady horizontal turns

3.2.1 Turns using the ailerons only, $\delta_r = 0$

Now, what are the required ϕ, β and δ_a to fly at a certain non-dimensional turn rate $rb/(2V)$, if we don't deflect the rudder, i.e. $\delta_r = 0$? Well, in this case, equation (3.11) reduces to

$$\begin{aligned} C_{n_\beta}\beta + C_{n_r}\frac{rb}{2V} + C_{n_{\delta_r}}\delta_r = C_{n_\beta}\beta + C_{n_r}\frac{rb}{2V} &= 0 \\ \beta &= -\frac{C_{n_r}}{C_{n_\beta}}\frac{rb}{2V} \end{aligned} \tag{3.13}$$

We can substitute this value into equation (3.10), to obtain

$$\begin{aligned} C_{l_\beta}\beta + C_{l_r}\frac{rb}{2V} + C_{l_{\delta_a}}\delta_a &= 0 \\ C_{l_\beta}\cdot -\frac{C_{n_r}}{C_{n_\beta}}\frac{rb}{2V} + C_{l_r}\frac{rb}{2V} + C_{l_{\delta_a}}\delta_a &= 0 \\ \delta_a &= \frac{1}{C_{l_{\delta_a}}}\frac{C_{l_\beta}C_{n_r} - C_{l_r}C_{n_\beta}}{C_{n_\beta}}\frac{rb}{2V} \end{aligned}$$

[1] Like, set $\delta_a = 10°$, what are the corresponding values of ϕ, etc., but we only discuss the cases here where we set it to 0.

Similarly, we can substitute equation (3.13) into equation (3.9) to obtain

$$C_L\phi + C_{Y_\beta}\cdot -\frac{C_{n_r}}{C_{n_\beta}}\frac{rb}{2V} - 4\mu_b\frac{rb}{2V} = 0 \tag{3.14}$$

$$\phi = \frac{4\mu_b + \frac{C_{n_r}}{C_{n_\beta}}C_{Y_\beta}}{C_L}\frac{rb}{2V} \tag{3.15}$$

I think the principle is very clear now: you assume that $rb/(2V)$ is a known value, you disregard δ_r, and then solve whichever equation is easiest to solve (there'll always be one equation with only one unknown, so you can solve that one first), and then solve the other equations using substitution.

Now, one thing that I have to show is how the derivatives behave: from equations (3.13), (3.14) and (3.15) we obtain

$$\frac{d\beta}{d\frac{rb}{2V}} = -\frac{C_{n_r}}{C_{n_\beta}} \tag{3.16}$$

$$\frac{d\delta_a}{d\frac{rb}{2V}} = \frac{1}{C_{l_{\delta_a}}}\frac{C_{l_\beta}C_{n_r} - C_{l_r}C_{n_\beta}}{C_{n_\beta}} \tag{3.17}$$

$$\frac{d\phi}{d\frac{rb}{2V}} = \frac{4\mu_b + \frac{C_{n_r}}{C_{n_\beta}}C_{Y_\beta}}{C_L} \tag{3.18}$$

Now, let's focus first on equation (3.17). We want this derivative to be *negative*: remember that a positive δ_a meant that the right aileron is deflected downward (so increase s lift over right wing), and the left aileron upward (so decreases lift over the left wing). If we want to turn to the right, we have to roll towards the right. This requires a *negative* aileron deflection, as we want the right wing to produce less lift than the left wing. If this is not the case, and if our stability derivatives are such that a *positive* aileron deflection is required for the pilot to turn right, it becomes very unnatural for the pilot, and in fact we get control instability, which was previously discussed in detail in part I of the summary (but for this part, it's much less important fortunately. It's nice that you know it now but they never ask for it on exams). Now, bearing in mind that $C_{l_{\delta_a}}$ is negative, and C_{n_β} is positive for conventional aircraft, this means that the requirement for stability is

$$C_{l_\beta}C_{n_r} - C_{l_r}C_{n_\beta} > 0$$

In fact, this is called **spiral stability**, which we'll discuss in more detail in the final part of the summary. Furthermore, note that since $C_{n_r} < 0$, $C_{n_\beta} > 0$ and $C_{Y_\beta} < 0$, we have $d\beta/d(rb/(2V)) > 0$ and $d\phi/d(rb/(2V)) > 0$: in other words, to get a positive turn rate when we don't deploy the rudder (i.e. we turn to the right rather than to the left), we get positive sideslip and roll towards the right. Positive sideslip means that our nose points *inwards* of the turn, it's sort of having oversteer in a car. Okay again Idk why I'm telling this it's never asked on exams.

3.2.2 *Turns using the rudder only, $\delta_a = 0$*

You can do pretty much the same derivation for analysing the case where you don't deflect the aileron. Then you end up at the derivatives

$$\frac{d\delta_r}{d\frac{rb}{2V}} = -\frac{1}{C_{n_{\delta_r}}}\frac{C_{l_\beta}C_{n_r} - C_{l_r}C_{n_\beta}}{C_{l_\beta}} \tag{3.19}$$

$$\frac{d\phi}{d\frac{rb}{2V}} = \frac{4\mu_b + C_{Y_\beta}\frac{C_{l_r}}{C_{l_\beta}}}{C_L} \tag{3.20}$$

$$\frac{d\beta}{d\frac{rb}{2V}} = -\frac{C_{l_r}}{C_{l_\beta}} \tag{3.21}$$

Here, since $C_{n_{\delta_r}}$ and C_{l_β} are both negative, the $d\delta_r/d(rb/(2V))$ in equation (3.19) will also be negative if the same condition is satisfied as in the previous subsection. In other words, to turn right, you have to turn the rudder right. The pilot is happy with this. Furthermore, with C_{Y_β} negative, C_{l_r} positive and C_{l_β} negative we have $d\phi/d(rb/(2V))$ and $d\beta/d(rb/(2V))$ both positive. In other words, we'll have to roll to the right to go right (makes sense), but we'll oversteer the aircraft a little.

3.2.3 Coordinated turns, $\beta = 0$

COORDINATED TURNS

A **coordinated turn** is a turn with no sideslipping, i.e. $\beta = 0$.

You can do the same derivations once more, ending up at

$$\frac{d\phi}{d\frac{rb}{2V}} = \frac{4\mu_b}{C_L} \tag{3.22}$$

$$\frac{d\delta_a}{d\frac{rb}{2V}} = -\frac{C_{l_r}}{C_{l_{\delta_a}}} \tag{3.23}$$

$$\frac{d\delta_r}{d\frac{rb}{2V}} = -\frac{C_{n_r}}{C_{n_{\delta_r}}} \tag{3.24}$$

The first derivative will obviously be positive. However, with C_{l_r} positive and $C_{l_{\delta_a}}$, C_{n_r} and $C_{n_{\delta_r}}$ all negative, this means that $d\delta_a/d(rb/(2V))$ is positive (whilst $d\delta_r/d(rb/(2V))$ is still negative, as it should)! This is undesired, and the pilot hates this (as it means he has to deflect the ailerons to the right to go to the right, which is counterintuitive as it puts your aircraft in a roll to the left), but nothing we can do about it lol.

3.2.4 Flat turns

COORDINATED TURNS

A **flat turn** is a turn with no roll, i.e. $\phi = 0$.

If we set $\phi = 0$, we obtain

$$\frac{d\beta}{d\frac{rb}{2V}} = \frac{1}{C_{Y_\beta}}4\mu_b \tag{3.25}$$

$$\frac{d\delta_a}{d\frac{rb}{2V}} = -\frac{1}{C_{l_{\delta_a}}}\frac{C_{l_\beta}\mu_b + 4C_{l_r}C_{Y_\beta}}{C_{Y_\beta}} \tag{3.26}$$

$$\frac{d\delta_r}{d\frac{rb}{2V}} = -\frac{1}{C_{n_{\delta_r}}}\frac{C_{n_\beta}\mu_b + 4C_{n_r}C_{Y_\beta}}{C_{Y_\beta}} \tag{3.27}$$

Remember that all of C_{Y_β}, $C_{l_{\delta_a}}$ and C_{l_β} are negative, whereas C_{l_r}, $C_{n_{\delta_r}}$ and C_{n_r} are all positive. Consequentially, the first derivative is negative, meaning that you have to sideslip out of the corner to turn right (so you get understeer so to say), $C_{l_{\delta_a}}$ is positive (again, pilot hates this but what's he gonna do about it), and $C_{n_{\delta_r}}$ is negative as it should.

3.3 Steady, straight, sideslipping flight

Now, we can also take the case where we set $rb/(2V)$ equal to zero, and see what happens to ϕ, δ_a and δ_r for various values of β. This corresponds to straight flight (as you are no longer turning) with constant sideslip, which is very common during landing and takeoff and thus is interesting to analyse.

Literally nothing changes is our approach though. We now set $rb/(2V)$ equal to zero in equations (3.9) to (3.11), and express everything in terms of β. This leads to

$$\begin{aligned}
C_L\phi + C_{Y_\beta}\beta = 0 \quad &\rightarrow \quad \phi = -\frac{C_{Y_\beta}}{C_L}\beta \quad \rightarrow \quad \frac{d\phi}{d\beta} = -\frac{C_{Y_\beta}}{C_L} \\
C_{l_\beta}\beta + C_{l_{\delta_a}}\delta_a = 0 \quad &\rightarrow \quad \delta_a = -\frac{C_{l_\beta}}{C_{l_{\delta_a}}}\beta \quad \rightarrow \quad \frac{d\delta_a}{d\beta} = -\frac{C_{l_\beta}}{C_{l_{\delta_a}}} \\
C_{n_\beta}\beta + C_{n_{\delta_r}}\delta_r = 0 \quad &\rightarrow \quad \delta_r = -\frac{C_{n_\beta}}{C_{n_{\delta_r}}}\beta \quad \rightarrow \quad \frac{d\delta_r}{d\beta} = -\frac{C_{n_\beta}}{C_{n_{\delta_r}}}
\end{aligned}$$

Remembering that $C_{Y_\beta} < 0$, $C_{l_\beta} < 0$, $C_{l_{\delta_a}} < 0$, $C_{n_\beta} > 0$ and $C_{n_{\delta_r}} < 0$, this means that $d\phi/d\beta > 0$: if you're at a positive sideslip, you roll a bit to the right. Note that it is beneficial to have C_{Y_β} be large in magnitude: this makes it easier for the pilot to detect sideslip, as the aircraft will experience a larger lateral force when it starts slipping (and you easily notice this force when it appears as you get pushed towards the side of your chair). The derivative $d\delta_a/d\beta$ will be negative (as desired), whereas $d\delta_r/d\beta$ will be positive (so you have to deflect your rudder to the left). Once more, it's more important that you can do the exam questions at the end of this chapter rather than anything else.

3.4 *Steady straight flight with one or more engines inoperative*

Okay so one last thing before we start doing old exam questions after all this boring reading.

What happens when one engine shuts off for whatever reason? To be specific, what happens if the right engine shuts off? Well, obviously, a very large positive yawing moment is generated, as the left engine will cause an enormous positive yawing moment that is no longer counteracted by the right engine. This yawing moment is given by

$$C_{n_e} = k\frac{\Delta T_p y_e}{\frac{1}{2}\rho V^2 Sb} \tag{3.28}$$

where y_e is the spanwise position of the engine, ΔT_p the difference in thrust from the operative engine and the drag of the inoperative engine and k a scaling factor. This scaling factor is often *larger* than 1! Why? Well, consider a propeller engine: the rotation of the propellers speeds up the flow near the propeller, which means the pressure in the slipstream of an engine is lower than usual. Thus, if the right engine shuts down, there's no longer an area of lower pressure behind the right engine. The flow will move over the fuselage to the other side, to the slipstream of the operative engine. This crossflow results in sidewash from right to left, resulting in a force being generated by the vertical tailplane in negative Y_B-direction. As a consequence, the vertical tailplane further increases the positive yawing moment created by the inoperative engine; this is why k is even larger than 1.

Furthermore, it should be noted that a rolling moment is created when an engine becomes inoperative. After all, as I said, the flow has a higher velocity near an engine, thus increasing the lift there. If only the left engine is working, this means the left wing produces more lift, creating a positive rolling moment C_{l_e}.

These modifications can be straightforwardly included in the equations of motion of equations (3.9)-(3.11). We assume straight flight, i.e. $r = 0$, so that we get

$$\begin{aligned}
C_L\phi + C_{Y_\beta}\beta &= 0 \\
C_{l_\beta}\beta + C_{l_{\delta_a}}\delta_a + C_{l_e} &= 0 \\
C_{n_\beta}\beta + C_{n_{\delta_r}}\delta_r + C_{n_e} &= 0
\end{aligned}$$

or, in matrix format,

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 & C_{Y_{\delta_r}} \\ 0 & C_{l_\beta} & C_{l_{\delta_a}} & 0 \\ 0 & C_{n_\beta} & 0 & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \phi \\ \beta \\ \delta_a \\ \delta_r \end{bmatrix} = - \begin{bmatrix} 0 \\ C_{l_c} \\ C_{n_e} \end{bmatrix}$$

Again, we have one free variable so just set one arbitrarily (based on what the exam question asks) and solve for the rest.

One thing I should note: for propeller aircraft, $T_p V$ is often constant. This means that in equation (3.28), C_{n_e} varies inversely to the cube of the airspeed, V_e. This means that the slower you fly, the larger C_{n_e}, and thus the harder it becomes to control the aircraft. This is why the **minimum control speed** $V_{m.c.}$ is defined; it's the minimum speed at which the aircraft can still be controlled with one engine inoperative, with $|\phi| < 5°$.

3.5 *Exam questions*

As promised, finally some exam questions. A lot of these questions require you to solve a 3 × 3 system of equations. You can either do this analytically, i.e. manually substituting everything, leaving everything purely symbolic until the very end, or you can do it quickly by just inverting the matrix on your calculator. Personally, I prefer the latter as it's much faster for me. However, I should note: if you make a typo on your calculator such that your numerical results are wrong, you get all points subtracted[2]. On the other hand, if you do it analytically, you receive quite a bit of partial points, although you'll still get subtracted points for sign errors etc., but they're a bit more lenient as it's visible for them where the error comes from. However, personally, between solving a system of equations symbolically and just typing it in my calculator, I vastly prefer the latter. You can just type it in multiple times separately, check whether it's indeed the same and get all the points easily since honestly, typing stuff in your calculator shouldn't be something that you have trouble with calculators (although of course, if you have a Casio calculator it's automatically a bit harder since they're inherently more difficult to work with compared to Texas Instruments, but that was your own choice when buying one ofc).

Exam April 2013: problem 4 (20p)

COnsider a twin-engined propeller-driven aircraft, from which the aircraft data is summarized in figure 3.2.

m = 16200 kg	C_L = 0.445	V = 130 m s^{-1}	S = 70 m^2	b = 29 m
ρ = 0.7 kg m^{-3}	C_{Y_β} = -0.90	C_{ℓ_β} = -0.13	C_{n_β} = 0.11	$C_{Y_{\delta_a}}$ = 0
$C_{\ell_{\delta_a}}$ = -0.11	$C_{n_{\delta_a}}$ = 0	$C_{Y_{\delta_r}}$ = 0.31	$C_{\ell_{\delta_r}}$ = 0	$C_{n_{\delta_r}}$ = -0.086

Figure 3.2: Aircraft data.

(a) Derive, using the asymmetric equations of motions shown below, the equations of motion for *stationary*, *straight* flight conditions. Clearly indicate the assumptions you make. Provide your final answer in matrix format.

$$C_L\phi + C_{Y_\beta}\beta + \left(C_{Y_r} - 4\mu_b\right)\frac{rb}{2V} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r = 0$$
$$C_{l_\beta}\beta + C_{l_r}\frac{rb}{2V} + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r = 0$$
$$C_{n_\beta}\beta + C_{n_r}\frac{rb}{2V} + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r = 0$$

[2] As in, if it's a three by three system so that there are three unknowns, and the question is worth 6 points, then you get subtracted 2 points for each incorrect numerical result, even if it was just one sign error you made when typing it in your calculator.

Due to an engine failure there is a loss of engine thrust (ΔT) from the right engine. As a result of this, a moment N_e is acting on the aircraft. Its non-dimensional coefficient is,

$$C_{n_e} = k \cdot \frac{\Delta T \cdot y_e}{\frac{1}{2}\rho V^2 S b}$$

With $k = 1.13$, $\Delta T = 27.5\,\text{kN}$, and $y_e = 4.7\,\text{m}$. y_e is the lateral distance between the right engine and center of gravity. The rolling moment L_e due to the engine failure is being neglected.

(b) Append the equations of motion from question (a) with the yawing moment induced by the asymmetric thrust. Provide your final answer in matrix format.

(c) Calculate the roll angle, the rudder angle and the aileron deflection angle for the case that the pilot manages to keep the side slip angle at zero angles.

The pilot wants to turn the aircraft to fly to the nearest airport on the remaining engine.

(d) Derive an expression for the additional aileron and rudder deflections in case the pilot flies a rate one turn (yaw rate of 180 degrees in one minute) to the right or to the left. The pilot maintains zero side slip and the roll angle found under (c). You do *not* have to compute the actual deflections. Is one direction significantly preferable over the other?

Okay maybe it's an intimidating question at first, but it's actually not so difficult, especially once you've seen one or two of these exercises (questions about this part of the summary are very similar each time). First, for a), we have that for stationary (or steady), straight flight that both the roll and yaw rate are zero. Thus, these terms drop out from the equations. Furthermore, $C_{n_{\delta_a}}$, $C_{l_{\delta_r}}$ and $C_{Y_{\delta_a}}$ are all zero, meaning that our equations of motion simplify to

$$\begin{aligned} C_L\phi + C_{Y_\beta}\beta + C_{Y_{\delta_r}}\delta_r &= 0 \\ C_{l_\beta}\beta + C_{l_{\delta_a}}\delta_a &= 0 \\ C_{n_\beta}\beta + C_{n_{\delta_r}}\delta_r &= 0 \end{aligned}$$

In matrix format, this can be written as

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 & C_{Y_{\delta_r}} \\ 0 & C_{l_\beta} & C_{l_{\delta_a}} & 0 \\ 0 & C_{n_\beta} & 0 & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \phi \\ \beta \\ \delta_a \\ \delta_r \end{bmatrix} = \mathbf{0}$$

That's all we had to do. Note: in the previous sections, we assumed $C_{Y_{\delta_r}} \approx 0$. You are not allowed to make that assumption here! After all, it's given in the table to equal 0.31, and then obviously can't be like hey bitches I'm gonna assume it's zero cause 0.31 isn't 0.

For b), we simply take the yawing moment equation, which will become

$$C_{n_\beta}\beta + C_{n_{\delta_r}}\delta_r + C_{n_e} = 0$$

In matrix format:

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 & C_{Y_{\delta_r}} \\ 0 & C_{l_\beta} & C_{l_{\delta_a}} & 0 \\ 0 & C_{n_\beta} & 0 & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \phi \\ \beta \\ \delta_a \\ \delta_r \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ C_{n_e} \end{bmatrix}$$

If we set $\beta = 0°$, our matrix equation simplifies to

$$\begin{bmatrix} C_L & 0 & C_{Y_{\delta_r}} \\ 0 & C_{l_{\delta_a}} & 0 \\ 0 & 0 & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \phi \\ \delta_a \\ \delta_r \end{bmatrix} = -\begin{bmatrix} 0 \\ 0 \\ C_{n_e} \end{bmatrix}$$

From the third row, we have

$$C_{n_{\delta_r}}\delta_r = -C_{n_e}$$
$$\delta_r = -\frac{C_{n_e}}{C_{n_{\delta_r}}}$$

From the second row, we have $C_{l_{\delta_a}}\delta_a = 0$, so $\delta_a = 0$. From the first row, we obtain

$$C_L\phi + C_{Y_{\delta_r}}\delta_r = 0$$
$$\phi = -\frac{C_{Y_{\delta_r}}}{C_L}\delta_r = \frac{C_{Y_{\delta_r}}}{C_L}\frac{C_{n_e}}{C_{n_{\delta_r}}}$$

Substituting in the values leads to

$$C_{n_e} = 1.13 \cdot \frac{27500 \cdot 4.7}{\frac{1}{2} \cdot 0.7 \cdot 130^2 \cdot 70 \cdot 29} = 0.01216$$
$$\phi = \frac{0.31}{0.445}\frac{0.01216}{-0.086} = -0.0985\,\mathrm{rad}$$
$$\delta_a = 0$$
$$\delta_r = -\frac{0.01216}{-0.086} = 0.1414\,\mathrm{rad}$$

Note: you could have also just calculated this by inverting the matrix on your calculator. Just do whatever you want, I'm personally a fan of inverting matrices so personally I just stick to that.

If we want a yaw rate of 180° in one minute, then that corresponds to $r = \pi/60 = 0.052\,36\,\mathrm{rad/s}$. We must now rederive our equations of motion starting from the initial equations they give. We get, by setting $p = 0$ and $C_{Y_{\delta_a}}$, $C_{n_{\delta_a}}$ and $C_{l_{\delta_r}}$ equal to zero as well,

$$C_L\phi + C_{Y_\beta}\beta + \left(C_{Y_r} - 4\mu_b\right)\frac{rb}{2V} + C_{Y_{\delta_r}}\left(\delta_r + \Delta\delta_r\right) = 0$$
$$C_{l_\beta}\beta + C_{l_r}\frac{rb}{2V} + C_{l_{\delta_a}}\left(\delta_a + \Delta\delta_a\right) = 0$$
$$C_{n_\beta}\beta + C_{n_r}\frac{rb}{2V} + C_{n_{\delta_r}}\left(\delta_r + \Delta\delta_r\right) + C_{n_e} = 0$$

Here, ϕ, β, r, C_{n_e}, δ_a and δ_r are already known[a], and thus should be moved to the right-hand side:

$$C_{Y_{\delta_r}}\Delta\delta_r = -C_L\phi - C_{Y_\beta}\beta - \left(C_{Y_r} - 4\mu_b\right)\frac{rb}{2V} - C_{Y_{\delta_r}}\delta_r$$
$$C_{l_{\delta_a}}\Delta\delta_a = -C_{l_\beta}\beta - C_{l_r}\frac{rb}{2V} - C_{l_{\delta_a}}\delta_a$$
$$C_{n_{\delta_r}}\Delta\delta_r = -C_{n_\beta}\beta - C_{n_r}\frac{rb}{2V} - C_{n_{\delta_r}}\delta_r - C_{n_e}$$

In matrix format, this can be written as

$$\begin{bmatrix} 0 & C_{Y_{\delta_r}} \\ C_{l_{\delta_a}} & 0 \\ 0 & C_{n_{\delta_r}} \end{bmatrix}\begin{bmatrix} \Delta\delta_a \\ \Delta\delta_r \end{bmatrix} = \begin{bmatrix} -C_L & -C_{Y_\beta} & -\left(C_{Y_r} - 4\mu_b\right) & 0 & -C_{Y_{\delta_r}} & 0 \\ 0 & -C_{l_\beta} & -C_{l_r} & -C_{l_{\delta_a}} & 0 & 0 \\ 0 & -C_{n_\beta} & -C_{n_r} & 0 & -C_{n_{\delta_r}} & -Cn_e \end{bmatrix}\begin{bmatrix} \phi \\ \beta \\ \frac{rb}{2V} \\ \delta_a \\ \delta_r \\ C_{n_e} \end{bmatrix}$$

This equation can be solved for $\Delta\delta_a$ and $\Delta\delta_r$. By switching the sign of r, the additional deflections for the right (when taking r positive) and left (when taking r negative) are obtained. Okay no it can't

actually cause it's overdetermined, but this is how the official solution has it (and I'm not disagreeing with it). There was just a fault in the design of the question: they should have allowed the roll angle or sideslip angle to change, then it'd have been a determined system. But you could have analysed a priori that the system would not have a solution. To be honest I have the feeling that more exam questions are poorly designed and have solutions that show serious flaws, this is just one of them.
In any case, since the right engine is inoperative, the airplane will naturally want to turn right. Thus, it's probably preferable turn to the right, and so to the right is probably preferable.

[a] δ_a and δ_r are the old values of the deflections. We are merely interested in the differences to the new values, so only interested in $\Delta\delta_a$ and $\Delta\delta_r$.

Exam August 2011: problem 5 (20p)

Consider a twin-engined jet aircraft, from which the aircraft data is summarized in figure 3.3.

m = 65000 kg	C_L = 0.46	V = 100 m s^{-1}	S = 125 m^2	b = 35 m
ρ = 1.225 kg m^{-3}	C_{Y_β} = -0.90	C_{ℓ_β} = -0.25	C_{n_β} = 0.1	$C_{Y_{\delta_a}}$ = 0
$C_{\ell_{\delta_a}}$ = -0.08	$C_{n_{\delta_a}}$ = -0.01	$C_{Y_{\delta_r}}$ = 0.55	$C_{\ell_{\delta_r}}$ = -0.04	$C_{n_{\delta_r}}$ = -0.25

Figure 3.3: Aircraft data.

During a landing approach, the aircraft is hit by lighting, which leads to an electrical failure in the rudder actuator system causing the rudder to lock up at an angle of $\delta_{r_{fail}}$ = +7.0°. As a result of this, an unwanted yawing and rolling moment is acting on the aircraft.

(a) Derive, using the asymmetric equations of motions shown below, the equations of motion for steady, straight, sideslipping flight. Clearly indicate the assumptions you make. Provide your final answer in matrix format.

$$C_L\phi + C_{Y_\beta}\beta + \left(C_{Y_r} - 4\mu_b\right)\frac{rb}{2V} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r = 0$$
$$C_{l_\beta}\beta + C_{l_r}\frac{rb}{2V} + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r = 0$$
$$C_{n_\beta}\beta + C_{n_r}\frac{rb}{2V} + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r = 0$$

(b) Calculate the required sideslip angle β, the roll angle ϕ and the aileron deflection δ_a in order to continue the landing approach.

(c) It is given that the maximum aileron deflection is $\delta_{a_{max}} = \pm 25°$. It is now clear that the pilot is unable to continue the landing approach. In order to reduce aileron deflections, the pilot sets the engines to generate a differential thrust. The non-dimensional moment produced by the differential thrust is $C_{n_e} \cdot \Delta T$ with the coefficient C_{n_e} given by:

$$C_{n_e} = \frac{y_e}{\frac{1}{2}\rho V^2 S b}$$

with y_e = 6.5 the lateral distance between the engines and the center of gravity of the aircraft. Rewrite the equations of motion for steady, straight, sideslipping flight from part (a) such that they also include the differential thrust ΔT.

(d) Calculate the required differential thrust ΔT, the new sideslip angle β and the new roll angle ϕ such that the ailerons have a deflection of 80% of their maximum deflection.

This question is already a bit harder than the previous one I think. However, it's still very doable if you're a bit handy with matrices. First, for a), for steady, straight, sideslipping flight, we have zero roll and yaw rate, so $b = p = 0$. Furthermore, $C_{Y_{\delta_a}} = 0$, so we obtain

$$C_L\phi + C_{Y_\beta}\beta + C_{Y_{\delta_r}}\delta_r = 0$$
$$C_{l_\beta}\beta + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r = 0$$
$$C_{n_\beta}\beta + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r = 0$$

In matrix format, this becomes

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 & C_{Y_{\delta_r}} \\ 0 & C_{l_\beta} & C_{l_{\delta_a}} & C_{l_{\delta_r}} \\ 0 & C_{n_\beta} & C_{n_{\delta_a}} & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \phi \\ \beta \\ \delta_a \\ \delta_r \end{bmatrix} = \mathbf{0}$$

Now, δ_r is known to equal $\delta_{r_{fail}} = 7°$ and thus must be moved to the right-hand side. This is very easy if you have the matrix form: simply delete the δ_r from the vector containing the unknowns, and remove the corresponding *column* of the matrix (δ_r is the fourth entry of the vector, so we remove the fourth column) and move it to the other side, not forgetting that we include a minus sign:

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 \\ 0 & C_{l_\beta} & C_{l_{\delta_a}} \\ 0 & C_{n_\beta} & C_{n_{\delta_a}} \end{bmatrix} \begin{bmatrix} \phi \\ \beta \\ \delta_a \end{bmatrix} = - \begin{bmatrix} C_{Y_{\delta_r}} \\ C_{l_{\delta_r}} \\ C_{n_{\delta_r}} \end{bmatrix} \delta_{r_{fail}}$$

For b), yeah lol I'm not gonna solve this analytically, that's way too much effort considering there's not a single row that's nice. Just plugging in the numbers yields:

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 \\ 0 & C_{l_\beta} & C_{l_{\delta_a}} \\ 0 & C_{n_\beta} & C_{n_{\delta_a}} \end{bmatrix} \begin{bmatrix} \phi \\ \beta \\ \delta_a \end{bmatrix} = - \begin{bmatrix} C_{Y_{\delta_r}} \\ C_{l_{\delta_r}} \\ C_{n_{\delta_r}} \end{bmatrix} \delta_{r_{fail}}$$
$$\begin{bmatrix} 0.46 & -0.9 & 0 \\ 0 & -0.25 & -0.08 \\ 0 & 0.1 & -0.01 \end{bmatrix} \begin{bmatrix} \phi \\ \beta \\ \delta_a \end{bmatrix} = - \begin{bmatrix} 0.55 \ -0.04 \\ -0.25 \end{bmatrix} \cdot \frac{7\pi}{180}$$
$$\begin{bmatrix} \phi \\ \beta \\ \delta_a \end{bmatrix} = \begin{bmatrix} 0.300\,13\,\text{rad} \\ 0.228\,06\,\text{rad} \\ -0.7737\,\text{rad} \end{bmatrix}$$

For c), the differential thrust only affects the equation for the yawing moment, i.e. only the third row of the matrix equation, where it adds $C_{n_e}\Delta T$ to the left-hand side. We can straightforwardly add this to matrix equation:

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 & 0 \\ 0 & C_{l_\beta} & C_{l_{\delta_a}} & 0 \\ 0 & C_{n_\beta} & C_{n_{\delta_a}} & C_{n_e} \end{bmatrix} \begin{bmatrix} \phi \\ \beta \\ \delta_a \\ \Delta T \end{bmatrix} = - \begin{bmatrix} C_{Y_{\delta_r}} \\ C_{l_{\delta_r}} \\ C_{n_{\delta_r}} \end{bmatrix} \delta_{r_{fail}}$$

In case your memory of linear algebra is a bit rusty, this operation is perfectly fine (and it's an operation that we'll see more often). You just add an additional entry to the vector of unknowns, and add the corresponding column in the matrix. This is a much faster way of doing it then rederiving the equations of motion, obviously, so that's why it's helpful to remember this 'trick'. Furthermore, we know that the aileron deflection will now be $\delta_{a_{max}} = -25°$ (negative, cause the answer to (b) was negative as well, so the deflection will now be -25°. It won't be +25° cause that would be like, I wanna go 130 km/h to arrive in Amsterdam asap, but my car can only go 100 km/h, so I'll just drive 100 km/h in reverse, that genuinely doesn't make sense. In any case, it means we can remove the third entry of the vector of unknowns, and the third column of the matrix, and move it to the other side:

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 \\ 0 & C_{l_\beta} & 0 \\ 0 & C_{n_\beta} & C_{n_e} \end{bmatrix}\begin{bmatrix} \phi \\ \beta \\ \Delta T \end{bmatrix} = -\begin{bmatrix} C_{Y_{\delta_r}} \\ C_{l_{\delta_r}} \\ C_{n_{\delta_r}} \end{bmatrix}\delta_{r_{fail}} - \begin{bmatrix} 0 \\ C_{l_{\delta_a}} \\ C_{n_{\delta_a}} \end{bmatrix}\delta_a = -\begin{bmatrix} C_{Y_{\delta_r}} & 0 \\ C_{l_{\delta_r}} & C_{l_{\delta_a}} \\ C_{n_{\delta_r}} & C_{n_{\delta_a}} \end{bmatrix}\begin{bmatrix} \delta_{r_{fail}} \\ \delta_a \end{bmatrix}$$

Note that you can straightforwardly combine the vectors on the right-hand side as I showed.

So we set $\delta_a = -20°$, and then again we just do it by matrix inversion cause honestly doing it symbolically sucks balls:

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 \\ 0 & C_{l_\beta} & 0 \\ 0 & C_{n_\beta} & C_{n_e} \end{bmatrix}\begin{bmatrix} \phi \\ \beta \\ \Delta T \end{bmatrix} = -\begin{bmatrix} C_{Y_{\delta_r}} & 0 \\ C_{l_{\delta_r}} & C_{l_{\delta_a}} \\ C_{n_{\delta_r}} & C_{n_{\delta_a}} \end{bmatrix}\begin{bmatrix} \delta_{r_{fail}} \\ \delta_a \end{bmatrix}$$

$$\begin{bmatrix} 0.46 & -0.9 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0.1 & \frac{6.5}{\frac{1}{2}\cdot 1.225\cdot 100^2\cdot 125\cdot 35} \end{bmatrix}\begin{bmatrix} \phi \\ \beta \\ \Delta T \end{bmatrix} = -\begin{bmatrix} 0.55 & 0 \\ -0.04 & -0.08 \\ -0.25 & -0.01 \end{bmatrix}\begin{bmatrix} \frac{7\pi}{180} \\ \frac{-20\pi}{180} \end{bmatrix}$$

$$\begin{bmatrix} \phi \\ \beta \\ \Delta T \end{bmatrix} = \begin{bmatrix} 0.342\,\text{rad} \\ 0.0922\,\text{rad} \\ 73\,536\,\text{N} \end{bmatrix}$$

One thing I like to point out: it is essential that you include the values for $\delta_{r_{fail}}$ and δ_a in *radians*. If you don't, the result for ΔT will have used degrees rather than radians, meaning it's off by a factor $180/pi$. So always input it as radians.

Exam April 2008: problem 5 (25p)

Consider a twin-engined propeller-driven aircraft, from which the aircraft data is summarized in figure 3.4.

m = 17200 kg	C_L = 0.531	V = 130 m s^{-1}
S = 75 m^2	b = 29 m	ρ = 0.653 kg m^{-3}
C_{Y_β} = -0.90	C_{ℓ_β} = -0.090	C_{n_β} = 0.16
$C_{Y_{\delta_a}}$ = 0	$C_{\ell_{\delta_a}}$ = -0.089	$C_{n_{\delta_a}}$ = 0
$C_{Y_{\delta_r}}$ = 0.30	$C_{\ell_{\delta_r}}$ = 0	$C_{n_{\delta_r}}$ = -0.095

Figure 3.4: Aircraft data.

Due to an engine failure there is a loss of engine thrust from the right engine, so it has become inoperative. As a result of this a yawing moment N_e is acting on the aircraft and its non-dimensional coefficient is:

$$C_{n_e} = k \cdot \frac{\Delta T \cdot y_e}{\frac{1}{2}\rho V^2 S b}$$

with $k = 1.15$, $\Delta T = 30\,\mathrm{kN}$ and $y_e = 6.0\,\mathrm{m}$. The non-dimensional rolling moment coefficient due to the engine failure C_{l_e} is 0.01. The steady state equations of motion can be written as:

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 & C_{Y_{\delta_r}} \\ 0 & C_{l_\beta} & C_{l_{\delta_a}} & 0 \\ 0 & C_{n_\beta} & 0 & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \phi \\ \beta \\ \delta_a \\ \delta_r \end{bmatrix} = - \begin{bmatrix} 0 \\ C_{l_e} \\ C_{n_e} \end{bmatrix}$$

(a) After the failure, the pilot levels the wings, $\phi = 0$. Calculate the rudder angle δ_r, the aileron deflection δ_a and the sideslip angle β.

(b) Next the pilot decides to 'lean' on the live engine and establishes a roll angle to the left such that $\beta = 0$. Compute the roll angle and the new rudder and aileron deflections.

In the situation of $\phi = 0$, the pilot has to apply an aileron deflection δ_a. To reduce the required aileron deflection, fuel is being used from only one of the wing tanks.

(c) Now derive the equations of motion taking into account of Δm_f, which indicates the fuel mass difference between the left wing and the right, i.e. $\Delta m_f = m_{f_{\text{right wing}}} - m_{f_{\text{left wing}}}$, the center of mass of the fuel in each wing being located at 7.5 m from the plane of symmetry.

(d) Calculate β, δ_r and Δm_f such that $\delta_a = 0$ and $\phi = 0$. Which of the wings should the pilot have selected to consume fuel from?

For a), we simply set $\phi = 0$ so that we obtain the matrix equation

$$\begin{bmatrix} C_{Y_\beta} & 0 & C_{Y_{\delta_r}} \\ C_{l_\beta} & C_{l_{\delta_a}} & 0 \\ C_{n_\beta} & 0 & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \beta \\ \delta_a \\ \delta_r \end{bmatrix} = - \begin{bmatrix} 0 \\ C_{l_e} \\ C_{n_e} \end{bmatrix}$$

From the first row, we then have

$$C_{Y_\beta}\beta + C_{Y_{\delta_r}}\delta_r = 0$$
$$\beta = -\frac{C_{Y_{\delta_r}}}{C_{Y_\beta}}\delta_r$$

From the third row we obtain

$$C_{n_\beta}\beta + C_{n_{\delta_r}}\delta_r = -C_{n_e}$$
$$-C_{n_\beta}\frac{C_{Y_{\delta_r}}}{C_{Y_\beta}}\delta_r + C_{n_{\delta_r}}\delta_r = -C_{n_e}$$
$$\delta_r = -\frac{C_{n_e}}{C_{n_{\delta_r}} - C_{n_\beta}\frac{C_{Y_{\delta_r}}}{C_{Y_\beta}}}$$

From the second row, we also have

$$\delta_a = \frac{-C_{l_e} - C_{l_\beta}\beta}{C_{l_{\delta_a}}}$$

We can now plug in the numbers:

$$C_{n_e} = k \cdot \frac{\Delta T \cdot y_e}{\frac{1}{2}\rho V^2 S b} = 1.15 \cdot \frac{30000 \cdot 6}{\frac{1}{2} \cdot 0.653 \cdot 130^2 \cdot 75 \cdot 29} = 0.017248$$

$$\delta_r = -\frac{C_{n_e}}{C_{n_{\delta_r}} - C_{n_\beta}\frac{C_{Y_{\delta_r}}}{C_{Y_\beta}}} = \frac{-0.017248}{-0.095 - 0.16 \cdot \frac{0.3}{-0.9}} = 0.413\,952\,\mathrm{rad}$$

$$\beta = -\frac{C_{Y_{\delta_r}}}{C_{Y_\beta}}\delta_r = -\frac{0.3}{-0.9} \cdot 0.413952 = 0.137\,984\,\mathrm{rad}$$

$$\delta_a = \frac{-C_{l_e} - C_{l_\beta}\beta}{C_{l_{\delta_a}}} = \frac{-0.01 - -0.9 \cdot 0.137984}{-0.089} = -0.027\,17\,\mathrm{rad}$$

For b), we now set $\beta = 0$ to obtain the matrix equation

$$\begin{bmatrix} C_L & 0 & C_{Y_{\delta_r}} \\ 0 & C_{l_{\delta_a}} & 0 \\ 0 & 0 & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \phi \\ \delta_a \\ \delta_r \end{bmatrix} = -\begin{bmatrix} 0 \\ C_{l_e} \\ C_{n_e} \end{bmatrix}$$

From the third row, we have the equation

$$C_{n_{\delta_r}}\delta_r = -C_{n_e}$$
$$\delta_r = \frac{C_{n_e}}{C_{n_{\delta_r}}}$$

From the second row:

$$C_{l_{\delta_a}}\delta_a = -C_{l_e}$$
$$\delta_a = \frac{C_{l_e}}{C_{l_{\delta_a}}}$$

From the first row we also obtain:

$$C_L\phi + C_{Y_{\delta_r}}\delta_r = 0\phi \qquad = -\frac{C_{Y_{\delta_r}}}{C_L}\delta_r$$

Substituting in the numbers yields

$$\delta_r = \frac{C_{n_e}}{C_{n_{\delta_r}}} = -\frac{0.017248}{-0.095} = 0.181\,56\,\mathrm{rad}$$

$$\delta_a = \frac{C_{l_e}}{C_{l_{\delta_a}}} = -\frac{0.01}{-0.089} = 0.112\,36\,\mathrm{rad}$$

$$\phi = -\frac{C_{Y_{\delta_r}}}{C_L}\delta_r = -\frac{0.3}{0.531} \cdot 0.18156 = -0.102\,58\,\mathrm{rad}$$

For c), you may think, did I miss out on something? Did we ever discuss this before? The answer is no, indeed this is new, but you kinda have to be able to deal with new stuff on the exam as well. It's not so difficult when you understand the concept. Having more mass in your right wing then in your left wing (so when Δm_f is positive) will create a *positive* rolling moment. Furthermore, non-dimensionalising it is very similar to what happens to the thrust differential:

$$C_{m_f} = \frac{\Delta m_f \cdot y_f}{\frac{1}{2}\rho V^2 S b}$$

where Δm_f is the weight difference in *Newton* and $y_f = 7.5\,\mathrm{m}$. This only affects the equation for the rolling moment, and thus can be straightforwardly appended to our matrix equation. We simply add the new unknown Δm_f into the vector of unknowns, and add the corresponding column:

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 & C_{Y_{\delta_r}} & 0 \\ 0 & C_{l_\beta} & C_{l_{\delta_a}} & 0 & \frac{y_f}{\frac{1}{2}\rho V^2 S b} \\ 0 & C_{n_\beta} & 0 & C_{n_{\delta_r}} & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \beta \\ \delta_a \\ \delta_r \\ \Delta m_f \end{bmatrix} = -\begin{bmatrix} 0 \\ C_{l_e} \\ C_{n_e} \end{bmatrix}$$

Not so hard in the end.

For d), we set $\phi = 0$ and $\delta_a = 0$, obtaining the matrix equation

$$\begin{bmatrix} C_{Y_\beta} & C_{Y_{\delta_r}} & 0 \\ C_{l_\beta} & 0 & \frac{y_f}{\frac{1}{2}\rho V^2 S b} \\ C_{n_\beta} & C_{n_{\delta_r}} & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \delta_r \\ \Delta m_f \end{bmatrix} = -\begin{bmatrix} 0 \\ C_{l_e} \\ C_{n_e} \end{bmatrix}$$

Okay I'm kinda done with finding analytical expressions so I'll just invert the matrix:

$$\begin{bmatrix} C_{Y_\beta} & C_{Y_{\delta_r}} & 0 \\ C_{l_\beta} & 0 & \frac{y_f}{\frac{1}{2}\rho V^2 S b} \\ C_{n_\beta} & C_{n_{\delta_r}} & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \delta_r \\ \Delta m_f \end{bmatrix} = -\begin{bmatrix} 0 \\ C_{l_e} \\ C_{n_e} \end{bmatrix}$$

$$\begin{bmatrix} -0.9 & 0.3 & 0 \\ -0.09 & 0 & \frac{7.5}{\frac{1}{2}\cdot 0.653\cdot 130^2\cdot 75\cdot 29} \\ 0.16 & -0.095 & 0 \end{bmatrix} \begin{bmatrix} \beta \\ \delta_r \\ \Delta m_f \end{bmatrix} = -\begin{bmatrix} 0 \\ 0.01 \\ 0.017248 \end{bmatrix} = \begin{bmatrix} 0.137\,984\,\mathrm{rad} \\ 0.413\,952\,\mathrm{rad} \\ 3870.123\,\mathrm{N} \end{bmatrix}$$

In other words, the mass difference is $\Delta m_f = 394.51\,\mathrm{kg}$. As this is positive, this means the right wing contains more fuel, so the *left* tank needs to have been selected by the pilot.

Exam August 2013: problem 4 (25p)

Consider a twin-engined jet aircraft with a conventional configuration, from which the aircraft data is summarized in figure 3.5. During a cross-wind landing approach, the aircraft has to maintain a sideslip angle of $\beta = 6°$ to stay aligned with the runway. Due to a hydraulics failure the right wing aileron is stuck at an angle of $\delta_{fail} = +20°$. The left wing aileron is still fully functional. As a result of this, an unwanted rolling and yawing moment is acting on the aircraft.

m = 4800 kg	C_L = 0.46	V = 55 m s^{-1}	S = 31.80 m^2	b = 15.90 m
ρ = 1.225 kg m^{-3}	C_{Y_β} = -0.90	C_{ℓ_β} = -0.07	C_{n_β} = 0.13	$C_{Y_{\delta_a}}$ = 0
$C_{\ell_{\delta_a}}$ = -0.35	$C_{n_{\delta_a}}$ = -0.11	$C_{Y_{\delta_r}}$ = 0.21	$C_{\ell_{\delta_r}}$ = 0	$C_{n_{\delta_r}}$ = -0.05

Figure 3.5: Aircraft data.

The total non-dimensional rolling and yawing moment coefficients, which include the influence of both the damaged and the undamaged aileron, are given as follows:

$$C_{l_{\delta_a}}\delta_a \xrightarrow{failure} \frac{1}{2}C_{l_{\delta_a}}\delta_a + \frac{1}{2}C_{l_{\delta_a}}\delta_{a_{fail}}$$
$$C_{n_{\delta_a}}\delta_a \xrightarrow{failure} \frac{1}{2}C_{n_{\delta_a}}\delta_a + \frac{1}{4}C_{n_{\delta_a}}\delta_{a_{fail}}$$

(a) Derive, using the asymmetric equations of motions shown below, the equations of motion for steady, straight, sideslipping flight with one aileron stuck. Clearly indicate the assumptions you make. Provide your final answer in matrix format.

$$C_L\phi + C_{Y_\beta}\beta + \left(C_{Y_r} - 4\mu_b\right)\frac{rb}{2V} + C_{Y_{\delta_a}}\delta_a + C_{Y_{\delta_r}}\delta_r = 0$$
$$C_{l_\beta}\beta + C_{l_r}\frac{rb}{2V} + C_{l_{\delta_a}}\delta_a + C_{l_{\delta_r}}\delta_r = 0$$
$$C_{n_\beta}\beta + C_{n_r}\frac{rb}{2V} + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r = 0$$

(b) Calculate the required roll angle ϕ, the left wing aileron deflection δ_a and the rudder deflection δ_r in order to continue the cross-wing landing approach with $\beta = 6°$.

(c) It is given that the maximum aileron deflection is $\delta_{a_{max}} = \pm 25°$ and the maximum rudder deflection is $\delta_{r_{max}} = \pm 25°$. It is now clear that the pilot is unable to continue the cross-wind landing approach. Which control surface(s) is/are the limiting factor in this case?

(d) In order to reduce the deflections of the control surfaces, and to continue the cross-wind landing approach, the pilot sets the engines to generate a differential thrust. The non-dimensional moment produced by the differential thrust is $C_{n_e}\Delta T$ with C_{n_e} give by

$$C_{n_e} = \frac{y_e}{\frac{1}{2}\rho V^2 Sb}$$

with $y_e = 1.5$ the lateral distance between the engines and the center of gravity of the aircraft. Calculate the required differential thrust ΔT, the new roll angle ϕ and the new aileron deflection δ_a so that the control surface with the largest deflection under (b) has a deflection of 80% of its maximum deflection. The sideslip angle remains unchanged at $\beta = 6°$. The rolling effect due to differential thrust is neglected.

This is the hardest question I could find about this topic. However, it's once more mostly knowing your ways with matrix operations and you'll be fine. First, for a), we have for straight flight that $r = 0$; furthermore, from the data, $C_{Y_{\delta_a}} = 0$ and $C_{l_{\delta_r}} = 0$. Thus, our equations of motion become

$$C_L\phi + C_{Y_\beta}\beta + C_{Y_{\delta_r}}\delta_r = 0$$
$$C_{l_\beta}\beta + C_{l_{\delta_a}}\delta_a = 0$$
$$C_{n_\beta}\beta + C_{n_{\delta_a}}\delta_a + C_{n_{\delta_r}}\delta_r = 0$$

or in matrix format

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 & C_{Y_{\delta_r}} \\ 0 & C_{l_\beta} & C_{l_{\delta_a}} & 0 \\ 0 & C_{n_\beta} & C_{n_{\delta_a}} & C_{n_{\delta_r}} \end{bmatrix}\begin{bmatrix} \phi \\ \beta \\ \delta_a \\ \delta_r \end{bmatrix} = \mathbf{0}$$

Now, how do we deal with that aileron deflection? Well, the following happens: take the second equation, for the rolling moment: there, we actually have that basically, $\delta_a = \frac{1}{2}\delta_a + \frac{1}{2}\delta_{a_{fail}}$. Similarly, for the third equation, for the yawing moment, we have that basically, $\delta_a = \frac{1}{2}\delta_a + \frac{1}{4}\delta_{a_{fail}}$. In both of these, δ_a is the remaining unknown, but $\delta_{a_{fail}}$ is a known value and thus needs to be moved to the right-hand side of the equation. We can do this relatively easily however. Taking into account the minus sign when we move stuff to the right-hand side, we simply obtain

$$\begin{bmatrix} C_L & C_{Y_\beta} & 0 & C_{Y_{\delta_r}} \\ 0 & C_{l_\beta} & \frac{1}{2}C_{l_{\delta_a}} & 0 \\ 0 & C_{n_\beta} & \frac{1}{2}C_{n_{\delta_a}} & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \phi \\ \beta \\ \delta_a \\ \delta_r \end{bmatrix} = -\begin{bmatrix} 0 \\ \frac{1}{2}C_{l_{\delta_a}} \\ \frac{1}{4}C_{n_{\delta_a}} \end{bmatrix} \delta_{a_{fail}}$$

Furthermore, note that we said before that $\beta = 6°$, so we have to move that to the other side as well:

$$\begin{bmatrix} C_L & 0 & C_{Y_{\delta_r}} \\ 0 & \frac{1}{2}C_{l_{\delta_a}} & 0 \\ 0 & \frac{1}{2}C_{n_{\delta_a}} & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \phi \\ \delta_a \\ \delta_r \end{bmatrix} = -\begin{bmatrix} 0 & C_{Y_\beta} \\ \frac{1}{2}C_{l_{\delta_a}} & C_{l_\beta} \\ \frac{1}{4}C_{n_{\delta_a}} & C_{n_\beta} \end{bmatrix} \begin{bmatrix} \delta_{a_{fail}} \\ \beta \end{bmatrix}$$

It's nothing more complicated than that. Fair enough, maybe you wouldn't have come up with it, but now that you've seen it, it's nothing too difficult.

For b), I'll just invert the matrix. Plugging in the numbers yields

$$\begin{bmatrix} C_L & 0 & C_{Y_{\delta_r}} \\ 0 & \frac{1}{2}C_{l_{\delta_a}} & 0 \\ 0 & \frac{1}{2}C_{n_{\delta_a}} & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} \phi \\ \delta_a \\ \delta_r \end{bmatrix} = -\begin{bmatrix} 0 & C_{Y_\beta} \\ \frac{1}{2}C_{l_{\delta_a}} & C_{l_\beta} \\ \frac{1}{4}C_{n_{\delta_a}} & C_{n_\beta} \end{bmatrix} \begin{bmatrix} delta_{a_{fail}} \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} 0.46 & 0 & 0.21 \\ 0 & \frac{1}{2}\cdot -0.35 & 0 \\ 0 & \frac{1}{2}\cdot -0.11 & -0.05 \end{bmatrix} \begin{bmatrix} \phi \\ \delta_a \\ \delta_r \end{bmatrix} = -\begin{bmatrix} 0 & -0.9 \\ \frac{1}{2}\cdot -0.35 & -0.07 \\ \frac{1}{4}\cdot -0.11 & 0.13 \end{bmatrix} \begin{bmatrix} \frac{20\pi}{180} \\ \frac{6\pi}{180} \end{bmatrix}$$

$$\begin{bmatrix} \phi \\ \delta_a \\ \delta_r \end{bmatrix} = \begin{bmatrix} -0.0281\,\text{rad} \\ -0.3910\,\text{rad} \\ 0.5103\,\text{rad} \end{bmatrix} = \begin{bmatrix} -1.6095° \\ -22.4° \\ 29.24° \end{bmatrix}$$

For c), evidently it's the rudder deflection that's the limiting factor as that one exceeds the maximum rudder deflection whereas the aileron deflection doesn't.

For d), we simply append the ΔT to the matrix equation:

$$\begin{bmatrix} C_L & 0 & C_{Y_{\delta_r}} & 0 \\ 0 & \frac{1}{2}C_{l_{\delta_a}} & 0 & 0 \\ 0 & \frac{1}{2}C_{n_{\delta_a}} & C_{n_{\delta_r}} & C_{n_e} \end{bmatrix} \begin{bmatrix} \phi \\ \delta_a \\ \delta_r \\ \Delta T \end{bmatrix} = -\begin{bmatrix} 0 & C_{Y_\beta} \\ \frac{1}{2}C_{l_{\delta_a}} & C_{l_\beta} \\ \frac{1}{4}C_{n_{\delta_a}} & C_{n_\beta} \end{bmatrix} \begin{bmatrix} delta_{a_{fail}} \\ \beta \end{bmatrix}$$

Furthermore, we set $\delta_r = 0.8 \cdot 25 = 20°$, so we can move it to the other side:

$$\begin{bmatrix} C_L & 0 & 0 \\ 0 & \frac{1}{2}C_{l_{\delta_a}} & 0 \\ 0 & \frac{1}{2}C_{n_{\delta_a}} & C_{n_e} \end{bmatrix} \begin{bmatrix} \phi \\ \delta_a \\ \Delta T \end{bmatrix} = -\begin{bmatrix} 0 & C_{Y_\beta} & C_{Y_{\delta_r}} \\ \frac{1}{2}C_{l_{\delta_a}} & C_{l_\beta} & 0 \\ \frac{1}{4}C_{n_{\delta_a}} & C_{n_\beta} & C_{n_{\delta_r}} \end{bmatrix} \begin{bmatrix} delta_{a_{fail}} \\ \beta \\ \delta_r \end{bmatrix}$$

$$\begin{bmatrix} 0.46 & 0 & 0 \\ 0 & \frac{1}{2}\cdot -0.35 & 0 \\ 0 & \frac{1}{2}\cdot -0.11 & \frac{1.5}{\frac{1}{2}\cdot 1.225\cdot 55^2\cdot 31.8\cdot 15.9} \end{bmatrix} \begin{bmatrix} \phi \\ \delta_a \\ \Delta T \end{bmatrix} = -\begin{bmatrix} 0 & -0.9 & 0.21 \\ \frac{1}{2}\cdot -0.35 & -0.07 & 0 \\ \frac{1}{4}\cdot -0.11 & 0.13 & -0.05 \end{bmatrix} \begin{bmatrix} \frac{20\pi}{180} \\ \frac{6\pi}{180} \\ \frac{20\pi}{180} \end{bmatrix}$$

$$\begin{bmatrix} \phi \\ \delta_a \\ \Delta T \end{bmatrix} = \begin{bmatrix} 0.045\,553\,\text{rad} \\ -0.390\,95\,\text{rad} \\ -5036\,\text{N} \end{bmatrix}$$

Again, plug the angles in in radians. Although the results for ϕ and δ_a would still be correct (their units would simply be degrees), ΔT would be hopelessly wrong so it's a really stupid way of losing points.

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