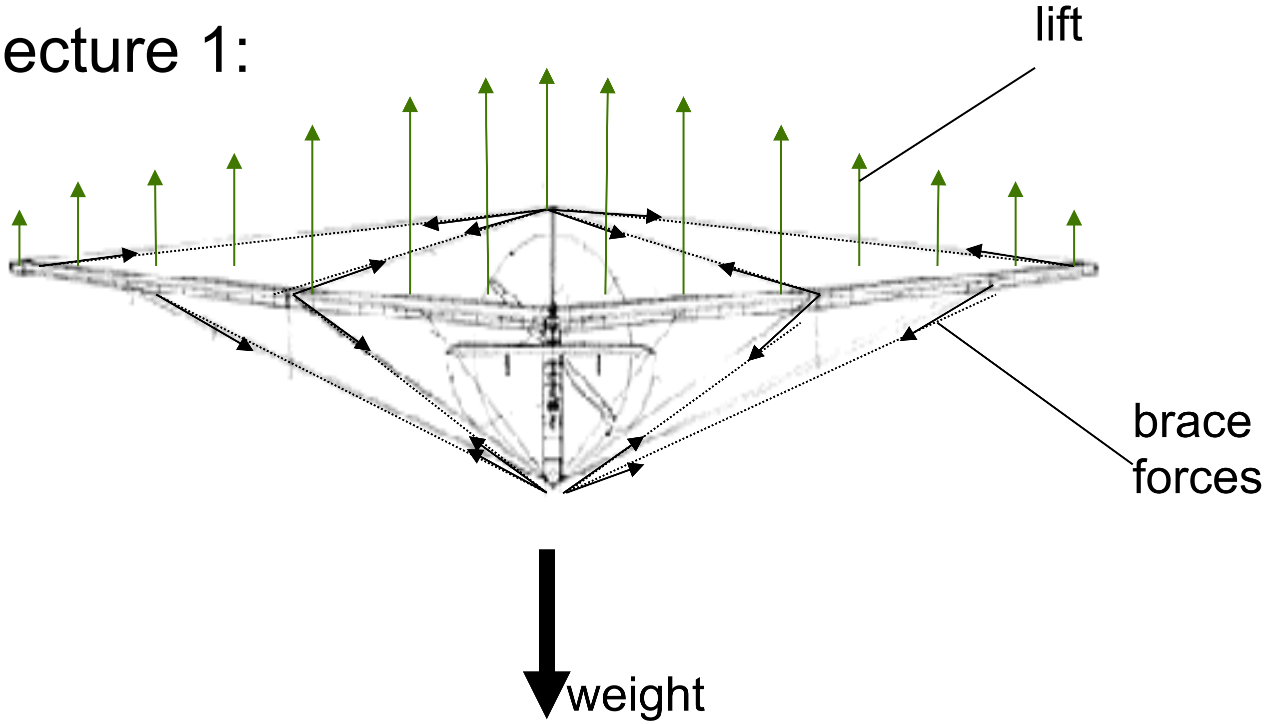
Megson: 8.1, 8.3, 8.5

Buckling of beams

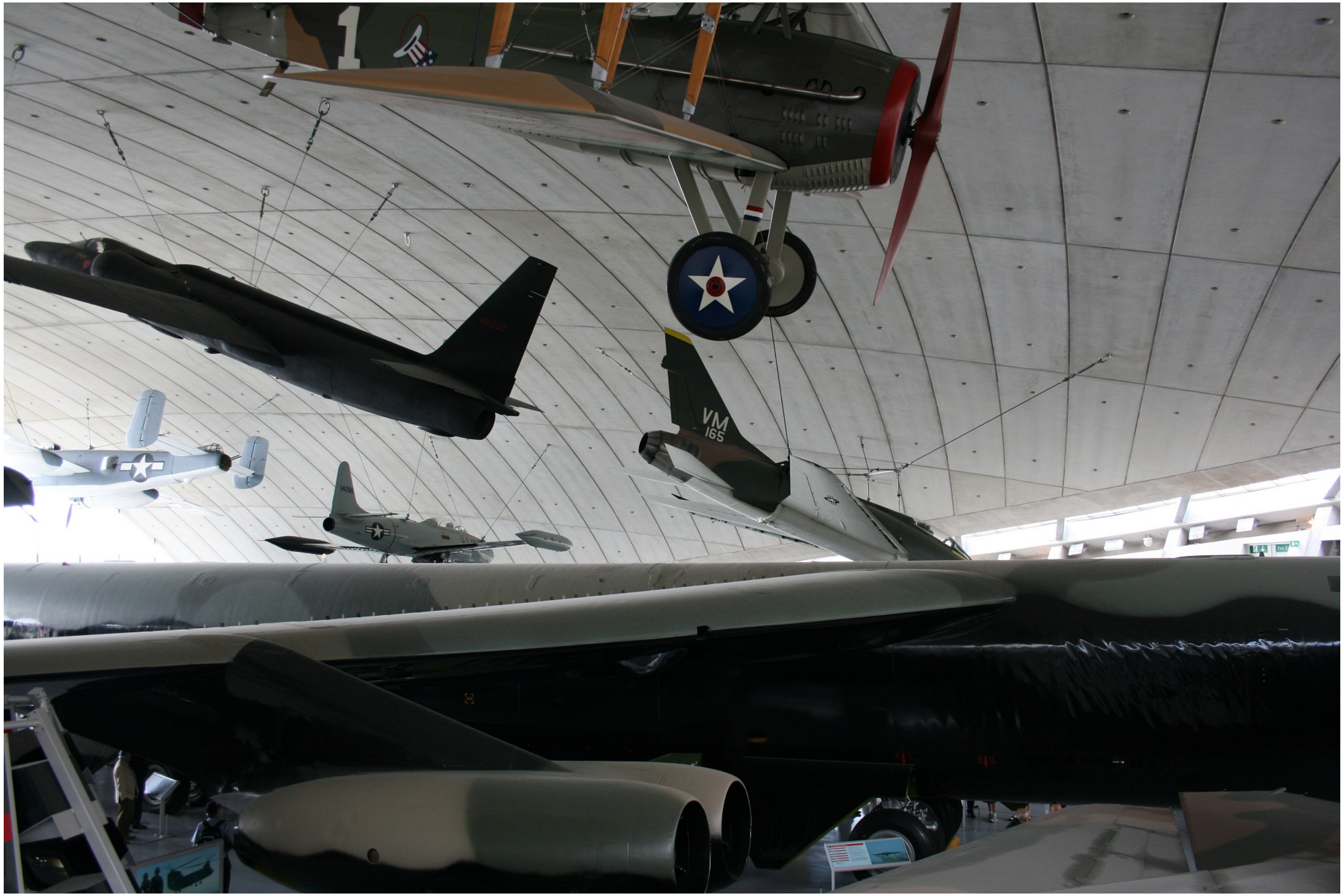
• back in lecture 1:

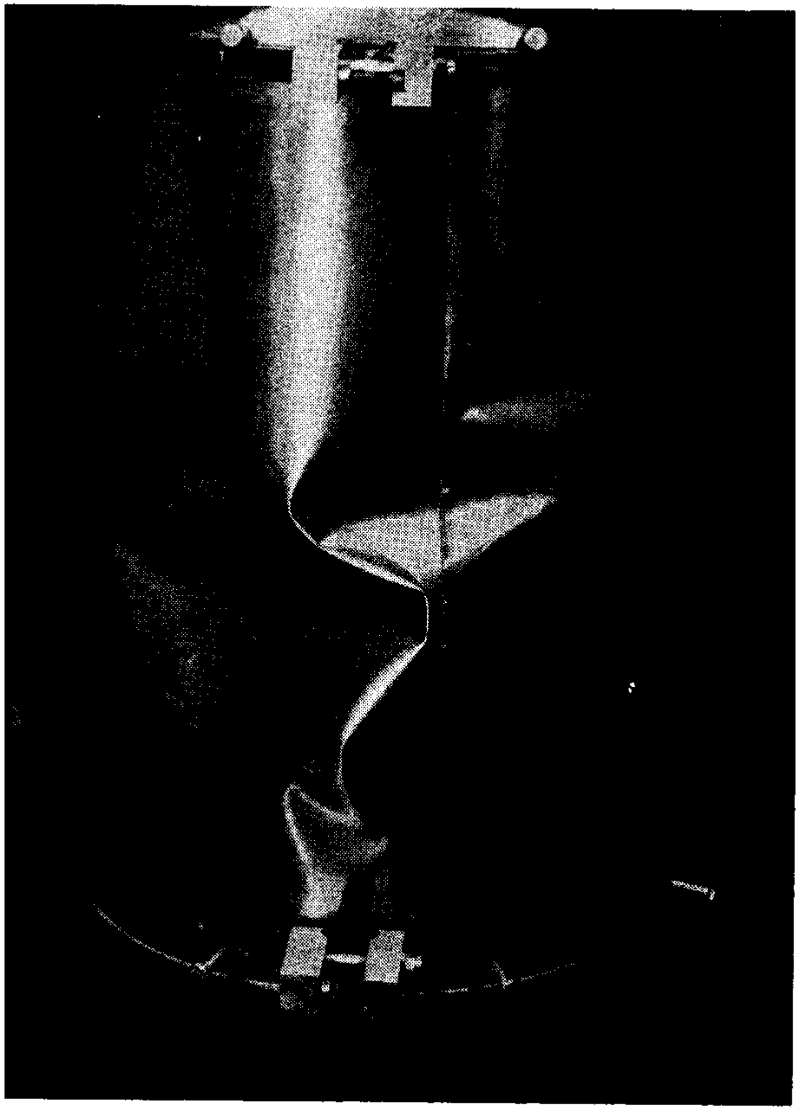


• (some of) the braces are in tension with a force component parallel to the wing spar putting it under compression

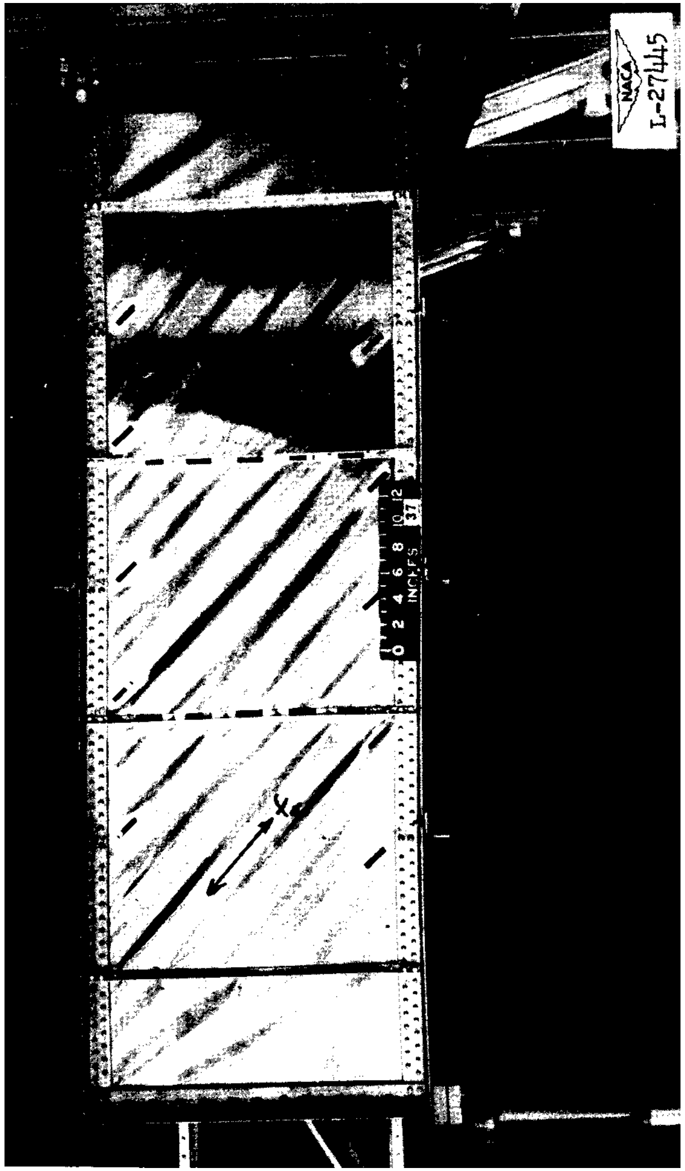
• this can cause failure not by failure strength but (more often) by instability or buckling

B-52 buckling on the ground





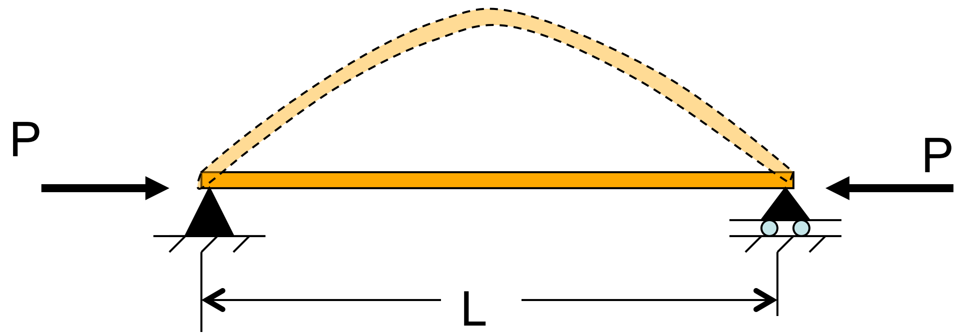
Buckling of cylinder under compression (Bruhn C8.2)



Buckling of panel under shear (Bruhn C11.3)

Megson: 8.1, 8.3, 8.5

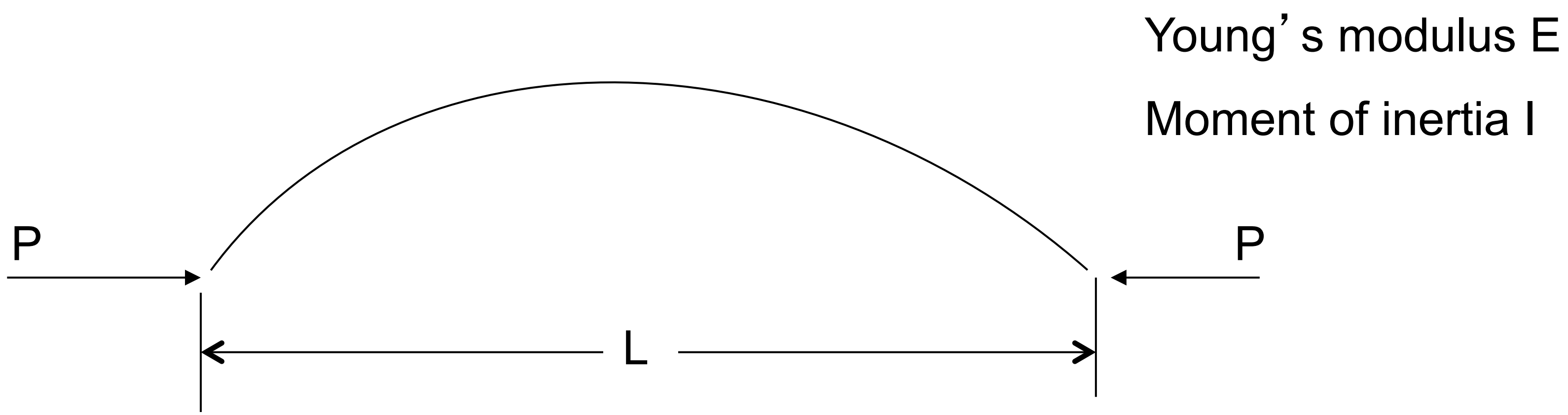
Buckling of beams



both ends pinned

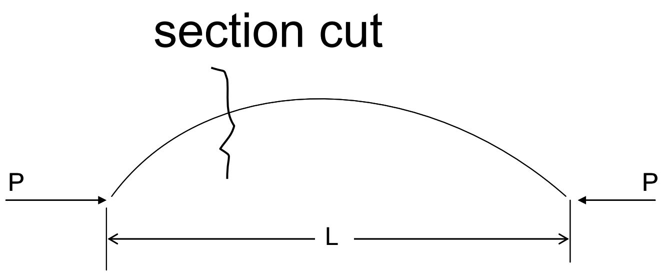
- a short beam under compression fails by material failure
- a long beam under compression deflects away from its axis (buckles) and can support no more load

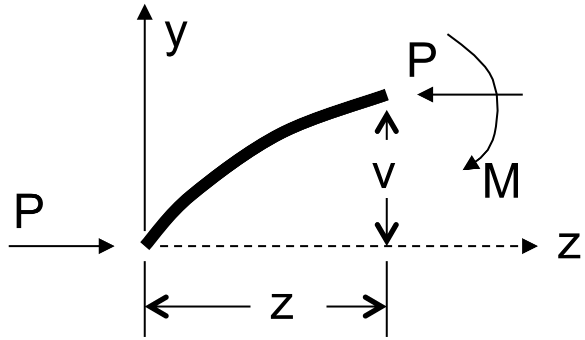
Euler column buckling (governing eq)



- consider a beam of length L under compressive load P at its two ends; P is acting at the neutral axis
- for sufficiently high value of P, the beam buckles (displaces away from its axis); let I the moment of inertia about the axis the beam is bending
- for simplicity, assume the beam is simply supported

Euler column buckling (governing eq)





• moment equilibrium of the cut section gives (taking moments about the left end of the beam):

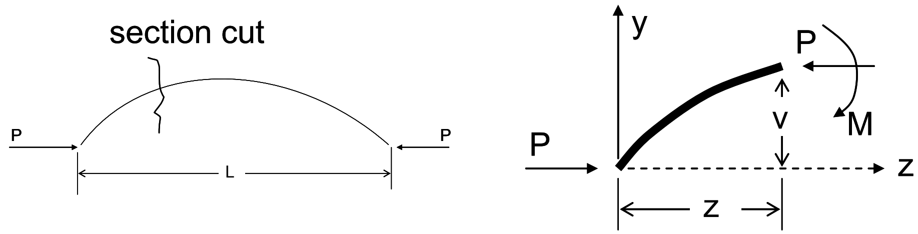
$$M = Pv \tag{13.1}$$

with v the displacement along y

• from standard beam theory, the moment is related to the local radius of curvature via (see lecture 2)

$$M = -\frac{EI}{R} \tag{13.2}$$

Euler column buckling (governing eq)



$$M = Pv \qquad (13.1)$$

$$M = -\frac{EI}{R} \qquad (13.2)$$

• in addition, from basic calculus, the radius of curvature R is related to the second derivative of the deflection v:

$$\frac{1}{R} = \frac{d^2v}{dz^2} \longrightarrow M = -EI\frac{d^2v}{dz^2} \qquad (13.3)$$

• combining (13.1)-(13.3), the governing equation for v is obtained (eliminating M and R):

$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = 0 \qquad (13.4)$$

Euler column buckling (solution to governing eq)

$$\frac{d^2 v}{dz^2} + \frac{P}{EI} v = 0 \qquad (13.4)$$

- to solve, assume v=Ce^{pz} and substitute in eq (13.4)
- cancelling common factors we get the following eq for p:

$$p^2 + \frac{P}{EI} = 0 \Rightarrow p^2 = -\frac{P}{EI} \Rightarrow p = \pm i\sqrt{\frac{P}{EI}} \qquad (13.5)$$

that is, p is pure imaginary because P/EI is positive

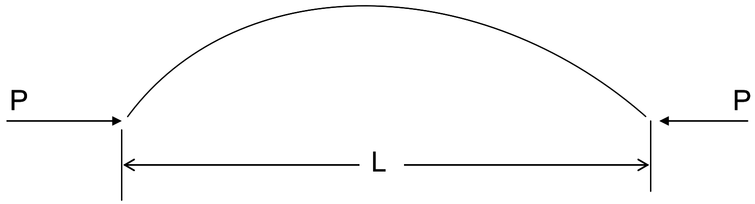
- then, the assumed solution becomes:

$$v = C_1 e^{i\sqrt{\frac{P}{EI}}z} + C_2 e^{-i\sqrt{\frac{P}{EI}}z} \qquad (13.6)$$

- from the theory of complex variables, $e^{i\theta} = \cos\theta + i\sin\theta$
- using that to substitute in (13.6) and collecting terms:

$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z$$ with A and B new unknown constants (13.7)

Euler column buckling (solution to governing eq – BC's)



$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z \tag{13.7}$$

- note the solution is in terms of two unknown constants, A, and B, as it should since it is a 2nd ODE

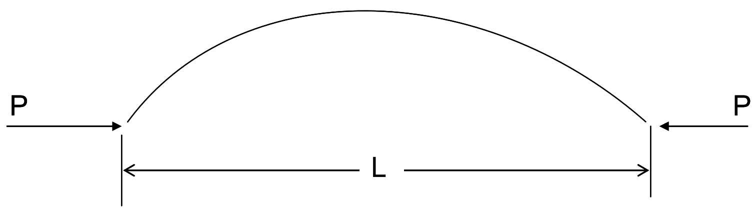
- assume now (as already mentioned) that the beam is simply supported at its ends; this means that

$$v(z = 0) = 0$$

$$v(z = L) = 0$$

- using the first of the BC's: $A = 0$ (13.8)
- use that to substitute in the second of the BC's: $B\sin\sqrt{\frac{P}{EI}}L = 0$

Euler column buckling (solution to governing eq – BC's)



$$v = B\sin\sqrt{\frac{P}{EI}}z \tag{13.7}$$

$$B\sin\sqrt{\frac{P}{EI}}L = 0 \tag{13.8}$$

- there are two possibilities from eq. (13.8):
  - either B=0 which means v=0; this is a "trivial" but true solution; it corresponds to the case where the beam stays straight (only contracts under P)
  - or $\sin\sqrt{\frac{P}{EI}}L = 0 \Rightarrow \sqrt{\frac{P}{EI}}L = n\pi \Rightarrow \boxed{P_{crit} = \frac{n^2\pi^2 EI}{L^2}}$ for any n (13.9)
- that is, there is a set of distinct values of P, P_{cr} given by eq. (13.9) for which v is non-zero and the beam (column) buckles

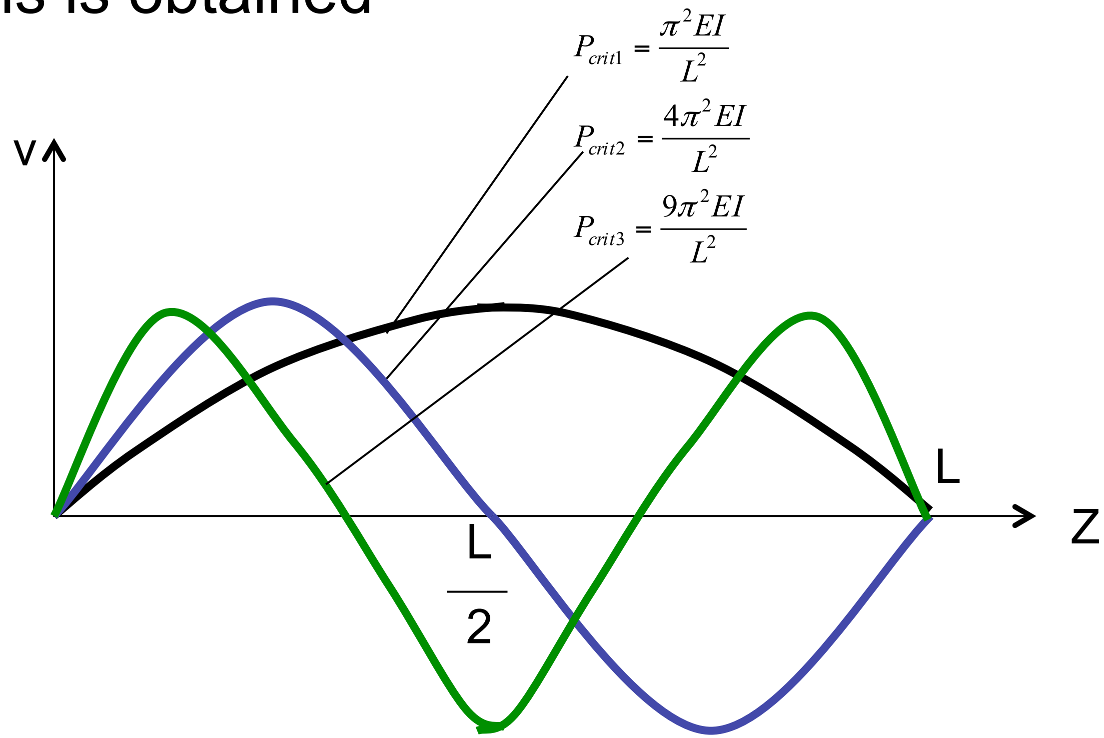
Euler column buckling – implications of solution

$$v = B\sin\sqrt{\frac{P}{EI}}z \qquad (13.7)$$

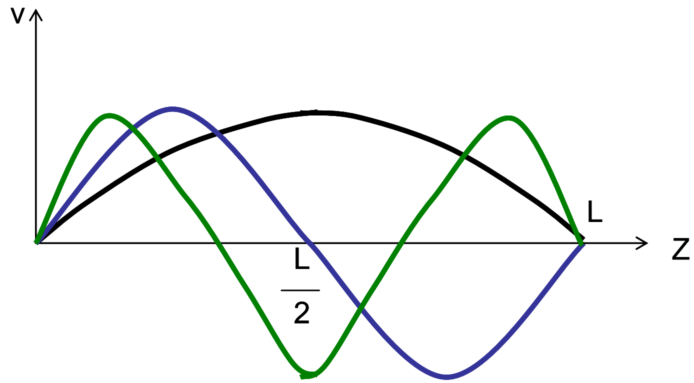
(Note that our solution cannot determine B!)

$$P_{crit} = \frac{n^2\pi^2 EI}{L^2} \qquad (13.9)$$

- for each value of n, a different solution for the beam deflections is obtained



Euler column buckling – implications of solution

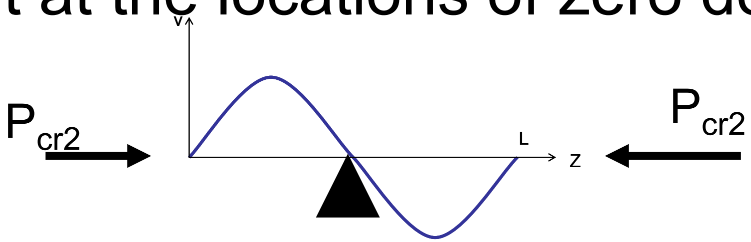


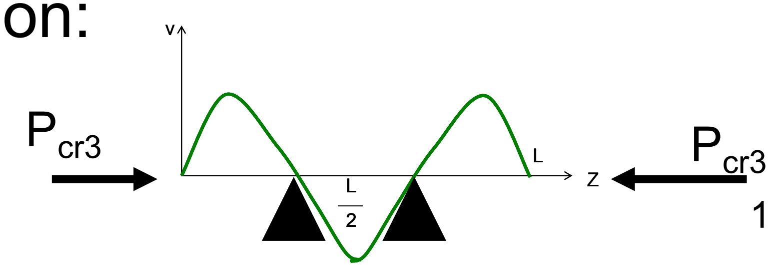
$$v = B\sin\sqrt{\frac{P}{EI}}z \tag{13.7}$$

$$P_{crit} = \frac{\pi^2 EI}{L^2} \tag{13.9}$$

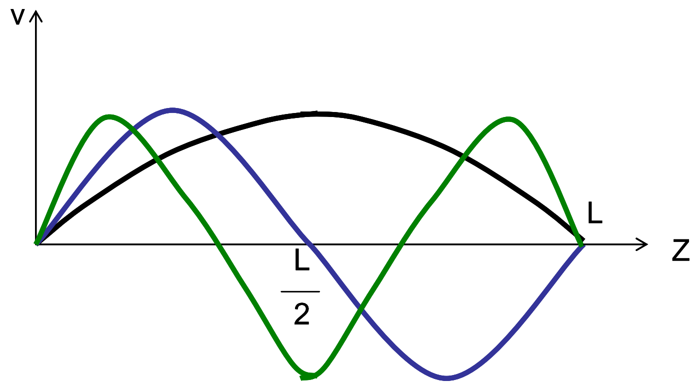
• in practice, it is impossible to have a perfect column with perfect boundary conditions and perfect load introduction; as a result, **all columns in nature** buckle in the first mode (n=1)

• to "force" a beam to buckle in higher modes, need to "hold" it at the locations of zero deflection:





Euler column buckling – implications of solution



$$v = B\sin\sqrt{\frac{P}{EI}}z \qquad (13.7)$$

$$P_{crit} = \frac{\pi^2 EI}{L^2} \qquad (13.9)$$

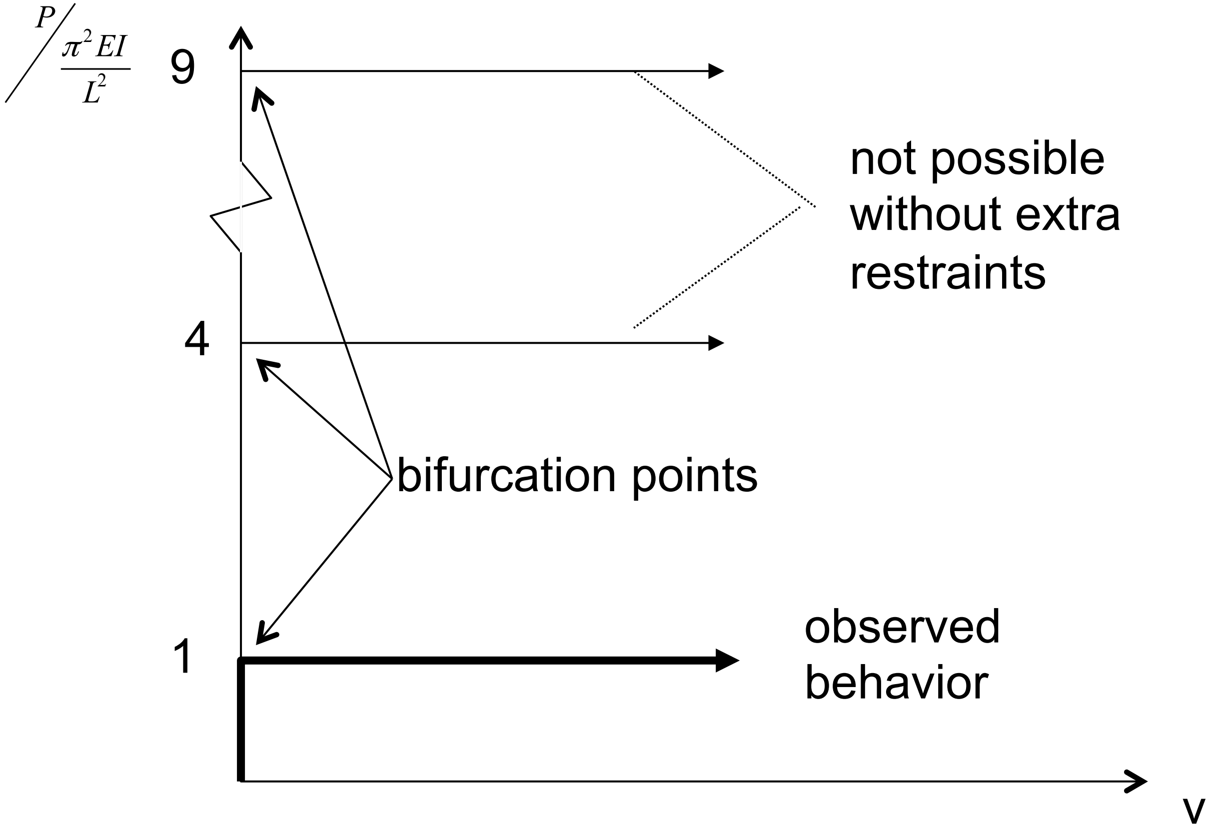
• it is also important to note that the solution for v still has one unknown coefficient, B

• the mathematical reason is that this is an eigenvalue problem in the form of a Sturm-Liouville problem

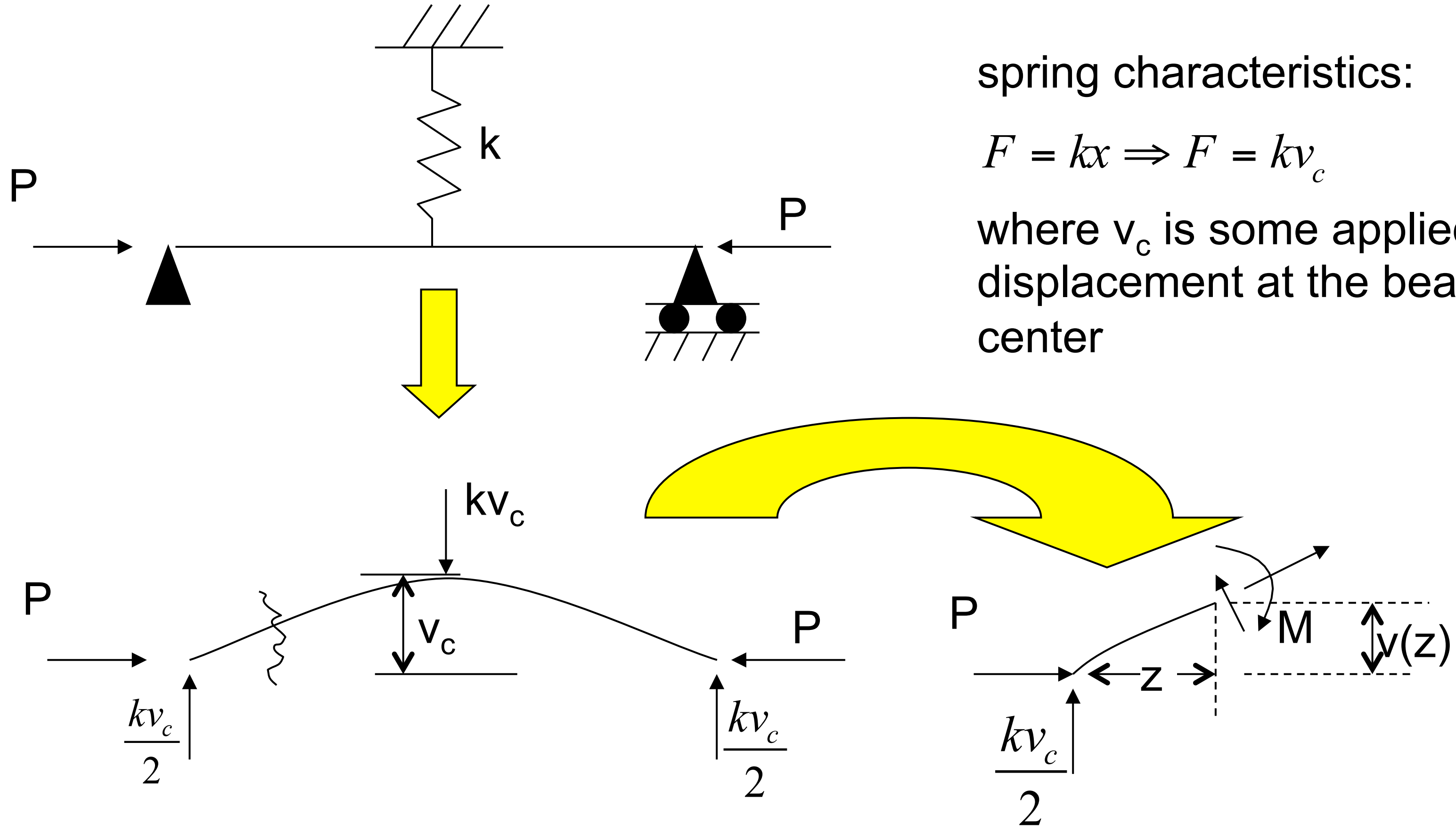
• the physical reason is that one more condition is needed to determine B related to the fact that as the beam deflects its projection on the z axis is no longer L but something less (=shortening condition)

Euler column buckling – implications of solution

- in a graphical representation of P versus v:



Buckling of simply-supported beam with an elastic support at its center



spring characteristics:

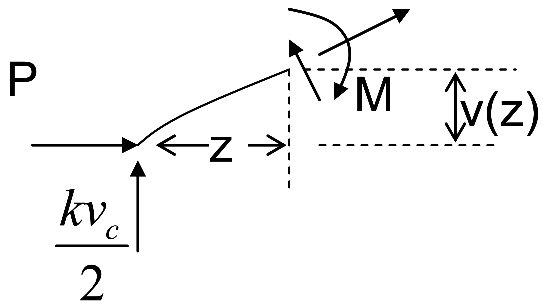
$$F = kx \Rightarrow F = kv_c$$

where v_c is some applied displacement at the beam center

taking moments about the cut:

$$M = Pv - \frac{kv_c}{2} z \qquad (13.10)$$

Buckling of simply-supported beam with an elastic support at its center



$$M = Pv - \frac{kv_c}{2} z \quad (13.10)$$

$$M = -EI \frac{d^2 v}{dz^2} \quad (13.3)$$

- combining (13.10) and (13.3):

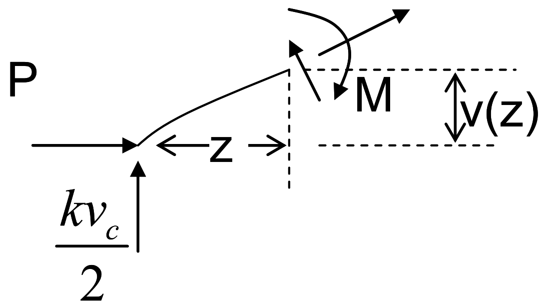
$$\frac{d^2 v}{dz^2} + \frac{P}{EI} v = \frac{kv_c}{2EI} z \quad (13.11)$$

note that v_c here is the displacement of the beam at the center; this is imposed externally (from the spring) so, unlike the previous solution, the beam deflections are completely determined here

- the solution to (13.11) consists of the homogeneous and the particular solution

- the homogeneous solution is identical to the solution to (13.4) (**before** imposing the BC's): $v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z$ (13.7)

Buckling of simply-supported beam with an elastic support at its center



$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = \frac{kv_c}{2EI}z \qquad (13.11)$$

$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z \qquad (13.7)$$

- for the particular solution to (13.11), try:

$$v_p = Cz \qquad (13.12)$$

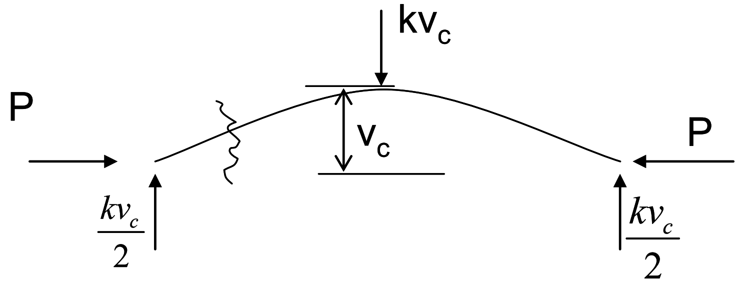
- substituting in (13.11) and solving for C:

$$\frac{P}{EI}Cz = \frac{kv_c}{2EI}z \Rightarrow C = \frac{kv_c}{2P} \qquad (13.13)$$

- combining particular and homogeneous solutions:

$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z + \frac{kv_c}{2P}z \qquad (13.14)$$

Buckling of simply-supported beam with an elastic support at its center



$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z + \frac{kv_c}{2P}z \quad (13.14)$$

- note that we really have three unknowns: A, B, and v_c; or, if we fix v_c, the unknowns are A, B, and k (k and v_c are not independent)

- so we need three boundary conditions:

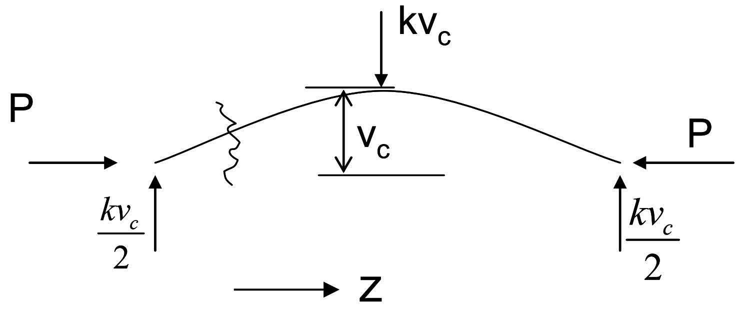
$$v(z=0)=0$$

$$v\left(z=\frac{L}{2}\right)=v_c$$

$$\frac{dv}{dz}\left(z=\frac{L}{2}\right)=0$$

note that v(z=L)=0 is another condition but it corresponds to the 2nd half of the beam for which the governing equation is slightly different so it is not applicable to the present differential equation

Buckling of simply-supported beam with an elastic support at its center



$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z + \frac{kv_c}{2P}z \qquad (13.14)$$

$$v(z=0) = 0$$

$$v\left(z = \frac{L}{2}\right) = v_c$$

$$\frac{dv}{dz}\left(z = \frac{L}{2}\right) = 0$$

note Megson has a typo in this eq!

- the 1st boundary condition gives A=0
- the 2nd boundary condition gives $$B = \frac{v_c}{\sin\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)}\left(1 - \frac{kL}{4P}\right) \qquad (13.15)$$
- the 3d boundary condition gives $$B\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) + \frac{kv_c}{2P} = 0$$
- and using (13.15): $$\left(1 - \frac{kL}{4P}\right)\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) + \frac{k}{2P}\sin\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) = 0$$
- rearranging: $$P = \frac{kL}{4}\left[1 - \frac{\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)}{\sqrt{\frac{P}{EI}}\frac{L}{2}}\right] \qquad (13.16)$$

the rearranging part…

$$\left(1-\frac{kL}{4P}\right)\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)+\frac{k}{2P}\sin\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)=0\Rightarrow$$

$$\left(1-\frac{kL}{4P}\right)\sqrt{\frac{P}{EI}}\cos\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)=-\frac{k}{2P}\sin\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\Rightarrow$$

$$\left(1-\frac{kL}{4P}\right)=-\frac{k}{2P\sqrt{\frac{P}{EI}}}\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\Rightarrow$$

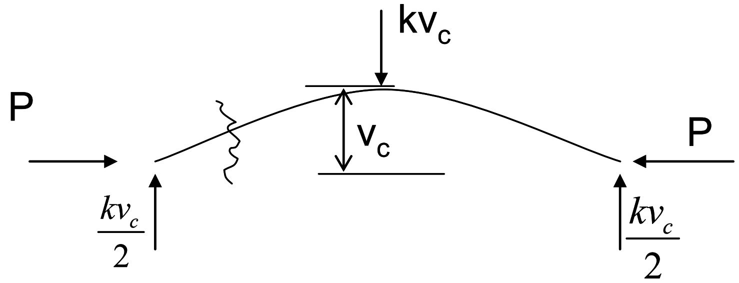
$$P-\frac{kL}{4}=-\frac{k}{2\sqrt{\frac{P}{EI}}}\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\Rightarrow$$

$$P=\frac{kL}{4}-\frac{kL}{2L\sqrt{\frac{P}{EI}}}\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\Rightarrow$$

$$P=\frac{kL}{4}-\frac{2kL}{4L\sqrt{\frac{P}{EI}}}\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)\Rightarrow$$

$$P=\frac{kL}{4}\left[1-\frac{\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)}{\sqrt{\frac{P}{EI}}\frac{L}{2}}\right]$$

Buckling of simply-supported beam with an elastic support at its center



$$P = \frac{kL}{4}\left[1 - \frac{\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)}{\sqrt{\frac{P}{EI}}\frac{L}{2}}\right] \qquad (13.16)$$

- Discussion:

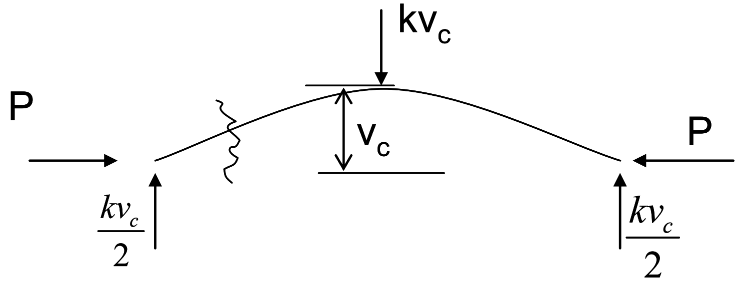
  – eq (13.16) for the buckling load has P on both sides of the equation and one must solve it by iteration

  – by changing the value of k one can obtain some special cases:

    – $k = \dfrac{4P}{L}$

    – $k = \infty$

Beam with elastic support at midspan – special cases



$$P = \frac{kL}{4}\left[1 - \frac{\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)}{\sqrt{\frac{P}{EI}}\frac{L}{2}}\right] \qquad (13.16)$$

- case 1: $k = \dfrac{4P}{L}$ substituting in (13.16)

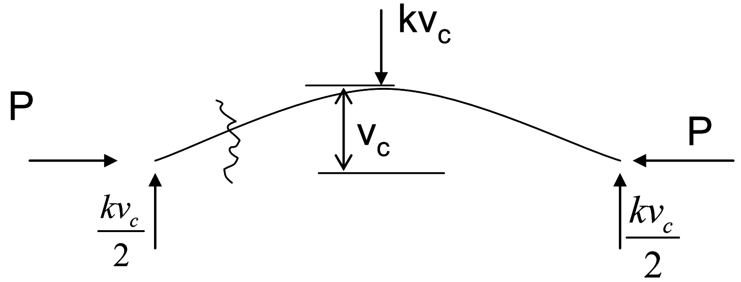
$$1 = 1 - \frac{\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)}{\sqrt{\frac{P}{EI}}\frac{L}{2}} \Rightarrow \tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) = 0 \Rightarrow \sqrt{\frac{P}{EI}}\frac{L}{2} = \begin{cases} 0 \\ \pi \\ 2\pi \\ \ldots \end{cases}$$

P=0; trivial

$$P = \frac{4\pi^2 EI}{L^2}$$

(1st) buckling mode

Beam with elastic support at midspan – special cases



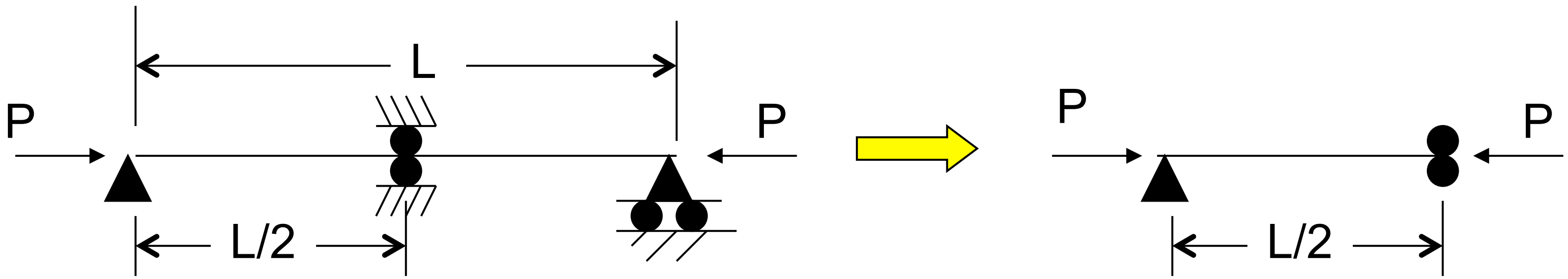
$$P = \frac{kL}{4}\left[1 - \frac{\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)}{\sqrt{\frac{P}{EI}}\frac{L}{2}}\right] \qquad (13.16)$$

- case 2: k=∞

- in this case, if the spring stiffness is infinite, the center of the beam does not move: $v_c=0$

- this would correspond to a case where the left end of the beam is simply supported (z=0) and the right end (which is the middle of the original beam at z=L/2!) has v=0 AND dv/dz=0 i.e. it is clamped):

Beam with elastic support at midspan – special cases



$$P = \frac{kL}{4}\left[1 - \frac{\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right)}{\sqrt{\frac{P}{EI}}\frac{L}{2}}\right] \qquad (13.16)$$

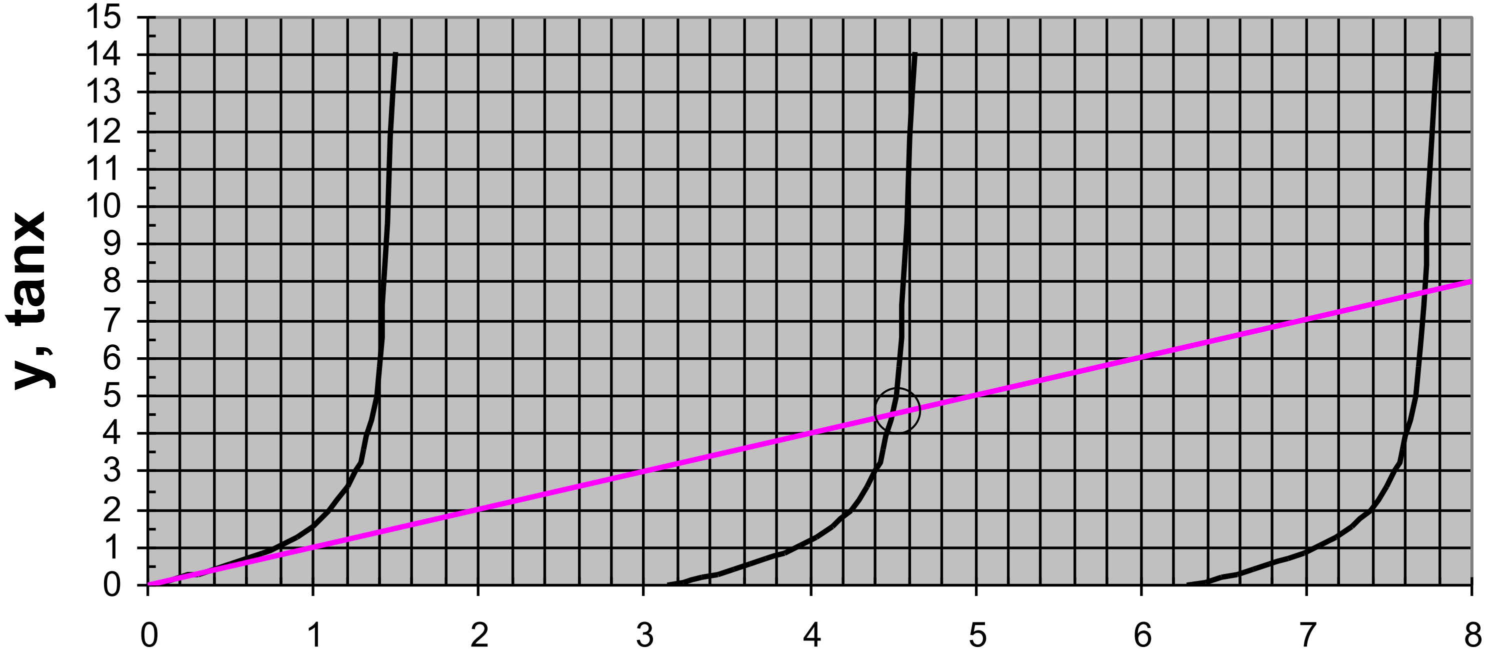
if k=∞, the RHS of (13.16) is ∞x(constant) while the LHS is a constant **finite** number; the only way this equation is still valid, is if the constant in the RHS equals zero (if it were non-zero and finite RHS=∞ and if it were ∞, RHS=∞: But ∞x (0)= indeterminate and can therefore equal a finite number)

- therefore, $\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) = \sqrt{\frac{P}{EI}}\frac{L}{2}$ (13.17)

gives an equation that can be solved numerically to obtain the buckling load of a column of length L/2 with one end simply supported (or pinned) and the other fixed (or clamped)

Beam with elastic support at midspan – special cases

solving graphically:

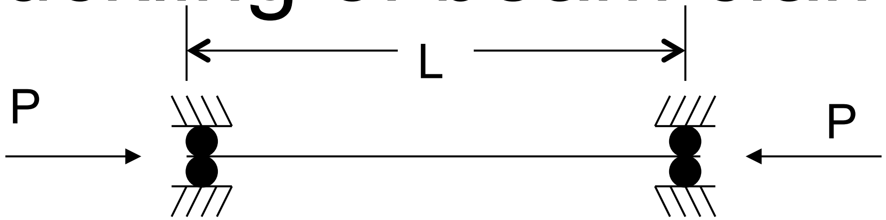


$$\tan\left(\sqrt{\frac{P}{EI}}\frac{L}{2}\right) = \sqrt{\frac{P}{EI}}\frac{L}{2} = \mathbf{x}$$

the first crossing between y=x and y=tanx is at approximately x=4.493; then

$$P = \frac{4(4.493)^2 EI}{L^2}$$

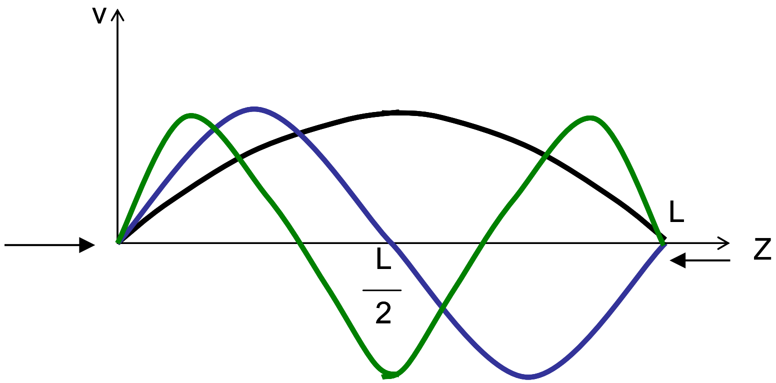
Buckling of beam clamped at both ends

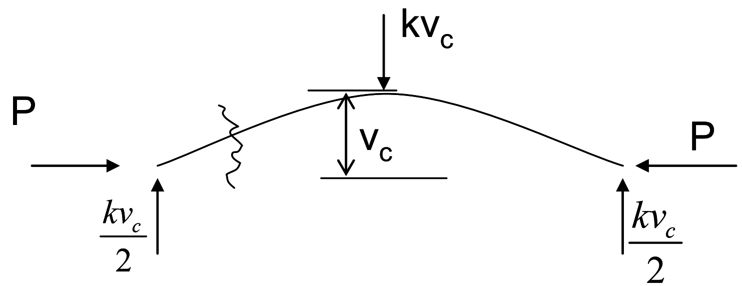


• before trying to solve the governing differential equation, we check if we can solve this as a special case from previous problems; what we need is a buckling solution that has

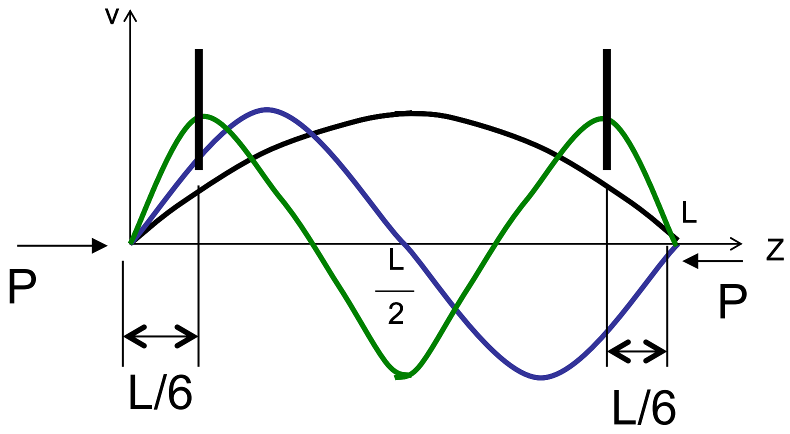
$$v = \frac{dv}{dz} = 0 \quad \text{at z=0 AND z=L}$$

• previous solutions:

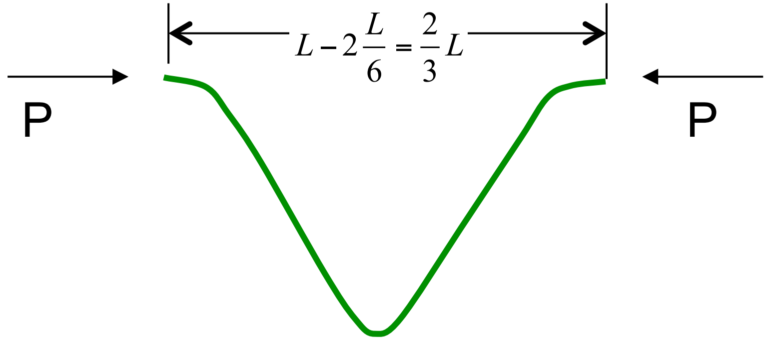




Buckling of beam clamped at both ends



isolate the "green" beam between the two vertical lines"



- so a simply supported beam of length L buckling in its third mode has the same buckling load as a clamped beam of length 2L/3

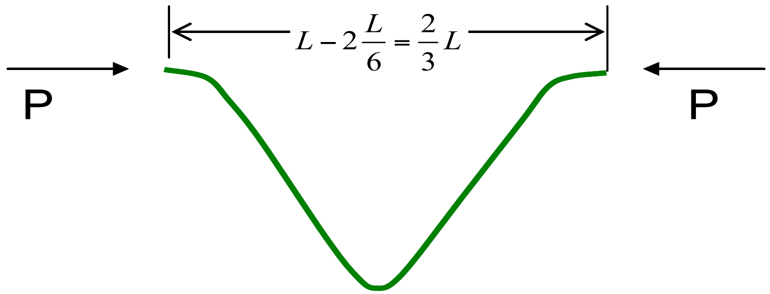
- the buckling load of a ss beam in 3d mode was found as:

$$P_{crit} = \frac{9\pi^2 EI}{L^2}$$

- this is the buckling load of a clamped beam with length 2L/3

- to get the buckling load of a clamped beam with length L,

Buckling of beam clamped at both ends



$$P_{crit}=\frac{9\pi^2 EI}{L^2}$$

- substitute for L in the above expression, 3L/2 (to increase length from 2/3L to L); then the buckling load for a clamped beam of length L is:

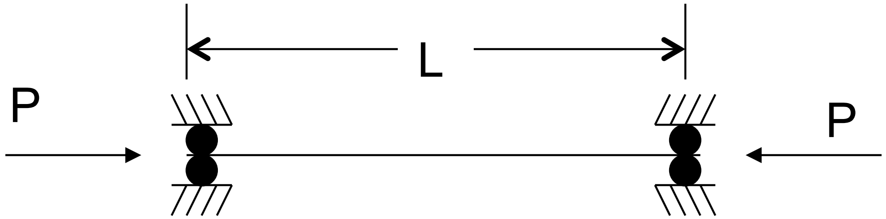
$$P_{crit}=\frac{9\pi^2 EI}{\left(\frac{3L}{2}\right)^2}\Rightarrow \boxed{P_{crit}=\frac{4\pi^2 EI}{L^2}} \qquad (13.18)$$

- note that, compared to a ss beam, eq.(13.9) with n=1 , a clamped beam has four times higher buckling load

$$P_{crit}=\frac{\pi^2 EI}{L^2} \qquad (13.9)$$

Buckling of beam clamped at both ends

• or, for the mathematically inclined, one can solve the governing differential equation



• the boundary conditions, as already mentioned, are

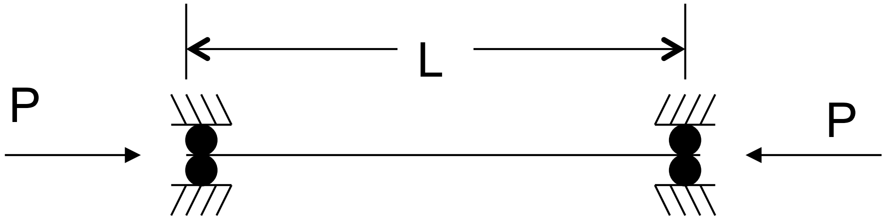
$$v = \frac{dv}{dz} = 0$$ at z=0 AND z=L; i.e. four conditions => we need an expression for v that has four unknown constants

• our original differential equation, (13.4)

$$\frac{d^2 v}{dz^2} + \frac{P}{EI} v = 0 \qquad (13.4)$$

being second order, allows ONLY two unknown constants

Buckling of beam clamped at both ends



- a more general (higher order) equation describing the beam deflections for this case is needed

- recall from basic beam bending (lecture 2)

$$S_y = \frac{\partial M_x}{\partial z} \qquad -p = \frac{\partial S_y}{\partial z} \qquad (2.10)$$

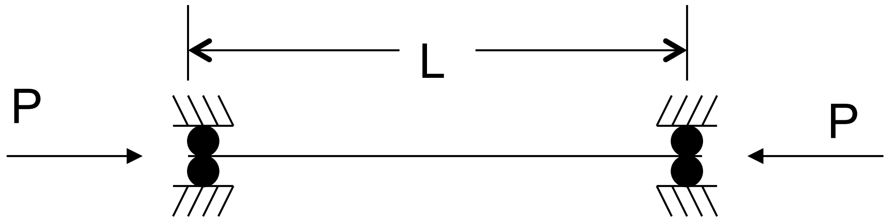
- also from earlier $M = -EI\dfrac{d^2 v}{dz^2}$ (13.3)

- from (2.10) and (2.10a), eliminating S_y: $-p = \dfrac{\partial^2 M_x}{\partial z^2}$

- using this to substitute in (13.3): $\boxed{EI\dfrac{d^4 v}{dz^4} = p}$ (13.19)

where subscript x was dropped and the partial derivatives became total for 1-D problem (dependence on only one variable)

Buckling of beam clamped at both ends



$$EI\frac{d^4v}{dx^4} = p \qquad (13.19)$$

$$-p = \frac{\partial^2 M_x}{\partial z^2}$$

$$M = Pv$$

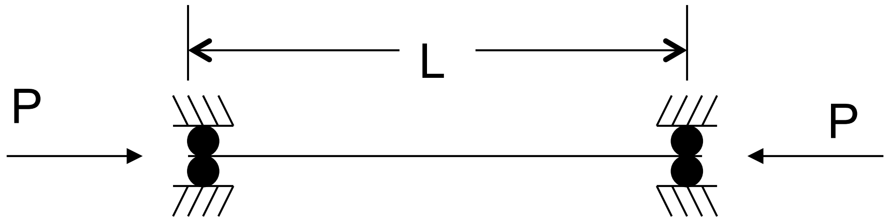
- eq. (13.19) is the more general equation for beam deflections of which our previous eq. (13.3) is a special case

- combining the three equations above, (13.19) becomes

$$\boxed{\frac{d^4v}{dz^4} + \frac{P}{EI}\frac{d^2v}{dz^2} = 0} \qquad (13.20)$$

- this is the general equation for a beam under compressive load P **irrespective** of the boundary conditions; it is equivalent to (13.19) when only axial load is applied and is more general than (13.3)

Buckling of beam clamped at both ends



$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = 0 \qquad (13.4)$$

$$\frac{d^4v}{dz^4} + \frac{P}{EI}\frac{d^2v}{dz^2} = 0 \qquad (13.20)$$

- integrating (13.20) once:

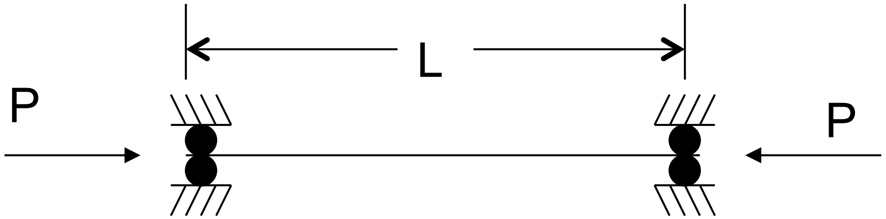
$$\frac{d^3v}{dz^3} + \frac{P}{EI}\frac{dv}{dz} = C_1$$ C_1 is an unknown constant

- integrating this equation once more:

$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = C_1 z + D_1 \qquad (13.21)$$ D_1 is an unknown constant

- if we compare (13.21) to the (13.4) the equation we had for buckling of a simply supported beam, we see that the only difference is the non-homogeneous term on the RHS

Buckling of beam clamped at both ends



$$\boxed{\frac{d^2v}{dz^2}+\frac{P}{EI}v} = 0 \qquad (13.4)$$

$$\frac{d^4v}{dz^4}+\frac{P}{EI}\frac{d^2v}{dz^2} = 0 \qquad (13.20)$$

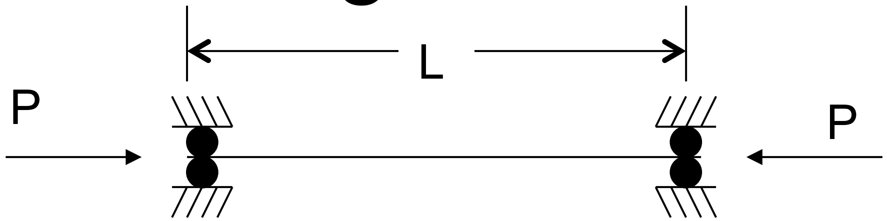
$$\boxed{\frac{d^2v}{dz^2}+\frac{P}{EI}v} = C_1 z + D_1 \qquad (13.21)$$

• the solution to (13.20) is then the same as the solution to (13.21); in turn, the solution to (13.21) is the sum of the homogeneous solution and the particular solution

• but the homogeneous solution is exactly the solution obtained earlier for eq. (13.4):

$$v_h = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z \qquad (13.7)$$

Buckling of beam clamped at both ends



$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = C_1 z + D_1 \qquad (13.21)$$

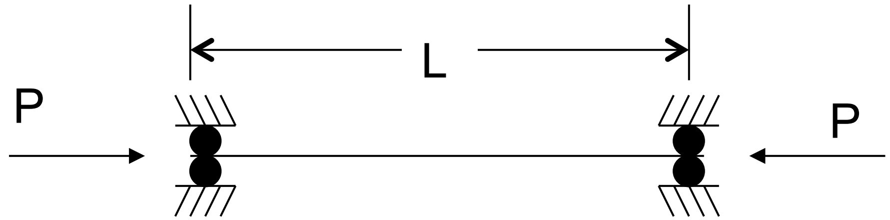
$$v_h = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z \qquad (13.7)$$

- for the particular solution, it is assumed that $v_p = Cz + D$
- if we substitute in (13.21), v_p satisfies it provided $C = C_1 EI/P$ and $D = D_1 EI/P$; so the complete solution (homogeneous +particular) is:

$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z + Cz + D \qquad (13.22)$$

- we can now impose the boundary conditions; v=0 at z=0 implies, A+D=0 (13.23)
- dv/dz=0 at z=0 implies, $B\sqrt{\frac{P}{EI}} + C = 0$ (13.24)

Buckling of beam clamped at both ends



$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z + Cz + D \quad (13.22)$$

$$A + D = 0 \quad (13.23)$$

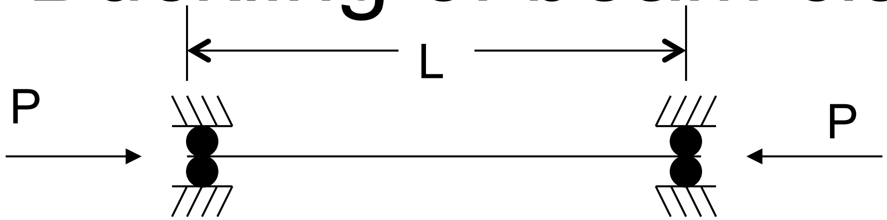
$$B\sqrt{\frac{P}{EI}} + C = 0 \quad (13.24)$$

- the third BC, v(L)=0 implies: $$A\cos\sqrt{\frac{P}{EI}}L + B\sin\sqrt{\frac{P}{EI}}L + CL + D = 0 \quad (13.25)$$
- the fourth BC, dv/dz=0 at z=L implies: $$-A\sqrt{\frac{P}{EI}}\sin\sqrt{\frac{P}{EI}}L + B\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}L + C = 0 \quad (13.26)$$

- from (13.23), D=-A
- from (13.24), $C = -B\sqrt{\frac{P}{EI}}$

- then, D and C can be replaced in eq (13.25) and (13.26)

Buckling of beam clamped at both ends



$$A\cos\sqrt{\frac{P}{EI}}L + B\sin\sqrt{\frac{P}{EI}}L + CL + D = 0 \qquad (13.25)$$

$$-A\sqrt{\frac{P}{EI}}\sin\sqrt{\frac{P}{EI}}L + B\sqrt{\frac{P}{EI}}\cos\sqrt{\frac{P}{EI}}L + C = 0 \qquad (13.26)$$

- to give the following two eqns:

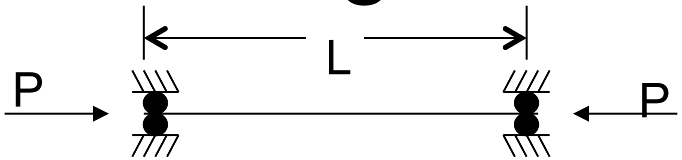
D=-A $\qquad C = -B\sqrt{\frac{P}{EI}}$

$$A\left(\cos\sqrt{\frac{P}{EI}}L - 1\right) + B\left(\sin\sqrt{\frac{P}{EI}}L - \sqrt{\frac{P}{EI}}L\right) = 0 \qquad (13.25a)$$

$$-A\sin\sqrt{\frac{P}{EI}}L + B\left(\cos\sqrt{\frac{P}{EI}}L - 1\right) = 0 \qquad (13.26a)$$

- this is a system of two equations in two unknowns A and B; however, the RHS in both equations is zero; this means either A=B=0 (no buckling) or one equation is a multiple of the other, which can be achieved if the determinant of the coefficient matrix is zero

Buckling of beam clamped at both ends



$$A\left(\cos\sqrt{\frac{P}{EI}}L-1\right)+B\left(\sin\sqrt{\frac{P}{EI}}L-\sqrt{\frac{P}{EI}}L\right)=0 \qquad (13.25a)$$

$$-A\sin\sqrt{\frac{P}{EI}}L+B\left(\cos\sqrt{\frac{P}{EI}}L-1\right)=0 \qquad (13.26a)$$

$$\det\begin{vmatrix}\left(\cos\sqrt{\frac{P}{EI}}L-1\right) & \left(\sin\sqrt{\frac{P}{EI}}L-\sqrt{\frac{P}{EI}}L\right)\\ -\sin\sqrt{\frac{P}{EI}}L & \left(\cos\sqrt{\frac{P}{EI}}L-1\right)\end{vmatrix}=0$$

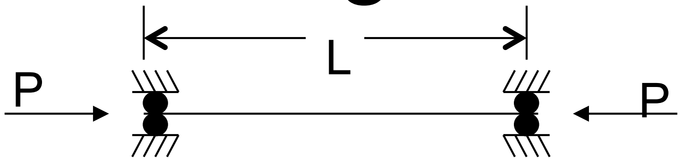
which leads to

$$\left(\cos\sqrt{\frac{P}{EI}}L-1\right)^2+\sin\sqrt{\frac{P}{EI}}L\left(\sin\sqrt{\frac{P}{EI}}L-\sqrt{\frac{P}{EI}}L\right)=0 \qquad (13.27)$$

which for convenience, can be rewritten with x= $\sqrt{\frac{P}{EI}}L$

$$\left(\cos x-1\right)^2+\sin x\left(\sin x-x\right)=0 \qquad (13.27a)$$

Buckling of beam clamped at both ends



$$\left(\cos x - 1\right)^2 + \sin x\left(\sin x - x\right) = 0 \qquad (13.27a)$$

• after some thinking, we can see that a solution to (13.27a) is $x = 2n\pi$ and since x=$\sqrt{\frac{P}{EI}}L$, the buckling load is given by

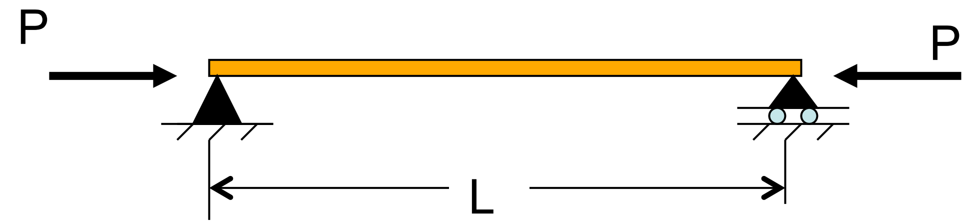
$$P = \frac{4n^2\pi^2 EI}{L^2} \qquad (13.28)$$

• as for the case of the simply-supported beam, in reality only n=1 occurs (supports are needed at the node locations to force a beam to buckle in a higher mode or n value); so:

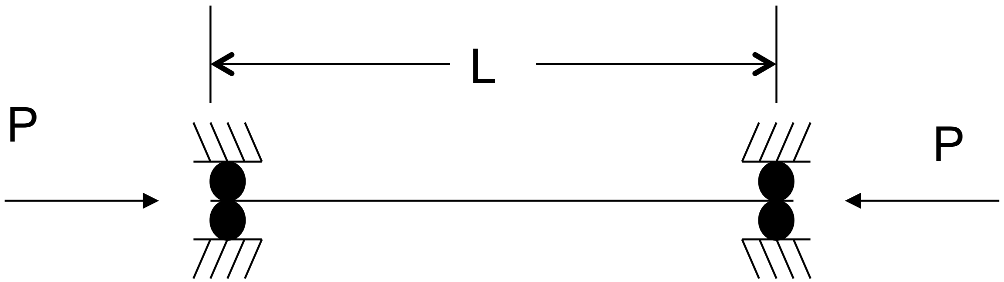
$$\boxed{P_{cr} = \frac{4\pi^2 EI}{L^2}} \qquad (13.29)$$

• this is exactly what we found earlier using the ss beam solution, eq. (13.18), and finding where its shape has zero deflection and slope (with considerable more effort this time!)

Simply-supported versus clamped beam buckling loads



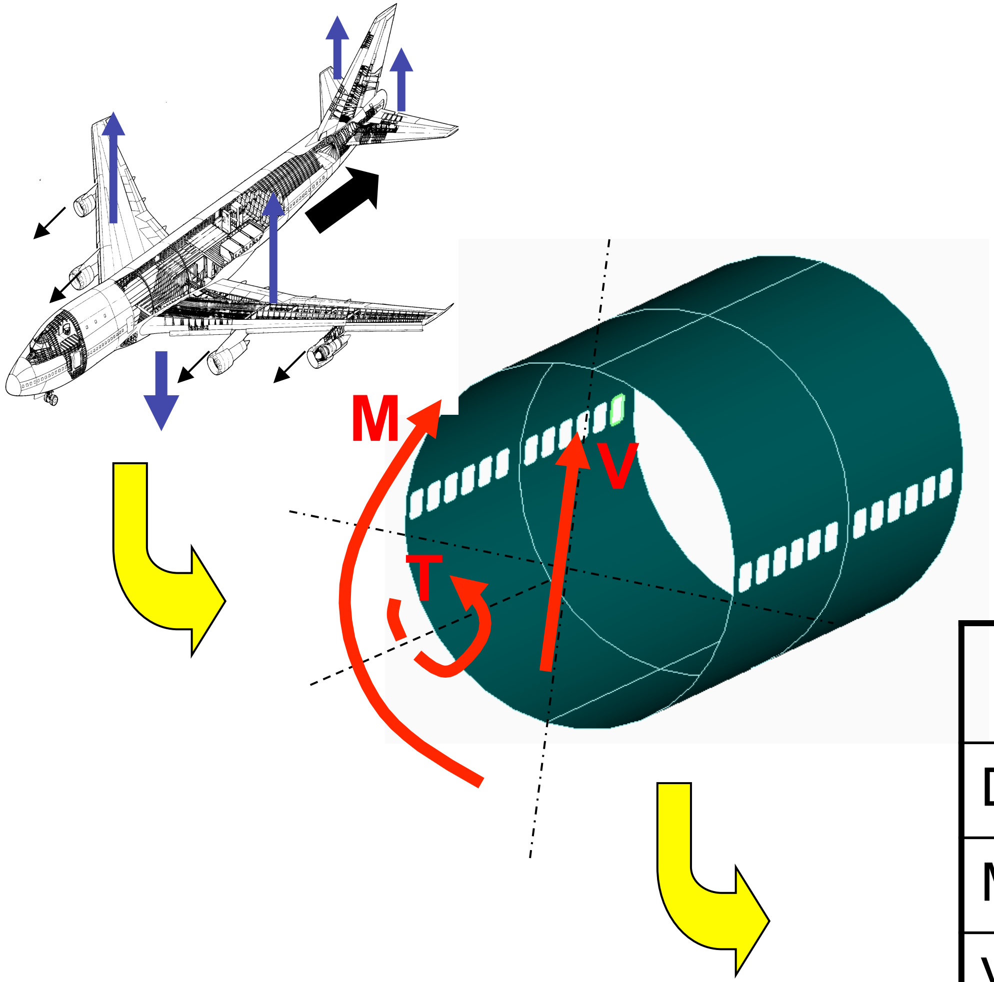
$$P_{crit} = \frac{\pi^2 EI}{L^2} \qquad (13.9)$$



$$P_{cr} = \frac{4\pi^2 EI}{L^2} \qquad (13.29)$$

- as already mentioned, the clamped beam has 4 times the buckling load of a simply supported beam

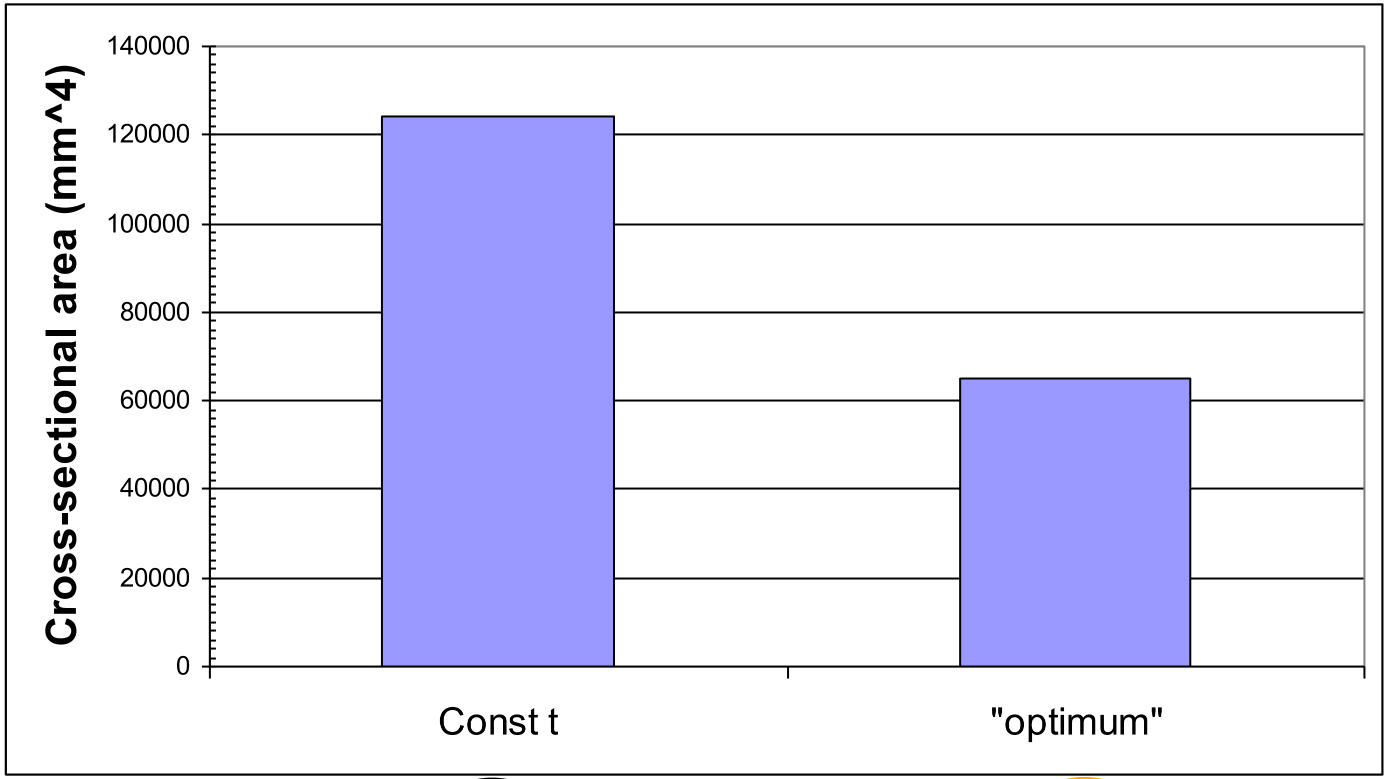
“Running” example – Fuselage cross-section

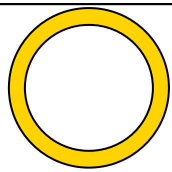


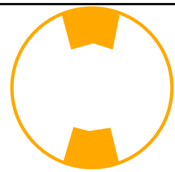
Property	Value
Diameter(m)	4.0
M (MNm)	60
V (kN)	660
T (kNm)	30

Using beam bending considerations, we found before:

- Cross-sectional Area = weight/ρ per unit length=W/(ρL)

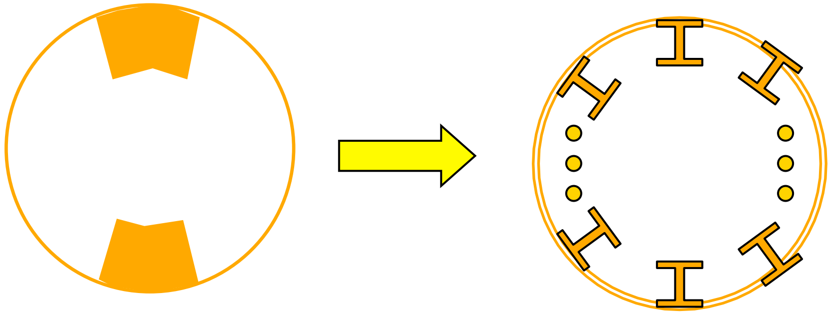






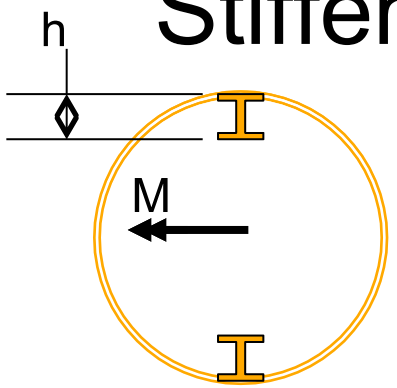
168.5mm
over 5^o

Can we do better with a stiffened structure?



- in a sense, our previous solution pointed to the use of stiffeners by suggesting to use two single stiffeners at the top and bottom of the fuselage
- at this point there is not sufficient information to determine whether we need more than two stiffeners (one on top and one on bottom)
- let’s design these two stiffeners and see what information we can gather

Stiffened fuselage: Stiffener design



Max bending stress: $\sigma_{\max,\min} = \dfrac{MR_o}{I}$

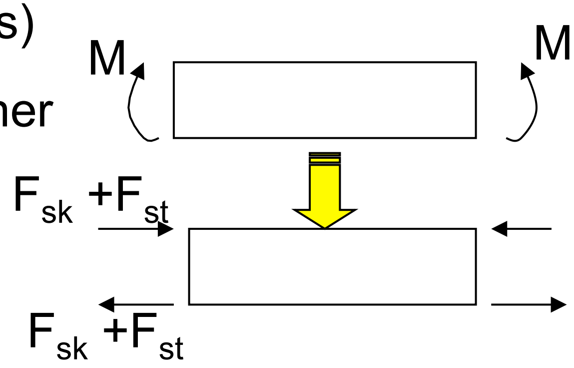
from before

Mom of Inertia $I = \pi R_o^{\,3} t + 2A_{st}\left(R_o - \dfrac{h}{2}\right)^2 \approx \pi R_o^{\,3} t + 2A_{st} R_o^{\,2}$

A_{st}= Area of stiffener

valid for h<<R: to be checked later

- both tension and compression sides should be checked for strength
- need to determine stress in skin and stress in stiffener(s)
- under the applied total force F_{sk}+F_{st}, the skin and stiffener (at top of bottom) move **together** => same axial strain

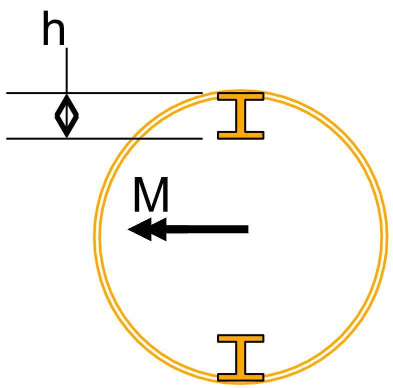


$$\varepsilon_{sk} = \varepsilon_{st} \Rightarrow \frac{F_{sk}}{\cancel{E}A_{sk}} = \frac{F_{st}}{\cancel{E}A_{st}} = \frac{F_{sk}+F_{st}}{\cancel{E}\left(A_{sk}+A_{st}\right)}$$

- from which: $\dfrac{F_{sk}}{A_{sk}} = \dfrac{F_{sk}+F_{st}}{A_{sk}+A_{st}} = \sigma_{\max,\min}$

$$\frac{F_{st}}{A_{st}} = \frac{F_{sk}+F_{st}}{A_{sk}+A_{st}} = \sigma_{\max,\min}$$

Stiffened fuselage: Stiffener design



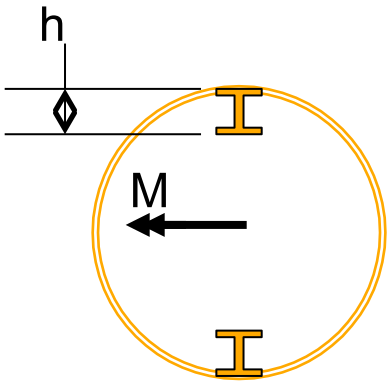
$$\frac{F_{sk}}{A_{sk}} = \frac{F_{sk} + F_{st}}{A_{sk} + A_{st}} = \sigma_{\max,\min}$$

$$\frac{F_{st}}{A_{st}} = \frac{F_{sk} + F_{st}}{A_{sk} + A_{st}} = \sigma_{\max,\min}$$

So the max (or min) stress in the skin equals the max or min stress in the skin/stiffener combination and the max(or min) stress in the stiffener equals the max or min stress in the skin/stiffener combination

- This may appear obvious but it is not. If the stiffness of the stiffener were different than that of the skin, the stiffnesses would not cancel out and the individual stresses in skin and stiffener would not be equal
- Check for material failure of 5^o arc.
- Recall that for 7075-T6 Al, σ_y^t = 482.6 MPa and σ_y^c = 489.5 MPa

Stiffened fuselage: Stiffener design



$$\frac{F_{sk}}{A_{sk}} = \frac{F_{sk}+F_{st}}{A_{sk}+A_{st}} = \sigma_{\max,\min}$$

$$\frac{F_{st}}{A_{st}} = \frac{F_{sk}+F_{st}}{A_{sk}+A_{st}} = \sigma_{\max,\min}$$

$$\sigma_{\max,\min} = \frac{MR_o}{I}$$

$$I \approx \pi R_o^{\,3} t + 2A_{st}R_o^{\,2}$$

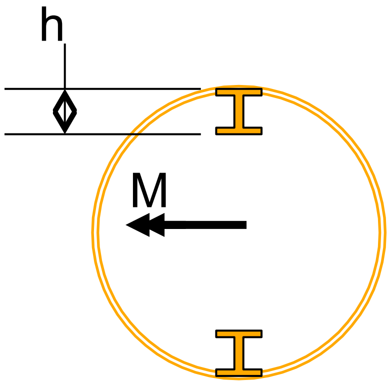
σ_y^t = 482.6 MPa and σ_y^c = 489.5 MPa

- strength check: failure of skin or stiffener occurs when the corresponding stress exceeds the yield stress of the material
- since the tension yield is slightly smaller, use that (it would be conservative for the compression side):

$$482.6MPa = \frac{MR_o}{\pi R_o^{\,3} t + 2A_{st}R_o^{\,2}} \Rightarrow A_{st} = \frac{M - \pi R_o^{\,2} t (482.6E6)}{2R_o(482.6E6)}$$

- substituting values: $A_{st} = 29500mm^2$
- note this depends on t which was assumed at 0.5mm!

Stiffened fuselage: Stiffener design



$$A_{st} = 29500mm^2$$

This stiffener area guarantees that skin and stiffener fail at the same time when max M value is reached

- buckling check (compr. side only):

$$\sigma_{crit} = \frac{\pi^2 EI_{st}}{A_{st}L^2}$$

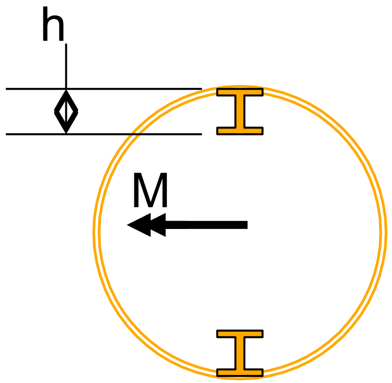
simply supported ends; note A_{st} in the denominator to make our usual equation from force to stress

• require that buckling occurs when the maximum compressive stress is reached and solve for the moment of inertia of the stiffener :

$$\frac{MR_o}{\pi R_o^{\,3}t + 2A_{st}R_o^{\,2}} = \frac{\pi^2 EI_{st}}{A_{st}L^2} \Rightarrow I_{st} = \frac{A_{st}L^2 M}{\pi^2 E\left(\pi R_o^{\,2}t + 2A_{st}R_o\right)}$$

- this relation allows determination of I_{st} if t, A_{st}, and L are known

Stiffened fuselage: Stiffener design



$$A_{st} = 29500mm^2$$

$$I_{st} = \frac{A_{st} L^2 M}{\pi^2 E\left(\pi R_o{}^2 t + 2A_{st} R_o\right)}$$

- for E= 69 GPa, t=0.5 mm, A_{st}=29500mm^2 and L=0.5m (typical frame spacing):

$$I_{st} = 5.228x10^6 mm^4$$

but is this the lightest design?

is there another value of t that will give a lighter design?

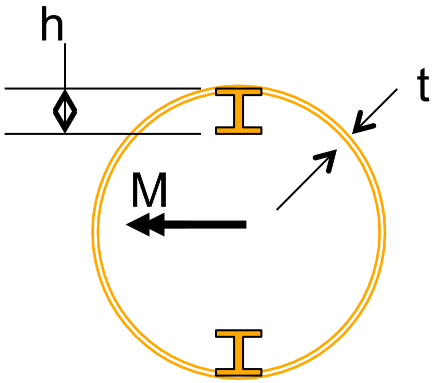
- the weight/density for length L of fuselage is given by

$$\frac{W}{\rho} = \left(2\pi R_o t + 2A_{st}\right) \text{L}$$

- and using A_{st} from before:

$$\frac{W}{\rho} = \left(2\pi R_o t + 2\frac{M - \pi R_o{}^2 t(482.6E6)}{2R_o(482.6E6)}\right) L \Rightarrow \frac{W}{\rho} = \left(\pi R_o t + \frac{M}{R_o(482.6E6)}\right) L$$

Stiffened fuselage: Stiffener design



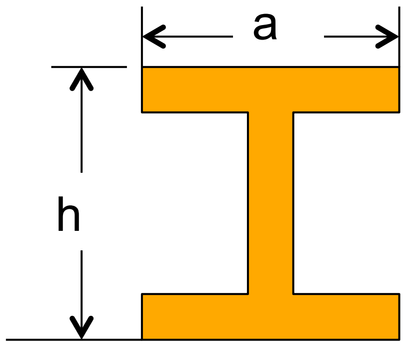
$$\frac{W}{\rho} = \left(\pi R_o t + \frac{M}{R_o (482.6E6)} \right) L$$

- so for a given stiffener length L (frame spacing in this case) the weight is minimized when t is minimized

- so we keep the value of t at 0.5 as the lowest skin thickness available

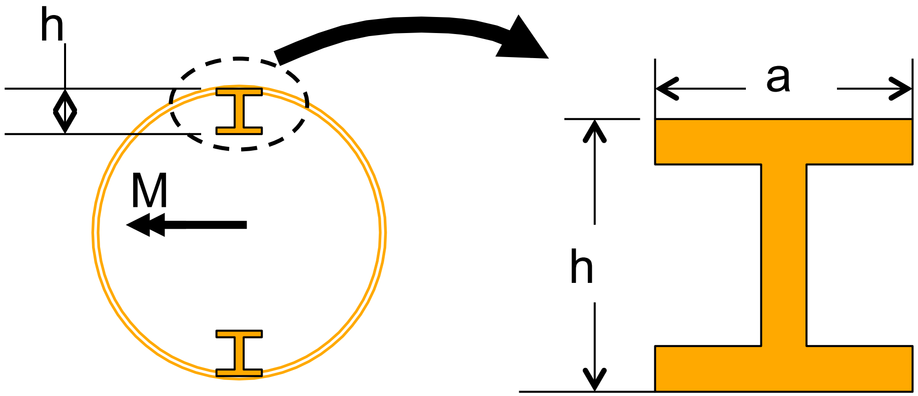
- to finalize the design we need to determine the stiffener cross section such that A_{st}=29500 mm^2 and I_{st}=5.228x10^6 mm^4

- select the following basic shape:



thickness t_{st} everywhere

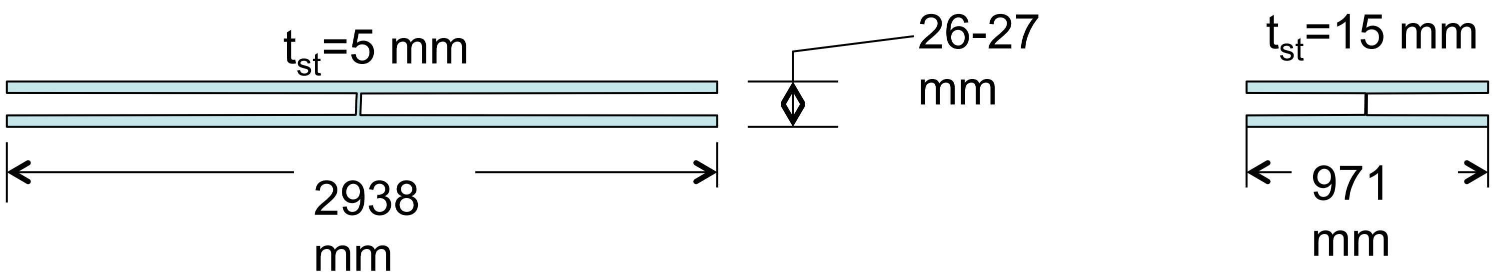
Stiffened fuselage: Stiffener design



thickness t_{st} everywhere

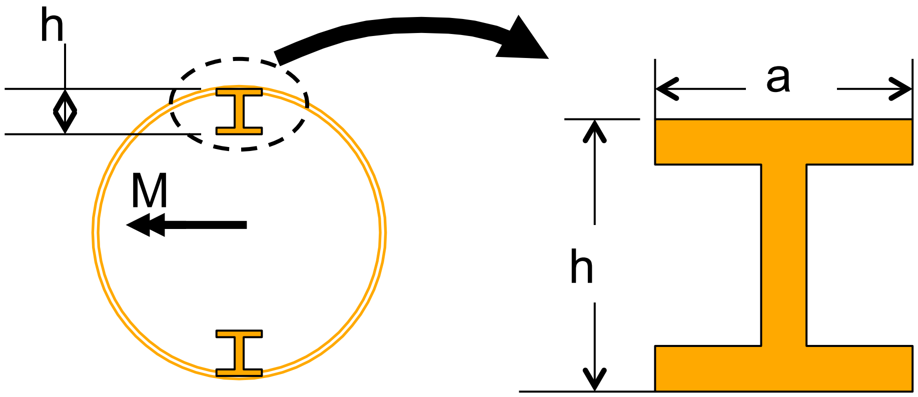
A_{st}=29500 mm^2
I_{st}=5.228x106 mm^4

- approximately to scale, the solution is:



- clearly, this solution is, at best, ridiculous!
  - not manufacturable (a too big, h too small)
  - does not fit fuselage
  - stiffener is not a beam any more (0.5m long but almost 1-3 m wide!)

Stiffened fuselage: Stiffener design



thickness t_{st} everywhere

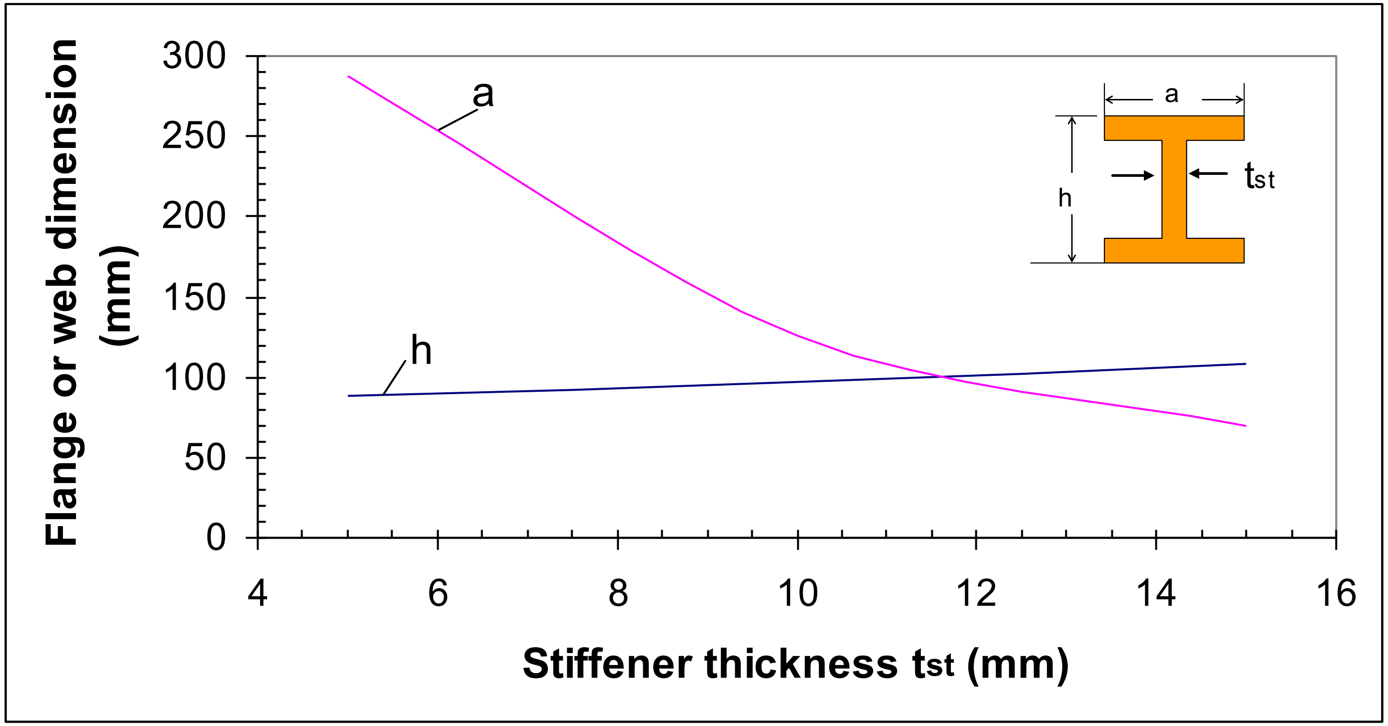
A_{st}=29500 mm^2
I_{st}=5.228x106 mm^4

• we can create a producible design by "splitting" the required area into more stiffeners but keeping I_{st} the same

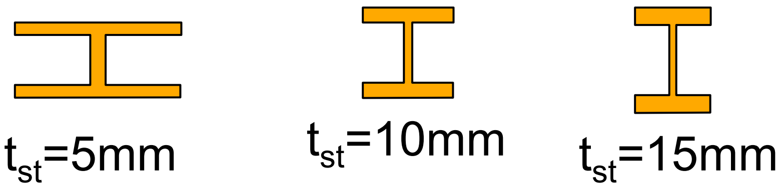
• this is acceptable because as we add stiffeners, to the left or right of the top stiffener, the (bending) stress on the additional stiffeners is lower; by keeping the bending stress the same for all stiffeners (and equal to the max value) we are slightly conservative

Stiffened fuselage: Stiffener design

- for **nine** stiffeners at top and bottom, a and h as a function of t_{st} are given by

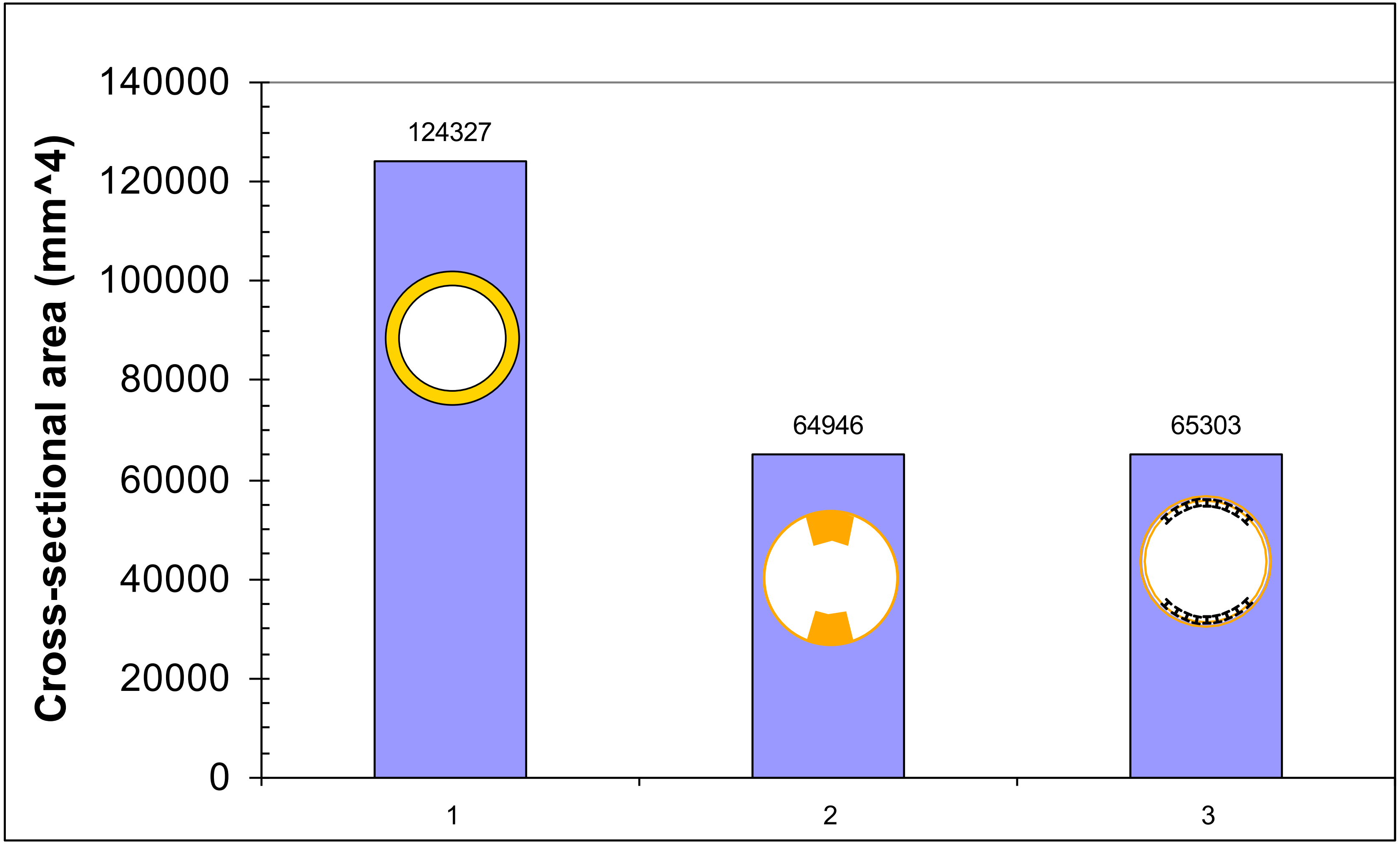


- approx to scale (except thickness)



much better designs!!

Comparing to previous designs…



Keep in mind that:

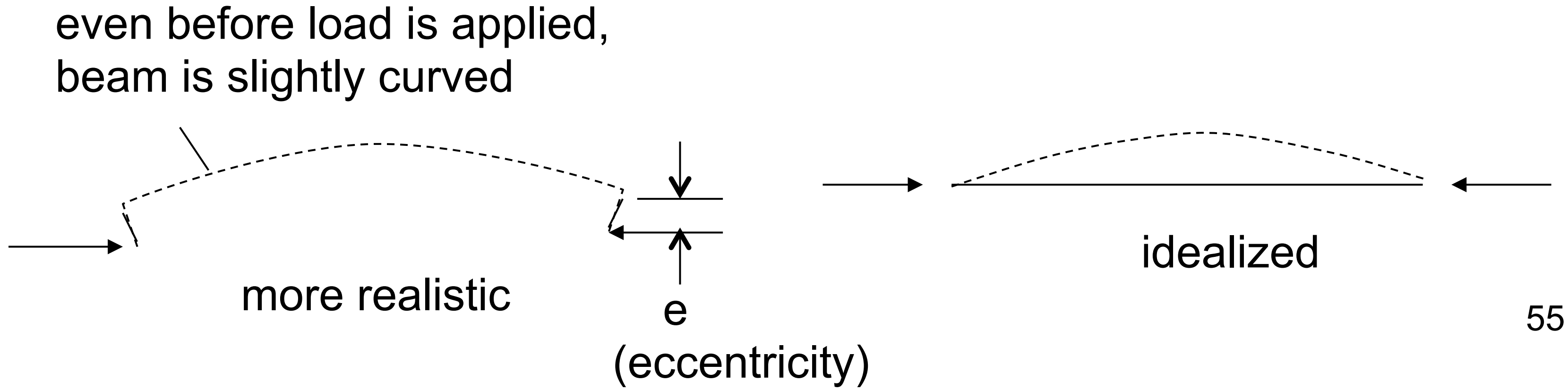
- bottom stiffener is under compression and top under tension so we could have two different designs (tension for strength and compression for buckling)
- spacing is not decided yet and not for this course

Effect of eccentricities & initial imperfections

• it was already mentioned that it is impossible to create a perfect straight beam with exactly the same cross-section everywhere, and apply the load exactly at the neutral axis on both ends

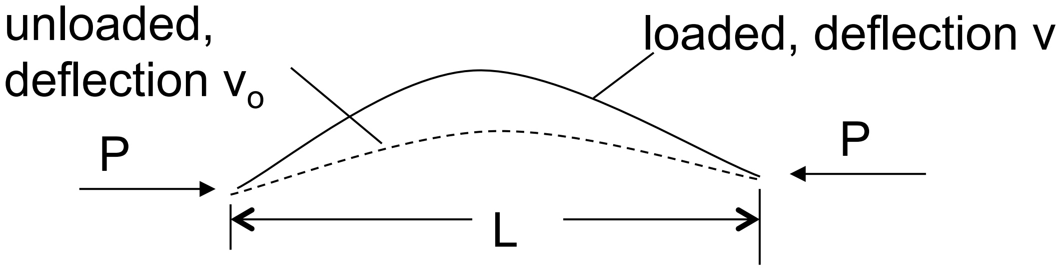
• the most common deviations from "perfection" are (a) the beam is already slightly bent into some shape and (b) the load is not acting at the neutral axis of the cross-section

• let's include some of the effects of reality into our solution



Effect of initial imperfections

- (a) before load is applied, beam is bent

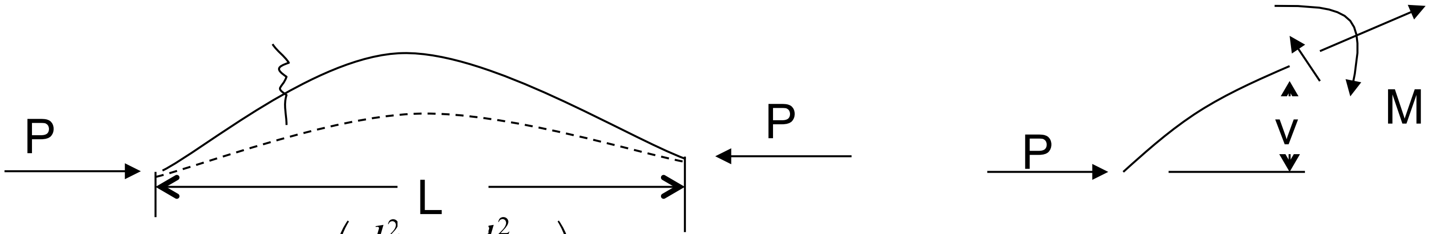


- in general, the bending moment in the loaded beam is proportional to the curvature: $M = -EI\dfrac{d^2v}{dz^2}$ (curvature=-d^{2}v/dz^2)

- in this case, since there is no moment applied in the unloaded condition, the moment created when P is exerted is proportional to the **change** in curvature from the unloaded position:

$$M = -EI\left(\frac{d^2v}{dz^2} - \frac{d^2v_o}{dz^2}\right) \tag{13.30}$$

Effect of initial imperfections



$$M = -EI\left(\frac{d^2v}{dz^2} - \frac{d^2v_o}{dz^2}\right) \qquad (13.30)$$

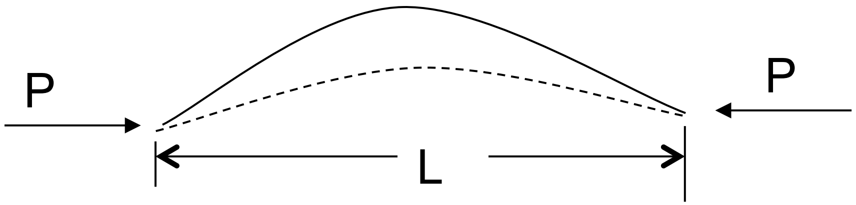
- taking moments about the cut:

$$M = Pv \qquad (13.31)$$

- combining (13.30) and (13.31) and rearranging:

$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = \frac{d^2v_o}{dz^2} \qquad (13.32)$$

Effect of initial imperfections



$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = \frac{d^2v_o}{dz^2} \qquad (13.32)$$

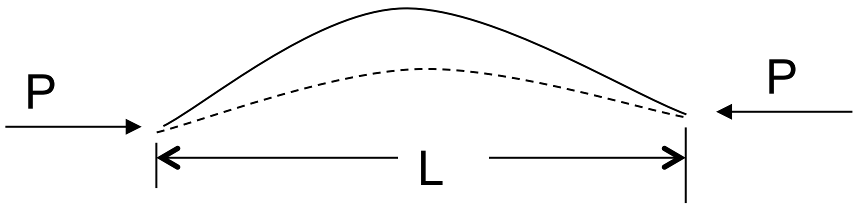
- we' ve seen this eq twice before, for the simply supported beam and for the ss beam with a spring in the middle; the only difference between all these cases is the right hand side

- the solution will always be the sum of the homogeneous and the particular solution.

- the homogeneous solution (when RHS=0) is the same as for the ss beam, eq. (13.7):

$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z \qquad (13.7)$$

Effect of initial imperfections



$$\frac{d^2v}{dz^2}+\frac{P}{EI}v=\frac{d^2v_o}{dz^2} \qquad (13.32)$$

$$v=A\cos\sqrt{\frac{P}{EI}}z+B\sin\sqrt{\frac{P}{EI}}z \qquad (13.7)$$

- for the particular solution, we must know the right hand side (which is an input in this problem); this means we must know the initial curved shape of the beam v_o
- we know from the theory of Fourier series, that any function v_o can be expressed as an infinite series:

$$v_o=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi z}{L} \qquad (13.33)$$

where A_n are appropriate constants determined from the theory of Fourier series