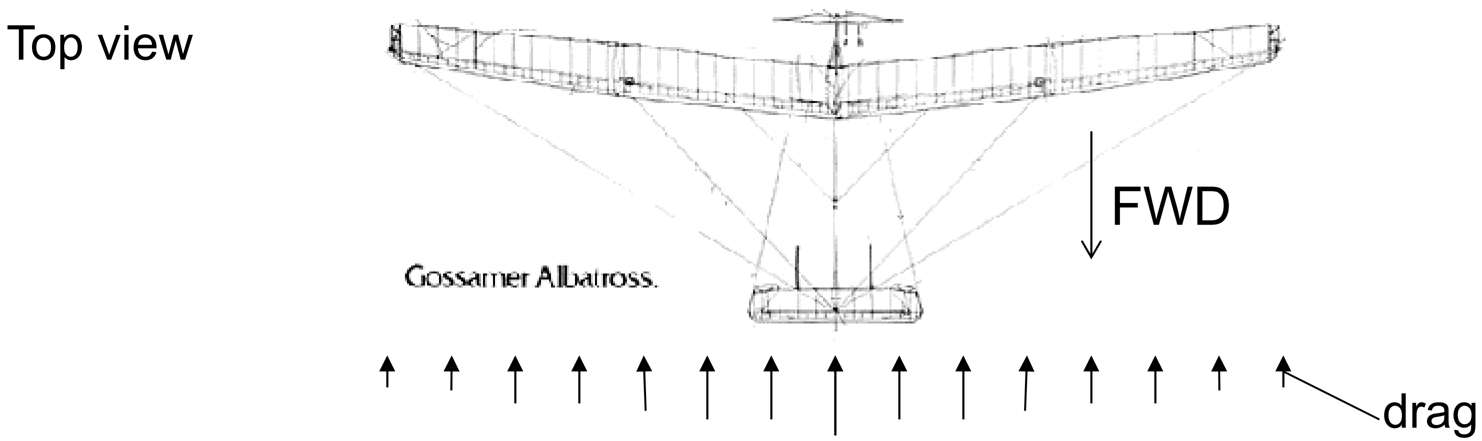
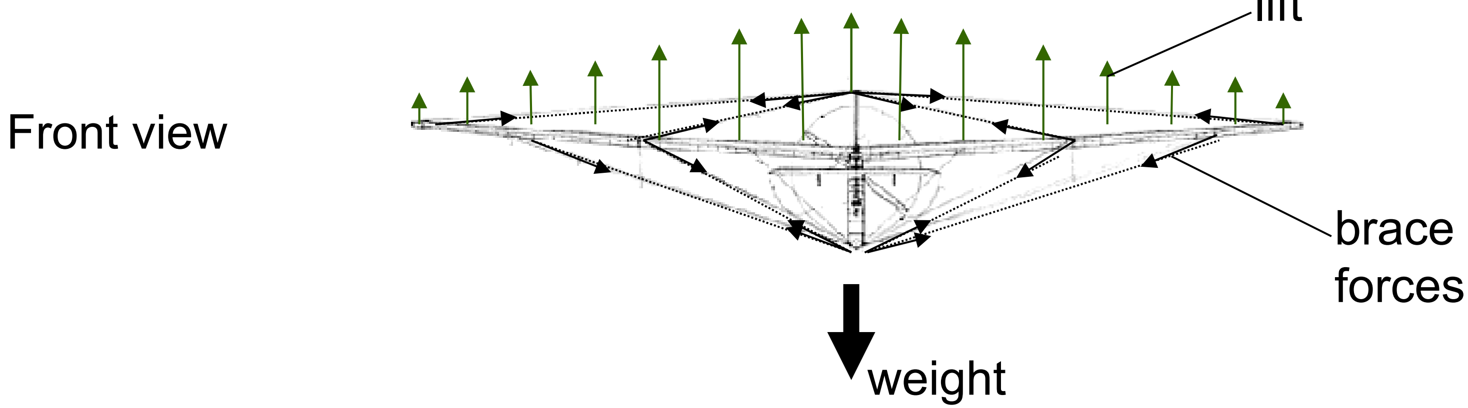
Human Powered Gossamer Albatross in Flight

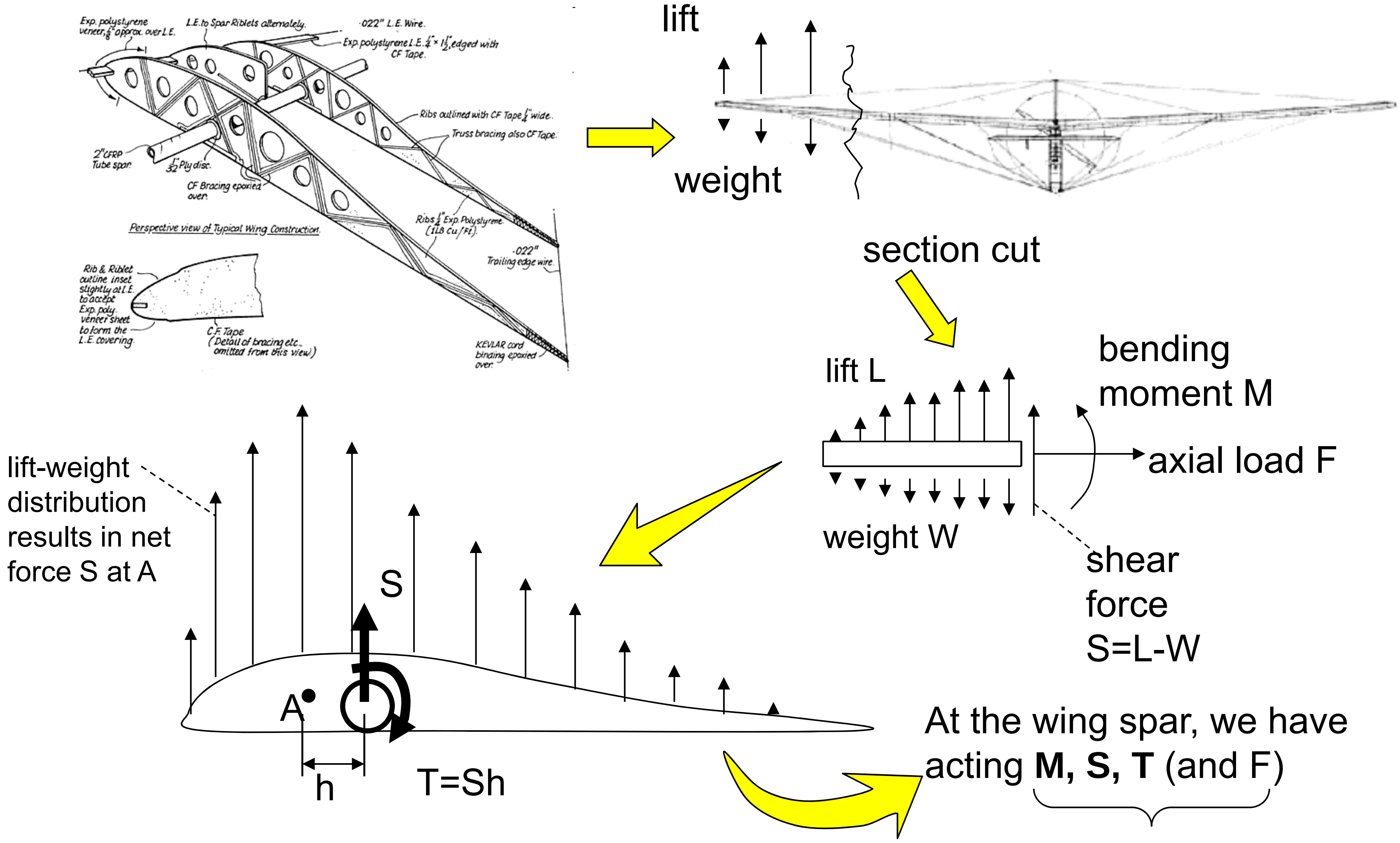
Dryden Flight Research Center ECN 12604 Photographed 1980
Testing the Gossamer Albatross NASA photo

Forces in flight





Isolate the loads on a wing section



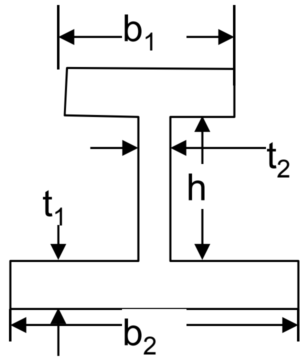
Course subject matter

• the entire course is about the effect of M, S, T (and F) and how to come up with structures that do not fail in the presence of these loads:

– thickness

– other dimensions

ex:

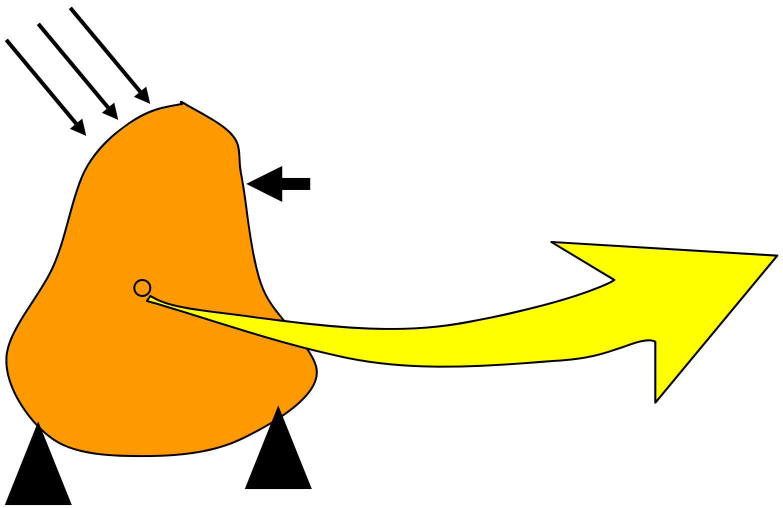


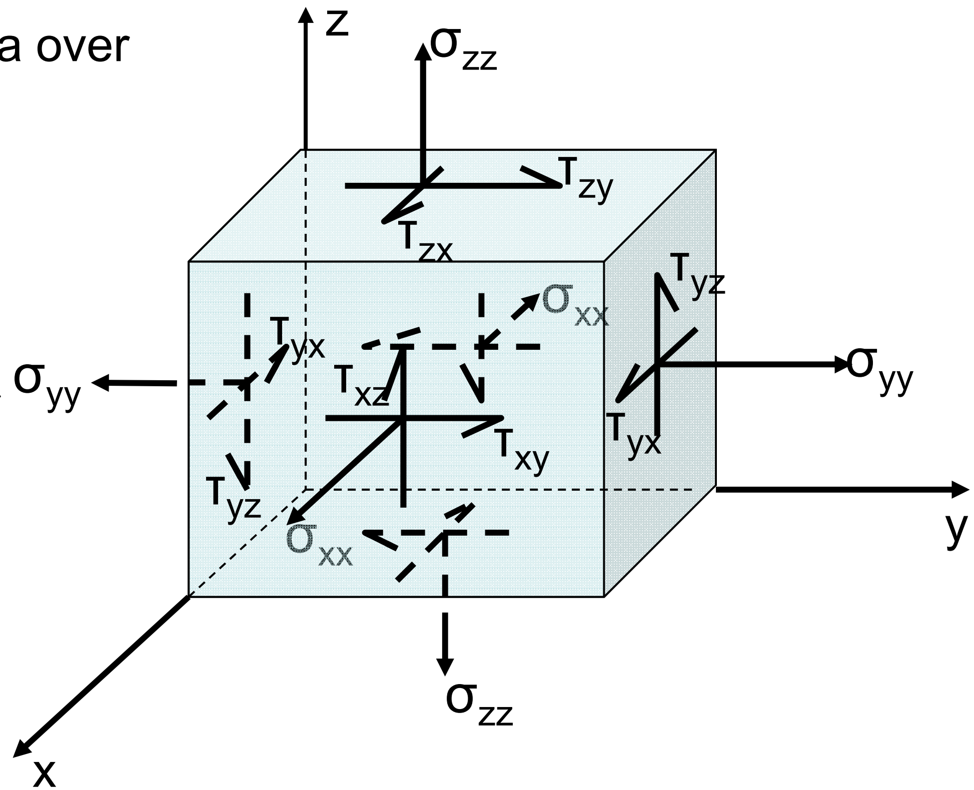
t_1, t_2, b_1, b_2… = ???

• to do this we need to go back to the basics and define some important quantities: stresses, strains, displacements, and how they relate to applied loads and, eventually, failure

Stresses: σ_{xx} (or σ_x) σ_{yy} (or σ_y) σ_{zz} (or σ_z) τ_{yz} τ_{xz} τ_{xy}

(stress= force divided by area over which it acts)

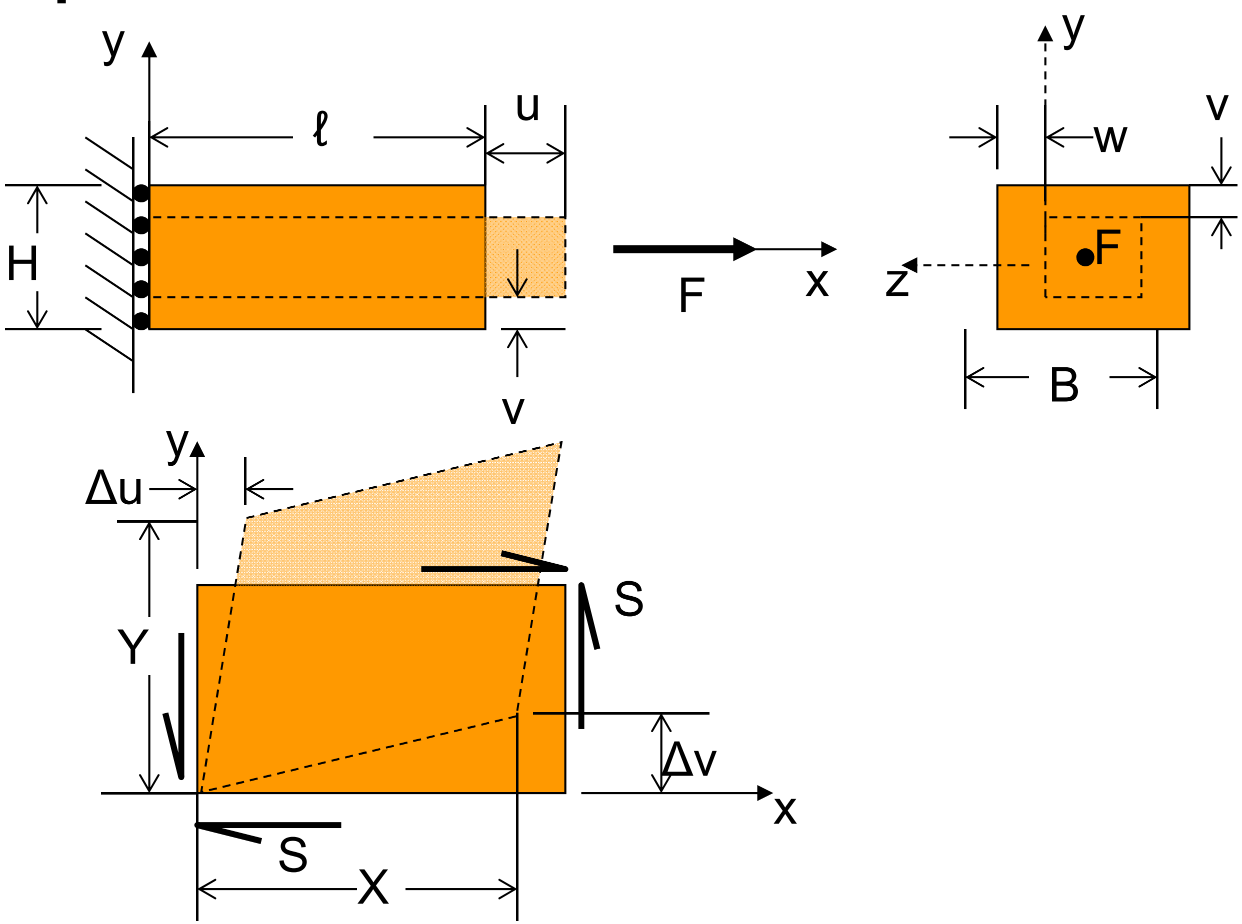




- notation: 1st index the face, 2nd index the direction
- from moment equilibrium of an elemental cube, stress tensor is symmetric => $\tau_{mn} = \tau_{nm}$
- while the stress state is unique, the stress values change with coordinate system

Strains: ε_{xx} (or ε_x), ε_{yy} (or ε_y), ε_{zz} (or ε_z), γ_{yz}, γ_{xz}, γ_{xy}

Displacements : u, v, w



normal or direct strains

$$\varepsilon_x = \frac{u}{\ell}$$

$$\varepsilon_y = \frac{v}{\frac{H}{2}}$$

$$\varepsilon_z = \frac{w}{\frac{B}{2}}$$

shear strains

$$\gamma_{xy} = \frac{\Delta u}{Y} + \frac{\Delta v}{X}$$

undeformed deformed

Strains: ε_{xx} (or ε_x), ε_{yy} (or ε_y), ε_{zz} (or ε_z), γ_{yz}, γ_{xz}, γ_{xy}
Displacements : u, v, w

- when a body deforms under load, each point moves to a new location; the change in position measured along the x, y, and z axes gives the u, v, and w displacements
- normal strains ε_x, ε_y, ε_z measure the % length change in any direction
- shear strains $\gamma_{yz,}$ γ_{xz}, γ_{xy} measure the change in angle

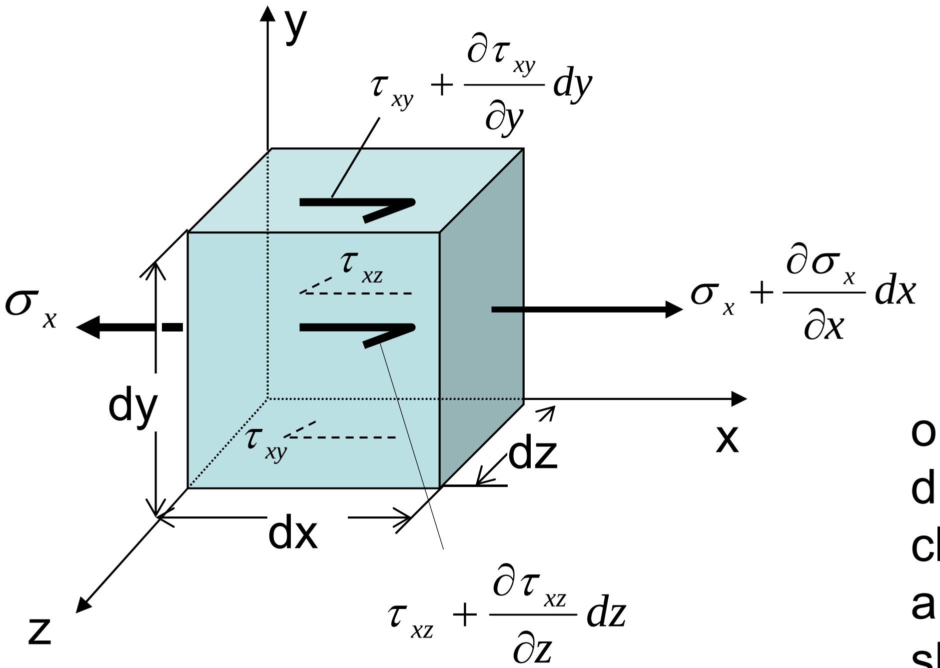
strain-displacement equations

• in the general case, for a 3-D body under deformation, the strains at any point are related to the displacements through the equations:

$$\boxed{\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} & \qquad \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\ \varepsilon_y &= \frac{\partial v}{\partial y} & \qquad \gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \\ \varepsilon_z &= \frac{\partial w}{\partial z} & \qquad \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}} \qquad (1.1)\text{-}(1.6)$$

Equilibrium equations

- force equilibrium of a small cube within a body under load



only stresses in x dir are shown for clarity; there is also a body force X not shown for clarity

- force equilibrium in x direction:

$$\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dydz - \sigma_x dydz + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} dy\right) dxdz - \tau_{xy} dxdz + \left(\tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} dz\right) dxdy - \tau_{xz} dxdy$$
$$+ Xdxdydz = 0$$

Equilibrium equations

• canceling out terms gives the first equilibrium equation

• the remaining two equations (equilibrium in y and z directions) are obtained in an analogous manner

• in the end:

$$\boxed{\begin{aligned} &\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X = 0 \\ &\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + Y = 0 \\ &\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0 \end{aligned}} \qquad (1.7)\text{-}(1.9)$$

where X, Y, and Z, are (body) forces per unit volume

Stress-strain relations

- since a force acting on a body causes a deflection, there is a relation between forces and deflections or, in a more useful configuration, stresses (which relate to forces) and strains (which relate to deflections)

- the most common force-deflection equation is Hooke's law:

$$F = ku$$

where k (the spring constant) is a constant of proportionality

Stress-strain relations

• generalizing this one-dimensional relation and using stresses and strains (isotropic material):

$$
\begin{aligned}
\sigma_x &= \frac{1-\nu}{(1+\nu)(1-2\nu)}E\varepsilon_x + \frac{\nu}{(1+\nu)(1-2\nu)}E\varepsilon_y + \frac{\nu}{(1+\nu)(1-2\nu)}E\varepsilon_z \\
\sigma_y &= \frac{\nu}{(1+\nu)(1-2\nu)}E\varepsilon_x + \frac{1-\nu}{(1+\nu)(1-2\nu)}E\varepsilon_y + \frac{\nu}{(1+\nu)(1-2\nu)}E\varepsilon_z \\
\sigma_z &= \frac{\nu}{(1+\nu)(1-2\nu)}E\varepsilon_x + \frac{\nu}{(1+\nu)(1-2\nu)}E\varepsilon_y + \frac{1-\nu}{(1+\nu)(1-2\nu)}E\varepsilon_z \\
\tau_{yz} &= G\gamma_{yz} \\
\tau_{xz} &= G\gamma_{xz} \\
\tau_{xy} &= G\gamma_{xy}
\end{aligned}
\qquad (1.10)\text{-}(1.15)
$$

E is Young's modulus,

G is shear modulus and

v is Poisson's ratio

with $\boxed{G = \frac{E}{2(1+\nu)}}$ (1.16)

• note that these equations indicate that a normal stress in one direction can cause strains in any of the three dir. x,y,z

Equations of elasticity

• the strain-displacement equations (1.1)-(1.6), the equilibrium equations (1.7)-(1.9), and the stress-strain equations (1.10)-(1.15) form a system of 15 equations in the 15 unknowns:

$$\begin{cases} u,\ v,\ w, \\ \varepsilon_x,\ \varepsilon_y,\ \varepsilon_z,\ \gamma_{yz},\ \gamma_{xz},\ \gamma_{xy}, \\ \sigma_x,\ \sigma_y,\ \sigma_z,\ \tau_{yz},\ \tau_{xz},\ \tau_{xy} \end{cases}$$

• if we could solve this system of equations in the general case this course (and all others in structures) would be done in two lectures

Simplifications: Plane stress

- but we cannot, so we simplify as much as possible and try to deal with more tractable problems
- one such simplification is based on the recognition that most aerospace structures have one dimension much smaller than the other two (plates, shells, etc)
- it is reasonable to assume (and verified by tests) that stresses in that direction are negligible compared to the others; if z is the direction of smallest dimension:

$$\sigma_z = \tau_{yz} = \tau_{xz} \approx 0 \qquad (1.17)$$

- then, from eqs (1.13) and (1.14):

$$\gamma_{yz} = \gamma_{xz} \approx 0 \qquad (1.18)$$

(and 5 of the original unknowns are eliminated)

Plane stress

- as a result, the strain-displacement equations (1.1)-(1.6) become:

$$\boxed{\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} && (1.1a) \\ \varepsilon_y &= \frac{\partial v}{\partial y} && (1.2a) \\ \varepsilon_z &= \frac{\partial w}{\partial z} && (1.3a) \end{aligned}}$$

$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}} \qquad (1.6a)$$

- the equilibrium equations (1.7)-(1.9) become:

$$\boxed{\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X &= 0 && (1.7a) \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + Y &= 0 && (1.8a) \end{aligned}}$$

Plane stress

• finally, for the stress-strain equations, using (1.17) to substitute in (1.12), we can solve for ε_z:

$$\varepsilon_z = -\frac{\nu}{1-\nu}\varepsilon_x - \frac{\nu}{1-\nu}\varepsilon_y \tag{1.19}$$

• which, in turn, can be substituted in the remaining stress-strain equations; after rearranging:

$$\boxed{\begin{aligned}
\sigma_x &= \frac{E}{1-\nu^2}\varepsilon_x + \frac{\nu E}{1-\nu^2}\varepsilon_y && \text{(1.10a)}\\
\sigma_y &= \frac{\nu E}{1-\nu^2}\varepsilon_x + \frac{E}{1-\nu^2}\varepsilon_y && \text{(1.11a)}\\
\tau_{xy} &= G\gamma_{xy} && \text{(1.15a)}
\end{aligned}}$$

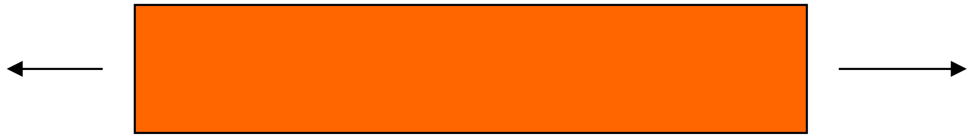
Plane stress

• sometimes, it is useful to invert equations (1.10a), (1.11a) and (1.15a) to have the strains as the unknowns:

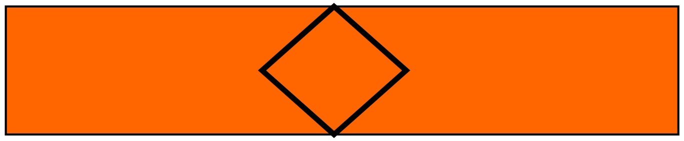
$$\boxed{\begin{aligned} \varepsilon_x &= \frac{1}{E}\sigma_x - \frac{\nu}{E}\sigma_y && (1.20) \\ \varepsilon_y &= -\frac{\nu}{E}\sigma_x + \frac{1}{E}\sigma_y && (1.21) \\ \gamma_{xy} &= \frac{1}{G}\tau_{xy} && (1.22) \end{aligned}}$$

Stress transformation: Why bother?

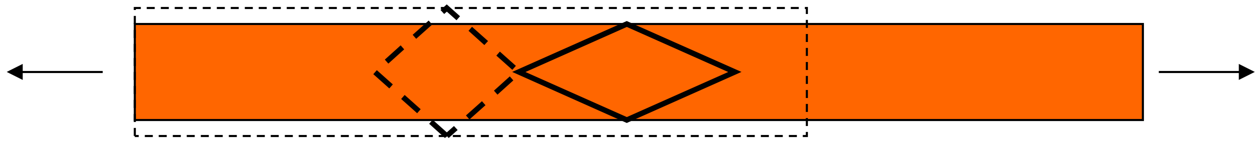
• consider a rectangular bar under tension



• on which we mark (scratch) a diamond shape prior to loading



• applying uniaxial tension leads to the deformed pattern



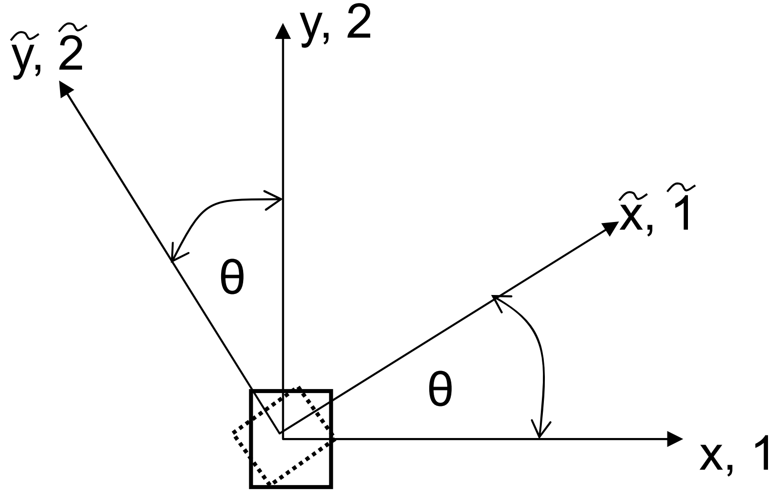
• in a coord. system aligned with the bar (outside rectangle) angles (i.e. strains and thus stresses do not change; in the system of the diamond they do; yet the stress state is the same

• **given a loading system the stress state is unique but the stresses measured are a function of the coord system chosen**

Stress-transformation

• the values of stresses or strains depend on the coordinate system

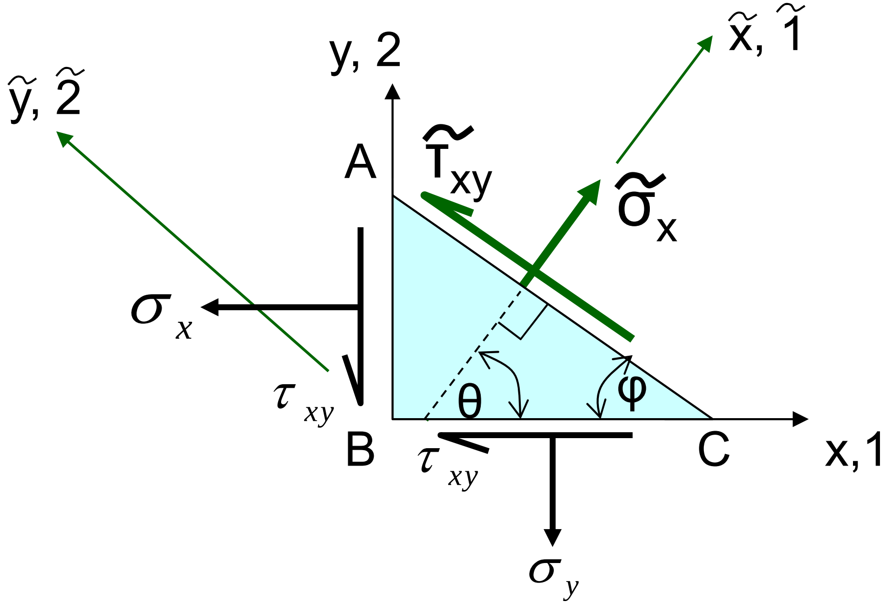
• if stresses or strains are known in one coordinate system, it is very useful (and important) to be able to determine stresses or strains in any other coordinate system



Stresses at different coordinate systems

• for simplicity, consider a 2-D case first

• the stresses σ_x, σ_y, τ_{xy}, are known in the coordinate system x,y (or 1,2) and we need to determine them in the coordinate system $\tilde{x},\tilde{y}$ (or $\tilde{1},\tilde{2}$)



BA=dy

BC=dx

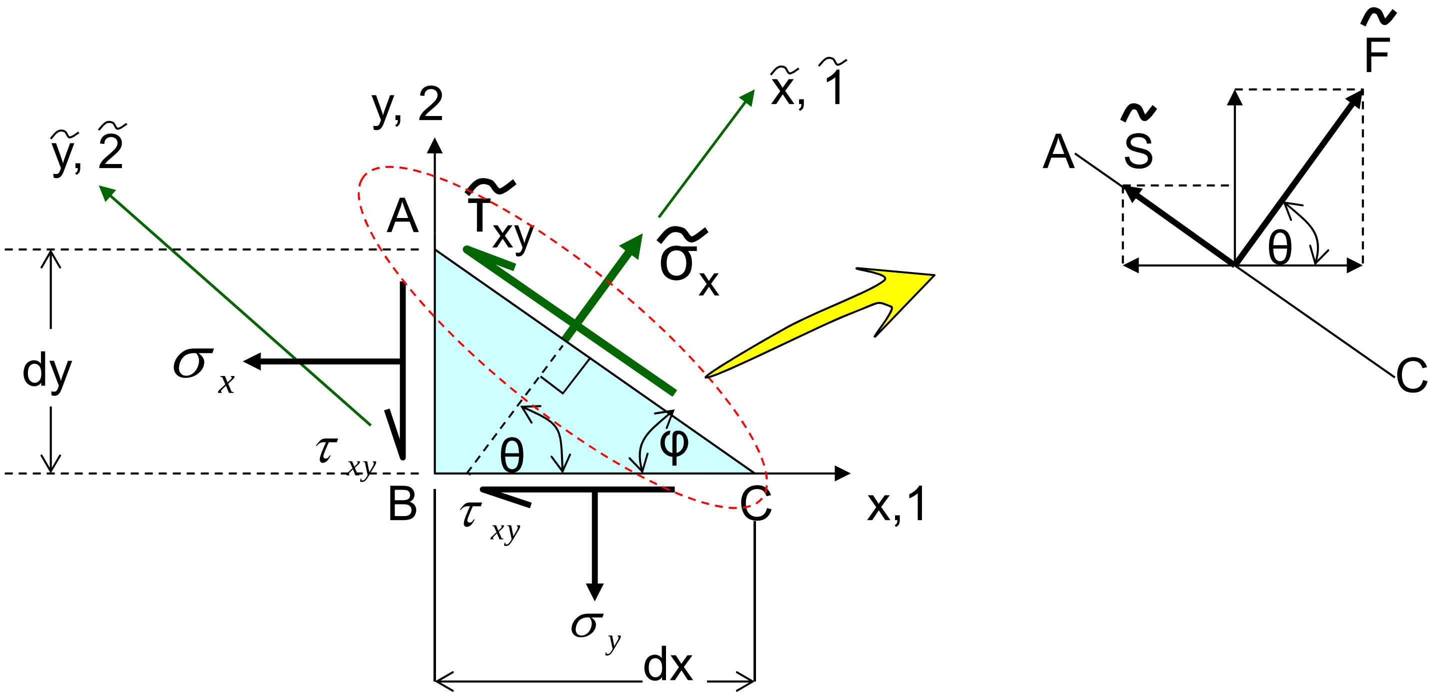
AC=BC/cosφ=dx/cosφ

dz=dimension perpendicular to the page

$\theta+\varphi=90^o$

• isolate an elemental triangle and expose the stresses acting on its surfaces

Stresses at different coordinate systems



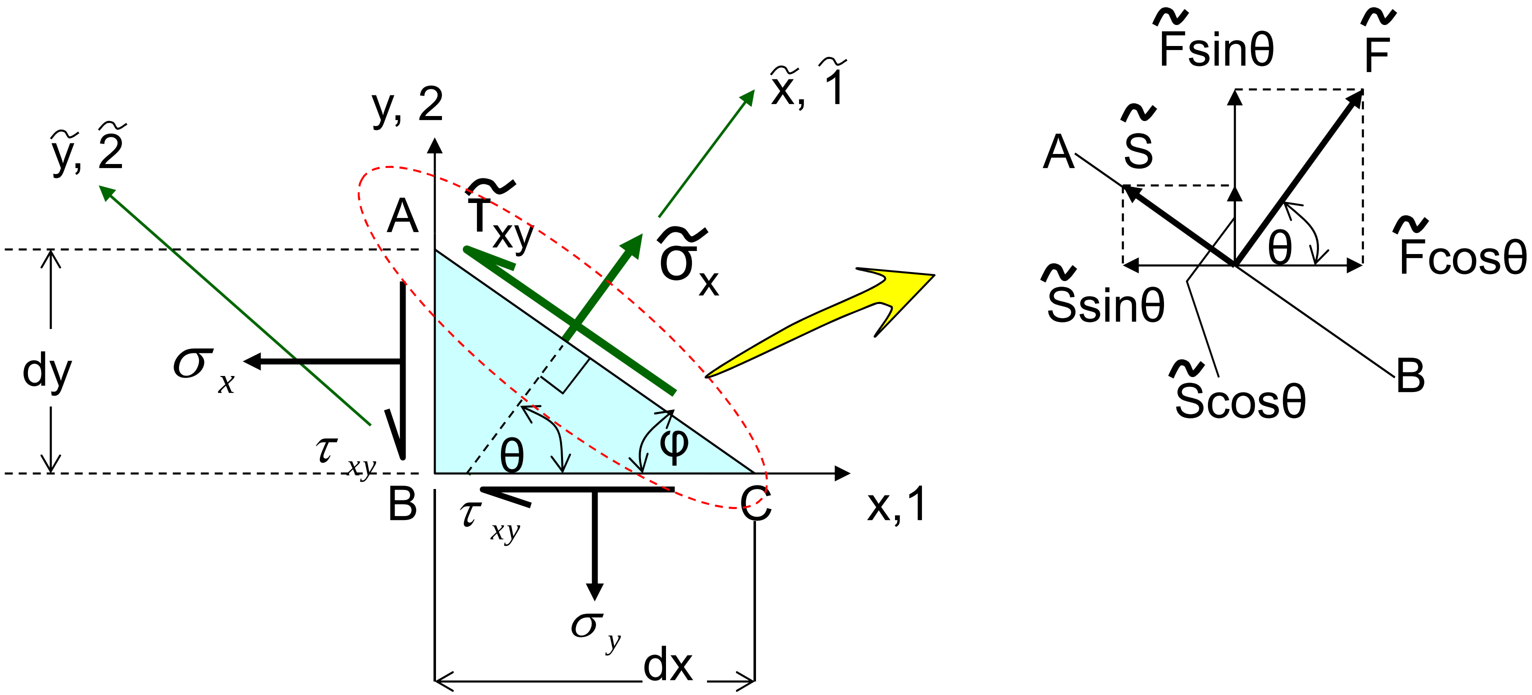
- determine forces caused by stresses on AC:

  - $\tilde{F} = \tilde{\sigma}_x (AC)dz = \tilde{\sigma}_x \dfrac{dx}{\cos\phi} dz$ (1)

  - $\tilde{S} = \tilde{\tau}_{xy} (AC)dz = \tilde{\tau}_{xy} \dfrac{dx}{\cos\phi} dz$ (2)

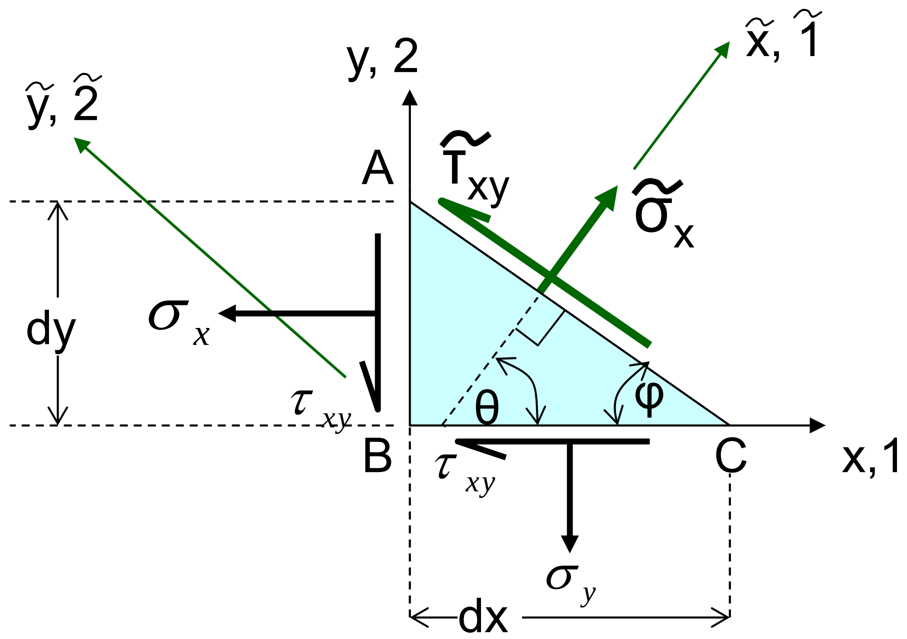
similarly for vertical surfaces

Stresses at different coordinate systems



- Force equilibrium in x and y directions:
  - $\sum F_{xi} = 0 \Rightarrow \tilde{F}\cos\theta - \tilde{S}\sin\theta - \sigma_x dydz - \tau_{xy} dxdz = 0$ (3)
  - $\sum F_{yi} = 0 \Rightarrow \tilde{F}\sin\theta + \tilde{S}\cos\theta - \tau_{xy} dydz - \sigma_y dxdz = 0$ (4)

Stresses at different coordinate systems

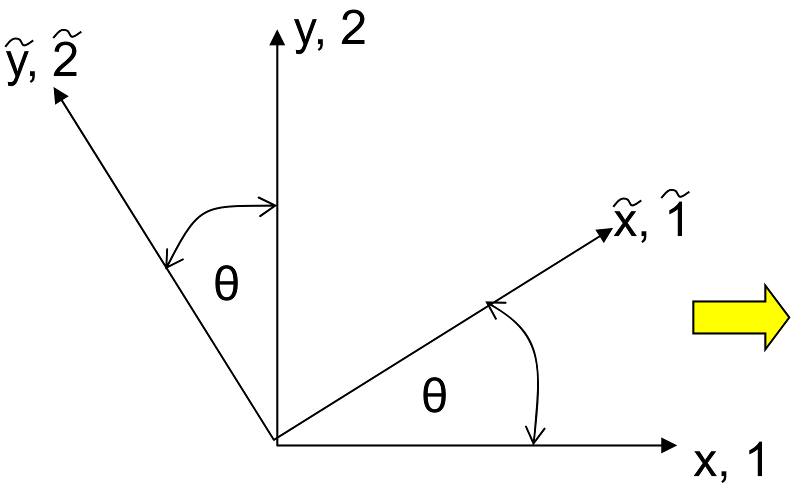


- substituting in (3) and (4) and using the fact that tanφ=dy/dx:

$$\left.\begin{aligned}\tilde{\sigma}_x\cos\theta-\tilde{\tau}_{xy}\sin\theta-\sigma_x\sin\phi-\tau_{xy}\cos\phi=0\\ \tilde{\sigma}_x\sin\theta+\tilde{\tau}_{xy}\cos\theta-\sigma_y\cos\phi-\tau_{xy}\sin\phi=0\end{aligned}\right\}$$

two equations in the two unknowns $\tilde{\sigma}_x$ and $\tilde{\tau}_{xy}$

Stress transformation equations

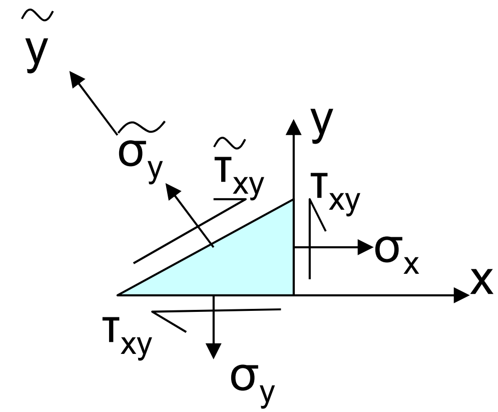


Recall:

θ+φ=90=>

sinφ =cosθ

cosφ =sinθ



- Solving:

$$\tilde{\sigma}_x = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \tag{5}$$

$$\tilde{\sigma}_y = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \tag{6}$$

$$\tilde{\tau}_{xy} = -\sigma_x \sin\theta\cos\theta + \sigma_y \sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right) \tag{7}$$

Stress transformation equations

- in matrix notation:

$$\begin{Bmatrix} \tilde{\sigma}_x \\ \tilde{\sigma}_y \\ \tilde{\tau}_{xy} \end{Bmatrix} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

- in terms of direction cosines and tensor notation:

$$\boxed{\tilde{\sigma}_{mn} = \ell_{\tilde{m}p}\,\ell_{\tilde{n}q}\,\sigma_{pq}} \qquad (7a)$$

m,n,p,q = 1,2

ℓ_{ij} = direction cosine between axis i and axis j

Principal directions, planes, stresses

- the fact that a given stress state gives different stress values in different directions gives rise to the question of what are the maximum and minimum values and in what direction they occur

- then, differentiating eq. (5) w.r.t θ and setting = 0, $\tilde{\sigma}_x$ is maximized or minimized when:

$$\tan 2\theta = -\frac{2\tau_{xy}}{\sigma_y - \sigma_x} \tag{8}$$

- i.e. there are two values of θ that maximize the normal stress:

$$\theta = \frac{1}{2}\tan^{-1}\left(-\frac{2\tau_{xy}}{\sigma_y - \sigma_x}\right)$$

$$\theta = \frac{1}{2}\tan^{-1}\left(-\frac{2\tau_{xy}}{\sigma_y - \sigma_x}\right) + \frac{\pi}{2}$$

Principal directions, planes, stresses

• the normal stress attains its maximum (or minimum) values along two directions defined by these two values of θ, which, differing by 90 degrees, are mutually perpendicular

• the normal stresses along these two directions are called **principal stresses**

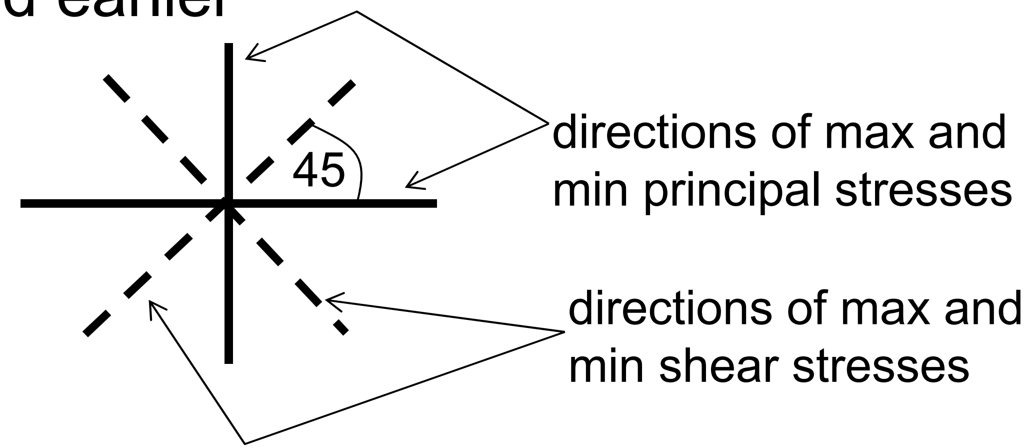
• substituting eq (8) into eq. (7), it can be shown that $\tilde{\tau}_{xy}=0$ in the two planes on which the principal stresses act

• finally, to determine where the shear stress is maximized or minimized we differentiate eq. (7) w.r.t. θ and set =0, which gives:

Principal directions, planes, stresses

$$\tan 2\theta = \frac{\sigma_y - \sigma_x}{2\tau_{xy}} \qquad (9)$$

• from basic calculus (lines with reciprocal slopes of opposite sign are perpendicular to each other) the orientation 2θ defined by (9) is perpendicular to that defined by (8); thus the θ orientations defined by these two equations differ by 45° and **the plane of max/min shear stress is at 45° to the max and min principal planes** defined earlier



Values of principal stresses

- what are the maximum and minimum values of $\tilde{\sigma}_x, \tilde{\tau}_{xy}$?

use eq. (8) to substitute into (5) (along with some trigonometry) to obtain:

$$\tilde{\sigma}_{x\max} = \sigma_I = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^{\ \ 2}} \quad \text{max principal stress} \qquad (10)$$

$$\tilde{\sigma}_{x\min} = \sigma_{II} = \frac{\sigma_x + \sigma_y}{2} - \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^{\ \ 2}} \quad \text{min principal stress} \qquad (11)$$

similarly, use eq. (9) to substitute into (7) to obtain:

$$\tilde{\tau}_{xy\max,\min} = \tau_{\max,\min} = \pm\frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^{\ \ 2}} \qquad (12)$$

also, subtracting (11) from (10) it can be shown that

$$\tau_{\max} = \frac{\sigma_I - \sigma_{II}}{2} \qquad (13)$$

Summary on stress transformation

- given a state of stress in a body, two (three for 3-D) **perpendicular directions** (given by eq.8) can be found along which the **normal stresses attain their maximum and minimum values** given by eqs. 10 and 11.

- **the shear stresses in the principal planes defined by the principal axes are zero**

- **at 45 degrees to the principal planes**, two mutually perpendicular planes exist **in which the shear stresses attain their maximum and minimum values** given (for 2-D) by eq.12; the max shear stress equals half the difference of the max and min principal stresses

Mohr's circle: obtaining transformed stresses graphically

- determine the curve that relates the transformed ($\tilde{\sigma}_x$ $\tilde{\sigma}_y$, $\tilde{\tau}_{xy}$) to the original stresses (σ_x σ_y, τ_{xy})

- square equations (5) and (7) and combine them to obtain (after manipulation)

$$\left(\tilde{\sigma}_x - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tilde{\tau}_{xy}^{\ 2} = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^{\ 2} \qquad (14)$$

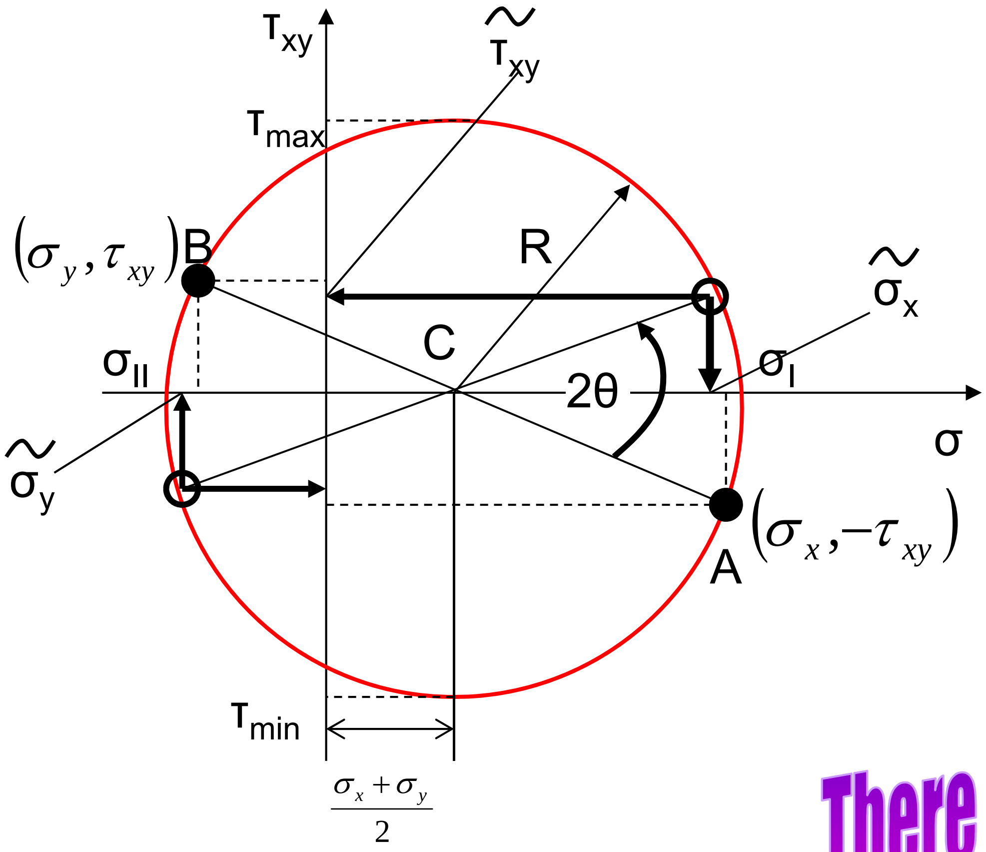
which is the equation of a circle with center at

$$\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$$

and radius

$$R = \frac{1}{2}\sqrt{\left(\sigma_x - \sigma_y\right)^2 + 4\tau_{xy}^{\ 2}}$$

Mohr's circle



1. Determine pt A (σ_x,$-\tau_{xy}$)
2. Determine pt B (σ_y,$+\tau_{xy}$)
3. Draw circle of diameter AB
4. Rotate AB (about center C) counterclockwise by 2θ
5. The intersections with the Mohr's circle define the rotated $\tilde{\sigma}_x$, $\tilde{\sigma}_y$, and $\tilde{\tau}_{xy}$

There is no Mohr's "sphere" in 3 dimensions!

Analogy with strains

• as the strains are related to the stresses through the constitutive (stress-strain) relations

$$\boxed{\sigma_{mn} = E_{mnpq}\varepsilon_{pq}} \qquad \text{(linear elastic material)} \tag{15}$$

• in an exactly analogous way, there are **max and min principal strains and max and min shear strains**, their directions coinciding with the corresponding ones for stresses

• also **<u>tensorial</u>** (as opposed to engineering) **strains transform in exactly the same way as stresses**:

$$\boxed{\varepsilon_{mn} = \ell_{mp}\ell_{nq}\varepsilon_{pq}} \tag{16}$$

• and **there is a Mohr's circle for strain transformation** exactly analogous to that for stress

Note: tensorial versus engineering strains

• unlike stress where there is a one-to-one correspondence between engineering and tensor stress,

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \Longleftrightarrow \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}$$

a factor of 2 is needed in the shear strains to maintain symmetry of the strain tensor

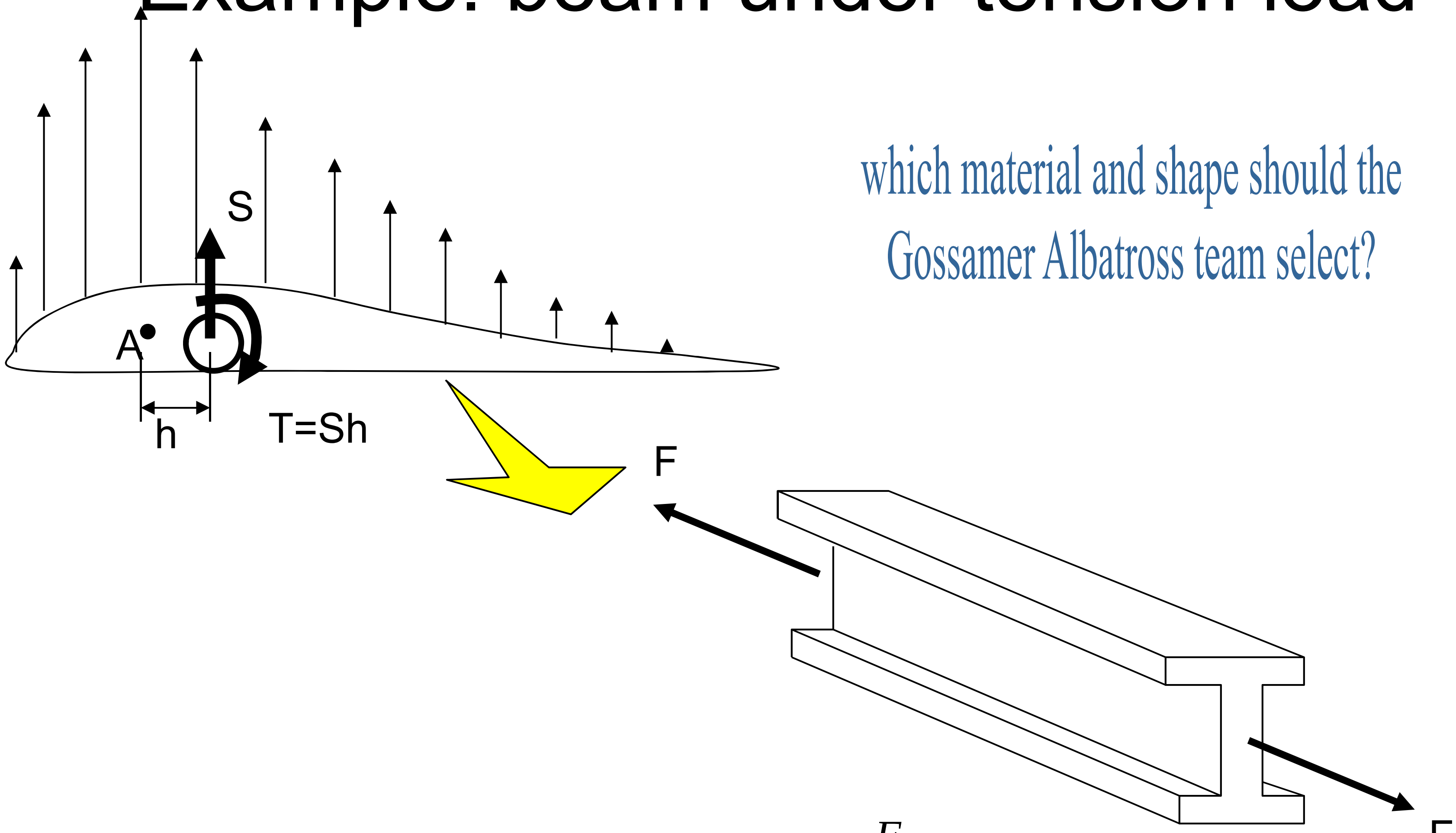
Putting it all together…

- so we now have a plate or shell with stresses, strains, displacements what good are they?
- there are two main goals in structures:
  - given a structural configuration and applied loads determine if/when it fails (“forward” problem)
  - given applied loads come up with a structural configuration that does not fail (“reverse” or design problem)
- it turns out that failure of materials is directly related to stresses and strains and NOT loads
  - materials have ultimate stress or strain capability and not ultimate force or moment capability

Basic structural material

Material	Ultimate tension strength (MPa)	Density (kg/m^3)	Mat'l cost (€/kg)
Steel (AM-350)	1262	7822	4.4
Aluminium (7075-T6)	552	2801	6.6
Titanium (Ti-6Al-4V)	958	4438	22
Quasi-Isotropic composite	483	1609	176

Example: beam under tension load



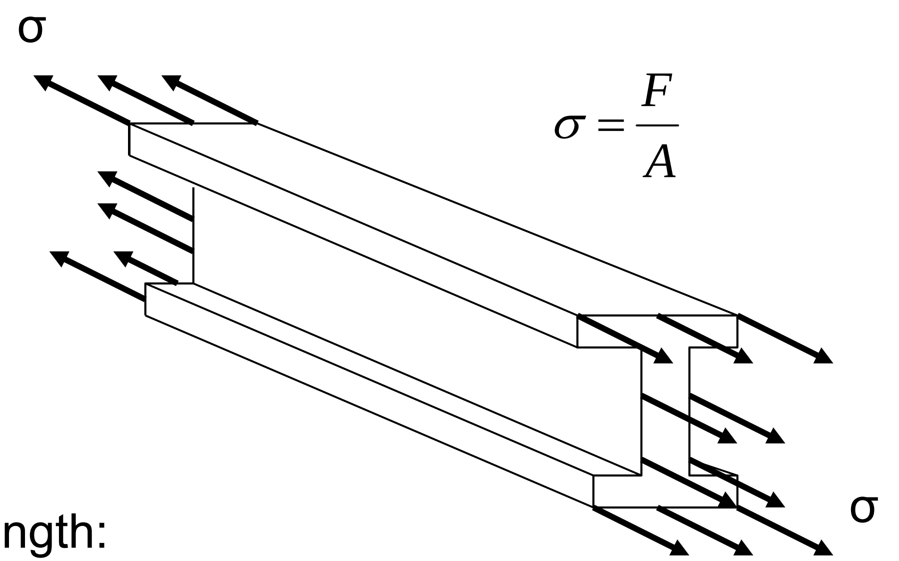
which material and shape should the Gossamer Albatross team select?

$$\sigma = \frac{F}{A}$$

Beam under tension

Assuming beam fails as soon as ultimate load F is reached:

$$\left.\begin{aligned} Weight &= \rho AL \\ A &= \frac{F}{\sigma_{ult}} \end{aligned}\right\} \quad \frac{Weight}{FL} = \frac{\rho}{\sigma_{ult}}$$



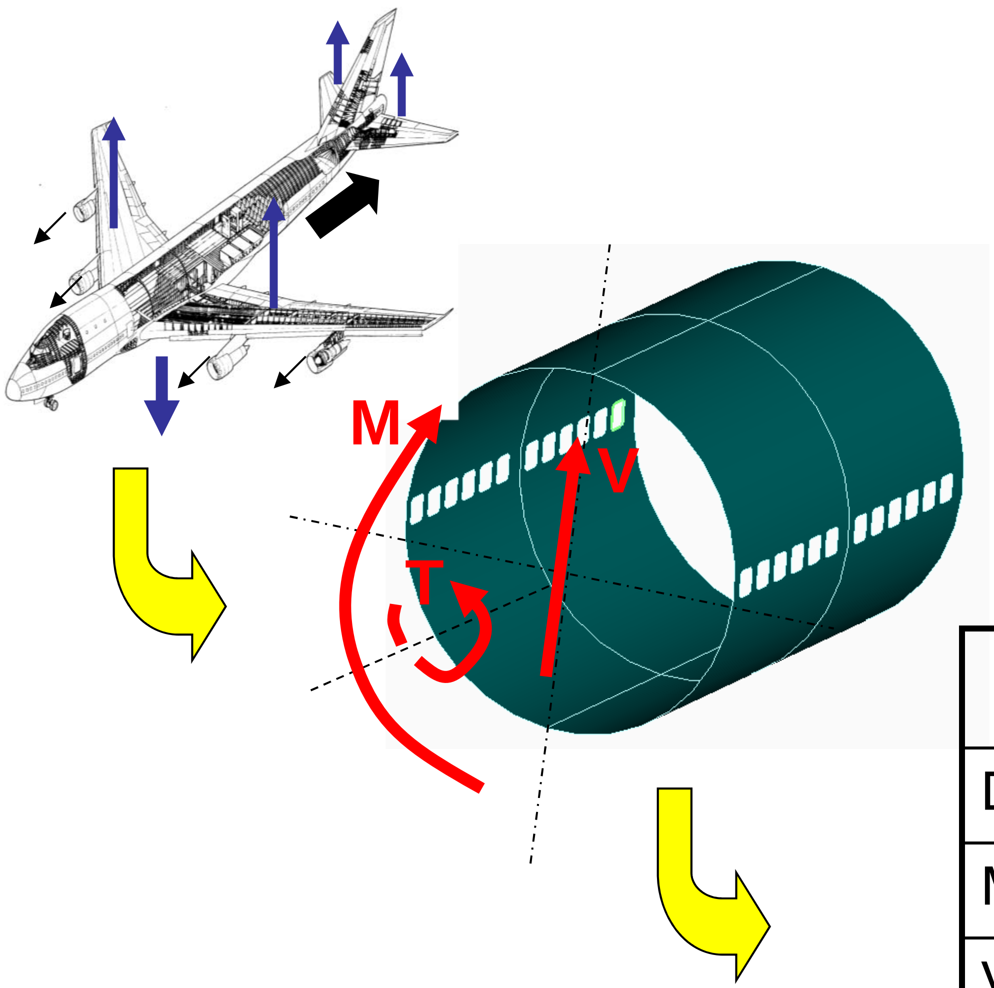
$$\sigma = \frac{F}{A}$$

For given applied load and beam length:

Material	Weight/(FL)	Cost/(FL)	
Steel (AM-350)	6.2	27.3	
Aluminium (7075-T6)	5.1	33.7	most airplanes
Titanium (Ti-6Al-4V)	4.6	101	
QI composite	**3.3**	**581**	GA team

QI=quasi-isotropic

“Running” example – Fuselage cross-section



Property	Value
Diameter(m)	4.0
M (MNm)	60
V (kN)	660
T (kNm)	30

to be continued…

Conclusion(s)

- stresses, strains (and displacements) must be known as a function of the applied loads in order to:
  - design or analyze a structure
  - perform trade-offs between materials and design concepts
  - optimize for minimum weight, cost, etc.
- rest of the course deals with how to get stresses, strains and displacements as a function of applied loads