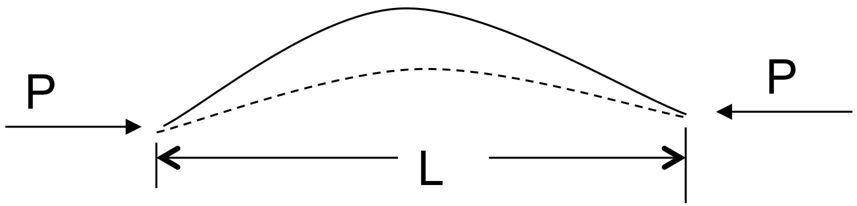
Effect of initial imperfections



$$\frac{d^2v}{dz^2}+\frac{P}{EI}v=\frac{d^2v_o}{dz^2} \qquad (13.32)$$

$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z \qquad (13.7)$$

$$v_o=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi z}{L} \qquad (13.33)$$

- substituting in eq (13.32):

$$\frac{d^2v}{dz^2}+\frac{P}{EI}v=-\sum_{n=1}^{\infty}\left(\frac{n\pi}{L}\right)^2A_n\sin\frac{n\pi z}{L} \qquad (14.1)$$

- to find the particular solution of (14.1) we try

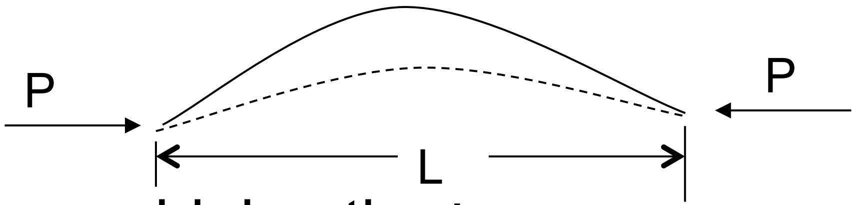
$$v_p=\sum_{n=1}^{\infty}C_n\sin\frac{n\pi z}{L} \qquad (14.2)$$

- substituting in (14.1) and solving for C_n:

$$C_n=-\frac{\left(\frac{n\pi}{L}\right)^2A_n}{\frac{P}{EI}-\left(\frac{n\pi}{L}\right)^2}=\frac{\left(\frac{n\pi}{L}\right)^2A_n}{\left(\frac{n\pi}{L}\right)^2-\frac{P}{EI}}=\frac{n^2A_n}{n^2-\frac{P}{\frac{\pi^2EI}{L^2}}} \qquad (14.3)$$

$=P_{cr}$ for ss beam!

Effect of initial imperfections



$$\frac{d^2v}{dz^2} + \frac{P}{EI}v = \frac{d^2v_o}{dz^2} \qquad (13.32)$$

- combining the homogeneous and particular solutions, eqs (13.7), (14.2) and (14.3), the complete expression for v is obtained:

$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z + \sum_{n=1}^{\infty}\frac{n^2A_n}{n^2 - \frac{P}{P_{cr}}}\sin\frac{n\pi z}{L} \qquad (14.4)$$

- apply now the BC's that v=0 at z=0 and z=L

of little interest

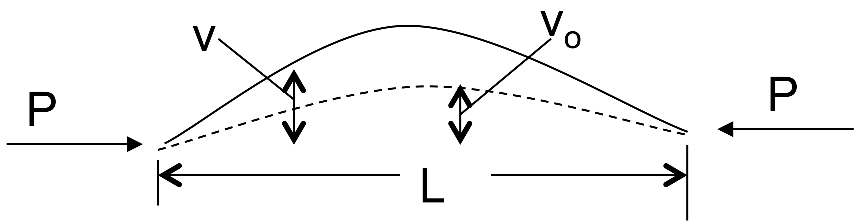
- at z=0: A=0; at z=L $B\sin\sqrt{\frac{P}{EI}}L = 0$ => either B=0 or P=0

or $\frac{P}{EI} = \frac{n^2\pi^2}{L^2}$ buckling condition but the beam is already bent

- so the final solution is:

$$v = \sum_{n=1}^{\infty}\frac{n^2A_n}{n^2 - \frac{P}{P_{cr}}}\sin\frac{n\pi z}{L} \qquad (14.5)$$

Effect of initial imperfections



$$v = \sum_{n=1}^{\infty} \frac{n^2 A_n}{n^2 - \dfrac{P}{P_{cr}}} \sin \frac{n\pi z}{L} \qquad (14.5)$$

- some points of interest:

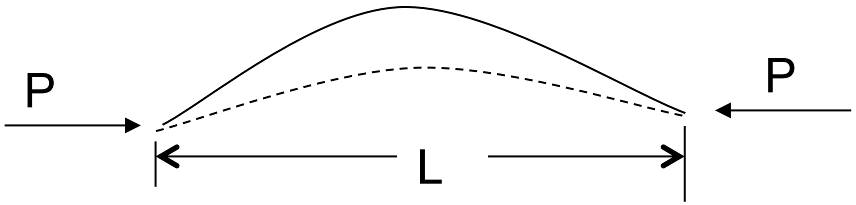
— this solution is valid for $P \le P_{cr}$; for $P > P_{cr}$ one must modify it to account for large deflections

— unlike the case of a straight ss beam where the solution for v has one unknown constant, here, v is completely determined; (for any value of P, the beam attains a stable equilibrium position) (A_n is known because v_o is known)

— the denominator in (14.5) is increasing very rapidly as n increases; so terms for n>1 are much smaller than for n=1; therefore:

$$v \approx \frac{A_1}{1 - \dfrac{P}{P_{cr}}} \sin \frac{\pi z}{L} \qquad (14.6)$$

Effect of initial imperfections



$$v \approx \frac{A_1}{1-\frac{P}{P_{cr}}}\sin\frac{\pi z}{L} \qquad (14.6)$$

- at the center of the beam, z=L/2

$$v_c = \frac{A_1}{1-\frac{P}{P_{cr}}} \qquad (14.7)$$

- when P=0, eq. (14.7) gives $v_c(P=0) = A_1$

i.e., A_1 is the center deflection of the unloaded beam, which can be measured experimentally

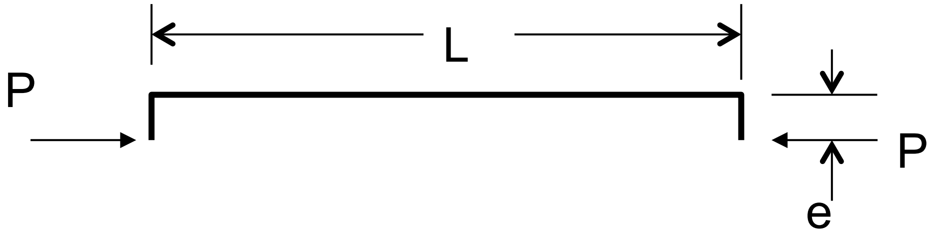
- if δ is the center deflection under load = v_c-A_1, using eq. (14.7) we get:

$$\delta = v_c - A_1 = \frac{A_1}{1-\frac{P}{P_{cr}}} - A_1 \Rightarrow \delta = \frac{A_1 P_{cr}}{P_{cr}-P} - A_1 \Rightarrow \delta P_{cr} - \delta P = A_1 P_{cr} - A_1 P_{cr} + A_1 P \Rightarrow \delta = \delta\frac{P_{cr}}{P} - A_1 \qquad (14.8)$$

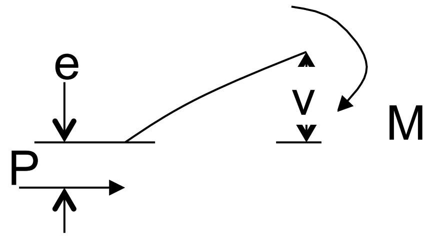
i.e. a plot of center deflection δ versus δ/P is a straight line with slope the buckling load P_{cr} (this kind of plot is called a Southwell plot)

Effect of eccentricities

- (b) the load is applied a small distance e from the neutral axis of the beam



- of primary importance here are the boundary conditions:
  - v=0 at z=0 and z=L
  - M=Pe at z=0 and z=L

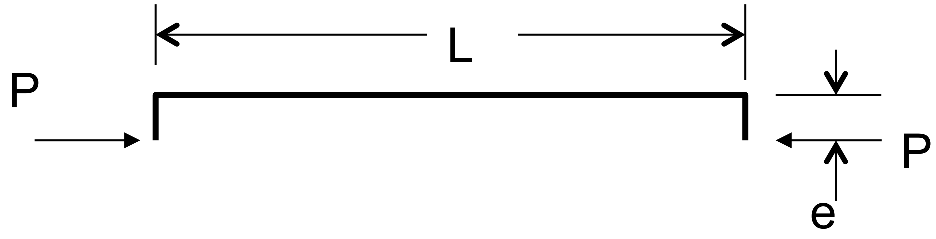


$$M = P(e + v(z))$$

- these are **four** conditions and, therefore, we need a fourth order differential equation!

note that Megson (section 8.3) uses a 2nd order differential equation but uses a reduced set of boundary conditions that he knows will work but it is a bit arbitrary

Effect of eccentricities



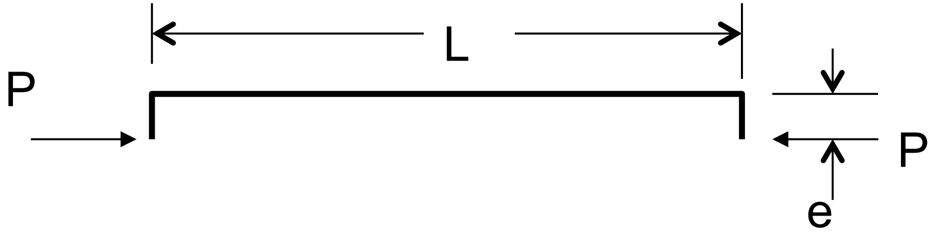
• we have already derived the fourth order diff. eqn. for such problems when we dealt with buckling of a clamped beam; eq (13.20):

$$\frac{d^4v}{dz^4} + \frac{P}{EI}\frac{d^2v}{dz^2} = 0 \tag{13.20}$$

• to which we also found the general solution, eq. (13.22)

$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z + Cz + D \tag{13.22}$$

Effect of eccentricities



$$\frac{d^4 v}{dz^4} + \frac{P}{EI}\frac{d^2 v}{dz^2} = 0 \qquad (13.20)$$

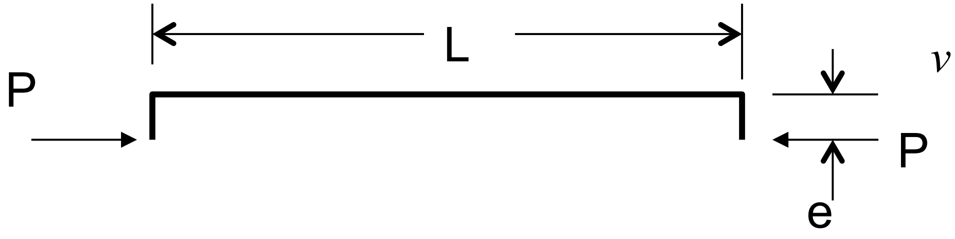
recall that for a beam,
M=-EI d^2v/dz^2

$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z + Cz + D \qquad (13.22)$$

- the BC v=0 at z=0 gives: $A + D = 0$ (14.9)
- the BC v=0 at z=L gives: $A\cos\sqrt{\frac{P}{EI}}L + B\sin\sqrt{\frac{P}{EI}}L + CL + D = 0$ (14.10)
- the BC M=Pe at z=0 gives: $-EI\frac{d^2v(0)}{dz^2} = Pe \Rightarrow -EI\left(-A\frac{P}{EI}\right) = Pe \Rightarrow A = e$ (14.11)
- the BC M=Pe at z=L gives: $-EI\left[-A\frac{P}{EI}\cos\sqrt{\frac{P}{EI}}L - B\frac{P}{EI}\sin\sqrt{\frac{P}{EI}}L\right] = Pe$ (14.12)

- now from (14.11) and (14.9): $D = -A = -e$ (14.13)

Effect of eccentricities



$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z + Cz + D \qquad (13.22)$$

$$A = e \qquad (14.11)$$

$$D = -e \qquad (14.13)$$

$$A\cos\sqrt{\frac{P}{EI}}L + B\sin\sqrt{\frac{P}{EI}}L + CL + D = 0 \qquad (14.10)$$

$$-EI\left[-A\frac{P}{EI}\cos\sqrt{\frac{P}{EI}}L - B\frac{P}{EI}\sin\sqrt{\frac{P}{EI}}L\right] = Pe \qquad (14.12)$$

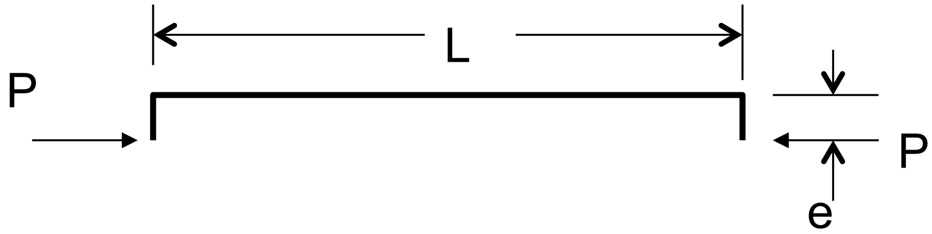
- from (14.12), knowing A from (14.11) can find B:

$$B = e\frac{\left(1-\cos\sqrt{\frac{P}{EI}}L\right)}{\sin\sqrt{\frac{P}{EI}}L} \qquad (14.13)$$

- then, substituting in (14.10) we find C: $C = 0$ (14.14)

- collecting everything in the expression for v (13.22):

Effect of eccentricities



$$v = A\cos\sqrt{\frac{P}{EI}}z + B\sin\sqrt{\frac{P}{EI}}z + Cz + D \quad (13.22)$$

$$v = e\left(-1 + \cos\sqrt{\frac{P}{EI}}z + \frac{\left(1 - \cos\sqrt{\frac{P}{EI}}L\right)}{\sin\sqrt{\frac{P}{EI}}L}\sin\sqrt{\frac{P}{EI}}z\right) \quad (14.15)$$

note that Megson has an expression that looks different but is, actually, identical to (14.15); the above expression is easier to use in practice but Megson's is simpler

Effect of eccentricities

• for the mathematically inclined, to show equivalence of our expression with that of Megson, the following are needed:

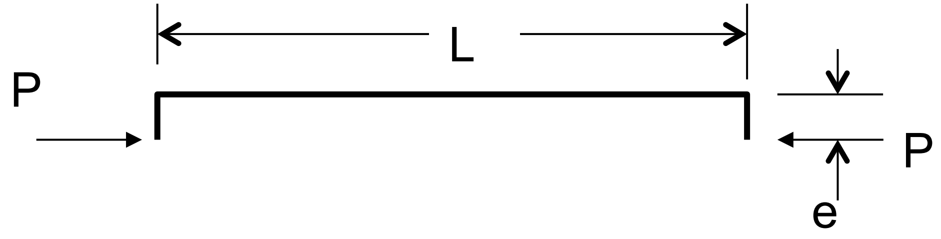
$$\cos x = 1 - 2\sin^2 \frac{x}{2}$$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

$$\therefore \frac{1-\cos x}{\sin x} = \tan\frac{x}{2}$$

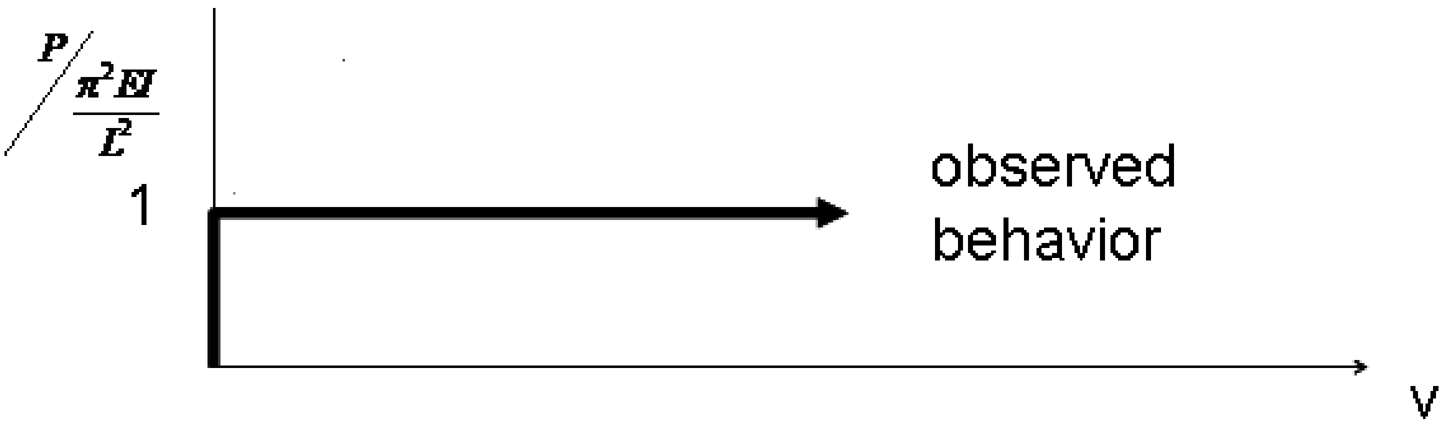
etc.

Effect of eccentricities

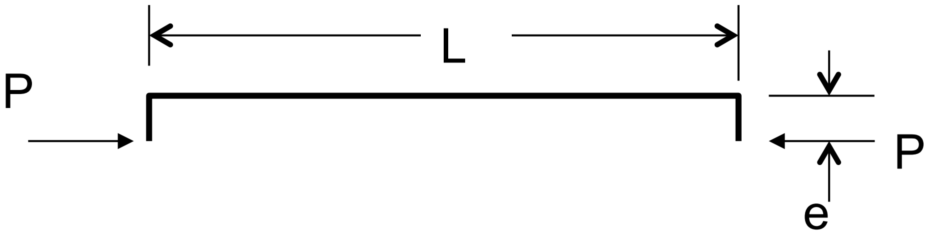


$$v = e\left(-1 + \cos\sqrt{\frac{P}{EI}}z + \frac{\left(1 - \cos\sqrt{\frac{P}{EI}}L\right)}{\sin\sqrt{\frac{P}{EI}}L}\sin\sqrt{\frac{P}{EI}}z\right) \tag{14.15}$$

- in the previous lecture we saw that for a perfect column (no eccentricities):



Effect of eccentricities



$$v = e\left(-1+\cos\sqrt{\frac{P}{EI}}z+\frac{\left(1-\cos\sqrt{\frac{P}{EI}}L\right)}{\sin\sqrt{\frac{P}{EI}}L}\sin\sqrt{\frac{P}{EI}}z\right) \quad (14.15)$$

- now, with eccentricities present, a plot of P/P_{cr} versus beam center deflection can be obtained

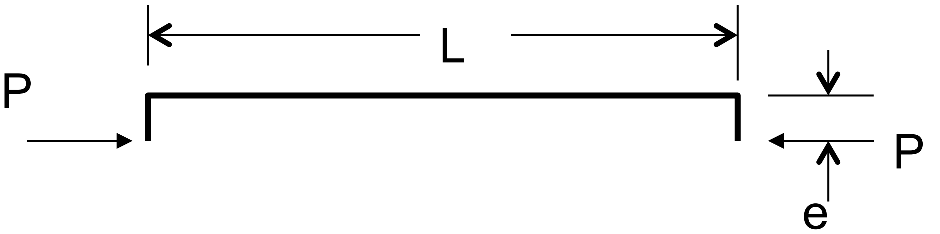
- to do that we need to rearrange the expression for v slightly; using

$$P_{crit}=\frac{\pi^2 EI}{L^2} \quad (13.9)$$

- we can write: $$\sqrt{\frac{P}{EI}}L=\sqrt{\frac{P}{EIP_{cr}}\frac{\pi^2 EI}{L^2}}L=\pi\sqrt{\frac{P}{P_{cr}}} \quad (14.16)$$

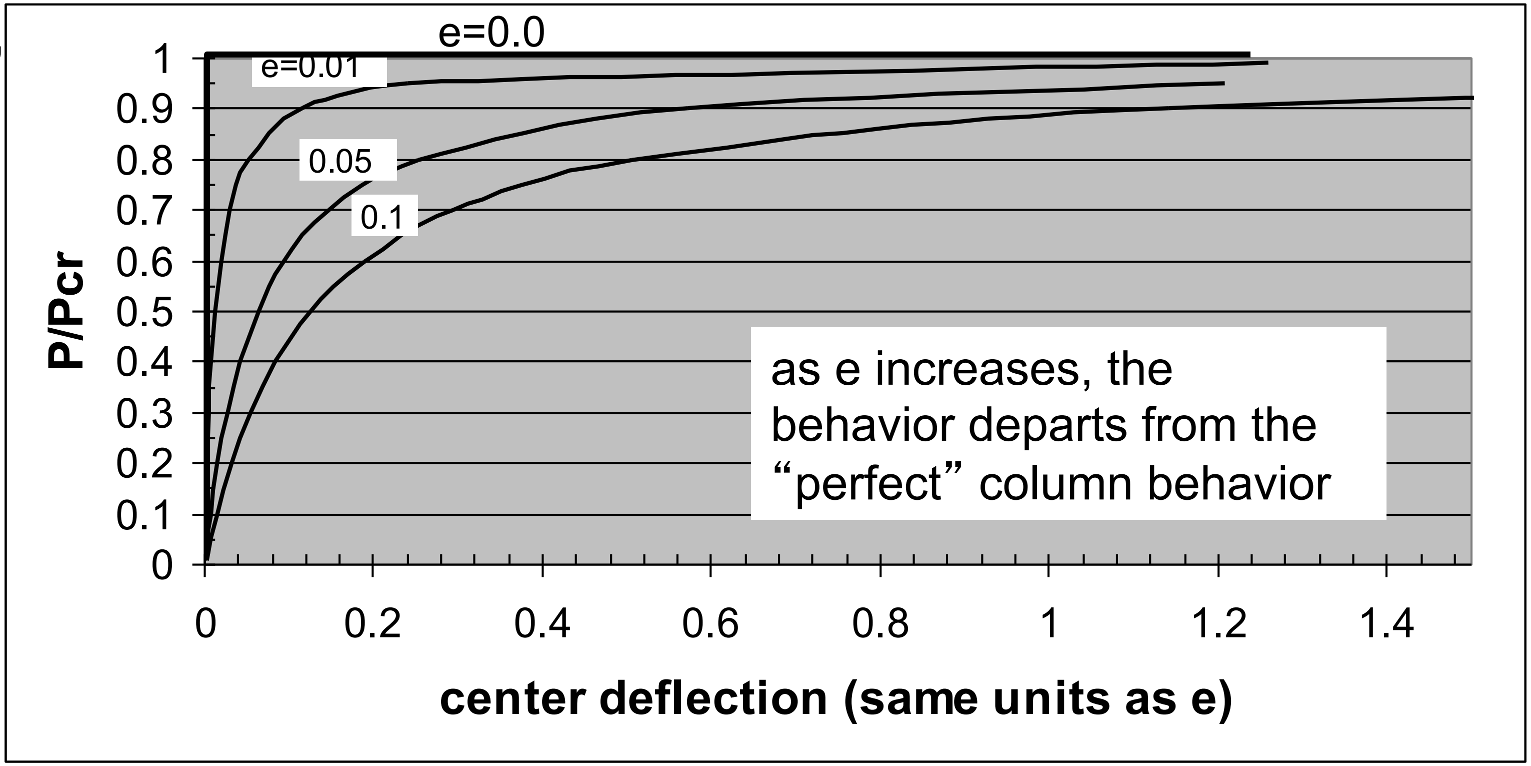
- and using that, we can rewrite eq. (14.15) as:

Effect of eccentricities



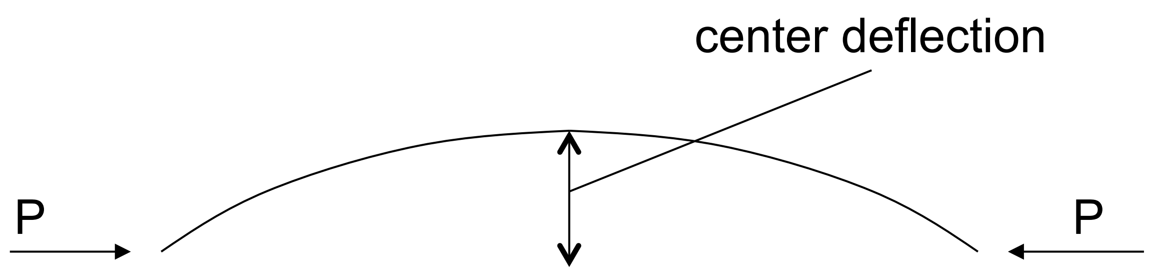
(14.15a)

• plotting,

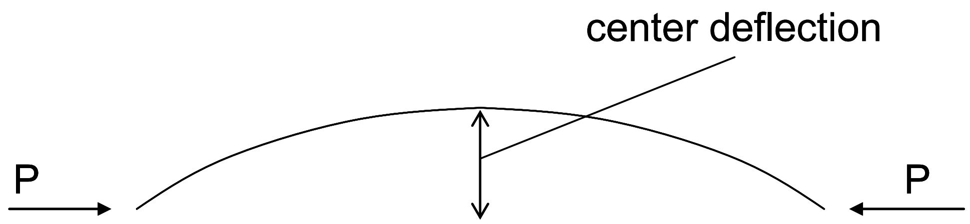


Application: Beam with eccentricity

- An aluminum beam (E=69 GPa) of length 1 m is under a compressive load P. When P=1000 N, the center deflection of the beam is 1 mm. When P= 2500 N, the center deflection is 3 mm. Determine the moment of inertia of the beam.



Application: Beam with eccentricity



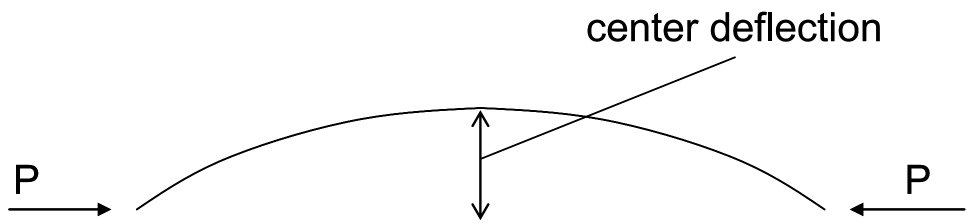
P_1=1kN, $v_1(L/2)$=0.001 m

P_2=3kN, $v_2(L/2)$=0.003 m

$$\left.\begin{aligned} v_1 &= e\left(-1+\cos\frac{\pi}{2}\sqrt{\frac{P_1}{P_{cr}}}+\frac{\left(1-\cos\pi\sqrt{\frac{P_1}{P_{cr}}}\right)}{\sin\pi\sqrt{\frac{P_1}{P_{cr}}}}\sin\frac{\pi}{2}\sqrt{\frac{P_1}{P_{cr}}}\right)\\ v_2 &= e\left(-1+\cos\frac{\pi}{2}\sqrt{\frac{P_2}{P_{cr}}}+\frac{\left(1-\cos\pi\sqrt{\frac{P_2}{P_{cr}}}\right)}{\sin\pi\sqrt{\frac{P_2}{P_{cr}}}}\sin\frac{\pi}{2}\sqrt{\frac{P_2}{P_{cr}}}\right)\end{aligned}\right\}$$

Divide one equation by the other to eliminate eccentricity e

Application: Beam with eccentricity



P_1=1kN, $v_1(L/2)$=0.001 m

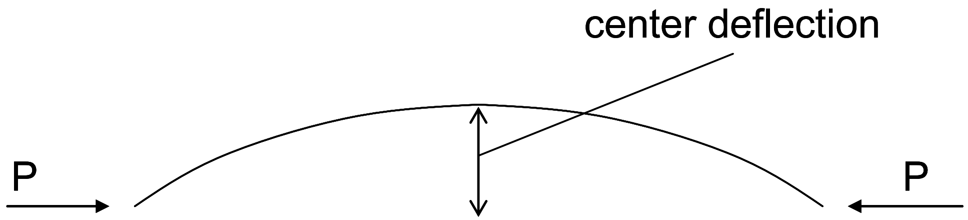
P_2=3kN, $v_2(L/2)$=0.003 m

$$\frac{v_1}{v_2} = \frac{-1+\cos\frac{\pi}{2}\sqrt{\frac{P_1}{P_{cr}}} + \frac{\left(1-\cos\pi\sqrt{\frac{P_1}{P_{cr}}}\right)}{\sin\pi\sqrt{\frac{P_1}{P_{cr}}}}\sin\frac{\pi}{2}\sqrt{\frac{P_1}{P_{cr}}}}{-1+\cos\frac{\pi}{2}\sqrt{\frac{P_2}{P_{cr}}} + \frac{\left(1-\cos\pi\sqrt{\frac{P_2}{P_{cr}}}\right)}{\sin\pi\sqrt{\frac{P_2}{P_{cr}}}}\sin\frac{\pi}{2}\sqrt{\frac{P_2}{P_{cr}}}}$$

In this equation we know v_1, v_2, P_1, and P_2. The only unknown is P_{cr}

- the equation is solved numerically (e.g. pick a value of P_{cr}, evaluate RHS, compare to LHS, and keep adjusting

Application: Beam with eccentricity



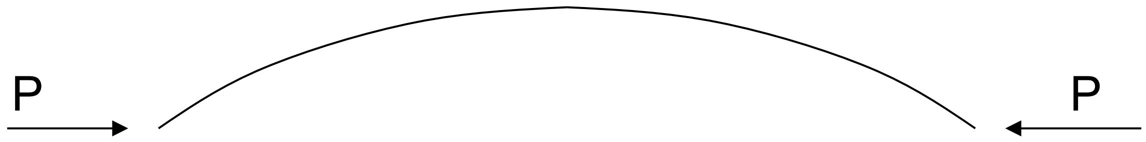
- solving: P_{cr} = 3594 N
- but, for a simply supported beam under compression, we know from eq (13.9):

$$P_{crit} = \frac{\pi^2 EI}{L^2} \qquad (13.9)$$

- setting P_{cr}=3594 N, E=69 GPa, and L=1 m, and solving for I:

$$\boxed{I = \frac{P_{cr} L^2}{\pi^2 E} = 5.278x10^{-9}\, m^4}$$

Alternate method to get buckling load using energy methods

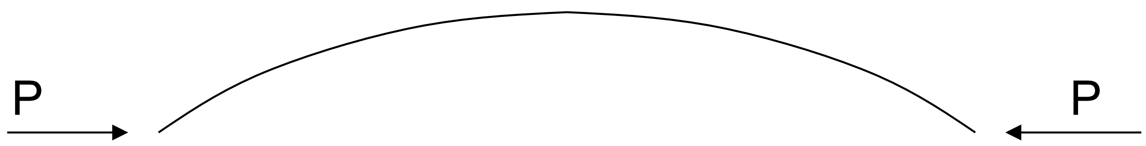


• the principle of minimum potential energy (revisited in lecture 13):

If the displacements in a body satisfy compatibility and the displacement boundary conditions then if the body is in equilibrium the total potential energy is minimized

• the idea is to get some approximate expression for the displacements of the beam during buckling, minimize the energy, and hope for the best…

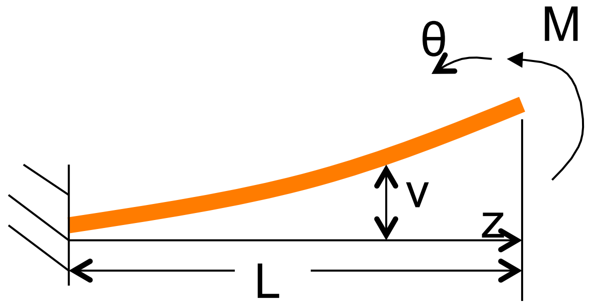
Alternate method to get buckling load using energy methods

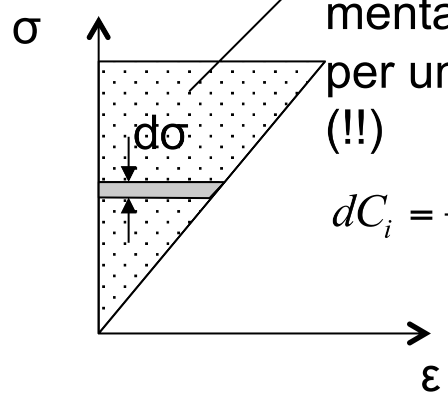


- first, we need to find an expression for the total potential energy of the beam
- total potential energy = internal strain energy – Work done by force P

Internal strain energy for a beam in bending

Beam under bending moment M





area = complementary energy per unit volume (!!)

$$dC_i = \frac{\sigma\varepsilon}{2}$$

we know $\sigma = E\varepsilon \Rightarrow \varepsilon = \frac{\sigma}{E}$ so $dC_i = \frac{\sigma^2}{2E}$

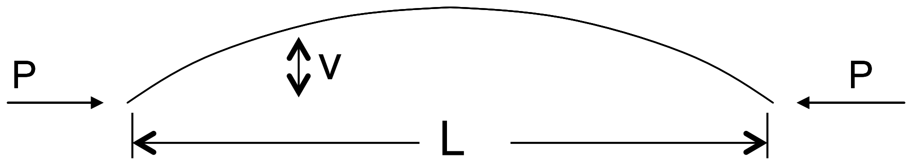
from bending theory, lecture 2, $\sigma_z = -\frac{My}{I}$

combining the two: $dC_i = \frac{M^2 y^2}{2EI^2}$

then, $C_i = \int_{vol} dC_i = \iiint \frac{M^2 y^2}{2EI^2} dxdydz$ but, by definition, $\iint y^2 dxdy = I$

therefore, $\boxed{C_i = \int_0^L \frac{M^2}{2EI} dz}$

Alternate method to get buckling load using energy methods



$$C_i = \int_0^L \frac{M^2}{2EI}\,dz$$

- this expression is in terms of the moment; as the theorem of min potential energy suggests, we must express everything in terms of displacements

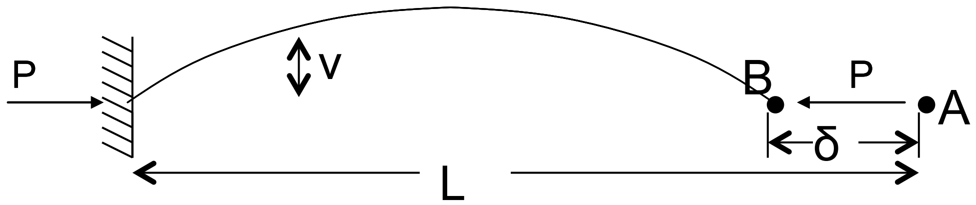
- from the basic moment-curvature relationship (13.3) for beams:

$$M = -EI\,\frac{d^2v}{dz^2} \qquad (13.3)$$

- using this to substitute:

$$C_i = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dz^2}\right)^2 dz \qquad (14.17)$$

Alternate method to get buckling load using energy methods



$$C_i = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dz^2}\right)^2 dz \qquad (14.17)$$

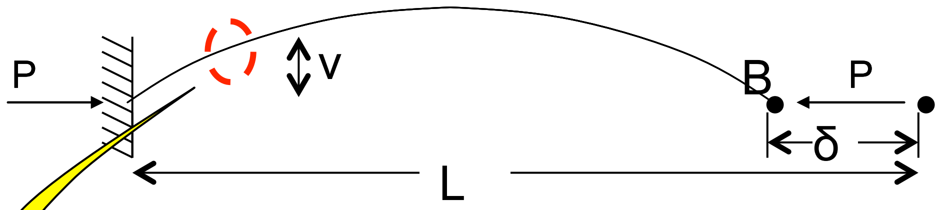
• need to also determine the work done by P

• assuming the beam is stationary at the left end, all the work is done by P acting on the right end; if the undeformed position of the right end was at A and, under load moved to B, a displacement δ, the work done by P is:

$$W = P\delta \qquad (14.18)$$

• which must be expressed in terms of v (in order to make use of the principle of minimum potential energy)

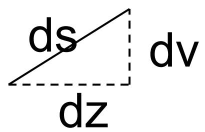
Alternate method to get buckling load using energy methods



$$C_i = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dz^2}\right)^2 dz \qquad (14.17)$$

$$W = P\delta \qquad (14.18)$$

- consider an element of arc length ds of the deformed beam:



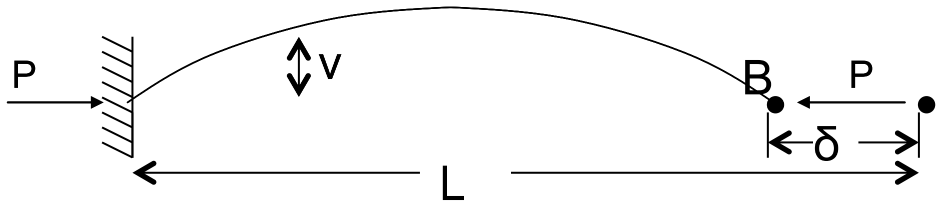
$$\delta = \int_0^L ds - L \qquad (14.19)$$

$$ds = \sqrt{dz^2 + dv^2} = dz\sqrt{1+\left(\frac{dv}{dz}\right)^2} \qquad (14.20)$$

- very close to the buckling load, the deflections are small and the quantity $(dv/dz)^2$ is very small; so (14.20) can be expanded in a Taylor series and use only the first term:

$$\sqrt{1+\left(\frac{dv}{dz}\right)^2} \approx 1 + \left[\frac{1}{2}\frac{1}{\sqrt{1+\left(\frac{dv}{dz}\right)^2}}\right]_{dv/dz=0}\left(\frac{dv}{dz}\right)^2 + \ldots = 1 + \frac{1}{2}\left(\frac{dv}{dz}\right)^2 \qquad (14.21)$$

Alternate method to get buckling load using energy methods



$$W = P\delta \qquad (14.18)$$

$$\delta = \int_0^L ds - L \qquad (14.19)$$

$$ds = \sqrt{dz^2 + dv^2} = dz\sqrt{1 + \left(\frac{dv}{dz}\right)^2} \qquad (14.20)$$

$$\sqrt{1 + \left(\frac{dv}{dz}\right)^2} \;;\; 1 + \left[\frac{1}{2}\frac{1}{\sqrt{1 + \left(\frac{dv}{dz}\right)^2}}\right]_{dv/dz=0} \left(\frac{dv}{dz}\right)^2 + \ldots = 1 + \frac{1}{2}\left(\frac{dv}{dz}\right)^2 \qquad (14.21)$$

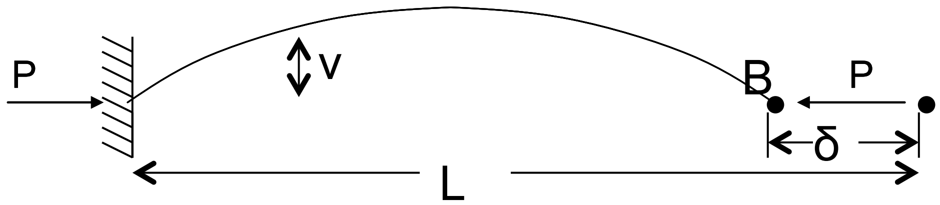
- combining (14.21), (14.20), and (14.19) we get:

$$\delta = \int_0^L \left[1 + \frac{1}{2}\left(\frac{dv}{dz}\right)^2\right] dz - L = \int_0^L \frac{1}{2}\left(\frac{dv}{dz}\right)^2 dz \qquad (14.22)$$

- and substituting in eq. (14.18),

$$W = P\int_0^L \frac{1}{2}\left(\frac{dv}{dz}\right)^2 dz \qquad (14.23)$$

Alternate method to get buckling load using energy methods



Total energy=C_i-W

$$C_i = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dz^2}\right)^2 dz \qquad (14.17)$$

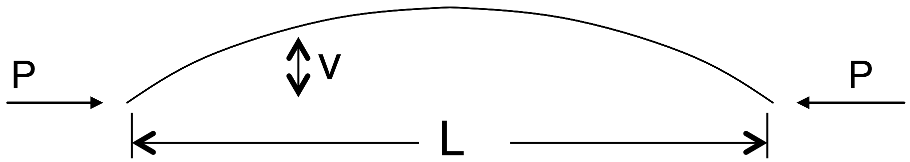
$$W = P\int_0^L \frac{1}{2}\left(\frac{dv}{dz}\right)^2 dz \qquad (14.23)$$

- finally, the total potential energy for a beam buckling under compressive load P is given by

$$\boxed{U - W = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dz^2}\right)^2 dz - P\int_0^L \frac{1}{2}\left(\frac{dv}{dz}\right)^2 dz} \qquad (14.24)$$

- if we knew v exactly, then (14.24) would be minimized; but if we knew it exactly we would not have to calculate the energy anyway (the problem would be solved)

Alternate method to get buckling load using energy methods



$$U - W = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dz^2}\right)^2 dz - P\int_0^L \frac{1}{2}\left(\frac{dv}{dz}\right)^2 dz \tag{14.24}$$

- suppose now that $v = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi z}{L}$ where A_n are unknown constants
  - note that this expression satisfies the simply-supported BC's that v=0 at z=0 and z=L
  - note also that, from Fourier series theory, this expression for v can reproduce any continuous function and, therefore, also the exact solution
  - since we do not know A_n, but we know that the energy must be minimized, we choose to minimize U-W with respect to A_n

Alternate method to get buckling load using energy methods

- need to evaluate the integrals in U-W:

$$\frac{dv}{dz} = \sum_{n=1}^{\infty} A_n \frac{n\pi}{L} \cos\frac{n\pi z}{L} \quad (14.25)$$

$$\left(\frac{dv}{dz}\right)^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n A_m \frac{n\pi}{L}\frac{m\pi}{L}\cos\frac{n\pi z}{L}\cos\frac{m\pi z}{L} \quad (14.26)$$

same as writing the two series next to each other with different indices

$$\int_0^L \left(\frac{dv}{dz}\right)^2 dz = \int_0^L \sum_{n=1}^{\infty} A_n^2 \frac{n^2\pi^2}{L^2}\left(\cos\frac{n\pi z}{L}\right)^2 dz \quad (14.27)$$

because

note Megson skips dozens of steps here!

$$\int_0^L \cos\frac{n\pi z}{L}\cos\frac{m\pi z}{L}\, dz = 0 \quad when \quad m \neq n \quad (14.28)$$

- then, using

$$\int_0^L \left(\cos\frac{n\pi z}{L}\right)^2 dz = \int_0^L \frac{1}{2}\left(1+\cos\frac{2n\pi z}{L}\right) dz = \frac{1}{2}\left[z + \frac{L}{2n\pi}\sin\frac{2n\pi z}{L}\right]_0^L = \frac{L}{2} \quad (14.29)$$

- we get: $$\int_0^L \left(\frac{dv}{dz}\right)^2 dz = \sum_{n=1}^{\infty} A_n^2 \frac{n^2\pi^2}{L^2}\frac{L}{2} = \sum_{n=1}^{\infty} A_n^2 \frac{n^2\pi^2}{2L} \quad (14.30)$$

Alternate method to get buckling load using energy methods

- similarly for the U term,

$$\frac{d^2v}{dz^2} = -\sum_{n=1}^{\infty} A_n \left(\frac{n\pi}{L}\right)^2 \sin\frac{n\pi z}{L} \tag{14.31}$$

$$\left(\frac{d^2v}{dz^2}\right)^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n A_m \left(\frac{n\pi}{L}\right)^2 \left(\frac{m\pi}{L}\right)^2 \sin\frac{n\pi z}{L}\sin\frac{m\pi z}{L} \tag{14.32}$$

- and

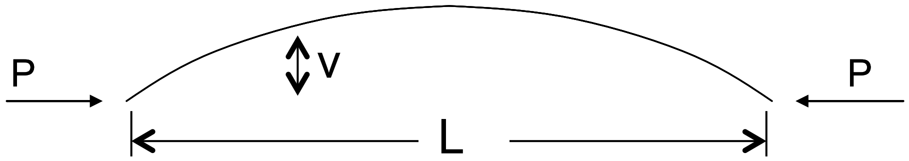
$$\int_0^L \left(\frac{d^2v}{dz^2}\right)^2 dz = \int_0^L \sum_{n=1}^{\infty} A_n^2 \left(\frac{n\pi}{L}\right)^4 \left(\sin\frac{n\pi z}{L}\right)^2 dz \tag{14.33}$$

because $\int_0^L \sin\frac{n\pi z}{L}\sin\frac{m\pi z}{L}\,dz = 0 \quad when \quad m \neq n$ (14.34)

- then, using $\int_0^L \left(\sin\frac{n\pi z}{L}\right)^2 dz = \int_0^L \frac{1}{2}\left(1-\cos\frac{2n\pi z}{L}\right)dz = \frac{1}{2}\left[z - \frac{L}{2n\pi}\sin\frac{2n\pi z}{L}\right]_0^L = \frac{L}{2}$ (14.35)

- we get: $\int_0^L \left(\frac{d^2v}{dz^2}\right)^2 dz = \sum_{n=1}^{\infty} A_n^2 \left(\frac{n\pi}{L}\right)^4 \frac{L}{2} = \sum_{n=1}^{\infty} A_n^2 \frac{n^4\pi^4}{2L^3}$ (14.36)

Alternate method to get buckling load using energy methods



$$U - W = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dz^2}\right)^2 dz - P\int_0^L \frac{1}{2}\left(\frac{dv}{dz}\right)^2 dz \qquad (14.24)$$

$$\int_0^L \left(\frac{dv}{dz}\right)^2 dz = \sum_{n=1}^{\infty} A_n^2 \frac{n^2\pi^2}{L^2}\frac{L}{2} = \sum_{n=1}^{\infty} A_n^2 \frac{n^2\pi^2}{2L} \qquad (14.30)$$

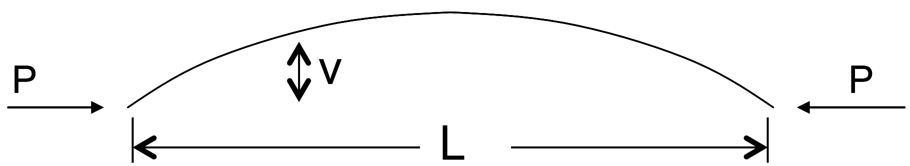
$$\int_0^L \left(\frac{d^2v}{dz^2}\right)^2 dz = \sum_{n=1}^{\infty} A_n^2 \left(\frac{n\pi}{L}\right)^4 \frac{L}{2} = \sum_{n=1}^{\infty} A_n^2 \frac{n^4\pi^4}{2L^3} \qquad (14.36)$$

- combining: $U - W = EI\sum_{n=1}^{\infty} A_n^2 \frac{n^4\pi^4}{4L^3} - P\sum_{n=1}^{\infty} A_n^2 \frac{n^2\pi^2}{4L}$ (14.37)

- in this expression, A_n are still unknown; to determine them, minimize the energy with respect to the A_n:

$$\frac{\partial(U - W)}{\partial A_n} = 0 \qquad (14.38)$$

Alternate method to get buckling load using energy methods



$$U - W = EI\sum_{n=1}^{\infty} A_n^{\,2}\frac{n^4\pi^4}{4L^3} - P\sum_{n=1}^{\infty} A_n^{\,2}\frac{n^2\pi^2}{4L} \qquad (14.37)$$

$$\frac{\partial(U-W)}{\partial A_n} = 0 \qquad (14.38)$$

- from (14.38), differentiating (14.37) (note that the summations disappear!):

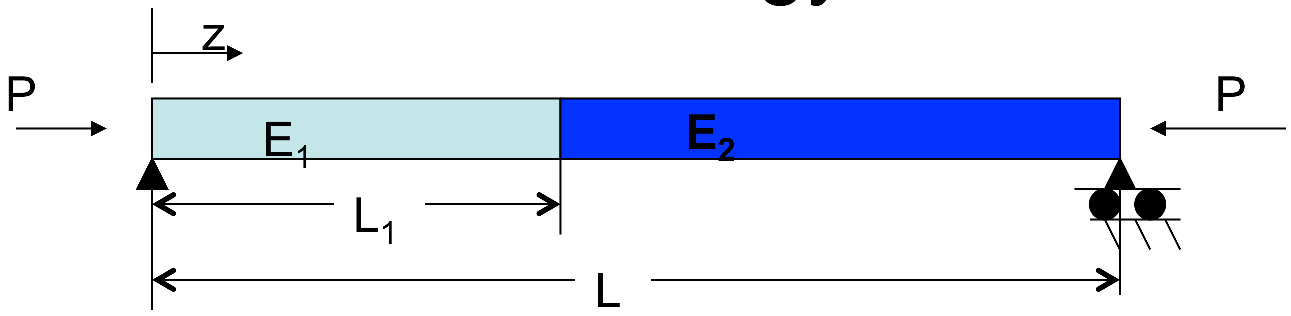
$$2EIA_n\frac{n^4\pi^4}{4L^3} - 2PA_n\frac{n^2\pi^2}{4L} = 0 \Rightarrow A_n\left(\frac{EIn^2\pi^2}{L^2} - P\right) = 0 \qquad (14.39)$$

- either A_n=0 which means the beam does not buckle, only compresses (trivial solution), or,

$$\left(\frac{EIn^2\pi^2}{L^2} - P\right) \Rightarrow P = \frac{n^2\pi^2 EI}{L^2} \qquad (13.9)$$

which is exactly the same as eq. (13.9) we got before for an ss beam!!

Alternate method to get buckling load using energy methods



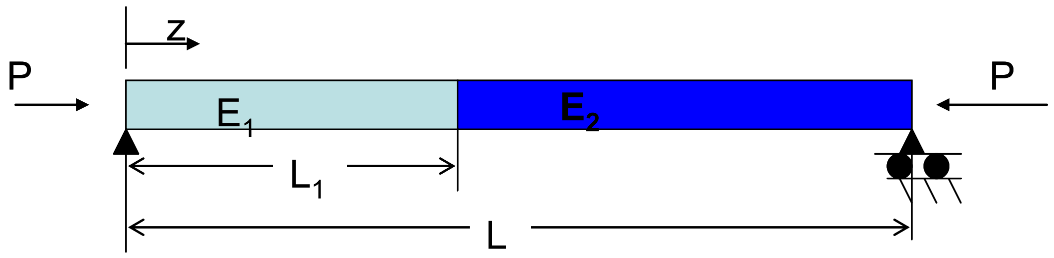
- consider now a beam of length L made of two materials welded together with Young's moduli E_1 and E_2 respectively but with moments of inertia I_1=I_2=I

- to determine the buckling load, assume an approximate expression for v:

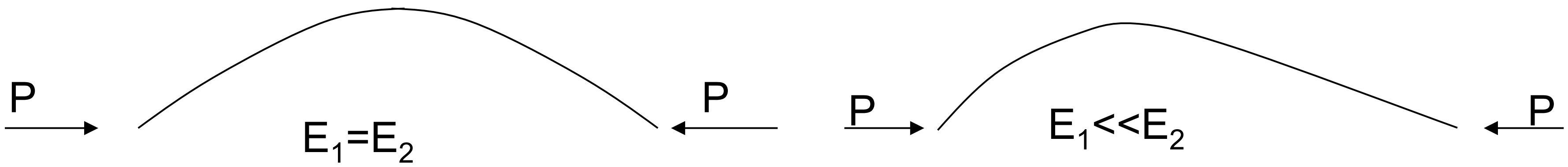
$$v = A_1 \sin\frac{\pi z}{L}$$ where A_1 is an unknown constant

- note that for the method to work, v must satisfy the boundary conditions (v=0 at z=0 and z=L)

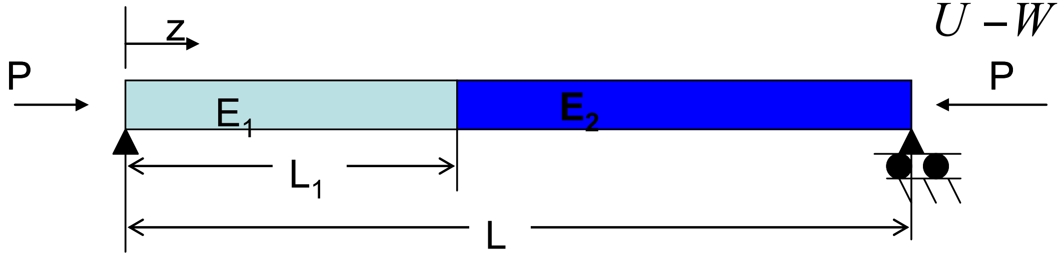
Buckling of beam consisting of two materials



• our assumed expression for the deflection is approximate because (for one thing), it is perfectly symmetric with respect to the beam mid-point which assumes the stiffnesses E_1 and E_2 are about the same



Buckling of beam consisting of two materials



$$U - W = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dz^2}\right)^2 dz - P\int_0^L \frac{1}{2}\left(\frac{dv}{dz}\right)^2 dz \qquad (14.24)$$

$$v = A_1 \sin\frac{\pi z}{L}$$

- to substitute in eq (14.24) we need:

$$\frac{dv}{dz} = A_1 \frac{\pi}{L}\cos\frac{\pi z}{L}$$

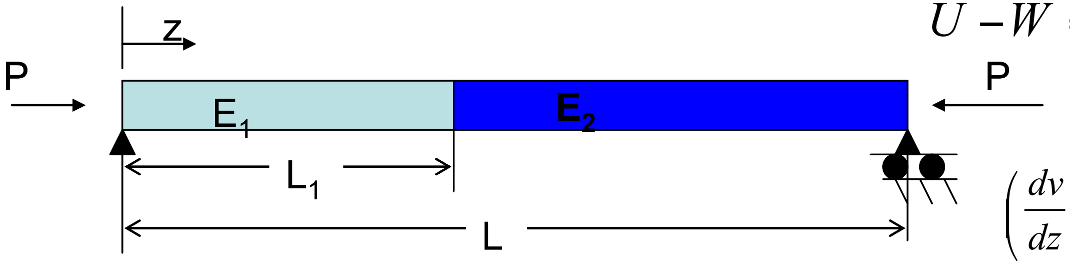
$$\frac{d^2v}{dz^2} = -A_1\left(\frac{\pi}{L}\right)^2 \sin\frac{\pi z}{L}$$

- squaring these terms:

$$\left(\frac{dv}{dz}\right)^2 = A_1^2 \frac{\pi^2}{L^2}\left(\cos\frac{\pi z}{L}\right)^2 = A_1^2 \frac{\pi^2}{2L^2}\left(1+\cos\frac{2\pi z}{L}\right)$$

$$\left(\frac{d^2v}{dz^2}\right)^2 = A_1^2 \frac{\pi^4}{L^4}\left(\sin\frac{\pi z}{L}\right)^2 = A_1^2 \frac{\pi^4}{2L^4}\left(1-\cos\frac{2\pi z}{L}\right)$$

Buckling of beam consisting of two materials



$$U - W = \frac{1}{2}\int_0^L EI\left(\frac{d^2v}{dz^2}\right)^2 dz - P\int_0^L \frac{1}{2}\left(\frac{dv}{dz}\right)^2 dz \tag{14.24}$$

$$\left(\frac{dv}{dz}\right)^2 = A_1^2\frac{\pi^2}{L^2}\left(\cos\frac{\pi z}{L}\right)^2 = A_1^2\frac{\pi^2}{2L^2}\left(1+\cos\frac{2\pi z}{L}\right)$$

$$\left(\frac{d^2v}{dz^2}\right)^2 = A_1^2\frac{\pi^4}{L^4}\left(\sin\frac{\pi z}{L}\right)^2 = A_1^2\frac{\pi^4}{2L^4}\left(1-\cos\frac{2\pi z}{L}\right)$$

- to integrate we notice the following:

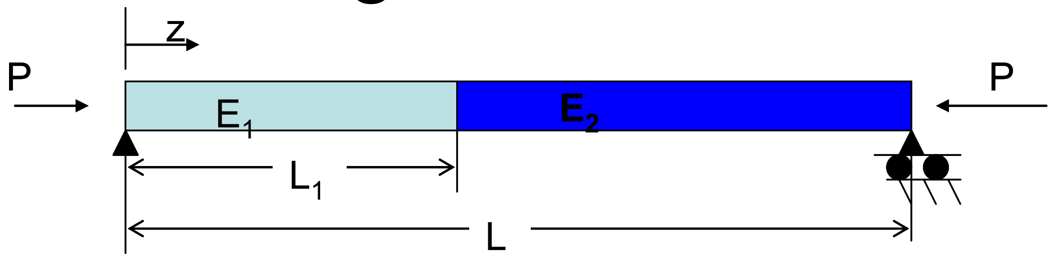
$$\int_a^b \left(\frac{dv}{dz}\right)^2 dz = A_1^2\frac{\pi^2}{2L^2}\int_a^b\left(1+\cos\frac{2\pi z}{L}\right)dz = A_1^2\frac{\pi^2}{2L^2}\left[z+\frac{L}{2\pi}\sin\frac{2\pi z}{L}\right]_a^b$$

$$\int_a^b \left(\frac{d^2v}{dz^2}\right)^2 dz = A_1^2\frac{\pi^4}{2L^4}\int_a^b\left(1-\cos\frac{2\pi z}{L}\right)dz = A_1^2\frac{\pi^4}{2L^4}\left[1-\frac{L}{2\pi}\sin\frac{2\pi z}{L}\right]_a^b$$

- substituting in the energy expression,

$$U - W = \frac{A_1^2}{4L}\left[E_1 I\frac{\pi^2}{L^3}\left(L_1 - \frac{L}{2\pi}\sin\frac{2\pi L_1}{L}\right) + E_2 I\frac{\pi^2}{L^3}\left(L - L_1 + \frac{L}{2\pi}\sin\frac{2\pi L_1}{L}\right) - P\right]$$

Buckling of beam consisting of two materials



$$U - W = \frac{A_1^2}{4L}\left[E_1 I \frac{\pi^2}{L^3}\left(L_1 - \frac{L}{2\pi}\sin\frac{2\pi L_1}{L}\right) + E_2 I \frac{\pi^2}{L^3}\left(L - L_1 + \frac{L}{2\pi}\sin\frac{2\pi L_1}{L}\right) - P\right]$$

- energy minimization implies:

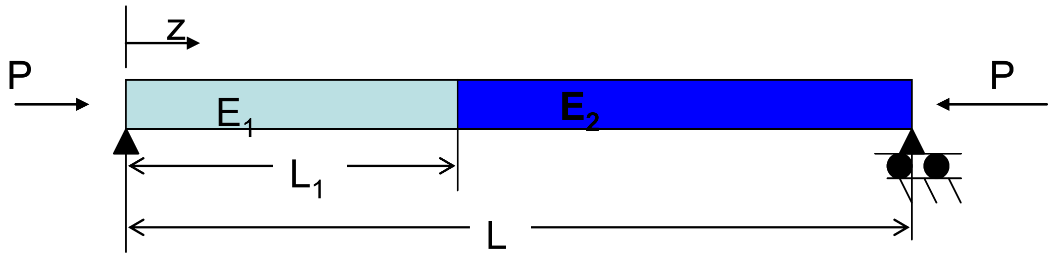
$$\frac{d(U-W)}{dA_1} = 0$$

- which, after some rearranging leads to,

$$\frac{2A_1}{4L}\left[\left(E_1 - E_2\right)\frac{\pi^2 I L_1}{L^3} + E_2\frac{L\pi^2 I}{L^3} - \left(E_1 - E_2\right)\frac{L\pi^2 I}{2\pi L^3}\sin\frac{2\pi L_1}{L} - P\right] = 0$$

- either A_1=0, (trivial solution implying no buckling) or the quantity in brackets is zero

Buckling of beam consisting of two materials



$$\frac{2A_1}{4L}\left[\left(E_1-E_2\right)\frac{\pi^2 IL_1}{L^3}+E_2\frac{L\pi^2 I}{L^3}-\left(E_1-E_2\right)\frac{L\pi^2 I}{2\pi L^3}\sin\frac{2\pi L_1}{L}-P\right]=0$$

- therefore, the buckling load is given by:

$$P=\frac{\pi^2 I}{L^3}\left[E_2 L+\left(E_1-E_2\right)\left(L_1-\frac{L}{2\pi}\sin\frac{2\pi L_1}{L}\right)\right]$$

(approximate buckling load)

- special case 1: $E_1=E_2=E$; then:

$$P_{crit}=\frac{\pi^2 EI}{L^2}$$

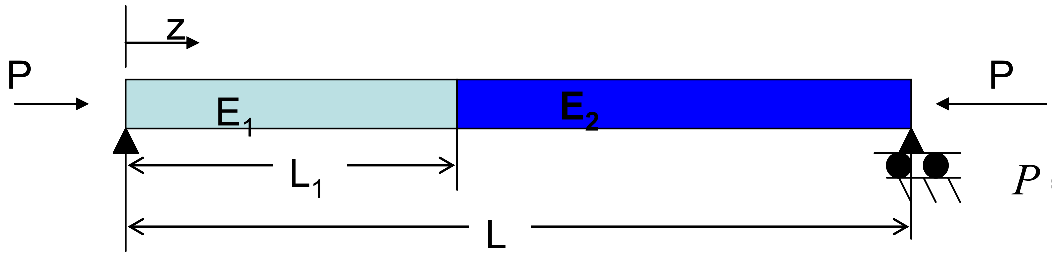
which is exactly the same expression we found before for the buckling load of a simply supported beam

- special case 2: $L_1=0$; then the entire beam is from mat'l 2

$$P_{crit}=\frac{\pi^2 E_2 I}{L^2}$$

exactly as expected

Buckling of beam consisting of two materials



$$P = \frac{\pi^2 I}{L^3}\left[E_2 L + \left(E_1 - E_2\right)\left(L_1 - \frac{L}{2\pi}\sin\frac{2\pi L_1}{L}\right)\right]$$

- special case 3: L_1=L; the entire beam is from mat'l 1

$$P = \frac{\pi^2 I}{L^3}\left[E_2 L + \left(E_1 - E_2\right)L\right] = \frac{\pi^2 E_1 I}{L^2}$$ exactly as expected