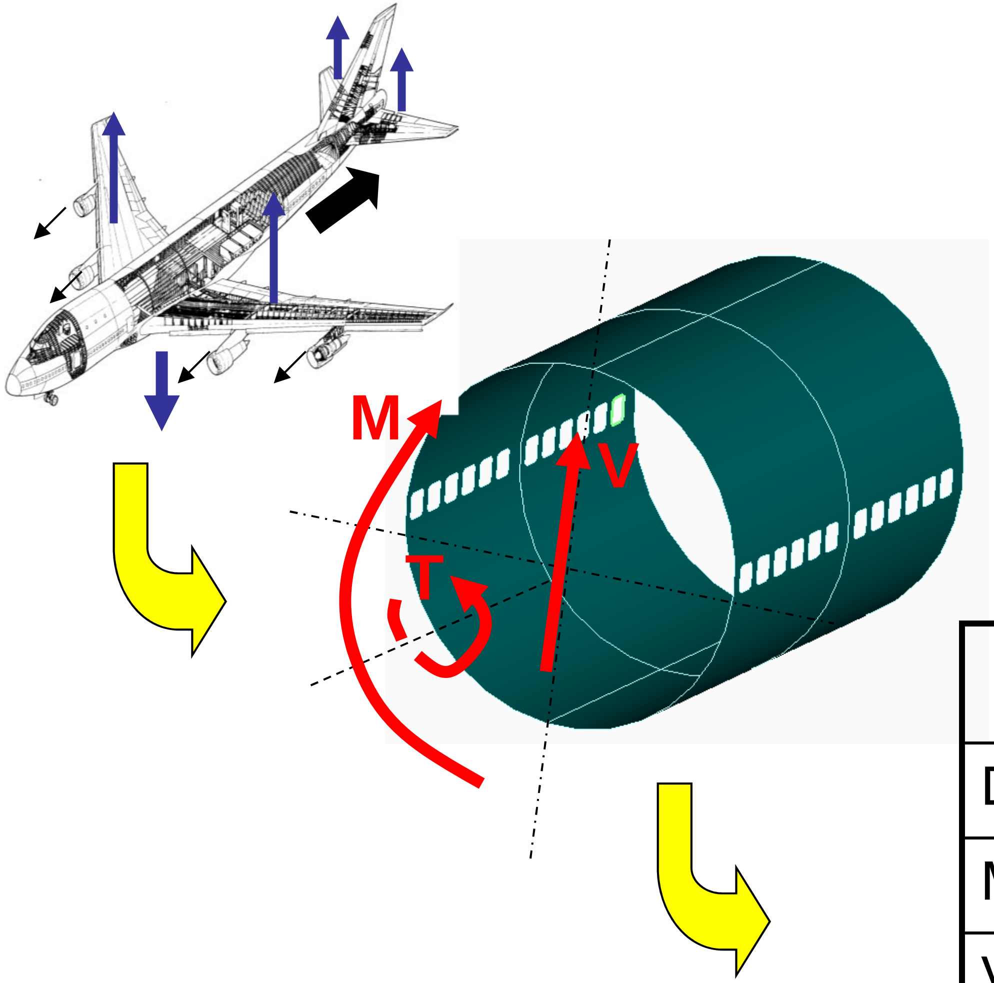
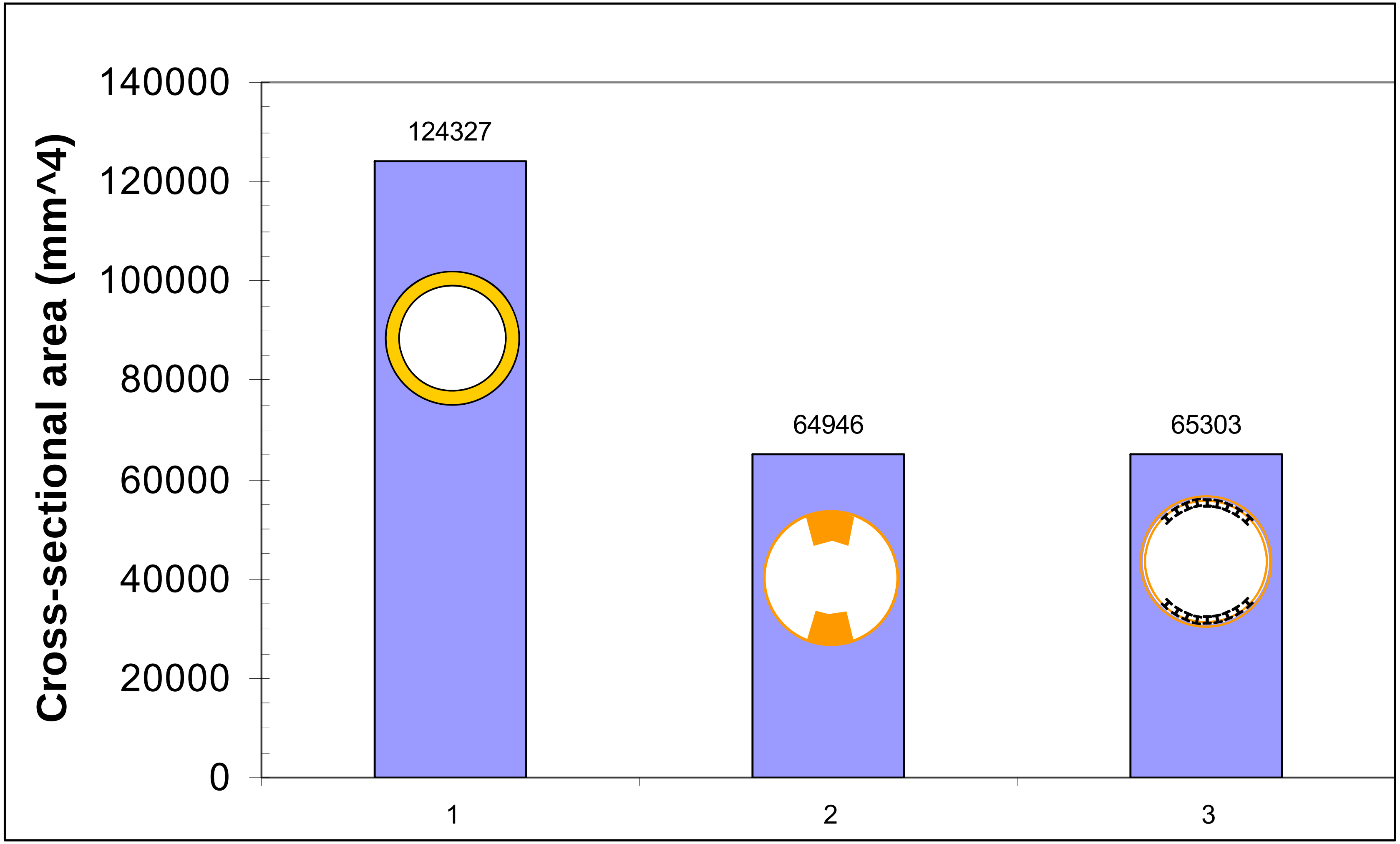
"Running" example – Fuselage cross-section



Property	Value
Diameter(m)	4.0
M (MNm)	60
V (kN)	660
T (kNm)	30

So far…


Cross-sectional area (mm^4)
140000
120000
100000
80000
60000
40000
20000
0
124327
64946
65303
1
2
3

Design now for torsion

- from previous lecture:
  - pure torsion results in constant shear flow q in the wall of closed section beam
  - $T = 2Aq$ (3.44)
- what happens when we have combined bending moment M and torque T? is q constant?

front view
edge view

• considering M alone, the normal stress σ_z is the same at any z => the load on each stiffener is constant!

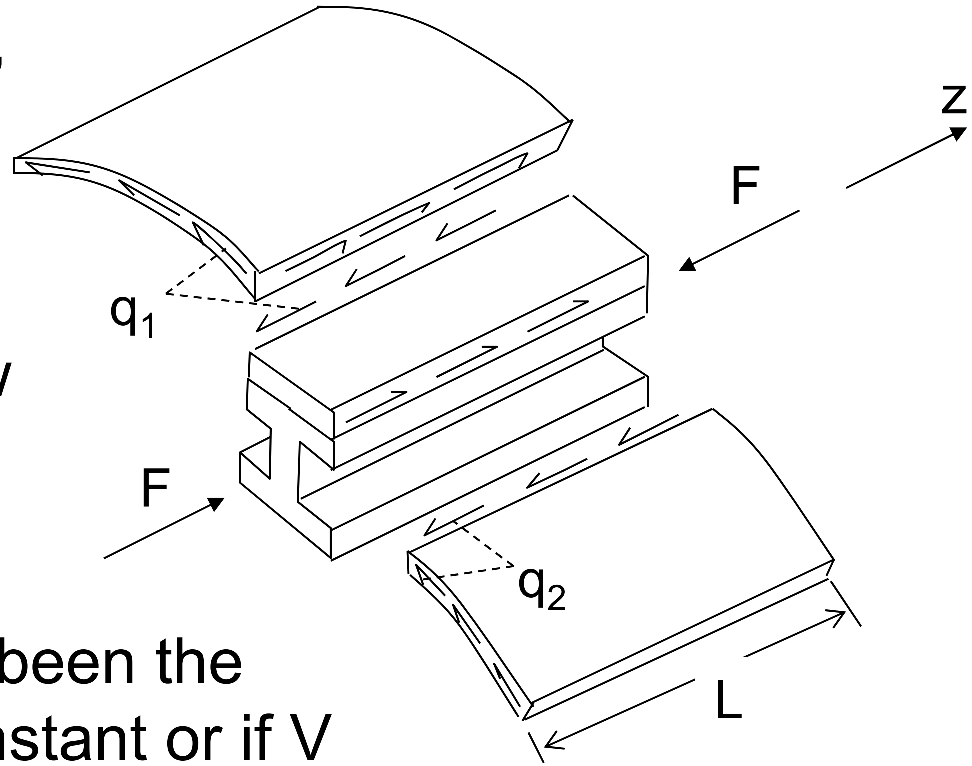
Design for torsion (cont'd)

- Now apply M and T at the same time, isolate the region around a single stiffener and place it in equilibrium:

for single stiffener,

$$\sum F_z = 0 \Rightarrow$$
$$-q_1 L - F + q_2 L + F = 0 \Rightarrow$$
$$q_1 = q_2$$

and the shear flow is still constant!



- This would not have been the case if M were not constant or if V were applied (the latter in a future lecture)

if M were not constant, F would not be constant!

Design for torsion…

- so our basic relationship, eq. (3.44) is still valid

$$T = 2Aq \qquad (3.44)$$

- we can now relate the shear flow to the shear stress, through eq. (3.43)

$$q = t\tau \qquad (3.43)$$

- and combining (3.43) and (3.44) we can solve for t:

$$t = \frac{T}{2A\tau}$$

- at failure, $\tau = \tau_y$ so

$$t = \frac{T}{2A\tau_y}$$

Design for torsion…

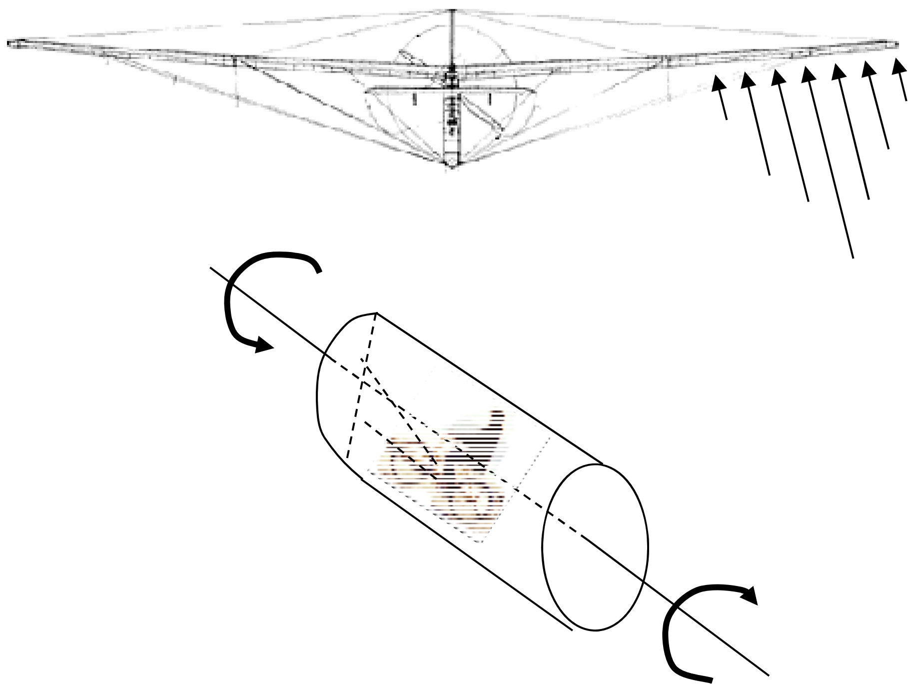
$$t = \frac{T}{2A\tau_y} = \frac{T}{2\pi R^2 \tau_y}$$

- Substituting values, t=0.004 mm !!
- This value is much smaller than the value currently used of 0.5 mm in our example so no change to our design

(Note: τ_y=275.1 MPa for 7075-T6 Al)

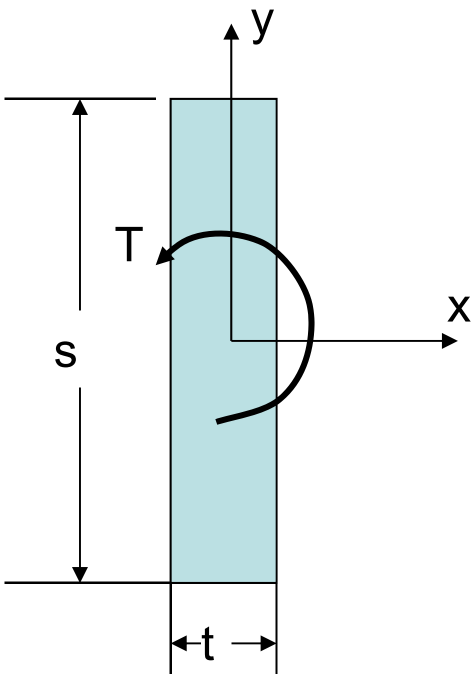
Megson (3.3), 3.4, 18.2

Torsion (cont'd)



e.g. wind gust acting on one wing tip twists the fuselage and puts it under torsion

Torsion of narrow rectangular cross-section



- assume s>>t (bar infinite in y dir)
- this means that if we fix x and move along y nothing changes
- if nothing changes with y, ∂/∂y=0
- then, the governing eq (3.18) can be combined with (3.30) to give

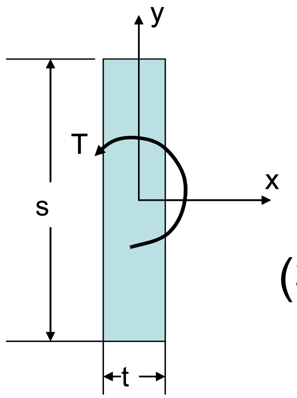
(3.18) $$\frac{\partial^2 \varphi}{\partial x^2} + \cancel{\frac{\partial^2 \varphi}{\partial y^2}} = \nabla^2 \varphi = F$$

(3.30) $$-2G\frac{d\theta}{dz} = F$$

$$\frac{\partial^2 \varphi}{\partial x^2} = -2G\frac{d\theta}{dz} \quad \text{or} \quad \overbrace{\frac{d^2 \varphi}{dx^2} = -2G\frac{d\theta}{dz}}^{\text{(since there is no dependence on y)}} \qquad (4.1)$$

- integrating twice with respect to x: $\varphi = -2G\frac{d\theta}{dz}\frac{x^2}{2} + Bx + C = -G\frac{d\theta}{dz}x^2 + Bx + C$

Torsion of narrow rectangular cross-section



$$\varphi = -G\frac{d\theta}{dz}x^2 + Bx + C$$

- to determine B and C, use the BC (3.22)

(3.22) φ = 0 on the boundary: x=± t/2
(how about y =± s/2 ??)

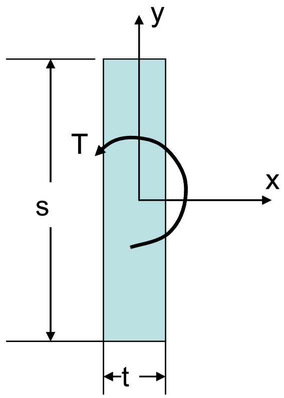
- applying the BC's:

$$\left.\begin{aligned} -G\frac{d\theta}{dz}\left(-\frac{t}{2}\right)^2 + B\left(-\frac{t}{2}\right) + C = 0 \\ -G\frac{d\theta}{dz}\left(\frac{t}{2}\right)^2 + B\left(\frac{t}{2}\right) + C = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} B &= 0 \\ C &= G\frac{d\theta}{dz}\left(\frac{t}{2}\right)^2 \end{aligned}$$

- and, therefore,

$$\varphi = -G\frac{d\theta}{dz}\left(x^2 - \left(\frac{t}{2}\right)^2\right) \qquad (4.2)$$

Torsion of narrow rectangular cross-section: stress determination



$$\varphi = -G\frac{d\theta}{dz}\left(x^2 - \left(\frac{t}{2}\right)^2\right) \qquad (4.2)$$

• stresses can now be determined using eqs (3.4) and (3.5)

(3.4) $\tau_{xz} = \dfrac{\partial\varphi}{\partial y}$

(3.5) $\tau_{yz} = -\dfrac{\partial\varphi}{\partial x}$

$\tau_{xz} \simeq 0$ (except near the ends y=±s/2 !!) (4.3)

$$\tau_{yz} = 2Gx\frac{d\theta}{dz} \qquad (4.4)$$

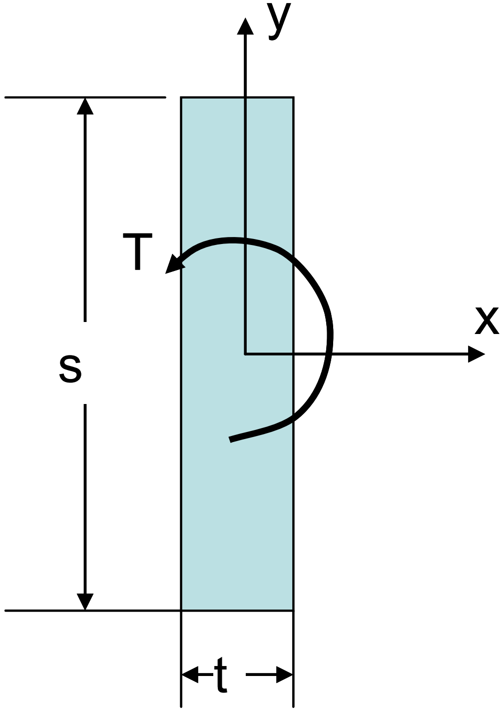
• now from (3.25), and using (4.2):

(3.25) $T = 2\iint \varphi dxdy \xrightarrow{(4.2)} T = 2\int_{-s/2}^{s/2} dy \int_{-t/2}^{t/2} -G\frac{d\theta}{dz}\left(x^2 - \left(\frac{t}{2}\right)^2\right)dx = -2sG\frac{d\theta}{dz}\left[\frac{x^3}{3} - x\left(\frac{t}{2}\right)^2\right]_{-t/2}^{t/2} = G\frac{d\theta}{dz}\frac{st^3}{3}$

• and comparing to (3.25), $T = GJ\dfrac{d\theta}{dz}$ we get $\boxed{J = \dfrac{st^3}{3}}$ (4.5)

(valid for long rectangular cross-sections)

Torsion of narrow rectangular cross-section: stress determination



$$\tau_{xz} \simeq 0 \qquad (4.3)$$

$$\tau_{yz} = 2Gx\frac{d\theta}{dz} \qquad (4.4)$$

- from (4.4), the max τ_{yz} is $\tau_{yz\max} = tG\dfrac{d\theta}{dz}$
- but from (3.25) $T = GJ\dfrac{d\theta}{dz}$
- combining: $\tau_{yz\max} = \dfrac{Tt}{J}$
- and using (4.5), $\boxed{J = \dfrac{st^3}{3}}$

$$\boxed{\tau_{yz\max} = \frac{3T}{st^2}} \qquad (4.6)$$

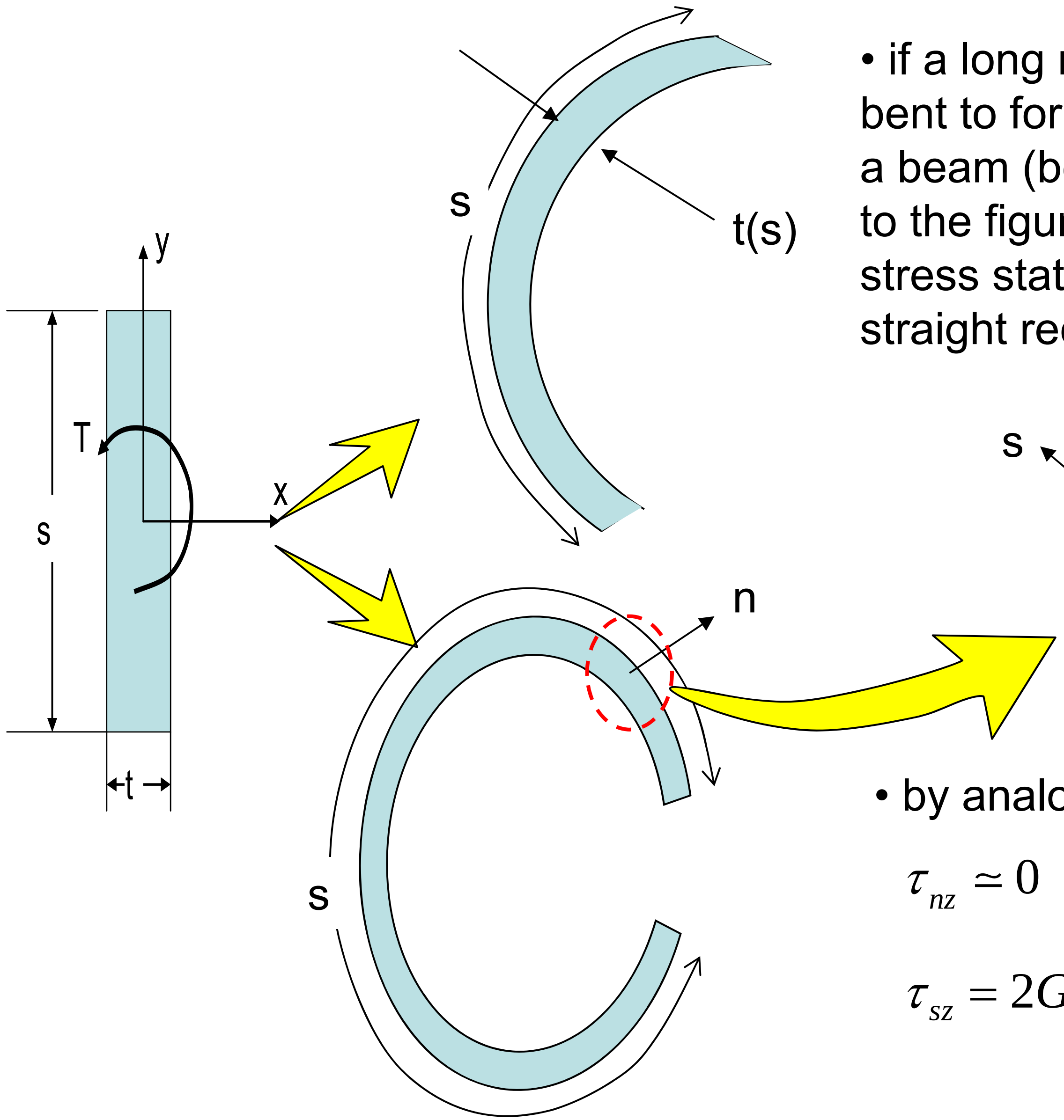
(4.7)

valid for long rectangular cross-sections

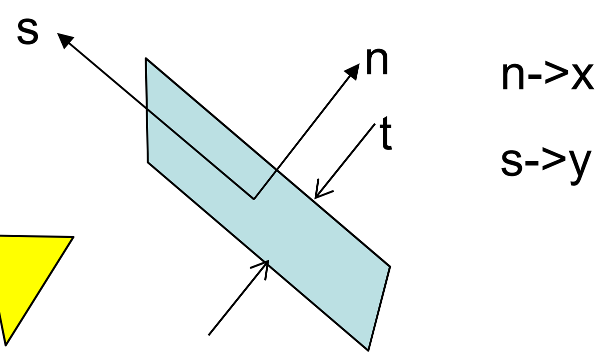
Notes: (1) s in eq (4.5) is always the longer dimension

(2) equations (4.5) and (4.6) are very accurate for s>10t; for smaller s, equation (4.5) is modified but (4.6) is still valid

Torsion of open section beams



- if a long rectangular cross-section is bent to form an open cross-section of a beam (beam axis is perpendicular to the figure) then, locally, the shear stress state is very similar to that of a straight rectangular cross-section:

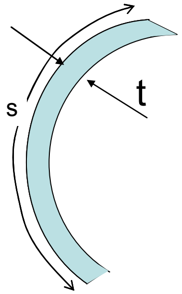


- by analogy to (4.3) and (4.4) we get:

$$\tau_{nz} \simeq 0 \qquad (4.8)$$

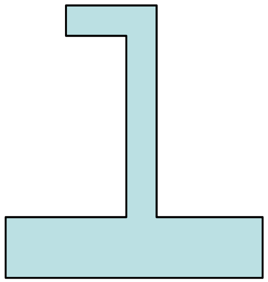
$$\tau_{sz} = 2Gn\frac{d\theta}{dz} \qquad (4.9) \qquad \tau_{sz\max} = tG\overbrace{\frac{d\theta}{dz}}^{=T/J} \qquad (4.10)$$

Torsion of open section beams



- Eq (4.7) for the torsion constant is still valid:

$$J = \frac{st^3}{3} \qquad (4.7)$$



- For open sections consisting of multiple segments and/or having variable thickness, eq. (4.7) can be generalized to:

$$J = \frac{1}{3}\sum_{i=1}^{N} s_i t_i^{\,3}$$ if the thickness t_i of each section is constant (4.11)

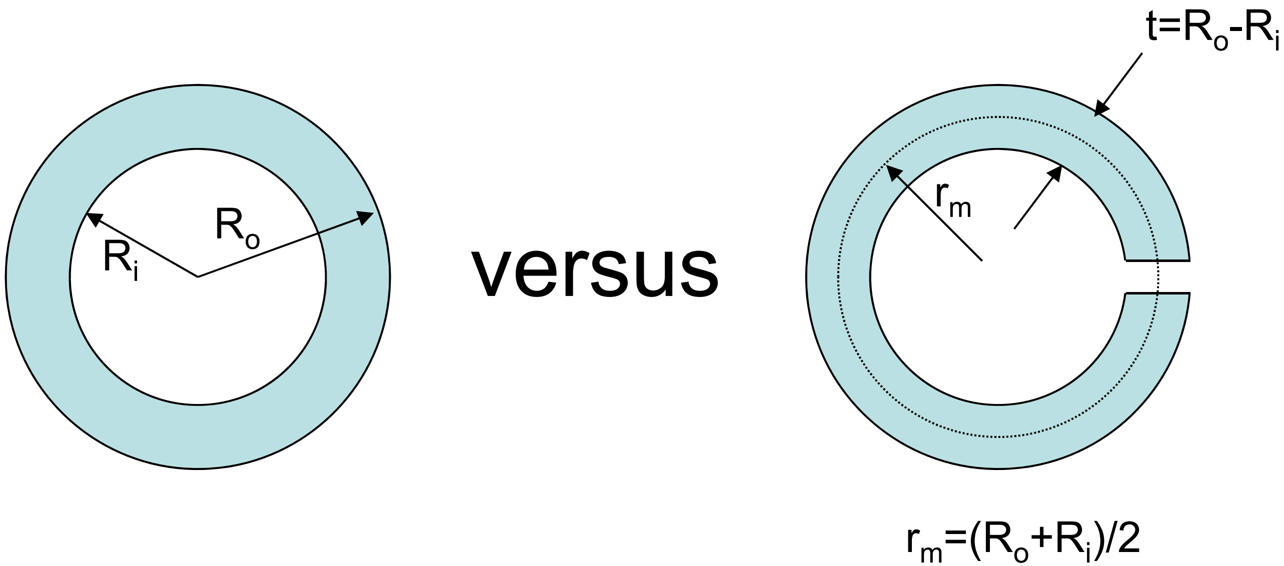
or

$$J = \frac{1}{3}\int_{section} t^3 ds$$ if the thickness t_i of each section is not constant (4.12)

note that if a segment is not "long and narrow" the 1/3 term is corrected as a function of t/s: $J=st^3(1/2+a(t/s)+b(t/s)^2+\ldots)$

Application: Open versus Closed section beams under torsion

• compare the strength and stiffness of a closed and open cylindrical cross-section beam



Torsion of closed and open circular section beams

- to compare the stiffness, compare the amount of twist for a given torque
- to compare the strength, compare the amount of torque needed to create **the same maximum shear** stress in each cross-section

Stiffness comparison

- we know from eq. (3.31)

$$T = GJ\frac{d\theta}{dz} \Rightarrow \theta = \frac{T}{GJ}z + \cancel{C}$$ (θ=0 at z=0)

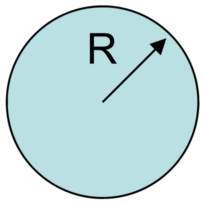
- so for a given applied torque T, since the shear modulus G is the same, it suffices to compare the J (torsion constant) values

- the configuration with higher J, will rotate less and is, therefore, stiffer

Stiffness comparison

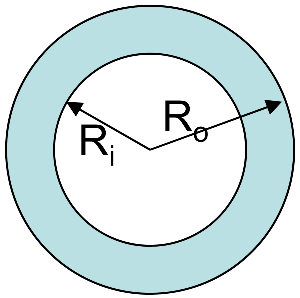
- for a closed circular cross-section, eq (3.35)

$$J = \frac{\pi R^4}{2}$$



- therefore, for a ring of outer radius R_o and inner radius R_i, $J=J_o-J_i$

$$J = \frac{\pi}{2}\left(R_o^{\ 4} - R_i^{\ 4}\right)$$



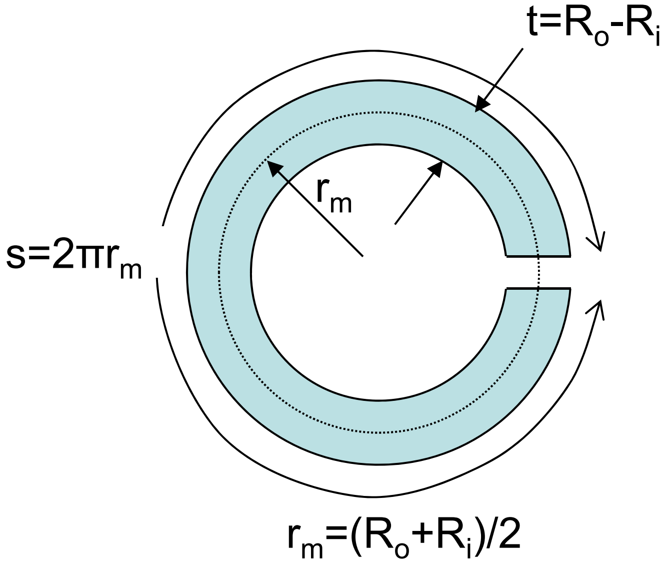
Note: this expression is valid for constant thickness, single material.
Otherwise, need to use: $\frac{d\theta}{dz} = \frac{1}{2A}\oint \frac{q}{tG} ds$ with q=const

$$T = 2Aq$$

$$\frac{d\theta}{dz} = \frac{T}{GJ}$$

Stiffness comparison

- for the open cross section, eq (4.7) $J = \dfrac{st^3}{3}$ becomes



$$J = \frac{1}{3} 2\pi r_m t^3$$

Stiffness comparison

- so, comparing the J values

$$\frac{J_{closed}}{J_{open}} = \frac{\frac{\pi}{2}\left(R_o{}^4 - R_i{}^4\right)}{\frac{1}{3}2\pi r_m t^3}$$

- but,

$$R_o = r_m + \frac{t}{2}$$

$$R_i = r_m - \frac{t}{2}$$

- and, therefore,

$$R_o{}^4 = r_m{}^4 + 4r_m{}^3\frac{t}{2} + 6r_m{}^2\left(\frac{t}{2}\right)^2 + 4r_m\left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4$$

$$R_i{}^4 = r_m{}^4 - 4r_m{}^3\frac{t}{2} + 6r_m{}^2\left(\frac{t}{2}\right)^2 - 4r_m\left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4$$

Stiffness comparison

- so that

≈ 0 for $t \ll r_m$

$$R_o^{\,4} - R_i^{\,4} = 4r_m^{\,3}t + \cancel{4r_m t^3}$$

- and finally,

$$\boxed{\frac{J_{closed}}{J_{open}} = 3\frac{r_m^{\,2}}{t^2}}$$

typical values: r_m=100 mm, t=3 mm =>J_{closed}/J_{open}=3333 !!!

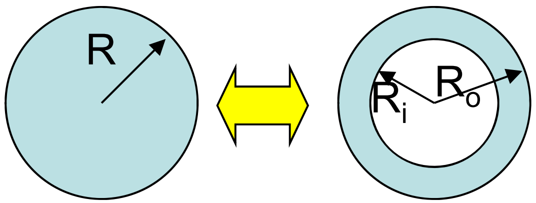
- the closed cross section is thousands of times stiffer and thus rotates thousands of times less under a given torque T (over the same length of beam)

Strength comparison

- the maximum shear stress for a closed circular cross-section is given by eq (3.40)

$$\tau = \frac{TR_o}{J}$$

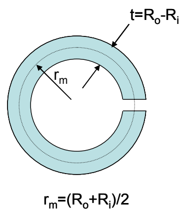
which is the same as for a hollow cylinder



(Note there is no difference in τ_{max} between solid and hollow circles, i.e., the center of a solid cylinder is NOT buying us anything in carrying shear stresses due to torsion)

- the maximum shear stress for the open cross-section is obtained by combining (4.10) and (3.31)

$$\tau = \frac{Tt}{J}$$



Note: there is a typo in Megson p. 80 right after eq 3.29, a t is missing from the numerator!

Strength comparison

- for either case (closed or open beam), failure would occur when the shear stress τ equals the shear strength τ_{ult} of the material

$$\tau_{ult} = \frac{T_{failc} R_o}{J_{closed}} \Rightarrow T_{failc} = \frac{\tau_{ult} J_{closed}}{(r_m + t/2)}$$

$$\tau_{ult} = \frac{T_{failo} t}{J_{open}} \Rightarrow T_{failo} = \frac{\tau_{ult} J_{open}}{t}$$

- therefore, the ratio of failure torques is

$$\frac{T_{failc}}{T_{failo}} = \frac{J_{closed}}{J_{open}} \frac{t}{r_m + \frac{t}{2}}$$

Strength comparison

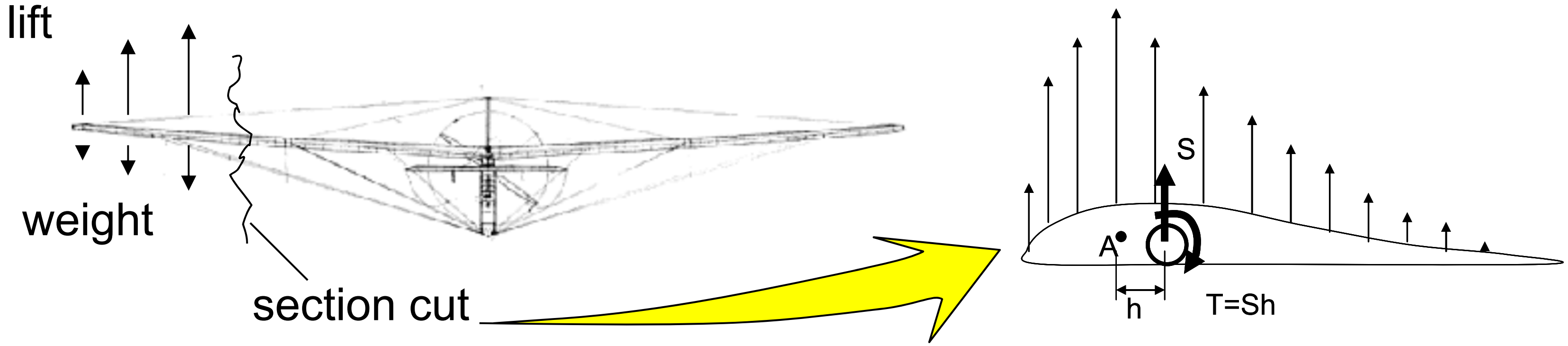
• but J_{closed}/J_{open} was calculated earlier; substituting,

$$\boxed{\frac{T_{failc}}{T_{failo}} = 3\frac{r_m^{\ 2}}{t^2}\frac{t}{r_m + \frac{t}{2}}}$$

for the same typical values as before (r_m=100 mm, t=3 mm), T_{failc}/T_{failo}=98.5 !!!

• the closed cross-section can take 100 times higher torque before it fails!

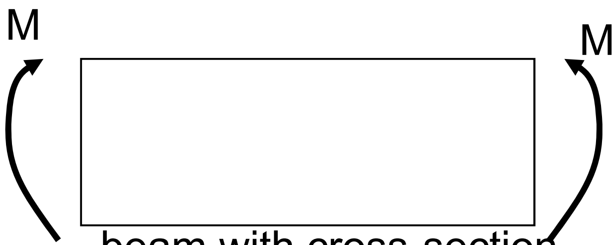
Bending revisited



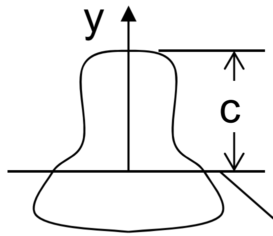
- structures under bending loads, such as a wing or fuselage, must be designed so that
  - the deflections are minimized
  - the resulting stresses do not cause failure
- the most efficient structural element to minimize deflections and stresses is a beam (with high moment of inertia I)

Closed versus open cross-sections in bending

• symmetrical bending about one axis (from



beam with cross-section with at least one axis of symmetry



y axis is axis of symmetry

neutral axis

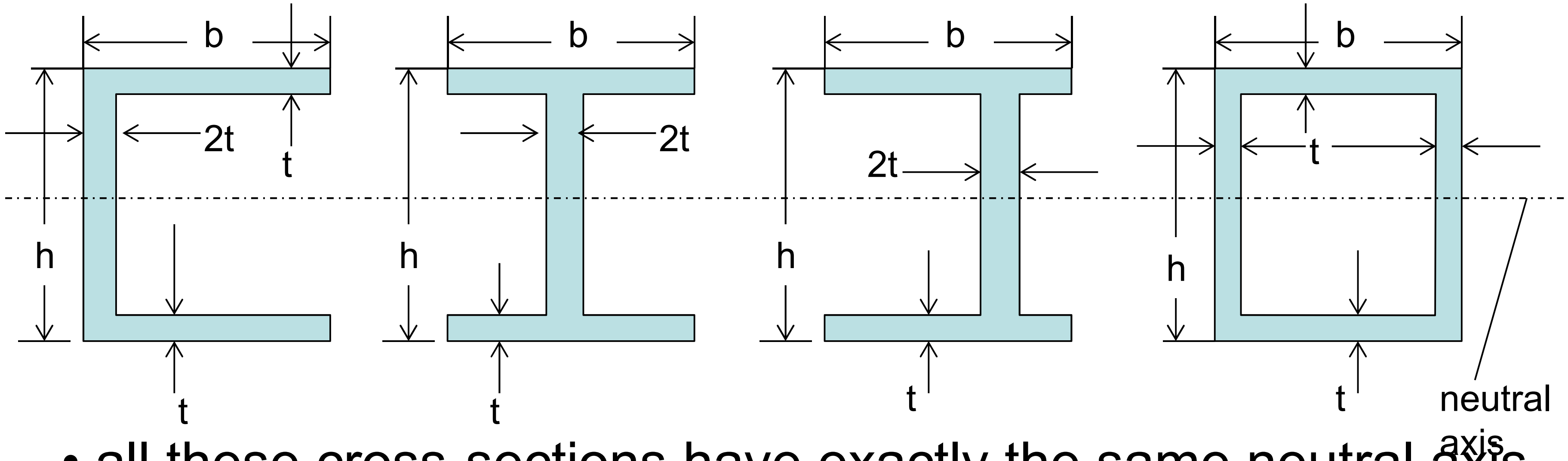
$$\boxed{\sigma_z = \frac{My}{I} \quad \sigma_{z\max} = \frac{Mc}{I}} \qquad (4.13)$$

• the normal stress σ_z depends on applied moment and cross-sectional properties (I,c)

• these properties do not change if the beam is closed or open; therefore, **there is no difference in bending stresses between closed and open cross sections**

• this is also valid for unsymmetric cross-sections

Carrying this one step further…

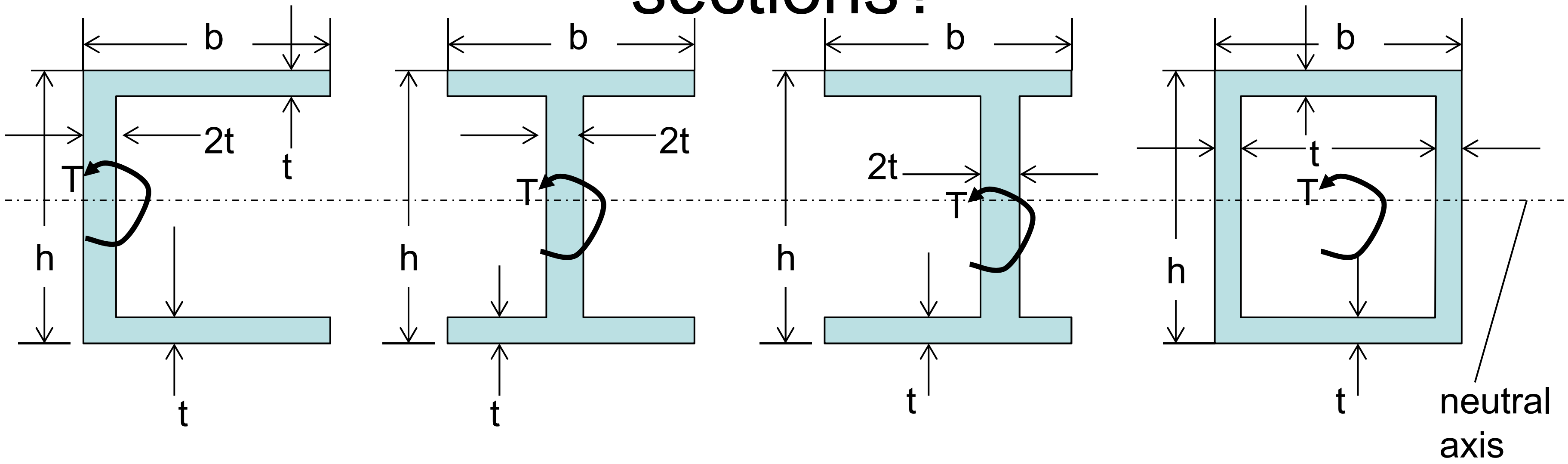


• all these cross-sections have exactly the same neutral axis location, at the mid-height of each cross-section (due to symmetry)

• all these cross-sections have exactly the same moment of inertia

• therefore, all these cross-sections have exactly the same maximum stress and, from a strength perspective **are equivalent**

What about pure torsion of the same cross-sections?

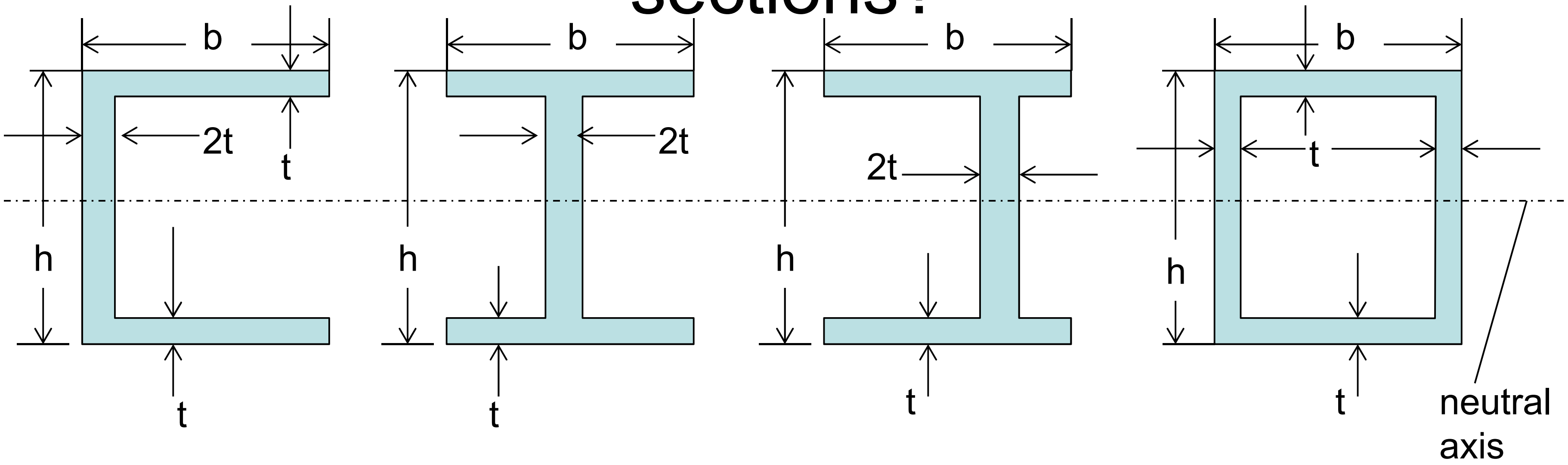


• from eq (4.11) the first three cross sections have exactly the same torsion constant J (as long as b,h>>t):

$$J = \frac{1}{3}\sum_{i=1}^{N} s_i t_i^{3}$$

• but the fourth one, being closed, has J that is thousands of times bigger than the other three!

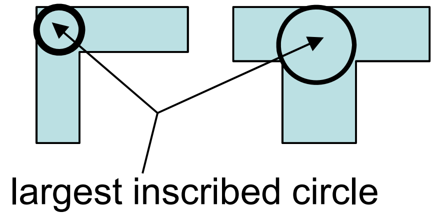
What about pure torsion of the same cross-sections?



• the maximum shear stress is a bit more complicated; it is given by a variation of eq. (4.10)

$$\tau_{\max} = \frac{Td}{J}$$

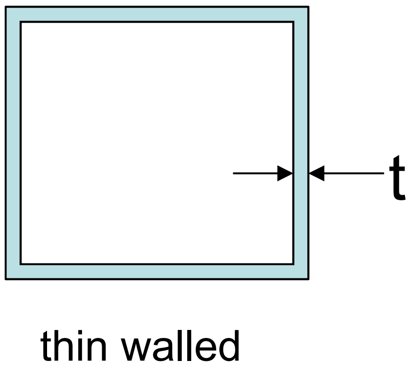
where d=2t and its location is at one of the points where the largest inscribed circle touches the boundary

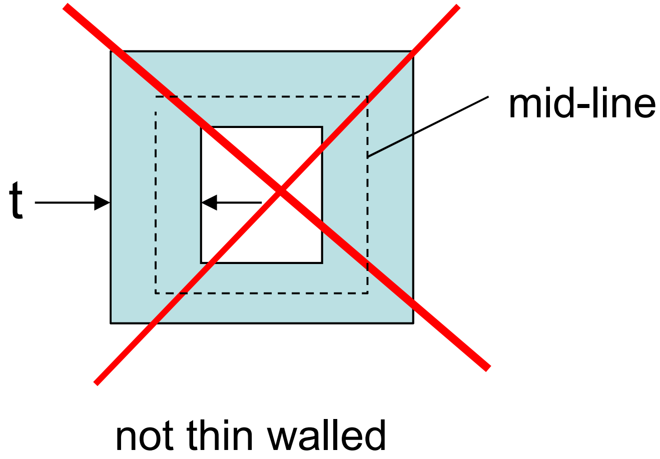


Megson 16.4.5,16.5

Approximations for thin-walled structure

- thin-walled structure => t << other dimensions of cross-section

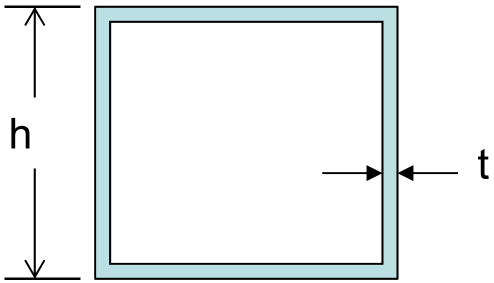




- implications of thin walled assumption:
  - stresses are constant through the thickness
  - cross-section can be represented by its mid-line
  - t^2, t^3 or higher order terms can be neglected in the calculations

Approximations for thin-walled structure

- example: calculate moment of inertia for following cross-section

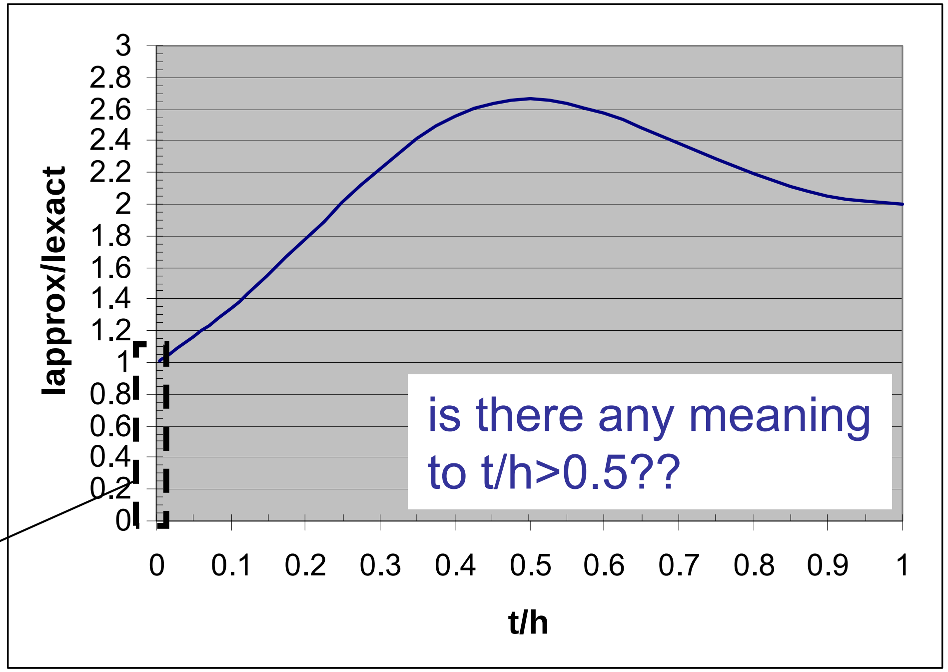


$$I = \frac{hh^3}{12} - \frac{(h-2t)(h-2t)^3}{12} =$$

$$\frac{h^4}{12} - \frac{1}{12}\left[h^4 - 4h^3(2t) + \overbrace{6h^2(2t)^2 - 4h^2(2t)^3 + (2t)^4}^{\text{neglected terms}}\right]$$

$$\simeq \frac{2}{3}h^3 t$$

typical aerospace structure has t/h<0.02 and the error is less than 6% (usually less than 1.5%); at the other extreme, for t/h=0.5 the error is 167%



typical thin-walled structure in this box

Approximations for thin-walled structure

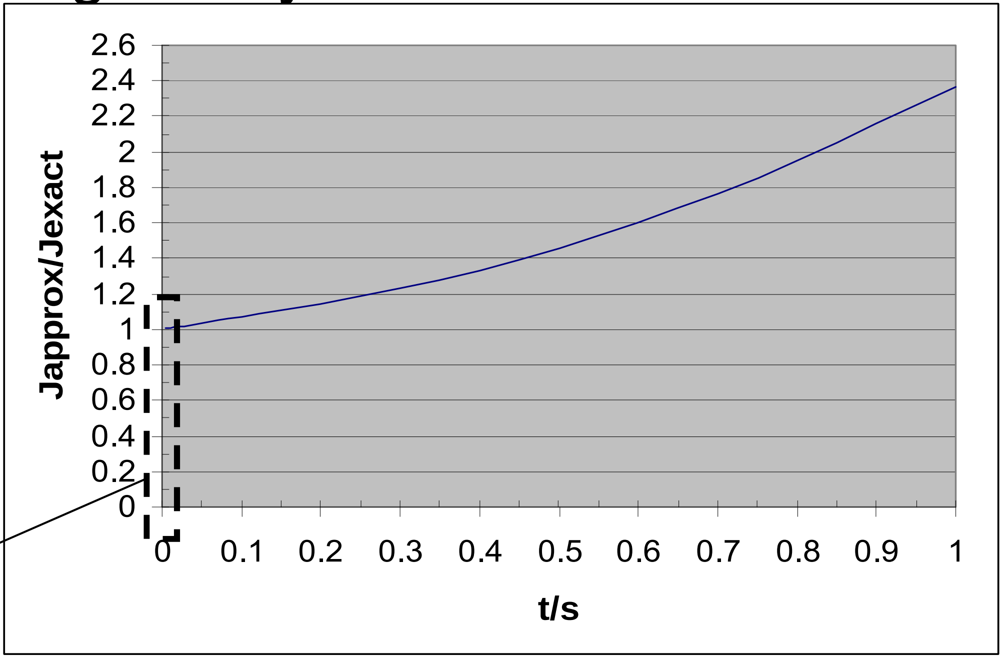
• for torsion, the same assumption of thin walled structure led to the expression for J in the form of eq. (4.7)

$$J = \frac{st^3}{3}$$

• for (open) sections where t<<s is not valid, a more accurate expression for J is given by

$$J = \frac{st^3}{3}\left[1 - 0.63\frac{t}{s}\left(1 - \frac{t^4}{12s^4}\right)\right]$$

typical aerospace structure has t/s<0.02 and the error is less than 1%; at the other extreme, for t/s=1 the error is 137%



typical thin-walled structure in this box

Application Session 1