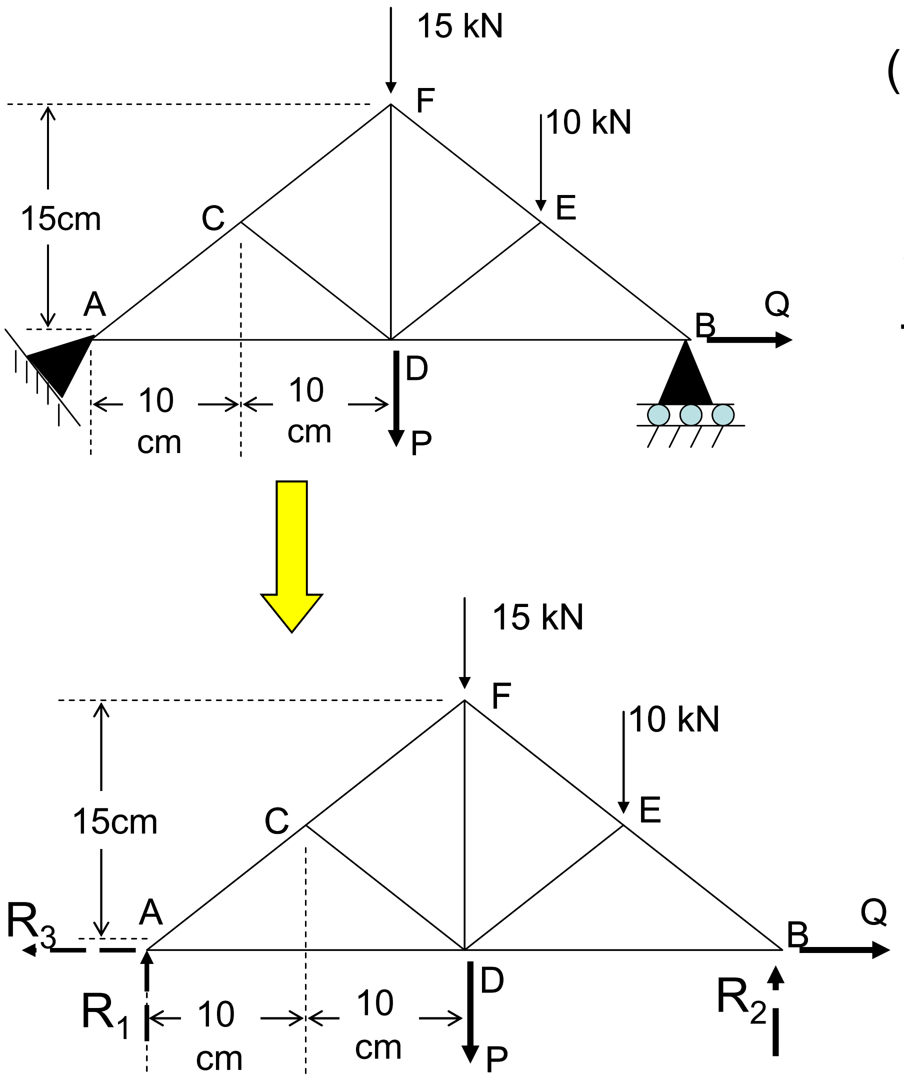
Castigliano's Second Theorem – Application 2



(1) Determination of deflection at pt D

- first, determine the reaction forces R_1, R_2, R_3
  - horizontal equilibrium:

$$R_3 = Q$$

  - vertical equilibrium:

$$R_1 + R_2 = P + 25kN$$

  - moments about point A:

$$P(20) + 15(20) + 10(30) = R_2(40) \Rightarrow$$

$$R_2 = \frac{P}{2} + 15$$

  - then, $R_1 = \frac{P}{2} + 10$

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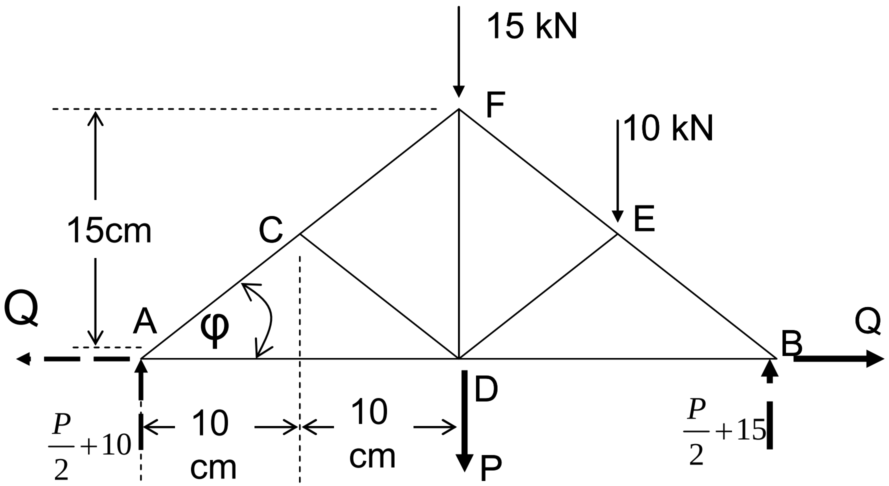

- second, determine the forces in each truss member as a function of P and Q
- note that

$$\tan\varphi = \frac{7.5}{10} \Rightarrow \varphi = 36.87^o$$

$$\left.\begin{array}{l} F_{AC}\sin\phi + \dfrac{P}{2} + 10 = 0 \Rightarrow F_{AC} = -\dfrac{P}{2\sin\phi} - \dfrac{10}{\sin\phi} \\ F_{AC}\cos\phi + F_{AD} - Q = 0 \end{array}\right\} \Rightarrow F_{AD} = Q + \left(\frac{P}{2} + 10\right)\cot\phi$$

$$\left.\begin{array}{l} F_{AC}\sin\phi + F_{CD}\sin\phi = F_{CF}\sin\phi \\ F_{AC}\cos\phi - F_{CD}\cos\phi = F_{CF}\cos\phi \end{array}\right\} \begin{array}{l} \Rightarrow F_{CD} = 0 \\ F_{CF} = F_{AC} = -\dfrac{P}{2\sin\phi} - \dfrac{10}{\sin\phi} \end{array}$$

$$\left.\begin{array}{l} F_{BE}\sin\phi + \dfrac{P}{2} + 15 = 0 \Rightarrow F_{BE} = -\dfrac{P}{2\sin\phi} - \dfrac{15}{\sin\phi} \\ F_{BE}\cos\phi + F_{BD} - Q = 0 \end{array}\right\} \Rightarrow F_{BD} = Q + \left(\frac{P}{2} + 15\right)\cot\phi$$

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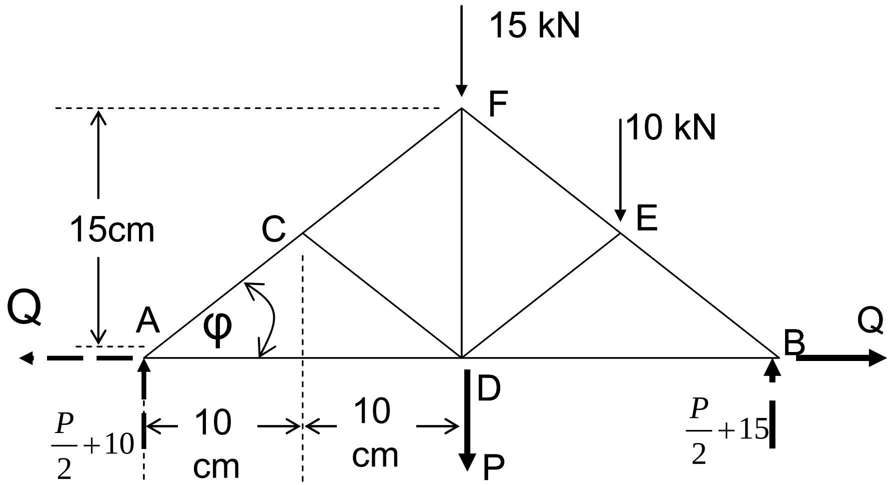

$$F_{AD} = Q + \left(\frac{P}{2} + 10\right)\cot\phi$$

$$F_{CD} = 0$$

$$F_{CF} = F_{AC} = -\frac{P}{2\sin\phi} - \frac{10}{\sin\phi}$$

$$F_{BD} = Q + \left(\frac{P}{2} + 15\right)\cot\phi$$

$$F_{BE} = -\frac{P}{2\sin\phi} - \frac{15}{\sin\phi}$$

$$\left.\begin{aligned} &F_{BE}\sin\phi + F_{ED}\sin\phi + 10 = F_{FE}\sin\phi \\ &F_{BE}\cos\phi - F_{ED}\cos\phi = F_{FE}\cos\phi \end{aligned}\right\} \begin{aligned} &2F_{ED}\sin\phi\cos\phi + 10\cos\phi = 0 \Rightarrow F_{ED} = -\frac{10}{2\sin\phi} \\ &F_{FE} = F_{BE} - F_{ED} = -\frac{P}{2\sin\phi} - \frac{10}{\sin\phi} \end{aligned}$$

$$F_{CF}\sin\phi + F_{FD} + F_{FE}\sin\phi + 15 = 0 \Rightarrow F_{FD} = -15 + \frac{P}{2} + 10 + \frac{P}{2} + 10 \Rightarrow F_{FD} = P + 5$$

as a check, horizontal equilibrium at point F:

$$F_{CF}\cos\phi \overset{?}{=} F_{FE}\cos\phi \Rightarrow -\frac{P}{2\sin\phi} - \frac{10}{\sin\phi} = -\frac{P}{2\sin\phi} - \frac{10}{\sin\phi}$$

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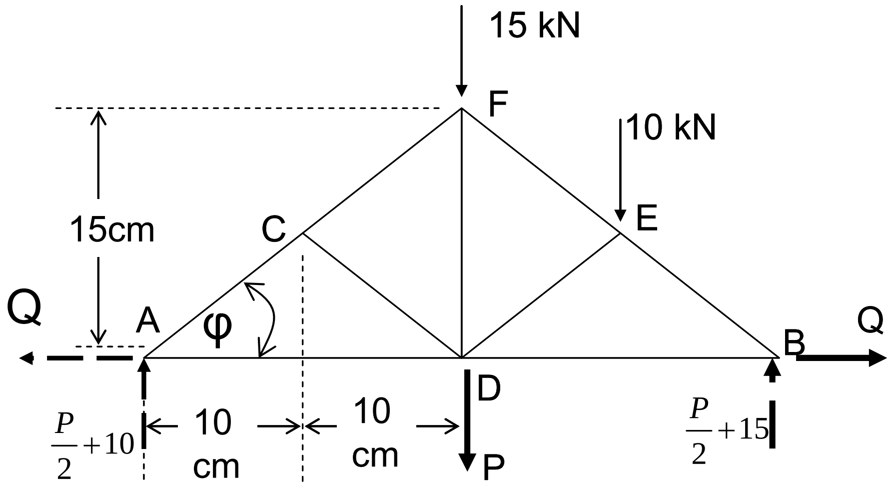

$$F_{AD} = Q + \left(\frac{P}{2} + 10\right)\cot\phi \qquad F_{BE} = -\frac{P}{2\sin\phi} - \frac{15}{\sin\phi}$$

$$F_{CD} = 0$$

$$F_{CF} = F_{AC} = -\frac{P}{2\sin\phi} - \frac{10}{\sin\phi} \qquad F_{ED} = -\frac{10}{2\sin\phi}$$

$$F_{BD} = Q + \left(\frac{P}{2} + 15\right)\cot\phi \qquad F_{FE} = -\frac{P}{2\sin\phi} - \frac{10}{\sin\phi}$$

$$F_{BE} = -\frac{P}{2\sin\phi} - \frac{15}{\sin\phi} \qquad F_{FD} = P + 5$$

- third, determine the total energy in the system from eq. (12.23) summed over all truss members

$$C_i = \sum_j \frac{F_j^{\,2} L_j}{2E_j A_j} \qquad (12.23)$$

$$C_i = \frac{1}{2EA}\left[\begin{aligned}&\left(-\frac{P}{2\sin\phi} - \frac{10}{\sin\phi}\right)^2 L_{AC} + \left(Q + \left(\frac{P}{2}+10\right)\cot\phi\right)^2 L_{AD} + \left(\frac{10}{2\sin\phi}\right)^2 L_{ED} + \left(-\frac{P}{2\sin\phi} - \frac{15}{\sin\phi}\right)^2 L_{BE} + \\ &\left(Q + \left(\frac{P}{2}+15\right)\cot\phi\right)^2 L_{BD} + \left(-\frac{P}{2\sin\phi} - \frac{10}{\sin\phi}\right)^2 L_{FC} + (P+5)^2 L_{FD} + \left(-\frac{P}{2\sin\phi} - \frac{10}{\sin\phi}\right)^2 L_{FE}\end{aligned}\right]$$

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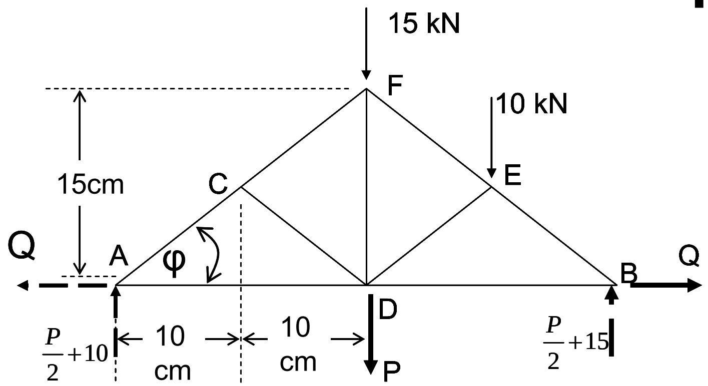

$$C_i = \frac{1}{2EA}\left[\begin{array}{l} \left(-\frac{P}{2\sin\phi}-\frac{10}{\sin\phi}\right)^2 L_{AC} + \left(Q+\left(\frac{P}{2}+10\right)\cot\phi\right)^2 L_{AD} + \left(\frac{10}{2\sin\phi}\right)^2 L_{ED} + \left(-\frac{P}{2\sin\phi}-\frac{15}{\sin\phi}\right)^2 L_{BE} + \\ \left(Q+\left(\frac{P}{2}+15\right)\cot\phi\right)^2 L_{BD} + \left(-\frac{P}{2\sin\phi}-\frac{10}{\sin\phi}\right)^2 L_{FC} + (P+5)^2 L_{FD} + \left(-\frac{P}{2\sin\phi}-\frac{10}{\sin\phi}\right)^2 L_{FE} \end{array}\right]$$

- fourth, obtain the required deflection using eq. (12.16)

$$\Delta_i = \frac{\partial C_i}{\partial P_i} \qquad (12.16)$$

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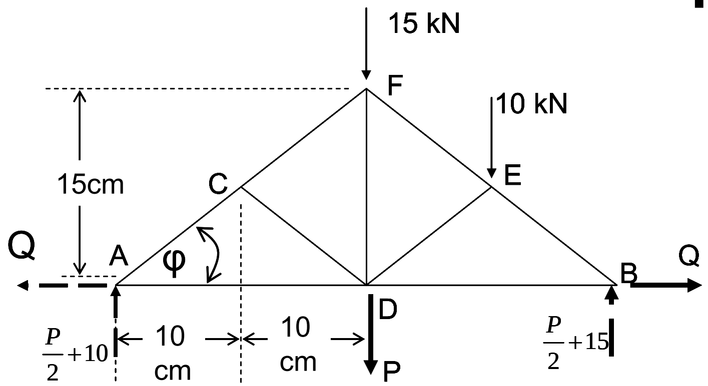


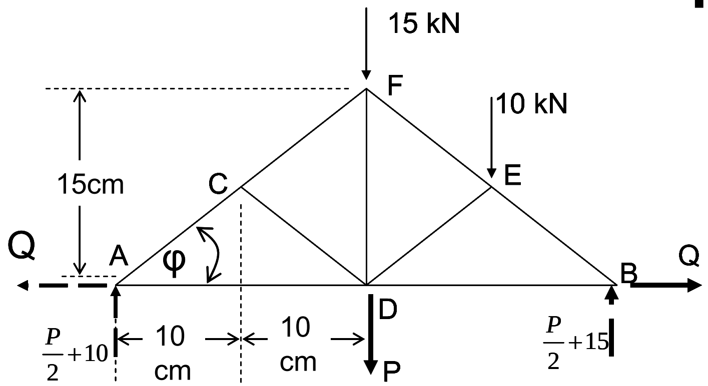
$$\Delta_i = \frac{\partial C_i}{\partial P_i} \qquad (12.16)$$

$$C_i = \frac{1}{2EA}\left[\begin{array}{l}\left(-\frac{P}{2\sin\phi}-\frac{10}{\sin\phi}\right)^2 L_{AC} + \left(Q+\left(\frac{P}{2}+10\right)\cot\phi\right)^2 L_{AD} + \left(\frac{10}{2\sin\phi}\right)^2 L_{ED} + \left(-\frac{P}{2\sin\phi}-\frac{15}{\sin\phi}\right)^2 L_{BE} + \\ \left(Q+\left(\frac{P}{2}+15\right)\cot\phi\right)^2 L_{BD} + \left(-\frac{P}{2\sin\phi}-\frac{10}{\sin\phi}\right)^2 L_{FC} + (P+5)^2 L_{FD} + \left(-\frac{P}{2\sin\phi}-\frac{10}{\sin\phi}\right)^2 L_{FE}\end{array}\right]$$

- the vertical displacement at point D is then:

$$\Delta_D = \frac{1}{2EA}\left[\begin{array}{l}2\left(\frac{P}{2\sin\phi}+\frac{10}{\sin\phi}\right)\frac{L_{AC}}{2\sin\phi} + 2\left(Q+\left(\frac{P}{2}+10\right)\cot\phi\right)\frac{L_{AD}\cot\phi}{2} + 2\left(\frac{P}{2\sin\phi}+\frac{15}{\sin\phi}\right)\frac{L_{BE}}{2\sin\phi} + \\ 2\left(Q+\left(\frac{P}{2}+15\right)\cot\phi\right)\frac{L_{BD}\cot\phi}{2} + 2\left(\frac{P}{2\sin\phi}+\frac{10}{\sin\phi}\right)\frac{L_{FC}}{2\sin\phi} + 2(P+5)L_{FD} + 2\left(\frac{P}{2\sin\phi}+\frac{10}{\sin\phi}\right)\frac{L_{FE}}{2\sin\phi}\end{array}\right]$$

Castigliano's Second Theorem – Application 2

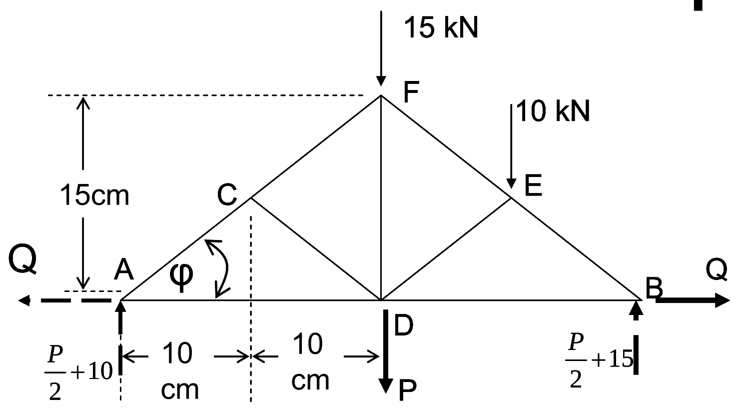


$$\Delta_D = \frac{1}{2EA}\left[\begin{array}{l} 2\left(\frac{P}{2\sin\phi}+\frac{10}{\sin\phi}\right)\frac{L_{AC}}{2\sin\phi}+2\left(Q+\left(\frac{P}{2}+10\right)\cot\phi\right)\frac{L_{AD}\cot\phi}{2}+2\left(\frac{P}{2\sin\phi}+\frac{15}{\sin\phi}\right)\frac{L_{BE}}{2\sin\phi}+ \\ 2\left(Q+\left(\frac{P}{2}+15\right)\cot\phi\right)\frac{L_{BD}\cot\phi}{2}+2\left(\frac{P}{2\sin\phi}+\frac{10}{\sin\phi}\right)\frac{L_{FC}}{2\sin\phi}+2(P+5)L_{FD}+2\left(\frac{P}{2\sin\phi}+\frac{10}{\sin\phi}\right)\frac{L_{FE}}{2\sin\phi} \end{array}\right]$$

- which simplifies to:

$$\Delta_D = \frac{1}{2EA}\left[\begin{array}{l} \left(\frac{P}{2\sin\phi}+\frac{10}{\sin\phi}\right)\frac{\left(L_{AC}+L_{FC}+L_{FE}\right)}{\sin\phi}+\left(Q+\frac{P}{2}\cot\phi\right)\left(L_{AD}+L_{BD}\right)\cot\phi+25\cot^2\phi\left(L_{AD}+L_{BD}\right)+ \\ \left(\frac{P}{2\sin\phi}+\frac{15}{\sin\phi}\right)\frac{L_{BE}}{\sin\phi}+2(P+5)L_{FD} \end{array}\right]$$

Castigliano's Second Theorem – Application 2



$$\Delta_D = \frac{1}{2EA}\left[\begin{array}{l}\left(\frac{P}{2\sin\phi}+\frac{10}{\sin\phi}\right)\frac{\left(L_{AC}+L_{FC}+L_{FE}\right)}{\sin\phi}+\left(Q+\frac{P}{2}\cot\phi\right)\left(L_{AD}+L_{BD}\right)\cot\phi+\\ 25\cot^2\phi\left(L_{AD}+L_{BD}\right)+\left(\frac{P}{2\sin\phi}+\frac{15}{\sin\phi}\right)\frac{L_{BE}}{\sin\phi}+2(P+5)L_{FD}\end{array}\right]$$

- fifth, set to zero any fictitious forces (in this case both P and Q); then, the required deflection at point D is

$$\Delta_D(P=0,Q=0)=\frac{1kN}{2EA}\left[\frac{10}{\sin^2\phi}\left(L_{AC}+L_{FC}+L_{FE}\right)+25\cot^2\phi\left(L_{AD}+L_{BD}\right)+15\frac{L_{BE}}{\sin^2\phi}+10L_{FD}\right]$$

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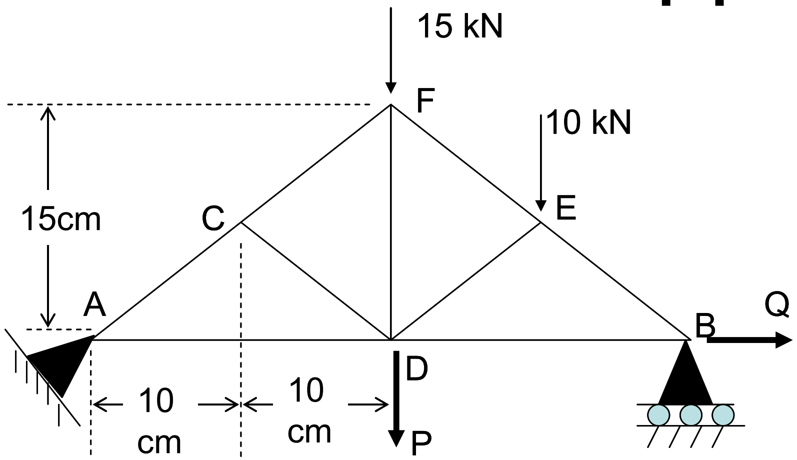


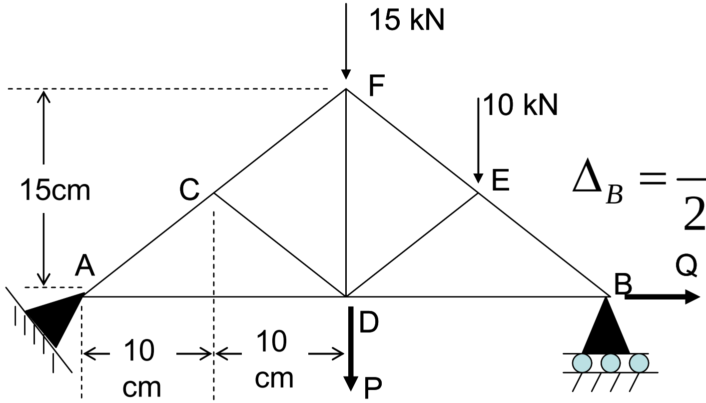
(2) Determination of force Q such that the horizontal deflection at point B is zero

$$\Delta_i = \frac{\partial C_i}{\partial Q}$$

• the first three steps, up to obtaining the total energy expression are the same as before; to determine the horizontal deflection at point B, differentiate the total energy with respect to Q to obtain

$$\Delta_B = \frac{1}{2EA}\left[2\left(Q+\left(\frac{P}{2}+10\right)\cot\phi\right)L_{AD} + 2\left(Q+\left(\frac{P}{2}+15\right)\cot\phi\right)L_{BD}\right]$$

Castigliano's Second Theorem – Application 2



$$\Delta_B = \frac{1}{2EA}\left[2\left(Q+\left(\frac{P}{2}+10\right)\cot\phi\right)L_{AD}+2\left(Q+\left(\frac{P}{2}+15\right)\cot\phi\right)L_{BD}\right]$$

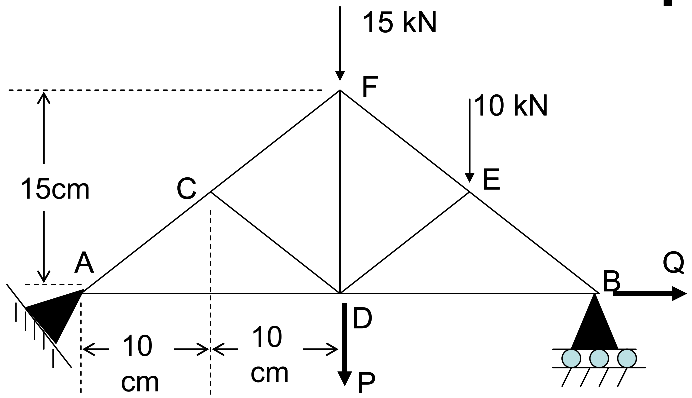
- recalling now that P is a fictitious force (but Q is no longer fictitious because we want to determine its value so that the displacement at B is zero) we can simplify:

$$\Delta_B = \frac{10kN}{2EA}\left[2\left(Q+10\cot\phi\right)L_{AD}+2\left(Q+15\cot\phi\right)L_{BD}\right]$$

- for the deflection at B to be zero, set Δ_B=0 and solve for Q:

$$Q = -5\cot\phi\frac{\left(2L_{AD}+3L_{BD}\right)}{L_{AD}+L_{BD}}$$

Castigliano's Second Theorem – Application 2



(3) With the Q obtained acting, determine the new vertical deflection at point D

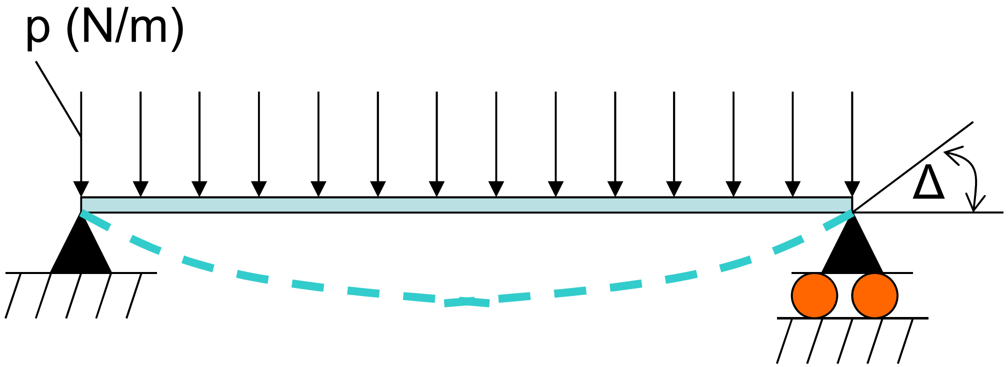
$$\Delta_D = \frac{1}{2EA}\left[\begin{array}{l}\left(\frac{P}{2\sin\phi}+\frac{10}{\sin\phi}\right)\frac{\left(L_{AC}+L_{FC}+L_{FE}\right)}{\sin\phi}+\left(Q+\frac{P}{2}\cot\phi\right)\left(L_{AD}+L_{BD}\right)\cot\phi+\\ 25\cot^2\phi\left(L_{AD}+L_{BD}\right)+\left(\frac{P}{2\sin\phi}+\frac{15}{\sin\phi}\right)\frac{L_{BE}}{\sin\phi}+2(P+5)L_{FD}\end{array}\right]$$

- the derivation is the same as in part (1) up to the expression for Δ_D with P and Q included
- now only P is fictitious, Q was determined in part (2); set P=0

$$\Delta_D = \frac{1}{2EA}\left[\frac{10kN}{\sin^2\phi}\left(L_{AC}+L_{FC}+L_{FE}\right)+Q\left(L_{AD}+L_{BD}\right)\cot\phi+25kN\cot^2\phi\left(L_{AD}+L_{BD}\right)+\frac{15kN}{\sin^2\phi}L_{BE}+10kNL_{FD}\right]$$

with Q as determined in part (2)

Castigliano's Second Theorem – Application 3

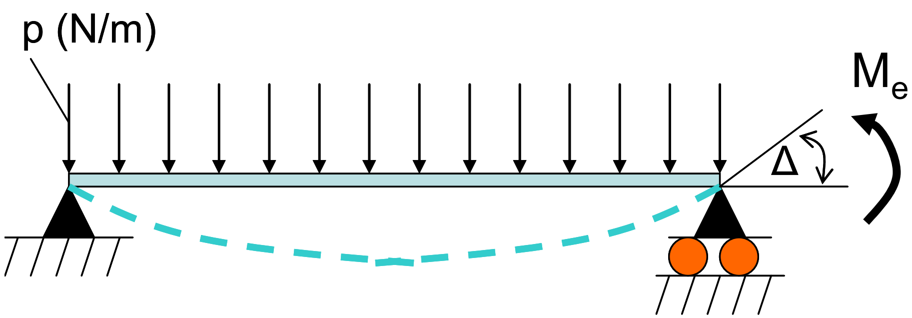


• for a simply supported beam under distributed load p, determine the angle of rotation Δ at the right end of the beam

• to do this we need to obtain the energy stored in the beam and differentiate it with respect to the load that causes the rotation Δ

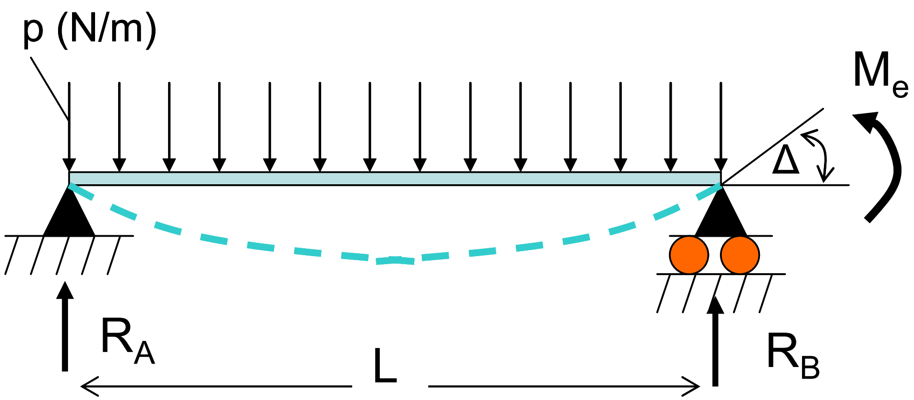
• but there is no such load applied

Castigliano's Second Theorem – Application 3



- since a rotation is caused by a moment, apply a fictitious end moment M_e and compute the energy in the beam due to both p and M_e acting

- first obtain the reaction forces



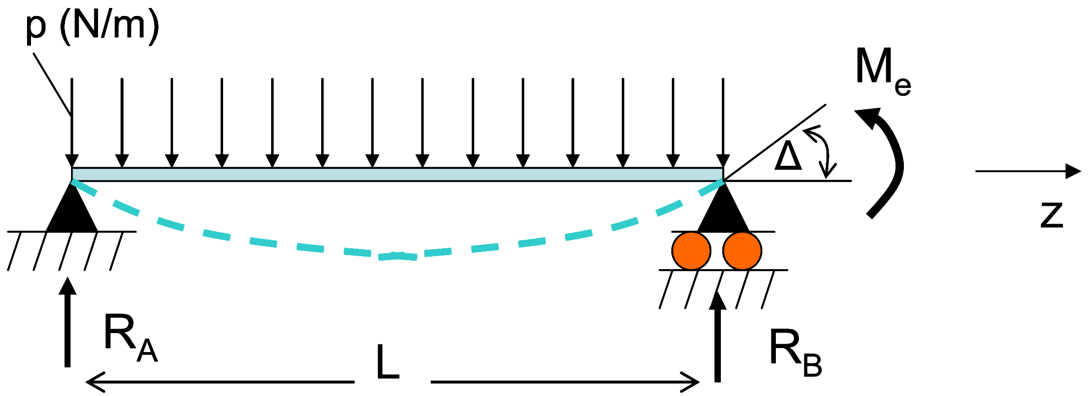
$$R_A + R_B = pL$$

$$R_A L - \int_0^L pz dz = M_e \Rightarrow R_A L - p\frac{L^2}{2} = M_e$$

$$\Rightarrow \quad R_A = \frac{pL}{2} + \frac{M_e}{L}$$

$$R_B = \frac{pL}{2} - \frac{M_e}{L}$$

Castigliano's Second Theorem – Application 3



$$R_A = \frac{pL}{2} + \frac{M_e}{L}$$

$$R_B = \frac{pL}{2} - \frac{M_e}{L}$$

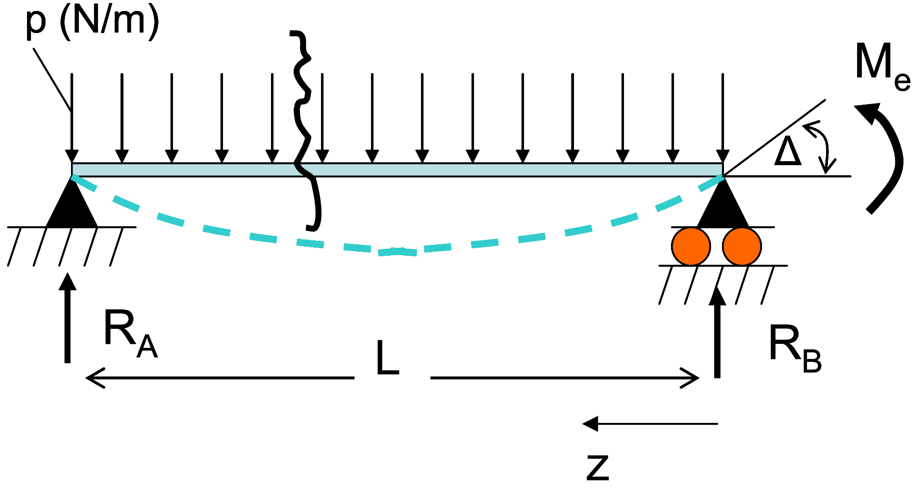
• to use Castigliano's theorem we need eq. (12.24)

$$C_i = \int_0^L \frac{M^2}{2EI} dz \qquad (12.24)$$

which expresses the energy in the beam in terms of the bending moment M along the beam

• therefore, the bending moment M must be determined as a function of z

Castigliano's Second Theorem – Application 3

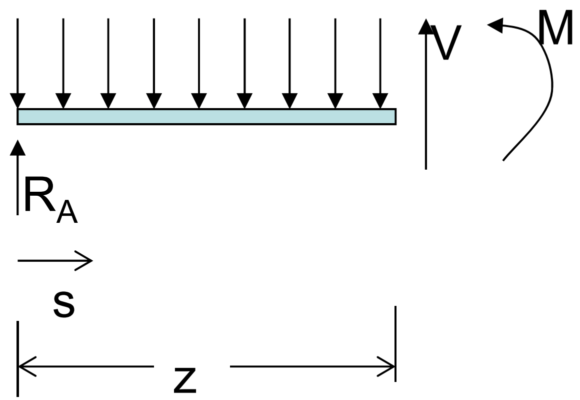


$$R_A = \frac{pL}{2} + \frac{M_e}{L}$$

$$R_B = \frac{pL}{2} - \frac{M_e}{L}$$

$$C_i = \int_0^L \frac{M^2}{2EI} dz \qquad (12.24)$$

- cutting the beam at some location z and applying moment equilibrium:

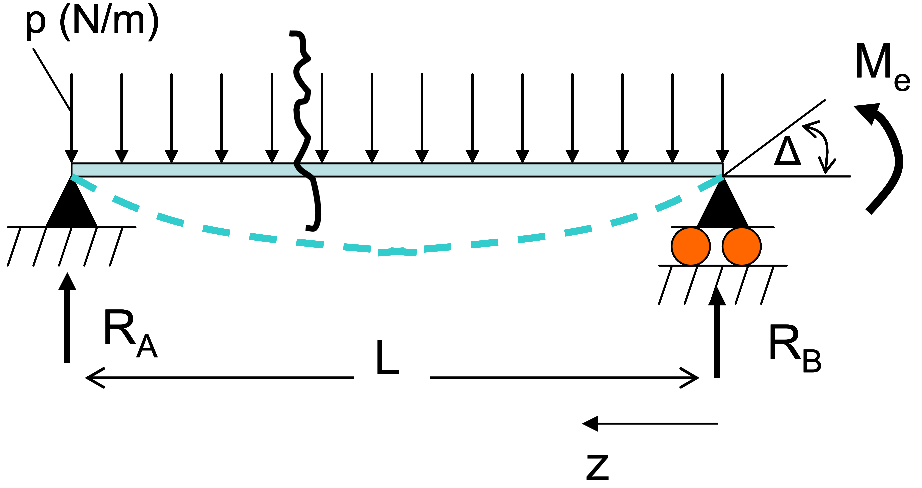


Taking moments about the cut:

$$M + \int_0^z p(z-s)ds - R_A z = 0 \Rightarrow M + \left(p\left(zs - \frac{s^2}{2} \right)_0^z \right) - \left(p\frac{L}{2} + \frac{M_e}{L} \right) z = 0 \Rightarrow$$

$$M = -p\frac{z^2}{2} + \left(p\frac{L}{2} + \frac{M_e}{L} \right) z$$

Castigliano’s Second Theorem – Application 3



$$C_i = \int_0^L \frac{M^2}{2EI}\,dz$$

$$M = -p\frac{z^2}{2} + \left(p\frac{L}{2} + \frac{M_e}{L}\right)z$$

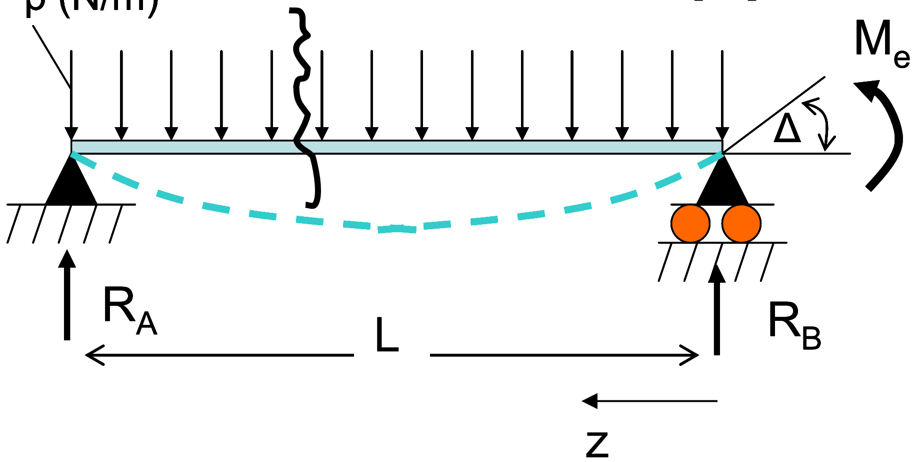
- then,

$$M^2 = p^2\frac{z^4}{4} + \left(\frac{p^2L^2}{4} + \frac{M_e^{\,2}}{L^2} + pM_e\right)z^2 - pz^3\left(p\frac{L}{2} + \frac{M_e}{L}\right)$$

- and substituting in the energy expression

$$C_i = \frac{1}{2EI}\left[p^2\frac{L^5}{20} + \left(\frac{p^2L^2}{4} + \frac{M_e^{\,2}}{L^2} + pM_e\right)\frac{L^3}{3} - p\frac{L^4}{4}\left(p\frac{L}{2} + \frac{M_e}{L}\right)\right]$$

Castigliano's Second Theorem – Application 3



$$C_i = \frac{1}{2EI}\left[\begin{array}{l} p^2 \frac{L^5}{20} + \left(\frac{p^2 L^2}{4} + \frac{M_e^{\ 2}}{L^2} + pM_e\right)\frac{L^3}{3} - \\ p\frac{L^4}{4}\left(p\frac{L}{2} + \frac{M_e}{L}\right) \end{array}\right]$$

- to determine the rotation Δ at the end, apply Castigliano's theorem:

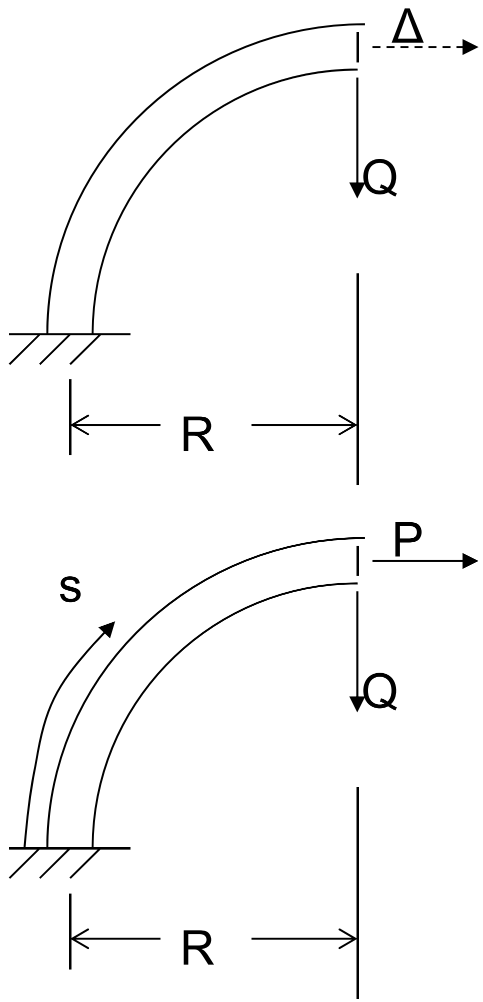
$$\Delta = \frac{\partial C_i}{\partial M_e}$$

- to obtain, $\Delta = \frac{1}{2EI}\left[\frac{2}{3}M_e L + p\frac{L^3}{12}\right]$

- finally, since M_e was a fictitious moment, setting M_e=0 gives the final answer:

$$\Delta = \frac{pL^3}{24EI}$$

Castigliano’s Second Theorem – Application 4



Determine the horizontal displacement Δ for a curved beam of radius R and bending stiffness EI under a vertical force Q

- we introduce a fictitious force P in the direction of the required displacement:

- the energy expression (12.24),

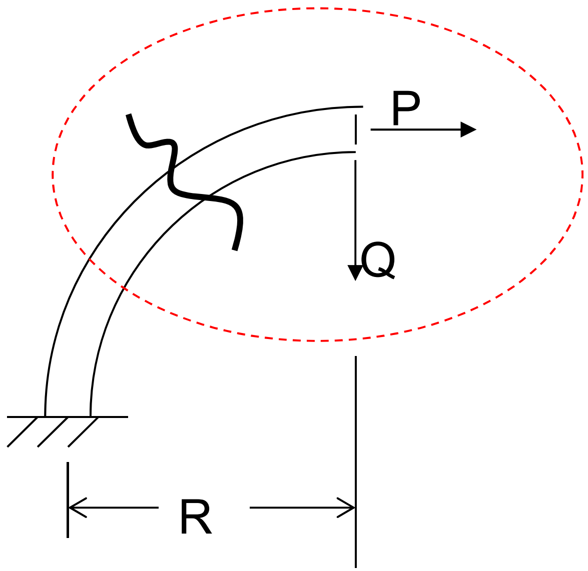
$$C_i = \int_0^L \frac{M^2}{2EI} dz$$

- becomes for a curved beam:

$$C_i = \int_0^{\pi R/2} \frac{M^2}{2EI} ds$$

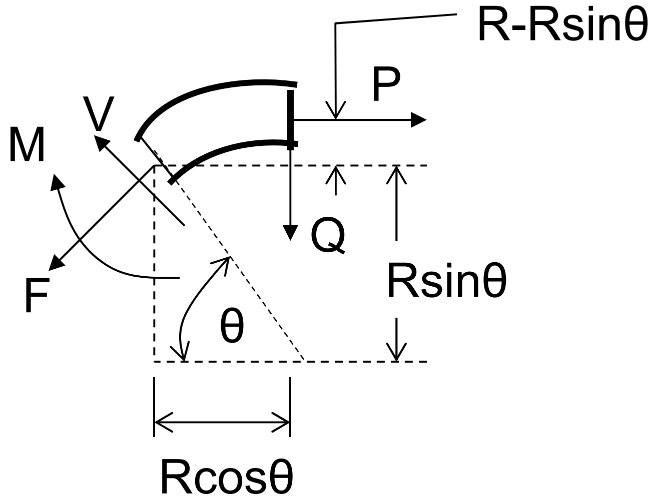
- and since ds=Rdθ, $C_i = \int_0^{\pi/2} \frac{M^2}{2EI} R d\theta$

Castigliano's Second Theorem – Application 4



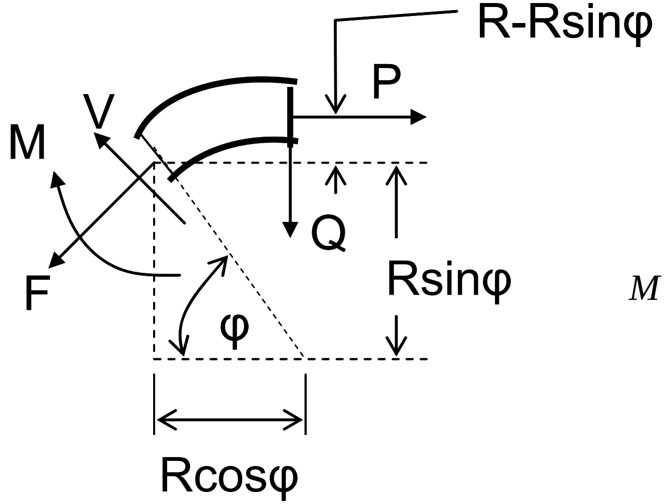
$$C_i = \int_0^{\pi/2} \frac{M^2}{2EI} R d\theta$$

- to determine the internal moment M, make a cut and place it in moment equilibrium:



$$M + PR(1-\sin\theta) + QR\cos\theta = 0 \Rightarrow$$
$$M = -PR(1-\sin\theta) - QR\cos\theta$$

Castigliano's Second Theorem – Application 4



$$C_i = \int_0^{\pi/2} \frac{M^2}{2EI} R d\theta$$

$$M = -PR(1-\sin\theta) - QR\cos\theta$$

- now

$$M^2 = P^2R^2(1-\sin\theta)^2 + Q^2R^2\cos^2\theta + 2PQR^2(1-\sin\theta)\cos\theta =$$

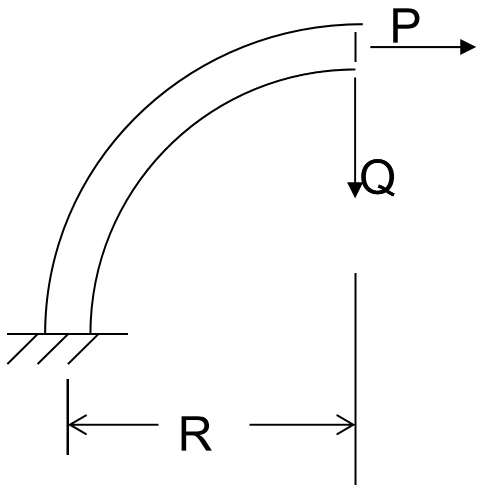
$$P^2R^2\left(1+\frac{1-\cos 2\theta}{2} - 2\sin\theta\right) + Q^2R^2\frac{1+\cos 2\theta}{2} + 2PQR^2\left(\cos\theta - \frac{1}{2}\sin 2\theta\right)$$

- substituting in the energy expression and evaluating the integrals:

$$C_i = \frac{R}{2EI}\left[\begin{array}{l} P^2R^2\left[\frac{\pi}{2} + \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right)_0^{\pi/2} + 2(\cos\theta)_0^{\pi/2}\right] + \frac{Q^2R^2}{2}\left[\frac{\pi}{4} + \frac{1}{2}(\sin 2\theta)_0^{\pi/2}\right] + \\ 2PQR^2\left[(\sin\theta)_0^{\pi/2} + \frac{1}{4}(\cos 2\theta)_0^{\pi/2}\right] \end{array}\right]$$

- which simplifies to: $C_i = \frac{R^3}{2EI}\left[P^2\left(\frac{3\pi}{4} - 2\right) + \frac{Q^2\pi}{4} + PQ\right]$

Castigliano's Second Theorem – Application 4



$$C_i = \frac{R^3}{2EI}\left[P^2\left(\frac{3\pi}{4} - 2\right) + \frac{Q^2\pi}{4} + PQ\right]$$

- to determine the horizontal displacement at the top right:

$$\Delta = \frac{\partial C_i}{\partial P}$$

- which gives: $\Delta = \dfrac{R^3}{2EI}\left[2P\left(\dfrac{3\pi}{4} - 2\right) + Q\right]$

- and since P is a fictitious force, setting P=0 gives the final answer:

$$\Delta = \frac{QR^3}{2EI}$$

not in Megson but read 5.1, 5.2 again

Castigliano's first theorem

• it is completely analogous to Castigliano's second theorem

• for a body under loads Q_1, Q_2, … (where Q_i can be either forces or moments), if the strain energy stored U can be expressed in terms of displacements q_1, q_2, … (where q_i can be either deflections or rotations) then:

$$\boxed{Q_i = \frac{\partial U}{\partial \Delta_i}} \qquad (12.29)$$

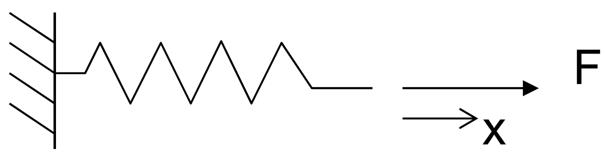
that is, to find the force applied to a body in a specific direction, need to differentiate the internal strain energy U with respect to the corresponding deflection

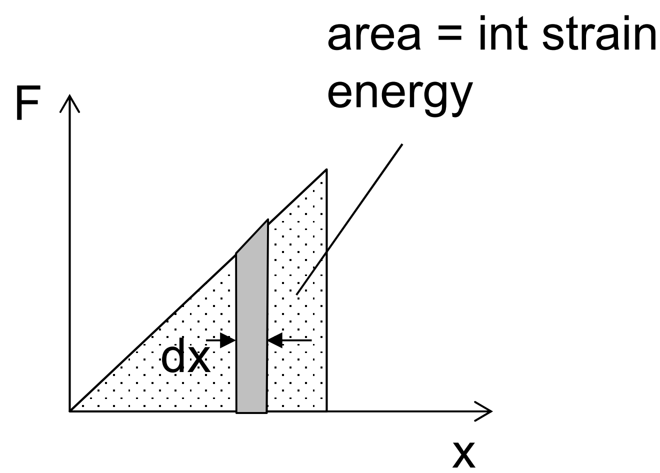
compare with Castigliano's 2nd theorem, eq (12.16): $\Delta_i = \frac{\partial C_i}{\partial Q_i}$ (12.16)

Castigliano's first theorem – linear spring

• what makes the use of this theorem more difficult is the need to express the internal strain energy in terms of deflections (unlike the 2nd theorem where expressing the complementary energy in terms of forces and moments was straightforward)

Linear Spring





linear spring with spring constant k;
force F causes displacement x=> F = kx

$$U = \int_0^x F dx$$ but F=kx substituting:

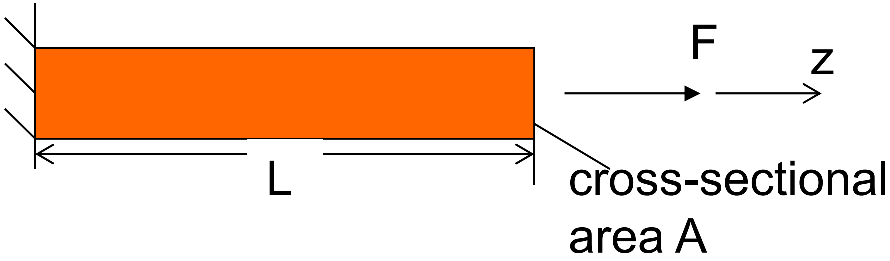
$$\boxed{U = \int_0^x kx dx = \frac{1}{2} kx^2}$$

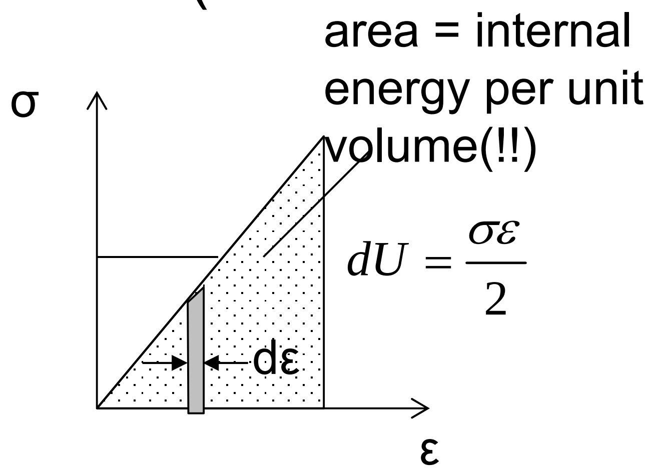
well known results from basic physics and dynamics

(12.30)

Castigliano's first theorem – truss member

Bar (or beam) in tension or compression (no buckling)





$$dU = \frac{\sigma\varepsilon}{2}$$

$$U = \iiint_{vol} dU = \iiint_{vol} \frac{\sigma\varepsilon}{2}\, dxdydz = \int_0^L \frac{A\sigma\varepsilon}{2}\, dz$$

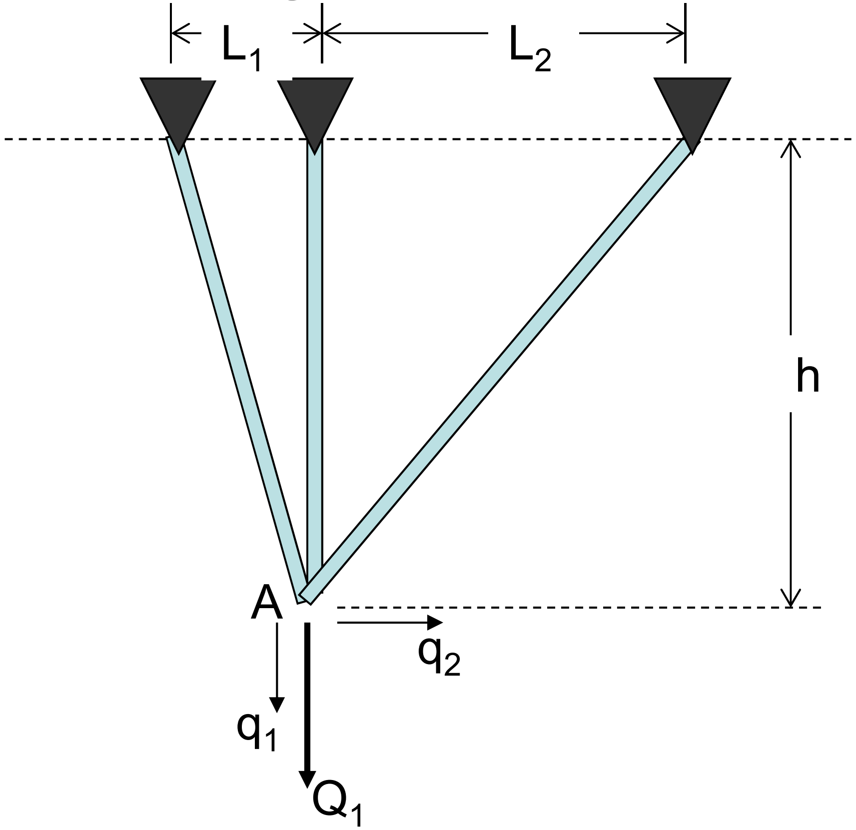
but $\sigma = E\varepsilon$

$$\varepsilon = \frac{\Delta}{L}$$

$$U = \int_0^L \frac{EA}{2}\frac{\Delta^2}{L^2} dz$$

- and since Δ, E, A, L are independent of z: $\boxed{U = \frac{EA\Delta^2}{2L}}$ (12.31)

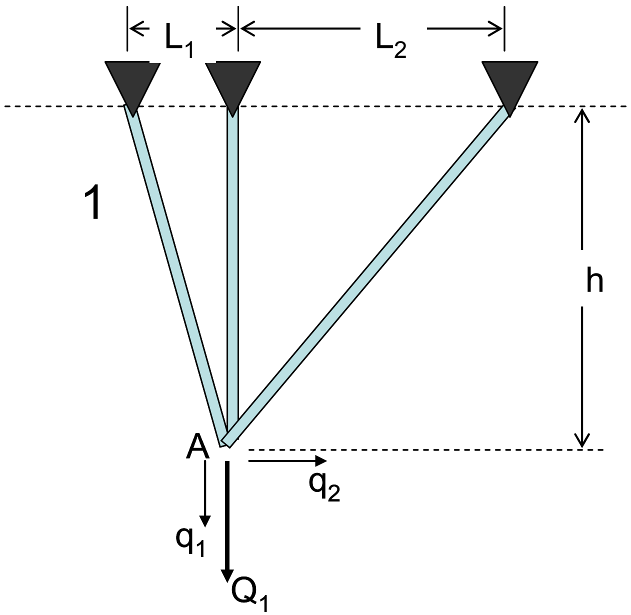
Castigliano's first theorem – Application



all 3 members have stiffness E and cross-sectional area A

- determine the displacements q_1 and q_2 of point A when a known load Q_1 is applied

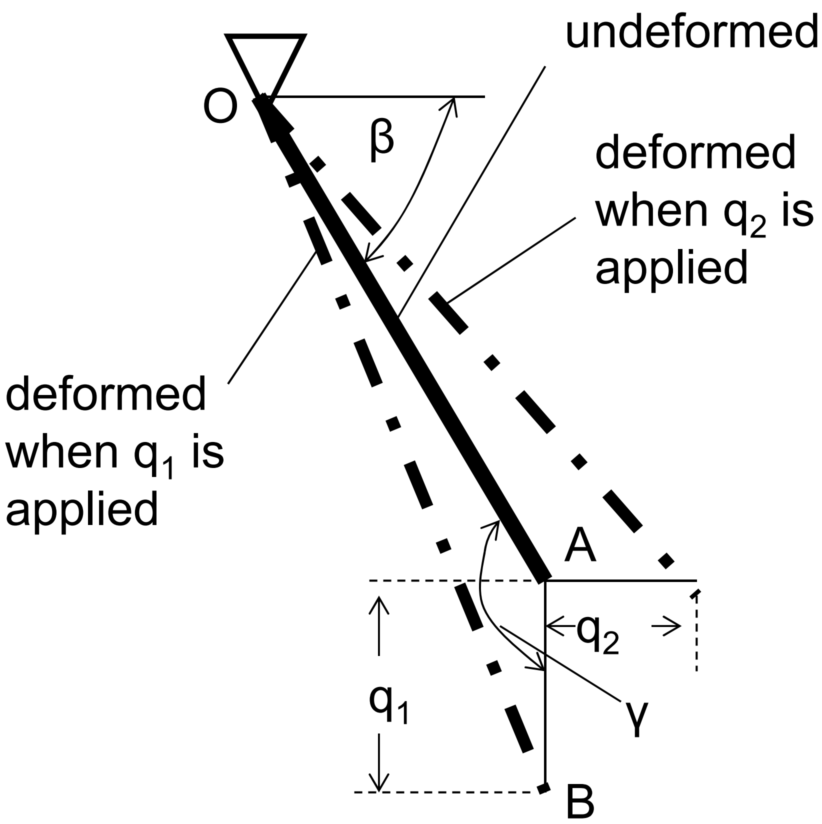
Castigliano’s first theorem – Application



all 3 members have stiffness E and cross-sectional area A

- first determine the deflections of the end of each truss member **along the member axis**, if the end moves by q_1 and q_2

Castigliano's first theorem – Application



Consider a beam of length L, stiff-ness E and cross-sectional area A pinned at one end with applied displacements q_1 and q_2 at the other

- analyze the effect of each deflection separately and superpose the results
- first, consider q_1 only
- from the law of cosines:

$$OB^2 = OA^2 + AB^2 - 2(OA)(AB)\cos\gamma$$

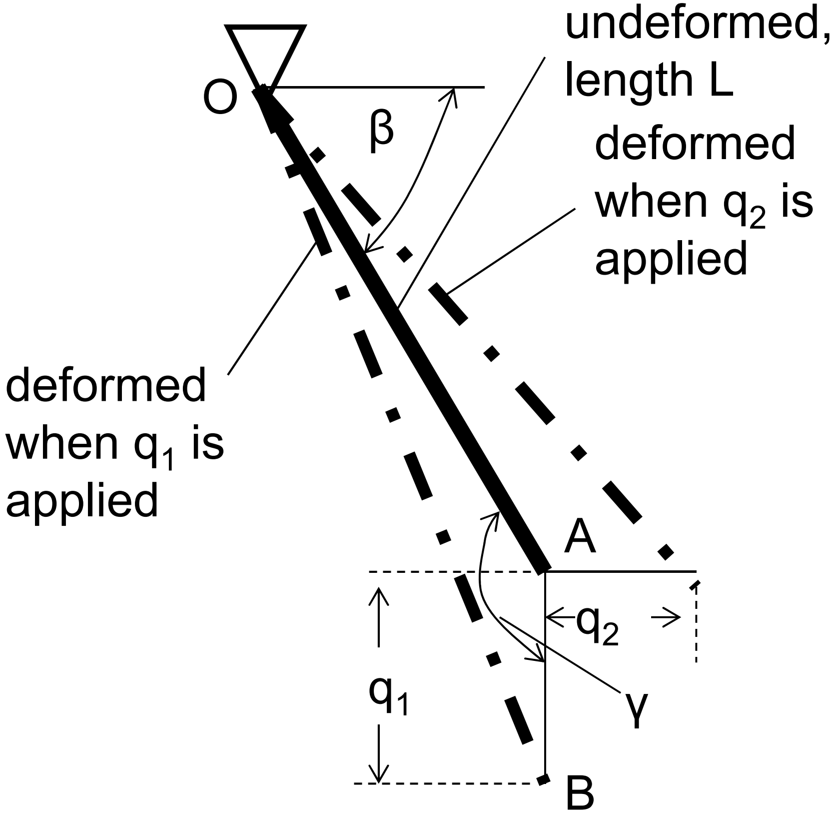
- But: OA=L, AB=q_1 and γ=90+β; So:

$$OB^2 = L^2 + q_1^{\,2} + 2Lq_1 \sin\beta$$

recall that:

$$\cos(90+\beta) = \cos 90 \cos\beta - \sin 90 \sin\beta$$

Castigliano's first theorem – Application



$$OB^2 = L^2 + q_1^{\,2} + 2Lq_1 \sin\beta$$

- the change in beam length is then,

$$\Delta_{q_1} = OB - OA = \sqrt{L^2 + q_1^{\,2} + 2Lq_1 \sin\beta} - L$$

- for small deflections, q_1 is small and q_1^2 can be neglected; then

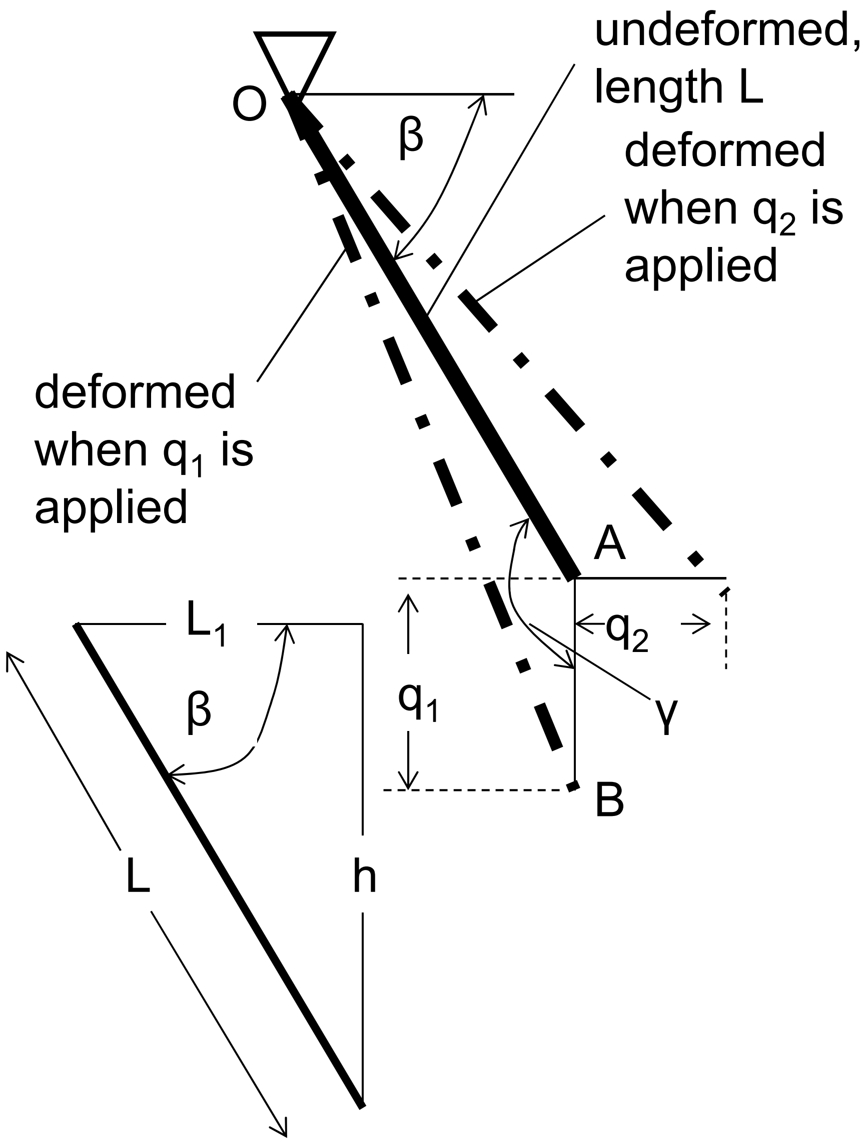
$$\Delta_{q_1} = OB - OA \doteq \sqrt{L^2 + 2Lq_1 \sin\beta} - L$$

- now, $2Lq_1\sin\beta$ is small compared to L^2 so the square root can be expanded in a Taylor series for small q_1:

$$\sqrt{L^2 + 2Lq_1 \sin\beta} \doteq \sqrt{L^2 + 2Lq_1 \sin\beta}\,\Big]_{q_1=0} + \frac{d}{dq_1}\left[\sqrt{L^2 + 2Lq_1 \sin\beta}\right]_{q_1=0} q_1 + \ldots =$$

$$L + \left[\frac{1}{2}\frac{2L\sin\beta}{\sqrt{L^2 + 2Lq_1 \sin\beta}}\right]_{q_1=0} q_1 + \ldots = L + q_1 \sin\beta$$

Castigliano's first theorem – Application



$$\Delta_{q_1} = OB - OA \doteq \sqrt{L^2 + 2Lq_1 \sin\beta} - L$$

$$\sqrt{L^2 + 2Lq_1 \sin\beta} \doteq L + q_1 \sin\beta$$

- substituting:

$$\Delta_{q_1} = q_1 \sin\beta$$

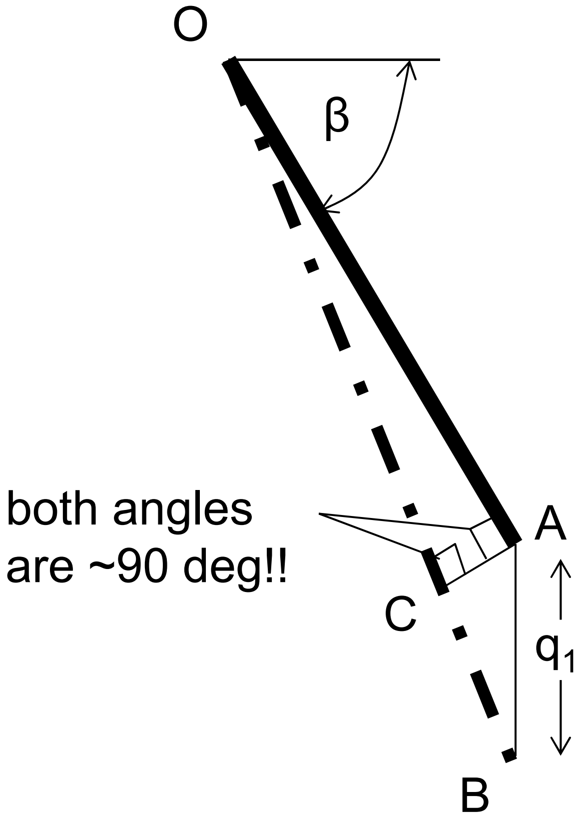
- and using the fact that

$$\sin\beta = \frac{h}{L}$$

- we get finally:

$$\boxed{\Delta_{q_1} = q_1 \frac{h}{L}}$$

Castigliano's first theorem – Application



• An alternate approach to get the same result. It is simpler but requires some assumptions associated with small deflection theory that can be confusing:

• With radius OA draw a circle segment that intersects OB at C. If the deflection q_1 is small, then the arc AC can be approximated by a straight line AC. Then:

$$OC = OA = L$$

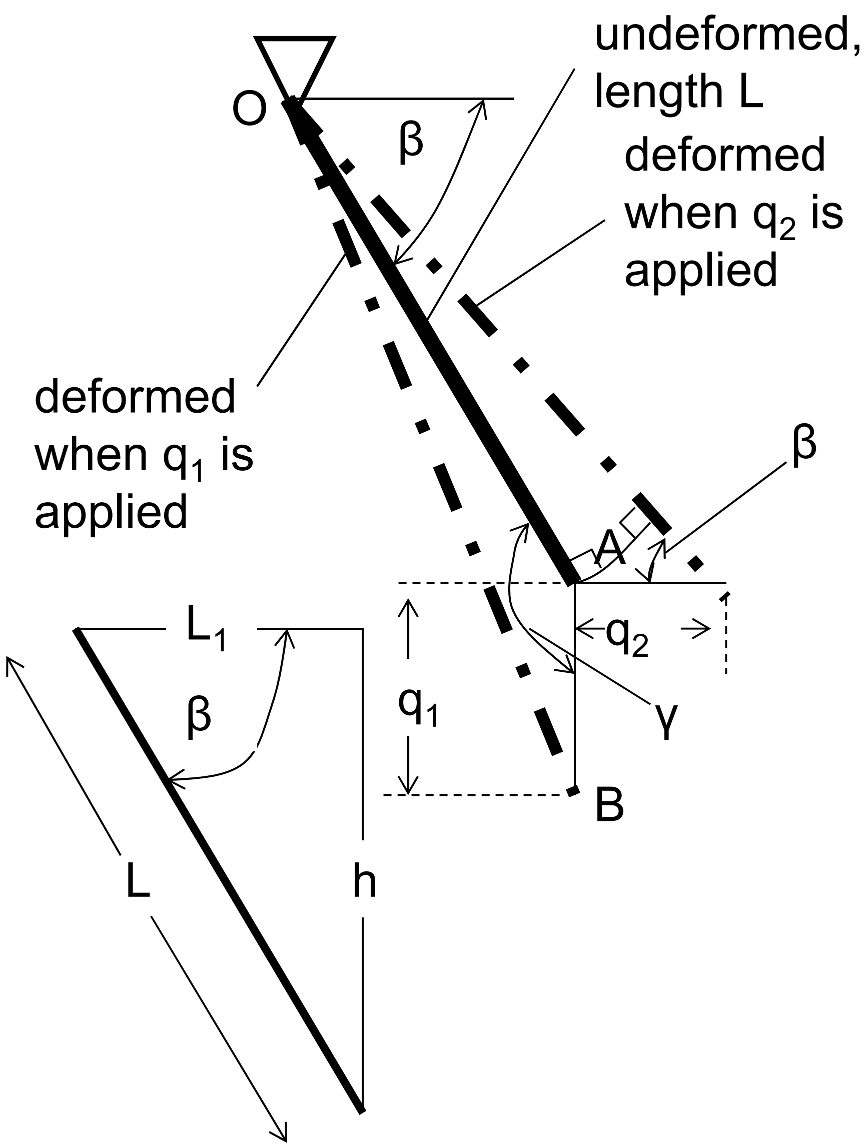
$$\Delta_{q_1} = OB - OC = BC$$

• Now angle CAB=β because it has mutually perpendicular sides with β. Then,

$$BC = q_1 \sin \sphericalangle CAB = q_1 \sin \beta$$

which is exactly what we got before!

Castigliano's first theorem – Application



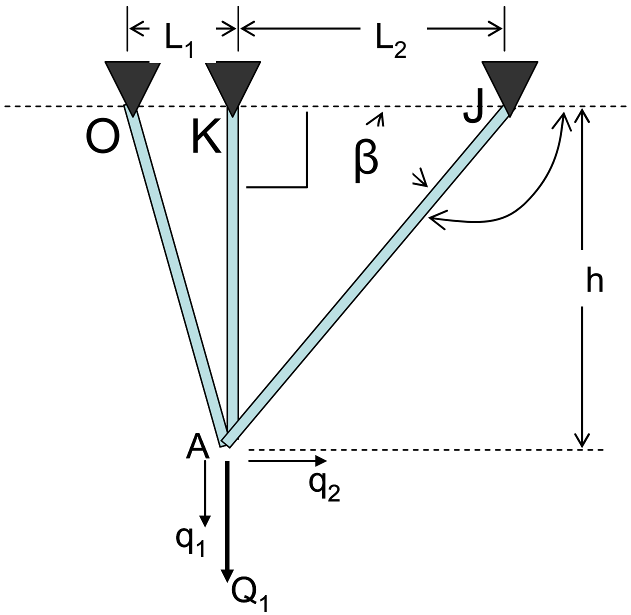
- in a completely analogous fashion, we can find the extension along the beam when it displaces horizontally by q_2:

$$\Delta_{q_2} = q_2 \cos\beta = q_2 \frac{L_1}{L}$$

- both Δq_1 and Δq_2 are along the length of the beam so the total deflection when the beam displaces by q_1 and q_2 is the sum of the two:

$$\Delta = \Delta_{q_1} + \Delta_{q_2} = q_1 \sin\beta + q_2 \cos\beta = q_1 \frac{h}{L} + q_2 \frac{L_1}{L}$$

Castigliano's first theorem – Application



$$\Delta = \Delta_{q_1} + \Delta_{q_2} = q_1 \sin\beta + q_2 \cos\beta = q_1 \frac{h}{L} + q_2 \frac{L_1}{L}$$

- for OA: $\Delta_{OA} = q_1 \sin\beta + q_2 \cos\beta = q_1 \dfrac{h}{L_{OA}} + q_2 \dfrac{L_1}{L_{OA}}$
- for KA: β=90°=> $\Delta_{KA} = q_1 \sin 90 = q_1$
- for JA: β→180-β=> $\Delta_{JA} = q_1 \sin(180-\beta) + q_2 \cos(180-\beta) =$

$$q_1 \sin\beta - q_2 \cos\beta = q_1 \frac{h}{L_{JA}} - q_2 \frac{L_2}{L_{JA}}$$

- use now eq (12.31) to determine the total strain energy,

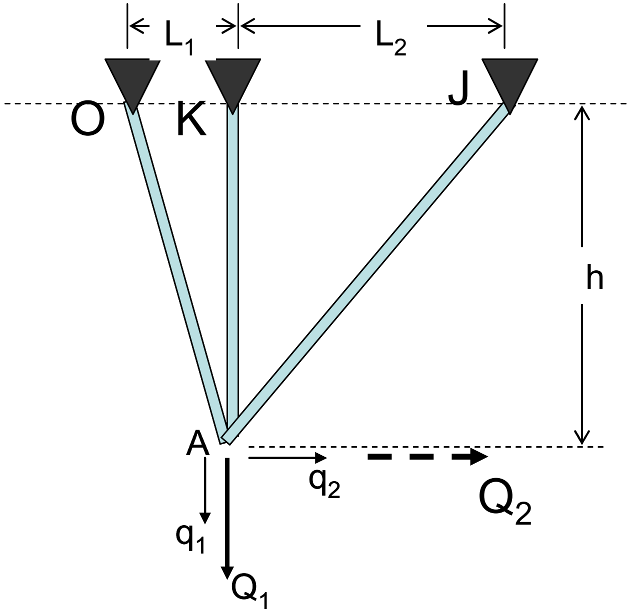
$$U = \frac{EA\Delta^2}{2L} \qquad (12.31)$$

- substituting:

$$U = \frac{EA}{2}\left[\frac{\Delta_{OA}{}^2}{L_{OA}} + \frac{\Delta_{KA}{}^2}{L_{KA}} + \frac{\Delta_{JA}{}^2}{L_{JA}}\right] = \frac{EA}{2}\left[q_1{}^2\left(\frac{1}{L_{KA}} + \frac{h^2}{L_{OA}{}^3} + \frac{h^2}{L_{JA}{}^3}\right) + q_2{}^2\left(\frac{L_1{}^2}{L_{OA}{}^3} + \frac{L_2{}^2}{L_{JA}{}^3}\right) + 2q_1q_2\left(\frac{hL_1}{L_{OA}{}^3} - \frac{hL_2}{L_{JA}{}^3}\right)\right]$$

Castigliano's first theorem – Application

$$U = \frac{EA}{2}\left[\frac{\Delta_{OA}^{\ 2}}{L_{OA}} + \frac{\Delta_{KA}^{\ 2}}{L_{KA}} + \frac{\Delta_{JA}^{\ 2}}{L_{JA}}\right] = \frac{EA}{2}\left[q_1^{\ 2}\left(\frac{1}{L_{KA}} + \frac{h^2}{L_{OA}^{\ 3}} + \frac{h^2}{L_{JA}^{\ 3}}\right) + q_2^{\ 2}\left(\frac{L_1^{\ 2}}{L_{OA}^{\ 3}} + \frac{L_2^{\ 2}}{L_{JA}^{\ 3}}\right) + 2q_1q_2\left(\frac{hL_1}{L_{OA}^{\ 3}} - \frac{hL_2}{L_{JA}^{\ 3}}\right)\right]$$



$$Q_i = \frac{\partial U}{\partial \Delta_i} \tag{12.29}$$

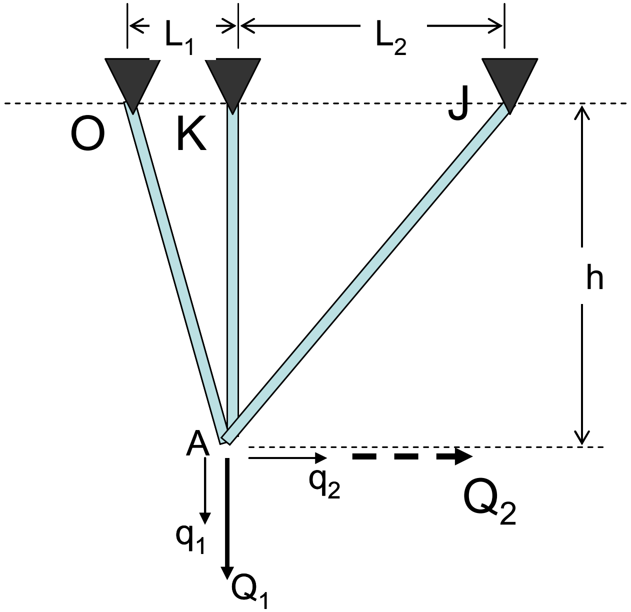
- use now eq (12.29) to obtain two equations for the loads Q_1 and Q_2:

$$Q_1 = EA\left[q_1\left(\frac{1}{L_{KA}} + \frac{h^2}{L_{OA}^{\ 3}} + \frac{h^2}{L_{JA}^{\ 3}}\right) + q_2\left(\frac{hL_1}{L_{OA}^{\ 3}} - \frac{hL_2}{L_{JA}^{\ 3}}\right)\right]$$

$$Q_2 = EA\left[q_1\left(\frac{hL_1}{L_{OA}^{\ 3}} - \frac{hL_2}{L_{JA}^{\ 3}}\right) + q_2\left(\frac{L_1^{\ 2}}{L_{OA}^{\ 3}} + \frac{L_2^{\ 2}}{L_{JA}^{\ 3}}\right)\right]$$

- in the present problem Q_2=0 and Q_1 is a given applied force

Castigliano's first theorem – Application



$$Q_1 = EA\left[q_1\left(\frac{1}{L_{KA}}+\frac{h^2}{L_{OA}{}^3}+\frac{h^2}{L_{JA}{}^3}\right)+q_2\left(\frac{hL_1}{L_{OA}{}^3}-\frac{hL_2}{L_{JA}{}^3}\right)\right]$$

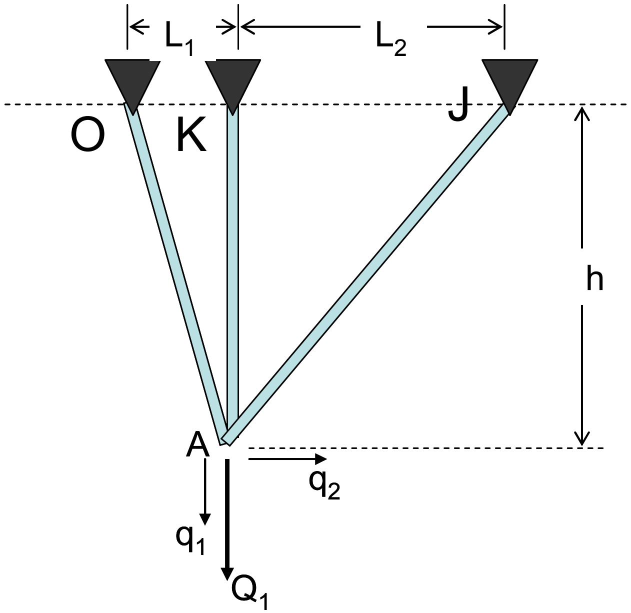
$$0 = EA\left[q_1\left(\frac{hL_1}{L_{OA}{}^3}-\frac{hL_2}{L_{JA}{}^3}\right)+q_2\left(\frac{L_1{}^2}{L_{OA}{}^3}+\frac{L_2{}^2}{L_{JA}{}^3}\right)\right]$$

- system of two equations in the two unknowns q_1 and q_2
- numerical example: $EA=10^7$ (appropriate units)

$L_1=50a$, $L_2=90a$, $h=120a$, $Q_1=10000$ (appr. units)

- then: $L_{OA}=130a$, $L_{KA}=120a$, $L_{JA}=150a$

Castigliano's first theorem – Application



$$Q_1 = EA\left[q_1\left(\frac{1}{L_{KA}}+\frac{h^2}{L_{OA}^{\ 3}}+\frac{h^2}{L_{JA}^{\ 3}}\right)+q_2\left(\frac{hL_1}{L_{OA}^{\ 3}}-\frac{hL_2}{L_{JA}^{\ 3}}\right)\right]$$

$$0 = EA\left[q_1\left(\frac{hL_1}{L_{OA}^{\ 3}}-\frac{hL_2}{L_{JA}^{\ 3}}\right)+q_2\left(\frac{L_1^{\ 2}}{L_{OA}^{\ 3}}+\frac{L_2^{\ 2}}{L_{JA}^{\ 3}}\right)\right]$$

•EA=10^7 (appropropriate units) L_1=50a, L_2=90a, h=120a, Q_1=10000 (app units)

- substituting values, the system to be solved becomes:

$$191544q_1 - 4690q_2 = 10000a$$

$$-4690q_1 + 35378q_2 = 0$$

- solving: q_1=0.0524a, q_2=0.0069a

Wrap-up

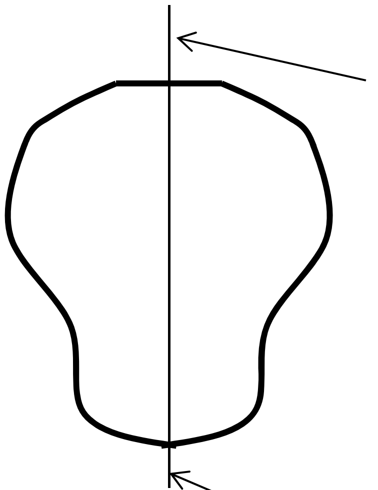
- bending of beams
  – symmetric and unsymmetric cross-sections
  – direct stresses in beams due to bending
- shear of beams
  – direct stresses due to shear
  – shear stresses due to shear
  – closed versus open sections
  – idealization: constant shear flow when skins take no bending loads
  – determination of flange/boom areas

Wrap-up

- Torsion of beams
  - stress functions
  - rate of twist
  - torsional rigidity
  - closed versus open sections
  - multi-cell beams
- Effect of taper
- Effect of cut-outs
- Castigliano's theorems (determination of deflections or forces as derivatives of strain energy)
- Buckling of beams
  - perfect
  - with initial imperfections or eccentricities
  - using energy methods

Things to remember in future lives

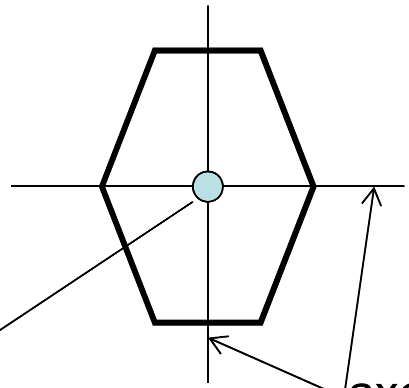
- if there is an axis of symmetry, the neutral axis is on it
- unsymmetric cross-sections have $I_{xy} \neq 0$; there must be at least one axis of symmetry (for the entire cross-section or for the booms if idealized) for I_{xy} to be zero
- if there is an axis of symmetry, the shear center is on it



neutral axis and shear center lie on this axis but do not necessarily coincide

$I_{xy}=0$ for this cross-section

axis of symmetry



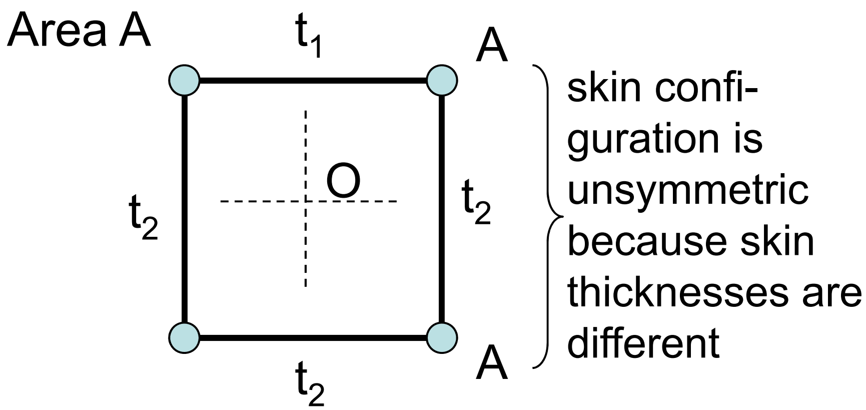
axes of symmetry

shear center and neutral axis at this location

$I_{xy}=0$

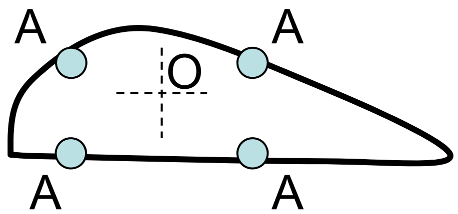
Things to remember in future lives

• if the booms or flanges have one axis of symmetry, in a problem where skin carries only shear, then the neutral axis is on that axis of symmetry and I_{xy}=0 even if the skin configuration is unsymmetric



neutral axis at O

I_{xy}=0



neutral axis at O

I_{xy}=0

Things to remember in future lives

- a shear force acting through the shear center causes only bending and no twist
- in bending problems where the skin is idealized to carry only shear, the shear flow between booms/flanges is constant and equal to the average of the shear flow that would be obtained if the skin carried direct stresses also

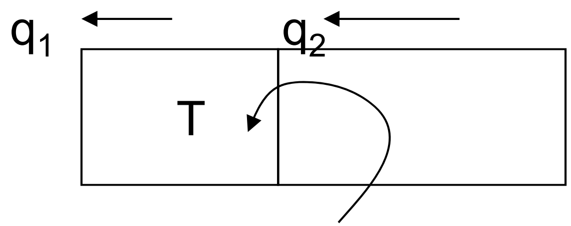
- for basic beam bending

$$\sigma_z = \frac{My}{I_{xx}}$$

Things to remember in future lives

- for a constant shear flow through a skin: $T = 2Aq$

- there is an important difference between moment equilibrium and torque equivalence

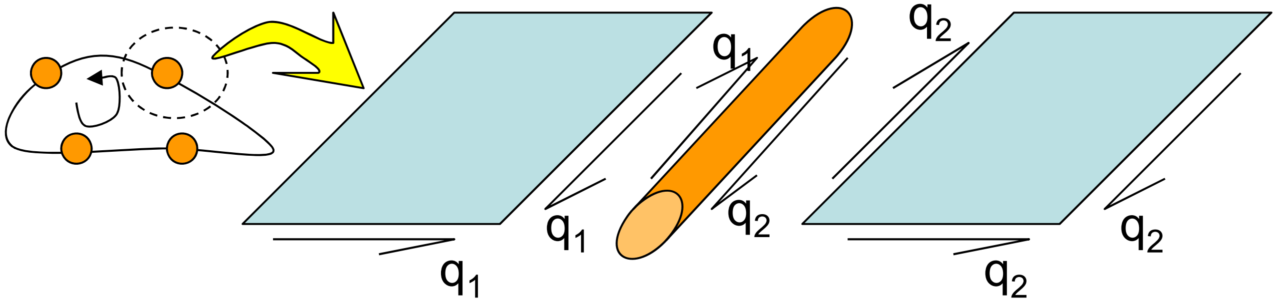


the shear flows are **caused** by T; then, the moment caused by T about a point is equal to the moment caused by the shear flows about the same point (torque equivalence) this is not the same as adding all the moments about a point and setting them equal to 0

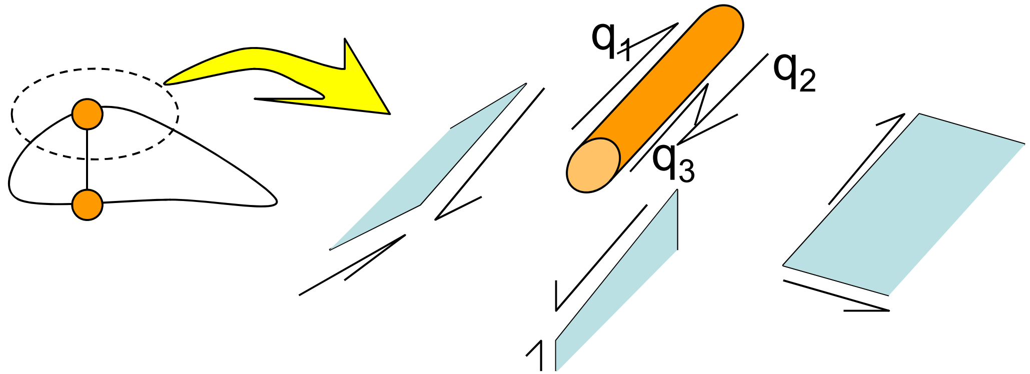
$T=2A_1q_1+2A_2q_2$ and NOT: $T+2A_1q_1+2A_2q_2=0$

Things to remember in future lives

- if a cross-section is under pure torsion, then for a single cell section, the booms do not change the shear flow; the shear flow is constant across booms
- for multi-cell beams under pure torsion, the shear flow changes across booms



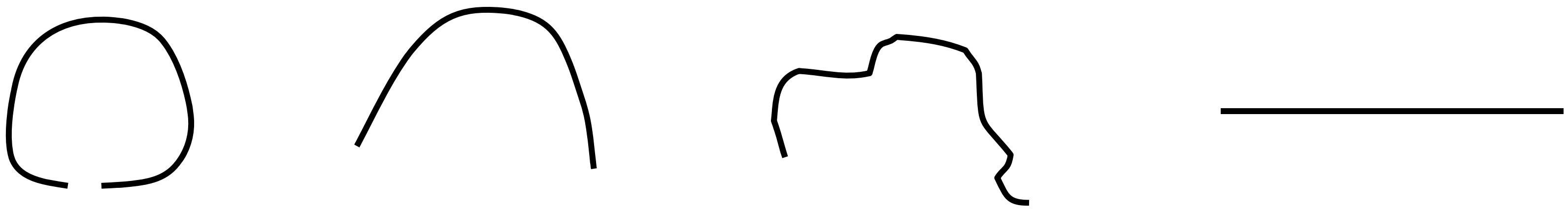
if there is NO force in the boom, then force equilibrium requires that q_1=q_2



even if there is no force in the boom, there are three shear flows at the intersection and they cannot be all equal (in general)

Things to remember in future lives

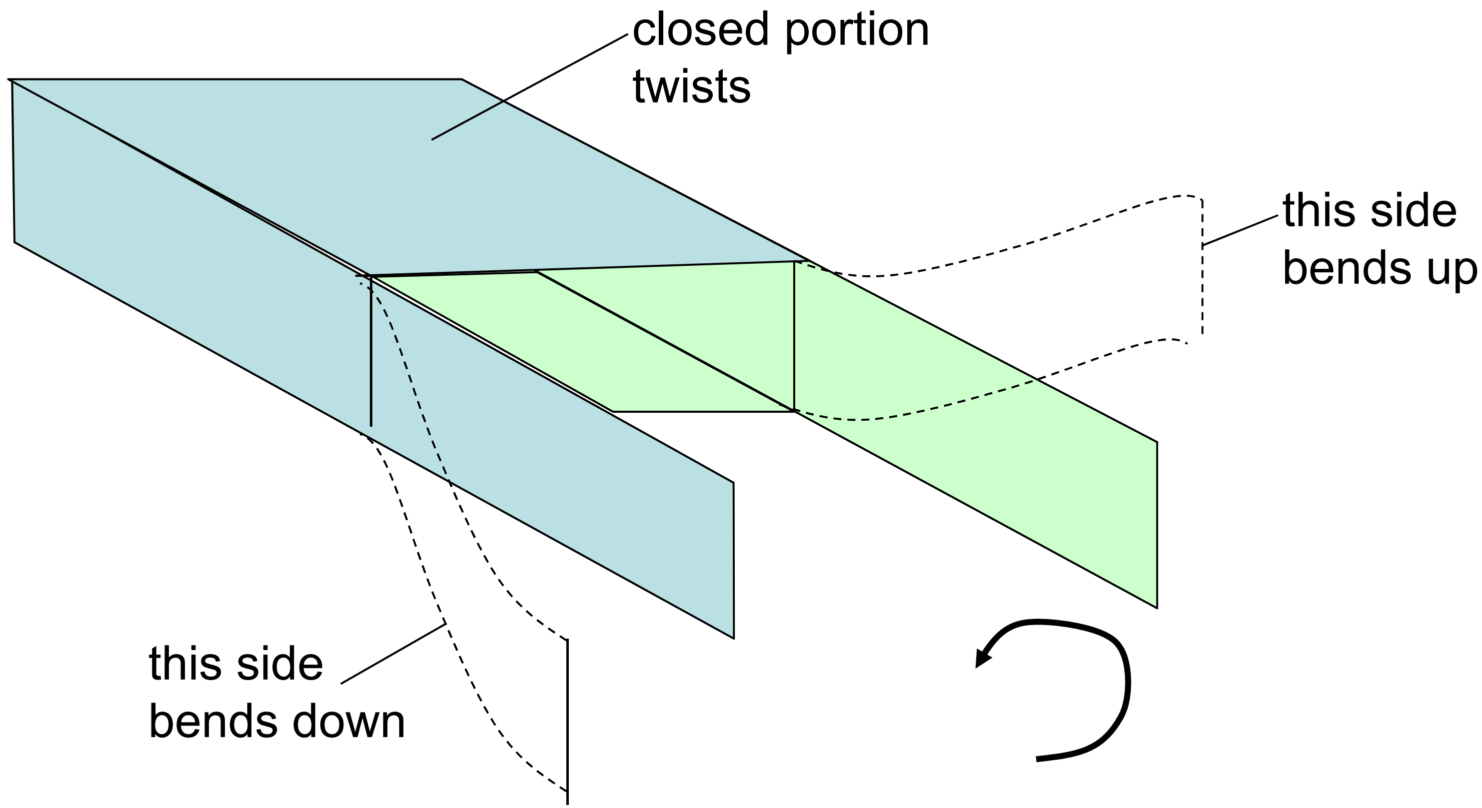
- for multi-cell beams, the rate of twist of each cell equals the rate of twist of the entire cross-section
- open sections are terrible in torsion; the shape of the open section has no effect on its torsional rigidity; it is always given by the thin rectangle approximation



same length same thickness for all => same J

Things to remember in future lives

- beams with cutouts take torsional loads at the cutout region through differential bending



Things to remember in future lives

- taper introduces additional (vertical) forces; the axial forces are still given by the standard equations
- cutouts are a pain to deal with
- Castigliano $\Delta_i = \dfrac{\partial C_i}{\partial P_i} \qquad Q_i = \dfrac{\partial U}{\partial \Delta_i}$
- the buckling load of a beam fixed (clamped) at both ends is four times the buckling load of a beam simply supported (pinned) at both ends

$$P_{crit} = \frac{\pi^2 EI}{L^2}$$

simply-supported or pinned

$$P_{cr} = \frac{4\pi^2 EI}{L^2}$$

fixed or clamped