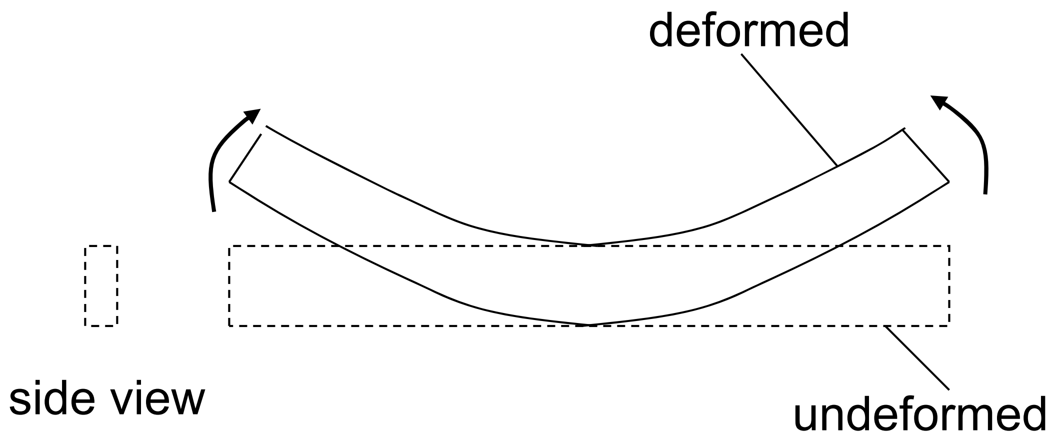
# Bending stresses in beams

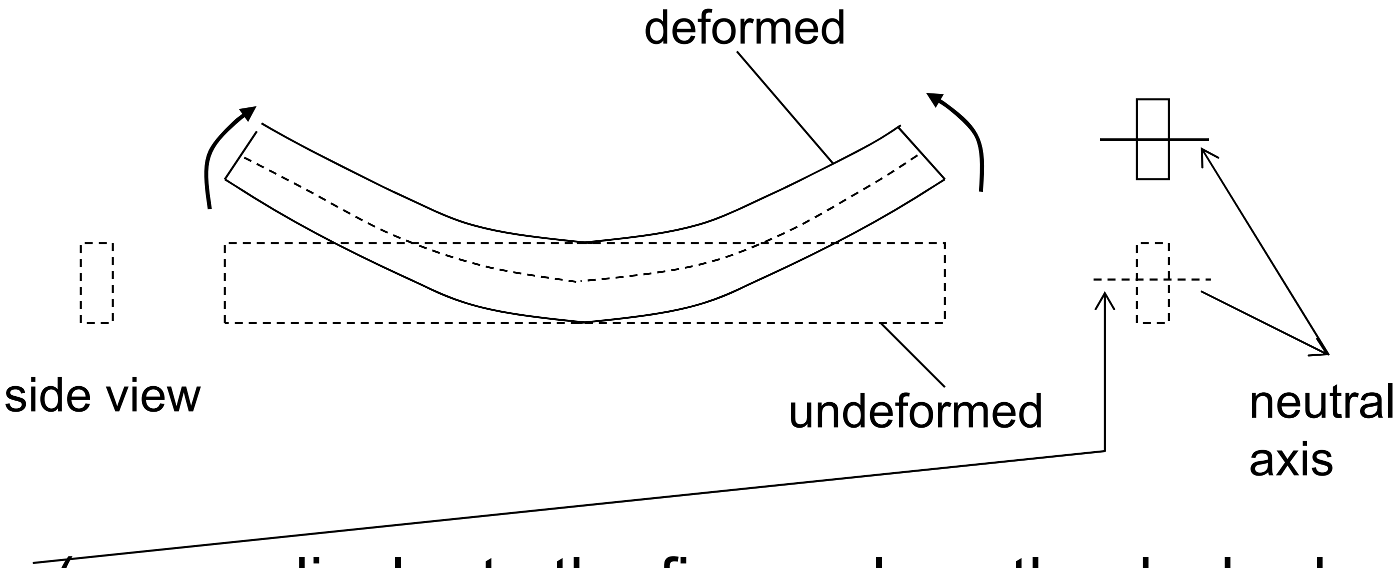
• determine stresses, strains and displacements in beams under (pure) bending loads




• upper side is under compression (thus under compressive stress) and lower side is under tension (tensile stress)

• since the stresses (more accurately strains) are continuous through the thickness, there must be a location (or plane) where direct stress is zero

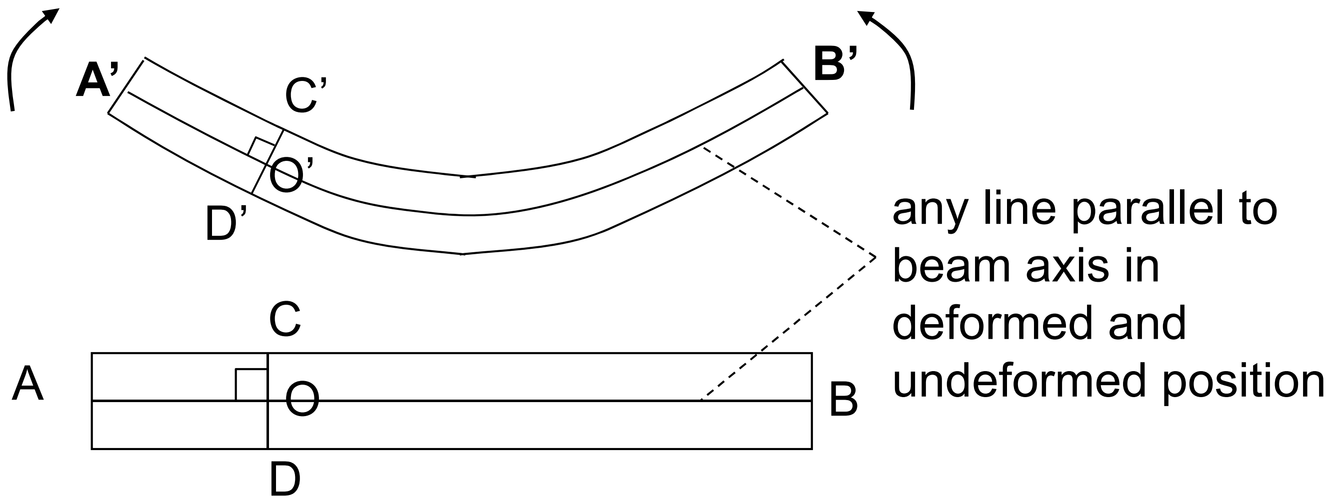
# Neutral axis




• the plane (perpendicular to the figure along the dashed line in the middle) in which direct stresses are zero, intersects any beam cross section along a horizontal line, the **neutral axis**

• the neutral axis is a convenient reference location when determining stresses under bending loads

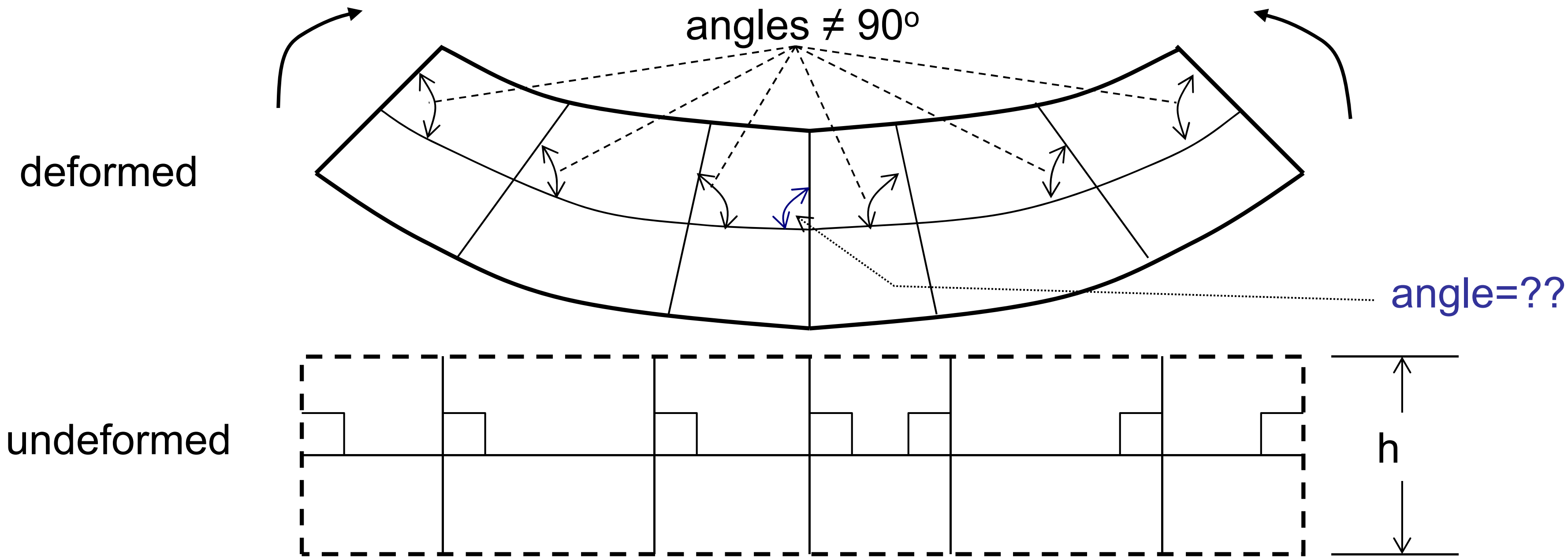
# Assumptions: Engineering bending theory




• plane sections (such as CD) remain plane after deformation (C'D' is still a straight line after deformation)

• plane sections remain perpendicular to the beam axis after deformation (angle AOC= angle A'O'C' = $90^o$)

# Assumptions: Engineering Bending theory

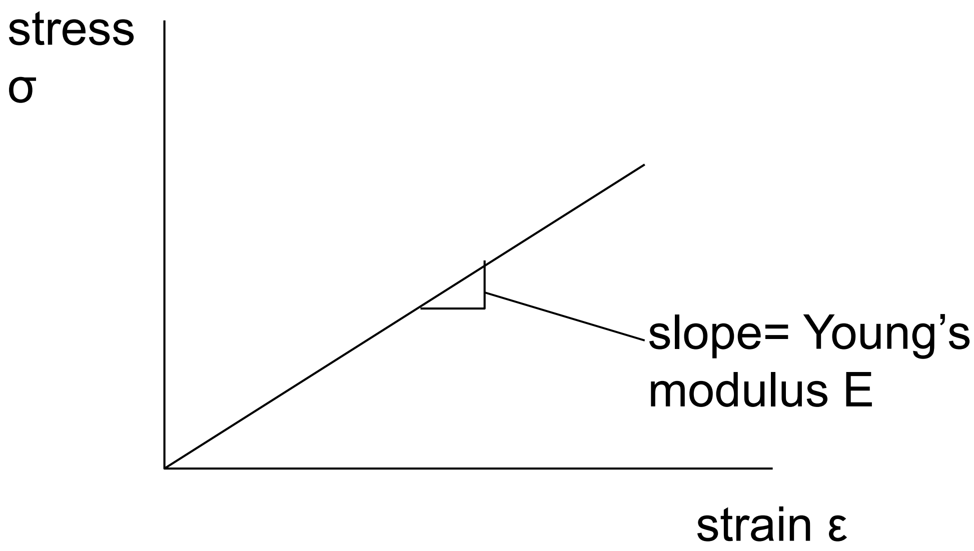



• the first assumption, “plane sections remain plane” is valid for most beams (except some non-homogeneous beams etc.)

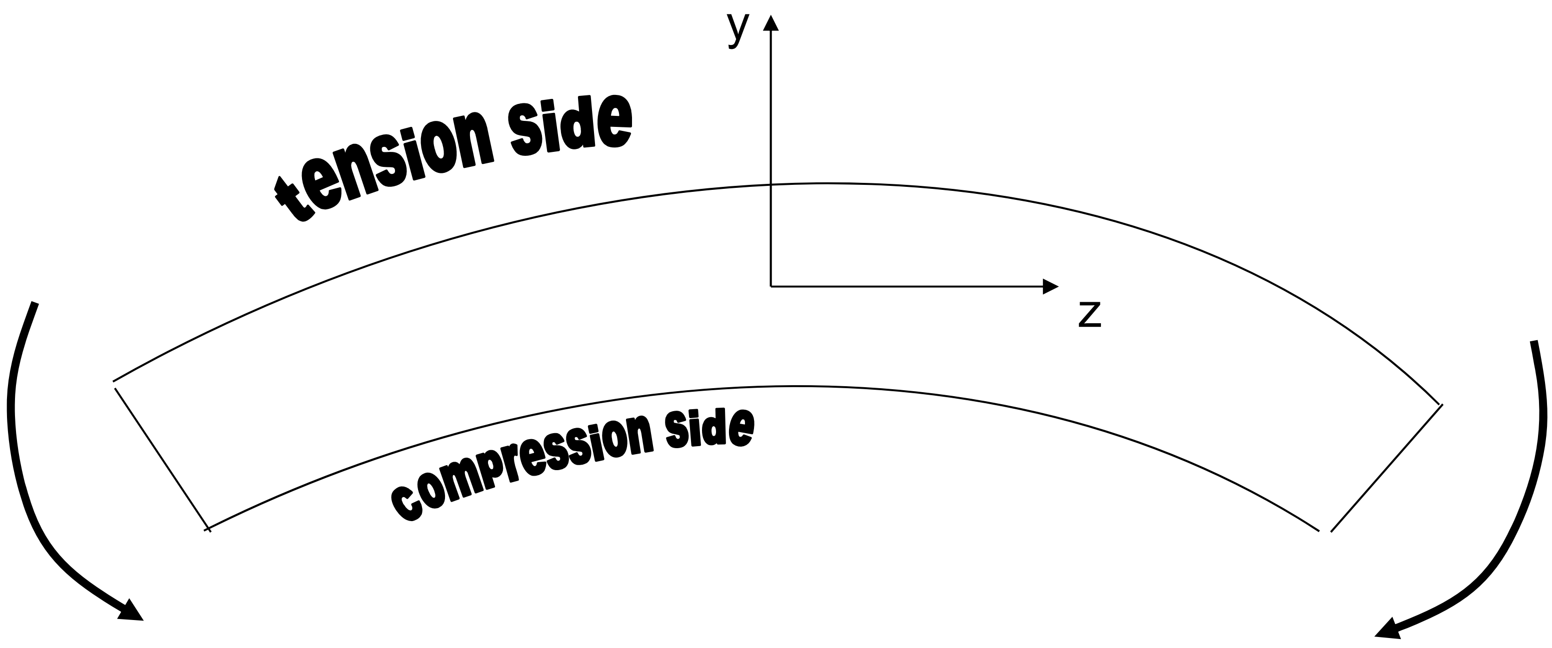
• the second assumption, “plane sections remain perpendicular to beam axis” is violated for deep beams (h large)

# Assumptions: Engineering Bending theory

- linear elastic beam: $\sigma = E\varepsilon$
- beam is homogeneous

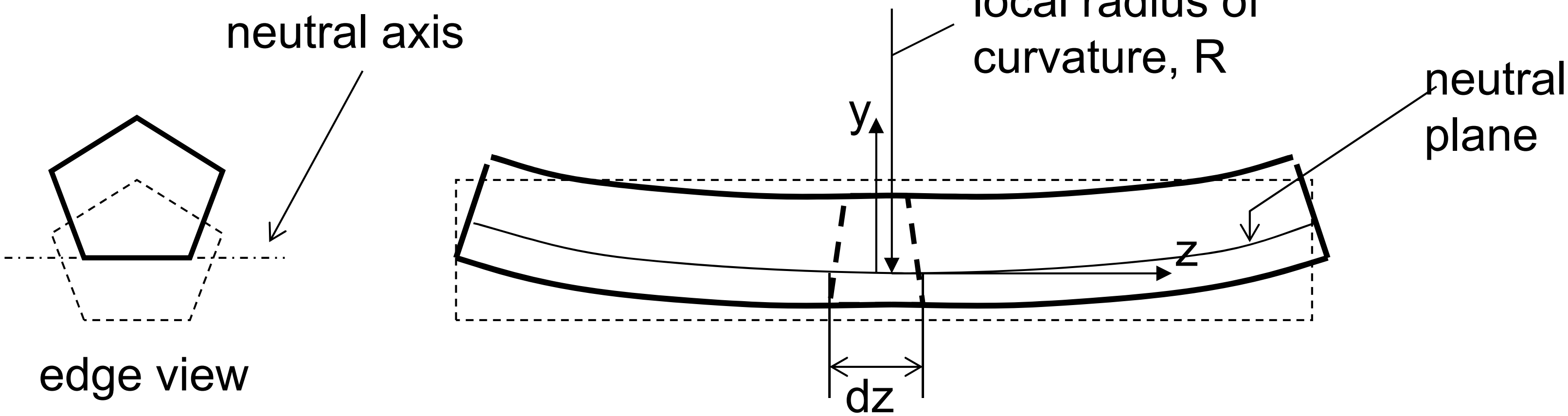


# Sign convention




• a positive moment M causes tension on the outer (extreme) fibers of the beam that lie in the region where y is positive

# Bending about one axis – (review)




• considering the deformations of a slice of beam of length dz:

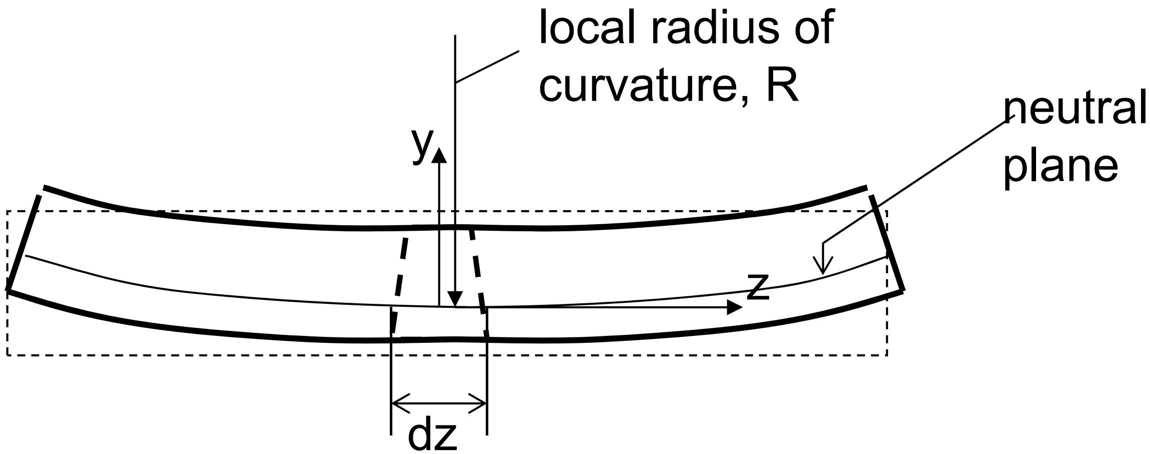
– locally the deformed beam shape can be approximated by a circular arc of radius R

– then strains are linear in y (and zero at the neutral axis)

$$\varepsilon_z = -\frac{y}{R}$$

(note negative sign to conform to the sign convention)

# Bending about one axis – (review)




• corresponding direct (or normal) stress is obtained by multiplying by Young's modulus (linear elastic assumption)

$$\sigma_z = -E\,\frac{y}{R}$$

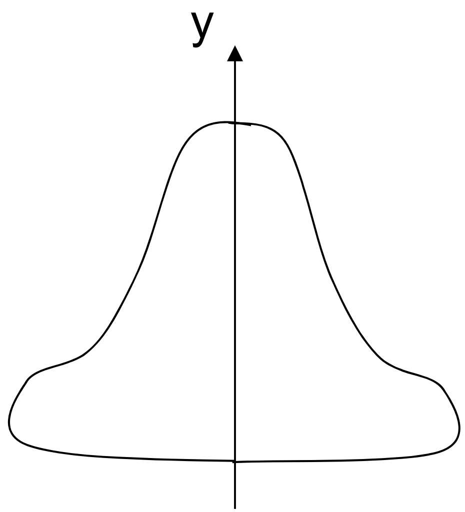
• and since the beam is under a pure moment, the total force in the z direction is zero:

$$\int_A \sigma_z \, dA = 0 \Rightarrow -\frac{E}{R}\int_A y\,dA = 0 \qquad (2.0)$$

**First moment of area is zero => neutral axis passes through the centroid** of the cross-section

# Bending about one axis – (review)

• if now the y axis is an axis of symmetry

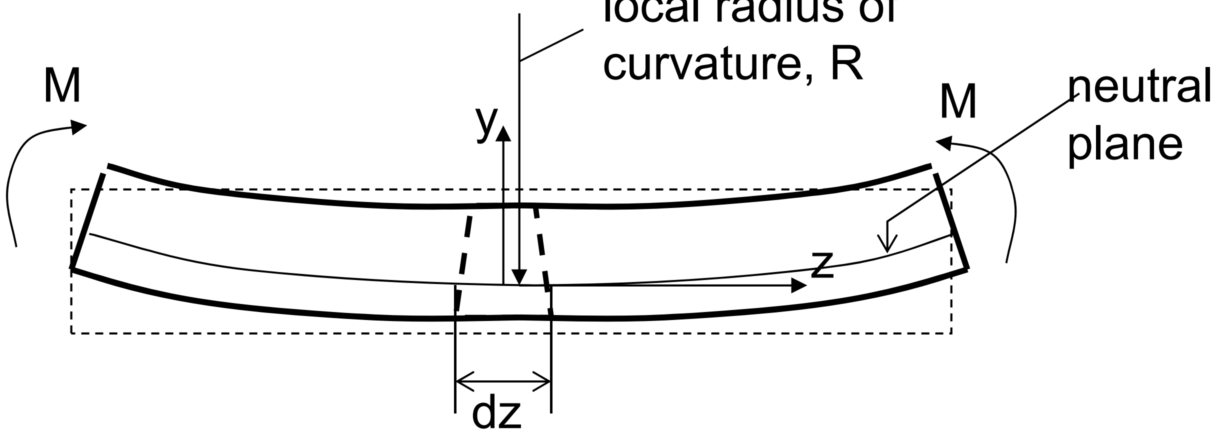



beam cross-section

(edge view)

• the origin of the coordinate system for determining stresses and strains coincides with the centroid of the beam cross section

# Bending about one axis – (review)




$$\sigma_z = -E\frac{y}{R}$$

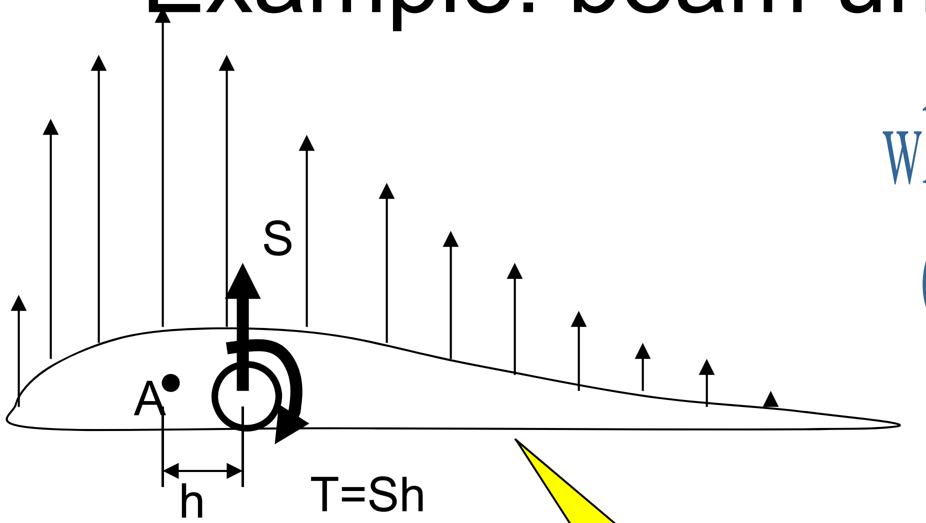
- moment equilibrium requires that at any cross-section, the moment caused by $\sigma_z$ is the result of the applied moment M:

second moment of area I

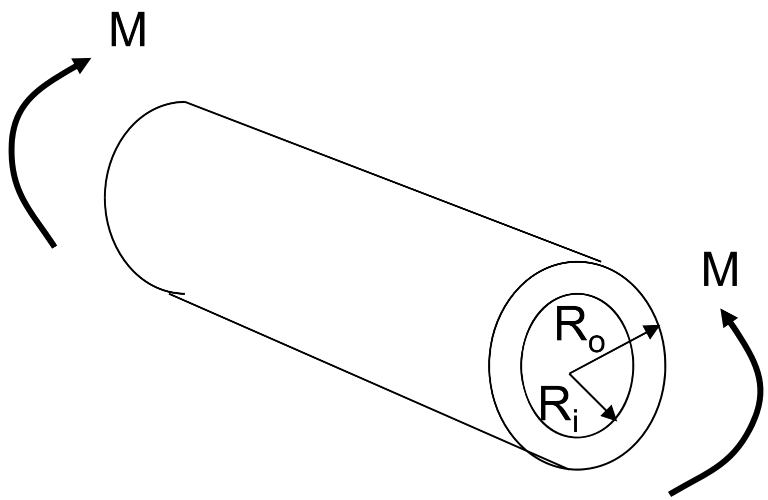
$$M = \int_A \sigma_z y dA = -\frac{E}{R}\int_A y^2 dA$$

$$\boxed{M = -\frac{EI}{R}} \Rightarrow \boxed{\sigma_z = -\frac{My}{I}}$$

# Example: beam under bending load



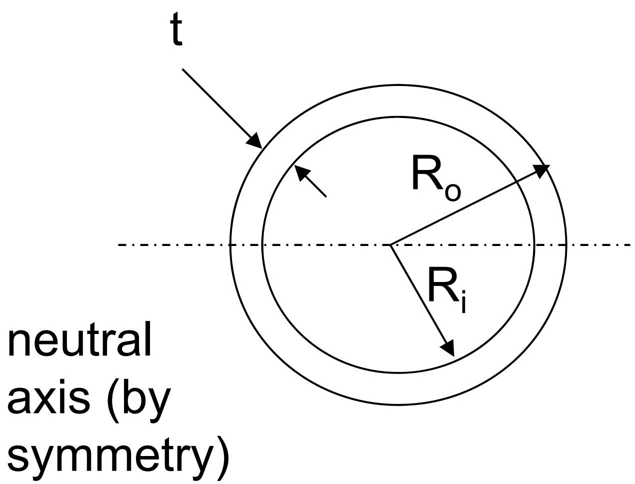

which material and shape should the Gossamer Albatross team select?




Problem definition and constraints:

- M=29430 Nm
- Airfoil shape limits $2R_o$ to 0.3m

# Example: beam under bending load




$$\sigma_z = -\frac{My}{I}$$

- The **circular shape** has the same moment of inertia in all directions; thus it is good to use if there are concerns about add'l bending moments in the plane of the wing (fwd/aft)
- Assume $t \ll R_i$, $R_o$; after solving the problem check if the resulting geometry violates this assumption
- From the $\sigma_z$ equation, to minimize the applied stress for a given M, we must maximize I; so set $R_o$=max allowable value = 0.15m (but if $R_o$ is large then y is also large which increases the normal stress; is this a good idea?)

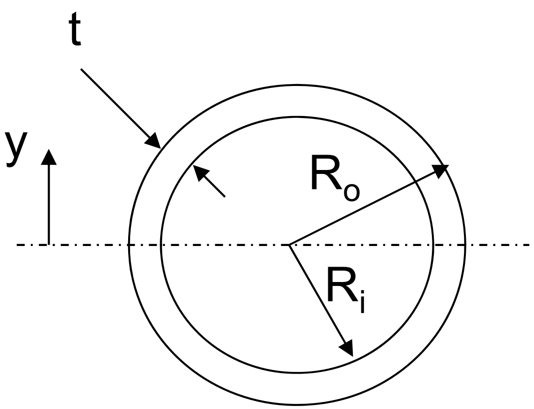
- moment of inertia I about the neutral axis is

negligible terms

$$I = \frac{\pi}{4}\left(R_o^4 - R_i^4\right) = \frac{\pi}{4}\left(R_o^4 - \left(R_o - t\right)^4\right) = \frac{\pi}{4}\left(\cancel{R_o^4} - \left(\cancel{R_o^4} - 4R_o^3 t + \cancel{6R_o^2 t^2} - \cancel{4R_o t^3} + \cancel{t^4}\right)\right)$$

- or, for small t: $I = \pi R_o^3 t$

# Example: beam under bending load




$$\sigma_z = -\frac{My}{I}$$

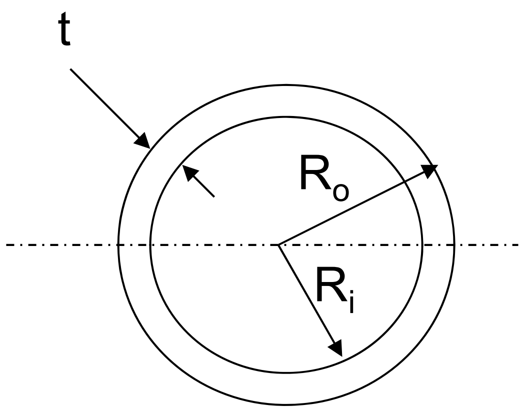
- therefore, the direct stress is:

$$\sigma_z = \frac{My}{\pi R_o^{\;3} t}$$

- to design this beam, we need to find the highest $\sigma_z$ stress and make sure it does not exceed the ultimate strength of the material selected; the highest stress is at $\pm y_{max} = \pm R_o$. Therefore,

$$\sigma_{z\,\max} = \frac{M}{\pi R_o^{\;2} t}$$

# Example: beam under bending load




$$\sigma_{z\max} = \frac{M}{\pi R_o^{\ 2} t}$$

• now failure occurs when $\sigma_{zmax} = \sigma_{ult}$:

$$\sigma_{ult} = \frac{M}{\pi R_o^{\ 2} t}$$

• in this equation, everything is known except for the thickness t. Solving for t,

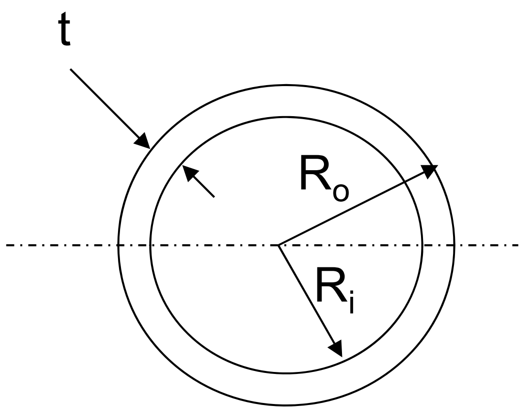
$$t = \frac{M}{\pi \sigma_{ult} R_o^{\ 2}}$$

• so if we pick a material (so $\sigma_{ult}$ is known) we can find the min value of t that will just fail the beam under applied M

# Basic structural material

| Material | Ultimate tension strength (MPa) | Density ($kg/m^3$) | Mat'l cost (€/kg) |
|---|---|---|---|
| Steel (AM-350) | 1262 | 7822 | 4.4 |
| Aluminium (7075-T6) | 552 | 2801 | 6.6 |
| Titanium (Ti-6Al-4V) | 958 | 4438 | 22 |
| Quasi-Isotropic composite | 483 | 1609 | 176 |

# Example: beam under bending load




$$t = \frac{M}{\pi \sigma_{ult} R_o^{\,2}}$$

$$Weight = \rho AL = \rho 2\pi RtL = \rho 2\pi (R_o - t)tL \Rightarrow \frac{Weight}{L} = \rho 2\pi R_o t$$

$$or \quad \frac{Weight}{L} = 2\rho \frac{M}{\sigma_{ult} R_o}$$

$$Also \quad \frac{Cost}{L} = \frac{€}{kg}\frac{Weight}{L} = \frac{€}{kg} 2\rho \frac{M}{\sigma_{ult} R_o}$$

| Material | thickness (mm) | Weight/L (kg/m) | Mat'l Cost/L (€/m) |
|---|---|---|---|
| Steel | 2.8 | 2.4 | 10.6 |
| Al | 6.4 | 2.0 | 13.2 |
| Ti | 3.7 | 1.8 | 39.6 |
| QI composite | 7.3 | 1.3 | 229 |

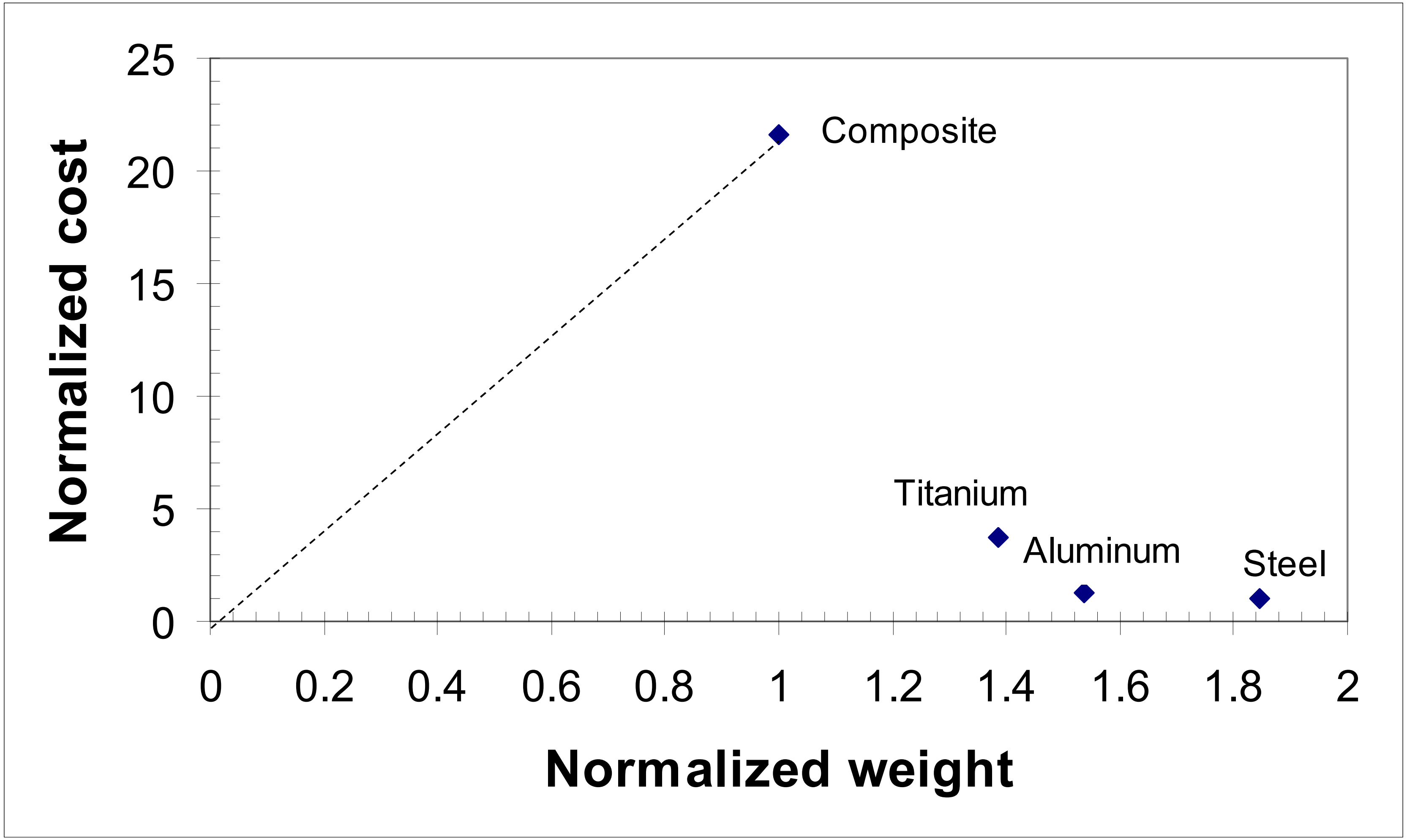
"sensible" thing to do

GA team

# Example: beam under bending load

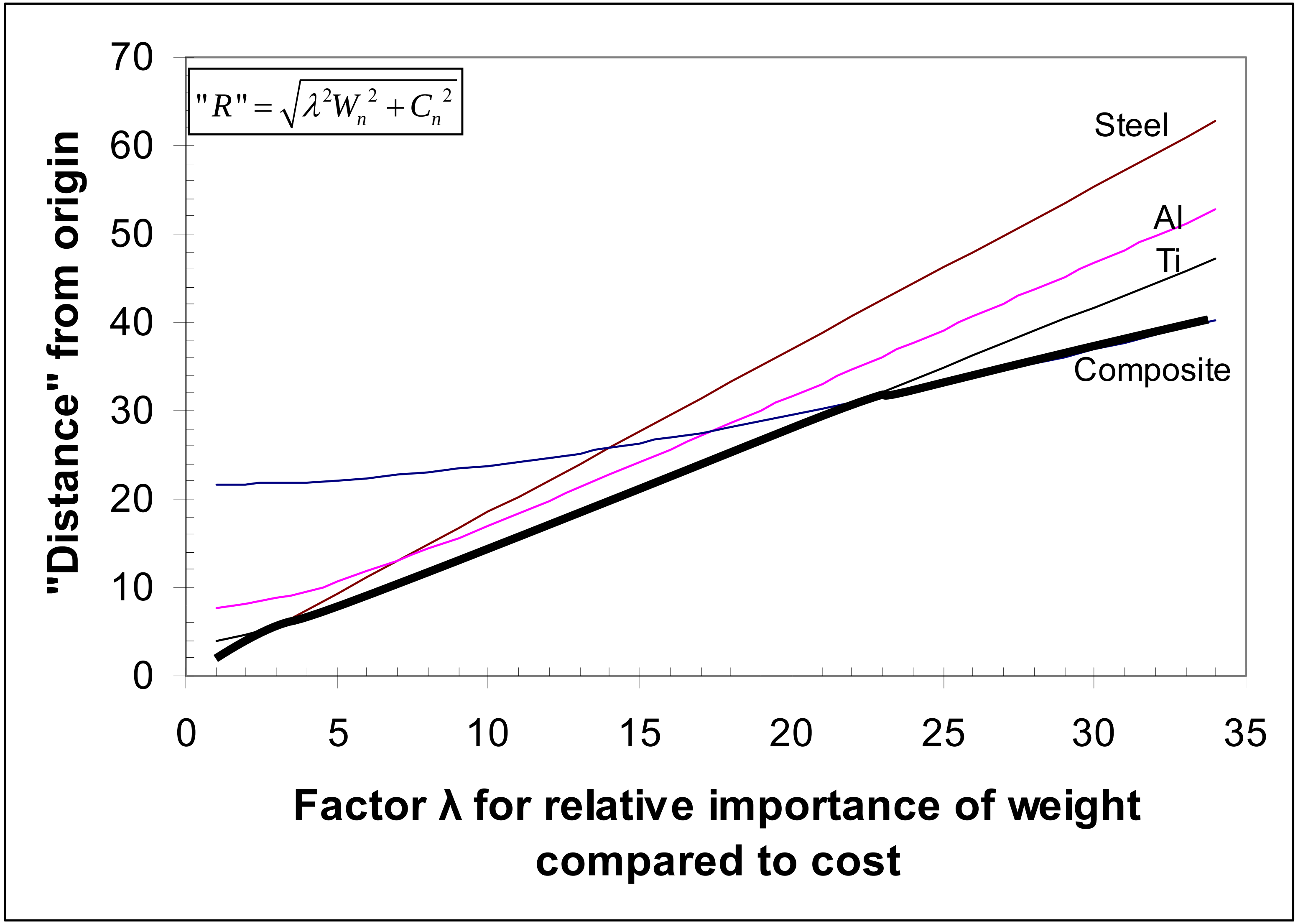
- a few final comments:
  - the calculated values of t are, indeed <<$R_o$ so the original assumption was justified
  - the material cost is just that: material; it does not include fabrication (labor) which would make the composite (and Ti) look even worse
  - the final choice is based on a cost versus weight trade: which is most important and by how much?

# Weight-Cost trades: Pareto analysis

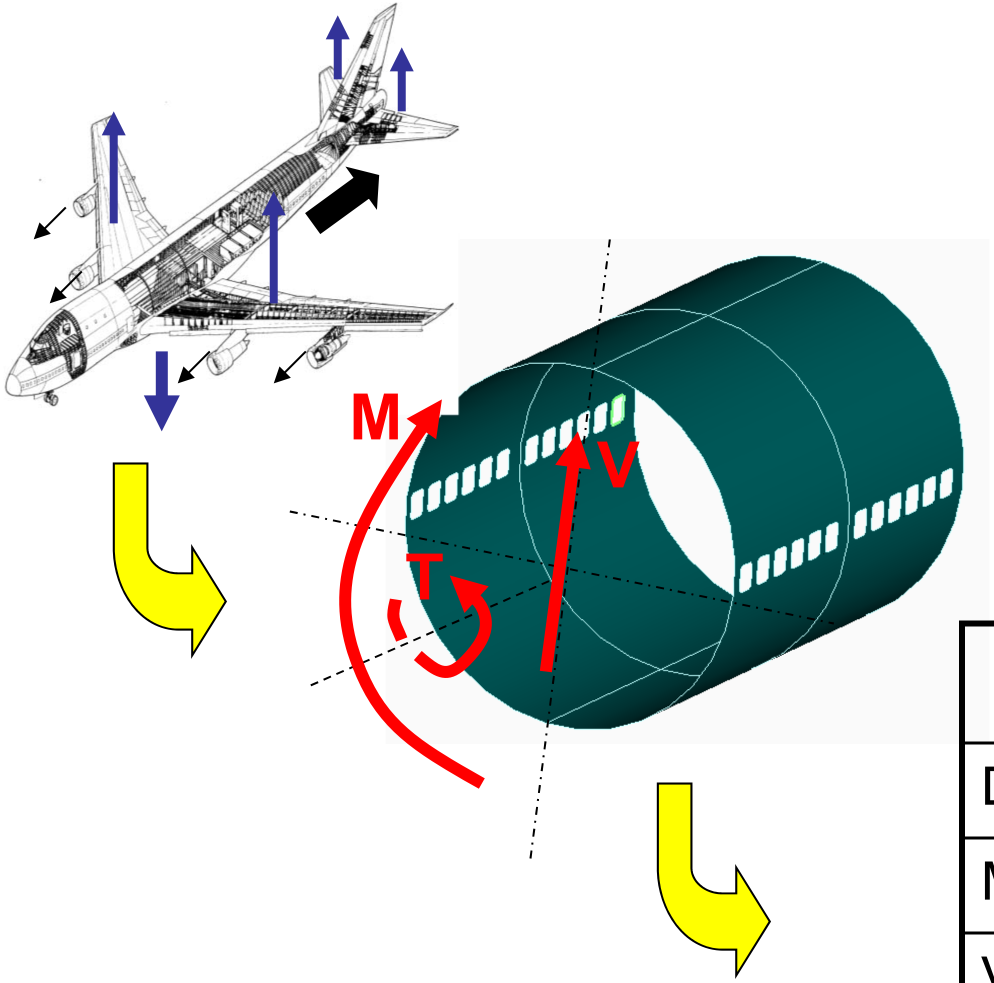



$$R = \sqrt{a_1 W_n{}^2 + a_2 C_n{}^2}$$ selection criterion: minimize R

# Weight-Cost trades: Relative importance of Cost and Weight



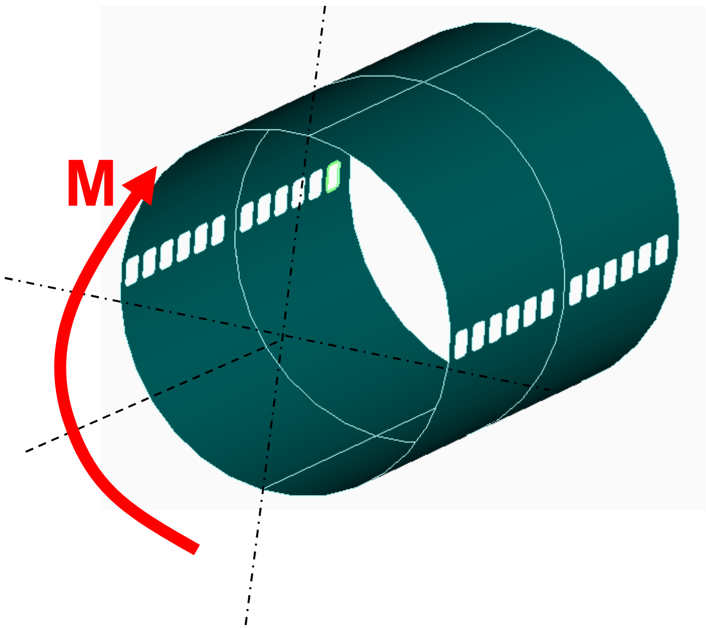
# “Running” example – Fuselage cross-section




Note: the bending moment M usually has opposite sign: the top is under tension and bottom under compression

| Property | Value |
|---|---|
| Diameter(m) | 4.0 |
| M (MNm) | 60 |
| V (kN) | 660 |
| T (kNm) | 30 |

# Concentrate on M first…




D = 4 m

M = 60x10$^6$ Nm

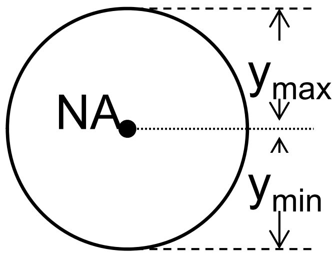
Also, from previous example:

$$\sigma_z = -\frac{My}{I}$$

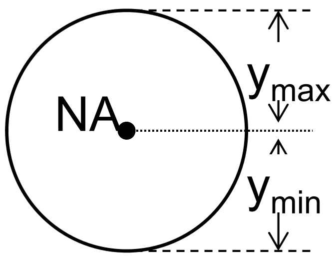
$I = \pi R_o{}^3 t$ approx.

- Calculate maximum tensile and compressive stresses
- For symmetric cross-section, neutral axis is at the center
- Max (or min) stress occurs at locations of $y_{max}$:

$$\sigma_{\max} = -\sigma_{\min} = \frac{My_{\max}}{\pi R^3 t}$$



# Required thickness to meet M load




D = 4 m

M = 60x10⁶ Nm

$$\sigma_{\max} = -\sigma_{\min} = \frac{My_{\max}}{\pi R^3 t}$$

- Failure (yielding) will just start when the max or min applied stress equals the yield stress $\sigma_y$ of the material
- For 7075-T6 Al, $\sigma_y^t$ = 482.6 MPa and $\sigma_y^c$ = 489.5 MPa
- Then for failure to occur:

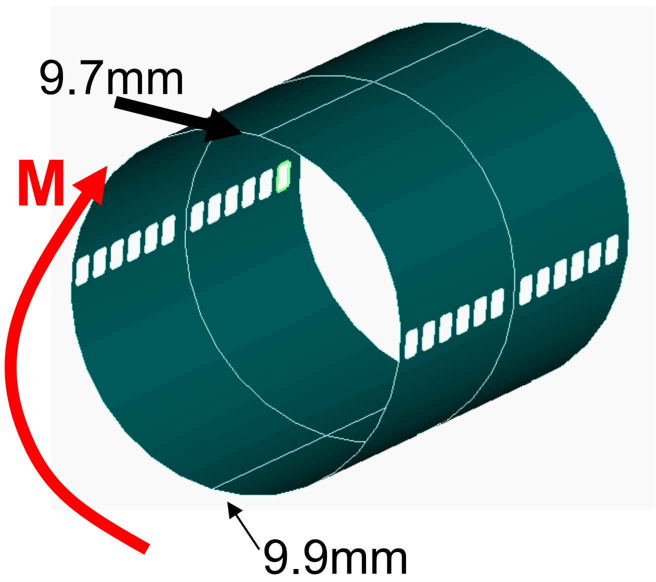
$$\sigma_y^{\ t} = \frac{My_{\min}}{\pi R^3 t} \quad \textit{tensile failure at bottom}$$

$$\sigma_y^{\ c} = \frac{My_{\max}}{\pi R^3 t} \quad \textit{compressive failure at top}$$

- Solving for t: $t_{bot} = \dfrac{My_{\min}}{\pi R^3 \sigma_y^{\ t}}$ Substituting: $t_{bot} = \dfrac{60x10^6(2)}{\pi 2^3 482.6x10^6} = 9.89mm$

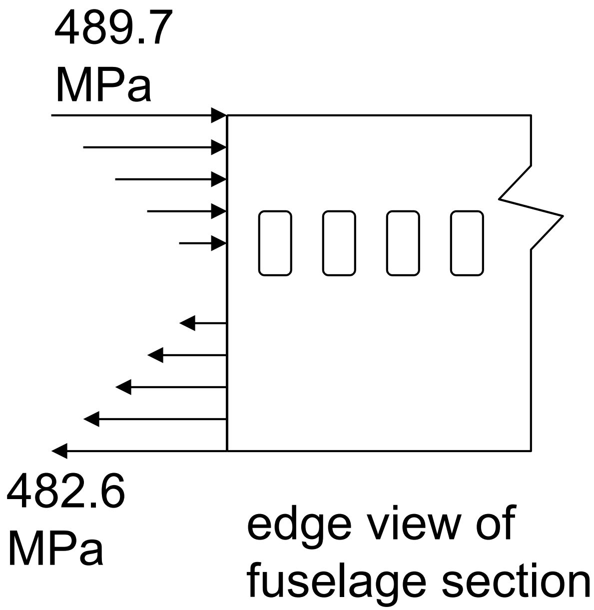
$$t_{top} = \frac{My_{\max}}{\pi R^3 \sigma_y^{\ c}} \qquad t_{top} = \frac{60x10^6(2)}{\pi 2^3 489.5x10^6} = 9.75mm$$

# Required thickness to meet M load




$t_{bot} = 9.89mm$

$t_{top} = 9.75mm$

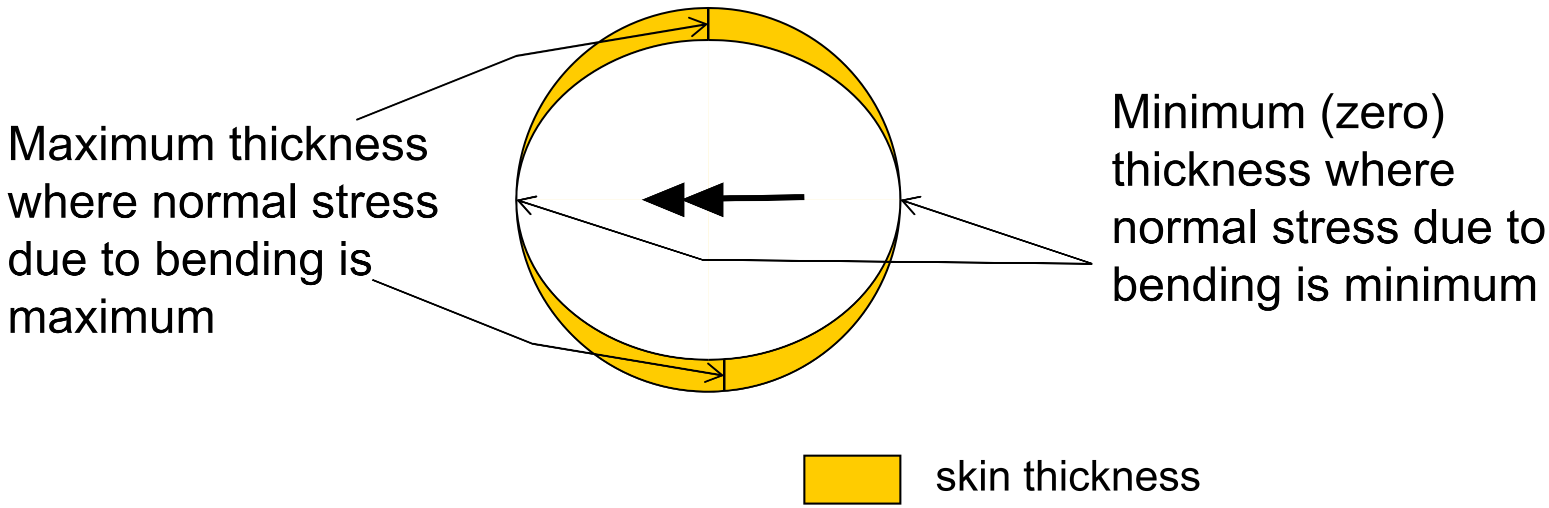



- setting the fuselage thickness equal to the highest of the two values gives the constant thickness solution with Mass/length:

$$\frac{M}{L} = \rho 2\pi R t_{\max} = 2774(2)\pi(2)(0.00989) = 344kg/m$$

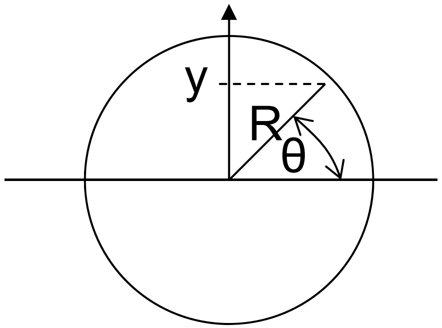
# Required thickness to meet M load

- But if the stress is not constant around the circum-ference, keeping a constant thickness is inefficient!
- The thickness can be reduced where the stress is lower
- As a first step, calculate the thickness at every location around the circumference that would just fail the fuselage skin at that location



# Required thickness- iteration 1

- use the bending stress equation with stress equal to the yield stress and back-solve for the skin thickness:




$$\sigma = \frac{My}{I} = \frac{MR\sin\theta}{\pi R^3 t} = \frac{M\sin\theta}{\pi R^2 t}$$

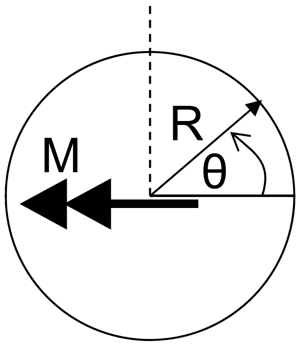
- from which:

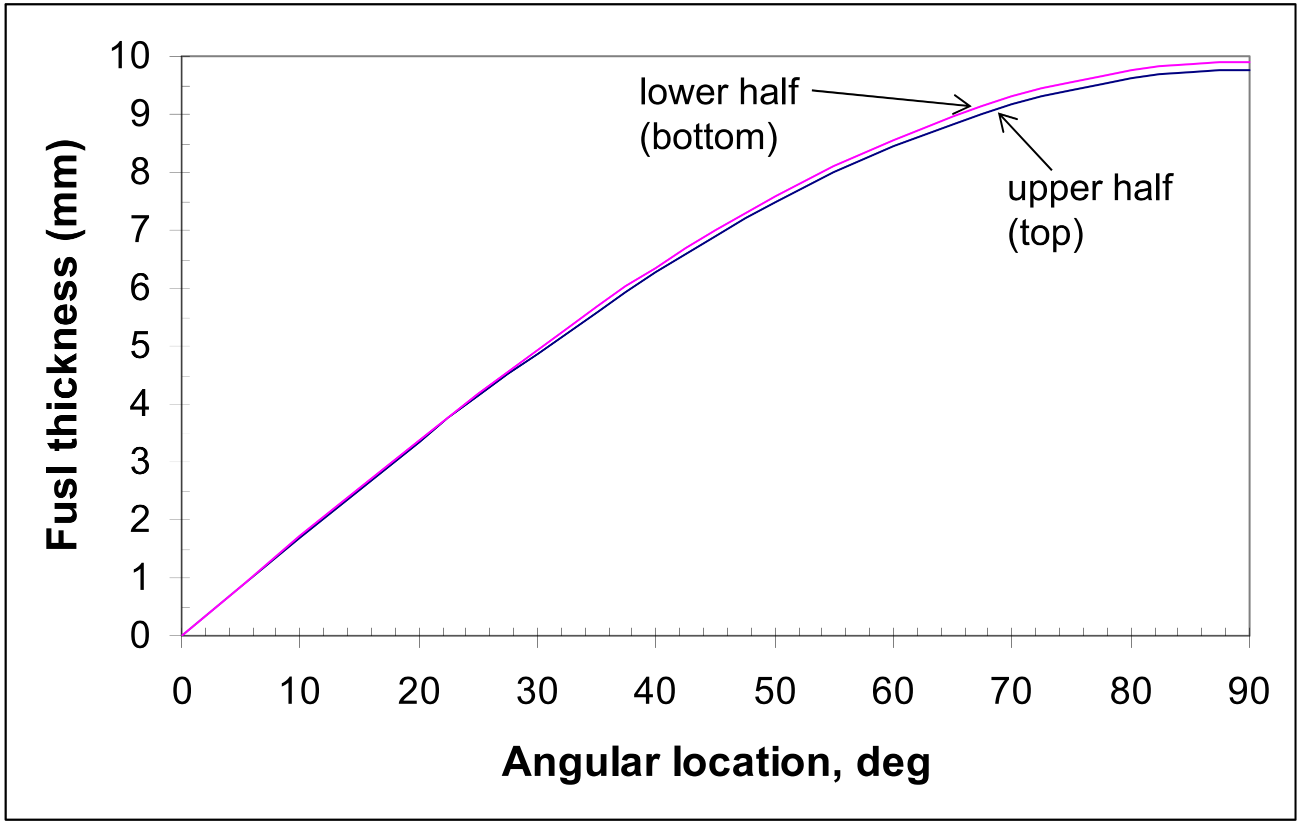
$$t = \frac{M\sin\theta}{\pi R^2 \sigma_y{}^c}$$ compression side

$$t = \frac{M\left|\sin\theta\right|}{\pi R^2 \sigma_y{}^t}$$ tension side

- substituting values gives:

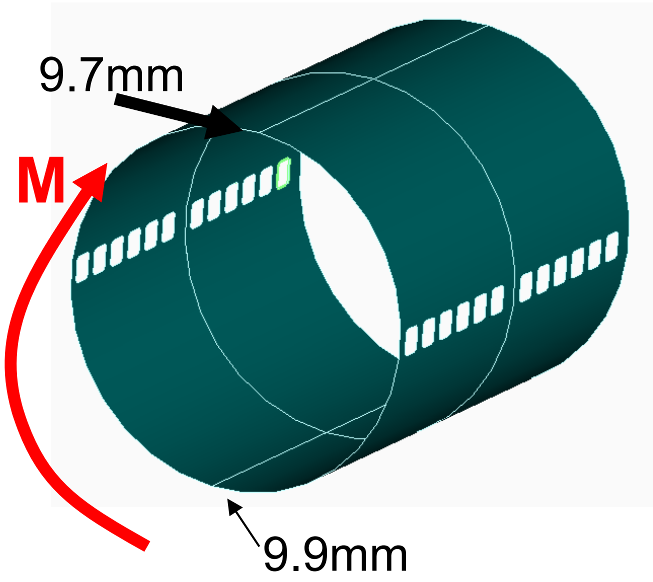
# Optimum thickness distribution – iteration 1




- note that the max thickness<10mm which implies that the thin-walled assumption is valid ($t^2$ terms and higher order terms are negligible)

# Optimum thickness distribution – iteration 2

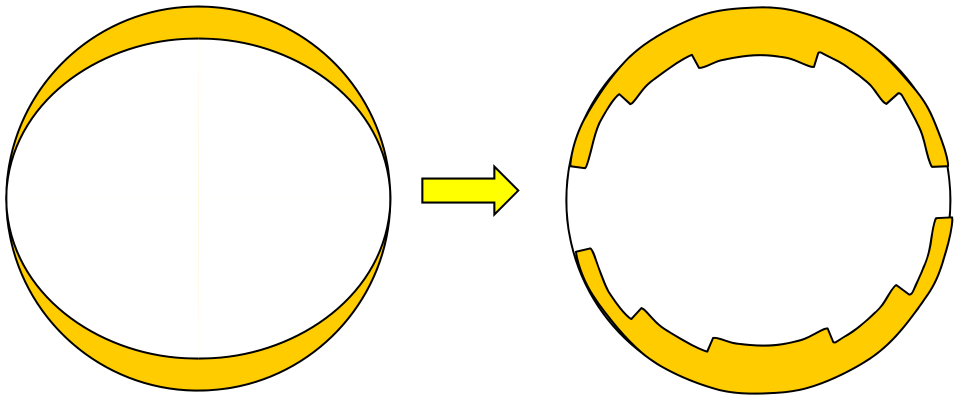



$$t_{bot} = 9.89mm$$

$$t_{top} = 9.75mm$$

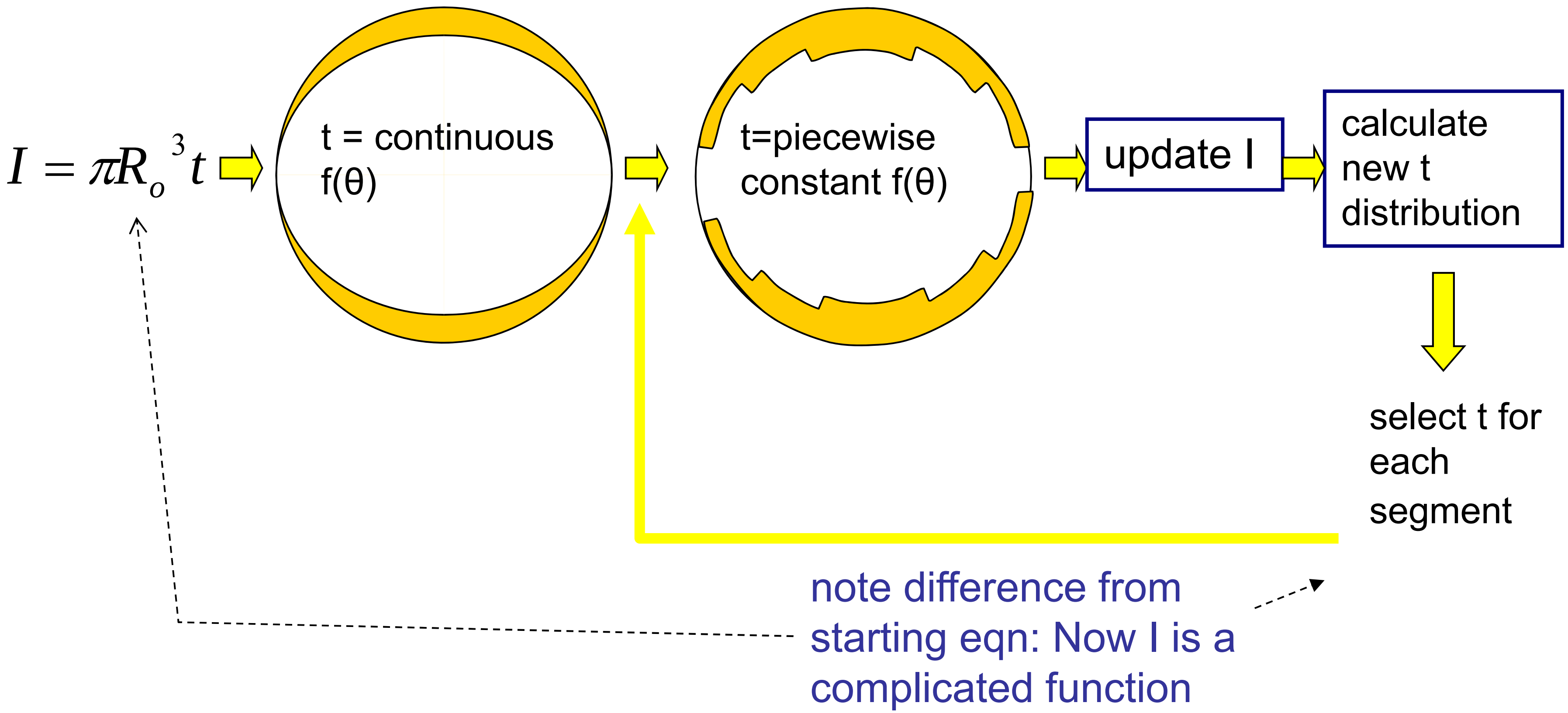
$$I = \pi R_o^{\ 3} t$$ approx.

- However, to determine I we assumed that the thickness was constant
- If the thickness varies as shown above, the expression for I used is no longer valid.
- Approximate continuous thickness distribution with a piecewise constant distribution to make the calculation of I easier
- Update I and repeat process until convergence



continuous piecewise linear

# Optimum thickness distribution - Iterations

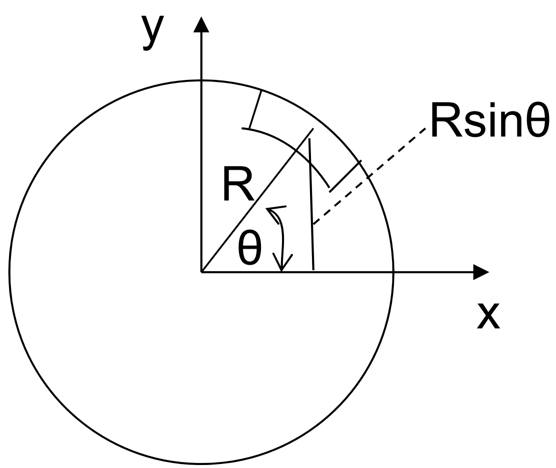



- iterations should lead to a converged value of I and distribution of t; note that the finer the segments of constant thickness are, the closer the design will be to the absolute minimum weight

# Optimum thickness distribution - Iterations

- The requirement at each iteration is that the bending stress My/I in each segment equals the yield stress of the material (use the lower yield stress of the two):

$$\boxed{\begin{aligned}\sigma_y^{\,c} &= \frac{MR\sin\theta}{I}\\ \sigma_y^{\,t} &= \frac{MR\left|\sin\theta\right|}{I}\end{aligned}}$$

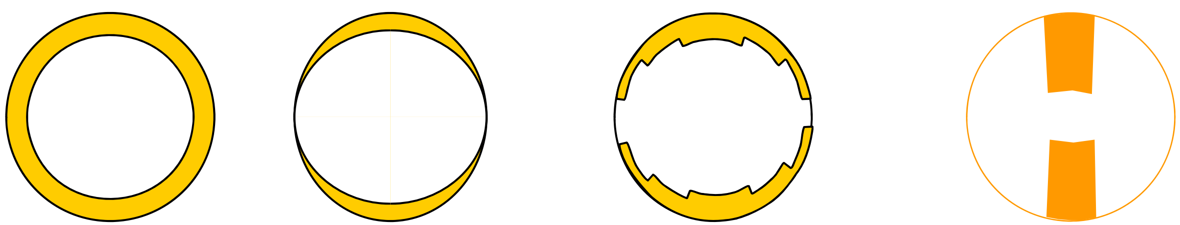


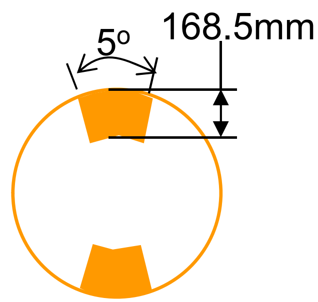

$$I = 4\int_0^{\pi/2} y^2 dA = 4\int_0^{\pi/2}\left(R\sin\theta\right)^2 t(\theta)Rd\theta \simeq 4\sum_{i=1} R^3\sin^2\theta_i t_i\Delta\theta$$

- Pick $\Delta\theta$ (e.g. $5^o$); find cg of segment i; find $\theta_i$; pick $t_i$
- Calculate I, check if local stress = yield stress; adjust $t_i$ and continue

# Optimum thickness distribution - Iterations

- The iterations remove material from the center of the beam and put more material at the top and bottom
- To avoid having zero thickness over most of the beam and infinite thickness at top and bottom, impose the condition that $t_i \geq 0.5$mm (thinner fuselage skin has handling problems, cannot be riveted, etc.)






const. t equal to max t of 9.9mm

variable t matching bending stress distribution

piecewise const t trying to make each segment fail at same time

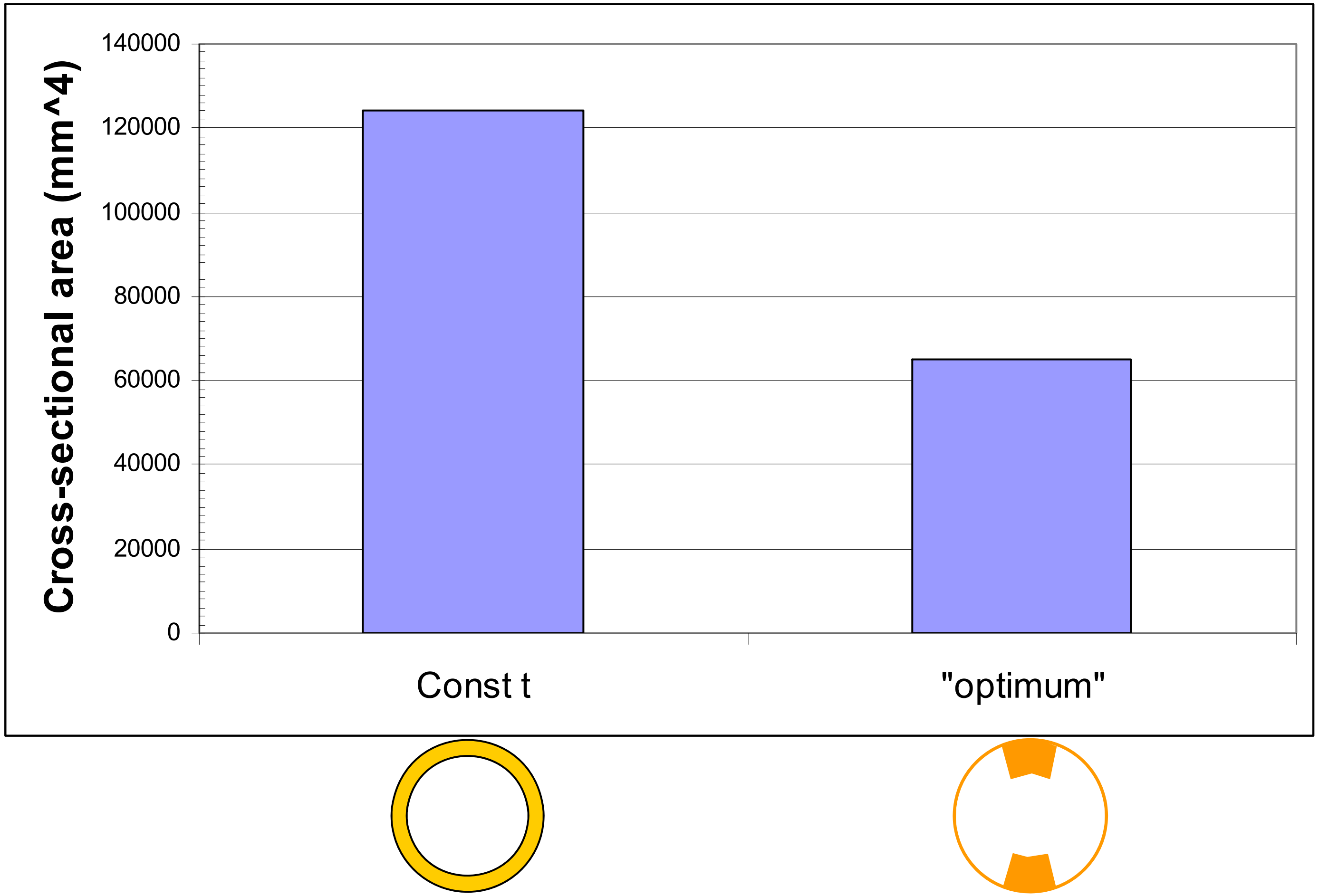
optimum solution => useless!!

compromise solution with 0.5 mm everywhere except top and bottom

not very good either

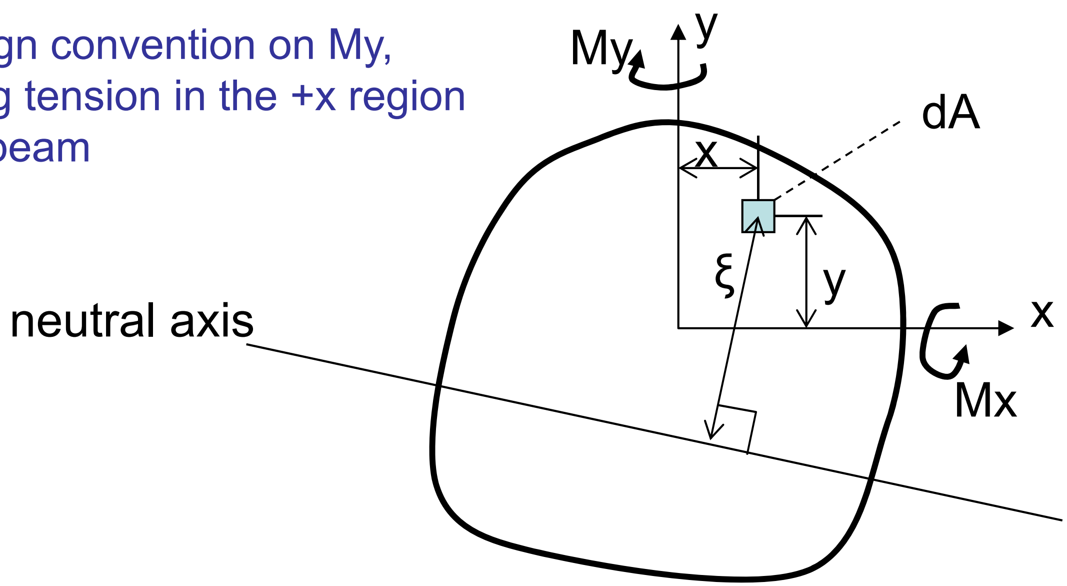
# Cross-sectional area

- Cross-sectional Area = weight/ρ per unit length=W/(ρL)



# Unsymmetrical bending

note sign convention on My, causing tension in the +x region of the beam

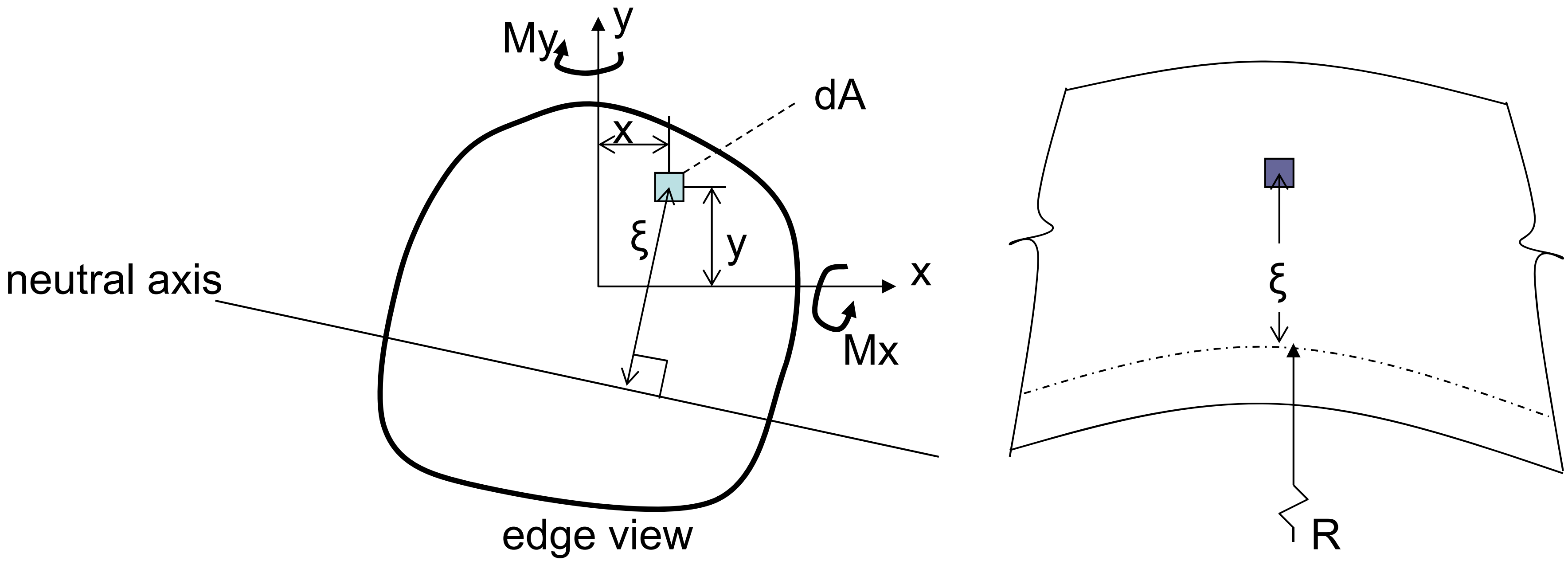



origin of coordinate system is located at centroid of the section

• random asymmetric cross-section under moments Mx and My

• the beam bends in space about some axis having compression on one side and tension on the other; so there is a “neutral axis” on which direct stress is zero

# Unsymmetrical bending strains and stresses

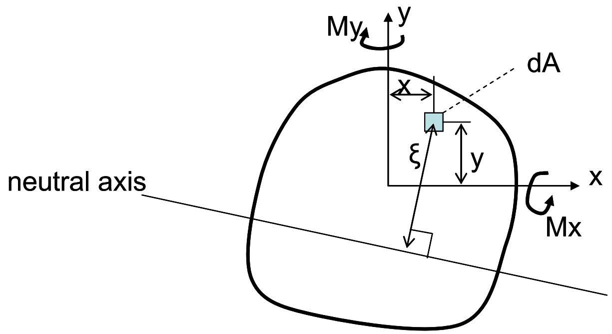



• because the origin is the centroid, the normal strain in the beam measured in that coordinate system is (as for symmetric bending)

$$\varepsilon_z = \frac{\xi}{R}$$

(notice no minus sign here because of the sign convention!)

# Unsymmetrical bending strains and stresses




$$\varepsilon_z = \frac{\xi}{R}$$

- and the corresponding normal stress is

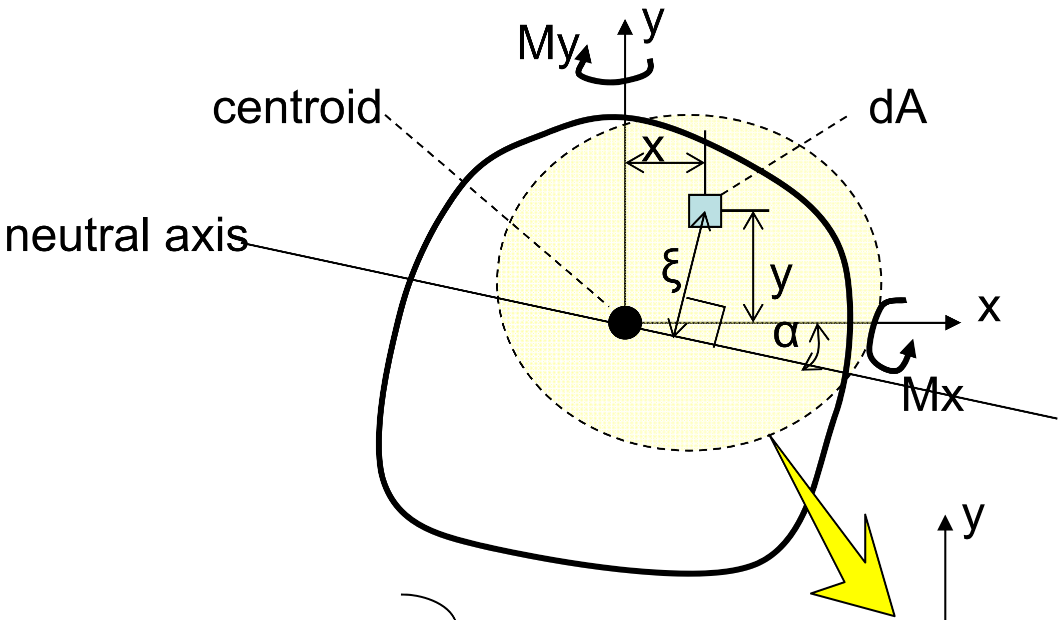
$$\sigma_z = E\frac{\xi}{R}$$

- again, because the beam is under pure bending, there is no net axial force so

$$\int_A \sigma_z dA = 0 \Rightarrow -\frac{E}{R}\int_A \xi dA = 0$$

- but this means that the moment of area of the beam cross-section about the neutral axis is zero => **the neutral axis passes through the centroid of the cross-section**

# Unsymmetrical bending strains and stresses




$\xi = AB + BC$

$AB = AD\cos\alpha = y\cos\alpha$

$BC = BE + EC$

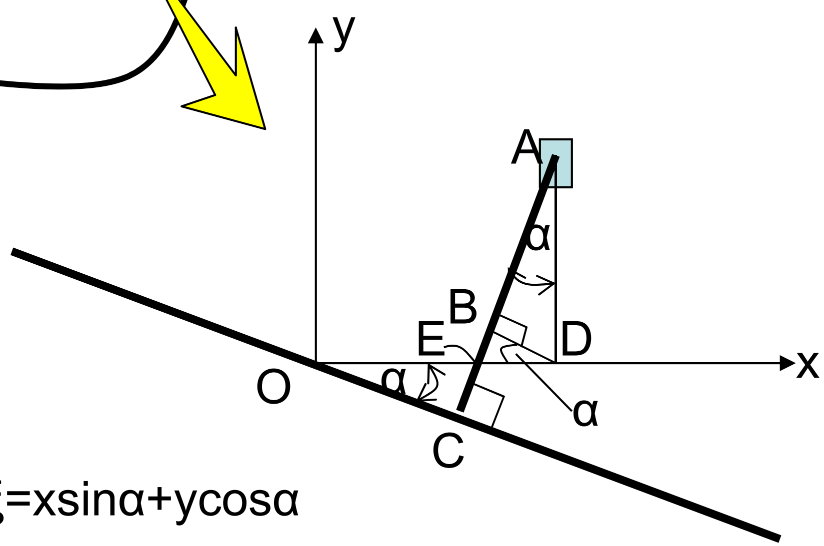
$BE = ED\sin\alpha$

$EC = OE\sin\alpha$

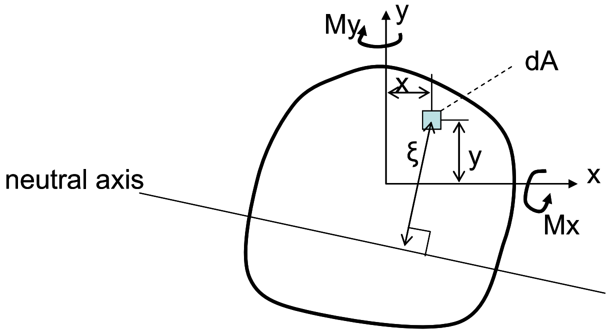
$BC = (ED + OE)\sin\alpha$

$ED + OE = x$

$\xi = x\sin\alpha + y\cos\alpha$



# Bending moments in unsymmetrical bending




ξ=xsinα+ycosα

$$\sigma_z = E\frac{\xi}{R}$$

- the normal stress is then given by:

$$\sigma_z = \frac{E}{R}\left(x\sin\alpha + y\cos\alpha\right)$$

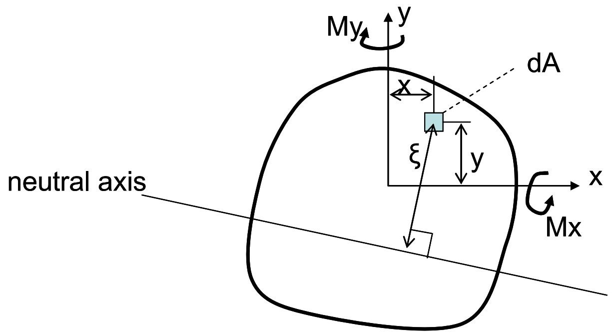
- this stress is caused by the moments Mx and My

$$M_x = \int_A \sigma_z y dA = \frac{E}{R}\left[\sin\alpha\int_A xydA + \cos\alpha\int_A y^2 dA\right]$$

- define $I_{xx} = \int_A y^2 dA$ $\quad I_{xy} = \int_A xydA$ Then:

$$M_x = \frac{E\sin\alpha}{R} I_{xy} + \frac{E\cos\alpha}{R} I_{xx}$$

# Bending moments in unsymmetrical bending




$$M_x = \frac{E\sin\alpha}{R} I_{xy} + \frac{E\cos\alpha}{R} I_{xx} \qquad (2.1)$$

- similarly,

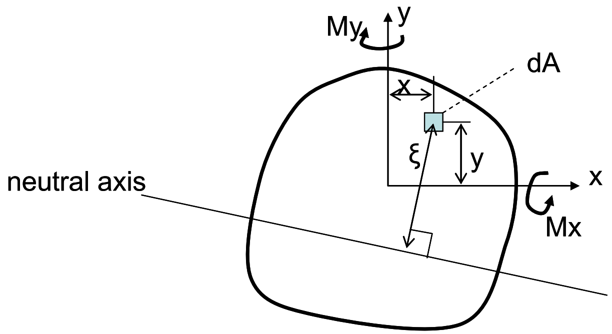
$$M_y = \frac{E\sin\alpha}{R} I_{yy} + \frac{E\cos\alpha}{R} I_{xy} \qquad \text{with} \qquad I_{yy} = \int_A x^2 dA \qquad (2.2)$$

- equations (2.1) and (2.2) form a system of two eqns in the two unknowns Esinα/R and Ecosα/R; Solving:

$$\frac{E\sin\alpha}{R} = \frac{1}{I_{xx}I_{yy} - I_{xy}^2}\left(-I_{xy}M_x + I_{xx}M_y\right)$$

$$\frac{E\cos\alpha}{R} = \frac{1}{I_{xx}I_{yy} - I_{xy}^2}\left(I_{yy}M_x - I_{xy}M_y\right) \qquad (2.3)$$

# Bending moments in unsymmetrical bending




$$\frac{E\sin\alpha}{R} = \frac{1}{I_{xx}I_{yy} - I_{xy}{}^{2}}\left(-I_{xy}M_x + I_{xx}M_y\right)$$

$$\frac{E\cos\alpha}{R} = \frac{1}{I_{xx}I_{yy} - I_{xy}{}^{2}}\left(I_{yy}M_x - I_{xy}M_y\right) \qquad (2.3)$$

$$\sigma_z = \frac{E}{R}\left(x\sin\alpha + y\cos\alpha\right) \qquad (2.4)$$

- using (2.3) to substitute in (2.4):

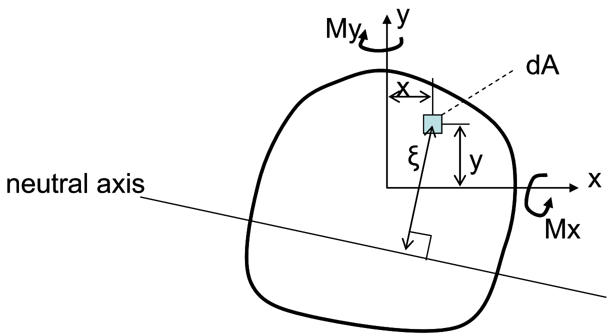
$$\boxed{\sigma_z = \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}{}^{2}}x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}{}^{2}}y} \qquad (2.5)$$

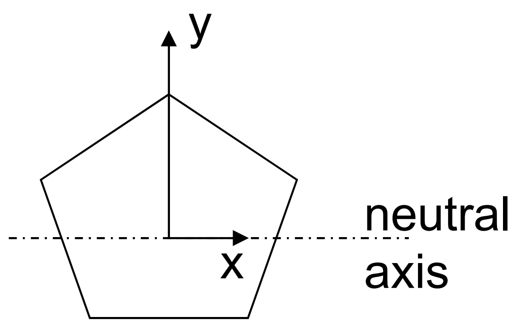
- so, given Mx and My, the direct bending stress $\sigma_z$ in a beam can be obtained from eq. (2.5)

# Unsymmetrical bending: implications of equation (2.5)

$$\sigma_z = \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2}y \tag{2.5}$$

- suppose one of the axes of the coordinate system is an axis of symmetry (e.g y axis)




- then the x integration in the expression for $I_{xy}$:

$I_{xy} = \int_A xydA$ is zero because: $\int_A xydA = \int y \underbrace{\left( \int_{lwr}^{upp} xdx \right)}dy \rightarrow = \left[ \frac{x^2}{2} \right]_{lwr\ lim}^{uppr\ lim} = 0$

because upper and lower limits (although function of y) are equal due to symmetry

# Unsymmetrical bending: implications of equation (2.5)

$$\sigma_z = \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2}y \tag{2.5}$$

• therefore, if one (or both) axes of the coordinate system with origin at the centroid are axes of symmetry, $I_{xy}=0$

• and as a result,

$$\boxed{\sigma_z = \frac{M_y x}{I_{yy}} + \frac{M_x y}{I_{xx}}} \tag{2.6}$$

• if either of the moments $M_x$ or $M_y$ are zero, we recover the basic beam bending equation from before:

$$\begin{aligned} \sigma_z &= \frac{M_y x}{I_{yy}} \quad for \quad M_x = 0 \\ \sigma_z &= \frac{M_x y}{I_{xx}} \quad for \quad M_y = 0 \end{aligned} \tag{2.7}$$

# Unsymmetrical bending: implications of equation (2.5)

$$\sigma_z = \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2}y \qquad (2.5)$$

- In the general case where $I_{xy} \neq 0$, if only $M_x$ (or only $M_y$) is applied, the direct stress is a function of <u>both</u> x and y

$$\sigma_z = \frac{-I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{I_{yy}M_x}{I_{xx}I_{yy} - I_{xy}^2}y \quad for \quad M_y = 0$$

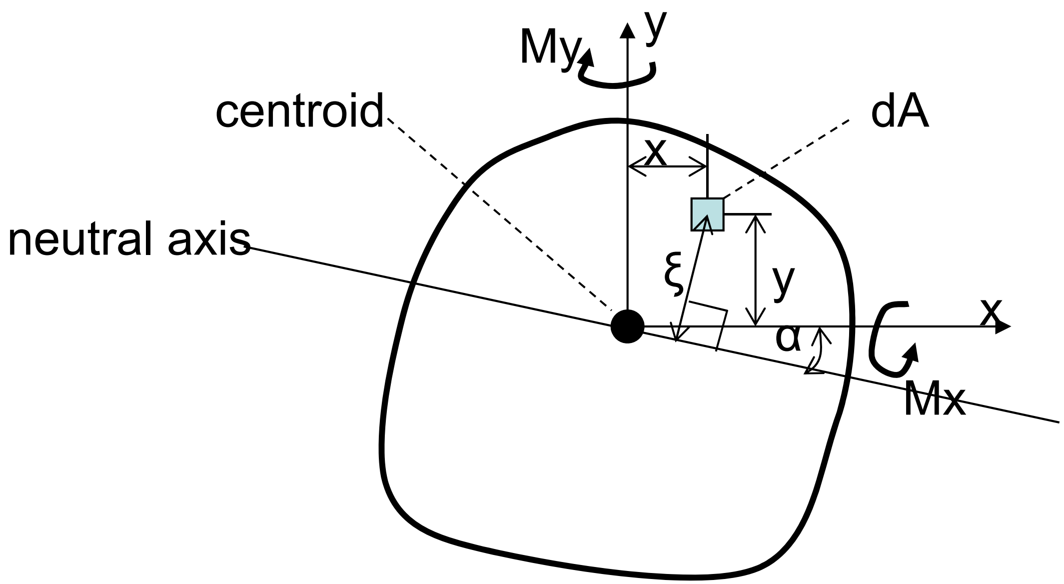
$$\sigma_z = \frac{I_{xx}M_y}{I_{xx}I_{yy} - I_{xy}^2}x + \frac{-I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2}y \quad for \quad M_x = 0$$

or:

$$\sigma_z = \frac{\left(-I_{xy}x + I_{yy}y\right)M_x}{I_{xx}I_{yy} - I_{xy}^2} \quad for \quad M_y = 0$$

$$\sigma_z = \frac{\left(I_{xx}x - I_{xy}y\right)M_y}{I_{xx}I_{yy} - I_{xy}^2} \quad for \quad M_x = 0 \qquad (2.8)$$

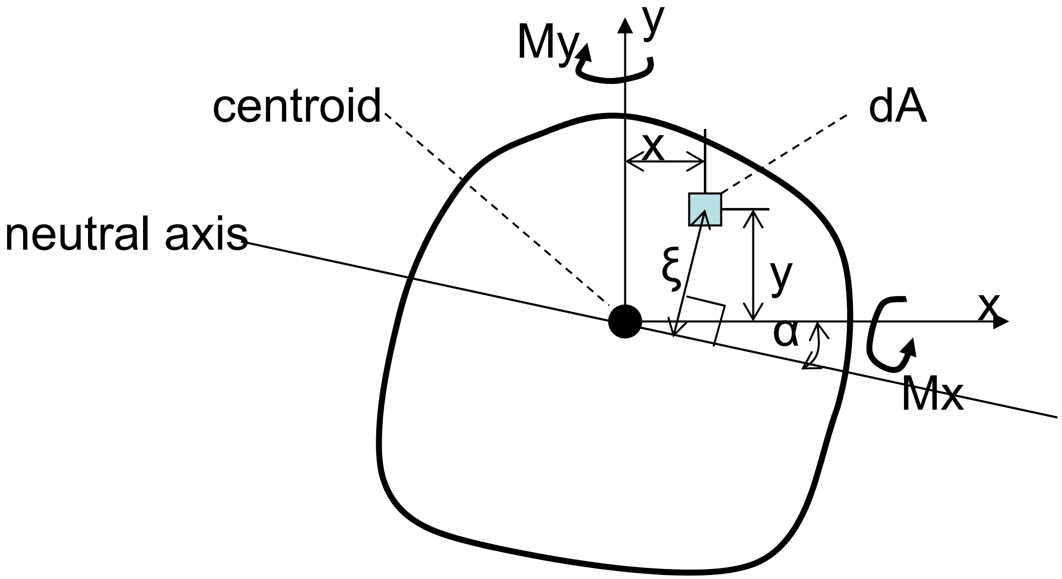
# Location of neutral axis




• as shown earlier, the neutral axis always passes through the centroid of the beam cross-section

• its orientation (angle α) is a function of applied loading and beam geometry

# Location of neutral axis



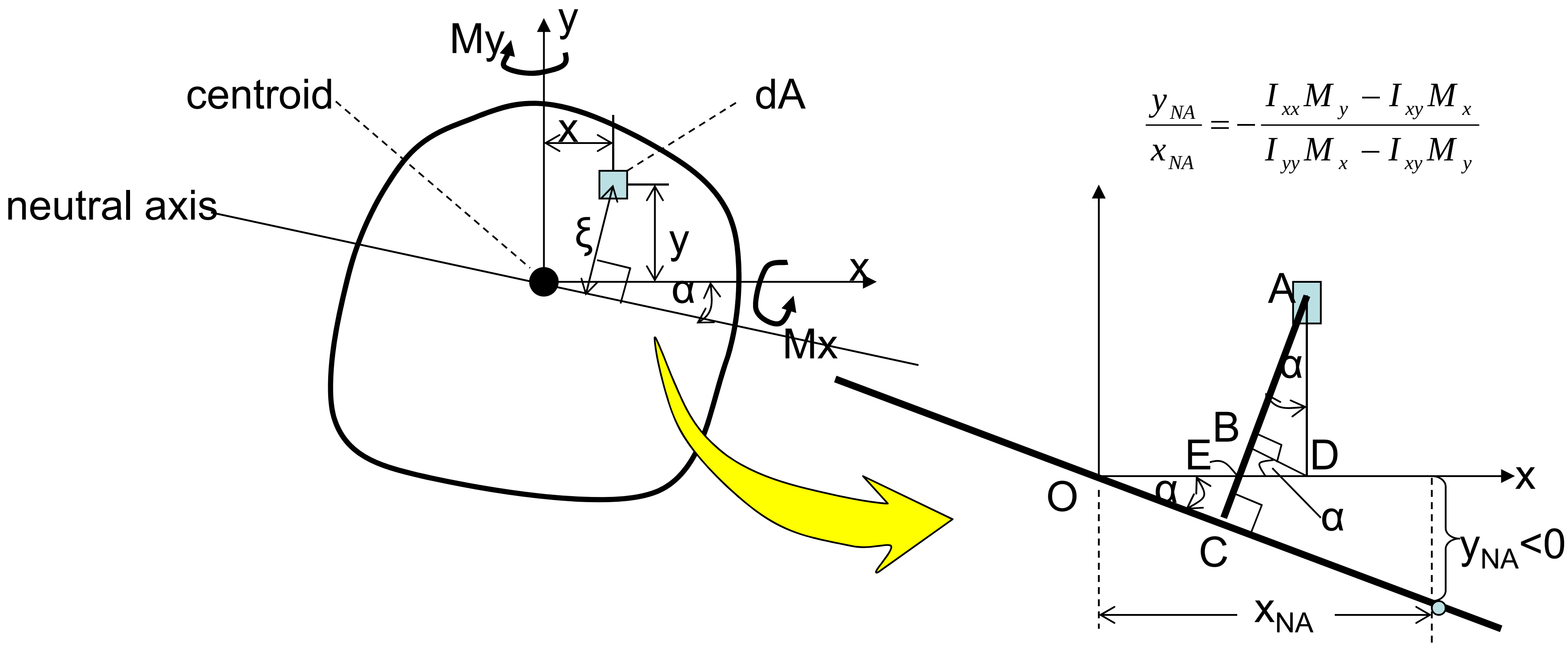

• as shown earlier, the direct stress $\sigma_z$ is zero at the neutral axis; using eq. (2.5):

$$0 = \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}^2} x_{NA} + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}^2} y_{NA} \qquad (2.5)$$

• from which we can solve for $y_{NA}/x_{NA}$:

$$\frac{y_{NA}}{x_{NA}} = -\frac{I_{xx}M_y - I_{xy}M_x}{I_{yy}M_x - I_{xy}M_y}$$

# Location of neutral axis



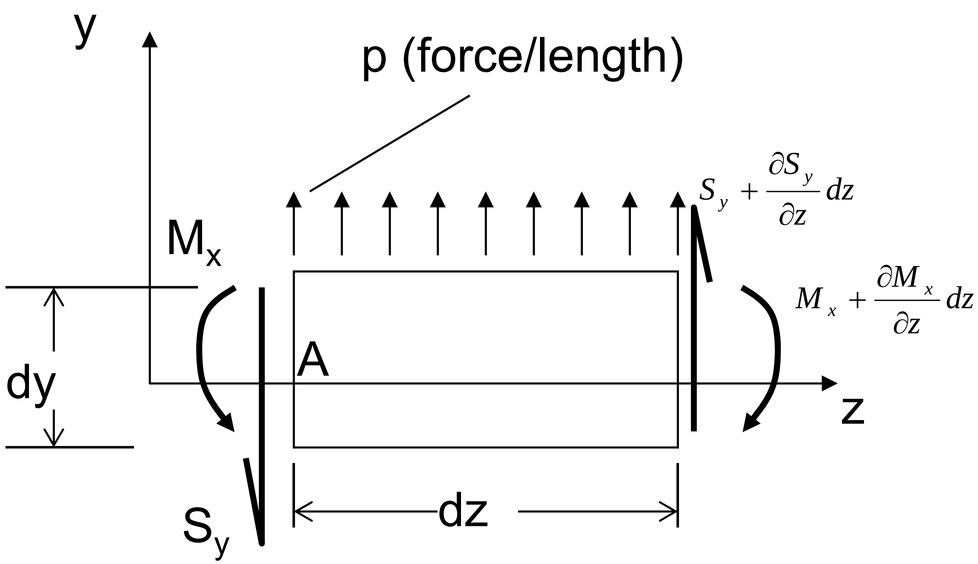

$$\frac{y_{NA}}{x_{NA}} = -\frac{I_{xx}M_y - I_{xy}M_x}{I_{yy}M_x - I_{xy}M_y}$$

- but $y_{NA}/x_{NA} = -\tan\alpha$ ($\alpha$ is defined positive clockwise so when it is positive x and y are of opposite sign); So:

$$\tan\alpha = \frac{I_{xx}M_y - I_{xy}M_x}{I_{yy}M_x - I_{xy}M_y} \qquad (2.9)$$

defines the orientation of the neutral axis

# Load intensity, shear, moment relations

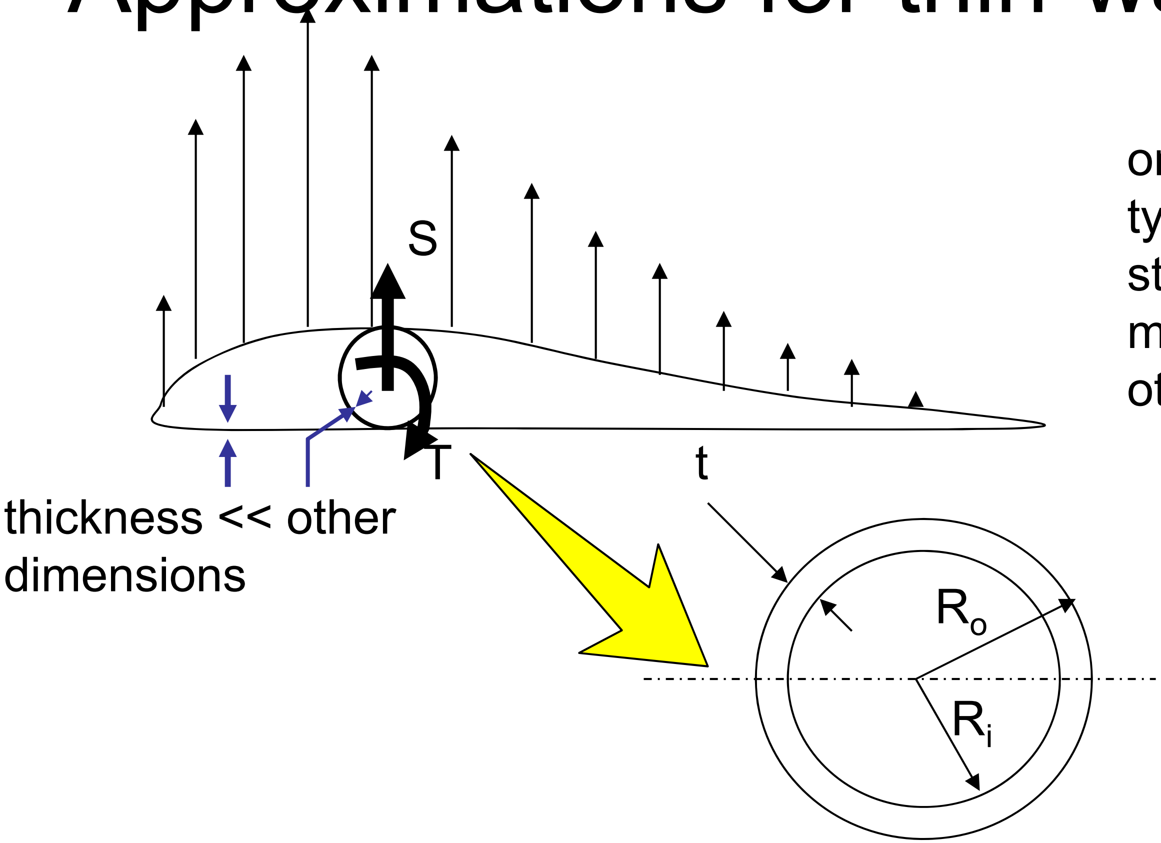



- force equilibrium in y direction (and moment equilibrium):

$$S_y + \frac{\partial S_y}{\partial z} dz - S_y + p dz = 0 \Rightarrow \boxed{p = -\frac{\partial S_y}{\partial z}}$$

$$-\left(M_x + \frac{\partial M_x}{\partial z} dz\right) + M_x + \left(S_y + \frac{\partial S_y}{\partial z} dz\right) dz + p dz \frac{dz}{2} = 0 \Rightarrow \boxed{S_y = \frac{\partial M_x}{\partial z}} \qquad (2.10)$$

# Approximations for thin-walled sections




one of the dimensions in typical aerospace structures is, usually, much smaller than the other dimensions

$$\sigma_z = -\frac{My}{I}$$

- recall the approximation(s) we made in calculating the moment of inertia of the tubular spar:

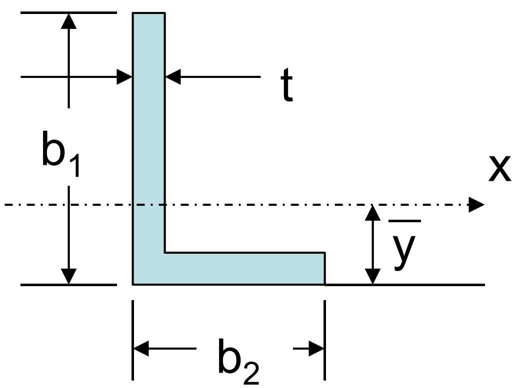
$$I = \frac{\pi}{4}\left(R_o^{\ 4} - R_i^{\ 4}\right) = \frac{\pi}{4}\left(R_o^{\ 4} - \left(R_o - t\right)^4\right) = \frac{\pi}{4}\left(\cancel{R_o^{\ 4}} - \left(\cancel{R_o^{\ 4}} - 4R_o^{\ 3}t + \cancel{6R_o^{\ 2}t^2} - \cancel{4R_o t^3} + \cancel{t^4}\right)\right)$$

# Approximations for thin-walled sections

(revisited in a future lecture)

- in general, second and third powers of thickness are neglected for thin-walled sections

- for calculating the moment of inertia $I_{xx}$ of the following section:




$$I_{xx} = \underbrace{\frac{tb_1^3}{12} + tb_1\left(\frac{b_1}{2} - \bar{y}\right)^2}_{\text{web}} + \underbrace{\frac{(b_2 - t)t^3}{12} + (b_2 - \overset{t}{\downarrow}{t})\left(\bar{y} - \frac{t}{2}\right)^2}_{\text{flange}}$$

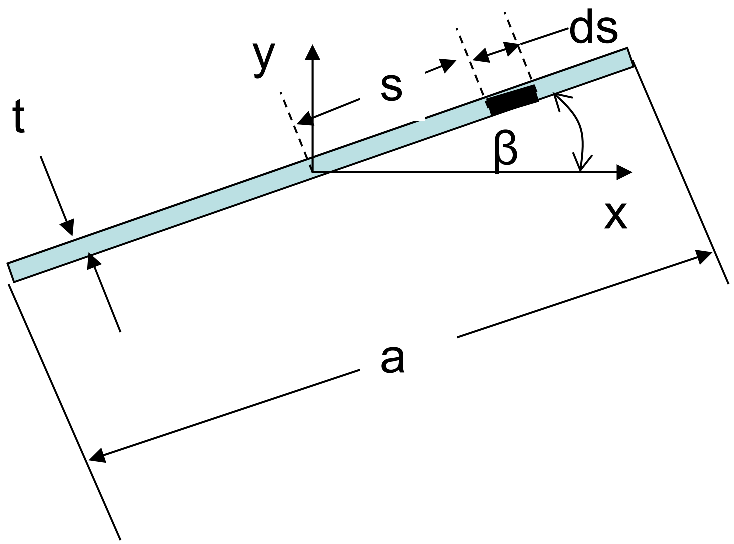
is approximated by

$$I_{xx} = \frac{tb_1^3}{12} + tb_1\left(\frac{b_1}{2} - \bar{y}\right)^2 + b_2 t\,\bar{y}^2$$

linear in t!

- it is really up to the analyst to decide when the approximation is accurate enough

# Thin-walled sections at an angle




$$I_{xx} = \int_A y^2 dA$$

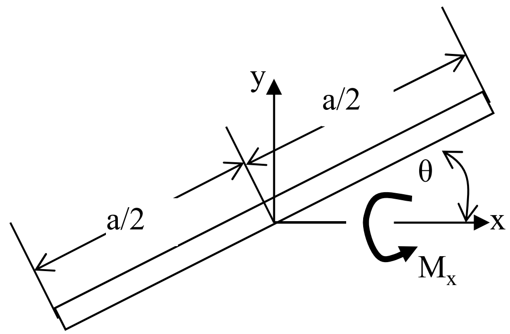
$$dA = dxdy = tds$$

$$y = s \sin \beta$$

Substituting:

$$I_{xx} = \int_{-a/2}^{a/2} ts^2 \sin^2 \beta ds = \frac{ts^3}{3}\Big|_{-a/2}^{a/2} \sin^2 \beta = \frac{ta^3 \sin^2 \beta}{12}$$

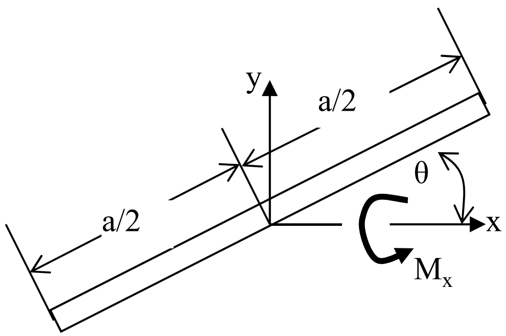
# Example: Unsymmetrical bending




Beam inclined at an angle θ to the x axis. Applied moment $M_x$. Determine the "worst" possible value of θ i.e. the value leading to the highest (or lowest) possible stresses in the beam

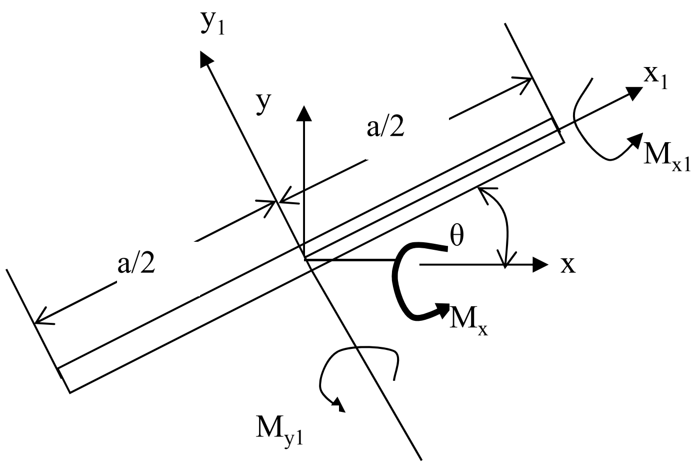
# Example: Unsymmetrical bending

• **important**: the x-axis is NOT the neutral axis. So simply calculating $I_{xx}$, $I_{yy}$, and $I_{xy}$ about the xy coordinate system is not correct. The beam will actually bend about its neutral axis which has some unknown orientation




• one can either determine the neutral axis and proceed from there or, which takes less calculations, resolve the applied moment $M_x$ to two components about the beam axes:

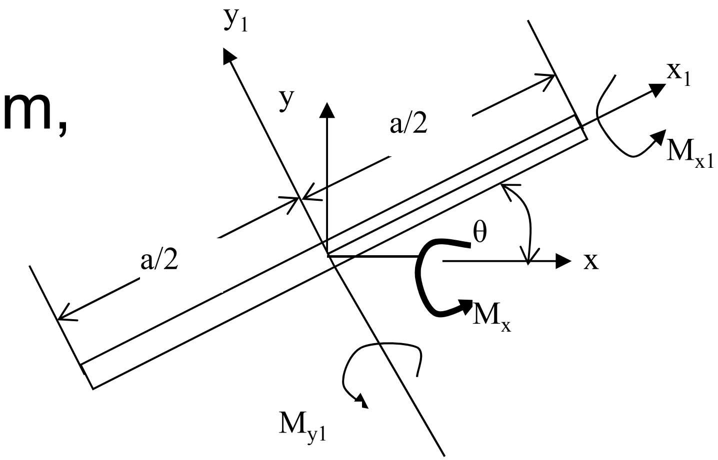
# Example: Unsymmetrical bending




$$M_{x1} = M_x \cos\theta$$
$$M_{y1} = M_x \sin\theta$$

note that for θ between 0 and 90 degrees, $M_{x1}$ and $M_{y1}$ are both positive according to the sign convention (a positive moment causes tension in the outer fibers lying on the positive half-plane)

# Example: Determination of moments of inertia and direct stress




- in the new coordinate system,

$$I_{x1x1} = \frac{at^3}{12}$$

$$I_{y1y1} = \frac{ta^3}{12}$$

$$I_{x1y1} = 0$$

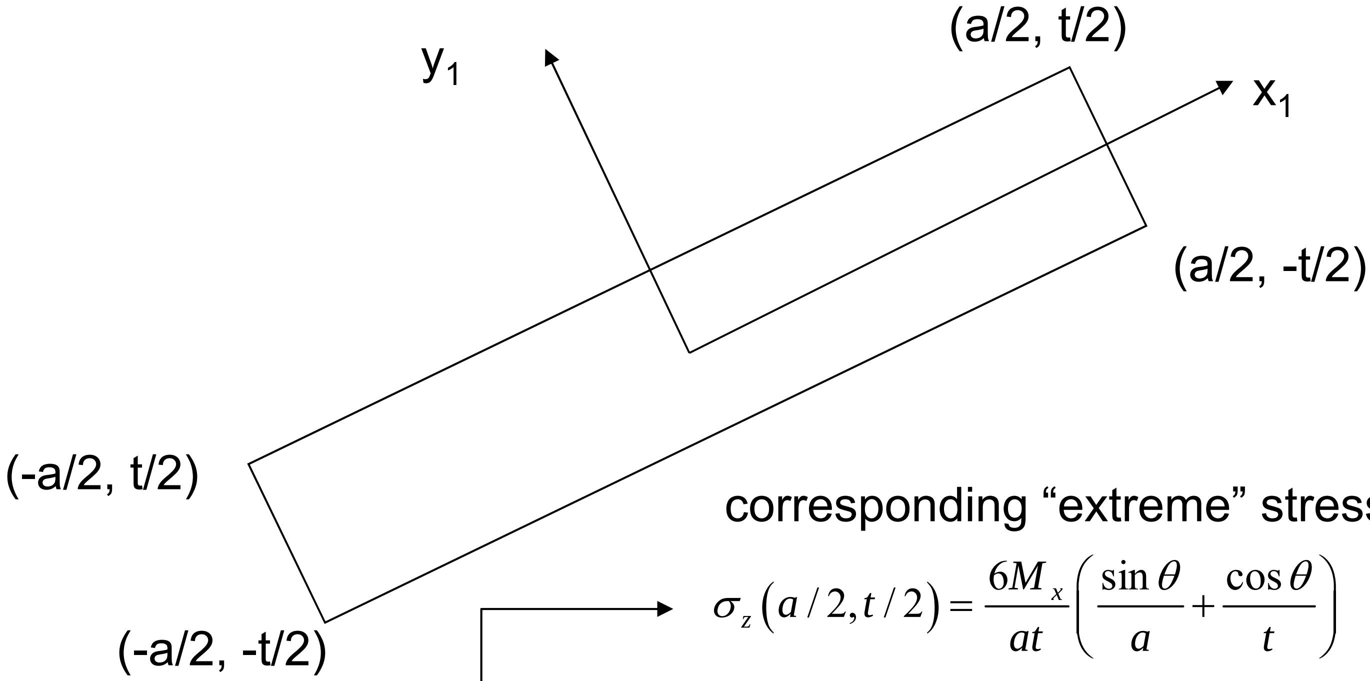
- substituting in (2.5)

$$\sigma_z = \frac{I_{xx}M_y - I_{xy}M_x}{I_{xx}I_{yy} - I_{xy}} x + \frac{I_{yy}M_x - I_{xy}M_y}{I_{xx}I_{yy} - I_{xy}} y$$

- leads to

$$\sigma_z = \frac{M_{y1}}{I_{y1y1}} x_1 + \frac{M_{x1}}{I_{x1x1}} y_1 = \frac{12M_x}{at}\left( \frac{x_1 \sin\theta}{a^2} + \frac{y_1 \cos\theta}{t^2} \right)$$

# Example: Critical direct stresses




corresponding “extreme” stresses

$$\sigma_z\left(a/2, t/2\right) = \frac{6M_x}{at}\left(\frac{\sin\theta}{a} + \frac{\cos\theta}{t}\right)$$

$$\sigma_z\left(a/2, -t/2\right) = \frac{6M_x}{at}\left(\frac{\sin\theta}{a} - \frac{\cos\theta}{t}\right)$$

opposite sign

$$\sigma_z\left(-a/2, t/2\right) = \frac{6M_x}{at}\left(-\frac{\sin\theta}{a} + \frac{\cos\theta}{t}\right)$$

$$\sigma_z\left(-a/2, -t/2\right) = \frac{6M_x}{at}\left(-\frac{\sin\theta}{a} - \frac{\cos\theta}{t}\right)$$

# Example: Critical direct stresses

$$\sigma_z(a/2, t/2) = \frac{6M_x}{at}\left(\frac{\sin\theta}{a} + \frac{\cos\theta}{t}\right)$$

$$\sigma_z(a/2, -t/2) = \frac{6M_x}{at}\left(\frac{\sin\theta}{a} - \frac{\cos\theta}{t}\right)$$

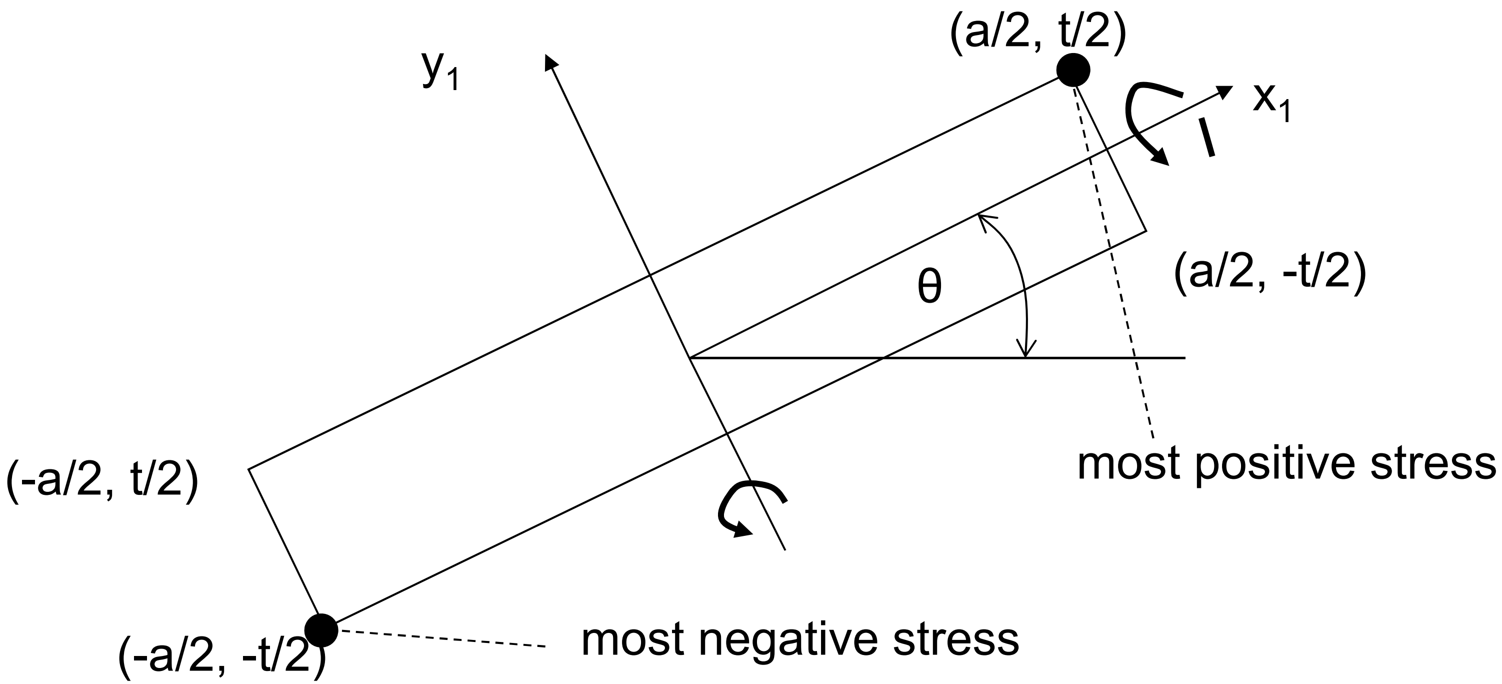
$$\sigma_z(-a/2, t/2) = \frac{6M_x}{at}\left(-\frac{\sin\theta}{a} + \frac{\cos\theta}{t}\right)$$

$$\sigma_z(-a/2, -t/2) = \frac{6M_x}{at}\left(-\frac{\sin\theta}{a} - \frac{\cos\theta}{t}\right)$$

- for $0 \leq \theta \leq 90$, sine and cosine are positive; so most positive stress is the first and most negative is the fourth (which is the negative of the first)

- to find critical $\theta$ suffices to find which $\theta$ maximizes the first or minimizes the last

# Example: Critical direct stresses




- therefore, for critical θ set

$$\frac{d}{d\theta}[\sigma_z]_{\max} = 0$$
$$or$$
$$\frac{d}{d\theta}[\sigma_z]_{\min} = 0$$

# Example: Critical direct stresses

$$\sigma_z(a/2, t/2) = \frac{6M_x}{at}\left(\frac{\sin\theta}{a} + \frac{\cos\theta}{t}\right)$$

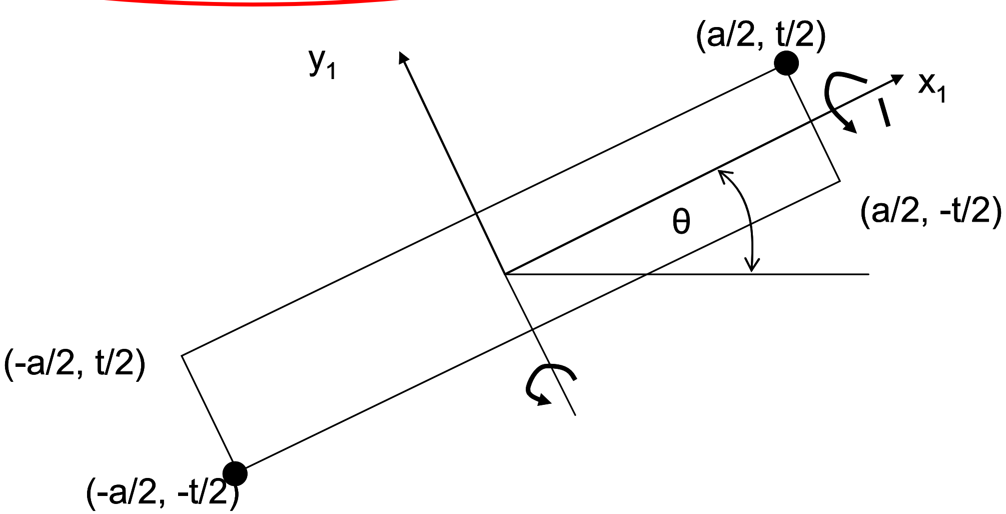
$$\sigma_z(a/2, -t/2) = \frac{6M_x}{at}\left(\frac{\sin\theta}{a} - \frac{\cos\theta}{t}\right)$$

$$\sigma_z(-a/2, t/2) = \frac{6M_x}{at}\left(-\frac{\sin\theta}{a} + \frac{\cos\theta}{t}\right)$$

$$\sigma_z(-a/2, -t/2) = \frac{6M_x}{at}\left(-\frac{\sin\theta}{a} - \frac{\cos\theta}{t}\right)$$

$$\frac{d}{d\theta}(\sigma_z)_{\max} = 0 \Rightarrow \frac{\cos\theta}{a} - \frac{\sin\theta}{t} = 0$$

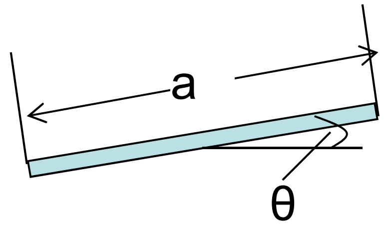
$$\tan\theta = \frac{t}{a}$$



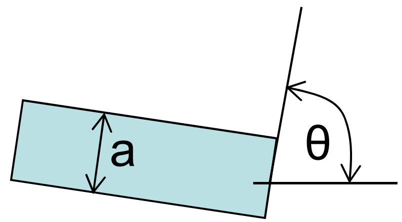
# Example: Implications of result

$$\tan\theta = \frac{t}{a}$$

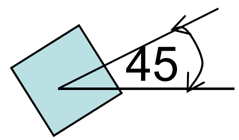
- if t<<a, $\theta \rightarrow 0$ (almost horizontal)




- if t>>a, $\theta \rightarrow 90^{o}$ (almost vertical)




- if t=a (square cross-section), $\theta = 45^{o}$




Note that the first two cases are essentially the same and they agree with intuition (“thin” beams bending about the long axis have high bending stresses)