Quantum Fundamentals: Single and Multiple Qubit Systems

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Outline I

The Qubit State & The Necessity of Complex Numbers

Premise: Quantum States Have Wave-like Properties

As established, quantum systems behave like waves. A key property of any wave is its **phase**. Therefore, our mathematical description of a quantum state must be able to account for this phase.

The General State of a Qubit

The most general way to write the state of a single qubit, $|\psi\rangle$, is:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

- The real parts control the **probability balance** between $|0\rangle$ and $|1\rangle$.

Why Complex Numbers are Essential

The Nature of the Phase Factor (Euler's Formula)

The phase factor is defined by Euler's formula, which is inherently complex:

$$e^{i\phi} = \cos(\phi) + i\sin(\phi)$$

Conclusion: To fully describe a qubit's state, we must include its relative phase ($e^{i\phi}$). Since this phase factor is fundamentally a complex number (containing i), the entire framework for describing quantum states **must use complex numbers**.

Combining Hilbert Spaces: The Tensor Product

How do we mathematically describe a two-qubit system? We combine their spaces using the **tensor product** ($\otimes$).

Rule: Forming the New Basis

The basis for the combined system is formed by the tensor product of **every** basis vector from each individual space.

- Qubit A basis: $\{|0\rangle_A, |1\rangle_A\}$
- Qubit B basis: $\{|0\rangle_B, |1\rangle_B\}$

The new 4D basis for the combined system is:

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

Tensor Product: Vector Representation

Example: Kronecker Product

The tensor product of column vectors is the **Kronecker product**:

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

This is why the Hilbert space dimensions multiply: $2 \times 2 = 4$.

Superposition: The Fundamental Rule

The "Rule of the Game"

The principle of superposition is a direct consequence of the linearity of the Schrödinger equation in Hilbert space.

- **Core Idea:** If a system can be in state $|A\rangle$ and also in state $|B\rangle$, then any linear combination $|\psi\rangle = \alpha |A\rangle + \beta |B\rangle$ is also a valid state.
- **Wave Analogy:** Multiple individual waves adding up to form a single composite wave, allowing for interference.

Superposition: Visual Example

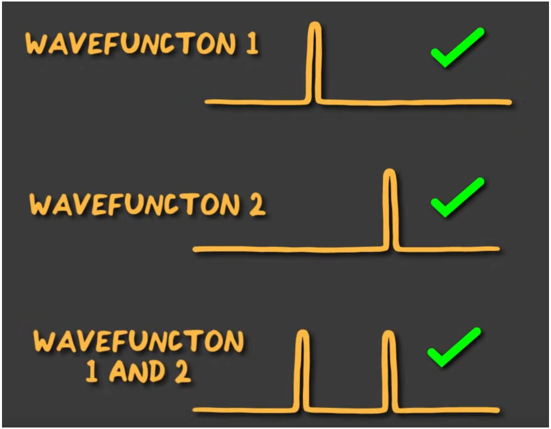

Figure: Combination of 2 wave functions

Entanglement: The Essence

What is an Entangled State?

An entangled state is a shared state for a multi-part system that **cannot be separated** into individual states for its components.

- **Separable State:** $|01\rangle = |0\rangle \otimes |1\rangle$.
  (We can describe each qubit individually).
- **Entangled State:** $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
  (The state of one qubit is undefined without the other).

Entanglement vs. Our Classical Worldview

Our Intuition: Local Realism

For centuries, physics was built on two intuitive assumptions:

1. **Realism:** Objects have definite, pre-existing properties, independent of observation.
   (A hidden coin is already heads or tails; we just don't know which.)
2. **Locality:** An object is only influenced by its immediate surroundings. No influence can travel faster than light.
   (Flipping a coin here cannot instantly flip a coin on the Moon.)

The "Spooky" Challenge

Entanglement seems to violate this. Measuring one particle appears to instantly determine the state of its distant partner.

The Verdict: Bell's Theorem

The Decisive Test

- In 1964, John Bell devised a mathematical test. He proved that if local realism were true, the correlations between entangled particles would have a strict limit.
- **Experiments have decisively and repeatedly shown that this limit is violated.**

Conclusion: Reality is Weirder Than We Think

The universe does **not** obey Local Realism. We must abandon at least one of our core classical assumptions about reality.
Bell's Inequality:
`https://www.youtube.com/watch?v=9OM0jSTeeBg`

Making Sense of It: The Many-Worlds Interpretation

One Possible "Story": The Branching Universe

This is a mathematically elegant and increasingly popular interpretation to explain the strangeness of entanglement.

- **Core Idea:** The wave function is real and never collapses. Instead, the universe **branches** for each possible measurement outcome.
- **How it explains entanglement:**
  - When you measure your entangled qubit, the universe splits.
  - In one branch, you get 0 and your distant partner also gets 0.
  - In another branch, another "you" gets 1 and your partner gets 1.
  - The perfect correlation is preserved across corresponding branches. The "spooky action" is the branching of reality itself.

Single-Qubit Gates: Unitary Operations

- **Overview:** Gates are 2x2 unitary matrices acting on qubit state: $|\psi'\rangle = U|\psi\rangle$, with $U^\dagger U = I$.

Pauli Gates

- **Pauli-X (NOT/bit-flip):**
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
Flips $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$.
- **Pauli-Z (phase flip):**
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
Leaves $|0\rangle$ unchanged, flips phase of $|1\rangle$ to $-|1\rangle$.
- **Pauli-Y (bit + phase flip):**
$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
Combines effects of X and Z.

Hadamard (H) Gate

- **Superposition Creator:**

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- **Examples:**
  - $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$ (equal probabilities for 0 and 1).
  - $H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$.

Phase Gates

- **General Phase Gate (P_θ):**

$$P_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}$$

  Leaves $|0\rangle$ unchanged, applies a phase $e^{i\theta}$ to $|1\rangle$.
- **S Gate:** A specific phase gate where $\theta = \frac{\pi}{2}$.

$$S = P_{\frac{\pi}{2}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

- **T Gate:** A specific phase gate where $\theta = \frac{\pi}{4}$.

$$T = P_{\frac{\pi}{4}} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i}{\sqrt{2}} \end{pmatrix}$$

Rotation Gates: Why We Need Them

Continuous Control

- Unlike Pauli gates (fixed 0/1 flips or 90/180-degree phase shifts), rotation gates allow **continuous control** over the qubit's state.
- Essential for fine-tuning quantum states and implementing algorithms that require parameter optimization.

Visualization

All single-qubit rotations can be visualized as rotating the state vector on the the **Bloch Sphere**.

Interactive Bloch Sphere Visualizer

Rotation Gates: X and Y Rotations

- **Rotation around X-axis (RX(θ)):**

$$R_x(\theta) = \begin{pmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

- **Rotation around Y-axis (RY(θ)):**

$$R_y(\theta) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$$

Rotation Gates: Z Rotation

- **Rotation around Z-axis (RZ(θ)):**

$$R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

- Rotates the state vector around the Z-axis by an angle θ.
- This gate primarily affects the phase of the $|1\rangle$ component.

Single-Qubit Calculations: Matrix Multiplication

- **Goal:** Apply X, then H on $|0\rangle$, compute via matrices.
- **Step 1 (X Gate):** $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $X|0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$.
- **Step 2 (H Gate):** $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$,

$$H|1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle.$$

Single-Qubit Calculations: Combined Operation

- **Combined Matrix:** To get the final state, we multiply the gate matrices in reverse order of application (right to left on the circuit diagram).

$$U = H \cdot X = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

- **Final State:** Applying the combined matrix U to the initial state $|0\rangle$:

$$U|0\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = |-\rangle.$$

Two-Qubit Systems: State Space I

- **Why Multi-Qubits?**
  - Single qubit: 2D Hilbert space ($|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$).
  - Two qubits: 4D Hilbert space via tensor product, which enables entanglement.

- **Tensor Product ($\otimes$):** Combines individual state spaces.
  - Example: $|0\rangle \otimes |0\rangle = |00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Two-Qubit Systems: State Space II

- **General State:** A superposition of all 4 basis states.

$$|\Psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

With normalization: $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$.

Two-Qubit Systems: Multi-Qubit Gates I

Example: Apply H on the first qubit

To apply a Hadamard gate to the first qubit while leaving the second unchanged (I), we use the tensor product of the gate matrices: $H \otimes I$. The resulting 4x4 matrix is:

$$H \otimes I = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

Two-Qubit Systems: Multi-Qubit Gates II

Applying

Applying this to the state $|00\rangle$:

$$(H \otimes I)\,|00\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

This creates a superposition between the two qubits.

Example Calculation On Circuit I

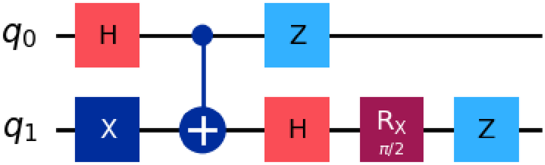

Figure: Sample Circuit: Two-Qubit Circuit with Multiple Gates

Example Calculation On Circuit II

Circuit Operations

- Qubit 0: $H \rightarrow Z$
- Qubit 1: $X \rightarrow H \rightarrow R_X(\pi/2) \rightarrow Z$
- CNOT between qubits

Initial: $|\psi_0\rangle = |00\rangle$

Example Calculation On Circuit III

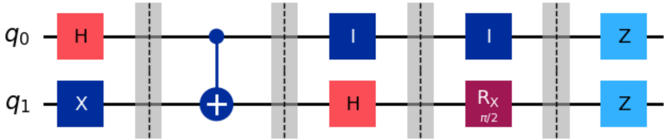

Figure: Clarified Circuit with Gate Labels

Example Calculation On Circuit IV

Step 1: Define Gate Matrices

Single-qubit gates:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Two-qubit operation matrix:

$$H \otimes X = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}$$

Example Calculation On Circuit V

Step 2: Apply First Layer ($H \otimes X$)

$$|\psi_1\rangle = (H \otimes X)|00\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$$

Example Calculation On Circuit VI

Step 3: Apply CNOT Gate

CNOT matrix:

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Example Calculation On Circuit VII

Applying CNOT

Apply CNOT:

$$|\psi_2\rangle = \text{CNOT} \cdot |\psi_1\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

Example Calculation On Circuit VIII

Step 4: Apply H to Qubit 1

Gate matrix: $(I \otimes H)$

$$I \otimes H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

Example Calculation On Circuit IX

Applying Hadamard

Apply operation:

$$|\psi_3\rangle = (I \otimes H) \cdot |\psi_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$$

Example Calculation On Circuit X

Step 5: Apply $R_X(\pi/2)$ to Qubit 1

Rotation gate:

$$R_X(\pi/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

Two-qubit matrix: $(I \otimes R_X(\pi/2))$

$$I \otimes R_X(\pi/2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & -i & 1 \end{pmatrix}$$

This introduces complex amplitudes to the state vector.

Example Calculation On Circuit XI

Step 6: Final Z Gates

Z gate matrix:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Apply to both qubits: $(Z \otimes Z)$

$$Z \otimes Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Example Calculation On Circuit XII

Final State Analysis

Matrix Representation Summary:

$$|\psi_{final}\rangle = (Z \otimes Z) \cdot (I \otimes R_X(\pi/2)) \cdot (I \otimes H) \cdot \text{CNOT} \cdot (H \otimes X) \cdot |00\rangle$$

Example Calculation On Circuit XIII

Final State Properties

Key Matrix Properties:

- All operations are **unitary**: $U^\dagger U = I$
- State vector norm is preserved: $\langle\psi|\psi\rangle = 1$
- CNOT creates **entanglement** by mixing computational basis states
- Rotation gate introduces **complex phases**
- Final state is a complex superposition of all four basis states

Computational Basis Amplitudes: The final state has the form:
$|\psi_{final}\rangle = c_0|00\rangle + c_1|01\rangle + c_2|10\rangle + c_3|11\rangle$
where $|c_0|^2 + |c_1|^2 + |c_2|^2 + |c_3|^2 = 1$.

Conclusion

- Summarize key concepts: Complex Numbers, Hilbert Space, Superposition, Entanglement.
- Emphasize the unique capabilities of quantum computing.
- Briefly mention the future outlook or applications.