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Methods in computational science

Homework 2

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1 Problem 1

First, lets assume that there exist two problems described by:

$$\mathbf{Au} = \mathbf{b} \qquad \text{and} \qquad \mathbf{A\hat{u}} = \mathbf{\hat{b}} \tag{1}$$

By substrating both:

$$\mathbf{Au} - \mathbf{A\hat{u}} = \mathbf{b} - \mathbf{\hat{b}} \tag{2}$$
$$\mathbf{A}(\mathbf{u} - \mathbf{\hat{u}}) = \mathbf{b} - \mathbf{\hat{b}} \tag{3}$$
$$\rightarrow (\mathbf{u} - \mathbf{\hat{u}}) = \mathbf{A}^{-1}(\mathbf{b} - \mathbf{\hat{b}}) \tag{4}$$

Considering $||\mathbf{Ax}|| \leq ||\mathbf{A}||||\mathbf{x}||$

$$||\mathbf{A}(\mathbf{u} - \mathbf{\hat{u}})|| \leq \|\mathbf{A}\|\|(\mathbf{u} - \mathbf{\hat{u}})\| = \|\mathbf{A}\|\|\mathbf{A}^{-1}(\mathbf{b} - \mathbf{\hat{b}})\| \leq \|\mathbf{A}\|\|\mathbf{A}^{-1}\|\|(\mathbf{b} - \mathbf{\hat{b}})\| \tag{5}$$

Thus:

$$\|\mathbf{A}\|\|(\mathbf{u} - \mathbf{\hat{u}})\| \leq \|\mathbf{A}\|\|\mathbf{A}^{-1}\|\|\|(\mathbf{b} - \mathbf{\hat{b}})\| = \kappa(\mathbf{A})\|(\mathbf{b} - \mathbf{\hat{b}})\| \tag{6}$$

Also, it can be stablished that $\|\mathbf{Au}\| = \|\mathbf{b}\| \leq \|\mathbf{A}\|\|\mathbf{u}\|$, and therefore $\frac{\|\mathbf{b}\|}{\|\mathbf{u}\|} \leq \|\mathbf{A}\|$
By multiplying both by $\|(\mathbf{u} - \mathbf{\hat{u}})\|$

$$\frac{\|\mathbf{b}\|}{\|\mathbf{u}\|}\|(\mathbf{u} - \mathbf{\hat{u}})\| \leq \|\mathbf{A}\|\|(\mathbf{u} - \mathbf{\hat{u}})\| \tag{7}$$

Therefore , by using equation 6 and equation 7:

$$\frac{\|\mathbf{b}\|}{\|\mathbf{u}\|}\|(\mathbf{u} - \mathbf{\hat{u}})\| \leq \kappa(\mathbf{A})\|(\mathbf{b} - \mathbf{\hat{b}})\| \tag{8}$$
$$\rightarrow \frac{\|\mathbf{u} - \mathbf{\hat{u}}\|}{\|\mathbf{u}\|} \leq \kappa(\mathbf{A})\frac{\|(\mathbf{b} - \mathbf{\hat{b}})\|}{\|\mathbf{b}\|} \tag{9}$$

2)

From equation 3:

$$||\mathbf{A}(\mathbf{u}-\hat{\mathbf{u}})|| \leq \|\mathbf{A}\|\|(\mathbf{u}-\hat{\mathbf{u}})\| \rightarrow ||\mathbf{r}|| \leq \|\mathbf{A}\|\|\mathbf{e}\| \tag{10}$$

And also as $\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$, then:

$$\|\mathbf{u}\| = \|\mathbf{A}^{-1}\mathbf{b}\| \leq \|\mathbf{A}^{-1}\|\|\mathbf{b}\| \tag{11}$$

By multiplying 10 by 11:

$$||\mathbf{r}||\|\mathbf{u}\| \leq \|\mathbf{A}^{-1}\|\|\mathbf{A}\|\|\mathbf{b}\|\|\mathbf{e}\| \tag{12}$$

$$||\mathbf{r}||\|\mathbf{u}\| \leq \kappa(\mathbf{A})\|\mathbf{b}\|\|\mathbf{e}\| \tag{13}$$

$$\frac{1}{\kappa(\mathbf{A})}\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \leq \frac{\|\mathbf{e}\|}{\|\mathbf{u}\|} \tag{14}$$

And as $\|\mathbf{e}\| = \|\mathbf{u}-\hat{\mathbf{u}}\|$ that means:

$$\frac{1}{\kappa(\mathbf{A})}\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \leq \frac{\|\mathbf{e}\|}{\|\mathbf{u}\|} \leq \kappa(\mathbf{A})\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \tag{15}$$

2 Problem 2

If $\mathbf{v}_j = \sin(j\pi x_i)$ is an eigenvector of $\mathbf{A}$, it should happen that $\mathbf{A}\mathbf{v}_j = \lambda_i \mathbf{v}_j$

$$(\mathbf{A}\mathbf{v}_j)_i = \frac{1}{h^2}(-v^{i-1} + 2v^i - v^{i+1}) \tag{16}$$

$$= \frac{1}{h^2}(-\sin(j\pi(i-1)h) + 2\sin(j\pi(i)h) - \sin(j\pi(i+1)h) \tag{17}$$

Lets redefine as the arguments as sums of angles $\theta = j\pi ih$ and $\phi = j\pi h$

$$\begin{aligned}
(\mathbf{A}\mathbf{v}_j)_i &= \frac{1}{h^2}(-\sin(\theta-\phi) + 2\sin(\theta) - \sin(\theta+\phi)) \\
&= \frac{1}{h^2}(-\sin(\theta)\cos(\phi) + \sin(\phi)\cos(\theta) + 2\sin(\theta) - \sin(\theta)\cos(\phi) - \sin(\phi)\cos(\theta)) \\
&= \frac{1}{h^2}(-2\sin(\theta)\cos(\phi) + 2\sin(\theta)) \\
&= \frac{1}{h^2}(2\sin(\theta)(1-\cos(\phi)))
\end{aligned}$$

And $1 - \cos(\phi) = 2\sin^2(\frac{\phi}{2})$

$$\begin{aligned}(\mathbf{A}\mathbf{v}_j)_i &= \frac{4}{h^2}(\sin^2(\frac{\phi}{2}))\sin(\theta) = \frac{4}{h^2}(\sin^2(\frac{j\pi h}{2}))\sin(j\pi i h) = \frac{4}{h^2}(\sin^2(\frac{j\pi h}{2}))\sin(j\pi x_i) \\ &= \frac{4}{h^2}(\sin^2(\frac{j\pi h}{2}))\mathbf{v_j}\end{aligned}$$

That $(\mathbf{A}\mathbf{v}_j)_i$ is the eigenvector of $\mathbf{A}$

2) The corresponding eigenvalue λ is the value infront of the eigenvector.

$$(\mathbf{A}\mathbf{v}_j)_i = \frac{4}{h^2}(\sin^2(\frac{j\pi h}{2}))\mathbf{v_j} = \lambda_j \mathbf{v_j}$$

Then:

$$\lambda_j = \frac{4}{h^2}\sin^2(\frac{j\pi h}{2}) = 4m^2\sin^2(\frac{j\pi}{2m}) \tag{18}$$

For $j = 1, ..., m-1$ (not including boundary conditions, where λ should be zero)

3) If $m \to \infty$:

- For j=1 is the lowest eigenvalue: $\frac{\pi}{2m}$ is small, then $\sin(\frac{\pi}{2m}) \approx \frac{\pi}{2m}$. Then

$$\lambda_{min} \approx 4m^2(\frac{\pi}{2m})^2 = \pi \tag{19}$$

- For j=m-1 is the lowest eigenvalue: $\frac{(m-1)\pi}{2m} = \frac{\pi}{2} - \frac{\pi}{2m} \approx \frac{\pi}{2}$, then $\sin(\frac{\pi}{2}) = 1$. Then:

$$\lambda_{max} \approx 4m^2(1)^2 = 4m^2 \tag{20}$$

Finally:

$$\kappa(\mathbf{A}) \approx \frac{4m}{\pi} = O(m^2) \tag{21}$$

3)

$$u(x) = \sin(k\pi x) + c(x - \frac{1}{2})^3 \tag{22}$$

$$u'(x) = k\pi \cos(k\pi x) + 3c(x - \frac{1}{2})^2 \tag{23}$$

$$u''(x) = -k^2\pi^2 \sin(k\pi x) + 6c(x - \frac{1}{2}) \tag{24}$$

Then, by susbtituting in $-u'' + \gamma u = f(x)$

$$k^2\pi^2 \sin(k\pi x) - 6c(x - \frac{1}{2}) + \gamma(\sin(k\pi x) + c(x - \frac{1}{2})^3) = f(x) \tag{25}$$

$$(k^2\pi^2 + \gamma) \sin(k\pi x) + \gamma c(x - \frac{1}{2})^3 - 6c(x - \frac{1}{2}) = f(x) \tag{26}$$