Columbia Engineering

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Methods in computational science

Homework 2

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1 Problem 1

First, lets assume that there exist two problems described by:

$$\mathbf{Au} = \mathbf{b} \qquad \text{and} \qquad \mathbf{A\hat{u}} = \hat{\mathbf{b}} \tag{1}$$

By substrating both:

$$\mathbf{Au} - \mathbf{A\hat{u}} = \mathbf{b} - \hat{\mathbf{b}} \tag{2}$$
$$\mathbf{A}(\mathbf{u} - \hat{\mathbf{u}}) = \mathbf{b} - \hat{\mathbf{b}} \tag{3}$$
$$\rightarrow (\mathbf{u} - \hat{\mathbf{u}}) = \mathbf{A}^{-1}(\mathbf{b} - \hat{\mathbf{b}}) \tag{4}$$

Considering $||\mathbf{Ax}|| \leq ||\mathbf{A}||||\mathbf{x}||$

$$||\mathbf{A}(\mathbf{u} - \hat{\mathbf{u}})|| \leq \|\mathbf{A}\|\|(\mathbf{u} - \hat{\mathbf{u}})\| = \|\mathbf{A}\|\|\mathbf{A}^{-1}(\mathbf{b} - \hat{\mathbf{b}})\| \leq \|\mathbf{A}\|\|\mathbf{A}^{-1}\|\|(\mathbf{b} - \hat{\mathbf{b}})\| \tag{5}$$

Thus:

$$\|\mathbf{A}\|\|(\mathbf{u} - \hat{\mathbf{u}})\| \leq \|\mathbf{A}\|\|\mathbf{A}^{-1}\|\|(\mathbf{b} - \hat{\mathbf{b}})\| = \kappa(\mathbf{A})\|(\mathbf{b} - \hat{\mathbf{b}})\| \tag{6}$$

Also, it can be stablished that $\|\mathbf{Au}\| = \|\mathbf{b}\| \leq \|\mathbf{A}\|\|\mathbf{u}\|$, and therefore $\frac{\|\mathbf{b}\|}{\|\mathbf{u}\|} \leq \|\mathbf{A}\|$
By multiplying both by $\|(\mathbf{u} - \hat{\mathbf{u}})\|$

$$\frac{\|\mathbf{b}\|}{\|\mathbf{u}\|}\|(\mathbf{u} - \hat{\mathbf{u}})\| \leq \|\mathbf{A}\|\|(\mathbf{u} - \hat{\mathbf{u}})\| \tag{7}$$

Therefore , by using equation 6 and equation 7:

$$\frac{\|\mathbf{b}\|}{\|\mathbf{u}\|}\|(\mathbf{u} - \hat{\mathbf{u}})\| \leq \kappa(\mathbf{A})\|(\mathbf{b} - \hat{\mathbf{b}})\| \tag{8}$$
$$\rightarrow \frac{\|\mathbf{u} - \hat{\mathbf{u}}\|}{\|\mathbf{u}\|} \leq \kappa(\mathbf{A})\frac{\|(\mathbf{b} - \hat{\mathbf{b}})\|}{\|\mathbf{b}\|} \tag{9}$$

2)

From equation 3:

$$||\mathbf{A}(\mathbf{u}-\hat{\mathbf{u}})|| \leq \|\mathbf{A}\|\|(\mathbf{u}-\hat{\mathbf{u}})\| \rightarrow ||\mathbf{r}|| \leq \|\mathbf{A}\|\|\mathbf{e}\| \tag{10}$$

And also as $\mathbf{u} = \mathbf{A}^{-1}\mathbf{b}$, then:

$$\|\mathbf{u}\| = \|\mathbf{A}^{-1}\mathbf{b}\| \leq \|\mathbf{A}^{-1}\|\|\mathbf{b}\| \tag{11}$$

By multiplying 10 by 11:

$$||\mathbf{r}||\|\mathbf{u}\| \leq \|\mathbf{A}^{-1}\|\|\mathbf{A}\|\|\mathbf{b}\|\|\mathbf{e}\| \tag{12}$$

$$||\mathbf{r}||\|\mathbf{u}\| \leq \kappa(\mathbf{A})\|\mathbf{b}\|\|\mathbf{e}\| \tag{13}$$

$$\frac{1}{\kappa(\mathbf{A})}\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \leq \frac{\|\mathbf{e}\|}{\|\mathbf{u}\|} \tag{14}$$

And as $\|\mathbf{e}\| = \|\mathbf{u}-\hat{\mathbf{u}}\|$ that means:

$$\frac{1}{\kappa(\mathbf{A})}\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \leq \frac{\|\mathbf{e}\|}{\|\mathbf{u}\|} \leq \kappa(\mathbf{A})\frac{\|\mathbf{r}\|}{\|\mathbf{b}\|} \tag{15}$$

2 Problem 2

If $\mathbf{v}_j = \sin(j\pi x_i)$ is an eigenvector of $\mathbf{A}$, it should happen that $\mathbf{A}\mathbf{v}_j = \lambda_i \mathbf{v}_j$

$$(\mathbf{A}\mathbf{v}_j)_i = \frac{1}{h^2}(-v^{i-1} + 2v^i - v^{i+1}) \tag{16}$$

$$= \frac{1}{h^2}(-\sin(j\pi(i-1)h) + 2\sin(j\pi(i)h) - \sin(j\pi(i+1)h) \tag{17}$$

Lets redefine as the arguments as sums of angles $\theta = j\pi ih$ and $\phi = j\pi h$

$$\begin{aligned}
(\mathbf{A}\mathbf{v}_j)_i &= \frac{1}{h^2}(-\sin(\theta-\phi) + 2\sin(\theta) - \sin(\theta+\phi)) \\
&= \frac{1}{h^2}(-\sin(\theta)\cos(\phi) + \sin(\phi)\cos(\theta) + 2\sin(\theta) - \sin(\theta)\cos(\phi) - \sin(\phi)\cos(\theta)) \\
&= \frac{1}{h^2}(-2\sin(\theta)\cos(\phi) + 2\sin(\theta)) \\
&= \frac{1}{h^2}(2\sin(\theta)(1-\cos(\phi)))
\end{aligned}$$

And $1 - \cos(\phi) = 2\sin^2(\frac{\phi}{2})$

$$\begin{aligned}(\mathbf{A}\mathbf{v}_j)_i &= \frac{4}{h^2}(\sin^2(\frac{\phi}{2}))\sin(\theta) = \frac{4}{h^2}(\sin^2(\frac{j\pi h}{2}))\sin(j\pi i h) = \frac{4}{h^2}(\sin^2(\frac{j\pi h}{2}))\sin(j\pi x_i) \\ &= \frac{4}{h^2}(\sin^2(\frac{j\pi h}{2}))\mathbf{v_j}\end{aligned}$$

That $(\mathbf{A}\mathbf{v}_j)_i$ is the eigenvector of $\mathbf{A}$

2) The corresponding eigenvalue λ is the value infront of the eigenvector.

$$(\mathbf{A}\mathbf{v}_j)_i = \frac{4}{h^2}(\sin^2(\frac{j\pi h}{2}))\mathbf{v_j} = \lambda_j\mathbf{v_j}$$

Then:

$$\lambda_j = \frac{4}{h^2}\sin^2(\frac{j\pi h}{2}) = 4m^2\sin^2(\frac{j\pi}{2m}) \tag{18}$$

For $j = 1, ..., m-1$ (not including boundary conditions, where λ should be zero)

3) If $m \to \infty$:

- For j=1 is the lowest eigenvalue: $\frac{\pi}{2m}$ is small, then $\sin(\frac{\pi}{2m}) \approx \frac{\pi}{2m}$. Then

$$\lambda_{min} \approx 4m^2(\frac{\pi}{2m})^2 = \pi \tag{19}$$

- For j=m-1 is the lowest eigenvalue: $\frac{(m-1)\pi}{2m} = \frac{\pi}{2} - \frac{\pi}{2m} \approx \frac{\pi}{2}$, then $\sin(\frac{\pi}{2}) = 1$. Then:

$$\lambda_{max} \approx 4m^2(1)^2 = 4m^2 \tag{20}$$

Finally:

$$\kappa(\mathbf{A}) \approx \frac{4m}{\pi} = O(m^2) \tag{21}$$

3)

$$u(x) = \sin(k\pi x) + c(x - \frac{1}{2})^3 \tag{22}$$

$$u'(x) = k\pi\cos(k\pi x) + 3c(x - \frac{1}{2})^2 \tag{23}$$

$$u''(x) = -k^2\pi^2\sin(k\pi x) + 6c(x - \frac{1}{2}) \tag{24}$$

Then, by susbtituting in $-u'' + \gamma u = f(x)$

$$k^2\pi^2\sin(k\pi x) - 6c(x - \frac{1}{2}) + \gamma(\sin(k\pi x) + c(x - \frac{1}{2})^3) = f(x) \tag{25}$$

$$(k^2\pi^2 + \gamma)\sin(k\pi x) + \gamma c(x - \frac{1}{2})^3 - 6c(x - \frac{1}{2}) = f(x) \tag{26}$$

3 Problem 3

For this problem, the code was created and its name is 'bvp.c'. Is added in the main file HW2. The file was made from tri.c code, and modfied by adding the term *gammau*, thus the matrix $\mathbf{A}$ has the form:

$$\mathbf{A}_i = \frac{1}{h^2}\begin{bmatrix}-1 & 2+\gamma & -1]\end{bmatrix} \tag{27}$$

And vector $b = f(x_i)$

By using