Data Structures 2018
Exercise 2, solutions (Week 38)

1

$$\begin{aligned}
\sqrt{19n} &= \sqrt{19}\sqrt{n} \in \mathcal{O}(\frac{1}{\sqrt{19}} * (\sqrt{19}\sqrt{n})) = \mathcal{O}(\sqrt{n}) \\
21n^2 &\in \mathcal{O}(n^2) \\
2^{100000} &\in \mathcal{O}(\frac{1}{2^{100000}} * (2^{100000} * 1)) = \mathcal{O}(1) \\
\log(4n^7) &= \log 4 + 7\log n \in \mathcal{O}(\log n) \\
\log\left(\frac{n\log n^2}{\log n}\right) &= \log\left(\frac{2n\log n}{\log n}\right) = \log 2n \in \mathcal{O}(\log n) \\
\log k^n &= \underbrace{\log k + \log k + \cdots + \log k}_{n \text{ kappaletta}} = n\underbrace{\log k}_{\text{vakio}} \in \mathcal{O}(n) \\
7(n^3+1)(n+1) &= 7n^4 + 7n^3 + 7n + 7 \in \mathcal{O}(n^4) \\
\log(10n) &= \underbrace{\log 10}_{\text{vakio}} + \log n \in \mathcal{O}(\log n) \\
11n\log n &\in \mathcal{O}(n\log n)
\end{aligned}$$

2 Correct order is as follows:

$$42, 7\log\log n, 3\log n, \sqrt{n}, \sqrt{n}\log n, 6n, n\log n, 5n^2, n!.$$

As for example:

$$\lim_{n\to\infty}\left(\frac{n!}{5n^2}\right) = \lim_{n\to\infty}\left(\frac{(n-1)!}{5n}\right) = \lim_{n\to\infty}\left(\frac{1}{5}\cdot\underbrace{\frac{(n-1)}{n}}_{\to 1}\cdot\underbrace{(n-2)!}_{\to\infty}\right) = \infty,$$

so $n!$ grows asymptotically faster than $5n^2$.

3 Check PrintValue.java.

4 Check SortingComparison.java.

5 Let $f(n) = a_0 + a_1 n + a_2 n^2 + \cdots + a_d n^d$ where $a_k \in \mathbb{R}$ and $a_d \neq 0$, $k = 0, 1, 2, \ldots, d$. Since for all $n \geq 1$

$$\begin{aligned} f(n) \leq |f(n)| &= |a_0 + a_1 n + a_2 n^2 + \cdots + a_d n^d| \\ &\leq |a_0| + |a_1| n + |a_2| n^2 + \cdots + |a_d| n^d \\ &= (\frac{|a_0|}{n^d} + \frac{|a_1|}{n^{d-1}} + \frac{|a_2|}{n^{d-2}} + \cdots + |a_d|) n^d \\ &\leq (|a_0| + |a_1| + |a_2| + \cdots + |a_d|) n^d \end{aligned}$$

holds and by choosing $c = |a_0| + |a_1| + |a_2| + \cdots + |a_d|$ and $n_0 = 1$, we obtain that $f(n) \leq cn^d$ when $n \geq n_0$, i.e. $f(n)$ is $\mathcal{O}(n^d)$.

6 Since $d(n)$ is $\mathcal{O}(f(n))$ and $e(n)$ is $\mathcal{O}(g(n))$, there exist $n_0, n_1 \in \mathbb{N}$ and $r, s > 0$ such that $d(n) \leq rf(n)$ and $e(n) \leq sg(n)$. Hence by choosing $n_2 = \max\{n_0, n_1\}$ and $b = rs$ we obtain $d(n)e(n) \leq r \cdot s \cdot f(n)g(n) = b \cdot f(n)g(n)$, when $n \geq n_2$. Thus, $d(n)e(n)$ is $\mathcal{O}(f(n)g(n))$.

7 Let $f(n) = 7 + \sqrt{5n} + n + 12n^3 + 6n^4$ and $g(n) = n^4$. Because

$$
\begin{aligned}
6n^4 &\leq 6n^4, \\
12n^3 &\leq 12n^4, \\
n &\leq n^4, \\
\sqrt{5n} &= \sqrt{5}\sqrt{n} \leq \sqrt{5}n \leq \sqrt{5}n^4, \\
7 &\leq 7n^4
\end{aligned}
$$

for all $n \geq 1$ we obtain

$$
\begin{aligned}
f(n) &= 7 + \sqrt{5n} + n + 12n^3 + 6n^4 \\
&\leq 7n^4 + \sqrt{5}n^4 + n^4 + 12n^4 + 6n^4 = (26 + \sqrt{5})n^4,
\end{aligned}
$$

when $n \geq 1$. Thus, we can choose $c = 26 + \sqrt{5}$ and $n_0 = 1$. Hence, there exist positive constants c and n_0 such that $f(n) \leq cg(n)$ when $n \geq n_0$. Thus, $f(n)$ is $\mathcal{O}(n^4)$.

8 Let $f(n) = 6 + 2\log\log\log n + 5n\sqrt{n} + n^3 + \sqrt[3]{2}n^4 + n^5$ and $g(n) = n^5$. Since

$$\begin{aligned} n^5 &\leq n^5, \\ \sqrt[3]{2}n^4 &\leq \sqrt[3]{2}n^5, \\ n^3 &\leq n^5, \\ 5n\sqrt{n} &\leq 5n \cdot n = 5n^2 \leq 5n^5 \\ 6 &\leq 6n^5 \end{aligned}$$

for all $n \geq 1$ and $2\log\log\log n \leq 2\log n \leq 2n \leq 2n^5$, when $n \geq 3$ ($n \geq 3$ ($2\log\log\log n$ is not defined when $n \in \{1, 2\}$ and logarithm is 2-base logarithm), we obtain

$$\begin{aligned} &6 + 2\log\log\log n + 5n\sqrt{n} + n^3 + \sqrt[3]{2}n^4 + n^5 \\ &\leq 6n^5 + 2n^5 + 5n^5 + n^5 + \sqrt[3]{2}n^5 + n^5 = (15 + \sqrt[3]{2})n^5. \end{aligned}$$

when $n \geq 3$. Thus, we can choose $c = 15 + \sqrt[3]{2}$ and $n_0 = 3$. Hence, there exist positive constants c and n_0 such that $f(n) \leq cg(n)$ when $n \geq n_0$. Hence, $f(n)$ is $\mathcal{O}(n^5)$.