Large Deviations of Cluster Sizes
Study notes
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1 Preliminaries

Some of the topics to develop are:

- Thermal properties of a ferromagnetic model over a random Poisson graph.
- The cavity and replica methods.
- Montecarlo methods, pertinent to this study.

2 Ising model over a random Poisson graph

A ferromagnetic Ising model over a finite graph $G_N := (V_N := \{1, \ldots, N\}, E_N)$ is defined by means of a Boltzmann-Gibbs density in the set $\mathbf{S} = \{s_i : i \in V_N,\, s_i = \pm 1\}$:

$$\mathcal{P}(\sigma_1, \ldots, \sigma_N) := \mathbb{P}\left[s_1 = \sigma_1, \ldots, s_N = \sigma_N\right] = \frac{1}{Z(\beta, \mathbf{J})} \exp\left[\beta\left(\sum_{(i,j)\in E_N} J_{ij}\sigma_i\sigma_j + h\sum_{i\in V_N}\sigma_i\right)\right] \quad (1)$$

where $\mathbf{J} = (J_{ij})$ and $Z = (\beta, \mathbf{J})$ is the corresponding partition function that guarantees the normalization of $\mathcal{P}$:

$$\sum_{\vec{\sigma}\in\{-1,1\}^N} \mathcal{P}(\sigma_1, \ldots, \sigma_N) = \sum_{\vec{\sigma}} \mathcal{P}(\sigma_1, \ldots, \sigma_N) = 1,$$

and $\vec{\sigma} := (\sigma_1, \ldots, \sigma_N)$.

If the adjacency matrix of the graph G_N, $\mathbf{A} = (a_{ij})$[1] is such that for each i:

$$\mathbb{P}(a_{ij} = 1) = \frac{c}{N}$$

where $c \geq 1$ is a constant, then in the limit $N \to \infty$ the degree X distribution of the graph becomes:

$$\mathbb{P}(X = k) \xrightarrow{N\to\infty} \frac{c^k}{k!}e^{-c} \quad (2)$$

In the context of Graph Theory, this is known as the *Erdős-Rényi model* or a *binomial graph*, where a graph is constructed by a random procedure. Formally speaking, given $0 < p < 1$, a finite set $\{1, \ldots, n\}$ and the space Ω of all the $\binom{n}{2}$ possible edge sets E_G form it, the probability space:

$$\mathbb{G}(n, p) := (\Omega, \mathcal{F}, \mathbb{P}) \quad (3)$$

where $\mathcal{F} = \mathcal{P}(\Omega)$ and $\forall G \in \mathcal{F}$:

$$\mathbb{P}(G) := p^{e_G}(1-p)^{\binom{n}{2} - e_G} \quad (4)$$

provided that $e_G = |E_G|$ stands for the number of edges of G. This can be viewed as a result of $\binom{n}{2}$ independent coin flippings, one for each pair of vertices, with the probability of success (i.e., drawing an edge) equal to p (see figure 2).

[1] Defined as $(a_{ij}) = 1$ if $\{i, j\} \in V_N$ and $(a_{ij}) = 0$ otherwise.

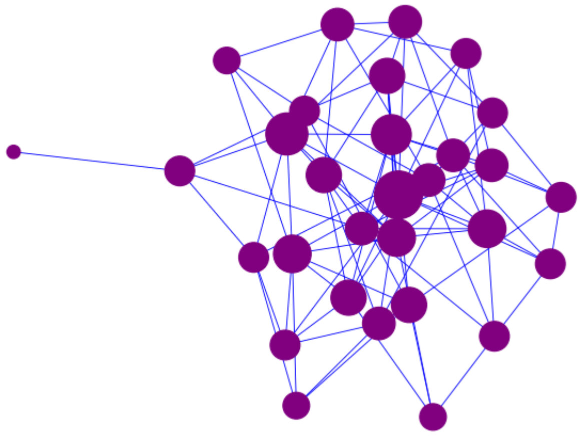

Figure 1: Erdős-Rényi graph with $n = 30$ $p = 1/5$, where each node size is proportional to its connectivity X. When computed, the mean connectivity $\mathbb{E}(X)$ equals $5.93 \approx np$.

2.1 The cavity method

Now let us introduce some useful definitions: for each $j \in V_N$, we define *the neighborhood of* j as:

$$\partial j := \{i \in V_N : a_{ij} = 1\}$$

and given any $A \subseteq V_N$, we define the following set of indices:

$$\vec{\sigma}_A := \{\sigma_i\}_{i \in A}$$

(with the particular case $\vec{\sigma}_{V_N} := \vec{\sigma}$).
So given a joint distribution $\mathcal{P}(\vec{\sigma})$, the marginal distribution $\mathcal{P}_A(\vec{\sigma}_A)$ is given by:

$$\mathcal{P}_A(\vec{\sigma}_A) := \sum_{\vec{\sigma}_{V_N \setminus A}} \mathcal{P}(\vec{\sigma}) \tag{5}$$

with the special case of the marginal density for a fixed spin σ_j:

$$\mathcal{P}_j(\sigma_j) := \mathcal{P}_{\{j\}}(\vec{\sigma}_{\{j\}}) \tag{6}$$

or alternatively:

$$\mathcal{P}_j(\sigma) := \sum_{\vec{\sigma}} \delta_{\sigma_j,\sigma} \mathcal{P}(\sigma_1, \ldots, \sigma_N)$$

So, let $i \in V_N$ fixed; it is possible to split the Hamiltonian inside the partition function 1:

$$\mathcal{H}(\vec{\sigma}, \mathbf{J}) = -\sum_{(i,j) \in E_N} J_{ij}\sigma_i\sigma_j - h\sum_{i \in V_N} \sigma_i \tag{7}$$

as the following sum:

$$\mathcal{H}(\vec{\sigma}, \mathbf{J}) = -h\sigma_i - \sigma_i \sum_{j \in \partial i} J_{ij}\sigma_j + \mathcal{H}^{(i)} \tag{8}$$

where $\mathcal{H}^{(i)}$ accounts for the Hamiltonian in the graph *without the contribution of the spin* s_i, with the corresponding Boltzmann-Gibbs density $\mathcal{P}^{(i)}(\vec{\sigma}) = \frac{1}{Z_i(\beta,\mathbf{J})} \exp(-\beta\mathcal{H}^{(i)})$. Thus we have:

$$\begin{aligned}
\mathcal{P}_i(\sigma_i) &= \frac{1}{Z(\beta, \mathbf{J})} e^{\beta h \sigma_i} \sum_{\vec{\sigma}_{\partial i}} \left[\exp\left(\beta\sigma_i \sum_{j \in \partial i} J_{ij}\sigma_j\right) \sum_{\vec{\sigma}_{V_N \setminus \partial i}} \exp(-\beta\mathcal{H}^{(i)}) \right] \\
&= \frac{1}{Z^{(i)}(\beta, \mathbf{J})} e^{\beta h \sigma_i} \sum_{\vec{\sigma}_{\partial i}} \left[\exp\left(\beta\sigma_i \sum_{j \in \partial i} J_{ij}\sigma_j\right) \mathcal{P}^{(i)}_{\partial i}(\vec{\sigma}_{\partial i}) \right]
\end{aligned} \tag{9}$$

for an appropriate normalization constant $Z^{(i)}(\beta, \mathbf{J})$.

Furthermore, for each $j \in \partial i$, we have the following formula for the marginal density $\mathcal{P}_i^{(j)}(\sigma_i)$:

$$
\begin{aligned}
\mathcal{P}_i^{(j)}(\sigma_i) &= \sum_{\vec{\sigma}_{V_N\setminus\{i\}}} \mathcal{P}^{(j)}(\sigma_1, \ldots \sigma_N) = \frac{1}{Z_i} e^{\beta h \sigma_i} \sum_{\vec{\sigma}_{\partial i}} \left[\exp\left(\beta\sigma_i \sum_{j\in\partial i} J_{ij}\sigma_j \right) \mathcal{P}_{\partial i}^{(i)}(\vec{\sigma_{\partial i}}) \right] \\
&= \frac{1}{Z^{(j)}} e^{\beta h \sigma_i} \sum_{\vec{\sigma}_{\partial i\setminus\{j\}}} \left[\exp\left(\beta\sigma_i \sum_{k\in\partial i\setminus\{j\}} J_{ik}\sigma_k \right) \sum_{\vec{\sigma}_{V_N\setminus\partial i}} \exp\left(-\beta\mathcal{H}^{(i,j)} \right) \right] \\
&= \frac{1}{Z_i^{(j)}} e^{\beta h \sigma_i} \sum_{\vec{\sigma}_{\partial i\setminus\{j\}}} \left[\exp\left(\beta\sigma_i \sum_{k\in\partial i\setminus\{j\}} J_{ik}\sigma_k \right) \mathcal{P}_{\partial i\setminus\{j\}}^{(i,j)}(\vec{\sigma}_{\partial i\setminus\{j\}}) \right]
\end{aligned}
\tag{10}
$$

Under the assumption that the graph is *locally tree-like*, we would have:

$$
\mathcal{P}_{\partial i\setminus\{j\}}^{(i,j)}(\vec{\sigma}_{\partial i\setminus\{j\}}) \equiv \mathcal{P}_{\partial i\setminus\{j\}}^{(i)}(\vec{\sigma}_{\partial i\setminus\{j\}}) = \prod_{k\in\partial i\setminus\{j\}} \mathcal{P}_k^{(i)}(\sigma_k) \tag{11}
$$

with the last equality named the *Bethe-Peierls approximation.* So returning to the equation 10:

$$
\begin{aligned}
\mathcal{P}_i^{(j)}(\sigma_i) &= \frac{1}{Z_i^{(j)}} e^{\beta h \sigma_i} \sum_{\vec{\sigma}_{\partial i\setminus\{j\}}} \left[\prod_{k\in\partial i\setminus\{j\}} \mathcal{P}_k^{(i)}(\sigma_k) \right] \\
&= \frac{1}{Z_i^{(j)}} e^{\beta h \sigma_i} \prod_{k\in\partial i\setminus\{j\}} \left[\sum_{\sigma_k\in\{-1,1\}} \exp\left(\beta J_{ik}\sigma_i\sigma_k\right) \mathcal{P}_k^{(i)}(\sigma_k) \right]
\end{aligned}
\tag{12}
$$

where we have used the summation identity $\sum_{\vec{\sigma}_{\partial i\setminus\{j\}}} \prod_{k\in\partial i\setminus\{j\}} \equiv \prod_{k\in\partial i\setminus\{j\}} \sum_{\sigma_k\in\{-1,1\}}$
As $\mathcal{P}_i^{(j)}(\sigma_i)$ is a probability measure in a two element set ($\sigma_i \in \{-1, 1\}$), we can parametrize it as:

$$
\mathcal{P}_i^{(j)}(\sigma_i) = \frac{\exp\left(\beta h_i^{(j)}\sigma_i\right)}{2\cosh\left(\beta h_i^{(j)}\right)} \tag{13}
$$

where the parameters $h_i^{(j)}$ are known as the *cavity fields*, which can be obtained by means of the following calculation:

$$
\begin{aligned}
\sum_{\sigma_i} \sigma_i \log\left(\mathcal{P}_i^{(j)}(\sigma_i)\right) &= \sum_{\sigma_i} \sigma_i \left\{ \beta h_i^{(j)}\sigma_i - \log\left[2\cosh\left(\beta h_i^{(j)}\right)\right] \right\} \\
&= 2\beta h_i^{(j)}
\end{aligned}
$$

so:[2]

$$
\therefore h_i^{(j)} = \frac{1}{2\beta} \sum_{\sigma_i} \sigma_i \log\left[\mathcal{P}_i^{(j)}(\sigma_i)\right] \tag{14}
$$

Combining this formula with 12:

[2]Similarly, if we parametrize $\mathcal{P}_i(\sigma_i)$ as $\frac{\exp(\beta h_i\sigma_i)}{2\cosh(\beta h_i)}$, we get $h_i^j = \frac{1}{2\beta}\sum_{\sigma_i} \sigma_i \log\left[\mathcal{P}_i(\sigma_i)\right]$.

$$
\begin{aligned}
2\beta h_i^{(j)} &= \sum_{\sigma_i} \sigma_i \log\left(\mathcal{P}_i^{(j)}(\sigma_i)\right) \\
&= \sum_{\sigma_i} \sigma_i \log\left\{\frac{e^{\beta h \sigma_i}}{Z_i^{(j)}} \prod_{k\in\partial i\setminus\{j\}} \left[\sum_{\sigma_k\in\{-1,1\}} \exp\left(\beta J_{ik}\sigma_i\sigma_k\right)\mathcal{P}_k^{(i)}(\sigma_k)\right]\right\} \\
&= \sum_{\sigma_i} \sigma_i \left\{\beta h \sigma_i + \sum_{k\in\partial i\setminus\{j\}} \log\left[\sum_{\sigma_k\in\{-1,1\}} \exp\left(\beta J_{ik}\sigma_i\sigma_k\right) \frac{\exp\left(\beta h_i^{(j)}\sigma_i\right)}{2\cosh\left(\beta h_i^{(j)}\right)}\right]\right\} \\
&= 2\beta h + \sum_{\sigma_i} \sigma_i \log\left\{\frac{\cosh\left[\beta\left(J_{ik}\sigma_i + h_k^{(i)}\right)\right]}{\cosh\left(\beta h_k^{(i)}\right)}\right\} \\
&= 2\beta h + \sum_{k\in\partial i\setminus\{j\}} \left\{\log\left[\cosh\beta\left(h_k^{(i)} + J_{ik}\right)\right] - \log\left[\cosh\beta\left(h_k^{(i)} - J_{ik}\right)\right]\right\} \\
&= 2\beta h + \sum_{k\in\partial i\setminus\{j\}} \log\left[\frac{\cosh\beta\left(h_k^{(i)} + J_{ik}\right)}{\cosh\beta\left(h_k^{(i)} - J_{ik}\right)}\right]
\end{aligned}
\tag{15}
$$

and using $\cosh(x\pm y) = \cosh(x)\cosh(y) \pm \sinh(x)\sinh(y)$, we get:

$$
\begin{aligned}
h_i^{(j)} &= h + \frac{1}{2\beta} \sum_{k\in\partial i\{j\}} \log\left[\frac{\cosh\left(\beta J_{ik}\right)\cosh\left(\beta h_k^{(i)}\right) + \sinh\left(\beta J_{ik}\right)\sinh\left(\beta h_k^{(i)}\right)}{\cosh\left(\beta J_{ik}\right)\cosh\left(\beta h_k^{(i)}\right) - \sinh\left(\beta J_{ik}\right)\sinh\left(\beta h_k^{(i)}\right)}\right] \\
&= h + \frac{1}{2\beta} \sum_{k\in\partial i\{j\}} \log\left[\frac{1 + \tanh\left(\beta J_{ik}\right)\tanh\left(\beta h_k^{(i)}\right)}{1 - \tanh\left(\beta J_{ik}\right)\tanh\left(\beta h_k^{(i)}\right)}\right] \\
&= h + \frac{1}{\beta} \sum_{k\in\partial i\setminus\{j\}} \operatorname{atanh}\left[\tanh\left(\beta J_{ik}\right)\tanh\left(\beta h_k^{(i)}\right)\right]
\end{aligned}
\tag{16}
$$

where we used the identity $\operatorname{atanh}(x) = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$.

2.2 Cavity equations

In sum, we got the following computational scheme: Once we parametrize $\mathcal{P}_i^{(j)}(\sigma_i)$ as $\frac{\exp\left(\beta h_i^{(j)}\sigma_i\right)}{2\cosh\left(\beta h_i^{(j)}\right)}$, we get the *cavity fields*

$$
h_i^{(j)} = \frac{1}{2\beta} \sum_{\sigma_i\in\{-1,1\}} \sigma_i \log \mathcal{P}_i^{(j)}(\sigma_i)
$$

which obey the auto-consistent field condition:

$$
h_i^{(j)} = h + \frac{1}{\beta} \sum_{k\in\partial i\setminus j} \operatorname{atanh}\left[\tanh\left(\beta h_k^{(i)}\right)\tanh\left(\beta J_{ki}\right)\right]
$$

and conform an asymptotic and recursive (fixed-point) method to get the *physical fields* h_i:

$$
h_i = h + \frac{1}{\beta} \sum_{k\in\partial i} \operatorname{atanh}\left[\tanh\left(\beta h_k^{(i)}\right)\tanh\left(\beta J_{ki}\right)\right]
\tag{17}
$$

that, in turn, make the marginal densities $\mathcal{P}_i(\sigma_i)$. Thus, the *magnetization* of this Ising model can be computed as follows:

$$
\begin{aligned}
M(\beta) &= \frac{1}{N}\sum_{i=1}^{N} m_i(\beta) \\
m_i(\beta) &= \sum_{\sigma_i\in\{-1,1\}} \sigma_i \mathcal{P}_i(\sigma_i) = \tanh\left(\beta h_i\right)
\end{aligned}
$$

Therefore, we arrived to a formula for the magnetization of this ferromagnetic Ising model over the finite graph random graph G_N:

$$M(\beta) = \frac{1}{N} \sum_{i=1}^{N} \tanh\left(\beta h_i\right) \tag{18}$$

2.3 Zero temperature limit

Recalling that $\beta = \frac{1}{k_B T}$ (where T is the temperature), we can compute the zero temperature limit for the equation 17 by taking $\beta \longrightarrow \infty$. First, we can use l'Hôpital's rule to show that:

$$\lim_{\beta \to \infty} \frac{1}{\beta} \log\left[\frac{\cosh(\beta x)}{\cosh(\beta y)}\right] = |x| - |y|$$

So we have

$$\lim_{\beta \to \infty} h_i^{(j)} = h + \frac{1}{2} \sum_{k \in \partial i \setminus j} \left\{ \left| h_k^{(i)} + J_{ki} \right| - \left| h_k^{(i)} - J_{ki} \right| \right\} \tag{19}$$

whose solution induces the fields h_i:

$$\lim_{\beta \to \infty} h_i = h + \frac{1}{2} \sum_{k \in \partial i} \left\{ \left| h_k^{(i)} + J_{ki} \right| - \left| h_k^{(i)} - J_{ki} \right| \right\} \tag{20}$$

Furthermore, these equations in turn induce the zero temperature magnetizations:

$$\lim_{\beta \to \infty} m_i(\beta) = \begin{cases} 1, & \text{if } h_i > 0 \\ 0, & \text{if } h_i = 0 \\ -1, & \text{if } h_i < 0 \end{cases} \tag{21}$$

2.4 Generalizing the Bethe-Peierls equations

As the exponential function satisfies $\exp(x + y) = \exp(x)\exp(y)$, it is possible to think on the relation 1 as a probability density of the form:

$$\mathcal{P}(\vec{x}) = \frac{1}{Z} \prod_{(i,j) \in E} \psi_{ij}(x_i, x_j) \prod_{k \in V} \phi_k(x_k) \tag{22}$$

over a finite graph $G = (V, E)$.[3]

If $(i, j) \in E$, the density:

$$\mathcal{P}^{(i,j)}(\vec{x}) := \frac{1}{Z^{(i,j)}} \prod_{(k,l) \in E \setminus (i,j)} \psi_{kl}(x_k, x_l) \prod_{k' \in V} \phi_l(x_{k'}) \tag{23}$$

(where $F \setminus (i, j) := F \setminus \{(i, j)\}$ for every $F \subseteq E$) represents a modified probability measure that *takes away the edge* (i, j) *contribution*; and if $i \in V$:

$$\mathcal{P}^{(i)}(\vec{x}) := \frac{1}{Z^{(i)}} \prod_{(k,l) \in E \setminus i} \psi_{kl}(x_k, x_l) \prod_{k \in V \setminus i} \phi_k(x_k) \tag{24}$$

(where $F \setminus i := \{(k, l) \in E : k \neq i \neq l\}$ for every $F \subseteq E$ and $W \setminus i := \{j \in V : j \neq i\}$) depict a density *without the vertex* i.

2.5 Monte Carlo integration of the model

As explained in B, we can use a Monte Carlo method to estimate any expectation $\mathbb{E}[f(\sigma_1, \ldots, \sigma_N)]$ from the density $\mathcal{P}(\vec{\sigma})$ in 1. Particularly, we are interested in the magnetization, given by:

$$M(\beta) = \frac{1}{N} \sum_{i \in V_N} \langle \sigma_i \rangle_\beta$$

where:

[3] In this case, $\psi_{ij}(x_i, x_j) = \exp(\beta J_{ij} s_i s_j)$, $\phi_k(x_k) = \exp(\beta h s_k)$.

$$\langle \sigma_i \rangle_\beta = \sum_{\vec{\sigma}} \sigma_i \mathcal{P}(\vec{\sigma})$$

is the *thermal average* of σ_i.

Therefore, taking a (*relaxed*) sample $\{\vec{\Sigma}_t = (\sigma_1^{(t)}, \ldots, \sigma_N^{(t)})\}_{i=1}^k$ with the Metropolis-Hastings algorithm, we can estimate $\langle \sigma_i \rangle$ as:

$$\langle \sigma_i \rangle_\beta \approx \frac{1}{k} \sum_{t=1}^{k} \sigma_i^{(t)} \tag{25}$$

In this case we update separately every spin value σ_i for each Monte Carlo step, where the acceptance probability $\alpha(\vec{\Sigma}_t, Y)$ is given by:

$$\alpha(\vec{\Sigma}_t, Y) = \min\left(1, \exp\left(-\beta \Delta \mathcal{H}\right)\right) \tag{26}$$

where $\Delta \mathcal{H}$ is the change in the Ising Hamiltonian given by updating the spin vector $\vec{\Sigma}_t$ at the Monte Carlo step t. In the special case of flipping separately each spin $\sigma_i^{(t)} \mapsto \sigma_i'$ for every t, we have an useful expression for $\Delta \mathcal{H}$

$$\Delta \mathcal{H} = \left(\sigma_i^{(t)} - \sigma_i'\right) \left(h + \sum_{j \in \partial i} J_{ij}\, \sigma_j^{(t)}\right) \tag{27}$$

In resume, for each Monte Carlo step, we flip every spin $\sigma_i^{(t)} \mapsto \sigma_i' = -\sigma_i^{(t)}$ with probability:

$$\alpha(\sigma_i^{(t)}, \sigma_i') = \min\left\{1, \exp\left[-2\beta\sigma_i^{(t)} \left(h + \sum_{j \in \partial i} J_{ij}\, \sigma_j^{(t)}\right)\right]\right\}$$

2.6 The replica method

For the Hamiltonian in equation 7, let us add the following *generating field*

$$\mathcal{H}_\lambda(\vec{\sigma}, \mathbf{J}) = \mathcal{H}(\vec{\sigma}, \mathbf{J}) - \lambda \mathcal{O}(\vec{\sigma}, \mathbf{J}) \tag{28}$$

for $\lambda \in \mathbb{R}$ and $\mathcal{O}(\vec{\sigma}, \mathbf{J})$ any observable that is a function of $\vec{\sigma}$. If $Z_\lambda(\beta, \mathbf{J})$ is the corresponding partition function, then it is straightforward to show that the thermal average of $\mathcal{O}(\vec{\sigma}, \mathbf{J})$ (that is to say, the expectation value under the Boltzmann-Gibbs distribution) is given by:

$$\langle \mathcal{O}(\vec{\sigma}, \mathbf{J}) \rangle_\beta = \frac{1}{\beta} \left.\frac{\partial}{\partial \lambda}\right|_{\lambda=0} \log Z_\lambda(\beta, \mathbf{J}) \tag{29}$$

If we suppose that $\mathbf{J}$ has a probability density $\mathrm{d}\mathcal{F}(\mathbf{J})$ over a space $\mathcal{K}$ (such that $\mathbb{P}(\mathbf{J} \in \mathcal{J}) = \int_{\mathcal{J}} \mathrm{d}\mathcal{F}(\mathbf{J})$ for every measurable subset $\mathcal{J} \subseteq \mathcal{K}$), we may also calculate the *configurational average* of the observable $\mathcal{O}(\vec{\sigma}, \mathbf{J})$:

$$[\mathcal{O}(\vec{\sigma}, \mathbf{J})]_{\mathbf{J}} := \int_{\mathcal{K}} \mathcal{O}(\vec{\sigma}, \mathbf{J})\, \mathrm{d}\mathcal{F}(\mathbf{J}) \tag{30}$$

So that the *typical value* for the observable $\mathcal{O}$ can be written as:

$$\overline{\mathcal{O}} := [\langle \mathcal{O}(\vec{\sigma}, \mathbf{J}) \rangle_\beta]_{\mathbf{J}} \tag{31}$$

Let us recall that the *Helmholtz free energy* F is given by the relationship:

$$-\beta F = \log Z(\beta, \mathbf{J}) \tag{32}$$

Thus:

$$-\beta\, [F]_{\mathbf{J}} = [\log Z(\beta, \mathbf{J})]_{\mathbf{J}} \tag{33}$$

We are going to take advantage from the relationship[4]

$$[\log Z(\beta, \mathbf{J})]_{\mathbf{J}} = \lim_{n \to 0} \frac{[Z^n(\beta, \mathbf{J})]_{\mathbf{J}} - 1}{n} = \lim_{n \to 0} \frac{\log\, [Z^n(\beta, \mathbf{J})]_{\mathbf{J}}}{n} \tag{34}$$

That is to say: one prepares n replicas of the original system, evaluates the configurational average of the product of their partition functions (that is, $[Z^n(\beta, \mathbf{J})]_{\mathbf{J}}$) , and then takes the limit $n \to 0$[5]. This *replica method*

[4] Which follows straightforwardly from $X^n = 1 + n \log X + \mathrm{O}(n^2)$: by taking expectations ($\frac{\mathbb{E}[X^n]-1}{n} = \mathbb{E}\,[\log X] + \mathrm{O}(n^2)$, which gives first member); and combining with $\log(1+x) = x + \mathrm{O}(x^2)$ (which gives the second member: $\log \mathbb{E}\,[X^n] = n\mathbb{E}\,[\log X] + \mathrm{O}(n^2)$).

[5] This step assumes a kind of analytic continuation because the replicas make sense only for $n \in \mathbb{N}$.

is useful because it is easier to evaluate $[Z^n(\beta,\mathbf{J})]_{\mathbf{J}}$ than $[\log Z(\beta,\mathbf{J})]_{\mathbf{J}}$.

The problem then consists in obtaining an explicit form for the following expression:

$$\overline{\mathcal{O}} = \frac{1}{\beta}\left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0}\lim_{n\to 0}\frac{[Z^n_\lambda(\beta,\mathbf{J})]_{\mathbf{J}}-1}{n} = \frac{1}{\beta}\left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0}\lim_{n\to 0}\frac{\log[Z^n(\beta,\mathbf{J})]_{\mathbf{J}}}{n} \tag{35}$$

Now we are going to focus on the $[Z^n_\lambda(\beta,\mathbf{J})]_{\mathbf{J}}$ for the Ising Hamiltonian, where the generating field term λ corresponds to the external field h:

$$Z^n(\beta,\mathbf{J}) = \left(\sum_{\vec{\sigma}} e^{-\beta\mathcal{H}(\vec{\sigma},\mathbf{J})}\right)^n = \sum_{\vec{\sigma}_1}\cdots\sum_{\vec{\sigma}_n}\exp\left(\beta J\sum_{i<j}\sum_{\alpha=1}^{n} a_{ij}\,\sigma_i^{(\alpha)}\sigma_j^{(\alpha)} + \beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\right) \tag{36}$$

where we have assumed that the spin coupling is constant for the whole edge set ($\mathbf{J} = J\mathbf{A} = J(a_{ij})$, where $\mathbf{A}$ is the adjacency matrix with the Erdős-Rényi measure $\mathbb{P}(a_{ij}=1)=\frac{c}{N}$) and adopted the notation $\vec{\sigma}_\alpha := (\sigma_1^{(\alpha)},\dots,\sigma_N^{(\alpha)})$ for the spin configuration of the replica $\alpha\in\{1,\dots,n\}$.

Now let us evaluate the configurational average of $Z^n(\beta,\mathbf{J})$ by taking into account that the term $\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}$ does not depend on $\mathbf{J}$, and that

$$\begin{aligned}
\left[\exp\left(\beta J\sum_{i<j}a_{ij}\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)}\right)\right]_{\mathbf{J}} &= \prod_{i<j}\left[\exp\left(\beta J\,a_{ij}\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)}\right)\right]_{(a_{ij})} = \prod_{i<j}\left[\frac{c}{N}e^{\beta J\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)}} + \left(1-\frac{c}{N}\right)\right]\\
&= \exp\left\{\frac{1}{2}\sum_{i\neq j}\log\left[1+\frac{c}{N}\left(e^{\beta J\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)}}-1\right)\right]\right\}\\
&= \exp\left[\frac{c}{2N}\sum_{i\neq j}\left(e^{\beta J\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)}}-1\right)+\mathrm{O}\left(N^0\right)\right]\\
&= \exp\left\{\frac{c}{2N}\left[\sum_{i,j=1}^{N}\left(e^{\beta J\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)}}-1\right)-\sum_{i=1}^{N}\left(e^{\beta Jn}-1\right)\right]+\mathrm{O}\left(N^0\right)\right\}\\
&\approx \exp\left[\frac{c}{2N}\sum_{i,j=1}^{N}\left(e^{\beta J\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)}}-1\right)\right]
\end{aligned}$$

which follows from independence of the matrix elements (a_{ij}), and where the last approximate equality is indeed exact in the thermodynamic limit $N\longrightarrow\infty$. Therefore:

$$[Z^n(\beta,\mathbf{J})]_{\mathbf{J}} = \sum_{\vec{\sigma}_1}\cdots\sum_{\vec{\sigma}_n}\exp\left[\frac{c}{2N}\sum_{i,j=1}^{N}\left(e^{\beta J\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\sigma_j^{(\alpha)}}-1\right)+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\right] \tag{37}$$

Let us introduce some notation to represent configurations in the *replica space*. Firstly:

$$\underline{\sigma}_i := (\sigma_i^{(1)},\dots,\sigma_i^{(n)}),\quad i=1,\dots,N;$$

secondly:

$$\mathcal{P}(\underline{\sigma}) := \mathcal{P}(\underline{\sigma},\{\underline{\sigma}_i\}) := \frac{1}{N}\sum_{i=1}^{N}\delta_{\underline{\sigma},\underline{\sigma}_i}$$

and finally:

$$\underline{\sigma}_i\cdot\underline{\tau}_j := \sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\tau_j^{(\alpha)}$$

This is useful since:

$$\begin{aligned}
\sum_{i,j=1}^{N}\sum_{\underline{\sigma}}\delta_{\underline{\sigma},\underline{\sigma}_i}e^{\beta J\underline{\sigma}_i\cdot\underline{\sigma}_j} &= \sum_{i,j=1}^{N}\sum_{\underline{\sigma},\underline{\tau}}\delta_{\underline{\sigma},\underline{\sigma}_i}\delta_{\underline{\tau},\underline{\sigma}_j}e^{\beta J\underline{\sigma}\cdot\underline{\tau}}\\
&= \sum_{\underline{\sigma},\underline{\tau}}\left(\sum_{i=1}^{N}\delta_{\underline{\sigma},\underline{\sigma}_i}\right)\left(\sum_{j=1}^{N}\delta_{\underline{\tau},\underline{\sigma}_j}\right) = N^2\sum_{\underline{\sigma},\underline{\tau}}\mathcal{P}(\underline{\sigma})\mathcal{P}(\underline{\tau})e^{\beta J\underline{\sigma}\cdot\underline{\tau}}
\end{aligned} \tag{38}$$

and $[Z^n(\beta,\mathbf{J})]_\mathbf{J}$ can be expressed more succinctly:

$$[Z^n(\beta,\mathbf{J})]_\mathbf{J} = \sum_{\vec{\sigma}_1\cdots\vec{\sigma}_n} \exp\left[\frac{cN}{2}\sum_{\underline{\sigma},\,\underline{\tau}}\mathcal{P}(\underline{\sigma})\mathcal{P}(\underline{\tau})\left(e^{\beta J\underline{\sigma}\cdot\underline{\tau}}-1\right)+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\right] \tag{39}$$

Let us name $\mathcal{F}\left[\mathcal{P}\left(\underline{\sigma},\{\underline{\sigma}_i\}\right)\right] := \exp\left[\frac{cN}{2}\sum_{\underline{\sigma},\,\underline{\tau}}\mathcal{P}(\underline{\sigma})\mathcal{P}(\underline{\tau})\left(e^{\beta J\underline{\sigma}\cdot\underline{\tau}}-1\right)\right]$. By using a Dirac delta (accompanied with its Fourier representation $\delta(x-a)=\int\frac{\mathrm{d}\hat{x}}{2\pi}e^{i\hat{x}(x-a)}$), we have:

$$\begin{aligned}\sum_{\vec{\sigma}_1\cdots\vec{\sigma}_n}\mathcal{F}\left[\mathcal{P}\left(\underline{\sigma},\{\underline{\sigma}_i\}\right)\right] &= \sum_{\vec{\sigma}_1\cdots\vec{\sigma}_n}\int\left[\prod_{\underline{\sigma}\in\{-1,1\}^n}\mathrm{d}\mathcal{Q}(\underline{\sigma})\right]\mathcal{F}\left[\mathcal{Q}\left(\underline{\sigma}\right)\right]\prod_{\underline{\sigma}\in\{-1,1\}^n}\delta\left[\mathcal{Q}(\underline{\sigma})-\mathcal{P}\left(\underline{\sigma},\{\underline{\sigma}_i\}\right)\right]\\ &= \sum_{\vec{\sigma}_1\cdots\vec{\sigma}_n}\int\left[\prod_{\underline{\sigma}\in\{-1,1\}^n}\frac{\mathrm{d}\mathcal{Q}(\underline{\sigma})\mathrm{d}\hat{\mathcal{Q}}(\underline{\sigma})}{2\pi/N}\right]\mathcal{F}\left[\mathcal{Q}\left(\underline{\sigma}\right)\right]\exp\left\{iN\sum_{\underline{\sigma}}\hat{\mathcal{Q}}(\underline{\sigma})\left[\mathcal{Q}(\underline{\sigma})-\frac{1}{N}\sum_{i=1}^{N}\delta_{\underline{\sigma},\underline{\sigma}_i}\right]\right\}\end{aligned} \tag{40}$$

Rewriting the partition function[6]:

$$\begin{aligned}[Z^n(\beta,\mathbf{J})]_\mathbf{J} &= \sum_{\vec{\sigma}_1\cdots\vec{\sigma}_n}\int\left[\prod_{\underline{\sigma}\in\{-1,1\}^n}\frac{\mathrm{d}\mathcal{Q}(\underline{\sigma})\mathrm{d}\hat{\mathcal{Q}}(\underline{\sigma})}{2\pi/N}\right]\exp\left[\frac{cN}{2}\sum_{\underline{\sigma},\,\underline{\tau}}\mathcal{Q}(\underline{\sigma})\mathcal{Q}(\underline{\tau})\left(e^{\beta J\underline{\sigma}\cdot\underline{\tau}}-1\right)\right]\cdot\\ &\quad\cdot\exp\left\{iN\sum_{\underline{\sigma}}\hat{\mathcal{Q}}(\underline{\sigma})\left[\mathcal{Q}(\underline{\sigma})-\frac{1}{N}\sum_{i=1}^{N}\delta_{\underline{\sigma},\underline{\sigma}_i}\right]\right\}\cdot\exp\left(\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\right)\\ &= \sum_{\vec{\sigma}_1\cdots\vec{\sigma}_n}\int\mathcal{D}\left[\mathcal{Q},\hat{\mathcal{Q}}\right]\exp\left\{iN\sum_{\underline{\sigma}}\hat{\mathcal{Q}}(\underline{\sigma})\mathcal{Q}(\underline{\sigma})+\frac{cN}{2}\sum_{\underline{\sigma},\,\underline{\tau}}\mathcal{Q}(\underline{\sigma})\mathcal{Q}(\underline{\tau})\left(e^{\beta J\underline{\sigma}\cdot\underline{\tau}}-1\right)+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}-\right.\\ &\quad\left.-i\sum_{\underline{\sigma}}\hat{\mathcal{Q}}(\underline{\sigma})\sum_{i=1}^{N}\delta_{\underline{\sigma},\underline{\sigma}_i}\right\}\\ &= \int\mathcal{D}\left[\mathcal{Q},\hat{\mathcal{Q}}\right]\exp\left[iN\sum_{\underline{\sigma}}\hat{\mathcal{Q}}(\underline{\sigma})\mathcal{Q}(\underline{\sigma})+\frac{cN}{2}\sum_{\underline{\sigma},\,\underline{\tau}}\mathcal{Q}(\underline{\sigma})\mathcal{Q}(\underline{\tau})\left(e^{\beta J\underline{\sigma}\cdot\underline{\tau}}-1\right)\right]\cdot\\ &\quad\cdot\sum_{\vec{\sigma}_1,\ldots,\vec{\sigma}_N}\exp\left[-i\sum_{\underline{\sigma}}\hat{\mathcal{Q}}(\underline{\sigma})\sum_{i=1}^{N}\delta_{\underline{\sigma},\underline{\sigma}_i}+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\right]\end{aligned} \tag{41}$$

where $\mathcal{D}\left[\mathcal{Q},\hat{\mathcal{Q}}\right]$ stands for $\prod_{\underline{\sigma}\in\{-1,1\}^n}\frac{\mathrm{d}\mathcal{Q}(\underline{\sigma})\mathrm{d}\hat{\mathcal{Q}}(\underline{\sigma})}{2\pi/N}$. And if we note that:

$$\begin{aligned}\sum_{\vec{\sigma}_1,\ldots,\vec{\sigma}_N}\exp\left[-i\sum_{\underline{\sigma}}\hat{\mathcal{Q}}(\underline{\sigma})\sum_{i=1}^{N}\delta_{\underline{\sigma},\underline{\sigma}_i}+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\right] &= \sum_{\underline{\sigma}_1,\cdots,\underline{\sigma}_N}\exp\left[-i\sum_{j=1}^{N}\hat{\mathcal{Q}}(\underline{\sigma}_j)+\beta h\sum_{i=1}^{N}\sum_{\alpha=1}^{n}\sigma_i^{(\alpha)}\right]\\ &= \sum_{\underline{\sigma}}\exp\left[-i\hat{\mathcal{Q}}(\underline{\sigma})+\beta\sum_{\alpha=1}^{n}\sigma^{(\alpha)}\right]\end{aligned}$$

then we get the following nice formula:

$$\begin{aligned}[Z^n(\beta,\mathbf{J})]_\mathbf{J} = \int\mathcal{D}\left[\mathcal{Q},\hat{\mathcal{Q}}\right]\exp&\left[iN\sum_{\underline{\sigma}}\hat{\mathcal{Q}}(\underline{\sigma})\mathcal{Q}(\underline{\sigma})+\frac{cN}{2}\sum_{\underline{\sigma},\,\underline{\tau}}\mathcal{Q}(\underline{\sigma})\mathcal{Q}(\underline{\tau})\left(e^{\beta J\underline{\sigma}\cdot\underline{\tau}}-1\right)+\right.\\ &\left.+N\log\sum_{\underline{\sigma}}e^{-i\hat{\mathcal{Q}}(\underline{\sigma})+\beta\sum_{\alpha=1}^{n}\sigma^{(\alpha)}}\right]\end{aligned} \tag{42}$$

[6]This is, in essence, having used the functional Dirac delta: $\mathcal{F}\left[f(x)\right]=\int\mathcal{D}g\,\mathcal{F}\left[g(x)\right]\,\delta\left[g(x)-f(x)\right]$

which has the form of a path integral partition function[7]:

$$[Z^n(\beta,\mathbf{J})]_{\mathbf{J}} = \int \mathcal{D}\left[\mathcal{Q},\hat{\mathcal{Q}}\right] e^{-N\mathcal{S}[\mathcal{Q},\hat{\mathcal{Q}}]} \asymp e^{-N\mathcal{S}[\mathcal{Q}_0,\hat{\mathcal{Q}}_0]} \tag{43}$$

with the *saddle point method* conditions:

$$\left.\frac{\delta\mathcal{S}}{\delta\mathcal{Q}}\right|_{\mathcal{Q}=\mathcal{Q}_0} = 0, \quad \left.\frac{\delta\mathcal{S}}{\delta\hat{\mathcal{Q}}}\right|_{\hat{\mathcal{Q}}=\hat{\mathcal{Q}}_0} = 0. \tag{44}$$

over the (*action*) function:

$$\mathcal{S}\left(\mathcal{Q},\hat{\mathcal{Q}}\right) := -i\sum_{\underline{\sigma}} \hat{\mathcal{Q}}(\underline{\sigma})\mathcal{Q}(\underline{\sigma}) - \frac{c}{2}\sum_{\underline{\sigma},\underline{\tau}} \mathcal{Q}(\underline{\sigma})\mathcal{Q}(\underline{\tau})\left(e^{\beta J\underline{\sigma}\cdot\underline{\tau}}-1\right) - \log\sum_{\underline{\sigma}} e^{-i\hat{\mathcal{Q}}(\underline{\sigma})+\beta\sum_{\alpha=1}^{n}\sigma^{(\alpha)}} \tag{45}$$

But since we have the following relations for the *Helmholtz free energy per particle* $f := F/N$:

$$\begin{aligned}\lim_{N\to\infty}\frac{-\beta\,[F]_{\mathbf{J}}}{N} &= \lim_{N\to\infty}\frac{1}{N}\left[\log Z(\beta,\mathbf{J})\right]_{\mathbf{J}} = \lim_{N\to\infty}\frac{1}{N}\lim_{n\to0}\frac{1}{n}\log\left[Z^n(\beta,\mathbf{J})\right]_{\mathbf{J}} \\ &= \lim_{n\to0}\frac{1}{n}\lim_{N\to\infty}\frac{1}{N}\log\left[Z^n(\beta,\mathbf{J})\right]_{\mathbf{J}} = -\lim_{n\to0}\frac{1}{n}\mathcal{S}\left(\mathcal{Q}_0,\hat{\mathcal{Q}}_0\right)\end{aligned} \tag{46}$$

then:

$$\therefore \beta f = \lim_{n\to0}\frac{1}{n}\mathcal{S}\left(\mathcal{Q}_0,\hat{\mathcal{Q}}_0\right) \tag{47}$$

Now we apply the first part of 44 to 45 to obtain:

$$\begin{aligned}0 &= -i\sum_{\underline{\sigma}}\hat{\mathcal{Q}}(\underline{\sigma})\,\delta_{\underline{\sigma},\underline{\gamma}} - \frac{c}{2}\sum_{\underline{\sigma},\underline{\tau}}\left(\delta_{\underline{\sigma},\underline{\gamma}}\mathcal{Q}(\underline{\tau}) + \delta_{\underline{\tau},\underline{\gamma}}\mathcal{Q}(\underline{\sigma})\right)\left(e^{\beta J\underline{\sigma}\cdot\underline{\tau}}-1\right) \\ &= -i\hat{\mathcal{Q}}(\underline{\gamma}) - \frac{c}{2}\sum_{\underline{\tau}}\mathcal{Q}(\underline{\tau})\left(e^{\beta J\underline{\gamma}\cdot\underline{\tau}}-1\right) - \frac{c}{2}\sum_{\underline{\sigma}}\mathcal{Q}(\underline{\sigma})\left(e^{\beta J\underline{\sigma}\cdot\underline{\gamma}}-1\right) \\ &= -i\hat{\mathcal{Q}}(\underline{\gamma}) - c\sum_{\underline{\sigma}}\mathcal{Q}(\underline{\sigma})\left(e^{\beta J\underline{\sigma}\cdot\underline{\gamma}}-1\right)\end{aligned}$$

$$\therefore \quad c\sum_{\underline{\sigma}}\mathcal{Q}(\underline{\sigma})\left(e^{\beta J\underline{\sigma}\cdot\underline{\gamma}}-1\right) = -i\hat{\mathcal{Q}}(\underline{\gamma}) \tag{48}$$

where we have used $\frac{\delta\mathcal{Q}(\underline{\sigma})}{\delta\mathcal{Q}(\underline{\gamma})} = \frac{\partial\mathcal{Q}(\underline{\sigma})}{\partial\mathcal{Q}(\underline{\gamma})} = \delta_{\underline{\sigma},\underline{\gamma}}$. Analogously, we use the second part of 44 to the relation 45 and this yields:

$$0 = -i\mathcal{Q}(\underline{\gamma}) - \frac{-i\sum_{\underline{\sigma}}\delta_{\underline{\sigma},\underline{\gamma}}e^{-i\hat{\mathcal{Q}}(\underline{\sigma})}}{\sum_{\underline{\sigma}}e^{-i\hat{\mathcal{Q}}(\underline{\sigma})}}$$

$$\therefore \quad \mathcal{Q}(\underline{\gamma}) = \frac{e^{-i\hat{\mathcal{Q}}(\underline{\gamma})}}{\sum_{\underline{\sigma}}e^{-i\hat{\mathcal{Q}}(\underline{\sigma})}} \tag{49}$$

By combining those two results we get:

$$\mathcal{Q}(\underline{\sigma}) \propto \exp\left[c\sum_{\underline{\tau}}\left(e^{\beta J\underline{\tau}\cdot\underline{\sigma}}-1\right)\mathcal{Q}(\underline{\tau})\right] \tag{50}$$

By means of the *replica symmetric ansatz* (RS), where we assume that interchanging two spins in the replica space does not affect the density $\mathcal{Q}(\underline{\sigma})$, we can propose the generic expression:

$$\mathcal{Q}(\underline{\sigma}) = \int w(h)\prod_{\alpha=1}^{n}\rho\left(\sigma^{(\alpha)}\middle|\,h\right)\mathrm{d}h \tag{51}$$

[7]The saddle point asymptotic relation can be accomplished by taking the thermodynamic limit $N \longrightarrow \infty$.

for a density $\rho\left(\sigma^{(\alpha)}|\, h\right)$ that depends on a parameter h distributed according to $w(h)$. We plug this into equation 50, together with the exponential Taylor series:

$$\begin{aligned}\mathcal{Q}(\underline{\sigma}) &= \int w(h)\prod_{\alpha=1}^{n}\rho\left(\sigma^{(\alpha)}|\, h\right)\mathrm{d}h \propto e^{-c}\sum_{l=0}^{\infty}\frac{c^l}{l!}\left(\sum_{\underline{\tau}} e^{\beta J\underline{\tau}\cdot\underline{\sigma}}\int w(h)\prod_{\alpha=1}^{n}\rho\left(\tau^{(\alpha)}|\, h\right)\mathrm{d}h\right)^l\\ &= e^{-c}\sum_{l=0}^{\infty}\frac{c^l}{l!}\int \mathrm{d}h_1\cdots\mathrm{d}h_l\, w(h_1)\cdots w(h_l)\sum_{\underline{\tau}_1,\cdots,\underline{\tau}_l}\left(e^{\beta J\sum_{k=1}^{l}\underline{\tau}_k\cdot\underline{\sigma}}\right)\prod_{k=1}^{l}\prod_{\alpha=1}^{n}\rho\left(\tau^{(\alpha)}|\, h_k\right)\\ &= e^{-c}\sum_{l=0}^{\infty}\frac{c^l}{l!}\int\left(\prod_{k=1}^{l}\mathrm{d}h_k\, w(h_k)\right)\left(\prod_{k=1}^{l}\sum_{\underline{\tau}}\prod_{\alpha=1}^{n}e^{\beta J\sigma^{(\alpha)}\tau^{(\alpha)}}\right)\prod_{k=1}^{l}\prod_{\alpha=1}^{n}\rho\left(\tau^{(\alpha)}|\, h_k\right)\\ &= e^{-c}\sum_{l=0}^{\infty}\frac{c^l}{l!}\int\left(\prod_{k=1}^{l}\mathrm{d}h_k\, w(h_k)\right)\prod_{\alpha=1}^{n}\prod_{k=1}^{l}\sum_{\tau}\rho\left(\tau|\, h_k\right)e^{\beta J\sigma^{(\alpha)}\tau}\end{aligned} \tag{52}$$

where the proportionality constant being precisely the normalization factor $\sum_{\sigma^{(\alpha)}}\prod_{k=1}^{l}\sum_{\tau}\rho\left(\tau|\, h_k\right)\exp(\beta J\sigma^{(\alpha)}\tau)$. This can be thought as an useful constant α for which, particularly, holds the functional relation $\rho^* = \int \alpha \mathrm{d}\rho\delta\left(\rho-\frac{\rho^*}{\alpha}\right)$. So we use this and then it follows that:

$$\begin{aligned}\mathcal{Q}(\underline{\sigma}) = \int w(h)\prod_{\alpha=1}^{n}\rho\left(\sigma^{(\alpha)}|\, h\right)\mathrm{d}h = \int \mathrm{d}h\sum_{l=0}^{\infty}\frac{c^l e^{-c}}{l!}\int\left(\prod_{k=1}^{l}\mathrm{d}h_k\, w(h_k)\right)\prod_{\alpha=1}^{n}\rho\left(\tau^{(\alpha)}|\, h\right)\cdot\\ \cdot\,\delta\left(\rho\left(\sigma^{(\alpha)}|\, h\right)-\frac{\prod_{k=1}^{l}\sum_{\tau}\rho\left(\tau|\, h_k\right)e^{\beta J\sigma^{(\alpha)}\tau}}{\sum_{\sigma^{(\alpha)}}\prod_{k=1}^{l}\sum_{\tau}\rho\left(\tau|\, h_k\right)e^{\beta J\sigma^{(\alpha)}\tau}}\right)\left(\sum_{\sigma^{(\alpha)}}\prod_{k=1}^{l}\sum_{\tau}\rho\left(\tau|\, h_k\right)e^{\beta J\sigma^{(\alpha)}\tau}\right)^n\end{aligned} \tag{53}$$

and therefore:

$$\begin{aligned}w(h) = \sum_{l=0}^{\infty}\frac{c^l e^{-c}}{l!}\int\left(\prod_{k=1}^{l}\mathrm{d}h_k\, w(h_k)\right)\delta\left(\rho\left(\sigma^{(\alpha)}|\, h\right)-\frac{\prod_{k=1}^{l}\sum_{\tau}\rho\left(\tau|\, h_k\right)e^{\beta J\sigma^{(\alpha)}\tau}}{\sum_{\sigma^{(\alpha)}}\prod_{k=1}^{l}\sum_{\tau}\rho\left(\tau|\, h_k\right)e^{\beta J\sigma^{(\alpha)}\tau}}\right)\cdot\\ \cdot\left(\sum_{\sigma^{(\alpha)}}\prod_{k=1}^{l}\sum_{\tau}\rho\left(\tau|\, h_k\right)e^{\beta J\sigma^{(\alpha)}\tau}\right)^n\end{aligned} \tag{54}$$

As before (equation 13), we make the following parametrization:

$$\rho\left(\sigma|\, h\right) = \frac{\exp\left(\beta\sigma h\right)}{2\cosh\left(\beta h\right)} \tag{55}$$

from which follows straightforwardly:

$$h = \frac{1}{2\beta}\sum_{\sigma}\sigma\log\left[\rho\left(\sigma|\, h\right)\right] \tag{56}$$

We introduce this into equation 54 to get:

$$h = \frac{1}{2\beta}\sum_{\sigma}\sigma\log\left[\frac{\prod_{k=1}^{l}\sum_{\tau}\rho\left(\tau|\, h_k\right)e^{\beta J\sigma^{(\alpha)}\tau}}{\sum_{\sigma^{(\alpha)}}\prod_{k=1}^{l}\sum_{\tau}\rho\left(\tau|\, h_k\right)e^{\beta J\sigma^{(\alpha)}\tau}}\right] \tag{57}$$

We can rearrange and remember that $\sigma\in\{-1,1\}$, so the previous equation simplifies to (analogously to equation 15):

$$h = \frac{1}{\beta}\sum_{k=1}^{l}\mathrm{atan}\left[\tanh\left(\beta J\right)\tanh\left(\beta h_k\right)\right] \tag{58}$$

Plugging into 54 and taking the limit $n\to 0$:

$$w(h) = \sum_{l=0}^{\infty}\frac{c^l e^{-c}}{l!}\int\left(\prod_{k=1}^{l}\mathrm{d}h_k\, w(h_k)\right)\delta\left(h-\frac{1}{\beta}\sum_{k=1}^{l}\mathrm{atan}\left[\tanh\left(\beta J\right)\tanh\left(\beta h_k\right)\right]\right) \tag{59}$$

3 Percolation transition of a stochastic graph

Appendix A The saddle point method

In this section we will present a way to approximate integrals of the form $I(\lambda) = \int_a^b \mathrm{d}y \exp\left(-S(y)/\lambda\right)$ in the limit of $\lambda \to 0^+$, which is equivalent to analyze expressions of the form $J(x) = \int_a^b \mathrm{d}y \exp\left(-xS(y)\right)$ in the regime of $x \to \infty$. Then, we will show how this extends to integrals in the complex plane of the form $I(\alpha) = \int_{\mathcal{C}} e^{\alpha f(z)}\mathrm{d}z$.

A.1 Introductory example

Let us recall the definition of the Gamma function:

$$\Gamma(x+1) \doteq \int_0^\infty \mathrm{d}t\, t^x e^{-t} = \int_0^\infty \mathrm{d}t\, e^{x \log t}$$

and make the change of variables $t = xy$ to get the following integral expression:

$$\Gamma(x+1) = x\int_0^\infty \mathrm{d}y\, e^{-xy + x\log y + x\log x} = x^{x+1}\int_0^\infty \mathrm{d}y\, e^{-x(y-\log y)} = x^{x+1}\int_0^\infty \mathrm{d}y\, e^{-xS(y)}$$

A way to obtain an asymptotic expression for this function in the limit of large x, is to calculate a Taylor's series expansion of $S(y) := y - \log y$ near its maximum (in this case, $y_0 = 1$), and then try to reduce the whole integral. We have:

$$S(y) = S(1) + S'(1)(y-1) + \frac{1}{2!}(y-1)^2 + \cdots = 1 + \frac{1}{2}(y-1)^2 + \cdots$$

Substituting in the above integral keeping up to the second order terms:

$$\Gamma(x+1) \simeq x^{x+1}\int_0^\infty \mathrm{d}y\, e^{-x(1+\frac{1}{2}(y-1)^2)} = x^{x+1}e^{-x}\int_0^\infty \mathrm{d}y\, e^{-\frac{(y-1)^2}{2x^{-1}}}$$

We can recognize the last integral as a Gaussian: $\int_{-\infty}^{\infty} \mathrm{d}y\, e^{-\frac{(y-\mu)^2}{2\sigma^2}} = \sqrt{2\pi\sigma^2}$ (with $\sigma^2 = \frac{1}{x}$), except for the integral limits. But if we assume that $x \gg 1$, we would have a very narrow Gaussian-like function around $y = 1$, so by taking the integral over the whole real line we will get:

$$I = x^{x+1}e^{-x}\sqrt{\frac{2\pi}{x}} = x^x e^{-x}\sqrt{2\pi x}$$

This is precisely the Stirling approximation.

The method of steepest descents is applicable, in general, to integrals of the form $I(\alpha) = \int_{\mathcal{C}} e^{\alpha f(z)}\mathrm{d}z$, where $\mathcal{C}$ is a path in the complex plane such that the ends of the path do not contribute significantly to the integral. This method usually gives the first term in an asymptotic expansion of $I(\alpha)$, valid for large a.

A.2 The complex case

Let $f(z) = f(x+iy) = u(x+iy) + iv(x+iy)$. The integral $I(\alpha)$ should have the greatest contributions from those points where $u(x+iy)$ has the greatest absolute value; that is to say, where $\nabla u = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = 0$. But if we assume that f is holomorphic, then the Cauchy-Riemann equations would imply that $\nabla v = 0$ and $\nabla^2 u = 0 = \nabla^2 v$, so the surfaces $u(x+iy)$, $v(x+iy)$ are *saddle-like* in the points $z = z_0 = x_0 + iy_0$ where $f'(z_0) = 0$.

In a neighborhood of a point like these, we have the Taylor expansion:

$$f(z) = f(z_0) + \frac{1}{2}f''(z_0)(z-z_0)^2 + \mathcal{O}\left((z-z_0)^3\right)$$

By means of a polar representation:

$$f''(z_0) = \rho e^{i\theta},\ z - z_0 = se^{i\phi}$$

we have the following expressions for u and v (up to third order corrections):

$$u \simeq (z_0) + \frac{1}{2}\rho s^2\cos(\theta + 2\phi)$$
$$v \simeq v(z_0) + \frac{1}{2}\rho s^2\sin(\theta + 2\phi)$$

The greatest absolute value in the u function can be reached for $\theta + 2\phi = k\pi$, with $k \in \mathbb{Z}$ (reason for which this is the method of *steepest descent*), which at the same time override the second degree term in v, making it constant. Thus we get the following expression for $I(\alpha)$ under the condition $\theta + 2\phi = \pm\pi$ (so we can get a convergent integral):

$$I(\alpha) = \int_{\mathcal{C}} e^{\alpha f(z)} \mathrm{d}z \simeq e^{\alpha f(z_0)} \int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}\rho s^2} e^{i\phi} \mathrm{d}s$$

whenever the path $\mathcal{C}$ is warped to follow the steepest descent: $\mathrm{d}z = \frac{z-z_0}{|z-z_0|}\mathrm{d}s = e^{i\phi}\mathrm{d}s$, with $\phi = \frac{\pm\pi-\theta}{2}$ ($\theta = \mathrm{ang}(f''(z_0))$) chosen as to follow the correct direction of integration.

Solving for the Gaussian integral, we have:

$$I(\alpha) \simeq \sqrt{\frac{2\pi}{\alpha\,|f''(z_0)|}} e^{\alpha f(z_0)} e^{\frac{i}{2}(\pm\pi - \mathrm{ang}(f''(z_0)))}$$

A.3 Further examples

1. Complex gamma function.
2. Hermite polynomials.

This method...

Appendix B The Metropolis-Hastings algorithm

B.1 Monte Carlo integration

Monte Carlo integration evaluates expressions of the form

$$\mathbb{E}\left[f(X)\right] = \frac{\int f(x)\pi(x)\,\mathrm{d}x}{\int \pi(x)\,\mathrm{d}x} \tag{60}$$

for some function of interest f (even with the possibility that $\int \pi(x)\,\mathrm{d}x$ is unknown) by drawing samples $\{X_t : t = 1, \ldots, n\}$ from π, and then making the following *sample mean estimation*[8]:

$$\mathbb{E}\left[f(X)\right] \approx \frac{1}{n}\sum_{t=1}^{n} f\left(X_t\right) \tag{61}$$

One way of doing this is through a Markov chain having $\pi(x)$ as its stationary distribution.

B.2 Markov Chains

Let $\{X_0, X_1, X_2, \ldots\}$ be a time-homogeneous Markov chain over $S \subseteq \mathbb{R}^m$ such that X_{t+1} given X_t has a density[9] $p(x_{t+1}|x_t)$. By the total probability formula, we have the following expression for the density $q_{t+1}(x)$ of X_{t+1}:

$$q_{t+1}(x) = \int p(x|y)q_t(y)\,\mathrm{d}y \qquad \forall x \in S \tag{62}$$

Under suitable conditions, the distribution of X_t will not depend on t nor X_0, and will be given by the stationary distribution $\pi(x)$ which satisfies:

$$\pi(x) = \int p(x|y)\pi(y)\,\mathrm{d}y \qquad \forall x \in S \tag{63}$$

Thus, after a sufficiently long *burn-in* of say m iterations, points $\{X_t : t = m+1, \ldots, n\}$ will be (possibly dependent) samples approximately from $\pi(x)$. So it is possible to use the output from the Markov chain to estimate the expectation $\mathbb{E}[f(X)]$, where X has distribution $\pi(x)$. Burn-in samples are usually discarded for this calculation, giving an estimator:

$$\mathbb{E}\left[f(X)\right] \approx \frac{1}{n-m}\sum_{t=m+1}^{n} f\left(X_t\right) \tag{64}$$

Now let us state a precise Markov Chain that accomplish this: set X_0 arbitrarily, and for each $t \geq 0$ draw Y according to $p(y|X_t)$. Let:

[8] This approximation is ensured by the laws of large numbers.
[9] Also called *stochastic kernel*.

$$X_{t+1} = \begin{cases} Y, \text{ with probability } \alpha(X_t, Y) := \min\left(1, \frac{\pi(Y)p(X_t|Y)}{\pi(X_t)p(Y|X_t)}\right) \\ X_t, \text{ with probability } 1 - \alpha(X_t, Y) \end{cases} \quad (65)$$

where $\alpha(X_t, Y)$ is the *acceptance probability*.
Written in pseudo code, we have the following *recipe*:

1. `Initialize` X_0 `and set` $t = 0$.
2. `Repeat:`
   - `Sample a point` Y `according to` $p(y|X_t)$.
   - `Sample a number` U `according to Uniform`$(0, 1)$.
   - `If` $U < \alpha(X_t, Y) := \min\left(1, \frac{\pi(Y)p(X_t|Y)}{\pi(X_t)p(Y|X_t)}\right)$`, set` $X_{t+1} = Y$`. Otherwise, set` $X_{t+1} = X_t$.
   - $t+ = 1$

Any proposal distribution $p(x|y)$ will ultimately deliver samples from the target distribution $\pi(x)$. However, the rate of convergence to the stationary distribution will depend crucially on the relationship between them. Moreover, having *converged*, the chain may still mix slowly (i.e. move slowly around the support of $\pi(x)$).

The Metropolis algorithm (1953) considers only symmetric proposals, having the form $p(x|y) = p(y|x)$ for all x and y, so the acceptance probability reduces to:

$$\alpha(X_t, Y) := \min\left(1, \frac{\pi(Y)}{\pi(X_t)}\right) \quad (66)$$

A particular case is $p(x|y) \equiv p(|x - y|)$ (*random-walk Metropolis*). For example, we can update the process by proposing $Y = X_t + x$ for each t, where $x \sim \mathrm{N}(0, 1)$.

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