Theeradon Sarawek 6680210

HW2

Problem 1

$$1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6} \quad \forall n \in \mathbb{Z}^+$$

Predicate

$$P(i) \Rightarrow 1^2+2^2+\dots+i^2 = \frac{i(i+1)(2i+1)}{6}$$

Base case

$$P(1) = 1^2 = \frac{1(1+1)(2+1)}{6} \Rightarrow \frac{1(2)(3)}{6} = \frac{6}{6} = 1 \checkmark$$

inductive step

$$p(k) = 1^2+\dots+k^2 = \frac{k(k+1)(2k+1)}{6}$$

$$p(k) = 1^2+\dots+k^2+(k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$\Rightarrow \frac{k(k+1)(2k+1)}{6} + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$$

$\downarrow$

$$\frac{(k^2+k)(2k+1)}{6} + (k+1)^2 = \frac{2k^3+3k^2+k}{6} + \frac{(k+1)^2}{1}$$

$$= \frac{2k^3+3k^2+k+6(k+1)^2}{6}$$

$(k+1)^2 = k^2+2k+1$

$(k+1)(k+1)$

$k^2+k+k+1$

$6(k^2+2k+1)$

$= 6k^2+12k+6$

$$= \frac{2k^3+3k^2+k+6k^2+12k+6}{6}$$

(1) $$= \frac{2k^3+9k^2+13k+6}{6}$$

consider

$$\frac{(k+1)(k+2)(2k+3)}{6} = \frac{(k^2+3k+2)(2k+3)}{6}$$

$$= \frac{2k^3+6k^2+4k+3k^2+9k+6}{6}$$

$$② \quad = \frac{2k^3+9k^2+13k+6}{6}$$

Therefore, since ① = ②, we have proven the statement by induction.

Problem 2

$$1^3+2^3+\ldots+n^3 = (1+2+\ldots+n)^2 \quad \forall n \in \mathbb{Z}^+$$

Predicate

$P(i) \Rightarrow 1^3+2^3+\ldots+i^3 \Rightarrow (1+2+\ldots+i)^2$

Base case

$P(1) = 1^3 = (1)^2 = 1$ ✓

Induction step

$P(k) = 1^3+\ldots+k^3 = (1+\ldots+k)^2$

$P(k+1) = 1^3+\ldots+k^3+(k+1)^3 = (1+\ldots+(k+1))^2$

Recall that sum of natural numbers $1 \to n = \frac{n(n+1)}{2}$

$$\Rightarrow ① \left(\frac{(k+1)(k+1+1)}{2}\right)^2 = ② \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

① $\left[\frac{(k+1)(k+2)}{2}\right]^2 = \frac{(k+1)^2(k+2)^2}{4}$

② $\frac{(k^2)(k+1)^2}{4} + (k+1)^3 \Rightarrow \frac{(k^2)(k+1)^2 + 4(k+1)^3}{4}$

$(k+1)^2 = k^2+2k+1$ $\quad$ $(k+2)^2 = k^2+4k+4$

$(k+1)^3 = k+1\,(k^2+2k+1)$

$= k^3+k^2+2k^2+2k+k+1$

$= k^3+3k^2+3k+1$

① $\frac{(k^2+2k+1)(k^2+4k+4)}{4}$

$=$ ② $\frac{(k^2)(k^2+2k+1) + 4k^3+12k^2+12k+4}{4}$

Expand ① $\frac{k^4+4k^3+4k^2+2k^3+8k^2+8k+k^2+4k+4}{4}$

$= \frac{k^4+6k^3+13k^2+12k+4}{4}$

Expand (2) $k^4+2k^3+k^2+4k^3+12k^2+12k+4$

$= \frac{k^4+6k^3+13k^2+12k+4}{4}$

equal

Therefore, through proof by induction, we've proven the original statement

Problem 3

$$F_{n+1} = F_n + F_{n-1}$$

$$F_2 + F_4 + F_6 + \ldots + F_{2n} = F_{2n+1} - 1$$

$\Rightarrow$ Even numbers e can be expressed as $e = 2n$ where $e \geq 1$ and $e \in \mathbb{Z}$

Predicate

$$\sum_{i=1}^{i} F_{2i} = F_{2i+1} - 1$$

Base case

$P(1) = F_2 = F_3 - 1$

$1 = 2 - 1$ ✓

consider

$F_{n+1} = F_n + F_{n-1}$

when $n \geq 2$

Inductive step

$P(k) = F_2 + \ldots + F_k = F_{k+1} - 1$

$P(k+2) = F_2 + \ldots + F_k + F_{k+2} = F_{(k+2)+1} - 1$

$\hookrightarrow \quad F_{k+1} - 1 + F_{k+2} = F_{k+3} - 1$

Rearrange $\Rightarrow$ $F_{k+1} + F_{k+2} - 1 = F_{k+3} - 1$

Cancel out 1 $\quad \boxed{F_{k+1} + F_{k+2} = F_{k+3}}$

$$\left.\begin{aligned} &\text{if } k+3 = n+1 \\ &\hookrightarrow k+2 = n \\ &\quad k+1 = n-1 \end{aligned}\right\} \text{applies to the formula given at the start.}$$

Therefore, we've proven the statement by induction.

Problem 4

a is divisible by b if $\exists n \in \mathbb{Z}$ where $a = nb$

↳ also denoted as $b \mid a$ or "b divides a".

4.1) Show $5 \mid 8^n - 3^n \ \forall n \in \mathbb{Z}^+$

$$5n = 8^n - 3^n \Rightarrow \overset{\text{predicate}}{p(i) \Rightarrow 5 \mid 8^i - 3^i}$$

Base case $\Rightarrow 8^1 - 3^1 = 5$

$5n = 5 \Rightarrow n = 1$ which $n \in \mathbb{Z}^+$ ✓

inductive step

$p(k) = 5 \mid 8^k - 3^k \Rightarrow 8^k - 3^k = 5a$

$p(k+1) = 5 \mid 8^{k+1} - 3^{k+1} \Rightarrow 8^{k+1} - 3^{k+1} = 5b$

$$8^{k+1} - 3^{k+1} = 8(8^k) - 3(3^k)$$

$$\Downarrow$$

$$= 8 \cdot 8^k \left(- 8 \cdot 3^k + 8 \cdot 3^k\right) - 3 \cdot 3^k$$

$$= 8(8^k - 3^k) + 8(3^k) - 3(3^k)$$

$$= 8(\underbrace{5a}_{\text{per IH}}) + 5(3^k)$$

$$= 40a + 5(3^k)$$

$$= 5(8a + 3^k)$$

where $b = 8a + 3^k$

$\therefore 5 \mid 8^n - 5^n \ \exists n \in \mathbb{Z}^+$

☆

4.2) Given $a, b \in I$ and $n \in I^+$

show that $a-b \mid a^n - b^n$

↳ $a^n - b^n = p(a-b)$ where $p \in I^+$

Base case

↳ $a-b \mid a^1 - b^1$

$P(1)$ $a - b = p(a-b)$

↳ $p = \dfrac{a-b}{a-b} = 1$

Inductive step

$P(k)$ $a-b \mid a^k - b^k \Rightarrow a^k - b^k = p(a-b)$

$P(k+1)$ $a-b \mid a^{k+1} - b^{k+1} \Rightarrow a^{k+1} - b^{k+1} = q(a-b)$

$$\Downarrow$$

$$
\begin{aligned}
a^{k+1} - b^{k+1} &= a(a^k) - b(b^k) \\
&= a(a^k) - a(b^k) + a(b^k) - b(b^k) \\
&= a(a^k - b^k) + a-b(b^k) \\
&= a\underbrace{(p(a-b))}_{\text{per IH}} + a-b(b^k) \\
&= ap(a-b) + b^k(a-b) \\
&= a-b(ap + b^k)
\end{aligned}
$$

valid

Where $q = ap + b^k$

4.3) Using 4.2 $\Rightarrow$ $a+b \mid a^n + b^n$

for some $n = 2k-1$, $k \in I^+$

and $a, b \in \mathbb{Z}$

$a^n + b^n$ if $a > 0$ then $a^{odd} > 0$

$\hookrightarrow$ suppose $b < 0$ then $b^{odd} < 0$

this can be written as $a^n - (-b)^n$

if we make $a = p$ and $-b = q$

then from 4.2 we know that

$p - q \mid p^n - q^n$

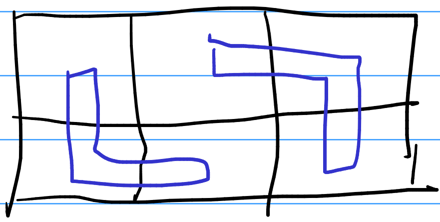
so $a^n + b^n$ is divisible by $a - (-b)$

which is $a+b$

Problem 5

Show that a $2m \times 3n$ checkerboard can be covered by L-shaped triminoes.

Suppose $m=1$, $n=1$ (base case)



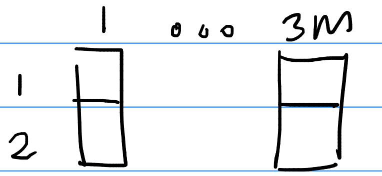
✓

For m dimension:

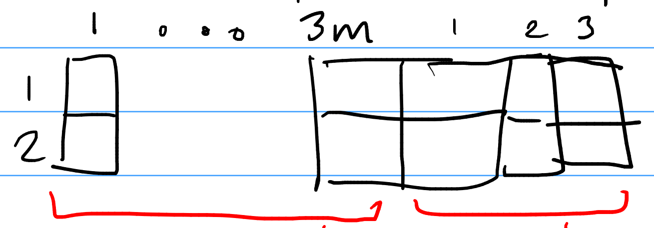
$2 \circ 3m$ works (based on base-case)

therefore $2 \circ 3(m+1)$ works.

$2 \circ 3m$ grid



$2 \circ 3(m+1)$ grid $\underbrace{}_{=3(m+1)}$



We know that $(2 \circ 3)$ covers ✓ base case

& $(2 \circ 3m)$ covers

$\Longrightarrow 2 \circ 3(m+1) = 2 \circ (3m+3)$

$= \underset{\text{assumption from base case}}{(2 \circ 3m)} + \underset{\text{base case}}{(2 \circ 3)}$

Therefore, $2 \circ (3m+1)$ can be filled w/ L-shape

For n dimension

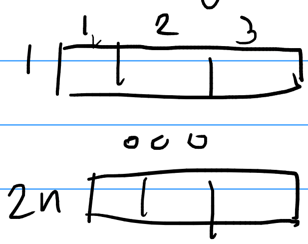
Base case $\Longrightarrow$ $(2 \circ 3)$

induction hypothesis

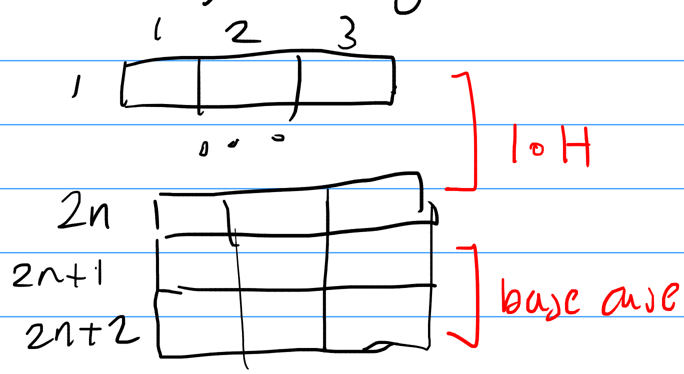
Assume $2n \circ 3$ is valid

Prove $2(n+1) \circ 3$ is valid

$2n \circ 3$ grid



$2(n+1) \circ 3$ grid



so same as the previous dimension..

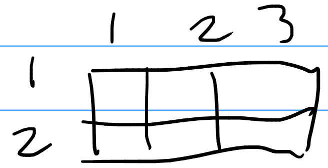
$= (2n \circ 3) + (2 \circ 3)$

Therefore, $2n \circ 3$ can be filled with L-shaped tiles.

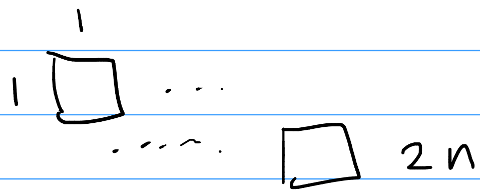
For both dimensions...

$\boxed{2n \circ 3m}$

consider



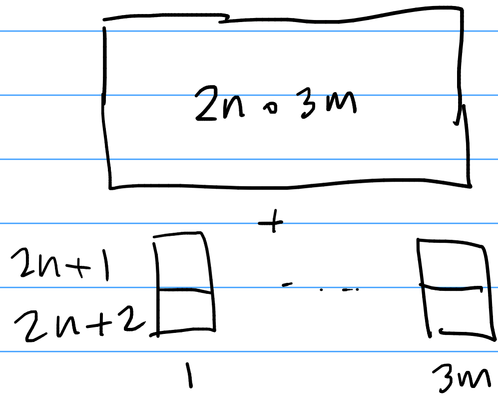
$2 \circ 3$
(b.c)



$2n \circ 3m$
$\boxed{\text{I.H}}$

inductive step

Try $2(n+1) \circ 3m$



Dimensions $= (2 \circ 3m)$

$$= (2n \circ 3m) + (2 \circ 3m)$$

I.H — proven previously

$\therefore$ You can fit L-shapes into a $2n \times 3m$ checkerboard ☆

Problem 6

$$n^3 \leq 3^n \quad \forall n \geq 1$$

Predicate

$P(i) \Rightarrow i^3 \leq 3^i$ for all $i \geq 1$

Base case

$P(3)$ per hint $\quad 3^3 \leq 3^3$ ✓

$P(2) \Rightarrow 2^3 \leq 3^2$

$8 \leq 9$ ✓

$P(1)$ $1^3 \leq 3^1$

$1 \leq 3$ ✓

Inductive step

$P(k) \Rightarrow k^3 \leq 3^k$

$P(k+1) \Rightarrow (k+1)^3 \leq 3^{k+1}$

For some $k \geq 4$

$\hookrightarrow \quad (k^2+2k+1)(k+1) \leq 3 \cdot 3^k$

$k^3 + 3k^2 + 3k + 1 \leq 3 \cdot 3^k$

$\Downarrow$ split + compare

$k^3 \leq 3^k$ ✓ base case

Take $k=4$

$3(4^2) \leq 3^4$ $\Leftarrow$ $3k^2 \leq 3^k$ ✓

$3(16) \leq 3^4$

$3(16) \leq 3(27)$ ✓

$3k \leq 3^k$ ✓

Take $k=4$ $\quad 3(4) \leq 3^4$ ✓ $\Leftarrow$

$1 \leq 0$ ✓

Applies for all $k \geq 4$ due to how positive numbers work.

So we know that $k^3 \leq 3^k$

which aligns w/ our original statement which we proved manually $\Leftarrow$ when $k \geq 4$ ☆

Problem 7

Predicate $\Rightarrow$ $\displaystyle\sum_{i=1}^{n} i \cdot 2^{i} = (n-1) \cdot 2^{n+1} + 2$

Base case

$P(1) = 1 \cdot 2^{1} = (1-1) \cdot 2^{2} + 2$

$= 2$ $\quad$ $= 0 + 2$

$\Leftrightarrow$ $= 2$ ✓

$P(k)$

$$(1 \cdot 2) + (2 \cdot 2^{2}) + \ldots + (k \cdot 2^{k}) = (k-1) 2^{k+1} + 2$$

$P(k+1)$

$$(1 \cdot 2) + (2 \cdot 2^{2}) + \ldots + (k \cdot 2^{k}) + ((k+1) + 2^{k+1})$$

$$= ((k+1)-1)\, 2^{(k+1)+1} + 2$$

$$= k \cdot 2^{k+2} + 2$$

$\therefore$ $(k-1) 2^{k+1} + 2 + (k+1) \cdot 2^{k+1}$ ①

$= k \cdot 2^{k+2} + 2$ ②

$\Downarrow \boxed{-2}$

$(k-1)(2^{k+1}) + (k+1) \cdot 2^{k+1} = k \cdot 2^{k+2}$

$$(2^{k+1})(k-\cancel{1}+k+\cancel{1}) = k(2^{k+2})$$

$$(2^{k+1})(2k) = k(2^{k+2})$$

$$\underline{(2)}\,\underline{(2^k)}\,\underline{(2)}\,\underline{(k)} = \underline{(k)}\,\underline{(2)}\,\underline{(2)}\,\underline{(2^k)}$$

Therefore, through proof by induction, our statement is true.

Problem 8

Pigeon hole theory

pigeons　　　holes

↳ if n items in m containers where $n > m$

hole　　　pigeon

then at least 1 container has more than 1 item

In this case, pigeon = number of socks pulled.

holes = color of sock

6　　where $n > m$

Since there are 5 colors, if we pull $5+1$ socks then we are bound to have at least one pair (i.e. two socks of the same color).

Problem 9

S = set of $n+1$ integer $\mathbb{Z}^{n+1}$

$a, b \in S$ where $(a-b)$ is a multiple of n

↳ $(a-b) = kn$ where $k \in \mathbb{Z}$

↳ $\frac{a-b}{n} \equiv 0$

Consider every element modulo n:

$$S^{n+1} = \{\overset{1}{a}, \overset{2}{b}, \overset{3}{c}, \overset{4}{d} \ldots, \overset{n+1}{foo}\}$$

where ≥ 1 pair could $\frac{p_1 - p_2}{n} \equiv 0$

Pigeon hole

pigeon = numbers in the set

hole = remainder from 0 to $n-1$ of $\frac{a-b}{n}$ (we want 0)

Since there should be n holes and a set has $n+1$ numbers. In theory, this means that at least two numbers must have a remainder of 0

We know that if $\frac{a}{n} \equiv 0$ and $\frac{b}{n} \equiv 0$ then

per principle $\frac{a-b}{n}$ should also be 0 since

since it is dividable by n

a can be written in the form kn where $k \in \mathbb{Z}$

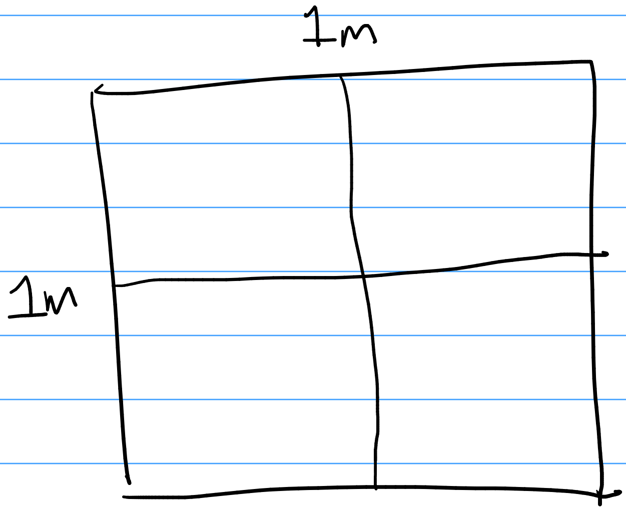
so can b. $\Rightarrow$ say ln where $l \in \mathbb{Z}$

Therefore, since $a-b$ can be written in the form $kn - ln = n(k-l)$

so can be a multiple of n

Problem 10

Any 5 points in a 1m x 1m.
At least 1 pair of points will have a distance of $\leq \frac{\sqrt{2}}{2}$



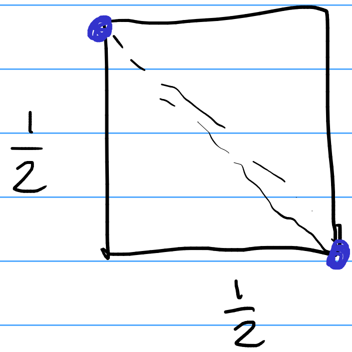
Pigeon hole

pigeon = dots
hole = subsection of square

So implies that at least 1 sub-square of area $\left(\frac{1}{2}m\right)^2$ will have ≥ 2 dots.

Consider the longest possible distance between two points of a square:

Must be sitting on the edges of the quadrants.



$\Downarrow$

Knowing this, even if five points are placed on each corner + center, the distance between them will not exceed $\frac{\sqrt{2}}{2}$.

$$\text{distance} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \sqrt{\frac{1}{2}} \quad \text{or} \quad \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

This corresponds with the idea that at least one pair of points will have a distance less than or equal to $\frac{\sqrt{2}}{2}$. In the case of this square, it should in theory apply to all five points.