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HW3

Problem 1

From my understanding of the question, whoever makes the last move, i.e being the last person to break the chocolate into 1x1 pieces, is the winner.

Using a previously done proof that a $m \times n$ chocolate bar, regardless of combination, will take $mn-1$ turns to break. The goal is to reach $mn-1$ first so that you are the one to break the chocolate into 1x1 pieces.

This would depend on the parity of $mn-1$:

	Turn Number				
P1	1		3		
				...	$mn-1$
P2		2		4	

$\Rightarrow$ whoever starts first will always have the odd-numbered turn. The other person will have an even-numbered turn.

Therefore, if $mn-1$ (turn required to break chocolate to 1x1 piece) is odd, you should start first so you reach this number. Likewise, if $mn-1$ is even then you should start last. This should be the winning strategy.

Problem 2

Predicate $P(i) = i-1$ steps.

Base case $P(1) = 1-1 = 0$ steps

Solving a 1-piece jigsaw puzzle takes 0 steps.

Inductive step + hyp. Assume $P(k) = k-1$ steps. for some integer k.

Let us prove that $P(k+1)$ will take k steps to solve.

Assume that we combine two puzzles a and b to make $k+1$ puzzle pieces in total.

$a + b = k+1$

Where a and b are the smaller solvable pieces.
We assume per I.H that a and b will take a−1 and b−1 steps to solve. Therefore, the total number of steps will be:

$(a-1) + (b-1) + 1$

↑ include the step we take to combine a and b

$= a+b-1$

$= (\overset{\Downarrow}{k+1}) - 1$

$= k$ steps

Problem 3

7 baht + 4 baht coins

Predicate $P(i) = 7a + 4b = i$ for some i

Base case $P(18) = \underset{\underset{a=2}{\Downarrow}}{7a} + \underset{\underset{b=1}{\Downarrow}}{4b}$

Inductive step + I.H Assume $P(k) = 7a + 4b$

Also assume that $P(k+4)$ is payable since we can only pay in 7-baht and 4-baht intervals.

$\Downarrow$

$P(k+4) \underset{\text{per I.H}}{=} 7a + \underset{\Downarrow}{4b} + 4$

$= P(k) + 4$ or $7a + 4(b+1)$

∴ it is payable.

However, we have to consider $18 < k < 18+4$ too ⇓

$P(19) = 7(1) + 4(3)$

$P(20) = 7(0) + 4(5)$

$P(21) = 7(3) + 4(0)$

Since we know that $k \leq i \leq k+4$ is payable, it is safe to assume any number $i \geq 18$ is payable in 7 or 4 baht coins.

To test, we know that $P(22) = \underbrace{\overset{\text{base case}}{P(18)} + 4}_{\to P(k+4) \text{ proven}}$

$= 7(2) + 4(2)$

and that $P(25) = \underbrace{\overset{\text{proven}}{P(21)} + 4}_{\to p(k+4) \text{ proven}}$

$= 7(3) + 4(1)$

Problem 4

Power of two $\Rightarrow$ 2^n for some $n \in \mathbb{Z}$

Odd number $\Rightarrow$ $n = 2x - 1$ for some $x \in \mathbb{Z}, x \geq 1$

Predicate $P(i) = 2^n + O$ where $n \in \mathbb{Z}$ and O = odd number

Base case $P(1) = 2^0 + 1$

I.H $P(k) = 2^n + O$

<u>Consider the parities of each number:</u>

If k is odd, then you can just do $2^0 \times k$

$= 1 \times k$

$= k$

If k is even, we'd want to assume that k can be written in the form $\underline{2a}$ where $a \in \mathbb{Z}^+$

Per I.H., we assume $a = 2^n \cdot O$ for some $n \in \mathbb{Z}$ and $O = \text{odd}$

meaning that k can be written as $2[2^n \cdot O]$

which is just $2^{n+1} \cdot O$

that fulfills the format we want.

Therefore, we've proven every number $k \geq 1$ can be written as the product betwee a power of two + odd number

Problem 5

$T_n = T_{n-1} + T_{n-2} + T_{n-3} \qquad n \geq 4$

Predicate $T_i < 2^i$ for some $i \geq 4$

Base case $T_1 < 2^1 \Rightarrow 1 < 2$ ✓

$T_2 < 2^2 \Rightarrow 1 < 4$ ✓

$T_3 < 2^3 \Rightarrow 1 < 8$ ✓

IH and IS Assume $T_K < 2^K$

T_{k+1} must therefore be less than 2^{k+1}

$$T_{k+1} = T_k + \underset{\Downarrow \text{ per I.H}}{T_{n-1}} + T_{k-2}$$

$$\therefore T_{k+1} = 2^k + 2^{k-1} + 2^{k-2} \qquad \text{assume } k \geq 4$$

$$2^{k+1}(2^{-1} + 2^{-2} + 2^{-3})$$

$$2^{k+1}\left(\tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8}\right)$$

$$2^{k+1}\left(\tfrac{4}{8} + \tfrac{2}{8} + \tfrac{1}{8}\right)$$

$$2^{k+1} \cdot \tfrac{7}{8}$$

since $\frac{7}{8} 2^{k+1} < 2^{k+1}$

via induction we have proven the statement. ☆

Problem 6

For $n \geq 1$

Predicate $P(i) = 2^{a_1} + 2^{a_2} + \dots + 2^{a_n}$

where $a \in \mathbb{Z}_{\geq 0}$

Base case $P(1) = 2^0$

IH $P(k) = 2^{a_1} + \dots + 2^{a_n}$ for some $n \in \mathbb{Z}^+$

Consider parity

If k = odd $\therefore$

$$\text{Assume } P(k-1) = \underbrace{[2^{a_1} + \dots + 2^{a_n}]}_{\substack{\Downarrow \\ \text{per I.H assume} \\ \text{that anything} < k \text{ is true}}} \text{ call this equation } X$$

$$\text{so } P(k) = X + \underbrace{1}_{\substack{\Downarrow \\ \text{base case}}}$$

$\therefore$ $P(k)$ is a sum of distinct powers of two

If k is even

Assume $k = 2b$ where $b \in \mathbb{Z}^+$

$$\text{Assume } b = \underbrace{[2^{a_1} + \dots + 2^{a_n}]}_{\substack{\Uparrow \\ \text{per IH}}} \text{ call this equation } Y$$

so k can be written as :

$$2[2^{a_1} + \dots + 2^{a_n}]$$

$$= 2^{a_1+1} + \dots + 2^{a_n+1}$$

which means statement is proven.

Problem 7

Predicate Length of periphery of i squares is even for n ≥ 1

Base case $P(1) = 4$ which is even so ✓

$\left[E = \forall x = 2n, \; n \in \mathbb{Z} \right]$

IH / IS Assume $P(k) = E$

$P(k+1) = E$

$\Downarrow$

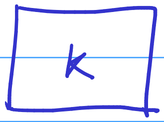
know that $P(k+1) = P(k) + x$

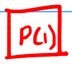
$\Downarrow$

x is dependent on how we place the new square.

Take $P(k)$

Take $P(1) = 4$

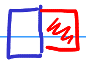




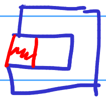
x cannot be 4 since at least one side will be amalgamated w/ the larger shape.

Possibilities:

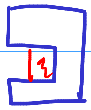
(suppose we are adding to a shape already)



1 side gone
3 sides added $\Rightarrow P(k) - 1 + 3 = P(k) + 2$



2 sides gone
2 sides added $\Rightarrow P(k) - 2 + 2 = P(k)$



3 sides gone
1 side added $\Rightarrow P(k) - 3 + 1 = P(k) - 2$

$\Downarrow$

know that even ± even = even

∴ statement proven via induction ☆

Theorem: If we have n straight line in a 2 dimensional plane($n \geq 2$), and none of the line are parallel to each other, then all lines must intersect at exactly one and the same point.

Proof: by induction-ish?.

Inductive Predicate:$P(n)$ = every set of n straight lines intersect at exactly one point.

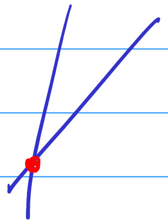
Base Case: $n = 2$. Every two lines intersect at one point. Therefore, $P(2)$ is true.

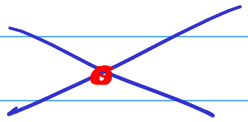
Inductive Step:

Assume that every n straight lines intersect at one point, we want to show that every $n + 1$ line intersect at one point.

- Let the set of $n + 1$ be $\{a_1, a_2, a_3, \ldots, a_n, a_{n+1}\}$
- From the inductive hypothesis $\{a_1, a_2, a_3, \ldots, a_n\}$ must intersect at one point since it is a set of n lines. Let us call the intersection point for these lines D.
- Similarly, $\{a_2, a_3, a_4 \ldots, a_n, a_{n+1}\}$ intersect at exactly one point since there are n lines in this set. Let us call this point E.
- Since both D and E are the point where a_2 intersects a_3. D and E are therefore the same point.
- Therefore, $\{a_1, a_2, a_3, \ldots, a_n, a_{n+1}\}$ intersect at exactly one point.■

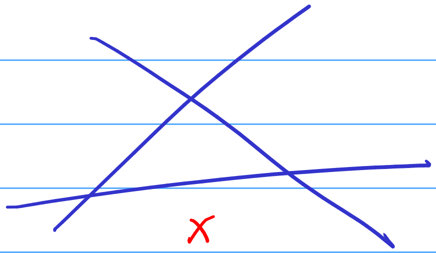
Consider $n=2$:

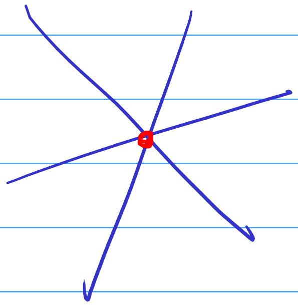




regardless of orientation

Consider $n=3$:





Whilst possible, not all lines will have a common intersect.

Additionally, the proof uses sets to determine intersect points. It assumes that a set has lines that will intersect ONCE at a specific point.

It assumes if two lines are in both sets, their intersect must also be true for the whole set, both sets included. This generalization creates a flaw in the proof. This, along with us showing $n = 3$ is not necessarily true for all configurations of the 3 lines, means the proposition is not true.