Linear Algebra
Problem set 4

Problem 0
a) Theeradon sarawek
b) 6680210
c) -
d) Lecture Note 4

Problem 1

$$A = \begin{bmatrix} -2 & 1 \\ 4 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \quad C = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

a) $A \circ B$

$$\begin{bmatrix} -2 & 1 \\ 4 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix}$$

$(1,1) = -2\cdot 1 + 1\cdot 2 = -2+2=0$
$(1,2) = -2\cdot 0 + 3\cdot 1 = 3$
$(2,1) = 4\cdot 1 + 0\cdot 2 = 4$
$(2,2) = 4\cdot 0 + 0\cdot 3 = 0$

b) $B\cdot C$

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \circ \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -5 & 2 \end{bmatrix}$$

$(1,1) = 1\cdot -1 + 0\cdot -1 = -1$
$(1,2) = 1\cdot 1 + 0\cdot 0 = 1$
$(2,1) = 2\cdot -1 + 3\cdot -1 = -5$
$(2,2) = 2\cdot 1 + 3\cdot 0 = 2$

c) $AB \circ C$

$$\begin{bmatrix} 0 & 3 \\ 4 & 0 \end{bmatrix} \circ \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -4 & 4 \end{bmatrix}$$

$(1,1) = 0\cdot -1 + 3\cdot -1 = -3$
$(1,2) = 0\cdot 1 + 3\cdot 0 = 0$
$(2,1) = 4\cdot -1 + 0\cdot -1 = -4$
$(2,2) = 4\cdot 1 + 0\cdot 0 = 4$

d) $A \cdot BC$

$$\begin{bmatrix} -2 & 1 \\ 4 & 0 \end{bmatrix} \circ \begin{bmatrix} -1 & 1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ -4 & 4 \end{bmatrix}$$

$(1,1) = -2 \cdot -1 + 1 \cdot -5 = 2 - 5 = -3$

$(1,2) = -2 \cdot 1 + 1 \cdot 2 = -2 + 2 = 0$

$(2,1) = 4 \cdot -1 + 5 \cdot 0 = -4$

$(2,2) = 4 \cdot 1 + 0 \cdot 2 = 4$

$\therefore AB \circ C = A \circ BC$

Problem 2

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 5 & 3 & 1 \end{bmatrix}$$

a) $A \cdot B$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \circ \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$(1,1) = 1 \cdot 2 + 0 \cdot 4 + 1 \cdot 3 = 5$

$(2,1) = 0 \cdot 2 + 1 \cdot 4 + -1 \cdot 3 = 1$

b) $B \cdot C$

$$\begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} \circ \begin{bmatrix} 1 & 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 10 & 6 & 2 \\ 4 & 20 & 12 & 4 \\ 3 & 15 & 9 & 3 \end{bmatrix}$$

c) $AB \circ C$

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 5 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 25 & 15 & 5 \\ 1 & 5 & 3 & 1 \end{bmatrix}$$

d) $A \cdot BC$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \circ \begin{bmatrix} 2 & 10 & 6 & 2 \\ 4 & 20 & 12 & 4 \\ 3 & 15 & 9 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 25 & 15 & 5 \\ 1 & 5 & 3 & 1 \end{bmatrix}$$

$(1,1) = 1\cdot 2 + 0\cdot 4 + 1\cdot 3 = 5$

$(1,2) = 10 + 15 = 25$

$(1,3) = 6 + 9 = 15$

$(1,4) = 2 + 3 = 5$

$(2,1) = 4 - 3 = 1$

$(2,2) = 20 - 15 = 5$

$(2,3) = 12 - 9 = 3$

$(2,4 = 4 - 3 = 1$

$\therefore A \circ BC = AB \circ C$

Problem 3

$A = \begin{bmatrix} 8 & 18 \\ -3 & -7 \end{bmatrix}$ $B = \begin{bmatrix} -3 & -12 \\ 2 & 7 \end{bmatrix}$

a) AB

$$\begin{bmatrix} 8 & 18 \\ -3 & -7 \end{bmatrix} \circ \begin{bmatrix} -3 & -12 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 12 & 30 \\ -5 & -13 \end{bmatrix}$$

$(1,1) = 8 \cdot -3 + 18 \cdot 2 = -24 + 36 = 12$

$(1,2) = 8 \cdot -12 + 18 \cdot 7 = -96 + 126 = 30$

$(2,1) = -3 \cdot -3 + -7 \cdot 2 = 9 - 14 = -5$

$(2,2) = -3 \cdot -12 + -7 \cdot 7 = 36 - 49 = -13$

b) $B \cdot A$

$$\begin{bmatrix} -3 & -12 \\ 2 & 7 \end{bmatrix} \circ \begin{bmatrix} 8 & 18 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 12 & 30 \\ -5 & -13 \end{bmatrix}$$

$(1,1) = -3 \cdot 8 + -12 \cdot -3 = -24 + 36 = 12$

$(1,2) = -3 \cdot 18 + -12 \cdot -7 = -54 + 84 = 30$

$(2,1) = 2 \cdot 8 + 7 \cdot -3 = 16 - 21 = -5$

$(2,2) = 2 \cdot 18 + 7 \cdot -7 = 36 - 49 = -13$

c) $AB = BA \therefore$ they commute.

d) $A^T = \begin{bmatrix} 8 & -3 \\ 18 & -7 \end{bmatrix}$ $B^T = \begin{bmatrix} -3 & 2 \\ -12 & 7 \end{bmatrix}$

$A^T \circ B^T$

$\begin{bmatrix} 8 & -3 \\ 18 & -7 \end{bmatrix} \circ \begin{bmatrix} -3 & 2 \\ -12 & 7 \end{bmatrix} = \begin{bmatrix} 12 & -5 \\ 30 & -13 \end{bmatrix}$

$(1,1) = 8 \cdot -3 + -3 \cdot -12 = -24 + 36 = 12$

$(1,2) = 8 \cdot 2 + -3 \cdot 7 = -5$

$(2,1) = 18 \cdot -3 + -7 \cdot -12 = 54 + 84 = 30$

$(2,2) = 18 \cdot 2 + -7 \cdot 7 = 36 - 49 = -13$

$B^T \circ A^T$

$\begin{bmatrix} -3 & 2 \\ -12 & 7 \end{bmatrix} \circ \begin{bmatrix} 8 & -3 \\ 18 & -7 \end{bmatrix} = \begin{bmatrix} 12 & -5 \\ 30 & -13 \end{bmatrix}$

$(1,1) = -3 \cdot 8 + 2 \cdot 18 = -24 + 36 = 12$

$(1,2) = -3 \cdot -3 + -7 \cdot 2 = 9 - 14 = -5$

$(2,1) = -12 \cdot 8 + 7 \cdot 18 = -96 + 126 = 30$

$(2,2) = -12 \cdot -3 + 7 \cdot -7 = 36 - 49 = -13$

Yes, A^T and B^T commute since $A^T B^T = B^T A^T$

e) $A = \begin{bmatrix} 8 & 18 \\ -3 & -7 \end{bmatrix}$ $B^T = \begin{bmatrix} -3 & 2 \\ -12 & 7 \end{bmatrix}$

$A \cdot B^T = \begin{bmatrix} -240 & 142 \\ 93 & -55 \end{bmatrix}$

$(1,1) = 8 \cdot -3 + 18 \cdot -12 = -240$

$(1,2) = 8 \cdot 2 + 18 \cdot 7 = 16 + 126 = 142$

$(2,1) = -3 \cdot -3 + -7 \cdot -12 = 9 + 84 = 93$

$(2,2) = -3 \cdot 2 + -7 \cdot 7 = -6 - 49 = -55$

$B^T \cdot A = \begin{bmatrix} -30 & -68 \\ & \end{bmatrix}$

$(1,1) = -3 \cdot 8 + 2 \cdot -3 = -24 + -6 = -30$

$(1,2) = -3 \cdot 18 + 2 \cdot -7 = -54 - 14 = -68$

No, A and B^T do not commute.

Problem 4

$$A = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix}$$

a) $AB = \begin{bmatrix} 22 & 79 \\ 17 & 61 \end{bmatrix}$

$(1,1) = 4\cdot 1 + 9\cdot 2 = 18 + 4 = 22$
$(1,2) = 4\cdot 4 + 9\cdot 7 = 16 + 63 = 79$
$(2,1) = 3\cdot 1 + 7\cdot 2 = 3 + 14 = 17$
$(2,2) = 3\cdot 4 + 7\cdot 7 = 12 + 49 = 61$

$BA = \begin{bmatrix} 16 & 37 \\ 29 & 67 \end{bmatrix}$

$(1,1) = 1\cdot 4 + 3\cdot 4 = 4 + 12 = 16$
$(1,2) = 1\cdot 9 + 4\cdot 7 = 9 + 28 = 37$
$(2,1) = 2\cdot 4 + 7\cdot 3 = 8 + 21 = 29$
$(2,2) = 2\cdot 9 + 7\cdot 7 = 18 + 49 = 67$

c) No, A and B do not commute

Problem 5

$$\overset{A}{\begin{bmatrix} 1 & 0 \\ 1 & x \end{bmatrix}} \text{ and } \overset{B}{\begin{bmatrix} 1 & 2 \\ y & 4 \end{bmatrix}}$$

$(1,1)\ AB = (1,1)\ BA$

$1 + 0y = 1 + 2$
$= 3$

$1 = 3$ which is a contradiction.

Therefore, impossible to have a pair of real numbers that make the two matrices commute.

Problem 6

$$\underset{A}{\begin{bmatrix} 1 & -1 & 1 \\ -2 & -3 & 2 \\ -2 & x & 4 \end{bmatrix}} \circ \underset{B}{\begin{bmatrix} 1 & 0 & y \\ -1 & -1 & 1 \\ -1 & -2 & 2 \end{bmatrix}}$$

$-x - 8 = 7 + 2x$
$5 - 8 = 7 - 10$
$-3 = -3$ ✓

$\boxed{x = -5}$
$-6 - x = -1$ ✓

	AB	BA	
(1,1)	$1 + 1 - 1 = 1$	$1 - 2y$	$1 - 2y = 1 \Rightarrow y = 0$ ✓
(1,2)	$1 - 2 = -1$	$-1 + xy$	$-1 = -1$ ✓
(1,3)	$y - 1 + 2 = y + 1$	$1 + 4y$	$1 = 1$ ✓
(2,1)	$-2 + 3 - 2 = -1$	$-1 + 2 - 2 = -1$	$-1 = -1$ ✓
(2,2)	$3 - 4 = -1$	$1 + 3 + x$	$4 + x = -1 \Rightarrow x = -5$ ✓
(2,3)	$-2y - 3 + 4 = -1$	$-1 - 2 + 4 = -1$	$-1 = -1$ ✓

	AB	BA
(3,1)	$-2 - x - 4$	$-1 + 4 - 4 = -1$
(3,2)	$-x - 8$	$1 + 6 + 2x$
(3,3)	$-2y + x + 8$	$-1 - 4 + 8$

$-5 + 8 = -5 + 8$ ✓

$\therefore$ solution set is $\overset{x}{(-5},\overset{y}{0)}$

Problem 7

$$M = \begin{bmatrix} -1 & 2 & -3 \\ 0 & 1 & -1 \\ 2 & 3 & -2 \end{bmatrix} \qquad N = \begin{bmatrix} 1 & -5 & 1 \\ -2 & 8 & -1 \\ -2 & 7 & -1 \end{bmatrix}$$

a) $MN = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$(1,1) = -1-4+6 = 1$ $\quad (2,1) = 0-2+2 = 0$ $\quad (3,1) = 2-6+4 = 0$

$(1,2) = 5+16-21 \; 0$ $\quad (2,2) = 0+8-7 = 1$ $\quad (3,2) = -10+24-14 = 0$

$(1,3) = -1-2+3 \; 0$ $\quad (2,3) = 0-1+1 = 0$ $\quad (3,3) = 2-3+2 = 1$

b) $NM = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$(1,1) = -1+0+2 = 1$ $\quad (2,1) \; 2+0-2 = 0$ $\quad (3,1) = 2+0-2 = 0$

$(1,2) = 2-5+3 = 0$ $\quad (2,2) \; -4+8-3 = 1$ $\quad (3,2) = -4+7-3 = 0$

$(1,3) = -3+5-2 = 0$ $\quad (2,3) \; 6-8+2 = 0$ $\quad (3,3) = 6-7+2 = 1$

c) Since $MN = NM = I_m$, they are inverses

$\Downarrow$

- in RREF
- square

Problem 8

$$U = \begin{bmatrix} 2 & 1 & 0 \\ 5 & 3 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} 3 & -1 \\ -5 & 2 \\ 0 & 0 \end{bmatrix}$$

$UV = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$(1,1) = 6-5 = 1$

$(1,2) = -2+2 = 0$

$(2,1) = 15-15 = 0$

$(2,2) = -5+6$

$VU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$(1,1) = 6-5 = 1$ $\quad (2,1) \; -10+10 = 0$ $\quad (3,1) = 0$

$(1,2) = 3-3 = 0$ $\quad (2,2) \; -5+6 = 1$ $\quad (3,2) = 0$

$(1,3) = 0$ $\quad (2,3) \; 0$ $\quad (3,3) = 0$

$UV = I_m$ but $VU \neq I_m$ $\therefore$ U and V are not inverses

Problem 9 $A = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$

a) $A^2 = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ 2 & -1 \end{bmatrix}$

$(1,1) = 0 - 2 = -2$
$(1,2) = 0 - 1 = -1$
$(2,1) = 2$
$(2,2) = -2 + 1 = -1$

b) $A^3 = \begin{bmatrix} -2 & -1 \\ 2 & -1 \end{bmatrix} \circ \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -2 & -3 \end{bmatrix}$

$(1,1) = 0 - 2 = -2$
$(1,2) = 2 - 1 = 1$
$(2,1) = 0 - 2 = -2$
$(2,2) = -2 - 1 = -3$

Problem 10

a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow$

2x2 matrix rule

i) $ad - bc \neq 0 \Rightarrow$ invertible

$1 - 0 = 1$ $\therefore$ invertible.

ii) Inverse $= \begin{bmatrix} \frac{1}{1} \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

recall inverse of $I_m = I_m$

b) $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

2x2 matrix rule

i) $ad - bc \neq 0 \Rightarrow$ invertible

$0 - (-1) = 1$ $\therefore$ invertible

ii) inverse $= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

c) $C = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$

2x2 matrix rule

i) $ad - bc \neq 0 \Rightarrow$ invertible

$-4 - (-4) = 0$

$\Rightarrow$ $\therefore$ not invertible

d) $D = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}$

per 2x2 matrix rule

i) $12 - 15 = -3 \neq 0$ ∴ invertible.

ii) $\frac{1}{-3} \circ \begin{bmatrix} 6 & -3 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{5}{3} & -\frac{2}{3} \end{bmatrix}$

e) $E = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{8} \end{bmatrix}$

per 2x2 matrix rule

i) $-\frac{1}{24} - \frac{1}{8} = -\frac{4}{24} = -\frac{1}{6}$

ii) $\frac{1}{-\frac{1}{6}} \circ \begin{bmatrix} -\frac{1}{8} & -\frac{1}{4} \\ -\frac{1}{2} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{-6}{-8} & \frac{-6}{-4} \\ \frac{-6}{-2} & \frac{-6}{3} \end{bmatrix}$

or -6

$= \begin{bmatrix} \frac{3}{4} & \frac{3}{2} \\ 3 & -2 \end{bmatrix}$

f) $F = \begin{bmatrix} \frac{6}{7} & -2 \\ 5/28 & -5/12 \end{bmatrix}$

Per 2x2 matrix rule

i) $\left(\frac{6}{7}\right)\left(\frac{-5}{12}\right) - (-2)\left(\frac{5}{28}\right) = 0$

∴ not invertible

g) $G = \begin{bmatrix} 0 & 0 \\ 0 & 25 \end{bmatrix}$

per 2x2 matrix rule

i) $(0)(25) - (0)(0) = 0$

∴ not invertible.

Problem 11

$C = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$

a) $CD = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$

$(1,1) = 0 + 4 = 4$

$(1,2) = -1 + 4 = 3$

$(2,1) = 0 + 2 = 2$

$(2,2) = 0 + 2 = 2$

$CD^T = \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}$

b) $C^T D^T$

$C^T = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ $D^T = \begin{bmatrix} 0 & 2 \\ -1 & 2 \end{bmatrix}$

$C^T D^T = \begin{bmatrix} 0 & 2 \\ -1 & 6 \end{bmatrix}$

$(1,1) = 0 + 0 = 0$
$(1,2) = 2 + 0 = 2$
$(2,1) = 0 - 1 = -1$
$(2,2)\ 4 + 2 = 6$

No, $C^T D^T \neq (CD)^T$

c) $D^T C^T$ $= \begin{bmatrix} 4 & 2 \\ 3 & 2 \end{bmatrix}$

$(1,1) = 0 + 4 = 4$
$(1,2) = 0 + 2 = 2$
$(2,1) = -1 + 4 = 3$
$(2,2) = 0 + 2 = 2$

Yes, $D^T C^T = (CD)^T$