Linear Algebra
problem set 6

Problem 0

a) Theeraulon Saraweh

b) 6680210

c) None

d) Lecture Notes Wk. 6

Problem 1

$$A = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}.$$

a) $[B \mid C] = \left[\begin{array}{cc|cc} 2 & 3 & 2 & 0 \\ 1 & 0 & 3 & -1 \end{array}\right]$

b) $A \cdot B = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \circ \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & -3 \end{bmatrix}$

$(1,1) = 2-3 = -1 \quad (1,2) = 3$

$(2,1) = -2+4 = 2 \quad (2,2) = -3$

c) $A \cdot C = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \circ \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} \overset{2-9}{-7} & 3 \\ \overset{-2+12}{10} & -4 \end{bmatrix}$

d) $A \cdot [B \mid C] = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \circ \begin{bmatrix} 2 & 3 & 2 & 0 \\ 1 & 0 & 3 & -1 \end{bmatrix}$

$$= \begin{bmatrix} \overset{2-3}{-1} & \overset{3}{3} & \overset{2-9}{-7} & \overset{3}{3} \\ \overset{-2+4}{2} & \overset{-3}{-3} & \overset{-2+12}{10} & \overset{-4}{-4} \end{bmatrix}$$

equal

$[AB \mid AC] = \begin{bmatrix} -1 & 3 & -7 & 3 \\ 2 & -3 & 10 & -4 \end{bmatrix}$

Problem 2

$$D = \begin{bmatrix} 0 & 1 & -1 \\ 3 & 2 & 1 \\ -1 & 0 & -1 \end{bmatrix}.$$

a) $[D \mid I_3] = \left[\begin{array}{ccc|ccc} 0 & 1 & -1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$

$R_2 = R_2 - 2R_1$ det = 1

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ 3 & 0 & 3 & -2 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 = R_2 + 3R_3$ det = 1

$$\begin{bmatrix} 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ -1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \Leftrightarrow R_3$ det = −1

$$\begin{bmatrix} -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 1 & -1 & 1 & 0 & 0 \end{bmatrix}$$

$R_2 \Leftrightarrow R_3$ det = −1

$$\begin{bmatrix} -1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$R_1 = -R_1$ det = −1

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$R_3 = \frac{1}{3}R_3$ det = $\frac{1}{3}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 = R_1 + R_3$ det = 1

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$ Tr in RREF ✓

G H

b) No, since G has an all-zero row. Therefore, det(D) = 0

c) tr(H) = 1

det(H) = 0 since there is an all-zero row.

$$K = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 3 & -3 \\ 0 & 0 & 6 & 1 \end{bmatrix}. \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Problem 3 suppose

a) $$[K \mid I_4] = \left[\begin{array}{cccc|cccc} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \Leftrightarrow R_1$ det $= -1$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & -3 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \Leftrightarrow R_2$ det $= -1$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & -3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_3 = R_3 + R_2$ det $= 1$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 3 & -3 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_1 = R_1 - 3R_3$ det $= 1$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -6 & -3 & -3 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 6 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$R_4 = R_4 - 6R_3$ det $= 1$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -6 & -3 & -3 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -5 & -6 & -6 & 0 & 1 \end{array}\right]$$

$R_1 = R_1 - R_4$ det $= 1$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 3 & 3 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -5 & -6 & -6 & 0 & 1 \end{array}\right]$$

$R_4 = \frac{1}{-5} R_4$ det $= -\frac{1}{5}$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & 3 & 3 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{6}{5} & \frac{6}{5} & 0 & -\frac{1}{5} \end{array}\right]$$

$R_1 = R_1 + R_4$ det $= 1$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{21}{5} & \frac{21}{5} & 1 & \frac{-6}{5} \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{6}{5} & \frac{6}{5} & 0 & -\frac{1}{5} \end{array}\right]$$

$R_2 = R_2 - R_4$ det $= 1$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{21}{5} & \frac{21}{5} & 1 & \frac{-6}{5} \\ 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{-6}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{6}{5} & \frac{6}{5} & 0 & -\frac{1}{5} \end{array}\right]$$

$R_3 = R_3 - R_4$ det $= 1$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{21}{5} & \frac{21}{5} & 1 & \frac{-6}{5} \\ 0 & 1 & 0 & 0 & -\frac{1}{5} & \frac{-6}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & -\frac{1}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 1 & \frac{6}{5} & \frac{6}{5} & 0 & -\frac{1}{5} \end{array}\right]$$

RREF ✓

$\frac{21}{5} - \frac{1}{5} = \frac{20}{5} = 4$ ✓

3 zeroes ✓

6 negatives ✓

b) Since $L = I_m$, K should be invertible.

$$\det(K) = \frac{1}{\det(M)}$$

$$\det(M) = (-1)(-1)$$

$$1 \quad (1)(1)(1)(1)\left(-\frac{1}{5}\right)(1)(1)(1)$$

$$= -\frac{1}{5}$$

$$\therefore \det(K) = \frac{1}{-\frac{1}{5}} = \boxed{-5}$$

since $L = I_4$, $K^{-1} = M$

$$= \begin{bmatrix} \frac{21}{5} & \frac{21}{5} & 1 & -\frac{6}{5} \\ -\frac{1}{5} & -\frac{6}{5} & 0 & \frac{1}{5} \\ -\frac{1}{5} & -\frac{1}{5} & 0 & \frac{1}{5} \\ \frac{6}{5} & \frac{6}{5} & 0 & -\frac{1}{5} \end{bmatrix}$$

c) $tr(M)$

$$= \frac{21}{5} - \frac{6}{5} - \frac{1}{5}$$

$$= \frac{14}{5} \text{ or } 2.8$$

$$\det(M) = -\frac{1}{5}$$

Problem 4

a) $\det\begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 1 & 1 & 4 \end{pmatrix} = \left|\begin{matrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 1 & 1 & 4 \end{matrix}\right| \begin{matrix} 1 & 2 \\ -1 & 0 \\ 1 & 1 \end{matrix}$

$$\begin{bmatrix} 1\cdot 0\cdot 4 + 2\cdot 3\cdot 1 + 1\cdot -1\cdot 1 \\ 0 + 6 - 1 \end{bmatrix} = 5$$

$$-\begin{bmatrix} 1\cdot 0\cdot 1 + 1\cdot 3\cdot 1 + 4\cdot -1\cdot 2 \\ 0 + 3 - 8 \end{bmatrix}$$

$$5 - (-5) = 10$$

b) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 1 & 1 & x \end{bmatrix}$ where matrix is singular. (not invertible)

$\therefore \det(\text{matrix}) = 0$

$\left|\begin{matrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 1 & 1 & x \end{matrix}\right| \begin{matrix} 1 & 2 \\ -1 & 0 \\ 1 & 1 \end{matrix}$

$$[0 + 3 - 2x] - [0 + 6 - 1] = 0$$

$$3 - 2x - 5 = 0$$

$$-2\,[-2x] = 0$$

$$-2x = 2$$

$$\boxed{x = -1}$$

where $x \in \mathbb{R}$

Problem 5

a) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ i) $tr(a) = -2 + \frac{2}{3} - 1 = -\frac{7}{3}$

ii) $det(a) = \begin{vmatrix} -2 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & -1 \end{vmatrix} \begin{matrix} -2 & 0 \\ 0 & \frac{2}{3} \\ 0 & 0 \end{matrix} = \left(\frac{4}{3} + 0 + 0\right) - (0 + 0 + 0)$

(or since all diagonals are nonzero) $= \frac{4}{3}$

iii) Matrix is invertible.

Inverse $= \begin{bmatrix} \frac{1}{-2} & 0 & 0 \\ 0 & \frac{1}{2/3} & 0 \\ 0 & 0 & \frac{1}{-1} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & -1 \end{bmatrix}$

b) $\begin{bmatrix} 4/7 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3/7 \end{bmatrix}$ i) $tr(b) = \frac{4}{7} + 0 + \frac{3}{7} = 1$

ii) $det(b) = \frac{4}{7} \cdot \frac{3}{7} \cdot 0 = 0$

iii) Matrix is not invertible.

c) $diag\left(-\frac{1}{3}, \frac{1}{2}, -\frac{1}{6}\right)$ i) $tr(c) = -\frac{1}{3} + \frac{1}{2} - \frac{1}{6} = 0$

ii) $det(c) = \frac{1}{36}$

iii) Matrix is invertible.

$c^{-1} = diag(-3, 2, -6)$

d) $\text{diag}(-2,-1,0,1,2)$

i) $tr(d) = -2-1+0+1+2 = 0$

ii) $\det(d) = 0$

since there is a zero.

iii) Not invertible.

e) $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

i) $tr(e) = 1+1 = 2$

ii) $\det(e) = 1 \cdot 1 = 1$

iii) $I_2^{-1} = I_2$ (itself) $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Problem 6

$$N = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

a) $$N \cdot P = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} \circ \begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & -7 \\ 0 & -6 & 5 \\ 0 & 0 & 4 \end{bmatrix}$$

$(1,1) = 2$ $\quad (2,1) = 0$ $\quad (3,1) = 0$

$(1,2) = -2-2 = -4$ $\quad (2,2) = -6$ $\quad (3,2) = 0$

$(1,3) = -6+1-2 = -7$ $\quad (2,3) = 3+2 = 5$ $\quad (3,3) = 4$

Yes, the product is upper-triangular.

$$P \cdot N = \begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \circ \begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 8 \\ 0 & -6 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$(1,1) = 2$ $\quad (2,1) = 0$ $\quad (3,1) = 0$

$(1,2) = 1-3 = -2$ $\quad (2,2) = -6$ $\quad (3,2) = 0$

$(1,3) = -1-1-6 = -8$ $\quad (2,3) = -2+2 = 0$ $\quad (3,3) = 4$

Yes, the product is upper-triangular.

c) $X := NP - PN$

$$= \begin{bmatrix} 2 & -4 & -7 \\ 0 & -6 & 5 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & -2 & 8 \\ 0 & -6 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 6 & -15 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Per definition, this is upper triangular since all elements below the diagonals are zero.

Problem 7

$$\det\left(\begin{array}{c} A \\ \begin{bmatrix} 2 & 0 & 3 & -1 & 9 \\ 0 & 1 & 4 & 2 & 5 \\ 0 & 0 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix} \end{array} \cdot \begin{array}{c} B \\ \begin{bmatrix} -5 & 1 & -1 & -1 & 4 \\ 0 & 3 & 2 & 1 & -4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix} \end{array} \right).$$

$$= (2 \cdot 1 \cdot -3 \cdot 1 \cdot -2) \circ (-5 \cdot 3 \cdot 1 \cdot -1 \cdot 5)$$

$$= 12 \circ 75$$

$$= 900$$

b) $tr(A \cdot B) = (2 \cdot -5) \quad -10$

$+ (1 \cdot 3) \quad 3$

$+ (-3 \cdot 1) \quad -3$

$+ (1 \cdot -1) \quad -1$

$+ (-2 \cdot 5) \quad -10$

$= \boxed{-21}$

$$Q := \begin{bmatrix} 1 & -2 & 0 \\ -4 & 2 & 3 \\ 8 & -7 & 3 \end{bmatrix} \qquad Q^T = \begin{bmatrix} 1 & -4 & 8 \\ -2 & 2 & -7 \\ 0 & 3 & 3 \end{bmatrix}$$

Problem 8

$$Q = \underbrace{\tfrac{1}{2}(Q + Q^T)}_{\text{symmetric}} + \underbrace{\tfrac{1}{2}(Q - Q^T)}_{\text{antisymmetric}}$$

$$\text{symmetric} = \frac{1}{2}\left(\begin{bmatrix} 1 & -2 & 0 \\ -4 & 2 & 3 \\ 8 & -7 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -4 & 8 \\ -2 & 2 & -7 \\ 0 & 3 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 2 & -6 & 8 \\ -6 & 4 & -4 \\ 8 & -4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 2 & -2 \\ 4 & -2 & 3 \end{bmatrix}$$

$$\text{antisymmetric} = \frac{1}{2}\left(\begin{bmatrix} 1 & -2 & 0 \\ -4 & 2 & 3 \\ 8 & -7 & 3 \end{bmatrix} - \begin{bmatrix} 1 & -4 & 8 \\ -2 & 2 & -7 \\ 0 & 3 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2 & -8 \\ -2 & 0 & 10 \\ 8 & -10 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -4 \\ -1 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}$$

$$\therefore Q = \begin{bmatrix} 1 & -3 & 4 \\ -3 & 2 & -2 \\ 4 & -2 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 1 & -4 \\ -1 & 0 & 5 \\ 4 & -5 & 0 \end{bmatrix}$$

☆