$F_n = \frac{1}{\sqrt{5}}(a^n - b^n)$ when $a = \frac{1+\sqrt{5}}{2}$

$b = \frac{1-\sqrt{5}}{2}$

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HW4 Discrete Maths

Problem 1 consider $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$

Let $F_n = x^n \Rightarrow x^n = x^{n-1} + x^{n-2}$

$$\Downarrow$$

$$\div x^{n-2} = x^{n-(n-2)} = x^{n-1-(n-2)} + x^{n-2-(n-2)}$$

$$= x^2 = x^1 + x^0$$

$$= x^2 = x+1$$

$$\Downarrow$$

$$x^2 - x - 1 = 0$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} \Rightarrow \frac{1 \pm \sqrt{(-1)^2-4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

which is the same as the values of $\underline{a}$ and $\underline{b}$

$\therefore F_n = xa^n + yb^n$ linear recurrence

consider that if $n=1$, $xa^1 + yb = 1$

if $n=2$, $xa^2 + yb^2 = 1$

$$\Downarrow$$

since a and b are solutions $x(a+1) + y(b+1) = 1$

for $x^2 - x - 1$, $a^2 = a+1$

$b^2 = b+1$

$$xa + yb = xa + x + yb + y$$

$\therefore xa + yb = \underline{xa + yb} + x + y$

$\therefore x + y = 0 \Rightarrow x = -y$

If $x = -y$, $y = -x$

$$\hookrightarrow \quad x\left[\frac{1+\sqrt{5}}{2}\right] - x\left[\frac{1-\sqrt{5}}{2}\right] = 1$$

$$x\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right] = 1$$

$$x\left[\frac{1+\sqrt{5}-(1-\sqrt{5})}{2}\right] = 1$$

$$x\left[\frac{2\sqrt{5}}{2}\right] = 1 \Rightarrow x\sqrt{5} = 1$$

$$x = \frac{1}{\sqrt{5}}$$

$\therefore$ we can write the formula as $\frac{1}{\sqrt{5}}\left[a^n - b^n\right]$ ☆

Problem 2 Assume cola $= 5$ State = no of grapes and no of cola.
grape $= 6$

Consider the 3 possible transitions:

$$P(G,G) \longrightarrow (G,C) \xRightarrow[\text{put other back}]{\text{eat 1 grape}} (G-1, C)$$

$$P(C,C) \longrightarrow (G,C) \xRightarrow[\text{put other back}]{\text{eat 1 cola}} (G, C-1)$$

$$P(G,C) \longrightarrow (G,C) \xRightarrow[\text{put other back}]{\text{eat 1 grape}} (G-1, C)$$

Consider (G_n, C_n) where $\overset{\text{invariant}}{G_n \geq 1}$ so the last candy must be grape

where $n =$ turn number

1) if $P(G,G)$ then $G_n \geq 2$ which also means $G_n \geq 1$

$G_{n+1} \geq 2-1 \Rightarrow$ meaning $G_{n+1} \geq 1$ $\therefore$ ✓

if $P(C, C)$ then $G_n \Rightarrow G_{n+1}$ should be unchanged meaning that $\boxed{G_n = G_{n+1}} \geq 1$

if $P(G, C)$ then $G_{n+1} = G_n - 1$. if $G_n = 1$ then $G_{n+1} = 0$

so invariant is not met.

Try with cola instead:

suppose (G_n, C_n) where $C_n \geq 1$ meaning last candy must be cola

if $P(G, G)$ then $C_{n+1} = C_n$ since it does not change. Therefore, invariant still preserved.

If $P(C, C)$ then $C_n \geq 2$ so $C_{n+1} \geq 1$

since $C_n \geq 2$ means $C_n \geq 1$ invariant preserved.

if $P(G, C)$ then $C_{n+1} = C_n$ since it does not change. Therefore, invariant still preserved.

Therefore, the flavor ajarn wants from this setup is cola.

Problem 3 pf: No matter what, last number is odd. $1+2+3+4+5+6 = \boxed{21 \text{ odd}}$

$\Downarrow$

$\therefore$ sum of remaining numbers is odd

state

Transitions:

Suppose we pick two numbers a and b. then we have 4 transitions

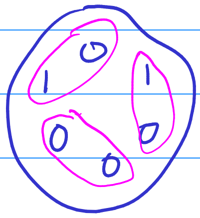
a	b	new sum
even	even	$= \underset{\text{odd}}{\text{old sum}} - \underset{\text{even} = \text{odd}}{(a+b)} + \underset{\text{even} = \text{odd}}{\lvert a-b \rvert} = \text{odd}$
even	odd	$= \underset{\text{odd}}{\text{old sum}} - \underset{\text{odd} = \text{even}}{(a+b)} + \underset{\text{odd} = \text{odd}}{\lvert a-b \rvert} = \text{odd}$

a	b	new sum
odd	even	$\underset{odd}{\text{old sum}} - \underset{\substack{odd \\ =even}}{(a+b)} + \underset{\substack{odd \\ =odd}}{\lvert a-b \rvert} = odd$
odd	odd	$\underset{odd}{\text{old sum}} - \underset{\substack{even \\ =odd}}{(a+b)} + \underset{\substack{even \\ =odd}}{\lvert a-b \rvert} = odd$

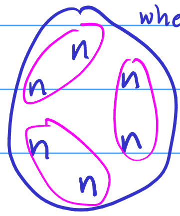
Since all 4 transitions lead to odd, we can conclude that no matter what we do, the final number will be odd (as denoted by the sum) ☆

Problem 4

Start



Goal



where $n \in \mathbb{Z}^+$

$[(n-n)+(n-n)+(n-n)=0]$ => call this equation G

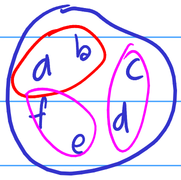
or

$(a-b)+(c-d)+(e-f)=0$ where $a=b=c=d=e=f$.

turn n Starting state $\Rightarrow (1-0)+(1-0)+(0-0)=2$

Transition:

suppose $(a-b)+(c-d)+(e-f)$



turn n+1

$\Rightarrow (a+1-(b+1)) + (c-d) + (e-f)$

$\Downarrow$

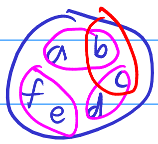
$= a + \cancel{1} - b - \cancel{1}$

$= (a-b)$

$= (a-b)+(c-d)+(e-f)$

preserved invariant

Suppose we try



$= (a-(b+1)) + ((c+1)-d) + (e-f)$

$= (a-b\cancel{-1}) + (c-d\cancel{+1}) + (e-f)$

$= (a-b)+(c-d)+(e-f)$

preserved invariant

Know that we can rearrange the sectors and the same result will be yielded since 1's cancel.

Meaning that suppose we start with some result $R=(a-b)+(c-d)+(e-f)$ then no matter what transition we do, R will not change. This means if we start with $R=2$ then it is not possible for $a=b=c=d=e=f$ since R must be 0 according to G.

☆

Problem 5 Prove (if possible) we can get $(\overset{W}{63}, \overset{B}{1})$ on an $8 \cdot 8$ board.

State: no. of W, no. of B.

Assume turn n, where (W_n, B_n) and B_n is even. (WLOG)
(so the starting turn applies to this)

Transition(s):

since even+even = 8 (even)

If we choose a row/column where B is even (so W is also even)
meaning that the total number of (W, B) will change accordingly

$$(\underset{\text{not relevant}}{W_n}, \underset{\text{even}}{B_n}) \qquad B_n \ e$$

when we swap, do -8 e first.
we are re-counting.

$$\text{swap color } (\underset{\text{even}}{B}, \underset{\text{even}}{W}) \qquad +\text{even } e$$

$$\Downarrow$$

$$(W_{n+1}, \underset{\text{even}}{B_{n+1}}) \qquad B_{n+1}$$

$$= \underset{=\text{even}}{\text{even} - \text{even}} + \underset{\text{even}}{\text{even}} = \text{even}$$

If we choose a row/column where B is odd (so W is odd too
since odd+odd = even
$n(B) + n(W) = 8$)

$$(W_n, \underset{\text{odd}}{B_n}) \qquad B_n \ o$$

$$\Downarrow \qquad -8 \text{ (same reason) } e$$

$$\text{swap color } (\underset{\text{odd}}{B}, \underset{\text{odd}}{W}) \qquad +\text{odd } o$$

$$\Downarrow \qquad B_{n+1}$$

$$(W_n, \underset{\text{even}}{B_{n+1}}) \qquad = \underset{=\text{odd}}{\text{odd} - \text{even}} + \underset{=\text{even}}{\text{odd}} = \text{even}$$

We notice a preserved invariant where no. of B stays even after each transition. Therefore, it is impossible to reach a state where $(\underset{n(W)}{63}, \underset{n(B)}{1})$ since $n(B)$ is odd.

Problem 6 Hint: mod 3

Pf: We can end up in a state where all Pokemon are the same color.
(or disprove)

so mod 3 of the colors $\Rightarrow$ $(0, 0, 0)$ order is arbitrary
0 n 0 where $n \bmod 3 = 0$

(per existing proof that is too long to write)

State: mod 3 of the 3 colors.

Start: Suppose we start in a state

b r g
(13, 15, 17)

$\Downarrow$

WLOG

13 mod 3, 15 mod 3, 17 mod 3

assume we then start with mod 3 $\Rightarrow$ (1, 0, 2)

b r g

Thm:

$\boxed{\begin{aligned} &\text{if } n \bmod 3 = 0: \\ &n-1 \bmod 3 = 2 \\ &n-2 \bmod 3 = 1 \\ &n-3 \bmod 3 = 0 \end{aligned}}$

$n-4 \bmod 3 = 2$

etc.

Transitions:

$\rightarrow$ if B and R meet then $n(G)$ += 2

$n(B)$ and $n(R)$ -= 1

b r g

apply thm $\Rightarrow$ (0 2 1)

$n(B)-1$ $n(R)-1$ $n(G)+2$

if $n \bmod 3 = 1$ if $n \bmod 3 = 0$ if $n \bmod 3 = 2$

$n-1 \bmod 3 = 0$ $n-1 \bmod 3 = 2$ $n+2 \bmod 3 = 1$

$\rightarrow$ if B and G meet then $n(R) \pm 2$

$n(B)$ and $n(G)$ -= 1

b r g

apply thm $\Rightarrow$ (0 2 1)

$n(B)-1$ $n(R)+2$ $n(G)-1$

$n \bmod 3 = 1$ $n \bmod 3 = 0$ $n \bmod 3 = 2$

$n-1 \bmod 3 = 0$ $n+2 \bmod 3 = 2$ $n-1 \bmod 3 = 1$

$\rightarrow$ if R and G meet then $n(B) \pm 2$

$n(R)$ and $n(G)$ -= 1

b r g

apply thm $\Rightarrow$ (0 2 1)

$n(B)+2$ $n(R)-1$ $n(G)-1$

$n \bmod 3 = 1$ $n \bmod 3 = 0$ $n \bmod 3 = 2$

$n+2 \bmod 3 = 0$ $n-1 \bmod 3 = 2$ $n-1 \bmod 3 = 1$

We observe that there is a preserved invariant - the mod 3 of all three colors are not the same (more specifically ending up in the same order too, but this is not needed)

$$\Downarrow$$

So it is impossible to end up in a state where some or all of mod3 of the colors are non-unique.

$$\Downarrow$$

An example of this happening would be when there is only one color since two colors will have the same mod 3 (0 mod 3 = 0)

$$\Downarrow$$

Therefore we will never reach a state where all Pokemon are the same color