6680210 Problem set 3

Collaborators: None

sources consulted: Week 3 Lecture Notes (Canvas)

Problem 1

$(i,j) = (2i) + j$

i rows, j columns

a)

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 3 & 4 & 5 \\ 5 & 6 & 7 \\ 7 & 8 & 9 \end{pmatrix} \end{array}$$

b) $A^T \Rightarrow \begin{pmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \\ 5 & 7 & 9 \end{pmatrix}$

c) $tr(A) = 3+6+9 = 18$

$tr(A^T) = 3+6+9 = 18$

yes, they are equal.

d) $\begin{pmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \\ 5 & 7 & 9 \end{pmatrix} \xrightarrow{R_1 \Leftrightarrow R_2} \begin{pmatrix} 4 & 6 & 8 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix} \xrightarrow{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 5 & 7 & 9 \end{pmatrix}$

$R_3 = R_3 - 5R_1$ $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 0 & 2 & 4 \end{pmatrix}$ $R_2 = R_2 - 0.5R_3$ $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{pmatrix}$ $R_3 = R_3 - R_2$ $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$ $R_1 = R_1 - R_3$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 1 \end{pmatrix}$

$R_2 = R_2 - R_3$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$ $R_2 \Leftrightarrow R_3$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ $R_3 = \frac{1}{2}R_3$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ $R_2 = R_2 - R_3$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

✓

Problem 2

$B \in \mathbb{R}^{3\times 4}$

a)

$$\begin{array}{cc} & \begin{array}{cccc} & & j & \\ 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ i\ 2 \\ 3 \end{array} & \begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 8 & 12 & 16 \\ 9 & 18 & 27 & 36 \end{Bmatrix} \end{array}$$

$i,j = i^2 \circ j$

b) $B^T = \begin{pmatrix} 1 & 4 & 9 \\ 2 & 8 & 18 \\ 3 & 12 & 27 \\ 4 & 16 & 36 \end{pmatrix}$

c) $\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 4 & 8 & 12 & 16 \\ 9 & 18 & 27 & 36 \end{Bmatrix}$

$R_2 = R_2 - 4R_1$

$\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 9 & 18 & 27 & 36 \end{Bmatrix}$

$R_3 = R_3 - 9R_1$

$\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Bmatrix}$ ✓

d) $\begin{Bmatrix} 1 & 4 & 9 \\ 2 & 8 & 18 \\ 3 & 12 & 27 \\ 4 & 16 & 36 \end{Bmatrix}$

$R_4 = R_4 - 2R_2$

$\begin{Bmatrix} 1 & 4 & 9 \\ 2 & 8 & 18 \\ 3 & 12 & 27 \\ 0 & 0 & 0 \end{Bmatrix}$

$R_3 = R_3 - 3R_1$

$\begin{Bmatrix} 1 & 4 & 9 \\ 2 & 8 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$

$R_2 = R_2 - 2R_1$

$\begin{Bmatrix} 1 & 4 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$ ✓

Problem 3

$A = [3] \quad B = [-2] \quad C = [9]$

a) $A + B$

$= 3 + (-2)$

$= 1$

b) $B - 2 \circ C$

$= -2 - (2 \circ 9)$

$= -20$

c) $-3 \circ A + B + 4 \circ C$

$= (-3 \circ 3) - 2 + (4 \circ 9)$

$= -9 - 2 + 36$

$= -11 + 36$

$= 25$

d) $A \circ B$

$= 3 \circ -2$

$= -6$

e) $B \circ C$

$= -2 \circ 9$

$= -18$

f) $tr(A + B) + tr(C C^T)$

$tr(1) = 1 \Rightarrow tr(C C^T) = tr(9) = 9$

$= 1 + 9$

$= 10$

g) $(A + B) \circ (B + C)$

$= 1 \circ 7$

$= 7$

h) $2AB - 3AC + 4BC$

$= (2)(3)(-2) - (3)(2)(9) + 4(-2)(9)$

$= -12 - 81 - 72$

$= -165$

Problem 4

$A = [0\ 1\ -1\ -2] \quad B = [2\ -1\ 3\ -5] \quad C = [3\ 0\ -2\ 3]$

a) $-A + B - 2C$

$-[0\ 1\ -1\ -2] + [2\ -1\ 3\ -5] - 2[3\ 0\ -2\ 3]$

$= [0\ -1\ 1\ 2] + [2\ -1\ 3\ -5] - [6\ 0\ -4\ 6]$

$= [-4\ -2\ 8\ -9]$

b) $-3A^T + 4C^T$

$A^T = \begin{pmatrix} 0 \\ 1 \\ -1 \\ -2 \end{pmatrix} \quad C^T = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 3 \end{pmatrix}$

$-3A^T = \begin{pmatrix} 0 \\ -3 \\ 3 \\ 6 \end{pmatrix} \quad 4C^T \begin{pmatrix} 12 \\ 0 \\ -8 \\ 12 \end{pmatrix} \quad -3A^T + 4C^T = \begin{pmatrix} 12 \\ -3 \\ -5 \\ 18 \end{pmatrix}$

c) 1x4 matrice with 1 row and 4 columns each

d) 4x1 matrice with 4 rows and 1 column each

e) $A \circ B^T$

$A = (0\ 1\ -1\ -2) \quad B^T = \begin{pmatrix} 2 \\ -1 \\ 3 \\ -5 \end{pmatrix} \quad A \circ B^T = (0 \cdot 2 + 1 \cdot -1 + -1 \cdot 3 + -2 \cdot -5)$

$= 0 - 1 - 3 + 10$

$= -4 + 10$

$= 6$

$\Downarrow$

(6)

f) $B \circ A^T$

$B = (2\ \ -1\ \ 3\ \ -5)$ $\quad A^T = \begin{pmatrix} 0 \\ 1 \\ -1 \\ -2 \end{pmatrix}$ $\quad B \circ A^T = (2\cdot 0 + -1\cdot 1 + 3\cdot -1 + -5\cdot -2)$

$= 0 - 1 - 3 + 10$

$= 6$

$= (6)$

g) $B^T \circ C$

$B^T = \begin{pmatrix} 2 \\ -1 \\ 3 \\ -5 \end{pmatrix}$ $\quad C = (3\ \ 0\ \ -2\ \ 3)$ $\quad B^T \circ C = \begin{pmatrix} 6 & 0 & -4 & 6 \\ -3 & 0 & 2 & -3 \\ 9 & 0 & -6 & 9 \\ -15 & 0 & 10 & -15 \end{pmatrix}$

h) $C^T \circ B$

$C^T = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 3 \end{pmatrix}$ $\quad B = (2\ \ -1\ \ 3\ \ -5)$ $\quad C^T \circ B = \begin{pmatrix} 6 & -3 & 9 & -15 \\ 0 & 0 & 0 & 0 \\ -4 & 2 & -6 & 10 \\ 6 & -3 & 9 & -15 \end{pmatrix}$

i) $(A+B) \circ (B+C)^T$

$A+B = [2\ \ 0\ \ 2\ \ -7]$ $\quad B+C = [5\ \ -1\ \ 1\ \ -2]$

$(B+C)^T = \begin{pmatrix} 5 \\ -1 \\ 1 \\ 2 \end{pmatrix}$

$(A+B) \circ (B+C)^T$

$= (2\cdot 5 + 0\cdot -1 + 2\cdot 1 + -7\cdot 2)$

$= 10 - 1 + 2 - 14$

$= -4 - 1 + 2$

$= (-3)$

j) $(A+B)^T \circ (B+C)$

$(A+B)^T = \begin{pmatrix} 2 \\ 0 \\ 2 \\ -7 \end{pmatrix}$ $(B+C) = [5 \ -1 \ 1 \ -2]$

$(A+B)^T \circ (B+C) = \begin{pmatrix} 10 & -2 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 10 & -2 & 2 & -4 \\ -35 & 7 & -7 & 14 \end{pmatrix}$

Problem 5

$$X = \begin{bmatrix} 2 & -5 & 0 & 1 \\ 0 & 7 & -3 & -2 \end{bmatrix} \quad Y = \begin{bmatrix} -1 & 8 & 0 & 4 \\ -2 & -6 & 3 & 2 \end{bmatrix}$$

a) $X + Y$

$$= \begin{bmatrix} 1 & 3 & 0 & 5 \\ -2 & 1 & 0 & 0 \end{bmatrix}$$

b) $-X^T + 2 \circ Y^T$

$$X^T = \begin{pmatrix} 2 & 0 \\ -5 & 7 \\ 0 & -3 \\ 1 & -2 \end{pmatrix} \quad -X^T = \begin{pmatrix} -2 & 0 \\ 5 & -7 \\ 0 & 3 \\ -1 & 2 \end{pmatrix} \quad Y^T = \begin{pmatrix} -1 & -2 \\ 8 & -6 \\ 0 & 3 \\ 4 & 2 \end{pmatrix} \quad 2Y^T = \begin{pmatrix} -2 & -4 \\ 16 & -12 \\ 0 & 6 \\ 8 & 4 \end{pmatrix}$$

$$\downarrow$$

$$-X^T + 2Y^T$$

$$\begin{pmatrix} -4 & -4 \\ 21 & -19 \\ 0 & 9 \\ 7 & 6 \end{pmatrix}$$

Problem 6

$$M = \begin{bmatrix} 0 & 2 & -3 \\ -1 & 0 & 2 \\ 2 & 0 & 1 \end{bmatrix} \qquad N = \begin{bmatrix} -3 & 0 & 1 \\ 0 & 2 & 2 \\ -1 & 0 & 0 \end{bmatrix}$$

a) $M \circ N$

$$= \begin{bmatrix} -3 & 4 & 4 \\ 1 & 0 & -1 \\ -7 & 0 & 2 \end{bmatrix}$$

$(1,1) = 0 \cdot -3 + 2 \cdot 0 + -3 \cdot 1 = -3$

$(1,2) = 0 \cdot 0 + 2 \cdot 2 + -3 \cdot 0 = 4$

$(1,3) = 0 \cdot 1 + 2 \cdot 2 + -3 \cdot 0 = 4$

$(2,1) = -1 \cdot -3 + 0 \cdot 0 + 2 \cdot -1 = 3 - 2 = 1$

$(2,2) = -1 \cdot 0 + 0 \cdot 2 + 2 \cdot 0 = 0$

$(2,3) = -1 \cdot 1 + 0 \cdot 2 + 2 \cdot 0 = -1$

$(3,1) = 2 \cdot -3 + 0 \cdot 0 + 1 \cdot -1 = -6 - 1 = -7$

$(3,2) = 2 \cdot 0 + 0 \cdot 2 + 1 \cdot 0 = 0$

$(3,3) = 2 \cdot 1 + 0 \cdot 2 + 1 \cdot 0 = 2$

b) $N \circ M$

$$N \begin{pmatrix} -3 & 0 & 1 \\ 0 & 2 & 2 \\ -1 & 0 & 0 \end{pmatrix} \qquad M \begin{pmatrix} 0 & 2 & -3 \\ -1 & 0 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

$N \cdot M$

$$\begin{pmatrix} 2 & -6 & 10 \\ 2 & 0 & 6 \\ 0 & -2 & 3 \end{pmatrix}$$

$(1,1) = -3 \cdot 0 + 0 \cdot -1 + 1 \cdot 2 = 2$

$(1,2) = -3 \cdot 2 + 0 \cdot 0 + 1 \cdot 0 = -6$

$(1,3) = -3 \cdot -3 + 0 \cdot 2 + 1 \cdot 1 = 10$

$(2,1) = 0 \cdot 0 + 2 \cdot -1 + 2 \cdot 2 = -2 + 4 = 2$

$(2,2) = 0 \cdot 2 + 2 \cdot 0 + 2 \cdot 0 = 0$

$(2,3) = 0 \cdot -3 + 2 \cdot 2 + 2 \cdot 1 = 4 + 2 = 6$

$(3,1) = -1 \cdot 0 + 0 \cdot -1 + 0 \cdot 2 = 0$

$(3,2) = -1 \cdot 2 + 0 \cdot 0 + 0 \cdot 0 = -2$

$(3,3) = -1 \cdot -3 + 0 \cdot 2 + 0 \cdot 1 = 3$

c) Don't feel like doing extra credit :(

Problem 7

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 4 & 3 \end{bmatrix}$$

a) $A \cdot A^T$

$$A^T = \begin{bmatrix} 5 & 0 \\ 2 & 4 \\ 1 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 4 & 3 \end{bmatrix} \qquad A \cdot A^T = \begin{pmatrix} 30 & 11 \\ 11 & 25 \end{pmatrix}$$

$(1,1) = 5 \cdot 5 + 2 \cdot 2 + 1 \cdot 1 = 30$

$(1,2) = 5 \cdot 0 + 4 \cdot 2 + 3 \cdot 1 = 11$

$(2,1) = 0 \cdot 5 + 4 \cdot 2 + 3 \cdot 1 = 11$

$(2,2) = 0 \cdot 0 + 4 \cdot 4 + 3 \cdot 3 = 25$

b) $A^T \cdot A$

$$A^T = \begin{bmatrix} 5 & 0 \\ 2 & 4 \\ 1 & 3 \end{bmatrix} \qquad A = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 4 & 3 \end{bmatrix} \qquad A^T \cdot A = \begin{pmatrix} 25 & 10 & 5 \\ 10 & 20 & 14 \\ 5 & 14 & 10 \end{pmatrix}$$

$(1,1) = 5 \cdot 5 + 0 \cdot 0 = 25$

$(1,2) = 5 \cdot 2 + 0 \cdot 4 = 10$

$(1,3) = 5 \cdot 1 + 0 \cdot 3 = 5$

$(2,1) = 2 \cdot 5 + 4 \cdot 0 = 10$

$(2,2) = 2 \cdot 2 + 4 \cdot 4 = 20$

$(2,3) = 2 \cdot 1 + 4 \cdot 3 = 14$

$(3,1) = 1 \cdot 5 + 3 \cdot 0 = 5$

$(3,2) = 1 \cdot 2 + 3 \cdot 4 = 14$

$(3,3) = 1 \cdot 1 + 3 \cdot 3 = 10$

c) $tr(A \cdot A^T) = 30 + 25 = 55$

$tr(A^T \cdot A) = 25 + 20 + 10 = 55$

Yes, they are the same.