6680210 | Theeradon Sarawek

Discrete Maths

HW1

1) $\exists m \in \mathbb{Z}$ such that $x = m^2$

if $x = m^2$, $y = n^2$ where $(m,n) \in I$

$xy = (m^2)(n^2) = (mn)^2$ where $o = mn$

and $o \in I$ as

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2) $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$

Proof by contradiction:

if: $a > \sqrt{n}$ and $b > \sqrt{n}$, $\therefore ab > n$

which is false as we've established $\boxed{ab = n}$,

Therefore, the original statement has to be true.

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3) If $x \in \mathbb{Q}$ and $y \in \mathbb{Q}$, $\therefore x+y \in \mathbb{Q}$

Consider that a rational number is $\frac{\text{integer}}{\text{integer}}$

so let $x = \frac{a}{b}$, $y = \frac{c}{d}$, $\therefore x+y = \frac{a}{b} + \frac{c}{d}$

$$= \frac{ad}{bd} + \frac{bc}{bd}$$

$$= \frac{ad+bc}{bd}$$

Since we know that integer addition results in an integer as well, so does integer multiplication, we know that $ad+bc$ and bd MUST be integers.

$\therefore x+y \in \mathbb{Q}$ so the statement is True

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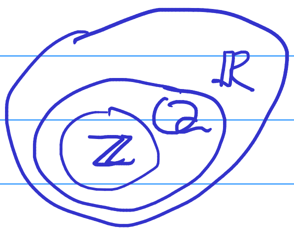
4) If $x \notin \mathbb{Q}$ then $\frac{1}{x} \notin \mathbb{Q}$

x is irrational as it cannot be written in the form $\frac{a}{b}$ where $(a, b) \in \mathbb{Z}$

recalling this definition

consider:



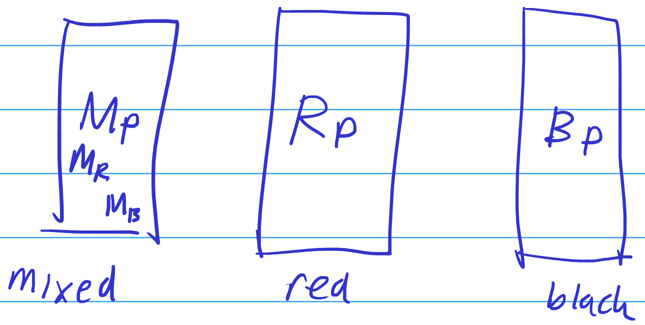
$\rightarrow$ if $x \notin \mathbb{Q}$ then $x \notin \mathbb{Z}$

meaning x cannot be an integer.

This means that $\frac{1}{x}$ $\leftarrow$ integer, $\leftarrow$ not integer

so it cannot be a rational number.

5) Proof of card trick



where no. of $R = 2R_p$

no. of $B = 2B_p$

We want to prove that $R_p = B_p$

A deck has 52 cards, 26 of each color:

① $M_R + M_B + 2R_p + 2B_p = 52$

No. of reds ② $M_R + 2R_p = 26$

No. of blacks ③ $M_B + 2B_p = 26$

Considering that since we're dealing with pairs, $M_R \overset{\text{must}}{=} M_B$.

Let's just call those variables x:

$$x + 2R_p = x + 2B_p = 26$$

Say x is anything, let's use 4 as a placeholder.

$$4 + 2R_p = 4 + 2B_p = 26$$

This must mean BOTH $2R_p$ and $2B_p = 26 - 4$
$= 22$

$\therefore R_p = B_p = 11$

Meaning that no matter what, the number of red pairs will be the same as the number of black pairs REGARDLESS of how many red/black (mixed) pairs there are.

6) $$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \quad \forall a, b \in \mathbb{R},\ b \neq 0$$

Consider all the possibilities:

	a^+	a^-
b^+	$\frac{\lvert a^+\rvert}{\lvert b^+\rvert} = \frac{a}{b}$	$\frac{\lvert a^-\rvert}{\lvert b^+\rvert} = \frac{-a}{b}$ or $-\left(\frac{a}{b}\right)$
b^-	$\frac{\lvert a^+\rvert}{\lvert b^-\rvert} = \frac{a}{-b}$ or $-\left(\frac{a}{b}\right)$	$\frac{\lvert a^-\rvert}{\lvert b^-\rvert} = \frac{-a}{-b}$ or $\frac{a}{b}$

And the original:

	a^+	a^-
b^+	$\frac{a^+}{b^+} \geq 0 \therefore = \frac{a}{b}$	$\frac{a^-}{b^+} < 0 \therefore = -\left(\frac{a}{b}\right)$
b^-	$\frac{a^+}{b^-} < 0 \therefore = -\left(\frac{a}{b}\right)$	$\frac{a^-}{b^-} \geq 0 \therefore = \frac{a}{b}$

Comparing with each box and the outputs are identical. Therefore, the original statement has been proven true.

7.1) If r is irrational, then $r^{1/5}$ is irrational.

contrapositive:

if $r^{\frac{1}{5}} \in \mathbb{Q} \implies r^{\frac{1}{5}} \in \mathbb{Q}$

consider if $r^{\frac{1}{5}} \in \mathbb{Q}$ then $r^{\frac{1}{5}} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$

then $r = \left(\frac{a}{b}\right)^5$

since x^n will always result in an integer if $n \in \mathbb{Z}$

we can assume that $\left(\frac{a}{b}\right)^5 = \frac{a^5}{b^5}$ where $a^5, b^5 \in \mathbb{Z}$

$\therefore \frac{a^5}{b^5} \in \mathbb{Q}$

7.2) if $r \in \mathbb{Q}$ then $r^{\frac{1}{5}} \in \mathbb{Q}$

know that a number is rational if it can be written as $\frac{a}{b}$ where $a, b \in \mathbb{Z}$ AND a and b aren't both even.

Take a rational number $\frac{3}{2}$: if $\frac{3}{2}$ is rational then we need to know if $\frac{3^{\frac{1}{5}}}{2^{\frac{1}{5}}}$ is rational.

We need to know if $3^{\frac{1}{5}} \in \mathbb{Z}$ and $2^{\frac{1}{5}} \in \mathbb{Z}$

This must mean that, say, $a^5 = 3$ and $b^5 = 2$ where both a and b are integers.

However, if we lsst out the result of the first few integers to the power of 5...

$$-2^5 = -32, \ -1^5 = -1, \ 0^5 = 0, \ 1^5 = 1, \ 2^5 = 32$$

$2 \quad 4^2 \quad 8^3 \quad 16^4 \quad 32^5$

None of them result in 3 or 2. This must mean that $3^{\frac{1}{5}}$ and $2^{\frac{1}{5}}$ are not integers.

Therefore, $\dfrac{3^{\frac{1}{5}}}{2^{\frac{1}{5}}}$ $\leftarrow$ non-integer
$\leftarrow$ non-integer

Does not meet the requirement of a rational number

Therefore, if $r = \frac{3}{2}$, $r^{\frac{1}{5}} = \dfrac{3^{\frac{1}{5}}}{2^{\frac{1}{5}}} \notin \mathbb{Q}$

so the statement is not true.

7.3) If $r^{\frac{1}{5}} \notin \mathbb{Q}$, $r \notin \mathbb{Q}$

The contrapositive of this is:

If $r \in \mathbb{Q} \implies r^{\frac{1}{5}} \in \mathbb{Q}$

We already proved that this statement is untrue in 7.2)
so by using contrapositivity, we can assume that this statement is also not true.

8) Zero-sum $\Rightarrow \sum$ of all gains $- \sum$ of all losses $= 0$

10 000 players, 1 million ฿ each:

Total money $= 10^4 \times 10^6 = 10^{10}$ ฿

Claim: After 20,000 rounds, everyone will profit.

Consider that a profit would mean everyone will have $10^6 + P$ baht, where p is the profit so $p > 0$.

The sum of everyone's money can be writen as

$$\sum_{n=1}^{10\,000} 10^6 + P_n$$

where each person's profit could be different so we differentiate them by using n.

This number should be equal to:

$$\sum_{n=1}^{10\,000} P_n + 10^{10}$$

However, we must assume that this game is played in a closed system. This means no money leaves or enters the game. Therefore the sum of everyone's money must be (still):

$$10^{10}$$

So we get the following dilemma:

$$\sum_{n=1}^{10\,000} P_n + 10^{10} = 10^{10}$$

but we know that $P_n > 0$ since everyone must profit.

The equation shows that $\displaystyle\sum_{n=1}^{10\,000} P_n = 10^{10} - 10^{10}$

$$= 0$$

meaning that such a scenario is impossible.

Such games can be considered as zero-sum games as one person's gain = another's loss.

9) Since $A_{Room} = 5m^2$ AND $\sum A_{Rugs} = 9m^2$

there will inevitably be $4m^2$ of area that will have overlapping rugs. Intouch wants the area of intersection to be less than $\frac{1}{9}m^2$ per two rugs.

The max number of overlapping pairs, P, can be given as $_9C_2$ since order does not matter.

$$_9C_2 = \frac{9!}{2!(9-2)!} = \frac{9!}{2!\,\cancel{7!}}$$

$$= \frac{9 \cdot 8}{2} = \frac{72}{2} = 36 \text{ pairs}$$

Therefore the max number of overlapping areas in this case must not exceed $36 \cdot \frac{1}{9} = 4m^2$

This means that $A_{overlap} < 4$

but a configuration where $9m^2$ of rugs in a $5m^2$ room leaves us with

$$A_{overlap} = 4$$

Since both those statements cannot be True at the same time, Their requirements are impossible to fulfill.

18) $\forall x \in O \ \exists (p,q) \in \mathbb{Z}$ such that $x = p^2 - q^2$
or "every odd integer can be written as a difference of two squares"

Consider an odd number x can be expressed as
$x = 2n+1$ where $n \in \mathbb{Z}$

This means $2n+1 = p^2 - q^2$
Recall that $p^2 - q^2 = (p+q)(p-q)$
and that odd $\times$ odd $=$ odd
and odd $\pm$ even $=$ odd
so if either p or q is even, the other should be odd

$\hookrightarrow$ so let's assume $p = q+1$ as consecutive numbers are never the same parity.

$$\Rightarrow (q+1)^2 - q^2 = 2n+1$$
$$\cancel{q^2} + 2q + 1 - \cancel{q^2} = 2n+1$$
$$2q+1 = 2n+1$$

this is exactly the form we want, where we can just substitute $n = q$ to prove our statement