Linear Algebra
Problem Set 5

Problem 0

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c) None

d) Lecture Note 5 on Canvas

Problem 1

a)
$$\begin{array}{c}3\end{array}\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}_{3}\circ\overset{4}{\begin{bmatrix}1&2&3&4\\5&6&7&8\\9&10&11&12\end{bmatrix}}\,3$$

$$=\begin{bmatrix}1&2&3&4\\9&10&11&12\\5&6&7&8\end{bmatrix}$$

(1,1) = 1, (1,2) = 2, (1,3) = 3, (1,4) = 4

(2,1) = 9, (2,2) = 10, (2,3) = 11, (2,4) = 12

(3,1) = 5, (3,2) = 6, (3,3) = 7, (3,4) = 8

Essentially the same as $R_2 \Leftarrow\Rightarrow R_3$

b)
$$4\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-4&0\\0&0&0&1\end{bmatrix}\circ\overset{2}{\begin{bmatrix}1&2\\3&4\\5&6\\7&8\end{bmatrix}}$$

$$=\begin{bmatrix}1&2\\3&4\\-20&-24\\7&8\end{bmatrix}$$

Essentially the same as $R_3 = -4R_3$

c)
$$\begin{bmatrix}1&0&0\\0&1&0\\0&2&1\end{bmatrix}\circ\begin{bmatrix}-1&0&1\\0&1&2\\1&2&3\end{bmatrix}$$

Essentially the same as $R_3 = R_3 + 2R_2$

$$=\begin{bmatrix}-1&0&1\\0&1&2\\1&4&7\end{bmatrix}$$

$$\begin{array}{ccc}1+0&2+2&3+4\end{array}$$

Problem 2

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

a) $A_1 \xrightarrow{R_2 = \frac{1}{4}R_2} A_2 \quad \therefore \quad E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A_2 \xrightarrow{R_2 = R_2 + R_4} A_3 \quad \therefore \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A_3 \xrightarrow{R_2 \Leftrightarrow R_4} A_4 \quad \therefore \quad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$A_4 \xrightarrow{R_3 = \frac{1}{5}R_3} A_5 \quad \therefore \quad E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b) $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ since it is a multiplication row operation.

$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ since it is an add/subtract operation.

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$ since it is a row-swapping operation.

$$E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ Since it is a multiplication operation.

c) $E_1^{-1} \circ E_2^{-1}$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_1^{-1} \circ E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Essentially $R_2 = 4R_2$ so

d) $(E_1^{-1} \circ E_2^{-1}) \circ E_3^{-1}$

$\Downarrow$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & -4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} =$$

$R_2 = 4R_2 - 4R_4$

$0 \quad 0 \quad 0 \quad 4$

$0 \quad -4 \quad 0 \quad 4$

$\Downarrow$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

e) $(E_1^{-1} \circ E_2^{-1} \circ E_3^{-1}) \circ E_4^{-1}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$(1,1) = 1$ $(2,1) = 0$ $(3,1) = \cancel{0}$ $(4,1) =$
$(1,2) = 0$ $(2,2) = -4$ $(3,2) = 0$ $(4,2) =$
$(1,3) = 0$ $(2,3) = 0$ $(3,3) = 5$ $(4,3) =$
$(1,4) = 0$ $(2,4) = 4$ $(3,4) = 0$ $(4,4) =$

Yes, this is the same matrix as A.

f) $E_4 \circ E_3$

Essentially just $R_3 = \frac{1}{5}R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

g) $(E_4 \circ E_3) \circ E_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Essentially $R_2 \Longleftrightarrow R_4$
and $R_3 = \frac{1}{5}R_3$

h) A_1^{-1}

since $A_5 = B = I_m$ we know that A is invertible.

$\therefore A_1^{-1} = E_4 \circ E_3 \circ E_2 \circ E_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \underset{}{\overset{\left[A_j^{-1}\right]}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & \frac{1}{4} & 0 & 1 \end{bmatrix}}}$$

i) $\det(E_1) = \frac{1}{4}$ (multiplication/division)

$\det(E_1^{-1}) = 4$

j) $\det(E_2) = 1$ (add/subtract)

$\det(E_2^{-1}) = 1$

k) $\det(E_3) = -1$ (row swap)

$\det(E_3^{-1}) = -1$

l) $\det(E_4) = \frac{1}{5}$ (multiplication/division)

$\det(E_4^{-1}) = 5$

m) $\det(A_1) = 4 \cdot 1 \cdot -1 \cdot 5 = -20$ $\quad \det(E_1^{-1}) \circ \det(E_2^{-1}) \circ \det(E_3^{-1}) \circ \det(E_4^{-1})$

$\det(A_1^{-1}) = \frac{1}{5} \cdot -1 \cdot 1 \cdot \frac{1}{4} \quad -\frac{1}{20}$ $\quad \det(E_4) \cdot \det(E_3) \cdot \det(E_2) \cdot \det(E_1)$

n) $\det(A_2) \Rightarrow$ consider that $A_2 = E_1 \circ A$

$$\therefore \det(A_2) = \det(E_1) \circ \det(A)$$
$$= \frac{1}{4} \cdot -20$$
$$= -5$$

$\det(A_3) \Rightarrow \det(E_2) \circ \det(A_2)$

$= 1 \cdot -5 \quad = -5$

$\det(A_4) \Rightarrow \det(E_3) \circ \det(A_3)$

$= -1 \cdot -5 \quad = 5$

$\det(A_5) \Rightarrow \det(E_4) \circ \det(A_4)$ $\quad$ which makes sense since $A_5 = I_m$

$= \frac{1}{5} \circ 5 \quad = 1$

$$A := \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 \\ 5 & -5 & 0 & 0 \\ -10 & 10 & 0 & -7 \end{bmatrix}.$$

Problem 3

a) $R_4 = R_4 + 2R_3$

$$\begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 \\ 5 & -5 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$

$R_3 = \frac{1}{5} R_3$

$$\begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -7 \end{bmatrix}$$

$R_4 = R_4 + R_1$

$$\begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \Leftrightarrow R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 = R_4 - \frac{1}{8} R_3$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

RREF ✓

so $X_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$ $X_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$X_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ $X_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $X_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{8} & 1 \end{bmatrix}$

where $A = X_1 \cdot X_2 \cdot X_3 \cdot X_4 \cdot X_5 \cdot B$

b) $\det(A) = 0$

since B has an all-zero row. No need to bother w/ further calculations.

Therefore, A is non-invertible.

Since A is a $4 \cdot 4$ square matrix,

we deduce that it is singular.

$$C := \begin{bmatrix} 2 & 0 & 6 & 1 & 0 \\ 3 & 0 & 9 & 1 & 1 \end{bmatrix}.$$

Problem 4

$R_1 = \frac{1}{2}R_1$ $\begin{bmatrix} 1 & 0 & 3 & \frac{1}{2} & 0 \\ 3 & 0 & 9 & 1 & 1 \end{bmatrix}$ $R_2 = R_2 - 3R_1$ $\begin{bmatrix} 1 & 0 & 3 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}$ $R_1 = R_1 + R_2$ $\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -\frac{1}{2} & 1 \end{bmatrix}$

$R_2 = 2R_2$ $\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$ $R_2 = -R_2$ $\begin{bmatrix} 1 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{bmatrix}$ $\boxed{= D}$

RREF

$\therefore$ $Y_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ $Y_2 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ $Y_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $Y_4 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$Y_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Where $D = Y_5 \cdot Y_4 \cdot Y_3 \cdot Y_2 \cdot Y_1 \cdot C$

meaning that $C = Y_1^{-1} \circ Y_2^{-1} \circ Y_3^{-1} \circ Y_4^{-1} \circ Y_5^{-1} \circ D$

$\Downarrow$

$Y_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $Y_2^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ $Y_3^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $Y_4^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

$Y_5^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Where we assume that these inverses are denoted as $Y_1 \ldots Y_5$ in the equation:

$$C = Y_1 \cdot Y_2 \cdot Y_3 \cdot Y_4 \cdot Y_5 \cdot D$$

Problem 5

$$M = \begin{bmatrix} -1 & 5 \\ -7 & 11 \end{bmatrix}$$

$$M = Z_1 \circ Z_2 \circ Z_3 \circ Z_4 \circ \left(I_m^{\,2\times 2} \right)$$

$R_1 = -R_1$ $\begin{bmatrix} 1 & -5 \\ -7 & 11 \end{bmatrix}$ $\quad R_2 = R_2 + 7R_1$ $\begin{bmatrix} 1 & -5 \\ 0 & -24 \end{bmatrix}$ $\quad R_2 = -\frac{1}{24} R_2$ $\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$ $\quad R_1 = R_1 + 5R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$Z_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad Z_2 = \begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix} \quad Z_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{24} \end{bmatrix} \quad Z_4 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

$$Z_1^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad Z_2^{-1} = \begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix} \quad Z_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -24 \end{bmatrix} \quad Z_4^{-1} = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$$

Meaning that $M = Z_1^{-1} \circ Z_2^{-1} \circ Z_3^{-1} \circ Z_4^{-1} \left[\circ I_m \right]$

but matrix $\circ\, I_m$ is just itself anyway.

b) Assume these inverses are Z_1, Z_2, Z_3, Z_4 respectively:

$\det(Z_1) = -1 \quad \det(Z_2) = 1$

$\det(Z_3) = -24 \quad \det(Z_4) = 1$

c) Basically do $Z_4 \circ Z_3 \circ Z_2 \circ Z_1$ from a)

$$\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{24} \end{bmatrix} \circ \underbrace{\begin{bmatrix} 1 & 0 \\ 7 & 1 \end{bmatrix} \circ \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{\begin{bmatrix} -1 & 0 \\ -7 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} -1 & 0 \\ \frac{7}{24} & \frac{-1}{24} \end{bmatrix}$$

$$\begin{bmatrix} \frac{11}{24} & \frac{-5}{24} \\ \frac{7}{24} & \frac{-1}{24} \end{bmatrix}$$

d) $M^{-1} = \dfrac{1}{(-11)-(-35)} \circ \begin{bmatrix} 11 & -5 \\ 7 & 1 \end{bmatrix}$

$= \dfrac{1}{24} \circ \begin{bmatrix} 11 & -5 \\ 7 & 1 \end{bmatrix}$

$= \begin{bmatrix} \frac{11}{24} & \frac{-5}{24} \\ \frac{7}{24} & \frac{1}{24} \end{bmatrix}$ so the same as my answer from part c.

d) $\det(M) = -1 \cdot -24 \cdot 1 \cdot 1 = 24$