ICCS208: Assignment 6

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1: Task 4: Quick Sort Recurrence

(i) This is the recurrence relation for quicksort:

$$f(n) = n + 1 + \frac{2}{n}(f(n-1) + f(n-2) + ... + f(1)) \qquad where f(0) = 0 \tag{1}$$

After a few operations, which I won't copy out exactly,

$$n \cdot f(n) = 2n + (n+1)f(n-1) \tag{2}$$

I understand this derivation!

(ii)

Suppose $g(n) = \frac{f(n)}{n+1}$, writing out what we have (per hint)

$$\begin{aligned} & \frac{n \cdot f(n)}{n(n+1)} && (3)\\ &= \frac{2n + (n+1)f(n-1)}{n(n+1)} && (4)\\ &= \frac{2n}{n(n+1)} + \frac{(n+1)f(n-1)}{n(n+1)} && (5)\\ &= \frac{2}{n+1} + \frac{f(n-1)}{n} && (6) \end{aligned}$$

Notice how RHS is essentially $g(n-1)$. This means that:

$$\begin{aligned} g(n) &= \frac{2}{n+1} + g(n-1) && (7)\\ &= g(n-1) + \frac{2}{n+1} && (8) \end{aligned}$$

(iii)

Now, solving for the recurrence - assuming $g(0) = 0$

$$\begin{aligned} g(n) &= g(n-1) + \frac{2}{n+1} && (9)\\ &= g(n-2) + \frac{2}{n} + \frac{2}{n+1} && (10)\\ &= g(n-3) + \frac{2}{n-1} + \frac{2}{n} + \frac{2}{n+1} && (11)\\ & ... && (12)\\ &= \frac{2}{1+1} + \frac{2}{2+1} + \frac{2}{3+1} + ... + \frac{2}{n-1} + \frac{2}{n} + \frac{2}{n+1} && (13)\\ &= 2 \cdot \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n-1} + \frac{1}{n} + \frac{1}{n+1}\right] && (14)\\ &= 2 \cdot (H_{n+1} - 1) && (15) \end{aligned}$$

(iv)

Now, to find the closed form:

$$g(n) = 2 \cdot (H_{n+1} - 1) = \frac{f(n)}{n+1} \tag{16}$$
$$\therefore f(n) = (n+1) \cdot 2 \cdot (H_{n+1} - 1) \tag{17}$$
$$= (2n+2)(H_{n+1} - 1) \tag{18}$$

Using the fact that $H_n \leq 1 + ln(n)$ - this means that $H_{n+1} - 1 \leq ln(n+1)$. If we subsitute this approximation into the closed-form, we get something like

$$(2n+2) \cdot ln(n+1) \tag{19}$$
$$= 2n \cdot ln(n+1) + 2 \cdot ln(n+1) \tag{20}$$
$$\approx n \cdot ln(n) + ln(n) \tag{21}$$

If we just consider the dominant terms (so we ignore the constants), we can conclude that $f(n) = O(n \cdot ln(n))$