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Rational Homotopy Groups of Spheres

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Abstract

Homotopy groups of spheres are generally quite difficult to compute, and they can behave quite erratic. We only focus on the non-torsion part of these groups and aim to show that for n even, the homotopy groups of S^n are torsion except for the nth and $2n-1$th group, while for n odd, the homotopy groups of S^n are torsion except for the nth group. The proof relies on finding a rational homotopy equivalence between S^n and $K(\mathbb{Z}, n)$ in the odd case and between S^n and a certain homotopy fibre F in the even case. We establish this link by using the Leray-Serre spectral sequence to calculate the rational cohomology algebra of $K(\mathbb{Z}, n)$ in the odd case and F in the even case.

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Contents

Chapter 1

Introduction

Homotopy groups of spheres are notoriously difficult to compute. This is in stark contrast to the homology groups of spheres that are trivial except for the zeroth dimension and the dimension of the sphere itself. Heinz Hopf provided in [Hop31] the first example of this difference via the construction of a non-trivial mapping $p : S^3 \to S^2$. Indeed, this mapping is a generator of the infinite cyclic group $\pi_3(S^2)$, as shown in Example 2.7. Considering this, one would very much like to be able to compute $\pi_{n+k}(S^n)$ for arbitrary $k, n > 0$. This is no easy task and has fueled the development of powerful tools in algebraic topology.

However, there is a much more definitive answer if one ignores torsion and takes the tensor product with $\mathbb{Q}$. The goal of this paper is to show that for n even, we have

$$\pi_k(S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & \text{if } k = n \text{ or } k = 2n - 1 \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

while for n odd, we have

$$\pi_k(S^n) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q}, & \text{if } k = n \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

The main ideas of the proof are taken from [Ber12]. We will use the rational cohomology algebra of the *Eilenber-Maclane spaces* $K(\mathbb{Z}, n)$ to find *rational homotopy equivalences* between the spheres and some spaces we already know the rational homotopy groups of.

Chapter 2 introduces some of the main concepts, including *Serre fibration* and the aforementioned Eilenberg-MacLane spaces. We also introduce *homotopy fibres* and compute the homotopy groups of some homotopy fibres of Eilenberg-MacLane spaces which will be necessary for the proof when n is even. At the end of Chapter 2, we state equivalent conditions for a rational homotopy equivalence which shall allow us to exploit the algebra structure of the rational cohomology of $K(\mathbb{Z}, n)$. Now, to compute $H^*(K(\mathbb{Z}, n); \mathbb{Q})$, we introduce, in Chapter 3, *spectral sequences* and state the Leray-Serre spectral sequence, which allows us to perform the computations. Then, using some of the properties of $K(\mathbb{Z}, n)$ introduced in Chapter 2, we find, in Chapter 4, the rational homotopy equivalences, which allow us to show the results in Equation (1.1) and Equation (1.2).

Chapter 2

Basic Constructions

2.1 Homotopy groups as loop spaces

One often defines the homotopy group $\pi_n(X, p)$ of a pointed topological space (X, p) to be the homotopy classes of maps $f : (I^n, \partial I^n) \to (X, p)$. However, we shall adopt the approach taken in [BT82] and define them in terms of *loop spaces*.

Definition 2.1 (Path space)**.** Given a based space (X, p), the *path space* $PX = X^I$ is the set of based maps from the unit interval $I = [0, 1]$ to X endowed with the compact open topology such that $f(0) = p$.

There is a canonical map $\chi : PX \to X$ given by $\chi(f) = f(1)$. The loop space of X can then be described as a fibre of this map.

Definition 2.2 (Loop space)**.** For a based space (X, p), we define the loop space of X as the space $\Omega X = \chi^{-1}(p)$ endowed with the subspace topology.

Definition 2.3 (Homotopy groups)**.** The 0th homotopy group, $\pi_0(X, p)$, of a pointed topological space is defined to be its path components. For the inductive step, we set the $n+1$th homotopy group to be $\pi_{n+1}(X, p) = \pi_n(\Omega_p X, \overline{p})$, where $\overline{p}$ is the constant map to the point p.

The operation of the composition of loops turns the homotopy groups into actual groups where the constant map acts as the identity, and the inverse is just the reversal of each loop.

2.2 Serre fibrations

Another crucial component of studying homotopy groups of spheres are long exact sequences of said groups. To this end, we need to understand when these can occur. As we shall see, this is the case when we have a *Serre fibration*. In order to understand the definition of a Serre fibration, we first need some preliminary notions, more specifically, the homotopy lifting property. We shall define this as it appears in [Mit01].

Definition 2.4 (Homotopy lifting property)**.** Given spaces X, E, and B, together with a map $p : E \to B$, we say that p has the homotopy lifting property if

$$\begin{tikzcd} X \arrow[r, "\tilde{F}_0"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "F"'] \arrow[ur, "\tilde{F}"] & B \end{tikzcd}$$

Figure 2.1: The map i_0 is taken to be the identification of X with $X \times \{0\}$ in $X \times I$.

(i) for any homotopy $F : X \times I \to B$ and

(ii) an initial map $\tilde{F}_0 : X \to E$ such that $F_0 = p \circ \tilde{F}_0$

there exists a map $\tilde{F}$ making the diagram in Figure 2.1 commute.

We say that $p : E \to B$ is a Serre fibration if p has the homotopy lifting property for all CW-complexes. One also has the notion of a general *fibration* in which the requirement is that p must have the homotopy extension property for all spaces and not just the CW-complexes.

For our purposes, a fundamental property of fibrations is that they admit a long exact sequence of homotopy groups.

Proposition 2.5. *Given a Serre fibration $p : E \to B$ with basepoint $b_0 \in B$ let $F = p^{-1}(b_0)$ and $x_0 \in F$. There is then a long exact sequence of homotopy groups*

$$\cdots \longrightarrow \pi_n(F, x_0) \longrightarrow \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \longrightarrow \pi_{n-1}(F, x_0) \longrightarrow \cdots \tag{2.1}$$

Proof. See section 4.2 of [Hat02]. ■

The *path space fibration* of X gives a canonical class of fibrations for any space X.

Proposition 2.6 (Path space fibration)**.** *Given any based space (X, x_0), the map $\chi : PX \to X$ is a Serre fibration with fibre ΩX.*

Proof. See [Fel+01]. ■

Example 2.7 (Hopf fibration)**.** An important historical example of a Serre fibration is the Hopf fibration

$$S^1 \longrightarrow S^3 \xrightarrow{p} S^2 \tag{2.2}$$

where $p : S^3 \to S^2$ is defined as $p(z, w) = [z : w]$, thinking of S^3 as the unit sphere in $\mathbb{C}^2$ and S^2 as $\mathbb{C}P^1$. For proof that this is a Serre fibration, the reader is referred to [Hat02]. We know that $\pi_i(S^n) = 0$ for $i < n$, see [Hat02], and so the only interesting part of the long exact sequence associated with the Hopf fibration is

$$0 \longrightarrow \pi_3(S^3) \xrightarrow{p_*} \pi_3(S^2) \longrightarrow 0 \tag{2.3}$$

Thus, $p_* : \pi_3(S^3) \to \pi_3(S^2)$ is an isomorphism. Moreover, by the Hurewicz theorem; $\pi_3(S^3) \cong \mathbb{Z}$ and one of the generators is represented by the identity map $\iota : S^3 \to S^3$. From this, we get that $\pi_3(S^2) \cong \mathbb{Z}$ and that one of the generators are

$$\begin{aligned} p_*([\iota]) &= [p \circ \iota] \\ &= [p], \end{aligned}$$

which means that the Hopf fibration $p : S^3 \to S^2$ represents the generator of $\pi_3(S^2)$. The historical importance of this result is that this showed that homotopy groups behave different compared to homology groups. So much more care is needed when thinking about $\pi_{k+n}(S^n)$ in general. For a more comprehensive discussion of the Hopf fibration, the reader is referred to [Lyo03].

2.3 Homotopy fibres

One of the main ingredients in our proof for the rational homotopy groups of even spheres is the use of *homotopy fibres*.

Definition 2.8 (Mapping fibration)**.** Let $f : Y \to X$ be a map between spaces. We then define the set $P(f)$ to be the fibreed product

$$P(f) = X^I \times_X Y = \{(\alpha, y) \mid \alpha(1) = f(y)\}. \tag{2.4}$$

The map $p : P(f) \to X$, defined by $p(\alpha, y) = \alpha(0)$, then gives the *mapping fibration* of f.

This gives us a way of associating a fibration to any map.

Proposition 2.9. *Given a map $f : Y \to X$, the mapping fibration $p : P(f) \to X$ is a Serre fibration.*

Proof. See [DK12]. ■

Definition 2.10 (Homotopy fibre)**.** Given a map $f : Y \to X$ and a point $x \in X$, we define the homotopy fibre of f over x as the fibre $p^{-1}(x)$ of the mapping fibration.

The fact that the mapping fibration is a Serre fibration lets us view any map $f : Y \to X$ as a fibration up to homotopy in light of the following proposition.

Proposition 2.11. *Let $f : Y \to X$ be any map, then there is a homotopy equivalence $\phi : Y \to P(f)$, which makes the following diagram commute*

$$\begin{tikzcd} Y \arrow[r, "\phi"] \arrow[dr, "f"'] & P(f) \arrow[d, "p"] \\ & X \end{tikzcd} \tag{2.5}$$

Proof. Define $\phi : Y \to P(f)$ by $\phi(y) = (c_{f(y)}, y)$ where $c_{f(y)}$ is the constant path at $f(y)$. Let now $\psi : P(f) \to Y$ be given by $\psi(\alpha, y) = y$. Then $\psi \circ \phi = \mathrm{id}_Y$, and so we only need to show that $\phi \circ \psi \simeq \mathrm{id}_{P(f)}$. Now, if $(\alpha, y) \in P(f)$, then

$(\phi \circ \psi)(\alpha, y) = (c_{f(y)}, y)$, and so all we need do is find a homotopy between the constant path $c_{f(y)}$ and α. This is easy enough; let α_t be the restriction of α to the interval $[t, 1]$ with the necessary re-parameterization. We then define the homotopy $F : P(f) \times [0,1] \to P(f)$ by $F((\alpha, y), t) = (\alpha_t, y)$. Since, by construction, we have $\alpha_1 = c_{f(y)}$ we see that $F((\alpha, y), 0) = (\alpha, y) = \mathrm{id}_{P(f)}(\alpha, y)$ and $F((\alpha, y), 1) = (c_{f(y)}, y) = (\phi \circ \psi)(\alpha, y)$. We thus see that ϕ is a homotopy equivalence, and it is also evident that ϕ makes the diagram commute.

■

Using this fact, we can replace $P(f)$ with Y in the long exact sequence associated with the mapping fibration $p : P(f) \to X$. This, in turn, is going to allow us to compute the homotopy groups of the homotopy fibre of a homotopy fibre which in the case of some *Eilenberg-MacLane spaces* shows that this is a new Eilenberg-MacLane space.

2.4 Eilenberg-MacLane spaces

Eilenberg-MacLane spaces first appeared in [EM45], and their rational cohomology will play a crucial role in the proof of the rational homotopy groups of spheres. So we shall now give an account of the critical features of these spaces, which we need to study the rational homotopy groups of spheres.

Definition 2.12 (Eilenberg-MacLane spaces)**.** For $n \in \mathbb{N}$ and G a group, we say that a space X is an Eilenberg-MacLane space of type $K(G, n)$ if its nth homotopy group is equal to G and otherwise trivial.

From this definition and Definition 2.3, it is clear that

$$\Omega K(G, n) \simeq \Omega K(G, n-1)$$

for $n > 1$. Noting the fact that $PK(G, n) \simeq *$ one gets a useful fibration by applying the path space fibration, which in this case becomes

$$K(G, n-1) \longrightarrow * \longrightarrow K(G, n) \tag{2.6}$$

Applying the *Serre spectral sequence* to this fibration shall allow us to compute the rational cohomology of $K(\mathbb{Z}, n)$.

Another crucial aspect of Eilenberg-MacLane spaces for our purposes is its connection to cohomology.

Proposition 2.13. *For an abelian group G and $n \in \mathbb{N}$ there is a natural bijection for all CW-complexes X between the space $[X, K(G, n)]$, the space of homotopy classes of maps from X to $K(G, n)$, and $H^n(X; G)$, the nth cohomology group of X with coefficients in G. Endowing $[X, K(G, n)]$ with a canonical group structure this turns into a natural isomorphism.*

Proof. See Section 4.2 in [Hat02]. ■

To conclude this section, let $f : K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n)$ be any map. If we choose some basepoint $x_0 \in K(\mathbb{Z}, 2n)$, we let A denote the homotopy fibre of f over x_0 with induced map $\alpha : A \to K(\mathbb{Z}, n)$. We then choose a basepoint $a_0 \in A$ and let B be the homotopy fibre of α over a_0 with induced map $\beta : B \to A$.

Proposition 2.14. *The homotopy groups $\pi_k(A)$ are infinite cyclic for $k = n, 2n-1$, and otherwise trivial. On the other hand, B is an Eilenberg-MacLane space of type $K(\mathbb{Z}, 2n-1)$.*

Proof. We first compute $\pi_k(A)$ using the long exact sequence of the fibration

$$A \xrightarrow{\quad\alpha\quad} K(\mathbb{Z}, n) \xrightarrow{\quad f\quad} K(\mathbb{Z}, 2n) \tag{2.7}$$

Looking at a section of the exact sequence, we have

$$\cdots \longrightarrow \pi_k(A) \xrightarrow{\alpha_*} \pi_k(K(\mathbb{Z}, n)) \xrightarrow{f_*} \pi_k(K(\mathbb{Z}, 2n)) \longrightarrow \cdots \tag{2.8}$$

From this we see that when $k = n$ we have $\pi_{n+1}(K(\mathbb{Z}, 2n)) = \pi_n(K(\mathbb{Z}, 2n)) = 0$ and so α_* is an isomorphism for $k = n$ and hence $\pi_n(A) = \mathbb{Z}$. A similar computation shows that $\pi_{2n-1}(A) = \mathbb{Z}$. Now, if $k \neq n, 2n-1$ then $\pi_{k+1}(K(\mathbb{Z}, 2n)) = \pi_k(K(\mathbb{Z}, n)) = 0$ and so, necessarily, $\pi_k(A) = 0$ showing the result for A.

Using the homotopy groups of A we then take a look at the fibration given by

$$B \xrightarrow{\quad\beta\quad} A \xrightarrow{\quad\alpha\quad} K(\mathbb{Z}, n) \tag{2.9}$$

Again, taking a section of the long exact sequence, we get

$$\cdots \longrightarrow \pi_k(B) \xrightarrow{\beta_*} \pi_k(A) \xrightarrow{\alpha_*} \pi_k(K(\mathbb{Z}, n)) \longrightarrow \cdots \tag{2.10}$$

For $k = 2n-1$ we have that β_* is an isomorphism so that $\pi_{2n-1}(B) \cong \mathbb{Z}$. The only non-obvious part of the sequence is

$$0 \longrightarrow \pi_n(B) \xrightarrow{\beta_*} \mathbb{Z} \xrightarrow{\alpha_*} \mathbb{Z} \xrightarrow{\partial_*} \pi_{n-1}(B) \longrightarrow 0 \tag{2.11}$$

By exactness we have that

$$\pi_{n-1}(B) \cong \frac{\mathbb{Z}}{\operatorname{im}(\alpha_*)}. \tag{2.12}$$

Furthermore,

$$\operatorname{im}(\alpha_*) \cong \frac{\mathbb{Z}}{\operatorname{im}(\beta_*)} \tag{2.13}$$

and so the only way for the quotient in 2.12 to make sense is if $\operatorname{im}(\beta_*) = 0$, but then $\pi_n(B) = 0$ and $\pi_{n-1}(B) \cong \mathbb{Z}/\mathbb{Z} \cong 0$, showing that the only non trivial homotopy group is $\pi_{2n-1}(B) = \mathbb{Z}$. We, therefore, must have that B is an Eilenberg-MacLane space of type $K(\mathbb{Z}, 2n-1)$ which was what we wanted to show and so we are done. ■

2.5 Rational homotopy equivalence

As mentioned in the introduction, one of the main ideas for the proof of Equation (1.1) and Equation (1.2) is to construct a rational homotopy equivalence between the spheres and some other space. We want to use the algebra structure in cohomology, so the following results will be essential to us.

Definition 2.15 (Rational homotopy equivalence)**.** Let $f : X \to Y$ be a map. We say that f is a *rational homotopy equivalence* if the induced map on rationalized homotopy groups, $\pi_*(f) \otimes \mathbb{Q} : \pi_*(X) \otimes \mathbb{Q} \to \pi_*(Y) \otimes \mathbb{Q}$, is an isomorphism.

This weakens the criterion of weak homotopy equivalence, which requires that $\pi_*(f) : \pi_*(X) \to \pi_*(Y)$ be an isomorphism. Thus, every homotopic and every weakly homotopic space is rationally homotopic. It turns out, due to a result of Serre, that the above definition is equivalent to requiring that the induced map in homology with rational coefficients is an isomorphism.

Proposition 2.16. *Let $f : X \to Y$ be a map between spaces. Then, f is a rational homotopy equivalence if and only if the induced map $f_* : H_*(X; \mathbb{Q}) \to H_*(Y; \mathbb{Q})$ is an isomorphism.*

Proof. See [Jea53]. ■

We also have a similar result for cohomology, which requires the notion of a *uniquely divisible group.*

Definition 2.17 (Uniquely divisible group)**.** An abelian group G is said to be *divisible* if for all $g \in G$, and all positive integers n, there exists $y \in G$ such that $ny = g$. The group is uniquely divisible if y is unique.

Proposition 2.18. *For any map $f : X \to Y$, we have that it is a rational homotopy equivalence if and only if for any uniquely divisible abelian group G the induced map $f^* : H^*(Y; G) \to H^*(X; G)$ is an isomorphism.*

Proof. See [Ber12]. ■

Chapter 3

Spectral sequences and the rational cohomology algebra of Eilenberg-MacLane spaces

This chapter introduces *spectral sequences* and uses the cohomological version of the *Serre spectral sequence* to compute the cohomology algebra of certain Eilenberg-MacLane spaces, more specifically, the cohomology algebra of $K(\mathbb{Z}, n)$ for $n \in \mathbb{N}$. This will then provide the final ingredient in our proof of the rational homotopy groups of spheres.

Our account of spectral sequences will closely follow [McC00] and [Ram17], which the reader should consult for a more thorough introduction.

3.1 Brief introduction to spectral sequences

Intuitively a spectral sequence is a collection of 'pages', which are two-dimensional lattices of abelian groups, together with differential maps between these such that each page is the (co)homology of the previous one. The abelian group structure of each of these pages can be captured by the following notion.

Definition 3.1 (Bigraded abelian group)**.** A *bigraded abelian group* $E_r = E_r^{*,*}$ is a doubly indexed direct sum of abelian groups,

$$E_r^{*,*} = \bigoplus_{p,q \in \mathbb{Z}} E_r^{p,q} \tag{3.1}$$

where the ordered pair (p, q) is called the *bidegree* of the abelian group $E_r^{p,q}$.

We shall only be interested in the first quadrant, i.e., the bidegrees (p, q) where $p, q \geq 0$, and assume that the groups $E_r^{p,q}$ are zero otherwise. We will refer to such a group as a *first quadrant bigraded abelian group.* There is also a notion of maps between bigraded abelian groups.

Definition 3.2. A morphism $d : A \to B$ between two bigraded abelian groups A and B in bidegree (m, n) is a collection of group homomorphisms

$$\{d^{p,q} : A^{p,q} \to B^{p+m,q+n}\}_{p,q\in\mathbb{Z}}$$

.

We can now give a more formal description of the cohomological version of a spectral sequence.

Definition 3.3 (Spectral sequence)**.** A spectral sequence E is a sequence of bigraded abelian groups E_r and endomorphisms $d_r : E_r \to E_r$ in bidegree $(r, r-1)$ for $r \geq 1$. Furthermore, each d_r is a differential in the sense that $d_r \circ d_r = 0$ and, letting

$$H^{p,q}(E_r) := \frac{\ker(d_r^{p,q})}{\mathrm{im}(d_r^{p-r,q+r-1})}, \tag{3.2}$$

we require that $E_{r+1}^{*,*} \cong H^{*,*}(E_r)$. In other words, E_{r+1} is the cohomology of E_r.

For a sequence of first quadrant bigraded abelian groups, there is also the notion of a *first quadrandt spectral sequence* which is just an ordinary spectral sequence E which has trivial components $E_r^{p,q} = 0$ for all $p, q < 0$ and all $r \geq 0$.

If there exists some $n \geq 1$ such that $d^r = 0$ for all $r \geq n$ we say that the spectral sequence collapses at the n–th term. In this case we get that $E_{r+1} \cong E_r$ for all $r \geq n$, and so one has some semblance of convergence of a spectral sequence. To make this notion more precise, we need to introduce the concept of a *filtration*.

Definition 3.4 (Filtration of an abelian group)**.** Given an abelian group G a filtration F^*G is a sequence of subgroups $F^iG \subset G$ indexed over the natural numbers such that $F^j \subset F^i$ for all $j \geq i$ and $F^0G = G$. When G is graded we also require that F^iG be graded for all i.

Analogous to ordinary analysis there is also a concept of a *limiting term* E_∞ of a spectral sequence. This comes about by first noticing that for any spectral sequence E we have a sequence of sub-objects

$$0 = B_0 \subset B_1 \subset \cdots \subset B_r \subset \cdots \subset Z_r \subset \cdots \subset Z_1 \subset Z_0 = E_1$$

where B_r and Z_r are defined inductively such that $Z_r/B_{r-1} = \ker(d_r)$ and $B_r/B_{r-1} = \mathrm{im}(d_r)$ with $Z_0 := E_1$ and $B_0 := 0$. We then set $Z_\infty = \cap Z_r$ and $B_\infty = \cup B_r$.

Definition 3.5 (Limiting term)**.** Given a spectral sequence E the limiting term of E, denoted E_∞, is given by

$$E_\infty = \frac{Z_\infty}{B_\infty}. \tag{3.3}$$

Definition 3.6 (Convergence of spectral sequence)**.** Given a spectral sequence E and a graded abelian group A^* with a filtration F^*A^* we say that the spectral sequence E converges to A^* if for all bidegrees (p, q) there is an isomorphism $E_\infty^{p,q} \cong F^pA^{p+q}/F^{p+1}A^{p+q}$. We also write this as

$$E_r^{p,q} \Longrightarrow A^{p+q}. \tag{3.4}$$

If a spectral sequence E converges to some graded abelian group A^* then there is the question of how to deal with the extension problems to determine what A^{p+q} is. However, we are only going to deal with vector spaces over $\mathbb{Q}$ and so the short exact sequence

$$0 \longrightarrow F_{i-1}A^* \longrightarrow F_iA^* \longrightarrow F_iA^*/F_{i-1}A^* \longrightarrow 0 \tag{3.5}$$

always splits. That is to say, $A^n \cong \bigoplus_{p+q=n} E_\infty^{p,q}$.

3.2 Spectral sequence of algebras

Since we are not only interested in the cohomology of Eilenberg-MacLane spaces as vector spaces but also as algebras over $\mathbb{Q}$, it would be nice if the spectral sequences could capture this.

Definition 3.7 (Spectral sequence of algebras)**.** A *spectral sequence of algebras* is a spectral sequence E such that each page (E_r, d_r) is a bigraded differential algebra. That is to say, for all $r \geq 1$ we have that d_r is an antiderivation, and there is a product $E_r \otimes E_r \to E_r$ such that the product of E_{r+1} coincides with the one induced by taking the cohomology of E_r.

With the additional structure of a product operation, we also need to impose an extra condition on what it means for a spectral sequence of algebras to converge compared to the ordinary convergence of spectral sequences. This is because if a spectral sequence of algebras E converges to some filtered algebra A^* then we inherently have two products, the one induced from $E_1, E_2 \cdots$ on E_∞ as well as the one induced from the filtration of A^*.

Definition 3.8 (Filtration of graded differential algebra)**.** Given a graded differential algebra A^* a filtration, F^*A^*, is an ordinary filtration of A^* as a graded abelian group such that $F^mA^* \cdot F^nA^* \subset F^{m+n}A^*$.

There is also a canonical induced product for $[a] \in F^pA^{p+q}/F^{p+1}A^{p+q}$ and $[b] \in F^{p'}A^{p'+q'}/F^{p'+1}A^{p'+q'}$ defined by

$$[a] \cdot [b] = [a \cdot b] \in F^{p+p'}A^{p+q+p'+q'}/F^{p+p'+1}A^{p+q+p'+q'} \tag{3.6}$$

which is well defined because of the compatibility of the filtration with the product.

Definition 3.9 (Convergence)**.** A spectral sequence of algebras, E, is said to converge to the graded differential algebra A^*, with a filtration F^*A^*, if it converges to A^* as a spectral sequence and the product induced on the E_∞ page coincides with the one in Equation (3.6).

3.3 The Leray-Serre spectral sequence

The *cohomological Leray-Serre spectral sequence* is the primary computational tool needed to compute the cohomology algebra of $K(\mathbb{Z}, n)$ and was discovered by Jean-Pierre Serre in his doctoral thesis [Ser51]. To fully state it, we briefly need to discuss a *system of local coefficients* on homology, which can be interpreted as a covariant functor.

Definition 3.10 (System of local coefficients)**.** Given a space X and a commutative ring R a system of local coefficients is a covariant functor

$$\mathcal{L}(X;R) : \Pi_1(X) \to \mathbf{Mod}_R$$

from the fundamental groupoid of X to the category of modules over R.

Given any space X with a system of local coefficients $\mathcal{L}(X;R)$, for some commutative ring R, there is then a modified version of the ordinary singular complex, which is defined by

$$C_n(X, \mathcal{L}(X;R)) = \bigoplus_{\sigma \in C_n(X)} \mathcal{L}(X;R)(\sigma(1,0,\dots,0)). \tag{3.7}$$

This also induces a cohomology theory with local coefficients obtained by dualizing the homology with local coefficient.

We are now able to state the cohomological Leray-Serre spectral sequence as it appears in [McC00] .

Theorem 3.11 (Cohomological Leray-Serre spectral sequence)**.** *Let R be a commutative ring with unit and $F \to E \to B$ a Serre fibration with B path-connected. There is then a first quadrant cohomological spectral sequence of algebras E converging to $H^*(E;R)$ as an algebra with $E_2^{p,q} \cong H^p(B, \mathcal{H}^q(F;R))$ where we consider $\mathcal{H}^q(F;R)$ as a system of local coefficients in the fibre F. Moreover, this spectral sequence is natural with respect to fibre-preserving maps of Serre fibrations. Finally, the cup product $\smile$ on $H^p(B, \mathcal{H}^q(F;R))$ is related to the product $\cdot_2 : E_2 \otimes E_2 \to E_2$ by $u \cdot_2 v = (-1)^{p'q} u \smile v$ for $u \in E_2^{p,q}$ and $v \in E_2^{p',q'}$.*

Proof. See [McC00]. ■

Since $\pi_1(B)$ induces an action on $H^*(F)$ in a fibration, the following proposition will be helpful when the action is trivial.

Proposition 3.12. *Let $F \to E \to B$ be a Serre fibration and R a commutative ring. Suppose $\pi_1(B)$ acts trivially on $H^*(F)$, then in the Leray-Serre spectral sequence, we have that*

$$E_2^{p,q} \cong H^p(B; H^q(F;R)). \tag{3.8}$$

Moreover, if $R = k$ is a field, then

$$E_2^{p,q} \cong H^p(B;k) \otimes_k H^q(F;k). \tag{3.9}$$

Proof. See [Hat04]. ■

3.4 The rational cohomology algebra of Eilenberg-MacLane spaces

Definition 3.13 (Exterior algebra on a single generator)**.** Let k be a field. The *exterior algebra* on a single generator x over k is defined as

$$\Lambda_k(x) = \frac{k[x]}{(x^2)}. \tag{3.10}$$

Theorem 3.14. *The rational cohomology algebra of $K(\mathbb{Z}, n)$ is isomorphic to an exterior algebra over $\mathbb{Q}$ on a single generator in degree n if n is odd, and a free graded commutative algebra over $\mathbb{Q}$ on a single generator if n is even.*

Proof. We proceed by induction. For $n = 1$ it is undoubtedly true that the rational cohomology of $K(\mathbb{Z}, 1) \simeq S^1$ is an exterior algebra on a generator in degree one. Therefore, we assume that the induction hypothesis holds for $n-1$ when $n \geq 1$ and want to show that this implies the statement for n. Assume first that n is even so that $n-1$ is odd. We now take the path-space fibration of $K(\mathbb{Z}, n)$ and get

$$K(\mathbb{Z}, n-1) \longrightarrow * \longrightarrow K(\mathbb{Z}, n) \tag{3.11}$$

Now, since $\pi_1(K(\mathbb{Z}, n)) = 0$, we get from Proposition 3.12 that the E_2–page of the Serre spectral sequence takes the form

$$E_2^{p,q} \cong H^p(K(\mathbb{Z}, n); \mathbb{Q}) \otimes H^q(K(\mathbb{Z}, n-1); \mathbb{Q}). \tag{3.12}$$

By assumption we have that $H^*(K(\mathbb{Z}, n-1); \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(y)$, where $|y| = n-1$, and so we have

$$H^q(K(\mathbb{Z}, n-1); \mathbb{Q}) = \begin{cases} \mathbb{Q}, & \text{if } q = 0, n-1 \\ 0, & \text{else.} \end{cases} \tag{3.13}$$

Since $*$ has trivial cohomology, the only component that survives is $E_2^{0,0}$ while all the other on the E_2–page must die. This implies that $E_n^{0,n-1} \cong E_2^{0,n-1} \cong \mathbb{Q}_y$ cannot survive to the next page and so the differential $d_n^{0,n-1} : E_2^{0,n-1} \to E_2^{n,0}$ must be injective. This ensures that $E_n^{n,0} \cong E_2^{n,0}$ contains at least a $\mathbb{Q}$. Furthermore, since $d_n^{n,0} = 0$ it also cannot contain more than $\mathbb{Q}$, and so $E_n^{n,0} \cong E_2^{n,0} \cong \mathbb{Q}$. We then have that $d_n^{0,n-1}$ is an isomorphism and so we set $x = d_n^{0,n-1}(y)$ to be the generator of $E_2^{n,0}$. From this we immediately have that

$$\begin{aligned} E_n^{n,n-1} &\cong E_2^{n,n-1} \\ &\cong \mathbb{Q}_x \otimes \mathbb{Q}_y \\ &= \mathbb{Q}_{xy}. \end{aligned}$$

Moreover, by similar reasoning as before, we must have that $d_n^{n,n-1}$ is an isomorphism. Indeed, we have that $E_2^{0,kn} \cong \mathbb{Q}$ for all $k \geq 0$. Furthermore, by the graded differential structure, we see that inductively

$$\begin{aligned} d_n^{kn,n-1}(x^k y) &= d_n^{kn,0}(x^k) y + (-1)^{|x^k|} x^k d_n^{0,n-1}(y) \\ &= 0 + x^k \cdot x \\ &= x^{k+1}. \end{aligned}$$

Thus the generator of $E_2^{kn,0}$ is x^k and so we finally get that

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong E_2^{*,0} \cong \mathbb{Q}[x] \tag{3.14}$$

which concludes the proof for even n.

Assume now that n is odd. We then have that $H^*(K(\mathbb{Z}, n-1)) \cong \mathbb{Q}[y]$ with $|y| = n-1$. Applying the same path-space fibration as before we have that $E_2^{p,q} \cong H^p(K(\mathbb{Z},n);\mathbb{Q}) \otimes H^q(K(\mathbb{Z},n-1);\mathbb{Q})$. By similar reasoning as in the even case we must have that $d_n^{0,n-1}$ is an isomorphism. We therefore set $x = d_n^{0,n-1}(y)$ to be the generator of $E_n^{n,0} \cong \mathbb{Q}$. Consequently, we have that xy^k is a generator of $E_2^{n,k(n-1)} \cong E_n^{n,k(n-1)} \cong \mathbb{Q}_{xy^k}$ for all $k \geq 0$. The claim now is that $E_n^{q,0} = 0$ for all $q > n$. To see this, note first that we inductively have

$$\begin{aligned}
d_n^{0,k(n-1)}(y^k) &= d_n^{0,n-1}(y)y^{k-1} + (-1)^{|y|} y d_n^{0,(k-1)(n-1)}(y^{k-1}) \\
&= xy^{k-1} + y(k-1)xy^{k-2} \\
&= kxy^{k-1}
\end{aligned}$$

which means that xy^{k-1} is killed by $\frac{1}{k}y^k$. However, this means that $d_n^{0,k(n-1)}$ is an isomorphism for all $k \geq 1$, and so we must have that $E_n^{2n,*} = 0$ for degree reasons. We then inductively have that $E_n^{q,*} = 0$ for $q > n$, which implies that

$$\begin{aligned}
H^*(K(\mathbb{Z},n);\mathbb{Q}) &\cong E_2^{*,0} \\
&= E_n^{*,0} \\
&\cong \mathbb{Q}[x]/(x^2) \\
&= \Lambda_{\mathbb{Q}}(x).
\end{aligned}$$

Hence we see that the rational cohomology algebra of $K(\mathbb{Z},n)$, for odd n, is an exterior algebra on one generator in degree n and so we are done. $\blacksquare$

Chapter 4

Proof of rational homotopy groups of spheres

Having computed the rational cohomology algebra of $K(\mathbb{Z}, n)$, we are now in a position to compute the rational homotopy groups of spheres. We shall concentrate on the even and odd-dimensional cases separately, and the main ideas of the proof are taken from [Ber12]. The objective in the odd case is to show that there is a rational homotopy equivalence between $K(\mathbb{Z}, n)$ and S^n, and a rational homotopy equivalence between a certain homotopy fibre F and S^n in the even case.

4.1 Odd-dimensional spheres

Theorem 4.1 (Rational homotopy groups of odd spheres)**.** *Let $n \in \mathbb{N}$ be odd, then*

$$\pi_k(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}, & \textit{if } k = n \\ 0, & \textit{otherwise.} \end{cases} \tag{4.1}$$

Proof. If we can find an isomorphism $f^* : H^*(K(\mathbb{Z}, n); \mathbb{Q}) \to H^*(S^n; \mathbb{Q})$ then by Proposition 2.18 we have that $f_* : \pi_*(S^n) \otimes \mathbb{Q} \to \pi_*(K(\mathbb{Z}, n)) \otimes \mathbb{Q}$ is an isomorphism and we will be done. We therefore choose $f : S^n \to K(\mathbb{Z}, n)$ to represent the generator of $\pi_n(K(\mathbb{Z}, n))$. The induced map $f_* : \pi_n(S^n) \to \pi_n(K(\mathbb{Z}, n))$ must then be an isomorphism. From the naturality of the Hurewicz homomorphism and Corollary 1.2.1. in [Moe15] we have a commutative diagram:

$$
\begin{tikzcd}
\pi_n(S^n)\otimes\mathbb{Q} \arrow[r, "f_*\otimes\mathrm{id}"] \arrow[d, "\cong"'] & \pi_n(K(\mathbb{Z},n))\otimes\mathbb{Q} \arrow[d, "\cong"] \\
H_n(S^n,\mathbb{Q}) \arrow[r, "f_*"] \arrow[d, "\cong"'] & H_n(K(\mathbb{Z},n),\mathbb{Q}) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathbb{Q}}(H_n(S^n,\mathbb{Q}),\mathbb{Q}) \arrow[d, "\cong"'] & \mathrm{Hom}_{\mathbb{Q}}(H_n(K(\mathbb{Z},n),\mathbb{Q}),\mathbb{Q}) \arrow[l, "f^*"] \arrow[d, "\cong"] \\
H^n(S^n;\mathbb{Q}) & H^n(K(\mathbb{Z},n);\mathbb{Q}) \arrow[l, "f^*"]
\end{tikzcd}
\tag{4.2}
$$

Since all the vertical maps and the top map are isomorphisms, it follows that the horizontal maps are isomorphisms. Hence, it follows that $f^* : H^n(K(\mathbb{Z},n);\mathbb{Q}) \to H^n(S^n;\mathbb{Q})$ is an isomorphism. Then, since both $H^n(K(\mathbb{Z},n);\mathbb{Q})$ and $H^n(S^n,\mathbb{Q})$ are exterior algebras on a single generator in degree n, we must have that f^* extends to an isomorphism in all degrees so that $f^* : H^*(K(\mathbb{Z},n);\mathbb{Q}) \to H^*(S^n;\mathbb{Q})$ is an isomorphism. Then, as remarked earlier, Proposition 2.18 implies that $f_* : \pi_*(S^n)\otimes\mathbb{Q} \to \pi_*(K(\mathbb{Z},n))\otimes\mathbb{Q}$ is an isomorphism and hence we are done. ■

4.2 Even-dimensional spheres

The rational cohomology algebra of $K(\mathbb{Z},n)$ for even n is, by Theorem 3.14, isomorphic to $\mathbb{Q}[x]$. There is therefore no map $f : S^n \to K(\mathbb{Z},n)$ which induces an isomorphism in cohomology and we cannot use the same strategy as in the odd dimensional case and we must take another approach.

Let $g : K(\mathbb{Z},n) \to K(\mathbb{Z},2n)$ represents the class $x^2 \in H^{2n}(K(\mathbb{Z},n)) \cong [K(\mathbb{Z},n), K(\mathbb{Z},2n)]$ and F the homotopy fibre of g.

Lemma 4.2. *The rational cohomology algebra of F is isomorphic to $\Lambda_{\mathbb{Q}}(x)$ with $|x| = n$.*

Proof. From Proposition 2.14 we know that the homotopy fibre of F with its projection map to $K(\mathbb{Z},n)$ is homotopically equivalent to $K(\mathbb{Z},2n-1) \simeq \Omega K(\mathbb{Z},2n)$. If we therefore also consider the path space fibration of $K(\mathbb{Z},2n)$ we have the following diagram:

$$
\begin{tikzcd}
\Omega K(\mathbb{Z},2n) \arrow[r] & PK(\mathbb{Z},2n) \arrow[r] & K(\mathbb{Z},2n) \\
\Omega K(\mathbb{Z},2n) \arrow[r] \arrow[u, "\mathrm{id}"] & F \arrow[r] \arrow[u] & K(\mathbb{Z},n) \arrow[u, "g"']
\end{tikzcd}
\tag{4.3}
$$

where the right square commutes and the left square commutes up to homotopy. Letting A denote the spectral sequence induced by the fibration in the top row and B the spectral sequence in the bottom row, we also have an induced map $\alpha : A \to B$. From Theorem 3.11 we know that

$$A_2^{p,q} = H^p(K(\mathbb{Z},2n-1);\mathbb{Q}) \otimes H^q(K(\mathbb{Z},2n);\mathbb{Q}) \Longrightarrow H^{p+q}(PK(\mathbb{Z},2n);\mathbb{Q}) = 0. \tag{4.4}$$

and so, for degree reasons, only $A_2^{0,0} = A_{2n}^{0,0}$ survives on to the A_{2n+1}–page. This means that all the differentials

$$d_{2n}^{2k,2n-1} : A_{2n}^{2k,2n-1} \to A_{2n}^{2k+2n,0}$$

for $k \in \mathbb{N}$ are isomorphisms. Of particular interest will be the differential

$$d_{2n}^{0,2n-1} : A_{2n}^{0,2n-1} \to A_{2n}^{2n,0}$$

since, by the naturality of the Serre spectral sequence, and the fact that $A_2 = A_{2n}$, $B_2 = B_{2n}$, we have a commutative diagram

$$\begin{tikzcd} B_{2n}^{0,2n-1} \arrow[r, "d_{2n}^{0,2n-1}"] & B_{2n}^{2n,0} \\ A_{2n}^{0,2n-1} \arrow[u, "\mathrm{id}"] \arrow[r, "d_{2n}^{0,2n-1}"] & A_{2n}^{2n,0} \arrow[u, "g^*"'] \end{tikzcd} \tag{4.5}$$

Now, the bottom map is an isomorphism. So if we can show that both of the vertical maps are isomorphisms then the top map must also be an isomorphism by the commutativity of the diagram. Since id is obviously an isomorphism, we need only show that g^* is an isomorphism. Remember that the map $g^* : H^*(K(\mathbb{Z}, 2n)) \to H^*(K(\mathbb{Z}, n))$ is induced from the class

$$x^2 \in B_{2n}^{2n,0} \cong H^{2n}(K(\mathbb{Z}, n)) \cong [K(\mathbb{Z}, n), K(\mathbb{Z}, 2n)]$$

and so if

$$y \in A_{2n}^{2n,0} \cong H^{2n}(K(\mathbb{Z}, 2n)) \cong [K(\mathbb{Z}, 2n), K(\mathbb{Z}, 2n)]$$

is the class of the identity map $\iota_{2n} : K(\mathbb{Z}, 2n) \to K(\mathbb{Z}, 2n)$, then

$$g^*(y) = [\iota_{2n} \circ g] = [g] = x^2. \tag{4.6}$$

Hence the generator of $A_{2n}^{2n,0}$ is taken to the generator of $B_{2n}^{2n,0}$, so the rightmost map must be an isomorphism. Thus, since both vertical maps are isomorphisms as well as the bottom map, we must have the top map is also an isomorphism. Then, by the graded differential structure of B_{2n} we have that all maps

$$d_{2n}^{2k,2n-1} : B_{2n}^{2k,2n-1} \to B_{2n}^{2k+2n,0}$$

are isomorphisms. Thus the only terms that survive to the B_∞–page are $B_{2n}^{0,0}$ and $B_{2n}^{n,0}$, showing that $B_\infty \cong H^*(F; \mathbb{Q})$ is an exterior algebra on a single generator in degree n which was what we wanted to show. ■

Theorem 4.3 (Rational homotopy groups of even spheres)**.** *Let $n \in \mathbb{N}$ be even, then*

$$\pi_k(S^n) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}, & \textit{if } k = n, 2n-1 \\ 0, & \textit{otherwise.} \end{cases} \tag{4.7}$$

Proof. Let $f : S^n \to K(\mathbb{Z}, n)$ denote the same generator as before and $g : K(\mathbb{Z}, n) \to K(\mathbb{Z}, 2n)$ the class $x^2 \in H^{2n}(K(\mathbb{Z}, n))$ with F its homotopy fibre.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	$\mathbb{Z}_{15}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{30}$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_{60}$	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	$\mathbb{Z}_2^3$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{120}$	$\mathbb{Z}_2^3$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{120}$

Table 4.1: Low dimensional homotopy groups of spheres. Adapted from https://en.wikipedia.org/wiki/Homotopy_groups_of_spheres.

Since $\pi_n(K(\mathbb{Z}, 2n)) = 0$ we have that f lifts to $h : S^n \to F$ making the following diagram commute:

$$\begin{tikzcd} F \arrow[r, "p"] & K(\mathbb{Z}, n) \arrow[r, "g"] & K(\mathbb{Z}, 2n) \\ & S^n \arrow[ul, dashed, "h"] \arrow[u, "f"] \arrow[ur, "g \circ f \simeq *"'] & \end{tikzcd} \tag{4.8}$$

We know that $H^*(F; \mathbb{Q}) \cong \Lambda_{\mathbb{Q}}(x)$ with $|x| = n$ and so if we can show that

$$h^* : H^*(F; \mathbb{Q}) \to H^*(S^n; \mathbb{Q})$$

is an isomorphisms then we will be done after applying Proposition 2.14 together with Proposition 2.18. Since f represents a generator we have that

$$f_* : \pi_n(S^n) \to \pi_n(K(\mathbb{Z}, n))$$

must be an isomorphism. Then, since $\pi_n(F) \cong \mathbb{Z}$, we have that

$$h_* : \pi_n(S^n) \to \pi_n(F)$$

is also an isomorphism. By replacing $K(\mathbb{Z}, n)$ with F in (4.2) we get that $h^* : H^n(F; \mathbb{Q}) \to H^n(S^n; \mathbb{Q})$ is an isomorphism. This means that the generator of $H^*(F; \mathbb{Q})$ is taken to the generator of $H^*(S^n; \mathbb{Q})$ which means

$$h^* : H^n(F; \mathbb{Q}) \to H^n(S^n; \mathbb{Q})$$

extends to an isomorphism $h^* : H^*(F; \mathbb{Q}) \to H^*(S^n; \mathbb{Q})$ and we have found our desired rational homotopy equivalence concluding the proof.

■

In Table 4.1 we can see some of the low dimensional homotopy groups of spheres. According to what we have just showed, we remark that the only non-torsion groups are exactly where we expect them to be in the table. Computing all of these groups goes well beyond the scope of this paper, but it is worth mentioning that the approach we have taken here with spectral sequences can be used to compute some of these groups. For example, the group $\pi_4(S^3) = \mathbb{Z}_2$ can be computed using the fibration $f : S^3 \to K(\mathbb{Z}, 3)$, with f the generator as before. Spectral sequences also appear once one wishes to compute *stable homotopy groups of spheres*, most notably through the Adams spectral sequence and the Adams-Novikov spectral sequence. Indeed, in studying homotopy groups of spheres, spectral sequences provide an invaluable tool. We should also mention that there are other approaches to computing the rational homotopy groups of spheres which don not use spectral sequences, see for example [KK04], and which develop other machinery for the proof.

A natural next step in studying homotopy groups of spheres might be to try to understand the torsion part of the low dimensional groups in Table 4.1. As mentioned above, we can use spectral sequences for some of them, but other groups have to be computed by other methods, and the interested reader is referred to Chapter 4 of [Hat02] for a treatment of some of these.

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