UiO : Department of Mathematics
University of Oslo

Mika Bohinen

Homotopy Groups of Spheres

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Supervisor: John Christian Ottem

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Abstract

Brief summary of the paper.

1 Introduction

Purpose of the paper, historical context, necessary background information and notation.

2 Basic Constructions

In this chapter our main goal will to be develop all the tools we need for what is to come. We start out by defining homotopy groups inductively as loop spaces before moving on to some other stuff.

2.1 Homotopy groups as loop spaces

One normally defines the homotopy group $\pi_n(X,p)$ of a pointed topological space (X,p) to be the homotopy classes of maps $f:(I^n,\partial I^n)\to(X,p)$. We shall take a slightly different approach in our definition, but one could just as well have used this definition and none of the results would change.

To start off we need the concept of a *loop space* as it appears in [BT82] .

Definition 2.1 (Loop space)**.** For a topological space X and a point $p\in X$ we define the loop space of X to be the space $\Omega_p X$ which consists of continuous maps f from the unit circle S^1(realized as the set $\{z\in\mathbb{C}:|z|=1\}$) into X such that $f(1)=p$. In order for $\Omega_p X$ to be a topological space we equip it with the compact open topology.

With the definition of a loop space we can then use this to inductively define $\pi_n(X,p)$ for arbitrary n.

Definition 2.2 (Homotopy groups)**.** The 0th homotopy group $\pi_0(X,p)$ of a pointed topological space is defined to be its path components. For the inductive step we set the $n+1$th homotopy group to be $\pi_{n+1}(X,p)=\pi_n(\Omega_p X,\overline{p})$ where $\overline{p}$ is the constant map to the point p.

With the operation of composition of loops this turns the homotopy groups into actual groups where the constant map acts as the identity, and the inverse is just the reversal of each loop.

2.2 Suspension and the smash product

With a firm grasp on the definition of homotopy groups we can begin to introduce some of the main tools and concept we shall need in studying them. Two of the most basic of these is the *suspension* of a space as well as the *smash product* of two spaces. We shall define both of these as they appear in chapter 0 of [Hat02].

Definition 2.3 (Suspension)**.** Given a space X the suspension of it, denoted SX, is the quotient of $X \times I$ where one collapse both $X \times \{0\}$ and $X \times \{1\}$ into one point each.

The prototypical example of a suspension is pictured in Figure **??** where we have taken the suspension of the unit circle S^1 and see how it is homeomorphic to S^2. One can also take the suspension of a map $f : X \to Y$ to be $Sf : SX \to SY$ defined as the quotient of $f \times \mathrm{id} : X \times I \to Y \times I$. It is also quite straightforward to check that S respects composition and we thus have that S is actually a covariant functor.

With the suspension out of the way we can now define the smash product of two spaces.

Definition 2.4 (Smash product)**.** Given two pointed spaces (X, x_0) and (Y, y_0) we define the smash product of them, written $X \wedge Y$, to be the quotient $X \times Y / X \vee Y$ where the wedge $X \vee Y \subset X \times Y$ is understood to be the part that contains the point (x_0, y_0).

The smash product may be a bit harder to visualize at first compared to the suspension. However, trying out on spheres provides useful examples as it can be shown in general that $S^n \wedge S^m \cong S^{n+m}$.

Both of the operators defined above are operators which plays an integral role in the theory of higher homotopy groups. We shall frequently encounter these in the following chapters.

2.3 Serre fibrations and fiber bundles

Another crucial component in the study of homotopy groups of spheres is long exact sequences of said groups. To this end we need to be able to understand when these can occur. As we shall see, this is the case when we have a *Serre fibration*. In order to understand the definition of a Serre fibration we first need some preliminary notions, more specifically, the homotopy lifting property. We shall define this as it appears in [Mit01].

Definition 2.5 (Homotopy lifting property)**.** Given spaces X, E, and B, together with a map $p : E \to B$, we say that p has the homotopy lifting property if

(i) for any homotopy $F : X \times I \to B$ and

(ii) an initial map $\tilde{F}_0 : X \to E$ such that $F_0 = p \circ \tilde{F}_0$

there exist a map $\tilde{F}$ making the diagram in Figure 1 commute.

With this definition in mind we say that $p : E \to B$ is a Serre fibration if p has the homotopy lifting property for all CW-complexes. One also has the notion of a general *fibration* in which the requirement is that p must have the homotopy extension property for all spaces and not just the CW-complexes.

An important class of Serre fibrations are *fiber bundles* which we now shall now define. We shall define a fiber bundle as it appears in [Hat02, p. 376–377]. To start off we need to understand the notion of a fiber.

$$
\begin{tikzcd}
X \arrow[r, "\tilde{F}_0"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\
X \times I \arrow[r, "F"'] \arrow[ur, "\tilde{F}"] & B
\end{tikzcd}
$$

Figure 1: The map i_0 is taken to be the identification of X with $X \times \{0\}$ in $X \times I$.

Definition 2.6 (Fiber)**.** For spaces E and B, a surjective map $p : E \to B$, and a point $b \in B$ we shall call the set $p^{-1}(b)$ the *fiber* of b. If there exists a space F such that for all points $b \in B$ we have $p^{-1}(b) \cong F$ then we simply refer to F as the fiber of B.

We now just need one more technical notion before introducing the definition of a fiber bundle.

Definition 2.7 (Local trivialization)**.** Let F, E, B be spaces with a map $p : E \to B$ such that F is a fiber of B. If there exists an open covering $\{U_i\}$ of B such that for all U_i there is a corresponding homeomorphism $\varphi_i : p^{-1}(U_i) \to U_i \times F$ which makes the diagram in Figure 2 commute, then the collection $\{(U_i, \varphi_i)\}$ is called a *local trivialization* of the spaces.

With that out of the way we are now ready for the definition of a fiber bundle.

Definition 2.8 (Fiber Bundle)**.** A fiber bundle, denoted (E, B, p, F), is a structure consisting of spaces E, B, and a fiber F of B, with the corresponding surjection $p : E \to B$. The space E is called the *total space*, B is called the *base space*, and the map p is called the *projection map.* We furthermore require that there exists a local trivialization of the spaces in the bundle. Another way to denote a fiber bundle is as

$$F \longrightarrow E \xrightarrow{p} B \tag{1}$$

Our claim now is that fiber bundles are examples of Serre fibrations.

Proposition 2.9. *A fiber bundle (F, E, p, B) satisfy the homotopy lifting property for all spaces X.*

$$
\begin{tikzcd}
p^{-1}(U) \arrow[rr, "\varphi_i"] \arrow[dr, "p"'] & & U \times F \arrow[dl] \\
& U &
\end{tikzcd}
$$

Figure 2: The unlabeled map is the projection onto the first factor.

References

[BT82] Bott, R. and Tu, L. W. *Differential Forms in Algebraic Topology.* Vol. 82. Graduate Texts in Mathematics. New York, NY: Springer New York, 1982.

[Hat02] Hatcher, A. *Algebraic Topology.* Algebraic Topology. Cambridge University Press, 2002.

[Mit01] Mitchell, S. A. *Notes on Serre fibrations.* 2001.