Integer Programming
Sheet 3

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Notation: We use $\subset$ to mean inclusion and $\subsetneq$ to mean strict inclusion.

Exercise B.1

(i) We first check that the proposed solution to the LP relaxation given by

$$\begin{aligned} x_j &= 1, && j = 1, \dots, r-1 \\ x_r &= \frac{b - \sum_{j=1}^{r-1} a_j}{a_r} \\ x_j &= 0, && j > r \end{aligned}$$

is feasible. To see this, note that

$$\begin{aligned} \sum_{j=1}^{n} a_j x_j &= \sum_{j=1}^{r-1} a_j + a_r \frac{b - \sum_{j=1}^{r-1} a_j}{a_r} \\ &= b \end{aligned}$$

and hence it is indeed a feasible solution.

We now construct a dual feasible solution by using complementary slackness. To do this, note first that the LP relaxation is given by

$$\begin{aligned} \max \ & c^T x \\ \text{s.t. } & a^T x \leq b \\ & x \leq 1 \\ & x \geq 0 \end{aligned}$$

and the dual is then

$$\begin{aligned} \min \ & b y_0 + \sum_{i=1}^{n} y_i \\ \text{s.t. } \ & a_i y_0 + y_i \geq c_i, \quad i = 1, \dots, n \\ & y \geq 0 \end{aligned}$$

where y_0 corresponds to the constraint $a^T x \leq b$. Using complementary slackness we can construct a dual optimal solution corresponding to the proposed primal solution.

The requirement from complementary slackness gives us that we must have

$$\begin{aligned} y_0 &= \frac{c_r}{a_r} \\ a_i y_0 + y_i &= c_i, \quad i = 1, \dots r-1 \\ (1-x_r)y_r &= 0 \\ y_i &= 0, \quad i > r. \end{aligned}$$

We therefore have that

$$\begin{aligned} y_0 &= \frac{c_r}{a_r} \\ y_i &= c_i - a_i \frac{c_r}{a_r}, \quad i = 1, \dots, r-1 \\ y_i &= 0, \quad i \geq r. \end{aligned}$$

This is a feasible solution as, for $i = 1, \dots, r-1$

$$\begin{aligned} a_i y_0 + y_i &= a_i \frac{c_r}{a_r} + c_i - a_i \frac{c_r}{a_r} \\ &= c_i \end{aligned}$$

and since $c_i/a_i > c_r/a_r$ for $i < r$ we have that $y_i \geq 0$ for all i. Moreover, for $i \geq r$ we have, by assumption, that

$$a_i y_0 + y_i = a_i \frac{c_r}{a_r} \geq c_i.$$

Hence the proposed dual solution is feasible. Moreover

$$\begin{aligned} b y_0 + \sum_{i=1}^{n} y_i &= b \frac{c_r}{a_r} + \sum_{i=1}^{r-1} \left(c_i - a_i \frac{c_r}{a_r} \right) \\ &= \sum_{i=1}^{r-1} c_j + c_r \frac{b - \sum_{i=1}^{r-1} a_i}{a_r} \\ &= \sum_{j=1}^{n} c_j x_j^*. \end{aligned}$$

which means x^* is optimal.

(ii) We can use the above result to do the branch-and-bound as we branch on whether to set $x_r = 1$ or $x_r = 0$. Now, since

$$\frac{17}{5} > \frac{10}{3} > \frac{25}{8} > \frac{17}{4}$$

we have that our incumbent is $x^* = (1, 1, 0, 0)$ and have a lower bound of $17 + 10 = 27$. Taking the LP-relaxation of $\mathscr{F}$ (our original problem) we have by the previous part that the optimal solution is given by $x^* = (1, 1, 0.5, 0)$. This gives a dual bound $\bar{z} = 39.5$. Since all coefficients are integer valued we can round down. The corresponding tree looks like the one in Figure 1.

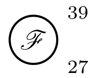

Figure 1: Step 1 in branch-and-bound.

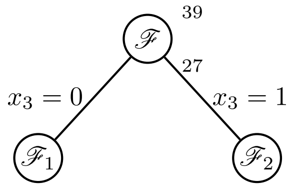

Figure 2: Branching on x_3.

We then branch on whether $x_3 = 0$ or $x_3 = 1$ and get the tree in Figure 2.

Focusing on $\mathscr{F}_1$ we take the LP-relaxation which is given by

$$\begin{aligned} \max \quad & 17x_1 + 10x_2 + 17x_4 \\ \text{s.t. } & 5x_1 + 3x_2 + 7x_4 \le 12 \\ & x_3 = 0 \\ & x \le 1 \\ & x \ge 0. \end{aligned}$$

The optimal solution is then, by part (i), given by $x^* = (1, 1, 0, 4/7)$ yielding the dual bound $\bar{z} = 36.71$ which we round down to $\bar{z} = 36$. We then bound on x_4 and get the tree in Figure 3. Taking the LP-relaxation of $\mathscr{F}_{11}$ we have the LP given by

$$\begin{aligned} \max \quad & 17x_1 + 10x_2 \\ \text{s.t. } & 5x_1 + 3x_2 \le 12 \\ & x_3 = 0 \\ & x_4 = 0 \\ & x \le 1 \\ & x \ge 0 \end{aligned}$$

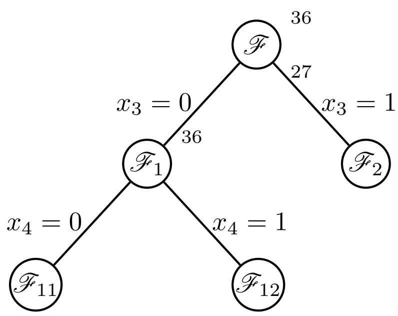

Figure 3: Branching on x_4.

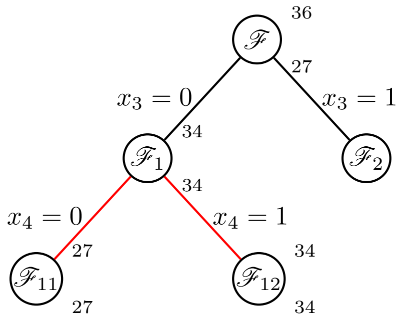

Figure 4: Pruning and bounding.

which has optimal solution given by $x^* = (1, 1, 0, 0)$ which was the incumbent we started with. We can prune this branch by optimality.

Taking the LP-relaxation of $\mathscr{F}_{12}$ gives the LP problem

$$\begin{aligned} \max \quad & 17x_1 + 10x_2 + 17 \\ \text{s.t. } & 5x_1 + 3x_2 \leq 5 \\ & x_3 = 0 \\ & x_4 = 1 \\ & x \leq 1 \\ & x \geq 0 \end{aligned}$$

which has optimal solution given by $x^* = (1, 0, 0, 1)$ and yields the dual bound $\bar{z} = 34$. We can then prune this branch by optimality. Our tree now looks like the one in Figure 4.

The next node to look at is $\mathscr{F}_2$ which has LP-relaxation given by

$$\begin{aligned} \max \quad & 17x_1 + 10x_2 + 25 + 17x_4 \\ \text{s.t. } & 5x_1 + 3x_2 + 7x_4 \leq 4 \\ & x_3 = 1 \\ & x \leq 1 \\ & x \geq 0. \end{aligned}$$

This has optimal solution given by $x^* = (\frac{4}{5}, 0, 1, 0)$ and a dual bound of $\bar{z} = 38.6$ which we round down to $\bar{z} = 38$. We then branch on x_1 and get the tree in Figure 5. The LP-relaxation of $\mathscr{F}_{21}$ is given by

$$\begin{aligned} \max \quad & 10x_2 + 25 + 17x_4 \\ \text{s.t. } & 3x_2 + 7x_4 \leq 4 \\ & x_1 = 0 \\ & x_3 = 1 \\ & x \leq 1 \\ & x \geq 0 \end{aligned}$$

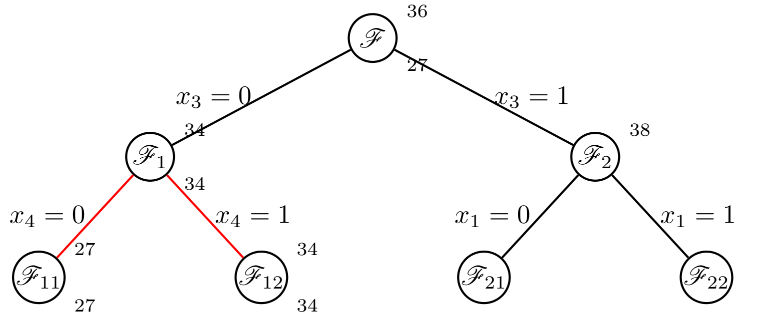

Figure 5: Branching on x_1.

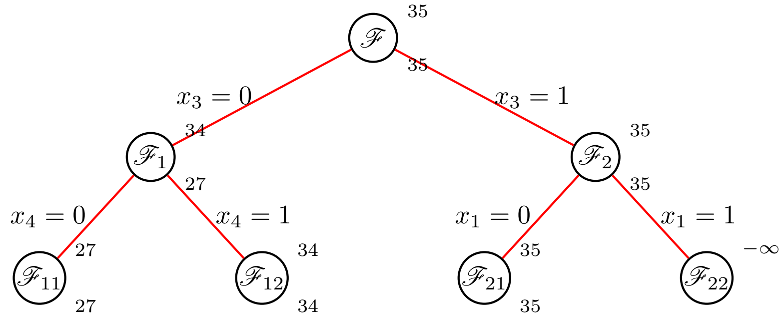

Figure 6: Final tree.

which has optimal solution given by $x^* = (0, 1, 1, 1/7)$, but we see that the solution $x^* = (0, 1, 1, 0)$ is the only integer valued feasible solution in $\mathscr{F}_{21}$ with dual and primal bound $\bar{z} = 35 = \underline{z}$.

The node $\mathscr{F}_{22}$ can be pruned by infeasibility. Hence we end up with the tree in Figure 6. We thus see that the optimal solution is $x^* = (0, 1, 1, 0)$ with corresponding optimal value $z^* = 35$.

Exercise B.2

(i) If $\mathscr{F} = \emptyset$ then there is nothing to prove. Hence, assume that

$$x_1^* = \text{argmax}_{x \in \mathscr{F}}(c^T x)$$
$$x_2^* = \text{argmax}_{x \in \mathscr{F}}(g(x))$$
$$x_3^* = \text{argmax}_{x \in \mathscr{R}}(g(x)).$$

We then have

$$z = c^T x_1^* \leq g(x_1^*) \leq g(x_2^*) \leq g(x_3^*) = w$$

as desired.

(ii) To show that MW1T is a relaxation of TSP we need to show two things:

(1) that $\mathscr{F} \subset \mathscr{R}$ where $\mathscr{F}$ is the feasible set for TSP and $\mathscr{R}$ is the feasible set for TSP,

(2) $c^T x \leq g(x)$ where $g(x)$ is the objective function for the MW1T.

Since $g(x) = c^T x$ we see that (2) is trivially true. We then need to show that every solution to the TSP is a 1-tree. However, this is also fairly trivially true as any Hamiltonian circuit is a subgraph of G which consists of the union of two edges to node 1 and a spanning tree on the rest (all paths are trees). Thus we see that MW1T is a relaxation of TSP.

(iii) We propose the following algorithm Suppose MinWeightOneTree does not

Algorithm 1 Minimum Weight 1-Tree Algorithm

Input: A connected graph $G = (V, E)$ with weight function $c : E \to \mathbb{R}$
Output: A minimum weight 1-tree
procedure MinWeightOneTree(G)
    Choose a node $v_1 \in V$
    Find two edges $e_1, e_2 \in E$ adjacent to v_1 with minimum weights
    $T \leftarrow \{e_1, e_2\}$ ▷ Initialize the 1-tree with the two minimum edges
    $G' \leftarrow G$ with v_1 and all edges incident to v_1 removed
    $T' \leftarrow$ Apply Kruskal's algorithm to G' to find MST
    $T \leftarrow T \cup T'$ ▷ Combine the edges to form the 1-tree
    return T
end procedure

produce a minimum weight 1-tree. Then there exists another tree T' such that, if x_e^1 denotes edges corresponding to T and x_e^2 denotes edges corresponding to T', then

$$\sum_{e \in \delta(1)} c_e x_e^2 + \sum_{e \in E \setminus \delta(1)} c_e x_e^2 < \sum_{e \in \delta(1)} c_e x_e^1 + \sum_{e \in E \setminus \delta(1)} c_e x_e^1.$$

We must then have either

$$\sum_{e \in \delta(1)} c_e x_e^2 < \sum_{e \in \delta(1)} c_e x_e^1$$

or

$$\sum_{e \in E \setminus \delta(1)} c_e x_e^2 < \sum_{e \in E \setminus \delta(1)} c_e x_e^1.$$

The first option violates the minimality of the edges adjacent to node 1 while the second option violates the optimality of the spanning tree returned by Kruskal's algorithm. Hence we see that the output of MinWeightOneTree must be a minimum 1-tree.

(iv) Seeing as a 1-tree has exactly one cycle we can branch by excluding one edge in the cycle. Each subproblem is then of the same type, i.e., a TSP that live in a subgraph of G and the union of all such subproblems is equal to original problem.

Exercise B.3

(i) We have that the block matrices A_k are given by

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A_7 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_8 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

We have that the D_k's which specify the $\mathscr{X}_k$'s are given by

$$D_k = \begin{bmatrix} 1 & 0 \\ \frac{d_k^2}{d_k^1 + d_k^2} & 1 \end{bmatrix}$$

so that the $\mathscr{X}_k$'s are given by

$$\mathscr{X}_k = \left\{ x_k \in \mathbb{Z}^2 \mid \begin{bmatrix} 1 & 0 \\ \frac{d_k^2}{d_k^1 + d_k^2} & 1 \end{bmatrix} x_k \leq \begin{bmatrix} d_k^1 + d_k^2 \\ d_k^2 + d_k^3 \end{bmatrix}, x_k \geq 0 \right\}.$$

Spelled out we then have

$$\mathscr{X}_1 = \left\{ x_1 \in \mathbb{Z}^2 \mid \begin{bmatrix} 1 & 0 \\ \frac{1}{3} & 1 \end{bmatrix} x_1 \leq \begin{bmatrix} 60 \\ 35 \end{bmatrix}, x_1 \geq 0 \right\}$$

$$\mathscr{X}_2 = \left\{ x_2 \in \mathbb{Z}^2 \mid \begin{bmatrix} 1 & 0 \\ \frac{5}{14} & 1 \end{bmatrix} x_2 \leq \begin{bmatrix} 70 \\ 45 \end{bmatrix}, x_2 \geq 0 \right\}$$

$$\mathscr{X}_3 = \left\{ x_3 \in \mathbb{Z}^2 \mid \begin{bmatrix} 1 & 0 \\ \frac{3}{11} & 1 \end{bmatrix} x_3 \leq \begin{bmatrix} 55 \\ 30 \end{bmatrix}, x_3 \geq 0 \right\}$$

$$\mathscr{X}_4 = \left\{ x_4 \in \mathbb{Z}^2 \mid \begin{bmatrix} 1 & 0 \\ \frac{2}{7} & 1 \end{bmatrix} x_4 \leq \begin{bmatrix} 70 \\ 50 \end{bmatrix}, x_4 \geq 0 \right\}$$

$$\mathscr{X}_5 = \left\{ x_5 \in \mathbb{Z}^2 \mid \begin{bmatrix} 1 & 0 \\ \frac{1}{4} & 1 \end{bmatrix} x_5 \leq \begin{bmatrix} 60 \\ 30 \end{bmatrix}, x_5 \geq 0 \right\}$$

$$\mathscr{X}_6 = \left\{ x_6 \in \mathbb{Z}^2 \mid \begin{bmatrix} 1 & 0 \\ \frac{3}{8} & 1 \end{bmatrix} x_6 \leq \begin{bmatrix} 80 \\ 40 \end{bmatrix}, x_6 \geq 0 \right\}$$

$$\mathscr{X}_7 = \left\{ x_7 \in \mathbb{Z}^2 \mid \begin{bmatrix} 1 & 0 \\ \frac{1}{5} & 1 \end{bmatrix} x_7 \leq \begin{bmatrix} 50 \\ 45 \end{bmatrix}, x_7 \geq 0 \right\}$$

$$\mathscr{X}_8 = \left\{ x_8 \in \mathbb{Z}^2 \mid \begin{bmatrix} 1 & 0 \\ \frac{4}{13} & 1 \end{bmatrix} x_8 \leq \begin{bmatrix} 65 \\ 40 \end{bmatrix}, x_8 \geq 0 \right\}.$$

We can then formulate the revenue maximisation problem of the airline as an IP in block-angular form by

$$\begin{aligned} \text{(RMP)}\ \max\ & \sum_{k=1}^{8} c_k^T x_k \\ \text{s.t}\ & \sum_{k=1}^{8} A_k x_k \leq \begin{bmatrix} 100 \\ 200 \\ 150 \\ 100 \\ 100 \end{bmatrix} \\ & x_k \in \mathscr{X}_k \text{ for } k = 1, \ldots, 8. \end{aligned}$$

(ii) We write each $\mathscr{X}_k$ as

$$\mathscr{X}_k = \{x_{k,t} \mid t \in \{1, \ldots, T_k\}\}.$$

From this we get the Dantzig-Wolfe reformulation given by

$$\begin{aligned} \text{(RMP)}\ \max_{\lambda} & \sum_{k=1}^{8} \sum_{t=1}^{T_k} (c_k^T x_{k,t}) \lambda_{k,t} \\ \text{s.t.}\ & \sum_{k=1}^{8} \sum_{t=1}^{T_k} (A_k x_{k,t}) \lambda_{k,t} \leq \begin{bmatrix} 100 \\ 200 \\ 150 \\ 100 \\ 100 \end{bmatrix} \\ & \sum_{t=1}^{T_k} \lambda_{k,t} = 1, \quad (k = 1, \ldots, 8) \\ & \lambda_{k,t} \in \{0, 1\}, \quad (t = 1, \ldots, T_k), \quad (k = 1, \ldots, 8) \end{aligned}$$

which has (LPM) given by

$$
\begin{aligned}
\text{(LPM)}\ \max_{\lambda} & \sum_{k=1}^{8}\sum_{t=1}^{T_k}(c_k^T x_{k,t})\lambda_{k,t} \\
\text{s.t.}\ & \sum_{k=1}^{8}\sum_{t=1}^{T_k}(A_k x_{k,t})\lambda_{k,t} \leq \begin{bmatrix}100\\200\\150\\100\\100\end{bmatrix} \\
& \sum_{t=1}^{T_k}\lambda_{k,t} = 1, \quad (k = 1, \dots, 8) \\
& \lambda_{k,t} \geq 0, \quad (t = 1, \dots, T_k), \quad (k = 1, \dots, 8).
\end{aligned}
$$

The dual of this (DM) is then given by

$$
\begin{aligned}
\text{(DM)}\ \min_{\mu,\pi}\ & 100\pi_1 + 200\pi_2 + 150\pi_3 + 100\pi_4 + 100\pi_5 + \sum_{k=1}^{8}\mu_k \\
\text{s.t.}\ & \pi^T A_k x_{k,t} + \mu_k \geq c_k^T x_{k,t}, \quad (k = 1, \dots, 8), \quad (t = 1, \dots, T_k) \\
& \pi \geq 0.
\end{aligned}
$$

The associated column generation subproblems (CGIP_k) are given by

$$
\begin{aligned}
(\text{CGIP})_k \quad \zeta_k = & \max_{x_k}(c_k - \pi^T A_k)x_k - \mu_k \\
\text{s.t.}\ & \begin{bmatrix}1 & 0\\ \frac{d_k^2}{d_k^1+d_k^2} & 1\end{bmatrix} x_k \leq \begin{bmatrix}d_k^1 + d_k^2\\ d_k^2 + d_k^3\end{bmatrix} \\
& x_k \geq 0 \\
& x_k \in \mathbb{Z}^2.
\end{aligned}
$$

Solving the (CGIP_k) is equivalent to finding $x_k^* = (x_k^{1*}, x_k^{2*})$ such that if $c_k - \pi^T A_k = (\tilde{c}_k^1, \tilde{c}_k^2)^T$ then

$$
\begin{aligned}
(x_k^{1*}, x_k^{2*}) = & \operatorname{argmax}_{x_k} \tilde{c}_k^1 x_k^1 + \tilde{c}_k^2 x_k^2 \\
\text{s.t.}\ & \begin{bmatrix}1 & 0\\ \frac{d_k^2}{d_k^1+d_k^2} & 1\end{bmatrix} x_k \leq \begin{bmatrix}d_k^1 + d_k^2\\ d_k^2 + d_k^3\end{bmatrix} \\
& x_k \geq 0 \\
& x_k \in \mathbb{Z}^2.
\end{aligned}
$$

There are then four cases to consider

(1) $\tilde{c}_k^1, \tilde{c}_k^2 > 0$: In this case we want both x_k^1 and x_k^2 to be as large as possible, but because of the constraints there is some interdependence between the two. We thus apply branch and bound in the usual way via LP-relaxation and branching on fractional values.

(2) $\tilde{c}_k^1 < 0$, $\tilde{c}_k^2 > 0$: In this case we want x_k^2 as large as possible and x_k^1 as small as possible. Thus, set $x_k^1 = 0$ and $x_k^2 = \lfloor d_k^2 + d_k^3 \rfloor$.

(3) $\tilde{c}_k^1 > 0$, $\tilde{c}_k^2 < 0$: In this case we want x_k^1 as large as possible and x_k^2 as small as possible. Thus, set $x_k^1 = \lfloor d_k^1 + d_k^2 \rfloor$ and $x_k^2 = 0$.

(4) $\tilde{c}_k^1, \tilde{c}_k^2 < 0$: Set $x_k^1 = 0 = x_k^2$.

(iii) If we are interested in the LP relaxation (P) of (RMP) then the $\mathscr{X}_k$ turn into $\mathscr{R}_k$ where

$$\mathscr{R}_k := \left\{ x_k \in \mathbb{R}^2 \mid \begin{bmatrix} 1 & 0 \\ \frac{d_k^2}{d_k^1 + d_k^2} & 1 \end{bmatrix} x_k \leq \begin{bmatrix} d_k^1 + d_k^2 \\ d_k^2 + d_k^3 \end{bmatrix}, x_k \geq 0 \right\}.$$

There is no possible way to enumerate all the points in $\mathscr{R}_k$, but we can enumerate the vertices of it and then express points inside as convex combinations of the vertices. In other words, by letting $\{v_{k,t}\}_{t \in \{1,\dots \mathcal{T}_k\}}$ be an enumeration of the vertices of $\mathscr{R}_k$, and writing $x_k = \sum_{t=1}^{\mathcal{T}_k} \lambda_{k,t} v_{k,t}$ where $\sum_{t=1}^{\mathcal{T}_k} \lambda_{k,t} = 1$ ($\lambda_{k,t} \geq 0$), we have the following representation

$$\begin{aligned} \text{(PM)} \quad \max_{\lambda} \sum_{k=1}^{8} \sum_{t=1}^{\mathcal{T}_k} &(c_k^T v_{k,t}) \lambda_{k,t} \\ \text{s.t.} \quad \sum_{k=1}^{8} \sum_{t=1}^{\mathcal{T}_k} &(A_k v_{k,t}) \lambda_{k,t} \leq \begin{bmatrix} 100 \\ 200 \\ 150 \\ 100 \\ 100 \end{bmatrix} \\ &\sum_{t=1}^{\mathcal{T}_k} \lambda_{k,t} = 1, \quad (k = 1, \dots, 8) \\ &\lambda_{k,t} \geq 0, \quad (t = 1, \dots, \mathcal{T}_k), \quad (k = 1, \dots, 8). \end{aligned}$$

The important difference between (PM) and (LPM) is that in the (LPM) we consider the convex hull of each $\mathscr{X}_k$ while for the (PM) we consider a possibly larger set since we first enlarge $\mathscr{X}_k$ and then consider the vertices of this enlarged set. More concretely, in (PM) we require that $x_k \in \mathscr{R}_k$ while in (LPM) we require that $x_k \in \text{conv}(\mathscr{X}_k)$ and we have that

$$\text{conv}(\mathscr{X}_k) \subset \mathscr{R}_k.$$

Since the objective function in both cases is the same we see that (PM) is a relaxation of (LPM). Hence we will generally get weaker dual bounds from (PM) as compared to (LPM).