Price of Anarchy of Simultaneous Auctions

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1 Introduction

Consider the following scenario: someone is holding a yard sale and have m items up for sale. There are n potential buyers, each interested in buying subsets of these items, and each of whom have some valuation of each subset of the items. What scheme should the seller come up with in order to efficiently allocate the items he his selling to the buyers.

In this mini project we will focus on two types of mechanism designs: simultaneous first-price auctions and simultaneous second-price auctions. Intuitively these are auctions in which every bidder places a bid for each item simultaneously and the one with the highest bid for a specific item receives that item. The only difference is in what he must pay. For first-price auctions the highest bidder pays his highest bid while for second-price auctions the highest bidder only pays the bid of the second highest bidder. To formalize this a bit we introduce the notion of "mechansims".

1.1 Auctions as mechanisms

Definition 1.1 (Mechanism [ST13]). *A **mechanism** $\mathcal{M}$ is a triple $\mathcal{M} = (\mathcal{A}, X, P)$, where $\mathcal{A} = \times_i \mathcal{A}_i$ is a set of actions $\mathcal{A}_i$ for each player i, $X : \mathcal{A} \to \mathcal{X}$ is an allocation function that maps each action profile $\mathbf{a} \in \mathcal{A}$ to an outcome $\mathbf{x} \in \mathcal{X} \subset \times_i \mathcal{X}_i$ (where $\mathcal{X}_i$ is the set of allocations for Player i), and $P : \mathcal{A} \to \mathbb{R}^n_{\geq 0}$ is a payment function that maps each action profile to a payment P_i for each player i. The **utility** of player i for some valuation $v_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$, and allocation $x_i \in \mathcal{X}_i$ is defined to be*

$$u_i^{v_i}(x_i) = v_i(x_i) - P_i(x_i) \tag{1.1}$$

Here the allocation function v_i is part of some set of valuations $\mathcal{V}_i$ which is a representation of how much a player i values a subset of outcomes. It would be strange to assume that a buyer knows exactly how the other buyers value a set of items. However, he could make an educated guess as to the distribution of potential valuations and assume another buyer gets his valuation from this distribution. We therefore say that a mechanism is in the **Bayesian setting** if for each player $i \in [n]$ the valuation v_i is drawn from $\mathcal{V}_i$ according to some known distribution $\mathcal{F}_i$. The $\mathcal{F}_i$ are assumed to be independent of each other. In this case we have a mechanism of **incomplete information**. Conversely, if each player's valuation v_i is fixed and known to all the other players then we are in the **full information setting**.

Now, in the case of our yard sale we are selling off each item individually and at the same time. Thus we are essentially looking at a "simultaneous composition" of single-item auctions. More formally, assume that there are n players and m mechanisms $\{\mathcal{M}_j = (\mathcal{A}^j, X^j, P^j)\}_{j=1}^m$. We make the extra assumption that each player has a valuation over vectors of outcomes from each mechanism, i.e., functions $v_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$ where $\mathcal{X}_i = \times_j \mathcal{X}_i^j$.

Definition 1.2 (Simultaneous composition of mechanisms [ST13]). *Given m mechanisms $\{\mathcal{M}_j\}_{j=1}^m$ as described above with n bidders/players we define the* ***simultaneous composition*** *of these mechanisms to be the mechanism $\mathcal{M} = (\mathcal{A}, X, P)$ where $\mathcal{A}_i = \times_j \mathcal{A}_i^j$, $\mathcal{X} = \times_j \mathcal{X}^j$, $X = \times_j X^j : \mathcal{A} \to \mathcal{X}$, and $P = \sum_j P^j : \mathcal{A} \to \mathbb{R}_{\geq 0}$. For $\mathbf{a} \in \mathcal{A}$ we therefore have that $X(\mathbf{a}) = (X^j(a^j))_j$ and $P(\mathbf{a}) = \sum_j P^j(a^j)$ with a^j denoting the jth component of $\mathbf{a}$.*

In the context of the above definitions we can formalize the notion of an auction as well as first/second-price auctions.

Definition 1.3 (Combinatorial auction). *Let $\mathcal{I}$ be a set of items. A* ***combinatorial auction*** *is a mechanism $\mathcal{M} = (\mathcal{A}, X, P)$ where $\mathcal{X}_i = 2^{\mathcal{I}}$ (the powerset of $\mathcal{I}$), $\mathcal{X}$ consists of partitions of $\mathcal{I}$, and $\mathcal{A}_i = \mathcal{B}_i$ is the set of bids player i can make. Moreover, each $v_i \in \mathcal{V}_i$ is assumed to be* ***normalized*** *and* ***monotone****, that is $v_i(\emptyset) = 0$ and $S \subseteq T \implies v_i(S) \leq v_i(T)$.*

Definition 1.4 (First-price auction). *A* ***first-price auction*** *is a combinatorial auction $\mathcal{M}$ in which the allocation function $X : \mathcal{A} \to \mathcal{X}$ is described by allocating items to the highest bidder. The price a player must pay for the allocation is equal to the bid if he has the highest bid and zero otherwise.*

Definition 1.5 (Second-price auction). *A* ***second-price auction*** *is a combinatorial auction $\mathcal{M}$ in which the allocation function $X : \mathcal{A} \to \mathcal{X}$ is described by allocating items to the highest bidder. The price a player must pay for the allocation is equal to the bid of the second highest bidder if he has the highest bid and zero otherwise.*

A **simultaneous first/second-price auction** is then just a simultaneous composition of first/second-price auctions. To more formally state our problem we also need a way to talk about how efficient or inefficient a particular allocation of items is.

1.2 Nash equilibria and the price of anarchy

Definition 1.6 (Strategy). *Let $\mathcal{M} = (\mathcal{A}, X, P)$ be a mechanism. A* ***strategy profile*** *$\mathbf{s} = (s_1, \ldots, s_n)$ is a probability distribution over $\mathcal{A}$ such that each s_i is a probability distribution over $\mathcal{A}_i$ and is independent of s_j for $j \neq i$. A* ***pure strategy*** *is a strategy profile $\mathbf{s}$ such that each component s_i has probability measure 1 on a single outcome $a_i \in \mathcal{A}_i$. Conversely, a* ***mixed strategy*** *is not a pure strategy. The utility for Player i of a strategy profile is defined to be $u_i(\mathbf{s}) = \mathbb{E}_{s\sim\mathbf{s}}[u_i(X_i(s))]$.*

Definition 1.7 (Nash equilibria). *Let $\mathcal{M} = (\mathcal{A}, X, P)$ be a mechanism and $\mathcal{V} = \times_i \mathcal{V}_i$ the corresponding valuations. Let $\mathcal{F} = \times_i \mathcal{F}_i$ be a distribution over $\mathcal{V}$. Let $\mathbf{s}$ be a strategy profile over $\mathcal{A}$. Then, $\mathbf{s}$ is called a*

- ***pure Nash equilibrium****, if $\mathbf{s}$ is a pure strategy and $u_i(\mathbf{s}) \geq u_i(s_i', \mathbf{s}_{-i})$ for all pure strategies s_i' over $\mathcal{A}_i$,*
- ***mixed Nash equilibrium****, if $\mathbf{s}$ is a strategy profile and $u_i(\mathbf{s}) \geq u_i(s_i', \mathbf{s}_{-i})$ for all strategies s_i' over $\mathcal{A}_i$,*
- ***Bayesian Nash equilibrium****, if $\mathbf{s}(\mathbf{v})$ is a strategy profile dependent upon $\mathbf{v} \sim \mathcal{F}$ and*

  $$\mathbb{E}_{\mathbf{v}_{-i}, s\sim\mathbf{s}(\mathbf{v})}[u_i(s)] \geq \mathbb{E}_{\mathbf{v}_{-i}, s\sim\mathbf{s}(\mathbf{v})}[u_i(s_i', \mathbf{s}_{-i})]$$

  for all strategies s_i' over $\mathcal{A}_i$.

It is reasonable to think that left to their own devices the buyers could end up choosing a strategy profile which is one of the equilibrium types mentioned above. However, this choice might be far from optimal.

Definition 1.8 (Social welfare). *Let $\mathcal{M}$ be a mechanism. The* ***social welfare*** *of a strategy $\mathbf{s}$ given a valuation profile $\mathbf{v} \in \mathcal{V}$ is defined to be*

$$SW^{\mathbf{v}}(\mathbf{s}) = \sum_{i=1}^{n} \mathbb{E}_{s\sim\mathbf{s}}[v_i(X_i(s))]. \tag{1.2}$$

Similarly, given a distribution $\mathcal{F}$ over valuations $\mathcal{V}$ the social welfare of a strategy $\mathbf{s}$ *is*

$$SW^{\mathcal{F}}(\mathbf{s}) = \sum_{i=1}^{n} \mathbb{E}_{\mathbf{v}\sim\mathcal{F},s\sim\mathbf{s}(\mathbf{v})}[v_i(X_i(s))]. \tag{1.3}$$

Note that $SW^{\mathbf{v}}$ is a special case of $SW^{\mathcal{F}}$ in which the probability measure of $\mathcal{F}$ is 1 for a specific valuation profile $\mathbf{v}$*.*

Definition 1.9 (Price of Anarchy (PoA))**.** *Let $\mathcal{M}$ be a mechanisms and let $\mathcal{E}_{\mathcal{M}}$ denote a corresponding set of equilibrium strategies. With $\mathcal{S}_{\mathcal{M}}$ denoting the set of all strategies, the **price of anarchy** (PoA) for a distribution $\mathcal{F}$ over $\mathcal{V}$ is*

$$PoA^{\mathcal{M},\mathcal{F}} = \frac{\max_{\mathbf{s}\in\mathcal{S}} SW^{\mathcal{F}}(\mathbf{s})}{\min_{\mathbf{s}'\in\mathcal{E}_{\mathcal{M}}} SW^{\mathcal{F}}(\mathbf{s}')}. \tag{1.4}$$

In the full information setting we restrict $\mathcal{F}$ to have probability measure 1 for a specific valuation profile $\mathbf{v}$*. Letting $\mathbb{M}$ denote the set of all mechanisms of a specific class and $\mathbb{F}_{\mathcal{M}}$ the set of all relevant distributions for $\mathcal{M} \in \mathbb{M}$ (depending upon whether we are in the full information or Bayesian setting). The **price of anarchy of** $\mathbb{M}$ is then*

$$PoA = \max_{\mathcal{M}\in\mathbb{M}} \left(\max_{\mathcal{F}\in\mathbb{F}} PoA^{\mathcal{M},\mathcal{F}} \right). \tag{1.5}$$

1.3 Different types of valuations

When defining our problem we are not interested in all types of valuations for a given mechanism $\mathcal{M}$. We want to restrict ourselves to a specific class of valuations $\mathcal{V}$.

Definition 1.10. *Let $\mathcal{M}$ be a mechanism and $v_i : \mathcal{X} \to \mathbb{R}_{\geq 0}$ a valuation for player $i \in [n]$. We say that v_i is*

- ***additive****, if $v_i(S) = \sum_{x\in S}$ for all $S \subset \mathcal{X}_i$,*
- ***submodular****, if $v_i(S \cup T) + v_i(S \cap T) \leq v_i(S) + v_i(T)$ for all $S, T \subset \mathcal{X}_i$,*
- ***fractionally subadditive*** *(or **XOS**), if v_i is determined by a finite set of additive valuations f_γ for $\gamma \in \Gamma$ such that $v_i(S) = \max_{\gamma\in\Gamma} f_\gamma(S)$,*
- ***subadditive****, if $v_i(S \cup T) \leq v_i(S) + v_i(T)$ for all $S, T \subset \mathcal{X}_i$.*

It is well known that each of the above classes are contained in the next one, i.e., additive valuations are submodular, submodular valuations are XOS, and XOS are subadditive.

1.4 The problem statement

Having set up necessary terminology we can now properly state the problem we want to investigate.

Problem 1. *Consider simultaneous first-price auctions. What are the currently known upper and lower bounds on the price of anarchy for submodular and subadditive valuations over mixed Bayesian and Bayesian-Nash equilibria respectively?*

2 Bounds on the Price of Anarchy

2.1 Smooth mechanisms

To derive the upper bounds for simultaneous first-price auctions we will outline the approach taken in [ST13] of defining smooth mechanisms and then deriving general bounds for smooth mechanisms and the composition of smooth mechanisms. In the following we let $u_i^{v_i}(\mathbf{a})$ denote the expected utility of a player $i \in [n]$ with valuation v_i when the randomized strategy vector $\mathbf{a}$ is played.

Definition 2.1 (Smooth mechanism [ST13])**.** *A mechanism $\mathcal{M}$ is (λ, μ)-smooth for some $\lambda, \mu \geq 0$, if there exists a randomized action profile $\mathbf{a}_i^*(\mathbf{v}, a_i)$ that is dependent on the whole valuation profile $\mathbf{v} \in \times_i \mathcal{V}_i$*

and the player's action $a_i \in \mathcal{A}_i$, such that for any valuation profile $\mathbf{v} \in \times_i \mathcal{V}_i$ and for any action profile $a \in \times_i \mathcal{A}_i$ we have

$$\sum_{i=1}^{n} u_i^{v_i}(\mathbf{a}_i^*(\mathbf{v}, a_i), a_{-i}) \geq \lambda OPT(\mathbf{v}) - \mu \sum_{i=1}^{n} P_i(a) \tag{2.1}$$

where $OPT(\mathbf{v}) = \sum_{i=1}^{n} v_i(x_i^*(\mathbf{v}))$ for some optimal allocation $x^*(\mathbf{v}) = argmax_{x \in \mathcal{X}} \sum_{i=1}^{n} v_i(x_i)$.

As we will see, first-price auctions are smooth and so we would like to say something about the simultaneous composition of smooth mechanisms.

Theorem 2.2 (Simultaneous composition [ST13])**.** *Let $\mathcal{M}$ be a simultaneous composition of m smooth mechanisms. Suppose that each component mechanism $\mathcal{M}_j$ is (λ, μ)-smooth when the mechanism restricted valuations of the players come from a set $(\mathcal{V}_i^j)_{i \in [n]}$. If the valuation $v_i : \mathcal{X}_i \to \mathbb{R}_{\geq 0}$ of each player across mechanisms is fractionally subadditive, and can be expressed as an XOS valuation by component valuations $v_{ij}^{\ell} \in \mathcal{V}_i^j$ then the global mechanism is also (λ, μ)-smooth.*

Proof. See [ST13, Theorem 5.1] □

This above theorem also holds for submodular valuations as submodular valuations are a strict subclass of XOS valuations.

Imposing the condition that a mechanism is smooth allows us to put concrete bounds on the price of anarchy.

Theorem 2.3 ([ST13])**.** *If a mechanism is $\mathcal{M}$ is (λ, μ)-smooth and players have the possibility to withdraw, then for any set of independent distributions $\{\mathcal{F}_i\}_{i=1}^n$ over the valuations sets $\{\mathcal{V}_i\}_{i=1}^n$, the price of anarchy over Bayes-Nash equilibria is at most* $\max(\mu, 1)/\lambda$.

Proof [ST13]. Note that the deviating strategy $\mathbf{a}_i^*(\mathbf{v}, a_i)$ of player i required by the smoothness property depends on the whole valuation profile $\mathbf{v}$ and not only on the valuation of player i. As a result $\mathbf{a}_i^*(\mathbf{v}, a_i)$ cannot be directly used as deviation for the player since the valuations of the other players are not known beforehand.

Consider the following randomized deviation for each player i that depends only on the information that he has which is own valuation v_i and the equilibrium strategy $\mathbf{s}$: he random samples a valuation profile $\mathbf{w} \sim \mathcal{F} = \times_i \mathcal{F}_i$ (he can do this as the distributions are common knowledge). Then he plays $\mathbf{a}_i^*((v_i, \mathbf{w}_{-i}), s_i(w_i))$, i.e., the player considers the equilibrium actions $\mathbf{s}(\mathbf{w})$, and deviates from this action profile using the action given by the smoothness property for his true type v_i, the random sample of the types of the others $\mathbf{w}_{-i}$, and the equilibrium action $s_i(w_i)$ of his randomly sampled type w_i. By assumption, this is not a profitable deviation and so

$$\begin{aligned}
\mathbb{E}_{\mathbf{v}} \left[u_i^{v_i}(\mathbf{s}(\mathbf{v})) \right] &\geq \mathbb{E}_{\mathbf{v},\mathbf{w}} \left[u_i^{v_i}(\mathbf{a}_i^*((v_i, \mathbf{w}_{-i}), s_i(w_i)), \mathbf{s}_{-i}(\mathbf{v}_{-i})) \right] \\
&= \mathbb{E}_{\mathbf{v},\mathbf{w}} \left[u_i^{w_i}(\mathbf{a}_i^*((w_i, \mathbf{w}_{-i}), s_i(v_i)), \mathbf{s}_{-i}(\mathbf{v}_{-i})) \right] \\
&= \mathbb{E}_{\mathbf{v},\mathbf{w}} \left[u_i^{w_i}(\mathbf{a}_i^*(\mathbf{w}, s_i(v_i)), \mathbf{s}_{-i}(\mathbf{v}_{-i})) \right]
\end{aligned}$$

where the first equality is an exchange of variable names, using independence. Summing over players and using smoothness then gives

$$\begin{aligned}
\mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^{n} u_i^{v_i}(\mathbf{s}(\mathbf{v})) \right] &\geq \mathbb{E}_{\mathbf{v},\mathbf{w}} \left[\sum_{i=1}^{n} u_i^{w_i}(\mathbf{a}_i^*(\mathbf{w}, s_i(v_i)), \mathbf{s}_{-i}(\mathbf{v}_{-i})) \right] \\
&\geq \mathbb{E}_{\mathbf{v},\mathbf{w}} \left[\lambda \mathrm{OPT}(\mathbf{w}) - \mu \sum_{i=1}^{n} P_i(\mathbf{s}(\mathbf{v})) \right] \\
&= \lambda \mathbb{E}_{\mathbf{w}} \left[\mathrm{OPT}(\mathbf{w}) \right] - \mu \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^{n} P_i(\mathbf{s}(\mathbf{v})) \right]
\end{aligned}$$

from which we end up with the expression

$$\mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^{n} v_i(X_i(\mathbf{s}(\mathbf{v}))) \right] \geq \lambda \mathbb{E}_{\mathbf{w}} \left[\mathrm{OPT}(\mathbf{w}) \right] + (1 - \mu) \mathbb{E}_{\mathbf{v}} \left[\sum_{i=1}^{n} P_i(\mathbf{s}(\mathbf{v})) \right]. \tag{2.2}$$

Now, if $\mu \leq 1$ then we immediately have

$$\frac{\mathbb{E}_{\mathbf{w}}\left[\mathrm{OPT}(\mathbf{w})\right]}{\mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} v_i(X_i(\mathbf{s}(\mathbf{v})))\right]} \leq \frac{1}{\lambda}. \tag{2.3}$$

Now, if $\mu > 1$, remember that players have the option to withdraw, meaning utility is never negative and hence $v_i(X_i(\mathbf{s}(\mathbf{v}))) \geq P_i(\mathbf{s}(\mathbf{v}))$ from which it follows that

$$\mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} v_i(X_i(\mathbf{s}(\mathbf{v})))\right] \geq \lambda \mathbb{E}_{\mathbf{w}}\left[\mathrm{OPT}(\mathbf{w})\right] + (1-\mu)\mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} v_i(X_i(\mathbf{s}(\mathbf{v})))\right] \tag{2.4}$$

and hence

$$\frac{\mathbb{E}_{\mathbf{w}}\left[\mathrm{OPT}(\mathbf{w})\right]}{\mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^{n} v_i(X_i(\mathbf{s}(\mathbf{v})))\right]} \leq \frac{\mu}{\lambda}. \tag{2.5}$$

Thus $\mathrm{PoA} \leq \max(\mu, 1)/\lambda$. □

2.2 Upper bounds

If we can show that first-price auctions are smooth mechanisms then Theorem 2.2 and Theorem 2.3 immediately gives us upper bound on simultaneous first-price auctions.

Proposition 2.4. *A first-price auction is a $(1-1/e, 1)$-smooth mechanism.*

Proof. We follow the argumentation in Section 6 of [ST13]. Note that we can prove smoothness both for full information setting and Bayesian setting simultaneously as the definition of a smooth mechanism gives us a specific valuation profile. To prove smoothness we must find a randomized bidding profile $\mathbf{b}^* = (b_1^*, \ldots, b_n^*)$ such that for any valuation profile $\mathbf{v} \in \times_i \mathcal{V}_i$ and for any bidding profile $\mathbf{b} \in \times_i \mathcal{B}_i$ we have

$$\sum_{i=1}^{n} u_i^{v_i}(b_i^*, \mathbf{b}_{-i}) \geq \left(1 - \frac{1}{e}\right) \mathrm{OPT}(\mathbf{v}) - \sum_{i=1}^{n} P_i(\mathbf{b}). \tag{2.6}$$

Note that each component in $\mathbf{v} = (v_1, \ldots, v_h)$ is a function from a singleton set and so we can think of v_i as representing the image of this single element. Let $v_h = \max_i v_i$. The above equation can then be rewritten as

$$\sum_{i=1}^{n} u_i^{v_i}(b_i^*, \mathbf{b}_{-i}) \geq \left(1 - \frac{1}{e}\right) v_h - \sum_{i=1}^{n} b_i. \tag{2.7}$$

Consider a distribution with density function $f(x) = \frac{1}{v_h - x}$ and support in $[0, (1-1/e)v_h]$. We then let b_h^* be drawn from this distribution and set $b_i^* = 0$ for all $i \neq h$. Note that the utility for player h if he bids b_k and wins is $u_h^{v_h}(b_k) = v_h - b_k$ (assuming no overbidding). Thus

$$\begin{aligned}
u_h^{v_h}(b_h^*, \mathbf{b}_{-h}) &= \int_0^{\left(1-\frac{1}{e}\right)v_h} u_h^{v_h}(x) f(x)\, dx \\
&\geq \int_{\max_{i \neq h} b_i}^{\left(1-\frac{1}{e}\right)v_h} (v_h - x) f(x)\, dx \\
&= \int_{\max_{i \neq h} b_i}^{\left(1-\frac{1}{e}\right)v_h} dx \\
&\geq \left(1 - \frac{1}{e}\right) v_h - \max_i b_i.
\end{aligned}$$

We also have $u_i^{v_i}(b_i^*) = 0$ for $i \neq h$ and hence

$$\sum_{i=1}^{n} u_i^{v_i}(b_i^*, \mathbf{b}_{-i}) = u_h^{v_h}(b_h^*, \mathbf{b}_{-h}) \geq \left(1 - \frac{1}{e}\right) v_h - \max_i b_i \geq \left(1 - \frac{1}{e}\right) v_h - \sum_{i=1}^{n} b_i \tag{2.8}$$

as desired. □

Corollary 2.5. *The price of anarchy for simultaneous first-price auctions with submodular valuations over mixed Nash and Bayesian-Nash equilibria is at most $e/(e-1)$.*

Proof. This immediately follows by applying the previous theorem in combination with Theorem 2.2 and Theorem 2.3 while noting that the class of mixed Nash equilibria is a subclass of Bayesian-Nash equilibria so that an upper bound for Bayesian-Nash equilibria is an upper bound for mixed Nash equilibria. Hence $PoA \leq \frac{e}{e-1}$. □

To get an upper bound in the case of subadditive valuations we give an overview of the approach taken in [FFGL13]. First some notation. Consider a bidding profile $\mathbf{b} = (b_1, \ldots, b_n)$ where the bid of Player i on item j is given by $b_i(j)$ (for a subset $S \subset [m]$ we shall also use the notation $b_i(S)$ instead of $\sum_{j\in S} b_i(j)$). We then let the **price vector perceived by** Player i be defined as the additive function $p_i : 2^{[m]} \to \mathbb{R}_{\geq 0}$ whose value on j is $\max_{k\neq i} b_k(j)$, i.e., the highest bid placed by the other bidders on each item. We use the notation $v_i(b_i, p_i)$ to denote the valuation of Player i on the allocation which results from bid b_i by Player i and prices p_i. Note that if the bids are randomized, then the price vector perceived by Player i, p_i, follows some distribution, call it $\mathcal{D}_i$.

Lemma 2.6 ([FFGL13])**.** *For any distribution of prices $\mathcal{D}$ and any subadditive valuation $v : \mathcal{X} \to \mathbb{R}_{\geq 0}$ there exists a bid b_0 such that*

$$\mathbb{E}_{p\sim\mathcal{D}}[v(b_0, p)] - \sum_{j=1}^{m} b_j(m) \geq \frac{1}{2}v([m]) - \mathbb{E}_{p\sim\mathcal{D}}[p([m])] \tag{2.9}$$

Proof due to [FFGL13]. Let $b \sim \mathcal{D}$ be a bidding profile drawn from the same distribution as the prices. We then have that

$$\begin{aligned}
\mathbb{E}_{b\sim\mathcal{D}}[\mathbb{E}_{p\sim\mathcal{D}}[v(p,b)]] &= \mathbb{E}_{p\sim\mathcal{D}}[\mathbb{E}_{b\sim\mathcal{D}}[v(p,b)]] \\
&= \frac{1}{2}\mathbb{E}_{p\sim\mathcal{D}}[\mathbb{E}_{b\sim\mathcal{D}}[v(p,b) + v(p,b)]] \\
&\geq \frac{1}{2}\mathbb{E}_{p\sim\mathcal{D}}[\mathbb{E}_{b\sim\mathcal{D}}[v([m])]] \\
&= \frac{1}{2}v([m])
\end{aligned}$$

where the inequality follows from the subadditivity of v. It thus follows that

$$\begin{aligned}
\mathbb{E}_{b\sim\mathcal{D}}[\mathbb{E}_{p\sim\mathcal{D}}[v(p,b)] - b([m])] &\geq \frac{1}{2}v([m]) - \mathbb{E}_{b\sim\mathcal{D}}[b([m])] \\
&= \frac{1}{2}v([m]) - \mathbb{E}_{p\sim\mathcal{D}}[p([m])].
\end{aligned}$$

A mean value argument then implies that there must exist a bid b_0 such that Equation 2.9 is satisfied. □

Theorem 2.7. *The price of anarchy for simultaneous first-price auctions over Bayes-Nash equilibria where bidders have subadditive valuations is at most 2.*

Proof due to [FFGL13]. The plan is to consider a fixed Bayes-Nash equilibrium and create a deviating strategy which we can use to get an upper bound as the deviating strategy must necessarily be worse.

To this end we first fix some notation. Let $\mathcal{F} = \times_i \mathcal{F}_i$ be the distribution over valuations and let $\mathbf{s}$ be a Bayes-Nash equilibrium. Fix a Player i and an arbitrary valuation v_i drawn from $\mathcal{F}_i$. Let $\mathbf{v}_{-i}$ be a valuation drawn from $\mathcal{F}_{-i}$ and set $\mathbf{v} = (v_i, \mathbf{v}_{-i})$. Then, draw another valuation profile $\mathbf{v}^*_{-i}$ from $\mathcal{F}_{-i}$ and set $\mathbf{v}^* = (v_i, \mathbf{v}^*_{-i})$. Let x^* be the corresponding optimal allocation for $\mathbf{v}^*$ which maximizes the social welfare. Consider the price vector perceived by Player i, given by $\tilde{p}_i$ and define a new price vector p_i which is equal to $\tilde{p}_i$ on the elements in x^*_i but zero elsewhere, i.e., p_i is the price vector only on those items which Player i should have in an optimal allocation. Let $\mathcal{D}_i$ be the corresponding distribution on prices $p_i = p_i(\mathbf{b}_{-i})$, where $\mathbf{b} \sim \mathbf{s}(\mathbf{v})$, meaning that $\mathcal{D}_i$ is precisely the distribution over maximum bids on the elements in x^*_i.

Using the previous lemma we have that there exists a bid vector b'_i over the items in x^*_i such that

$$\mathbb{E}_{p_i\sim\mathcal{D}_i}[v_i(b'_i, p_i)] - b'_i(x^*_i) \geq \frac{1}{2}v_i(x^*_i) - \mathbb{E}_{p_i\sim\mathcal{D}_i}[p(x^*_i)]. \tag{2.10}$$

Then, as $\mathbf{s}$ is a Bayes-Nash equilibrium, we must have that

$$\begin{aligned}
\mathbb{E}_{\mathbf{v}_{-i},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}[u_i^{v_i}(\mathbf{b})] &\geq \mathbb{E}_{\mathbf{v}_{-i},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}[u_i^{v_i}(b'_i, \mathbf{b}_{-i})] \\
&= \mathbb{E}_{\mathbf{v}_{-i},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}[v_i(b'_i, \tilde{p}_i)] - \mathbb{E}_{\mathbf{v}_{-i},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}[b'_i(X_i(b'_i, \mathbf{b}_{-i}))] \\
&\geq \mathbb{E}_{p_i\sim\mathcal{D}_i}[v_i(b'_i, p_i)] - b'_i(x^*_i),
\end{aligned}$$

where the last inequality follows from the definition of $\mathcal{D}_i$ and the fact that $X_i(b'_i, \mathbf{b}_{-i}) \subset x_i^*$ for all $\mathbf{b}_{-i}$. Using Equation 2.10 and the definition of $p_i \sim \mathcal{D}$ we then have that

$$\mathbb{E}_{\mathbf{v}_{-i},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}\left[u_i^{v_i}(\mathbf{s}(\mathbf{v}))\right] \geq \frac{1}{2}v_i(x_i^*) - \mathbb{E}_{\mathbf{v}_{-i},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}\left[\sum_{j\in x_i^*}\max_{k\neq i} b_k(j)\right]. \tag{2.11}$$

Taking the sum over all $v_i \sim \mathcal{F}_i$ and all $\mathbf{v}^*_{-i} \sim \mathcal{F}_{-i}$ we have

$$\sum_{i=1}^n \mathbb{E}_{\mathbf{v}_{-i},\mathbf{v}^*_{-i},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}[u_i^{v_i}(\mathbf{b})] \geq \frac{1}{2}\sum_{i=1}^n \mathbb{E}_{v_i,\mathbf{v}^*_{-i}}[v_i(x_i^*)] - \sum_{i=1}^n \mathbb{E}_{\mathbf{v},\mathbf{v}^*_{-i},\mathbf{b}_{-i}\sim\mathbf{s}_{-i}(\mathbf{v}_{-i})}\left[\sum_{j\in x_i^*}\max_{k\neq i} b_k(j)\right]. \tag{2.12}$$

Simplifying the above we get that

$$\mathbb{E}_{\mathbf{v},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}\left[\sum_{i=1}^n u_i^{v_i}(\mathbf{v})\right] \geq \frac{1}{2}\mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^n v_i(x_i^*)\right] - \mathbb{E}_{\mathbf{v},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}\left[\sum_{j=1}^m \max_k b_k(j)\right]. \tag{2.13}$$

Then, as we are in the first-price auction (so that prices are the highest bids) we have that

$$\mathbb{E}_{\mathbf{v},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}\left[\sum_{i=1}^n u_i^{v_i}(\mathbf{v})\right] = \mathbb{E}_{\mathbf{v},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}\left[\sum_{i=1}^n v_i(X_i(\mathbf{b}))\right] - \mathbb{E}_{\mathbf{v},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}\left[\sum_{j=1}^m \max_k b_k(j)\right]. \tag{2.14}$$

Combining this with Equation 2.13 we then get that

$$\mathbb{E}_{\mathbf{v},\mathbf{b}\sim\mathbf{s}(\mathbf{v})}\left[\sum_{i=1}^n v_i(X_i(\mathbf{b}))\right] \geq \frac{1}{2}\mathbb{E}_{\mathbf{v}}\left[\sum_{i=1}^n v_i(x_i^*)\right] \tag{2.15}$$

which implies the desired result. □

2.3 Lower bounds

The lower bound on the price of anarchy for simultaneous first-price auctions with OXS valuation (hence submodular) is the same as the upper bound.

Theorem 2.8 ([CKST16])**.** *The price of anarchy of simultaneous first-price auctions over mixed Nash equilibria and OXS valuation is at least $e/(e-1)$.*

Proof. We follow the approach taken in [CKST16, Theorem 3.4].

As we are interested in a lower bound it suffices to construct a simultaneous first-price auction instance where the PoA is arbitrarily close to $e/(e-1)$. Let $\mathcal{M}$ be a simultaneous first-price auction with $n+1$ players and n^n items, and let $\mathcal{I} = [n]^n$ denote the set of items.

Player 0 is a dummy player and has valuation $v_0(S) = 0$ for all $S \subset \mathcal{I}$. For $1 \leq i \leq n$ we associate Player i with one of the dimensions in the cube as follows: given $S \subset \mathcal{I}$ the valuation $v_i(S)$ is the cardinality of S projected along dimension i which more formally can be written as

$$v_i(S) = |\{w_{-i} \mid \text{ there exists } w_i \text{ such that } (w_i, w_{-i}) \in S\}|. \tag{2.16}$$

It is straightforward to check that these valuations are submodular. To give a lower bound on the price of anarchy we need to find a mixed Nash equilibrium $b^* = (b_0^*, b_1^*, \dots, b_n^*)$ such that the PoA is arbitrarily close to $e/(e-1)$. We let the strategy of Player 0 be to always bid 0. On the other hand, we let Player i pick a number $\ell \in [n]$ uniformly at random and an x according to the CDF provided by

$$G(x) = (n-1)\left(\frac{1}{(1-x)^{\frac{1}{n-1}}} - 1\right), \qquad x \in \left[0, 1 - \left(\frac{n-1}{n}\right)^{n-1}\right] \tag{2.17}$$

and then bids x for every item $w = (\ell, w_{-i})$, with $w_i = \ell$ and 0 for the rest. Tie-breaking is resolved by giving it to Player 0—which can only happen in case of 0 bids for an item. Fix a player i and an item j, we then let $F_{ij}(x)$ denote the probability that player i gets j given that he bids $b_i(j) = x$ and the rest of the bids b_{-i} are drawn from b^*_{-i}. For all other players $k \neq i$ the probability that Player k bids 0 for item

j is $(n-1)/n$, and the probability that j is in the slice Player k bids on, but for which his bid is lower than x, is $1/n \cdot G(x) = G(x)/n$. From this it follows that

$$F_{ij}(x) = \left(\frac{G(x)}{n} + \frac{n-1}{n}\right)^{n-1} = \frac{\left(\frac{n-1}{n}\right)^{n-1}}{1-x}, \qquad \forall i \in [n], \forall j \in \mathcal{I}. \tag{2.18}$$

Since F_{ij} is invariant of i and j we can denote it with a common distribution F. Notice that v_i restricted to the slice $\{(\ell, w_{-i}) \mid w \in \mathcal{I}, \ell \in [n]\}$ is additive. Hence, the utility for Player i of getting item j is $F(x)(1-x)$ and so the utility of getting all the items in the slice he bids on is $u_i(b_i^*) = n^{n-1} \cdot F(x)(1-x) = n^{n-1}\left(\frac{n-1}{n}\right)^{n-1}$.

We now show that b^* indeed is a mixed Nash equilibrium. To do this we first fix an arbitrary $w \in \mathcal{I}$ from which we define the column $C = \{(\ell, w_{-i}) \mid \ell \in [n]\}$ for Player i. Now, Player i has only bid on one item in this column, while his valuation is additive over items from different columns (follows from being additive over slices). In addition, in a fixed bidding profile b_{-i}, every other player k bids the same on each item in the column as either the column is in the slice of that player or it is not (follows from how the slice and column is constructed). Consider therefore a deviating bid for player i in which i bids a positive value for more than one item in C, i.e., at least $x \geq x' > 0$ where x is the highest bid for items in C. The expected utility for this column is then strictly less than $F(x)(1-x)$ as the value of the two items is only 1 while he might have to pay at least $x + x'$. It is therefore more optimal to bid x for only one item in C and 0 for the rest.

Secondly, if we restrict to any column C, submitting a bid $x \in \left[0, 1 - \left(\frac{n-1}{n}\right)^{n-1}\right]$ for one item (chosen arbitrarily), results in the utility $\left(\frac{n-1}{n}\right)^{n-1}$. Submitting a higher bid than $1 - \left(\frac{n-1}{n}\right)^{n-1}$ guarantees that particular item, but the utility is then strictly less than $\left(\frac{n-1}{n}\right)^{n-1}$. So, both bidding on more than one item in a column and bidding higher than $1 - \left(\frac{n-1}{n}\right)^{n-1}$ is a worse deviation compared to just bidding on one item according to $G(x)$ and hence b^* is a mixed Nash equilibrium.

To compute the social welfare of b^* we first introduce random variables Z_j where $Z_j = 1$ if one of the players $1 \leq i \leq n$ gets the item j and $Z_j = 0$ otherwise (Player 0 gets the item). By additivity along the slices we then have that the social welfare is $\sum_{j \in \mathcal{I}} Z_j$ and hence the expected social welfare is

$$SW^{\mathbf{v}}(b^*) = \sum_j \mathbb{E}\left[Z_j\right] = n^n \left(1 - \left(\frac{n-1}{n}\right)^n\right). \tag{2.19}$$

The maximum possible social welfare is n^n where all items are distributed among the real players in such a way that no two items allocated to one player is in the same column. This is possible to do by allocating item $(w_1, \ldots, w_1)$ to Player $(\sum_{i=1}^n w_i \bmod n) + 1$. The price of anarchy is therefore at least $\frac{1}{1-\left(\frac{n-1}{n}\right)^n}$ which converges to $e/(e-1)$ for large n. □

Theorem 2.9 ([CKST16]). *The price of anarchy over mixed Nash equilibria for simultaneous first-price with subadditive valuations is at least* 2.

Proof. We follow the approach taken in [CKST16, Theorem 4.1], but relabel players to be consistent with the AND-OR game in [HKMN11] and because not doing this leads to minor errors.

Our goal is to create scenario with a price of anarchy that is arbitrarily close to 2. To this end, consider a scenario with $n = 2$ players and m items, and the following valuations:

$$v_1(S) = \begin{cases} 1, & \text{if } 1 \leq |S| < m \\ 2, & \text{if } |S| = m \\ 0, & \text{else,} \end{cases}$$

$$v_2(S) = \begin{cases} v, & \text{if } 1 \leq |S| \\ 0, & \text{else} \end{cases}$$

where $v < 1$ is to be determined later. That v_1 and v_2 are subadditive is straightforward to check. We use the following cumulative distribution for the bids of Player 1 and Player 2 respectively

$$F(y) = \frac{v - 1/m}{v - y}, \quad y \in [0, 1/m]; \qquad G(x) = \frac{(m-1)x}{1-x}, \quad x \in [0, 1/m]. \tag{2.20}$$

The bidding strategy of Player 1 is to bid y for each of the items. On the other hand the bidding strategy of Player 2 is to pick one of the m items uniformly at random, and bid x for this item and 0 for all other.

In case of a tie, the item is allocated to Player 1. Let $b^* = (b_1^*, b_2^*)$ denote this randomized bidding profile. Our aim is to show that b^* is a mixed Nash equilibrium for all $v > 1/m$.

If Player 1 bids a common bid $y \in [0, 1/m]$ for all items then he gets m with probability $G(y)$ and $m-1$ items with probability $1 - G(y)$. His expected utility is thus $G(y)(2 - my) + (1 - G(y))(1 - (m-1)y) = G(y)(1-y) + 1 - (m-1)y = 1$. We must then show that Player 1 cannot get a higher utility by using a deviating strategy. Let y_i be the bid for item i that Player 2 makes for $1 \leq i \leq m$. Player 2 bids on item i with probability $1/m$ according to $G(x)$. Now, as G is a CDF it follows that $G(x) = 1 < \frac{(m-1)x}{1-x}$ for $x > 1/m$. With $y^* = (y_1, \ldots, y_n)$, the expected utility of Player 1 is therefore

$$
\begin{aligned}
u_1^{v_1}(y^*) =& \frac{1}{m} \sum_{i=1}^{m} \left(G(y_i) \left(2 - \sum_{j=1}^{m} y_j \right) + (1 - G(y_i)) \left(1 - \sum_{j \neq i} y_j \right) \right) \\
=& \frac{1}{m} \sum_{i=1}^{m} \left(G(y_i)(1 - y_i) + 1 - \sum_{j \neq i} y_j \right) \\
\leq& \frac{1}{m} \sum_{i=1}^{m} \left(\frac{(m-1)y_i}{1 - y_i}(1 - y_i) + 1 - \sum_{i \neq j} y_j \right) \\
=& \frac{1}{m} \sum_{i=1}^{m} \left(my_i + 1 - \sum_{j=1}^{m} y_j \right) \\
=& \frac{1}{m} \left(m \sum_{i=1}^{m} y_i + m - m \sum_{j=1}^{m} y_j \right) = 1.
\end{aligned}
$$

Similarly, if Player 2 bids any $x \in (0, 1/m]$ for the one item, he gets the item with probability $F(x)$. His expected utility is thus $F(x)(v - x) = v - 1/m$. Bidding something greater than $1/m$ results in a utility less than $v - 1/m$. What remains to show for Player 2 is that his utility while bidding for one item is at least his utility while bidding for more items. Suppose therefore that Player 2 bids x_i for $1 \leq i \leq m$. We can without loss of generality assume that $x_i \geq x_{i+1}$ for $1 \leq i \leq m-1$ (just reorder the items). It then follows that Player 2 gets no item if and only if $y \geq x_1$. So with probability $F(x_1)$, he gets at least one item and pays at least x_1. Consequently, his expected utility is at most $F(x_1)(v - x_1) = v - 1/m$. We thus see that b^* is a mixed Nash equilibrium.

The optimal allocation is to give all of the items to Player 1—this follows since $v < 1$. This allocation has social welfare 2. Now in the mixed Nash equilibrium b^*, Player 1 bids 0 with probability $1 - 1/mv$, and Player 2 thus gets at least one item with at least this probability. We thus have

$$
SW^{(v_1,v_2)}(b^*) \leq \frac{1}{mv} 2 + \left(1 - \frac{1}{mv}\right)(v+1) = 1 + v + \frac{1}{mv} - \frac{1}{m}. \tag{2.21}
$$

Setting $v = 1/\sqrt{m}$ we have $SW^{(v_1,v_2)}(b^*) \leq 1 + 2/\sqrt{m} - 1/m$ and hence $PoA \geq \frac{2}{1 + \frac{2}{\sqrt{m}} - \frac{1}{m}}$ which converges to 2 for large m. $\square$

3 Overview of PoA for Second-Price Auctions

We end our mini project by turning our attention to second-price auctions and give a short overview of the current known upper and lower bounds for second-price auctions. An overview of all the bounds (both first-price and second-price) can be found in Table 1. Before giving the bounds for simultaneous second-price auctions we formalize the notion of no-overbidding.

Definition 3.1 (Strongly no-overbidding, [FFGL13]). *Given a bidding profile* $\mathbf{b} = (b_1, \ldots, b_n)$, *we say that it is* ***strongly no-overbidding*** *if for all* $i \in [n]$ *and any* $S \subset [m]$ *we have* $b_i(S) \leq v_i(S)$.

Definition 3.2 (Weakly no-overbidding, [FFGL13]). *Given a price distribution* $\mathcal{D}_i$, *a bidder is said to be* ***weakly no-overbidding*** *if his bid vector* b_i *satisfies* $\mathbb{E}_{p_i \sim \mathcal{D}_i}[v_i(X(b_i, p_i))] \geq \mathbb{E}_{p_i \sim \mathcal{D}_i}[b_i(X_i(b_i, p_i))]$ *where* $X_i(b_i, p_i)$ *denotes the allocation of Player* i *given that he bids* b_i *against a perceived price profile* p_i.

Auction Type	Equilibrium type	Valuation	No-Overbidding	PoA Upper Bound	PoA Lower Bound
First-Price	Mixed Nash	Subadditive	Strong	2	2
First-Price	Bayesian-Nash	Subadditive	Strong	2	2
First-Price	Mixed Nash	Submodular	Strong	$\frac{e}{e-1}$	$\frac{e}{e-1}$
First-Price	Bayesian-Nash	Submodular	Strong	$\frac{e}{e-1}$	$\frac{e}{e-1}$
Second-Price	Bayesian-Nash	Subadditive	Strong & Weak	4	> 2.061
Second-Price	Bayesian-Nash	Submodular	Strong	2	Unknown

Table 1: A summary of the PoA for first-price and second-price simultaneous auctions over different valuation classes and equilibrium types.

3.1 Upper and lower bounds

The current best known upper bound for subadditive valuations is given in [FFGL13].

Theorem 3.3 ([FFGL13])**.** *In simultaneous second-price auctions where bidders have subadditive valuations, and every bidder is either strongly or weakly no-overbidding, the price of anarchy over Bayesian-Nash equilibria is at most 4.*

They also show the following best known lower bound:

Theorem 3.4 ([FFGL13])**.** *In simultaneous second-price auctions where bidders have subadditive valuations, and every bidder is weakly no-overbidding, the price of anarchy over Bayesian-Nash equilibria is at least* $4/1.94 > 2.061$*.*

For submodular valuations we have the following upper bound given in [CKS16]:

Theorem 3.5 ([CKS16])**.** *In simultaneous second-price auctions where bidders have submodular valuations, and every bidder is strongly no-overbidding, the price of anarchy over Bayesian-Nash equilibria is at most 2.*

They do not provide results for the case of the weakly no-overbidding condition. As mentioned by Feldman et al. in [FFGL13] the results for the strongly no-overbidding situation do not directly carry over to the weakly no-overbidding situation. A potential area of investigation could therefore be to study upper bounds for the PoA with submodular valuations under the weakly no-overbidding assumption.

For pure Nash equilibria, Christodoulou et al. [CKS16] showed the following related result.

Theorem 3.6 ([CKS16])**.** *If the valuation functions are submodular, then a pure Nash equilibrium that has PoA equal to 2 can be computed in polynomial time.*

It is currently not known if such an algorithm exists for Bayesian-Nash equilibria. Another area of investigation is therefore naturally to either try to prove or disprove such an algorithm.

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