7

Lie Algebra Homology and Cohomology

Lie algebras were introduced by Sophus Lie in connection with his studies of Lie groups; Lie groups are not only groups but also smooth manifolds, the group operations being smooth. If G is a Lie group, the tangent space $\mathfrak{g}$ of G at the identity $e \in G$ is a Lie algebra over $\mathbb{R}$. The vector space of left invariant vector fields on G is canonically isomorphic to $\mathfrak{g}$, and the Lie bracket $[X, Y]$ of vector fields X and Y may be defined as a vector field:

$$[X, Y]f = X(Yf) - Y(Xf), \quad f \text{ a smooth function on } G.$$

This rich interplay with Differential Geometry forms the original motivation for the abstract study of Lie algebras. More history is given in 7.8.14 below.

7.1 Lie Algebras

Let k be a fixed commutative ring. A *nonassociative algebra* A is a k-module equipped with a bilinear product $A \otimes_k A \to A$. Note that we do not assume the existence of a unit, so that 0 is the smallest possible nonassociative algebra. A *Lie algebra* $\mathfrak{g}$ is a nonassociative algebra whose product, written as $[xy]$ or $[x, y]$ and called the *Lie bracket*, satisfies (for $x, y, z \in \mathfrak{g}$):

Skew-symmetry: $[x, x] = 0$ (and hence $[x, y] = -[y, x]$);
Jacobi's Identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

An *ideal* of $\mathfrak{g}$ is a k-submodule $\mathfrak{h}$ such that $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$, that is, for all $g \in \mathfrak{g}$ and $h \in \mathfrak{h}$ we have $[g, h] \in \mathfrak{h}$. Note that an ideal is a Lie algebra in its own right, and that the quotient $\mathfrak{g}/\mathfrak{h}$ inherits the structure of a Lie algebra as well. There is a category whose objects are (k-)Lie algebras; a morphism $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is a product-preserving k-module homomorphism. Thus every ideal $\mathfrak{h} \subset \mathfrak{g}$ yields a short exact sequence (!) of Lie algebras:

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0.$$

Example 7.1.1 An *abelian* Lie algebra is one in which all the Lie brackets $[x, y] = 0$. Every k-module has the structure of an abelian Lie algebra.

If $\mathfrak{g}$ is any Lie algebra, define $[\mathfrak{g}, \mathfrak{g}]$ to be the k-submodule of $\mathfrak{g}$ generated by all Lie brackets $[x, y]$ with $x, y \in \mathfrak{g}$. Then $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$, and the quotient $\mathfrak{g}^{ab} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is an abelian Lie algebra. Obviously, $\mathfrak{g}^{ab}$ is the largest quotient Lie algebra of $\mathfrak{g}$ that is abelian.

Example 7.1.2 The primordial Lie algebra is the Lie algebra $\mathfrak{a} = \mathrm{Lie}(A)$ of an associative k-algebra A (even if A is an algebra without a unit). This is the underlying k-module A, given the commutator product $[x, y] = xy - yx$. We leave it to the reader (exercise!) to verify Jacobi's identify, that is, that $\mathfrak{a}$ is a Lie algebra, and to check that this defines a functor "Lie" from the category of (associative, possibly nonunital) k-algebras to the category of Lie algebras.

Examples 7.1.3 If A is an associative k-algebra, so is $M_m(A)$, the $m \times m$ matrices with coefficients in A. We write $\mathfrak{gl}_m(A)$ for the Lie algebra $\mathrm{Lie}(M_m(A))$. If $A = k$, we write $\mathfrak{gl}_m$ for $\mathfrak{gl}_m(k)$.

Here are some famous Lie subalgebras of $\mathfrak{gl}_m(A)$; if $A = k$, it is traditional to drop the reference to A, writing merely, $\mathfrak{o}_m$, $\mathfrak{sl}_m$, $\mathfrak{t}_m$, and $\mathfrak{n}_m$ instead of $\mathfrak{o}_m(k)$, $\mathfrak{sl}_m(k)$, and so on.

1. The *orthogonal algebra* $\mathfrak{o}_m(A)$ of all skew-symmetric matrices: $\{g : g_{ij} = -g_{ji}\}$.
2. The *special linear algebra* $\mathfrak{sl}_m(A)$. If A is commutative, this is the algebra of all matrices of trace 0. If A is not commutative, then we must consider the trace as taking values in $A/[A, A]$, because a matrix change of basis changes the trace $\sum g_{ii}$ by an element of $[A, A]$. Thus $\mathfrak{sl}_m(A)$ is the kernel of the trace map, yielding the short exact sequence of Lie algebras:

$$0 \to \mathfrak{sl}_m(A) \to \mathfrak{gl}_m(A) \xrightarrow{\text{trace}} A/[A, A] \to 0.$$

3. The *upper triangular matrices* $\mathfrak{t}_m(A) : \{g : g_{ij} = 0 \text{ if } i < j\}$.
4. The *strictly upper triangular matrices* $\mathfrak{n}_m(A) : \{g : g_{ij} = 0 \text{ if } i \leq j\}$.

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Example 7.1.4 (Derivation algebras) Let A be a nonassociative (= not necessarily associative) k-algebra. A *derivation* D of A (into itself) is a k-module endomorphism of A such that the Leibnitz formula holds

$$D(ab) = (Da)b + a(Db) \quad (a, b \in A).$$

The set $\mathrm{Der}(A)$ of derivations of A is clearly a k-submodule of $\mathrm{End}_k(A)$. Moreover, the commutator $[D_1, D_2]$ of two derivations is a derivation, since

$$\begin{aligned}[D_1 D_2]ab &= D_1(D_2(ab)) - D_2(D_1(ab)) \\ &= D_1((D_2a)b) + D_1(a(D_2b)) - D_2((D_1a)b) - D_2(a(D_1b)) \\ &= (D_1D_2a)b + a(D_1D_2b) - (D_2D_1a)b - a(D_2D_1b) \\ &= ([D_1D_2]a)b + a([D_1D_2]b).\end{aligned}$$

Hence $\mathrm{Der}(A)$ is a Lie algebra; it is called the *derivation algebra* of A.

Example 7.1.5 Given a k-module M, the *free Lie algebra on* M is a Lie algebra $\mathfrak{f}(M)$, containing M as a submodule, which satisfies the usual universal property: Every k-module map $M \to \mathfrak{g}$ into a Lie algebra extends uniquely to a Lie algebra map $\mathfrak{f}(M) \to \mathfrak{g}$. In other words, as a functor $\mathfrak{f}$ is left adjoint to the forgetful functor from Lie algebras to modules

$$\mathrm{Hom}_{k\text{-}\mathbf{mod}}(M, \mathfrak{g}) \cong \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{f}(M), \mathfrak{g}).$$

The existence of $\mathfrak{f}(M)$ follows from general considerations of category theory (the Adjoint Functor Theorem); a concrete construction will be given in section 7.3. Clearly $\mathfrak{f}(M)$ is unique up to isomorphism.

If X is a set, the *free Lie algebra on* X is $\mathfrak{f}(M)$, where M is the free k-module on the set X. Clearly

$$\mathrm{Hom}_{\mathbf{Sets}}(X, \mathfrak{g}) \cong \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{f}(X), \mathfrak{g}),$$

so there is a corresponding universal property for $\mathfrak{f}(X)$.

Exercise 7.1.1 Show that the free Lie algebra $\mathfrak{f}(\{x\}) = \mathfrak{f}(k)$ on the set $\{x\}$ is the 1-dimensional abelian Lie algebra k. Then show that $\mathfrak{f}(\{x, y\})$ is a graded, free k-module having an infinite basis of monomials

$$x, y, [xy], [x[xy]], [y[xy]], [x[x[xy]]], [x[y[xy]]], [y[y[xy]]], \cdots.$$

(There are 6 monomials of degree 5. In general, there are $\frac{1}{d}\sum_{i|d}\mu(i)2^{d/i}$ monomials of degree d, where μ denotes the Möbius function [Bour, ch. 2, sec. 3.3, thm. 2].)

Exercise 7.1.2 (Product Lie algebra) If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, we can make the k-module $\mathfrak{g} \times \mathfrak{h}$ into a Lie algebra by a slotwise product: $[(g_1, h_1), (g_2, h_2)] = ([g_1, g_2], [h_1, h_2])$. Show that $\mathfrak{g} \times \mathfrak{h}$ is the product in the category of Lie algebras.

Nilpotent Lie Algebras 7.1.6 In analogy with group theory, we define the *lower central series* of a Lie algebra $\mathfrak{g}$ to be the following descending sequence of ideals:

$$\mathfrak{g} \supseteq \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}] \supseteq \mathfrak{g}^3 = [\mathfrak{g}^2, \mathfrak{g}] \supseteq \cdots \supseteq \mathfrak{g}^n = [\mathfrak{g}^{n-1}, \mathfrak{g}] \supseteq \cdots.$$

We say that $\mathfrak{g}$ is a *nilpotent* Lie algebra if $\mathfrak{g}^n = 0$ for some n. For example, the strictly upper triangular Lie algebra $\mathfrak{n}_m(A)$ is nilpotent for every k-algebra A; $\mathfrak{n}_m(A)^n$ is the ideal of matrices (g_{ij}) with $g_{ij} = 0$ unless $i \geq j + n$. Abelian Lie algebras are another obvious class of nilpotent Lie algebras.

Solvable Lie Algebras 7.1.7 Again following group theory, we define the *derived series* of $\mathfrak{g}$ to be the descending sequence of ideals

$$\mathfrak{g} \supseteq \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] \supseteq \mathfrak{g}'' = (\mathfrak{g}')' \supseteq \cdots \supseteq \mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}] \supseteq \cdots.$$

We say that $\mathfrak{g}$ is a *solvable* Lie algebra if $\mathfrak{g}^{(n)} = 0$ for some n.

Lemma 7.1.8 *Every nilpotent Lie algebra is solvable.*

Proof It suffices to show that $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$, for then by induction we see that $\mathfrak{g}^{(n)} \subseteq \mathfrak{g}^n$. To see this we proceed by induction on j, the case $j = 1$ being the definition $\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}]$. Inductively, we compute

$$\begin{aligned}[\mathfrak{g}^i, \mathfrak{g}^{j+1}] = [\mathfrak{g}^i, [\mathfrak{g}^j, \mathfrak{g}]] &\subseteq [[\mathfrak{g}^i, \mathfrak{g}], \mathfrak{g}^j] + [[\mathfrak{g}^i, \mathfrak{g}^j], \mathfrak{g}] \\ &\subseteq [\mathfrak{g}^{i+1}, \mathfrak{g}^j] + [\mathfrak{g}^{i+j}, \mathfrak{g}] = \mathfrak{g}^{i+j+1}. \end{aligned}$$ ◇

Example 7.1.9 The upper triangular Lie algebra $\mathfrak{t}_m(A)$ of a commutative k-algebra A is solvable but not nilpotent.

7.2 $\mathfrak{g}$-Modules

Let $\mathfrak{g}$ be a Lie algebra over k. A (left) $\mathfrak{g}$-*module* M is a k-module equipped with a k-bilinear product $\mathfrak{g} \otimes_k M \to M$ (written $x \otimes m \mapsto xm$) such that

$$[x, y]m = x(ym) - y(xm) \text{ for all } x, y \in \mathfrak{g} \text{ and } m \in M.$$

Examples 7.2.1

1. If A is an associative algebra and $\mathfrak{g} = \text{Lie}(A)$, any left A-module may be thought of as a left $\mathfrak{g}$-module in an obvious way.
2. The Lie bracket makes $\mathfrak{g}$ itself into a left $\mathfrak{g}$-module (by Jacobi's identity). This module is usually called the *adjoint representation* of $\mathfrak{g}$.
3. A *trivial* $\mathfrak{g}$*-module* is a k-module M on which $\mathfrak{g}$ acts as zero: $xm = 0$ for all $x \in \mathfrak{g}$, $m \in M$.

A $\mathfrak{g}$*-module homomorphism* $f: M \to N$ is a k-module map that is product-preserving, that is, $f(xm) = xf(m)$. We write $\text{Hom}_{\mathfrak{g}}(M, N)$ for the set of all such $\mathfrak{g}$-module homomorphisms. If $\alpha \in k$, then αf is also a $\mathfrak{g}$-module map, so therefore $\text{Hom}_{\mathfrak{g}}(M, N)$ is a k-submodule of $\text{Hom}_k(M, N)$.

The left $\mathfrak{g}$-modules and $\mathfrak{g}$-module homomorphisms form a category called $\mathfrak{g}$–**mod**. By the above remarks, it is an additive category. The following exercise shows that it is in fact an abelian category.

Exercise 7.2.1

1. Let $f: M \to N$ be a $\mathfrak{g}$-module homomorphism. Show that the k-modules $\ker(f)$, $\text{im}(f)$, and $\text{coker}(f)$ are the kernel, image, and cokernel of f in $\mathfrak{g}$–**mod**.
2. Show that a monic (resp., epi) in $\mathfrak{g}$–**mod** is also a monic (resp., epi) in k–**mod**. By (1), this proves that $\mathfrak{g}$–**mod** is an abelian category.

Exercise 7.2.2 Let $E = \text{End}_k(M)$ be the associative algebra of k-module endomorphisms of a k-module M. Show that maps $\mathfrak{g} \otimes M \to M$ making M into a $\mathfrak{g}$-module are in 1–1 correspondence with Lie algebra homomorphisms $\mathfrak{g} \to \text{Lie}(E)$. Conclude that a $\mathfrak{g}$-module may also be described as a k-module M together with a Lie algebra homomorphism $\mathfrak{g} \to \text{Lie}(\text{End}_k(M))$.

Exercise 7.2.3 There is also a category **mod**–$\mathfrak{g}$ of right $\mathfrak{g}$-modules, whose definition should be obvious. If M is a right $\mathfrak{g}$-module, show that the product $xm = -mx \quad (x \in \mathfrak{g}, m \in M)$ makes M into a left $\mathfrak{g}$-module, and that this induces a natural isomorphism of categories: $\mathfrak{g}$–**mod** $\cong$ **mod**–$\mathfrak{g}$.

Many of the notions we introduced for G-modules in Chapter 6 have analogues for $\mathfrak{g}$-modules. For example, there is a *trivial* $\mathfrak{g}$*-module functor* from k–**mod** to $\mathfrak{g}$–**mod**; it is the exact functor obtained by considering a k-module as a trivial $\mathfrak{g}$-module. Consider the following two functors from $\mathfrak{g}$–**mod** to k–**mod**:

1. The *invariant submodule* $M^{\mathfrak{g}}$ of a $\mathfrak{g}$-module M,

$$M^{\mathfrak{g}} = \{m \in M : xm = 0 \quad \text{for all } \ x \in \mathfrak{g}\}.$$

   Considering k as a trivial $\mathfrak{g}$-module, we have $M^{\mathfrak{g}} \cong \mathrm{Hom}_{\mathfrak{g}}(k, M)$.
2. The *coinvariants* $M_{\mathfrak{g}}$ of a $\mathfrak{g}$-module M, $M_{\mathfrak{g}} = M/\mathfrak{g}M$.

Exercise 7.2.4 Let M be a $\mathfrak{g}$-module.

1. Show that $M^{\mathfrak{g}}$ is the maximal trivial $\mathfrak{g}$-submodule of M, and conclude that $-^{\mathfrak{g}}$ is right adjoint to the trivial $\mathfrak{g}$-module functor. Conclude that $-^{\mathfrak{g}}$ is a left exact functor.
2. Show that $M_{\mathfrak{g}}$ is the largest quotient module of M that is trivial, and conclude that $-_{\mathfrak{g}}$ is left adjoint to the trivial $\mathfrak{g}$-module functor. Conclude that $-_{\mathfrak{g}}$ is a right exact functor.

We will see in the next section that the category $\mathfrak{g}$–**mod** has "enough" projectives and injectives in the sense of Chapter 2. Therefore we can form the derived functors of $-^{\mathfrak{g}}$ and $-_{\mathfrak{g}}$.

Definition 7.2.2 Let M be a $\mathfrak{g}$-module. We write $H_*(\mathfrak{g}, M)$ or $H_*^{\mathrm{Lie}}(\mathfrak{g}, M)$ for the left derived functors $L_*(-_{\mathfrak{g}})(M)$ of $-_{\mathfrak{g}}$ and call them the *homology groups* of $\mathfrak{g}$ *with coefficients in* M. By definition, $H_0(\mathfrak{g}, M) = M_{\mathfrak{g}}$.

Similarly, we write $H^*(\mathfrak{g}, M)$ or $H^*_{\mathrm{Lie}}(\mathfrak{g}, M)$ for the right derived functors $R^*(-^{\mathfrak{g}})(M)$ of $-^{\mathfrak{g}}$ and call them the *cohomology groups* of $\mathfrak{g}$ *with coefficients* in M. By definition, $H^0(\mathfrak{g}, M) = M^{\mathfrak{g}}$.

Examples 7.2.3

0. If $\mathfrak{g} = 0$, $M_{\mathfrak{g}} = M^{\mathfrak{g}} = M$. Since the higher derived functors of an exact functor vanish, $H_*^{\mathrm{Lie}}(0, M) = H^*_{\mathrm{Lie}}(0, M) = 0$ for $* \neq 0$.
1. Let $\mathfrak{g}$ be the free k-module on basis $\{e_1, \cdots, e_n\}$, made into an (abelian) Lie algebra with zero Lie bracket. Since a $\mathfrak{g}$-module is just a k-module with n commuting endomorphisms $e_1, \cdots, e_n$, it follows that $\mathfrak{g}$–**mod** is isomorphic to the category R–**mod** of left modules over the polynomial ring $R = k[e_1, \cdots, e_n]$. If k is the trivial $\mathfrak{g}$-module, considered as an R-module on which the e_i act as zero, then $M_{\mathfrak{g}} = k \otimes_R M$ and $M^{\mathfrak{g}} = \mathrm{Hom}_R(k, M)$. Therefore we have

$$H_*^{\mathrm{Lie}}(\mathfrak{g}, M) = \mathrm{Tor}_*^R(k, M) \text{and} \ \ H^*_{\mathrm{Lie}}(\mathfrak{g}, M) = \mathrm{Ext}_R^*(k, M).$$

   These functors were discussed in Chapter 3.

2. Let $\mathfrak{f}$ be the free Lie algebra on a set X. In this case an $\mathfrak{f}$-module is just a k-module M with an arbitrary set $\{e_x : x \in X\}$ of endomorphisms. That is, the category $\mathfrak{f}$–**mod** is isomorphic to the category R–**mod** of left modules over the free ring $R = k\{X\}$ on the set X. If k denotes the trivial $\mathfrak{f}$-module, then $M_{\mathfrak{f}} = k \otimes_R M$ and $M^{\mathfrak{f}} = \mathrm{Hom}_R(k, M)$. Therefore

$$H_*^{\mathrm{Lie}}(\mathfrak{f}, M) = \mathrm{Tor}_*^R(k, M) \quad \text{and} \quad H_{\mathrm{Lie}}^*(\mathfrak{f}, M) = \mathrm{Ext}_R^*(k, M).$$

We end this section with a calculation of the H^* and H_* groups for $\mathfrak{f}$.

Proposition 7.2.4 *The ideal $\mathfrak{I} = Xk\{X\}$ of the free ring $k\{X\}$ is free as a right $k\{X\}$-module with basis the set X. Hence*

$$0 \to \mathfrak{I} \to k\{X\} \to k \to 0$$

is a free resolution of k as a right $k\{X\}$-module.

Proof As a free k-module, $k\{X\}$ has for basis the set $\mathcal{W}$ of words in the elements of the set X, and $\mathfrak{I}$ is a free k-module on basis $\mathcal{W} - \{1\}$. Every element of $\mathcal{W} - \{1\}$ has a unique expression of the form xw with $x \in X$ and $w \in \mathcal{W}$, so $\{xw : x \in X, w \in \mathcal{W}\}$ is another basis for $\mathfrak{I}$ as a k-module. For each $x \in X$ the k-span $xk\{X\}$ of the set $\{xw : w \in \mathcal{W}\}$ is isomorphic to $k\{X\}$, and $\mathfrak{I}$ is the direct sum of the $xk\{X\}$, both as k-modules and as right $k\{X\}$-modules. That is, $\mathfrak{I}$ is a free right $k\{X\}$-module with basis X, as claimed. $\diamond$

Corollary 7.2.5 *If $\mathfrak{f}$ is the free Lie algebra on a set X, then $H_n^{\mathrm{Lie}}(\mathfrak{f}, M) = H^n_{\mathrm{Lie}}(\mathfrak{f}, M) = 0$ for all $n \geq 2$ and all $\mathfrak{f}$-modules M. Moreover $H_0^{\mathrm{Lie}}(\mathfrak{f}, k) = H^0_{\mathrm{Lie}}(\mathfrak{f}, k) = k$, while*

$$H_1^{\mathrm{Lie}}(\mathfrak{f}, k) \cong \bigoplus_{x \in X} k \quad \text{and} \quad H^1_{\mathrm{Lie}}(\mathfrak{f}, k) = \prod_{x \in X} k.$$

Proof Using the given free resolution of k, $H_*^{\mathrm{Lie}}(\mathfrak{f}, M)$ is the homology of the complex $0 \to \mathfrak{I} \otimes_R M \to M \to 0$, and $H^*_{\mathrm{Lie}}(\mathfrak{f}, M)$ is the homology of the complex $0 \to M \to \mathrm{Hom}_{\mathfrak{f}}(\mathfrak{I}, M) \to 0$. For $M = k$, the differentials are zero. $\diamond$

Exercise 7.2.5 Let $\mathfrak{r}$ be an ideal of a free Lie algebra $\mathfrak{f}$ on a set X. Show that if $\mathfrak{r} \neq 0$, then $[\mathfrak{f}, \mathfrak{r}] \neq \mathfrak{r}$.

7.3 Universal Enveloping Algebras

The universal enveloping algebra $U\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ plays the same formal role as the group ring $\mathbb{Z}G$ of a group G does. In particular, $\mathfrak{g}$–**mod** is naturally isomorphic to the category $U\mathfrak{g}$–**mod** of left $U\mathfrak{g}$-modules. This isomorphism provides an easy proof that $\mathfrak{g}$–**mod** has enough projectives and injectives in the sense of Chapter 2, so that the derived functor definitions of $H_*(\mathfrak{g}, M)$ and $H^*(\mathfrak{g}, M)$ make sense.

In this section we will develop some of the ring-theoretic properties of $U\mathfrak{g}$. Since $U\mathfrak{g}$ will be a quotient ring of the tensor algebra $T(\mathfrak{g})$, we first describe the tensor algebra $T(M)$ of a k-module M.

Definition 7.3.1 If M is any k-module, the *tensor algebra* $T(M)$ is the following graded associative algebra with unit generated by M:

$$T(M) = k \oplus M \oplus (M \otimes M) \oplus (M \otimes M \otimes M) \oplus \cdots \oplus M^{\otimes n} \oplus \cdots.$$

Here $M^{\otimes n}$ denotes $M \otimes \cdots \otimes M$, the tensor product (over k) of n copies of M, whose elements are finite sums of terms $x_1 \otimes \cdots \otimes x_n$ $(x_i \in M)$. The product $\otimes$ in $T(M)$ amounts to concatenation of terms. Writing $i\colon M \to T(M)$ for the evident inclusion, this means that $T(M)$ is generated by $i(M)$ as a k-algebra. Clearly T is a functor from k–**mod** to the category of (associative, unital) k-algebras.

Here is a presentation of $T(M)$ as an algebra. $T(M)$ is the free algebra on generators $i(x)$, $x \in M$, subject only to the k-module relations on $i(M)$:

$$\alpha i(x) = i(\alpha x) \quad \text{and} \quad i(x) + i(y) = i(x+y) \quad (\alpha \in k;\, x, y \in M).$$

If M is a free module with basis $\{x_1, \ldots\}$, then $T(M)$ is the free k-algebra $k\{x_1, \ldots\}$. In particular, $T(k)$ is isomorphic to the polynomial ring $k[x]$. In general $T(M)$ is not a commutative algebra except when $M = k$ or $M \cong k/I$ for some ideal I of k.

Exercise 7.3.1 Show that T is the left adjoint of the forgetful functor from k-**alg** to k–**mod**, and that $i\colon M \to T(M)$ is the unit of this adjunction. That is, show that for every associative k-algebra A,

$$\mathrm{Hom}_{k\text{–}\mathbf{mod}}(M, A) \cong \mathrm{Hom}_{k\text{–}\mathbf{alg}}(T(M), A).$$

Exercise 7.3.2 (Free Lie algebras) Given a k-module M, consider the Lie algebra $\mathrm{Lie}(T(M))$ underlying the tensor algebra $T(M)$. Let $\mathfrak{f}$ denote the Lie subalgebra generated by M. That is, elements of $\mathfrak{f}$ are sums of iterated brackets $[x_1, [x_2[\cdots, x_n]]]$ of elements $x_i \in M$. Show that $\mathfrak{f}$ satisfies the universal property of a free Lie algebra of M (see 7.1.5). This provides a constructive proof of the existence of free Lie algebras.

Definition 7.3.2 If $\mathfrak{g}$ is a Lie algebra over k, the *universal enveloping algebra* $U(\mathfrak{g})$ is the quotient of $T(\mathfrak{g})$ by the 2-sided ideal generated by the relations

$$(*) \qquad i([x, y]) = i(x)i(y) - i(y)i(x) \quad (x, y \in \mathfrak{g}).$$

Alternatively, $U\mathfrak{g}$ is the free algebra on generators $i(x)$, $x \in \mathfrak{g}$, subject to the k-module relations on $\mathfrak{g}$ as well as the relation $(*)$. The relation $(*)$ guarantees that i preserves the Lie bracket, that is, that $i\colon \mathfrak{g} \to \mathrm{Lie}(U\mathfrak{g})$ is a Lie algebra homomorphism and that $U\mathfrak{g}$ is a left $\mathfrak{g}$-module. Since the construction is natural in $\mathfrak{g}$, U is a functor from Lie algebras to associative k-algebras. See [BAII, section 3.9] [JLA, ch. V].

Exercise 7.3.3 Show that U is the left adjoint of the "Lie" algebra functor described in 7.1.2 and that i is the unit of the adjunction. That is, for every associative k-algebra A, there is a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Lie}(A)) \cong \mathrm{Hom}_{k-\mathbf{alg}}(U\mathfrak{g}, A).$$

This isomorphism explains the term "universal"; any Lie algebra map $\mathfrak{g} \to \mathrm{Lie}(A)$ extends to a unique k-algebra map $U\mathfrak{g} \to A$.

Theorem 7.3.3 *If $\mathfrak{g}$ is a Lie algebra, then every left $\mathfrak{g}$-module is naturally a left $U\mathfrak{g}$-module, and conversely. The category $\mathfrak{g}$–**mod** is naturally isomorphic to the category $U\mathfrak{g}$–**mod** of left $U\mathfrak{g}$-modules.*

Proof Let M be a k-module and write $E = \mathrm{End}_k(M)$ for the k-algebra of all k-module endomorphisms of M. By adjointness,

$$\mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Lie}(E)) \cong \mathrm{Hom}_{k-\mathbf{alg}}(U\mathfrak{g}, \mathrm{End}_k(M)).$$

A $\mathfrak{g}$-module is a k-module M together with a Lie algebra map $\mathfrak{g} \to \mathrm{Lie}(E)$ (see exercise 7.2.2). But a $U\mathfrak{g}$-module is a k-module M together with an associative algebra map $U\mathfrak{g} \to \mathrm{End}_k(M)$, so the theorem follows. ◇

Corollary 7.3.4 *The category $\mathfrak{g}$–**mod** has enough projectives and enough injectives in the sense of Chapter 2. In particular, $U\mathfrak{g}$ is a projective object in $\mathfrak{g}$–**mod**.*

Here is a more concrete description of the correspondence between $\mathfrak{g}$-modules and $U\mathfrak{g}$-modules. Given a $\mathfrak{g}$-module M and a monomial $x_1 \cdots x_n$ in $U\mathfrak{g}$ $(x_i \in \mathfrak{g})$, the formula

$$(x_1 \cdots x_n)m = x_1(x_2(\cdots(x_n m))), m \in M,$$

makes M into a $U\mathfrak{g}$-module. Conversely, if M is a $U\mathfrak{g}$-module and $x \in \mathfrak{g}$, the formula $xm = i(x)m$ $(m \in M)$ makes M into a $\mathfrak{g}$-module because of the relation $(*)$ of 7.3.2.

Example 7.3.5 (Augmentation ideal) There is a unique k-algebra homomorphism $\varepsilon\colon U\mathfrak{g} \to k$, sending $i(\mathfrak{g})$ to zero, called the *augmentation*. This is clear from the presentation of $U\mathfrak{g}$, and ε corresponds to the zero Lie algebra map $\mathfrak{g} \to \mathrm{Lie}(k)$ under the adjunction. It is the analogue for Lie algebras of the augmentation map $\varepsilon\colon \mathbb{Z}G \to \mathbb{Z}$ of a group ring. Following that analogy, we define the *augmentation ideal* $\mathfrak{J}$ to be the kernel of ε; $\mathfrak{J}$ is evidently the (2-sided) ideal of $U\mathfrak{g}$ generated (as a left ideal) by $i(\mathfrak{g})$. Therefore $\mathfrak{J}$ is a $U\mathfrak{g}$-module and $k \cong U\mathfrak{g}/\mathfrak{J} = (U\mathfrak{g})_{\mathfrak{g}}$.

Corollary 7.3.6 *Let M be a $\mathfrak{g}$-module. Then*

$$H_*(\mathfrak{g}, M) \cong \mathrm{Tor}_*^{U\mathfrak{g}}(k, M) \text{ and}$$
$$H^*(\mathfrak{g}, M) \cong \mathrm{Ext}_{U\mathfrak{g}}^*(k, M).$$

Proof To show any two derived functors are isomorphic, we only need show the underlying functors are isomorphic. Therefore we need only observe

$$k \otimes_{U\mathfrak{g}} M = (U\mathfrak{g}/\mathfrak{J}) \otimes_{U\mathfrak{g}} M \cong M/\mathfrak{J}\, M = M/\mathfrak{g}M = M_{\mathfrak{g}};$$
$$\mathrm{Hom}_{U\mathfrak{g}}(k, M) = \mathrm{Hom}_{\mathfrak{g}}(k, M) = M^{\mathfrak{g}}. \qquad \diamond$$

We conclude this section by stating the Poincaré-Birkhoff-Witt Theorem, which gives the structure of $U\mathfrak{g}$ when k is a field (or more generally when $\mathfrak{g}$ is a free k-module). A proof may be found in [JLA, V.2] or [CE, XIII.3]. Let $\{e_\alpha\}$ be a fixed ordered k-basis of $\mathfrak{g}$. If $I = (\alpha_1, \cdots, \alpha_p)$ is a sequence of indices, we shall use the notation e_I for the product $e_{\alpha_1} \cdots e_{\alpha_p}$ in $U\mathfrak{g}$. The sequence I is called *increasing* if $\alpha_1 \le \cdots \le \alpha_p$. By convention, we regard the empty sequence ϕ as increasing, and set $e_\phi = 1$. If $I = (\alpha)$ is a single index, note that $e_\alpha \in \mathfrak{g}$, but $e_{(\alpha)} = i(e_\alpha)$ is in $U\mathfrak{g}$.

Poincaré-Birkhoff-Witt Theorem 7.3.7 *If $\mathfrak{g}$ is a free k-module, then $U\mathfrak{g}$ is also a free k-module. If $\{e_\alpha\}$ is an ordered basis of $\mathfrak{g}$, then the elements e_I with I an increasing sequence form a basis of $U\mathfrak{g}$.*

Corollary 7.3.8 *The map $i\colon \mathfrak{g} \to U\mathfrak{g}$ is an injection, so we may identify $\mathfrak{g}$ with $i(\mathfrak{g})$.*

Corollary 7.3.9 *If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, and k is a field, then $U\mathfrak{g}$ is a free $U\mathfrak{h}$-module.*

Proof First pick an ordered basis for $\mathfrak{h}$, and then complete it to an ordered basis of $\mathfrak{g}$. The e_I with increasing $I = (\alpha_1, \cdots, \alpha_p)$ such that no e_{α_i} is in $\mathfrak{h}$ will form a basis of $U\mathfrak{g}$ over $U\mathfrak{h}$. $\diamond$

Exercise 7.3.4 (Hom as a $\mathfrak{g}$-module) Let M and N be left $\mathfrak{g}$-modules. Then $\mathrm{Hom}_k(M, N)$ is a $\mathfrak{g}$-module by $(xf)(m) = xf(m) - f(xm)$, $x \in \mathfrak{g}$, $m \in M$. Show that there is a natural isomorphism $\mathrm{Hom}_{\mathfrak{g}}(M, N) \cong \mathrm{Hom}_k(M, N)^{\mathfrak{g}}$.

Exercise 7.3.5 (Cohomological dimension) Extend the natural isomorphism $\mathrm{Hom}_{\mathfrak{g}}(M, N) \cong \mathrm{Hom}_k(M, N)^{\mathfrak{g}}$ of exercise 7.3.4 to a natural isomorphism of δ-functors:

$$\mathrm{Ext}^*_{U\mathfrak{g}}(M, N) \cong H^*_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Hom}_k(M, N))$$

By the Global Dimension Theorem (4.1.2), this proves that the global dimension of $U\mathfrak{g}$ equals the Lie algebra cohomological dimension of $\mathfrak{g}$ (see 7.7.4).

Exercise 7.3.6 (Associated graded algebra) For any Lie algebra $\mathfrak{g}$, let $F_p = F_pU\mathfrak{g}$ be the k-submodule of $U\mathfrak{g}$ generated by all products $x_1 \cdots x_i$ of elements of $\mathfrak{g}$ with $i \le p$. By convention, $F_0U\mathfrak{g} = k$, and clearly $F_1U\mathfrak{g} = k + \mathfrak{g}$. Show that

$$k = F_0U\mathfrak{g} \subseteq F_1U\mathfrak{g} \subseteq F_2U\mathfrak{g} \subseteq \cdots$$

is an increasing filtration in the sense that $F_p \cdot F_q \subseteq F_{p+q}$. Then show that $A = k \oplus (F_1/F_0) \oplus (F_2/F_1) \oplus \cdots \oplus (F_p/F_{p-1}) \oplus \cdots$ is a commutative, associative graded k-algebra. Finally, if $\mathfrak{g}$ is a free k-module on basis $\{e_\alpha\}$, show that $F_1/F_0 = \mathfrak{g}$ and that A is a polynomial ring on the indeterminates e_α:

$$A = k[e_1, e_2, \cdots].$$

Exercise 7.3.7 (Hopf algebra) In this exercise we show that $U\mathfrak{g}$ is a Hopf algebra (see 6.7.15).

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1. Use the universal property of $U\mathfrak{g}$ to show that $U(\mathfrak{g} \times \mathfrak{h}) \cong U\mathfrak{g} \otimes_k U\mathfrak{h}$. In particular, $U(\mathfrak{g} \times \mathfrak{g}) \cong U\mathfrak{g} \otimes_k U\mathfrak{g}$.
2. Show that the diagonal map $\Delta\colon \mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ induces a ring homomorphism $\Delta\colon U\mathfrak{g} \to U\mathfrak{g} \otimes_k U\mathfrak{g}$ with $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$.
3. Show that there is an isomorphism $s : U\mathfrak{g} \cong (U\mathfrak{g})^{op}$, called the *antipode*, and that the resulting isomorphism between left and right $\mathfrak{g}$-modules
$$\textbf{mod–}\mathfrak{g} = \textbf{mod–}U\mathfrak{g} \cong (U\mathfrak{g})^{op}\textbf{–mod} \cong U\mathfrak{g}\textbf{–mod} = \mathfrak{g}\textbf{–mod}$$
is the correspondence $xm = -mx$ of 7.2.3.
4. Show that the maps Δ and s make $U\mathfrak{g}$ into a Hopf algebra.

Exercise 7.3.8 (Products) Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. Use the Künneth formula (3.6.3) as in 6.1.13 to construct split exact sequences

$$0 \to \bigoplus_{\substack{p+q\\=n}} H_p(\mathfrak{g}, k) \otimes H_q(\mathfrak{h}, k) \to H_n(\mathfrak{g} \times \mathfrak{h}, k) \to \bigoplus_{\substack{p+q\\=n-1}} \mathrm{Tor}_1^k(H_p(\mathfrak{g}), H_q(\mathfrak{h})) \to 0$$

$$0 \to \bigoplus_{\substack{p+q\\=n}} H^p(\mathfrak{g}, k) \otimes H^q(\mathfrak{h}, k) \xrightarrow{\times} H^n(\mathfrak{g} \times \mathfrak{h}, k) \to \bigoplus_{\substack{p+q\\=n-1}} \mathrm{Tor}_1^k(H^p(\mathfrak{g}), H^q(\mathfrak{h})) \to 0.$$

The map $\times$ is called the *cross product*. Composition with $\Delta^*\colon H^n(\mathfrak{g} \times \mathfrak{g}) \to H^n(\mathfrak{g})$ gives a graded bilinear product on $H^*(\mathfrak{g}, k)$, called the *cup product*. Show that the cup product makes $H^*(\mathfrak{g}, k)$ into an associative graded-commutative k-algebra (see 6.7.11). Dually, when k is a field, show that $H_*(\mathfrak{g}, k)$ is a coalgebra (6.7.13).

Exercise 7.3.9 (Restricted Lie algebras) Let k be a field of characteristic $p \neq 0$. A *restricted Lie algebra* over k is a Lie algebra $\mathfrak{g}$, together with a set map $x \mapsto x^{[p]}$ of $\mathfrak{g}$ such that $[x^{[p]}, y]$ equals the p-fold product $[x[x[\cdots[xy]]]]$; $(\alpha\ x)^{[p]} = \alpha^p x^{[p]}$ for all $\alpha \in k$; $(x+y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y)$, where $i \cdot s_i(x, y)$ is the coefficient of λ^{i-1} in the formal $(p-1)$-fold product $[\lambda x + y[\cdots[\lambda x + y, x]]]$. See [JLA, V.7].

1. If A is an associative k-algebra, show that $\mathrm{Lie}(A)$ is a restricted Lie algebra with $a^{[p]} = a^p$. In particular, this makes the abelian Lie algebra k into a restricted Lie algebra.
2. Let $u(\mathfrak{g})$ denote the quotient of $U\mathfrak{g}$ by the ideal generated by all elements $x^p - x^{[p]}$; $u(\mathfrak{g})$ is called the *restricted universal enveloping algebra* of $\mathfrak{g}$. If $\mathfrak{g}$ is n-dimensional over k, show that $u(\mathfrak{g})$ is n^p-dimensional as a vector space.
3. A *restricted* $\mathfrak{g}$*-module* M is a $\mathfrak{g}$-module in which the p-fold product $(x(x(\cdots(xm))))$ equals $x^{[p]}m$ for all $m \in M$ and $x \in \mathfrak{g}$. Show that the

category of restricted $\mathfrak{g}$-modules is equivalent to the category of $u(\mathfrak{g})$-modules.

4. Define the restricted cohomology groups $H^*_{\text{res}}(\mathfrak{g}, M)$ to be the right derived functors of $M^{\mathfrak{g}}$ on the category of restricted $\mathfrak{g}$-modules. Show that $H^*_{\text{res}}(\mathfrak{g}, M) \cong \mathrm{Ext}^*_{u(\mathfrak{g})}(k, M)$.
5. Show that there is a canonical map from $H^*_{\text{res}}(\mathfrak{g}, M)$ to the ordinary cohomology $H^*(\mathfrak{g}, M)$.

7.4 H^1 and H_1

The results in Chapter 6 for $H_1(G)$ and $H^1(G)$ have analogues for $H_1(\mathfrak{g})$ and $H^1(\mathfrak{g})$. As there, we begin with the exact sequence of $\mathfrak{g}$-modules:

$$0 \to \mathfrak{J} \to U\mathfrak{g} \to k \to 0.$$

If M is a $\mathfrak{g}$-module, applying $\mathrm{Tor}^{U\mathfrak{g}}_*(-, M)$ yields

$$H_n(\mathfrak{g}, M) = \mathrm{Tor}^{U\mathfrak{g}}_n(k, M) \cong \mathrm{Tor}^{U\mathfrak{g}}_{n-1}(\mathfrak{J}, M), \quad n \geq 2$$

and the exact sequence

$$(\dagger) \qquad 0 \to H_1(\mathfrak{g}, M) \to \mathfrak{J} \otimes_{U\mathfrak{g}} M \to M \to M_{\mathfrak{g}} \to 0.$$

Exercise 7.4.1 (Compare with exercise 6.1.4.)

1. Show that $i\colon \mathfrak{g} \to U\mathfrak{g}$ maps $[\mathfrak{g}, \mathfrak{g}]$ to $\mathfrak{J}^2$. Conclude that it induces a map $i\colon \mathfrak{g}^{ab} \to \mathfrak{J}/\mathfrak{J}^2$, where $\mathfrak{g}^{ab} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.
2. Show that there is a k-module map $\sigma\colon U\mathfrak{g} \to \mathfrak{g}^{ab}$ sending $\mathfrak{J}^2$ to zero and $i(x)$ to $\bar{x}$. *Hint:* First define a map from the tensor algebra $T(\mathfrak{g})$ to $\mathfrak{g}^{ab}$ sending $\mathfrak{g} \otimes_k \mathfrak{g}$ to zero and then pass to the quotient $U\mathfrak{g}$.
3. Deduce from (1) and (2) that $\mathfrak{J}/\mathfrak{J}^2 \cong \mathfrak{g}^{ab}$.

Theorem 7.4.1 *For any Lie algebra* $\mathfrak{g}$, $H_1(\mathfrak{g}, k) \cong \mathfrak{g}^{ab}$.

Proof Taking $M = k$ in $(\dagger)$ yields the exact sequence

$$0 \to H_1(\mathfrak{g}, k) \to \mathfrak{J} \otimes_{U\mathfrak{g}} k \to k \xrightarrow{\cong} k_{\mathfrak{g}} \to 0.$$

But for the right $\mathfrak{g}$-module $\mathfrak{J}$ the exercise 7.4.1 above yields

$$\mathfrak{J} \otimes_{U\mathfrak{g}} k = \mathfrak{J} \otimes_{U\mathfrak{g}} (U\mathfrak{g}/\mathfrak{J}) \cong \mathfrak{J}/\mathfrak{J}^2 \cong \mathfrak{g}^{ab}. \qquad \diamond$$

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Corollary 7.4.2 *If M is any trivial $\mathfrak{g}$-module, $H_1(\mathfrak{g}, M) \cong \mathfrak{g}^{ab} \otimes_k M$.*

Proof Since $M = M_{\mathfrak{g}}$, (†) yields $H_1(\mathfrak{g}, M) \cong \mathfrak{J} \otimes_{U\mathfrak{g}} M \cong (\mathfrak{J} \otimes_{U\mathfrak{g}} k) \otimes_k M \cong \mathfrak{g}^{ab} \otimes_k M$. ◇

Exercise 7.4.2 Let $\mathfrak{g}$ be a free k-module on basis $\{e_1, \cdots, e_n\}$, made into an abelian Lie algebra. Show that $H_p(\mathfrak{g}, k) \cong \Lambda^p \mathfrak{g} \cong k^{\binom{n}{p}}$, the p^{th} exterior power of the k-module $\mathfrak{g}$. *Hint:* $U\mathfrak{g} \cong k[e_1, \cdots, e_n]$.

Exercise 7.4.3 Consider the Lie algebra $\mathfrak{gl}_m(A)$ of $n \times n$ matrices over an associative k-algebra A.

1. Write e_{ij}^a for the matrix whose (i, j)-entry is a, all the other entries being 0. If i, j and k are distinct, show that

$$[e_{ij}^a, e_{jk}^b] = e_{ik}^{ab} \text{ and } [e_{ij}^a, e_{ji}^b] = e_{ii}^{ab} - e_{jj}^{ba}.$$

2. Recall from 7.1.3 that the special linear Lie algebra $\mathfrak{sl}_n(A)$ is the kernel of the trace map from $\mathfrak{gl}_n(A)$ to $A/[A, A]$. Show that for $n \geq 3$

$$H_1(\mathfrak{sl}_n(A), k) = 0 \text{ and } H_1(\mathfrak{gl}_n(A), k) \cong A/[A, A].$$

We now turn our attention to cohomology. Applying $\mathrm{Ext}^*_{U\mathfrak{g}}(-, M)$ to the sequence $0 \to \mathfrak{J} \to U\mathfrak{g} \to k \to 0$ yields

$$H^n(\mathfrak{g}, M) \cong \mathrm{Ext}^{n-1}_{U\mathfrak{g}}(\mathfrak{J}, M), n \geq 2$$

and the exact sequence

$$0 \to M^{\mathfrak{g}} \to M \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M) \to H^1(\mathfrak{g}, M) \to 0.$$

To describe $H^1(\mathfrak{g}, M)$, it remains to interpret $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M)$ as derivations and interpret the image of M as inner derivations.

Definition 7.4.3 If M is a $\mathfrak{g}$-module, a *derivation* from $\mathfrak{g}$ into M is a k-linear map $D: \mathfrak{g} \to M$ such that the Leibnitz formula holds

$$D([x, y]) = x(Dy) - y(Dx).$$

The set of all such derivations is denoted $\mathrm{Der}(\mathfrak{g}, M)$; it is a k-submodule of $\mathrm{Hom}_k(\mathfrak{g}, M)$. Note that if $\mathfrak{g} = M$, then $\mathrm{Der}(\mathfrak{g}, \mathfrak{g})$ is the derivation algebra $\mathrm{Der}(\mathfrak{g})$ of 7.1.4. If M is a trivial $\mathfrak{g}$-module, then $\mathrm{Der}(\mathfrak{g}, M) = \mathrm{Hom}_k(\mathfrak{g}^{ab}, M)$.

Example 7.4.4 (Inner derivations) If $m \in M$, define $D_m(x) = xm$. D_m is a derivation:

$$D_m([x, y]) = [x, y]m = x(ym) - y(xm).$$

The D_m are called the *inner derivations* of $\mathfrak{g}$ into M, and they form a k-submodule $\mathrm{Der}_{\mathrm{Inn}}(\mathfrak{g}, M)$ of $\mathrm{Der}(\mathfrak{g}, M)$.

Example 7.4.5 If $\varphi: \mathfrak{J} \to M$ is a $\mathfrak{g}$-map, let $D_\varphi: \mathfrak{g} \to M$ be defined by $D_\varphi(x) = \varphi(i(x))$. This too is a derivation:

$$D_\varphi([x, y]) = \varphi(i(x)i(y) - i(y)i(x)) = x\varphi(i(y)) - y\varphi(i(x)).$$

As in the analogous discussion for group cohomology (6.4.4), the next step is to show that every derivation is of the form D_φ.

Lemma 7.4.6 *The map $\varphi \mapsto D_\varphi$ is a natural isomorphism of k-modules:*

$$\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M) \cong \mathrm{Der}(\mathfrak{g}, M).$$

Proof The formula $\varphi \mapsto D_\varphi$ defines a natural homomorphism, so it suffices to show that it is an isomorphism. For this we use the fact (7.3.5) that the product map $U\mathfrak{g} \otimes_k \mathfrak{g} \to (U\mathfrak{g})\mathfrak{g} = \mathfrak{J}$ is onto, and that its kernel is the k-module generated by the terms $(u \otimes [xy] - ux \otimes y + uy \otimes x)$ with $u \in U\mathfrak{g}$ and $x, y \in \mathfrak{g}$.

Given a derivation $D: \mathfrak{g} \to M$, consider the map

$$f: U\mathfrak{g} \otimes_k \mathfrak{g} \to M, \qquad f(u \otimes x) = u(Dx).$$

Since D is a derivation, $f(u \otimes [xy] - ux \otimes y + uy \otimes x) = 0$ for all u, x, and y. Therefore f induces a map $\varphi: \mathfrak{J} \to M$, which is evidently a left $\mathfrak{g}$-module map. Since $D_\varphi(x) = \varphi(i(x)) = f(1 \otimes x) = Dx$, we have lifted D to an element of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M)$. On the other hand, given $D = D_h$ for some $h \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M)$, we have $\varphi(ux) = u(Dx) = uh(x) = h(ux)$ for all $u \in U\mathfrak{g}$, $x \in \mathfrak{g}$. Hence $\varphi = h$ as maps from $\mathfrak{J} = (U\mathfrak{g})\mathfrak{g}$ to M. ◇

Theorem 7.4.7 $H^1(\mathfrak{g}, M) \cong \mathrm{Der}(\mathfrak{g}, M)/\mathrm{Der}_{\mathrm{Inn}}(\mathfrak{g}, M)$.

Proof If $\varphi: \mathfrak{J} \to M$ extends to a $\mathfrak{g}$-map $U\mathfrak{g} \to M$ sending 1 to $m \in M$, then

$$D_\varphi(x) = \varphi(x \cdot 1) = xm = D_m(x).$$

Hence D_φ is an inner derivation. This shows that the image of

$$M \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{I}, M) = \mathrm{Der}(\mathfrak{g}, M)$$

is the submodule of inner derivations, as desired. ◇

Corollary 7.4.8 *If M is a trivial $\mathfrak{g}$-module*

$$H^1(\mathfrak{g}, M) \cong \mathrm{Der}(\mathfrak{g}, M) \cong \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, M) \cong \mathrm{Hom}_k(\mathfrak{g}^{ab}, M).$$

Semidirect Products 7.4.9 Given a Lie algebra $\mathfrak{g}$ and a (left) $\mathfrak{g}$-module M, we can form the *semidirect product* Lie algebra $M \rtimes \mathfrak{g}$, much as we did in group theory. The k-module underlying $M \rtimes \mathfrak{g}$ is the product $M \times \mathfrak{g}$, and the product is given by the formula

$$[(m, g), (n, h)] = (gn - hm, [gh]).$$

As in group theory, $M \rtimes \mathfrak{g}$ is a Lie algebra and both $M \times 0$ and $0 \times \mathfrak{g}$ are Lie subalgebras.

We will study other Lie algebra extensions of $\mathfrak{g}$ by M in section 7.6 below. But first, here is an interpretation of $H^1(\mathfrak{g}, M)$ in terms of automorphisms of $M \rtimes \mathfrak{g}$; it is the analogue of a result for semidirect products of groups (exercise 6.4.2). We say that a Lie algebra automorphism σ of $M \rtimes \mathfrak{g}$ *stabilizes* M and $\mathfrak{g}$ if $\sigma(m) = m$ for all m in $M = M \times 0$ and if the induced automorphism on the quotient $\mathfrak{g} \cong (M \rtimes \mathfrak{g})/M$ is the identity, that is, if there is a commutative diagram of Lie algebras:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & M \rtimes \mathfrak{g} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\
 & & \| & & \sigma\downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & M \rtimes \mathfrak{g} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0.
\end{array}$$

Exercise 7.4.4 If D is a derivation of $\mathfrak{g}$ into M, show that σ_D, defined by

$$\sigma_D(m, g) = (m + D(g), g),$$

is a Lie algebra automorphism of $M \rtimes g$ that stabilizes M and $\mathfrak{g}$. Then show that $\mathrm{Der}(\mathfrak{g}, M)$ is isomorphic to the subgroup of $\mathrm{Aut}(M \rtimes \mathfrak{g})$ of all automorphisms stabilizing M and $\mathfrak{g}$. Evidently the inner derivations correspond to the subgroup of all "inner" automorphisms of the form

$$\sigma(m, g) = (m + ga, g), a \in M.$$

In this way we can identify $H^1(\mathfrak{g}, M)$ with a subquotient of $\mathrm{Aut}(M \rtimes \mathfrak{g})$.

Exercise 7.4.5 (Extensions of $\mathfrak{g}$-modules) Use the natural isomorphism $\mathrm{Ext}^1_{U\mathfrak{g}}(M, N) \cong H^1(\mathfrak{g}, \mathrm{Hom}_k(M, N))$ of exercise 7.3.5 to interpret H^1 in terms of extensions of $\mathfrak{g}$-modules. In particular, show that $H^1(\mathfrak{g}, N)$ classifies extensions of $\mathfrak{g}$-modules of the form

$$0 \to N \to M \to k \to 0.$$

Exercise 7.4.6 Let $\mathfrak{g}$ be a restricted Lie algebra over a field of characteristic $p \neq 0$, and let N be a restricted $\mathfrak{g}$-module (exercise 7.3.9). Show that $H^1_{\mathrm{res}}(\mathfrak{g}, N)$ classifies extensions of restricted $\mathfrak{g}$-modules of the form

$$0 \to N \to M \to k \to 0.$$

Conclude that the natural map $H^1_{\mathrm{res}}(\mathfrak{g}, N) \to H^1(\mathfrak{g}, M)$ is an injection.

7.5 The Hochschild-Serre Spectral Sequence

In this section we develop the Hochschild-Serre spectral sequence, which is the analogue of the Lyndon/Hochschild-Serre spectral sequence for groups. The analogue of a normal subgroup of a group is an ideal of a Lie algebra. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{g}/\mathfrak{h}$ inherits a natural Lie algebra structure from $\mathfrak{g}$, and there is an exact sequence of Lie algebra homomorphisms

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0.$$

The proof of the following lemma is exactly the same as the proof of the corresponding result 6.8.4 for groups, and we omit it here.

Lemma 7.5.1 *If $\mathfrak{h}$ is an ideal of a Lie algebra $\mathfrak{g}$ and M is a $\mathfrak{g}$-module, then both $M_{\mathfrak{h}}$ and $M^{\mathfrak{h}}$ are $\mathfrak{g}/\mathfrak{h}$-modules. Moreover, the forgetful functor from $\mathfrak{g}/\mathfrak{h}$*–**mod** *to $\mathfrak{g}$*–**mod** *has $-_{\mathfrak{h}}$ as left adjoint and $-^{\mathfrak{h}}$ as right adjoint.*

Hochschild-Serre Spectral Sequence 7.5.2 For every ideal $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$, there are two convergent first quadrant spectral sequences:

$$E^2_{pq} = H_p(\mathfrak{g}/\mathfrak{h}, H_q(\mathfrak{h}, M)) \Rightarrow H_{p+q}(\mathfrak{g}, M)$$

$$E_2^{pq} = H^p(\mathfrak{g}/\mathfrak{h}, H^q(\mathfrak{h}, M)) \Rightarrow H^{p+q}(\mathfrak{g}, M).$$

Proof We claim that the functors $-_{\mathfrak{g}}$ and $-^{\mathfrak{g}}$ factor as follows.

$$\begin{tikzcd} \mathfrak{g}\text{–}\mathbf{mod} \arrow[r, "-_{\mathfrak{h}}"] \arrow[dr, "-_{\mathfrak{g}}"'] & \mathfrak{g}/\mathfrak{h}\text{–}\mathbf{mod} \arrow[d, "-_{\mathfrak{g}/\mathfrak{h}}"] \\ & k\text{–}\mathbf{mod} \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{g}\text{–}\mathbf{mod} \arrow[r, "-^{\mathfrak{h}}"] \arrow[dr, "-^{\mathfrak{g}}"'] & \mathfrak{g}/\mathfrak{h}\text{–}\mathbf{mod} \arrow[d, "-^{\mathfrak{g}/\mathfrak{h}}"] \\ & k\text{–}\mathbf{mod} \end{tikzcd}$$

The proof of this claim is the same as the proof of the corresponding claim for groups, and we leave the translation to the reader. To apply the Grothendieck spectral sequence (5.8.3), we need only see that $-_{\mathfrak{h}}$ preserves projectives and that $-^{\mathfrak{h}}$ preserves injectives. This follows from the preceding lemma (see 2.3.10): $-_{\mathfrak{h}}$ is left adjoint and $-^{\mathfrak{h}}$ is right adjoint to the forgetful functor, which is an exact functor. ◇

Low Degree Terms 7.5.3 The exact sequences of low degree terms in the Hochschild-Serre spectral sequence are

$$H_2(\mathfrak{g}, M) \to H_2(\mathfrak{g}/\mathfrak{h}, M_{\mathfrak{h}}) \xrightarrow{d} H_1(\mathfrak{h}, M)_{\mathfrak{g}/\mathfrak{h}} \to H_1(\mathfrak{g}, M) \to H_1(\mathfrak{g}/\mathfrak{h}, M_{\mathfrak{h}}) \to 0;$$

$$0 \to H^1(\mathfrak{g}/\mathfrak{h}, M^{\mathfrak{h}}) \to H^1(\mathfrak{g}, M) \to H^1(\mathfrak{h}, M)^{\mathfrak{g}/\mathfrak{h}} \xrightarrow{d} H^2(\mathfrak{g}/\mathfrak{h}, M^{\mathfrak{h}}) \to H^2(\mathfrak{g}, M).$$

Exercise 7.5.1

1. Show that there is an exact sequence

$$H_2(\mathfrak{g}/\mathfrak{h}, k) \oplus [\mathfrak{g}, \mathfrak{h}] \to \mathfrak{h}^{ab} \to \mathfrak{g}^{ab} \to (\mathfrak{g}/\mathfrak{h})^{ab} \to 0.$$

2. If M is a $\mathfrak{g}/\mathfrak{h}$-module, show that there is an exact sequence

$$0 \to \mathrm{Der}(\mathfrak{g}/\mathfrak{h}, M) \to \mathrm{Der}(\mathfrak{g}, M) \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{h}^{ab}, M) \to H^2(\mathfrak{g}/\mathfrak{h}, M) \to H^2(\mathfrak{g}, M).$$

3. Let $\mathfrak{n}_3$ be the nilpotent Lie algebra of strictly upper triangular 3×3 matrices over k (7.1.3). Using the extension

$$0 \to ke_{13} \to \mathfrak{n}_3(k) \to ke_{12} \oplus ke_{23} \to 0,$$

calculate $H^*(\mathfrak{n}_3, k)$ and $H_*(\mathfrak{n}_3, k)$.

4. Let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{gl}_3$ generated by e_{11}, e_{12}, e_{13}, and e_{23}. Use the extension $0 \to \mathfrak{n}_3 \to \mathfrak{g} \to k \to 0$ to compute $H^1(\mathfrak{g}, k)$ and $H^2(\mathfrak{g}, k)$.

Exercise 7.5.2 Suppose that $\mathfrak{f}$ is a free Lie algebra on a set of generators of a Lie algebra $\mathfrak{g}$ and that $\mathfrak{r}$ is the kernel of the natural surjection $\mathfrak{f} \to \mathfrak{g}$. Using the low degree sequence 7.5.3, show that the analogue of Hopf's theorem 6.8.8 holds, that is, that

$$H_2(\mathfrak{g}, k) \cong \frac{\mathfrak{r} \cap [\mathfrak{f}, \mathfrak{f}]}{[\mathfrak{f}, \mathfrak{r}]}.$$

Exercise 7.5.3 (Inflation and restriction) The forgetful map $\mathfrak{g}$–**mod**$\rightarrow \mathfrak{h}$–**mod** is exact for every Lie algebra homomorphism $\mathfrak{h} \rightarrow \mathfrak{g}$. Show that the natural injection $M^{\mathfrak{g}} \rightarrow M^{\mathfrak{h}}$ extends to a morphism $\text{res}^{\mathfrak{g}}_{\mathfrak{h}}: H^*(\mathfrak{g}, M) \rightarrow H^*(\mathfrak{h}, M)$ of δ-functors, called the *restriction map*. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, the *inflation map* is the composite

$$\text{inf}: H^*(\mathfrak{g}/\mathfrak{h}, M) \xrightarrow{\text{res}} H^*(\mathfrak{g}, m^{\mathfrak{h}}) \rightarrow H^*(\mathfrak{g}, M).$$

Show that the edge maps of the Hochschild-Serre spectral sequence for $H^*(\mathfrak{g}, M)$ are the inflation and restriction maps. (Cf. 6.7.1, 6.8.2.)

7.6 H^2 and Extensions

In Chapter 6 we showed that $H^2(G; A)$ classified extensions of groups. There is an analogous result for $H^2_{\text{Lie}}(\mathfrak{g}, M)$, which we shall establish in this section.

Definition 7.6.1 An *extension of Lie algebras* (of $\mathfrak{g}$ by M) is a short exact sequence of Lie algebras

$$0 \rightarrow M \rightarrow \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \rightarrow 0$$

in which M is an abelian Lie algebra. Such an extension makes M into a $\mathfrak{g}$-module in a well-defined way: If $g \in \mathfrak{g}$ and $m \in M$, define gm to be the product $[\tilde{g}, m]$ in $\mathfrak{e}$, where $\pi(\tilde{g}) = g$. Since M is abelian, gm is independent of the choice of $\tilde{g}$.

Exercise 7.6.1 Let M be a $\mathfrak{g}$-module, and form the semidirect product

$$0 \rightarrow M \rightarrow M \rtimes \mathfrak{g} \rightarrow \mathfrak{g} \rightarrow 0.$$

1. Show that the induced $\mathfrak{g}$-module structure on M agrees with the original $\mathfrak{g}$-module structure.
2. We say an extension *splits* if π has a Lie algebra section $\sigma: \mathfrak{g} \rightarrow \mathfrak{e}$. Show that an extension splits if and only if $\mathfrak{e}$ is isomorphic to the semidirect product Lie algebra $M \rtimes \mathfrak{g}$ constructed in 7.4.9, and that under this isomorphism π corresponds to the projection $M \rtimes \mathfrak{g} \rightarrow \mathfrak{g}$.

3. Let $\mathfrak{e} = \mathfrak{n}_3(k)$ to be the Lie algebra of strictly upper triangular matrices. Show that $[\mathfrak{e}, \mathfrak{e}]$ is the 1-dimensional subalgebra ke_{13} of matrices supported in the (1,3) spot, and that $\mathfrak{g} = \mathfrak{e}^{ab}$ is a 2-dimensional abelian Lie algebra. Finally, show that the following extension does not split:

$$0 \to ke_{13} \to \mathfrak{n}_3(k) \to \mathfrak{g} \to 0.$$

Extension Problem 7.6.2 Given a $\mathfrak{g}$-module M, we would like to determine how many extensions of $\mathfrak{g}$ by M exist in which the induced action of $\mathfrak{g}$ on M recovers the given $\mathfrak{g}$-module structure of M. As with groups (6.6.2), we say that two extensions $0 \to M \to \mathfrak{e}_i \to \mathfrak{g} \to 0$ are *equivalent* if there is an isomorphism $\varphi: \mathfrak{e}_1 \cong \mathfrak{e}_2$ so that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathfrak{e}_1 & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle\varphi}\downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & \mathfrak{e}_2 & \longrightarrow & \mathfrak{g} & \longrightarrow & 0
\end{array}$$

commutes, and we ask for a description of the set $\mathrm{Ext}(\mathfrak{g}, M)$ of equivalence classes of extensions.

Classification Theorem 7.6.3 *Let M be a $\mathfrak{g}$-module. The set* $\mathrm{Ext}(\mathfrak{g}, M)$ *of equivalence classes of extensions of* $\mathfrak{g}$ *by M is in 1–1 correspondence with* $H^2_{\mathrm{Lie}}(\mathfrak{g}, M)$.

The canonical approach to classifying extensions of groups (Chapter 6, section 6) has an analogue only for extensions in which $\mathfrak{g}$ is a free k-module (e.g., if k is a field). Rather than pursue that method, which calls for a canonical $\mathfrak{g}$-module resolution of k and a notion of 2-cocycle (see exercise 7.7.5), we shall resort to a more functorial method.

Suppose first that $0 \to M \to \mathfrak{e} \to \mathfrak{g} \to 0$ is an extension of $\mathfrak{g}$ by an abelian Lie algebra M. The low degree terms sequence of 7.5.3 with $\mathfrak{h} = M$ is

$$0 \to H^1(\mathfrak{g}, M) \to H^1(\mathfrak{e}, M) \to \mathrm{Hom}_{\mathfrak{g}}(M, M) \xrightarrow{d^2} H^2(\mathfrak{g}, M) \to H^2(\mathfrak{e}, M).$$

This sequence is natural with respect to extensions, so $d^2: \mathrm{Hom}_{\mathfrak{g}}(M,M) \to H^2(\mathfrak{g},M)$ depends only on the equivalence class of the extension in $\mathrm{Ext}(\mathfrak{g},M)$. Therefore assigning $d^2(\mathrm{id}_M)$ to the extension gives a well-defined set map from $\mathrm{Ext}(\mathfrak{g}, M)$ to $H^2(\mathfrak{g}, M)$, called the *classifying map*.

Before showing that the classifying map is a bijection, we consider a universal case. Choose a presentation of $\mathfrak{g}$: $0 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{g} \to 0$ with $\mathfrak{f}$ free on some set. Modding out by the ideal $[\mathfrak{r}, \mathfrak{r}]$ of $\mathfrak{f}$ gives an extension of $\mathfrak{g}$ by $\mathfrak{r}^{ab} = \mathfrak{r}/[\mathfrak{r}, \mathfrak{r}]$:

$$0 \to \mathfrak{r}^{ab} \to \mathfrak{f}/[\mathfrak{r}, \mathfrak{r}] \to \mathfrak{g} \to 0.$$

Let $u \in H^2(\mathfrak{g}, \mathfrak{r}^{ab})$ be the image of this extension under the classifying map.

If $\mathfrak{e} \to \mathfrak{g}$ is a central extension of $\mathfrak{g}$ by M, we can lift $\mathfrak{f} \to \mathfrak{g}$ to a map $\mathfrak{f} \to \mathfrak{e}$. This yields maps of Lie algebra extensions:

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{r} \arrow[r] \arrow[d] & \mathfrak{f} \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{r}^{ab} \arrow[r] \arrow[d, "\varphi"'] & \mathfrak{f}/[\mathfrak{r}, \mathfrak{r}] \arrow[r] \arrow[d] & \mathfrak{g} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathfrak{e} \arrow[r] & \mathfrak{g} \arrow[r] & 0
\end{tikzcd}$$

Comparing low degree term sequences (for the Hochschild-Serre spectral sequences 7.5.2) and using 7.2.5 yields a diagram

$$(*) \qquad \begin{tikzcd}
 & \mathrm{Hom}_{\mathfrak{g}}(M, M) \arrow[r, "d^2"] \arrow[d, "\varphi^*"'] & H^2(\mathfrak{g}, M) \arrow[d, equal] & \\
 & \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{r}^{ab}, M) \arrow[r, "d^2"] \arrow[d, "\cong"] & H^2(\mathfrak{g}, M) \arrow[d, equal] & \\
H^1(\mathfrak{f}, M) \arrow[r] & H^1(\mathfrak{r}, M)^{\mathfrak{g}} \arrow[r, "d^2"] & H^2(\mathfrak{g}, M) \arrow[r] & 0.
\end{tikzcd}$$

Exercise 7.6.2 In this exercise we show that $u \in H^2(\mathfrak{g}, \mathfrak{r}^{ab})$ is universal in the sense that the class of any extension of $\mathfrak{g}$ by M is $\varphi_*(u)$ for some $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{r}^{ab}, M)$. To do this, let $\varphi: \mathfrak{r}^{ab} \to M$ be the map induced from $\mathfrak{f} \to \mathfrak{e}$. Considered as an element of $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{r}^{ab}, M)$ we see from $(*)$ that $d^2(\varphi) = d^2(\mathrm{id}_M)$ in $H^2(\mathfrak{g}, M)$. Show that the corresponding map $\varphi_*: H^2(\mathfrak{g}, \mathfrak{r}^{ab}) \to H^2(\mathfrak{g}, M)$ sends u to $d^2(\mathrm{id}_M)$.

Lemma 7.6.4 *Every element of $H^2(\mathfrak{g}, M)$ arises as the class of an extension.*

Proof Since $\mathfrak{f}$ is free, $H^2(\mathfrak{f}, M) = 0$. By $(*)$, every element of $H^2(\mathfrak{g}, M)$ is $d^2(\varphi)$ for some element φ of

$$H^1(\mathfrak{r}, M)^{\mathfrak{g}} \cong \mathrm{Hom}_k(\mathfrak{r}^{ab}, M)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{r}^{ab}, M).$$

Regarding M as an $\mathfrak{f}$-module via $\mathfrak{f} \to \mathfrak{g}$, form the semidirect product $M \rtimes \mathfrak{f}$. The set $\mathfrak{h} = \{(\varphi(-r), r) : r \in \mathfrak{r}\}$ is an ideal of $M \rtimes \mathfrak{f}$; set $\mathfrak{e} = (M \rtimes \mathfrak{f})/\mathfrak{h}$. Evidently $\mathfrak{h} \cap M = 0$ so we have an extension

$$0 \to M \to \mathfrak{e} \to \mathfrak{g} \to 0$$

together with a map $\mathfrak{f} \to \mathfrak{e}$ over $\mathfrak{g}$. The resulting map from $\mathrm{Hom}_{\mathfrak{g}}(M, M)$ to $\mathrm{Hom}(\mathfrak{r}^{ab}, M)$ sends id_M to φ. By diagram $(*)$, the class of this extension is $d^2(\mathrm{id}_M) = d^2(\varphi)$ as desired. $\diamondsuit$

We are now ready to prove the classification theorem. The above lemma shows that the classifying map $\mathrm{Ext}(\mathfrak{g}, M) \to H^2(\mathfrak{g}, M)$ is onto; it suffices to show that this map is an injection. Suppose that $0 \to M \to \mathfrak{e}_i \to \mathfrak{g} \to 0$ $(i = 1, 2)$ are two extensions of $\mathfrak{g}$ by M that both map to $\theta \in H^2(\mathfrak{g}, M)$.

Choosing lifts $\tau_i : \mathfrak{f} \to \mathfrak{e}_i$, the above argument yields $\varphi_i \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{r}^{ab}, M)$ with $d^2(\varphi_i) = \theta$ in diagram $(*)$. By making $\mathfrak{f}$ larger if necessary, we may assume that $\mathfrak{f}$ maps onto both $\mathfrak{e}_1$ and $\mathfrak{e}_2$. (For this it suffices to add M to the set of generators of $\mathfrak{f}$.) Since $d^2(\varphi_2 - \varphi_1) = 0$, we see from $(*)$ that there is a derivation $D: \mathfrak{f} \to M$ such that the class of D in $H^1(\mathfrak{f}, M)$ maps to $\varphi_2 - \varphi_1$ in $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{r}^{ab}, M)$. Define a map $\tau : \mathfrak{f} \to \mathfrak{e}_1$ by sending $x \in \mathfrak{f}$ to $\tau_1(x) + D(x)$. This is a Lie algebra homomorphism, since

$$\begin{aligned}\tau([xy]) &= \tau_1([x, y]) + D([xy])\\ &= [\tau_1(x), \tau_1(y)] + x(Dy) - y(Dx)\\ &= [\tau_1(x) + D(x), \tau_1(y) + D(y)]\\ &= [\tau(x), \tau(y)].\end{aligned}$$

There is no harm in replacing τ_1 by τ, except that we replace φ_1 by $\varphi_1 + D = \varphi_2$ in $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{r}^{ab}, M)$. We are now in the situation

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{r} & \longrightarrow & \mathfrak{f} & \longrightarrow & \mathfrak{g} & \longrightarrow & 0 \\ & & {\scriptstyle\varphi}\downarrow & & {\scriptstyle\tau_i}\downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & \mathfrak{e}_i & \longrightarrow & \mathfrak{g} & \longrightarrow & 0. \end{array}$$

As $\mathfrak{f}$ maps onto $\mathfrak{e}_i$, we see that $\ker(\varphi)$ is an ideal of $\mathfrak{f}$ and that $\mathfrak{e}_1 \cong \mathfrak{f}/\ker(\varphi) \cong \mathfrak{e}_2$. As this isomorphism is a homomorphism over $\mathfrak{g}$, $\mathfrak{e}_1$, and $\mathfrak{e}_2$ define the same element of $\mathrm{Ext}(\mathfrak{g}, M)$. $\diamondsuit$

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Exercise 7.6.3 We saw in Corollary 7.2.5 that if $\mathfrak{f}$ is a free Lie algebra on a set, then $H^n(\mathfrak{f}, M) = 0$ for $n \geq 2$ and all $\mathfrak{f}$-modules M. Give a direct proof that $H^2(\mathfrak{f}, M) = 0$ by showing that all extensions $0 \to \mathfrak{h} \to \mathfrak{e} \to \mathfrak{f} \to 0$ split. Show that conversely if $\mathfrak{g}$ is free as a k-module and $H^2(\mathfrak{g}, M) = 0$ for all $\mathfrak{g}$-modules M, then $\mathfrak{g}$ is a free Lie algebra. *Hint:* Writing $\mathfrak{g} = \mathfrak{f}/\mathfrak{r}$ for some free Lie algebra $\mathfrak{f}$, with $\mathfrak{r} \subseteq [\mathfrak{f}, \mathfrak{f}]$, it suffices to show that $\mathfrak{r} = [\mathfrak{f}, \mathfrak{r}]$ (exercise 7.2.5). But $H^2(\mathfrak{g}, \mathfrak{r}/[\mathfrak{f}, \mathfrak{r}]) = 0$.

Exercise 7.6.4 (Restricted extensions) Let k be a field of characteristic $p \neq 0$. Let $\mathfrak{g}$ be a restricted Lie algebra and M a restricted $\mathfrak{g}$-module such that $M^{[p]} = 0$. (See exercise 7.3.9.) A *restricted extension* $\mathfrak{e}$ of $\mathfrak{g}$ by M is a restricted Lie algebra $\mathfrak{e}$ containing M as a restricted ideal, together with a restricted homomorphism $\mathfrak{e} \to \mathfrak{g}$ whose kernel is M. Let $\mathrm{Ext}_{\mathrm{res}}(\mathfrak{g}, M)$ denote the equivalence classes of restricted extensions of $\mathfrak{g}$ by M, $\mathfrak{e}$ and $\mathfrak{e}'$ being equivalent if there is a restricted homomorphism $\mathfrak{e} \to \mathfrak{e}'$ over $\mathfrak{g}$. Show that there is a natural isomorphism $\mathrm{Ext}_{\mathrm{res}}(\mathfrak{g}, M) \cong H^2_{\mathrm{res}}(\mathfrak{g}, M)$ compatible with the isomorphism $\mathrm{Ext}(\mathfrak{g}, M) \cong H^2(\mathfrak{g}, M)$ of the Classification Theorem 7.6.3.

7.7 The Chevalley-Eilenberg Complex

Throughout this section $\mathfrak{g}$ will denote a Lie algebra over k that is free as a k-module. We shall construct the $U\mathfrak{g}$-module chain complex $V_*(\mathfrak{g})$ originally used by C. Chevalley and S. Eilenberg [ChE] in 1948 to define $H^*_{\mathrm{Lie}}(\mathfrak{g}, M)$.

Let $\Lambda^p\mathfrak{g}$ denote the p^{th}-exterior product of the k-module $\mathfrak{g}$, which is generated by monomials $x_1 \wedge \cdots \wedge x_p$ with $x_i \in \mathfrak{g}$; see 4.5.1 above. Our chain complex has $V_p(\mathfrak{g}) = U\mathfrak{g} \otimes_k \Lambda^p\mathfrak{g}$; since $\Lambda^p\mathfrak{g}$ is a free k-module, $V_p(\mathfrak{g})$ is free as a left $U\mathfrak{g}$-module. By convention, $\Lambda^0\mathfrak{g} = k$ and $\Lambda^1\mathfrak{g} = \mathfrak{g}$, so $V_0 = U\mathfrak{g}$ and $V_1 = U\mathfrak{g} \otimes_k \mathfrak{g}$. We define $\varepsilon : V_0(\mathfrak{g}) = U\mathfrak{g} \to k$ to be the augmentation 7.3.5 and d: $V_1(\mathfrak{g}) \to V_0(\mathfrak{g})$ to be the product map $d(u \otimes x) = ux$ from $U\mathfrak{g} \otimes \mathfrak{g}$ to $U\mathfrak{g}$ whose image is the augmentation ideal $\mathfrak{J}$. By 7.3.5, we have an exact sequence

$$V_1(\mathfrak{g}) \xrightarrow{d} V_0(\mathfrak{g}) \xrightarrow{\varepsilon} k \to 0.$$

Definition 7.7.1 For $p \geq 2$, let d: $V_p(\mathfrak{g}) \to V_{p-1}(\mathfrak{g})$ be given by the formula $d(u \otimes x_1 \wedge \cdots \wedge x_p) = \theta_1 + \theta_2$, where (for $u \in U\mathfrak{g}$ and $x_i \in \mathfrak{g}$):

$$\theta_1 = \sum_{i=1}^{p}(-1)^{i+1}ux_i \otimes x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_p;$$

$$\theta_2 = \sum_{i<j}(-1)^{i+j}u \otimes [x_ix_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_p.$$

(The notation $\hat{x}_i$ indicates an omitted term.) For example, if $p = 2$, then

$$d(u \otimes x \wedge y) = ux \otimes y - uy \otimes x - u \otimes [xy].$$

$V_*(\mathfrak{g})$ with this differential is called the *Chevalley-Eilenberg complex*. It is sometimes also called the *standard complex*.

Exercise 7.7.1 Verify that $d^2 = 0$, so that V_* is indeed a chain complex of $U\mathfrak{g}$-modules. *Hint:* Writing $d(\theta_i) = \theta_{i1} + \theta_{i2}$, show that $-\theta_{11}$ is the $i = 1$ part of θ_{21} and that $\theta_{22} = 0$. Then show that $-\theta_{12}$ is the $i > 1$ part of θ_{21}.

Theorem 7.7.2 *$V_*(\mathfrak{g}) \xrightarrow{\varepsilon} k$ is a projective resolution of the $\mathfrak{g}$-module k.*

Proof (Koszul) It suffices to show that $H_n(V_*(\mathfrak{g})) = 0$ for $n \neq 0$.

Choose an ordered basis $\{e_\alpha\}$ of $\mathfrak{g}$ as a k-module. By the Poincaré-Birkhoff-Witt Theorem (7.3.7), $V_n(\mathfrak{g})$ is a free k-module with a basis consisting of terms

$$(*)\quad e_I \otimes (e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_n}),\ \ \alpha_1 < \cdots < \alpha_n \text{ and } I = (\beta_1, \cdots, \beta_m) \text{ increasing.}$$

We filter $V_*(\mathfrak{g})$ by k-submodules, letting F_pV_n be the submodule generated by terms $(*)$ with $m + n \leq p$. Since $[e_ie_j]$ is a linear combination of the e_α in $\mathfrak{g}$, this is actually a filtration by chain subcomplexes

$$0 \subset F_0V_* \subseteq F_1V_* \subseteq \cdots \subseteq V_*(\mathfrak{g}) = \cup F_pV_*.$$

This filtration is bounded below and exhaustive (see 5.4.2), so by 5.5.1 there is a convergent spectral sequence

$$E^0_{pq} = F_pV_{p+q}/F_{p-1}V_{p+q} \Rightarrow H_{p+q}(V_*(\mathfrak{g})).$$

This spectral sequence is concentrated in the octant $p \geq 0$, $q \leq 0$, $p + q \geq 0$. The first column is F_0V_*, which is zero except in the (0,0) spot, where E^0_{00} is $F_0V_0 = k$.

We claim that each column E^0_{p*} is exact for $p \neq 0$. This will prove that the spectral sequence collapses at E^1, with $E^1_{pq} = 0$ for $(p, q) \neq (0, 0)$, yielding the desired computation: $H_n(V_*) = 0$ for $n \neq 0$.

Let A_q be the free k-submodule of $U\mathfrak{g}$ on basis

$$\{e_I : I = (\beta_1, \cdots, \beta_q) \quad \text{is an increasing sequence}\}.$$

Then $A_q \cong F_qV_0/F_{q-1}V_0$ and $E^0_{pq} = A_{-q} \otimes_k \Lambda^{p+q}\mathfrak{g}$. Moreover, the formula for the differential in V_* shows that the differential $d^0\colon E^0_{pq} \to E^0_{p,q-1}$ is given by

$$d^0(a \otimes e_{\alpha_1} \wedge \cdots \wedge e_{\alpha n}) = \theta_1 = \sum_{i=1}^{n} (-1)^{i+1} a e_{\alpha_i} \otimes e_{\alpha_1} \wedge \cdots \wedge \hat{e}_{\alpha_i} \wedge \cdots \wedge e_{\alpha n}.$$

We saw in exercise 7.3.6 that $A = A_0 \oplus A_1 \oplus \cdots$ is a polynomial ring on the indeterminates $e_\alpha : A \cong k[e_1, e_2, \cdots]$. Comparing formulas for d, we see that the direct sum $\bigoplus E^0_{p*}$ of the chain complexes E^0_{p*} is identical to the Koszul complex

$$A \otimes_k \Lambda^* \mathfrak{g} = \Lambda^*(\oplus A e_\alpha) = K(\boldsymbol{x})$$

of 4.5.1 corresponding to the sequence $\boldsymbol{x} = (e_1, e_2, \cdots)$. Since $\boldsymbol{x}$ is a regular sequence, we know from *loc. cit.* that

$$H_n(\boldsymbol{x}, A) = H_n(A \otimes \Lambda^* \mathfrak{g}) = \bigoplus_{p=0}^{\infty} H_{n-p}(E^0_{p*}) = \bigoplus_{p=0}^{\infty} E^1_{p,n-p}$$

is zero for $n \neq 0$ and $A/\boldsymbol{x}A = k$ for $n = 0$. Since $E^1_{00} = k$, it follows that $E^1_{pq} = 0$ for $(p, q) \neq (0, 0)$, as claimed. ◇

Corollary 7.7.3 (Chevalley-Eilenberg) *If M is a right $\mathfrak{g}$-module, then the homology modules $H_*(\mathfrak{g}, M)$ are the homology of the chain complex*

$$M \otimes_{U\mathfrak{g}} V_*(\mathfrak{g}) = M \otimes_{U\mathfrak{g}} U\mathfrak{g} \otimes_k \Lambda^* \mathfrak{g} = M \otimes_k \Lambda^* \mathfrak{g}.$$

If M is a left $\mathfrak{g}$-module, then the cohomology modules $H^*(\mathfrak{g}, M)$ are the cohomology of the cochain complex

$$\mathrm{Hom}_{\mathfrak{g}}(V(\mathfrak{g}), M) = \mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g} \otimes_k \Lambda^* \mathfrak{g}, M) \cong \mathrm{Hom}_k(\Lambda^* \mathfrak{g}, M).$$

In this complex, an n-cochain $f\colon \Lambda^n \mathfrak{g} \to M$ is just an alternating k-multilinear function $f(x_1, \cdots, x_n)$ of n variables in $\mathfrak{g}$, taking values in M. The coboundary δf of such an n-cochain is the $(n+1)$-cochain

$$\begin{aligned}\delta f(x_1, \cdots, x_{n+1}) = &\sum (-1)^{i+1} x_i f(x_1, \cdots, \hat{x}_i, \cdots) \\ &+ \sum (-1)^{i+j} f([x_i x_j], x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots).\end{aligned}$$

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Application 7.7.4 (Cohomological dimension) If $\mathfrak{g}$ is n-dimensional as a vector space over a field k, then $H^i(\mathfrak{g}, M) = H_i(\mathfrak{g}, M) = 0$ for all $i > n$. Indeed, $\Lambda^i\mathfrak{g} = 0$ in this range. The following exercise shows that $H^n(\mathfrak{g}, M) \neq 0$ for some $\mathfrak{g}$-module M, so that $\mathfrak{g}$ has cohomological dimension $n = \dim_k(\mathfrak{g})$.

Exercise 7.7.2 If k is a field and $\mathfrak{g}$ is n-dimensional as a vector space, show that $U\mathfrak{g}$ has global dimension n (4.1.2). To do this, we proceed in several steps. First note that $pd_{U\mathfrak{g}}(k) \le n$ because $V_*(\mathfrak{g})$ is a projective resolution of k.

1. Let $\Lambda^n\mathfrak{g} \cong k$ be given the $\mathfrak{g}$-module structure

$$[y, x_1 \wedge \cdots \wedge x_n] = \sum_{i=1}^{n} x_1 \wedge \cdots \wedge [yx_i] \wedge \cdots \wedge x_n.$$

   Show that $H^n(\mathfrak{g}, \Lambda^n\mathfrak{g}) \cong k$. This proves that $pd_{U\mathfrak{g}}(\Lambda^n\mathfrak{g}) = n$ and hence that $gl.\dim(U\mathfrak{g}) \ge n$.
2. Use the natural isomorphism $\operatorname{Ext}^*_{U\mathfrak{g}}(M, N) \cong H^*_{\text{Lie}}(\mathfrak{g}, \operatorname{Hom}_k(M, N))$ (exercise 7.3.5) and the Global Dimension theorem 4.1.2 to show that $gl.\dim(U\mathfrak{g}) \le n$, and hence that $gl.\dim(U\mathfrak{g}) = n$.

Exercise 7.7.3 Use the Chevalley-Eilenberg complex to show that

$$H_3(\mathfrak{sl}_2, k) \cong H^3(\mathfrak{sl}_2, k) \cong k.$$

Exercise 7.7.4 (1-cocycles and module extensions) Let M be a left $\mathfrak{g}$-module. If $0 \to M \to N \xrightarrow{\pi} k \to 0$ is a short exact sequence of $\mathfrak{g}$-modules, and $n \in N$ is such that $\pi(n) = 1$, define $f: \mathfrak{g} \to M$ by $f(x) = xn$. Show that f is a 1-cocycle in the Chevalley-Eilenberg complex $\operatorname{Hom}_k(\Lambda^*\mathfrak{g}, M)$ and that its class $[f] \in H^1(\mathfrak{g}, M)$ is independent of the choice of n. Then show that $H^1(\mathfrak{g}, M)$ is in 1–1 correspondence with the equivalence classes of $\mathfrak{g}$-module extensions of k by M. (Compare to exercise 7.4.5.)

Exercise 7.7.5 (2-cocycles and algebra extensions) Let M be a left $\mathfrak{g}$-module, with $\mathfrak{g}$ free as a k-module.

1. If $0 \to M \to \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \to 0$ is an extension of Lie algebras, and $\sigma: \mathfrak{g} \to \mathfrak{e}$ is a k-module splitting of π, show that the Lie algebra structure on $\mathfrak{e} \cong M \times \mathfrak{g}$ may be described by an alternating k-bilinear function $f: \mathfrak{g} \times \mathfrak{g} \to M$ defined by

$$[\sigma(x), \sigma(y)] = \sigma([xy]) + f(x, y), \qquad x, y \in \mathfrak{g}.$$

Show that f is a 2-cocycle for the Chevalley-Eilenberg cochain complex $\mathrm{Hom}_k(\Lambda^*\mathfrak{g}, M)$. Also, show that if σ' is any other splitting of π, then the resulting 2-cocycle f' is cohomologous to f. This shows that such an extension determines a well-defined element $[f] \in H^2_{\mathrm{Lie}}(\mathfrak{g}, M)$.

2. Using part (1), show directly that $H^2_{\mathrm{Lie}}(\mathfrak{g}, M)$ is in 1–1 correspondence with equivalence classes of Lie algebra extensions of $\mathfrak{g}$ by M. This is the same correspondence as we gave in section 7.6 by a more abstract approach.

Exercise 7.7.6 If M is a right $\mathfrak{g}$-module and $g \in \mathfrak{g}$, show that the formula

$$\begin{aligned}(m \otimes x_1 \wedge \cdots \wedge x_p)g = [mg] \otimes x_1 \wedge \cdots \wedge x_p \\ + \sum m \otimes x_1 \wedge \cdots \wedge [x_i g] \wedge \cdots \wedge x_p\end{aligned}$$

makes $M \otimes V_*(\mathfrak{g})$ into a chain complex of right $\mathfrak{g}$-modules. Then show that the induced $\mathfrak{g}$-module structure on $H_*(\mathfrak{g}; M)$ is trivial.

7.8 Semisimple Lie Algebras

We now restrict our attention to finite-dimensional Lie algebras over a field k of characteristic 0. We will give cohomological proofs of several main theorems involving solvable and semisimple Lie algebras. First, however, we need to summarize the main notions of the classical theory of semisimple Lie algebras.

Definitions 7.8.1 An ideal of $\mathfrak{g}$ is called *solvable* if it is solvable as a Lie algebra (see 7.1.7). It is not hard to show that the family of all solvable ideals of $\mathfrak{g}$ forms a lattice, because the sum and intersection of solvable ideals is a solvable ideal [JLA, I.7]. If $\mathfrak{g}$ is finite-dimensional, there is a largest solvable ideal of $\mathfrak{g}$, called the *solvable radical* $\mathrm{rad}\,\mathfrak{g}$ of $\mathfrak{g}$. Every ideal $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathrm{rad}\,\mathfrak{g}$ is a solvable ideal.

A Lie algebra $\mathfrak{g}$ is called *simple* if it has no ideals except itself and 0, and if $[\mathfrak{g}, \mathfrak{g}] \neq 0$ (i.e., $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$). For example, $\mathfrak{sl}_n(k)$ is a simple Lie algebra for $n \geq 2$ (as $\mathrm{char}(k) \neq 2$).

A Lie algebra $\mathfrak{g}$ is called *semisimple* if $\mathrm{rad}\,\mathfrak{g} = 0$, that is, if $\mathfrak{g}$ has no nonzero solvable ideals. In fact, $\mathfrak{g}$ is semisimple iff $\mathfrak{g}$ has no nonzero abelian ideals; to see this, note that the last nonzero term $(\mathrm{rad}\,\mathfrak{g})^{(n-1)}$ in the derived series for $\mathrm{rad}\,\mathfrak{g}$ is an abelian ideal of $\mathfrak{g}$. Clearly simple Lie algebras are semisimple.

Lemma 7.8.2 *If $\mathfrak{g}$ is a finite-dimensional Lie algebra, then $\mathfrak{g}/(\mathrm{rad}\,\mathfrak{g})$ is a semisimple Lie algebra.*

Proof If not, $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$ contains a nonzero abelian ideal $\mathfrak{a} = \mathfrak{h}/\operatorname{rad}\mathfrak{g}$. But $[\mathfrak{a}, \mathfrak{a}] = 0$, so $\mathfrak{h}' = [\mathfrak{h}, \mathfrak{h}]$ must lie inside $\operatorname{rad}\mathfrak{g}$. Hence $\mathfrak{h}'$ is solvable, and therefore so is $\mathfrak{h}$. This contradicts the maximality of $\operatorname{rad}\mathfrak{g}$. ◇

Definition 7.8.3 (Killing form) If $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}_n$ we can use matrix multiplication to define the symmetric bilinear form $\beta(x, y) = \operatorname{trace}(xy)$ on $\mathfrak{g}$. This symmetric form is "$\mathfrak{g}$-invariant" in the sense that for $x, y, z \in \mathfrak{g}$ we have $\beta([xy], z) = \beta(x, [yz])$, or equivalently $\beta([xy], z) + \beta(x, [zy]) = 0$. (Exercise!)

Now suppose that $\mathfrak{g}$ is an n-dimensional Lie algebra over k. Left multiplication by elements of $\mathfrak{g}$ gives a Lie algebra homomorphism

$$ad\colon \mathfrak{g} \to \operatorname{Lie}(\operatorname{End}_k(\mathfrak{g})) = \mathfrak{gl}_n,$$

called the *adjoint representation* of $\mathfrak{g}$. The symmetric bilinear form on $\mathfrak{g}$ obtained by pulling back β is called the *Killing form* of $\mathfrak{g}$, that is, the Killing form is $\kappa(x, y) = \operatorname{trace}(ad(x)ad(y))$. The importance of the Killing form is summed up in the following result, which we cite from [JLA, III.4]:

Cartan's Criterion for Semisimplicity 7.8.4 *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic 0.*

1. *$\mathfrak{g}$ is semisimple if and only if the Killing form is a nondegenerate symmetric bilinear form on the vector space $\mathfrak{g}$.*
2. *If $\mathfrak{g} \subseteq \mathfrak{gl}_n$ and $\mathfrak{g}$ is semisimple, then the bilinear form $\beta(x,y) = \operatorname{trace}(xy)$ is also nondegenerate on $\mathfrak{g}$.*

Structure Theorem of Semisimple Lie Algebras 7.8.5 *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field of characteristic 0. Then $\mathfrak{g}$ is semisimple iff $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_r$ is the finite product of simple Lie algebras $\mathfrak{g}_i$. In particular, every ideal of a semisimple Lie algebra is semisimple.*

Proof If the $\mathfrak{g}_i$ are simple, every ideal of $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$ is a product of $\mathfrak{g}_i$'s and cannot be abelian, so $\mathfrak{g}$ is semisimple.

For the converse, it suffices to show that every minimal ideal $\mathfrak{a}$ of a semisimple Lie algebra $\mathfrak{g}$ is a direct factor: $\mathfrak{g} = \mathfrak{a} \times \mathfrak{b}$. Define $\mathfrak{b}$ to be the orthogonal complement of $\mathfrak{a}$ with respect to the Killing form. To see that $\mathfrak{b}$ is an ideal of $\mathfrak{g}$, we use the $\mathfrak{g}$-invariance of κ: for $a \in \mathfrak{a}$, $b \in \mathfrak{b}$, and $x \in \mathfrak{g}$,

$$\kappa(a, [x, b]) = \kappa([ax], b) = 0$$

because $[ax] \in \mathfrak{a}$. Hence $[xb] \in \mathfrak{b}$ and $\mathfrak{b}$ is an ideal of $\mathfrak{g}$.

To conclude, it suffices to show that $\mathfrak{a} \cap \mathfrak{b} = 0$, since this implies $\mathfrak{g} = \mathfrak{a} \times \mathfrak{b}$. Now $\mathfrak{a} \cap \mathfrak{b}$ is an ideal of $\mathfrak{g}$; since $\mathfrak{a}$ is minimal either $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}$ or $\mathfrak{a} \cap \mathfrak{b} = 0$. If $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}$, then $\kappa([a_1a_2], x) = \kappa(a_1, [a_2x]) = 0$ for every $a_1, a_2 \in \mathfrak{a}$ and $x \in \mathfrak{g}$. Since κ is nondegenerate, this implies that $[a_1a_2] = 0$. Thus $\mathfrak{a}$ is abelian, contradicting the semisimplicity of $\mathfrak{g}$. Hence $\mathfrak{a} \cap \mathfrak{b} = 0$, and we are done. ◇

Corollary 7.8.6 *If $\mathfrak{g}$ is finite-dimensional and semisimple (and char$(k) = 0$), then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Consequently,*

$$H_1(\mathfrak{g}, k) = H^1(\mathfrak{g}, k) = 0.$$

Proof If $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, then $\mathfrak{g}^{ab} = 0$. On the other hand, we saw in 7.4.1 and 7.4.8 that $H_1(\mathfrak{g}, k) \cong \mathfrak{g}^{ab}$ and $H^1(\mathfrak{g}, k) \cong \mathrm{Hom}_k(\mathfrak{g}^{ab}, k)$. ◇

Corollary 7.8.7 *If $\mathfrak{g} \subseteq \mathfrak{gl}_n$ is semisimple, then $\mathfrak{g} \subseteq \mathfrak{sl}_n = [\mathfrak{gl}_n, \mathfrak{gl}_n]$.*

Exercise 7.8.1 Suppose that k is an algebraically closed field of characteristic 0 and that $\mathfrak{g}$ is a finite-dimensional simple Lie algebra over k.

1. Use Schur's Lemma to see that $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathfrak{g}) \cong k$.
2. Show that $\mathfrak{g} \cong \mathrm{Hom}_k(\mathfrak{g}, k)$ as $\mathfrak{g}$-modules.
3. If $f: \mathfrak{g} \otimes \mathfrak{g} \to k$ is any $\mathfrak{g}$-invariant symmetric bilinear form, show that f is a multiple of the Killing form κ, that is, $f = \alpha\kappa$ for some $\alpha \in k$.
4. If V is any k-vector space and $f: \mathfrak{g} \otimes \mathfrak{g} \to V$ is any $\mathfrak{g}$-invariant symmetric bilinear map, show that there is a $v \in V$ such that $f(x, y) = \kappa(x, y)v$.

Exercise 7.8.2 (Counterexample to structure theorem in char. $p \neq 0$) Let k be a field of characteristic $p \neq 0$, and consider the Lie algebra $\mathfrak{gl}_n$, $n \geq 3$. Show that the only ideals of $\mathfrak{gl}_n$ are $\mathfrak{sl}_n = [\mathfrak{gl}_n, \mathfrak{gl}_n]$ and the center $k\cdot 1$. If $p|n$, show that the center is contained inside $\mathfrak{sl}_n$. This shows that $\mathfrak{pgl}_n = \mathfrak{gl}_n/k\cdot 1$ has only one ideal, namely $\mathfrak{psl}_n = \mathfrak{sl}_n/k\cdot 1$, and that $\mathfrak{psl}_n$ is simple. Conclude that $\mathfrak{pgl}_n$ is semisimple but not a direct product of simple ideals and show that $H_1(\mathfrak{pgl}_n, k) \cong H^1(\mathfrak{pgl}_n, k) \cong k$.

The Casimir Operator 7.8.8 Let $\mathfrak{g}$ be semisimple and let M be an m-dimensional $\mathfrak{g}$-module. If $\mathfrak{h}$ is the image of the structure map

$$\rho: \mathfrak{g} \to \mathrm{Lie}(\mathrm{End}_k(M)) \cong \mathfrak{gl}_m(k),$$

then $\mathfrak{g} \cong \mathfrak{h} \times \ker(\rho)$, $\mathfrak{h} \subseteq \mathfrak{gl}_m$, and the bilinear form β on $\mathfrak{h}$ is nondegenerate by Cartan's criterion 7.8.4. Choose a basis $\{e_1, \cdots, e_r\}$ of $\mathfrak{h}$; by linear algebra there is a dual basis $\{e^1, \cdots, e^r\}$ of $\mathfrak{h}$ such that $\beta(e_i, e^j) = \delta_{ij}$. The element $c_M = \sum e_i e^i \in U\mathfrak{g}$ is called the *Casimir operator* for M; it is independent of the choice of basis for $\mathfrak{h}$. The following facts are easy to prove and are left as exercises:

1. If $x \in \mathfrak{g}$ and $[e_i, x] = \sum c_{ij} e_j$, then $[x, e^j] = \sum c_{ij} e^i$.
2. c_M is in the center of $U\mathfrak{g}$ and $c_M \in \mathfrak{J}$. *Hint:* Use (1).
3. The image of c_M in the matrix ring $\mathrm{End}_k(M)$ is r/m times the identity matrix. In particular, if M is nontrivial as a $\mathfrak{g}$-module, then $r \neq 0$ and c_M acts on M as an automorphism. *Hint:* By (2) it is a scalar matrix, so it suffices to show that the trace is $r = \dim(\mathfrak{h})$.

Exercise 7.8.3 Let $\mathfrak{g} = \mathfrak{sl}_2$ with basis $x = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $h = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. If M is the canonical 2-dimensional $\mathfrak{g}$-module, show that $c_M = 2xy - h + h^2/2$, while its image in $\mathrm{End}(M)$ is the matrix $\begin{pmatrix} 3/2 & 0 \\ 0 & 3/2 \end{pmatrix}$.

Theorem 7.8.9 *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0. If M is a simple $\mathfrak{g}$-module, $M \neq k$, then*

$$H^i_{\mathrm{Lie}}(\mathfrak{g}, M) = H_i^{\mathrm{Lie}}(\mathfrak{g}, M) = 0 \quad \text{for all } i.$$

Proof Let C be the center of $U\mathfrak{g}$. We saw in 3.2.11 and 3.3.6 that $H_*(\mathfrak{g}, M) = \mathrm{Tor}_*^{U\mathfrak{g}}(k, M)$ and $H^*(\mathfrak{g}, M) = \mathrm{Ext}^*_{U\mathfrak{g}}(k, M)$ are naturally C-modules; moreover, multiplication by $c \in C$ is induced by $c\colon k \to k$ as well as $c\colon M \to M$. Since the Casimir element c_M acts by 0 on k (as $c_M \in \mathfrak{J}$) and by the invertible scalar r/m on M, we must have $0 = r/m$ on $H_*(\mathfrak{g}, M)$ and $H^*(\mathfrak{g}, M)$. This can only happen if these C-modules are zero. ◇

Corollary 7.8.10 (Whitehead's first lemma) *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0. If M is any finite-dimensional $\mathfrak{g}$-module, then $H^1_{\mathrm{Lie}}(\mathfrak{g}, M) = 0$. That is, every derivation from $\mathfrak{g}$ into M is an inner derivation.*

Proof We proceed by induction on $\dim(M)$. If M is simple, then either $M = k$ and $H^1(\mathfrak{g}, k) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = 0$ or else $M \neq k$ and $H^*(\mathfrak{g}, M) = 0$ by the theorem. Otherwise, M contains a proper submodule L. By induction, $H^1(\mathfrak{g}, L) = H^1(\mathfrak{g}, M/L) = 0$, so we are done via the cohomology exact sequence

$$\cdots H^1(\mathfrak{g}, L) \to H^1(\mathfrak{g}, M) \to H^1(\mathfrak{g}, M/L) \cdots. \qquad ◇$$

Weyl's Theorem 7.8.11 *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0. Then every finite-dimensional $\mathfrak{g}$-module M is completely reducible, that is, is a direct sum of simple $\mathfrak{g}$-modules.*

Proof Suppose that M is not a direct sum of simple modules. Since $\dim(M)$ is finite, M contains a submodule M_1 minimal with respect to this property. Clearly M_1 is not simple, so it contains a proper $\mathfrak{g}$-submodule M_0. By induction, both M_0 and $M_2 = M_1/M_0$ are direct sums of simple $\mathfrak{g}$-modules yet M_1 is not, so the extension M_1 of M_2 by M_0 must be represented (3.4.3) by a nonzero element of

$$\operatorname{Ext}^1_{U\mathfrak{g}}(M_2, M_0) \cong H^1_{\text{Lie}}(\mathfrak{g}, \operatorname{Hom}_k(M_2, M_0))$$

(see exercise 7.3.5), and this contradicts Whitehead's first lemma. $\diamond$

Corollary 7.8.12 (Whitehead's second lemma) *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0. If M is any finite-dimensional $\mathfrak{g}$-module, then $H^2_{\text{Lie}}(\mathfrak{g}, M) = 0$.*

Proof Since H^* commutes with direct sums, and M is a direct sum of simple $\mathfrak{g}$-modules, we may assume that M is simple. If $M \neq k$ we already know the result by 7.8.9, so it suffices to show that $H^2(\mathfrak{g}, k) = 0$, that is, that every Lie algebra extension

$$0 \to k \to \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \to 0$$

splits. We claim that $\mathfrak{e}$ can be made into a $\mathfrak{g}$-module in such a way that π is a $\mathfrak{g}$-map. To see this, let $\tilde{x}$ be any lift of $x \in \mathfrak{g}$ to $\mathfrak{e}$ and define $x \circ y$ to be $[\tilde{x}, y]$ for $y \in \mathfrak{e}$. This is independent of the choice of $\tilde{x}$ because k is in the center of $\mathfrak{e}$. The $\mathfrak{g}$-module axioms are readily defined (exercise!), and by construction $\pi(x \circ y) = [x, \pi(y)]$. This establishes the claim.

By Weyl's Theorem $\mathfrak{e}$ and $\mathfrak{g}$ split as $\mathfrak{g}$-modules, and there is a $\mathfrak{g}$-module homomorphism $\sigma\colon \mathfrak{g} \to \mathfrak{e}$ splitting π such that $\mathfrak{e} \cong k \times \mathfrak{g}$ as a $\mathfrak{g}$-module. If we choose $\tilde{x} = \sigma(x)$, then it is clear that σ is a Lie algebra homomorphism and that $\mathfrak{e} \cong k \times \mathfrak{g}$ as a Lie algebra. This proves that $H^2(\mathfrak{g}, k) = 0$, as desired. $\diamond$

Remark $H^3(\mathfrak{sl}_2, k) \cong k$ (exercise 7.7.3), so there can be no "third Whitehead lemma."

Levi's Theorem 7.8.13 *If $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field of characteristic zero, then there is a semisimple Lie subalgebra $\mathcal{L}$ of $\mathfrak{g}$ (called a*

Levi factor *of* $\mathfrak{g}$*) such that* $\mathfrak{g}$ *is isomorphic to the semidirect product*

$$\mathfrak{g} \cong (\text{rad } \mathfrak{g}) \rtimes \mathcal{L}.$$

Proof We know that $\mathfrak{g}/(\text{rad } \mathfrak{g})$ is semisimple, so it suffices to show that the following Lie algebra extension splits.

$$0 \to \text{rad } \mathfrak{g} \to \mathfrak{g} \to \mathfrak{g}/\text{rad } \mathfrak{g} \to 0$$

If rad $\mathfrak{g}$ is abelian then these extensions are classified by $H^2(\mathfrak{g}/(\text{rad } \mathfrak{g}), \text{rad } \mathfrak{g})$, which vanishes by Whitehead's second lemma, so every extension splits.

If rad $\mathfrak{g}$ is not abelian, we proceed by induction on the derived length of rad $\mathfrak{g}$. Let $\mathfrak{r}$ denote the ideal [rad $\mathfrak{g}$, rad $\mathfrak{g}$] of $\mathfrak{g}$. Since $\text{rad}(\mathfrak{g}/\mathfrak{r}) = (\text{rad } \mathfrak{g})/\mathfrak{r}$ is abelian, the extension

$$0 \to (\text{rad } \mathfrak{g})/\mathfrak{r} \to \mathfrak{g}/\mathfrak{r} \to \mathfrak{g}/(\text{rad } \mathfrak{g}) \to 0$$

splits. Hence there is an ideal $\mathfrak{h}$ of $\mathfrak{g}$ containing $\mathfrak{r}$ such that $\mathfrak{g}/\mathfrak{r} \cong (\text{rad } \mathfrak{g})/\mathfrak{r} \times \mathfrak{h}/\mathfrak{r}$ and $\mathfrak{h}/\mathfrak{r} \cong \mathfrak{g}/(\text{rad } \mathfrak{g})$. Now

$$\text{rad}(\mathfrak{h}) = \text{rad}(\mathfrak{g}) \cap \mathfrak{h} = \mathfrak{r},$$

and $\mathfrak{r}$ has a smaller derived length than rad $\mathfrak{g}$. By induction there is a Lie subalgebra $\mathcal{L}$ of $\mathfrak{h}$ such that $\mathfrak{h} \cong \mathfrak{r} \rtimes \mathcal{L}$ and $\mathcal{L} \cong \mathfrak{h}/\mathfrak{r} \cong \mathfrak{g}/\text{rad } \mathfrak{g}$. But then $\mathcal{L}$ is our desired Levi factor of $\mathfrak{g}$. ◇

Remark Levi factors are not unique, but they are clearly all isomorphic to $\mathfrak{g}/(\text{rad } \mathfrak{g})$ and hence to each other. Malcev proved (in 1942) that the Levi factors are all conjugate by nice automorphisms of $\mathfrak{g}$.

Historical Remark 7.8.14 (see [Bour]) Sophus Lie developed the theory of Lie groups and their Lie algebras from 1874 to 1893. Semisimple Lie algebras over $\mathbb{C}$ are in 1–1 correspondence with compact, simply connected Lie groups. In the period 1888–1894 much of the structure of Lie algebras over $\mathbb{C}$ was developed, including W. Killing's discovery of the solvable radical and semisimple Lie algebras, and the introduction of the "Killing form" in E. Cartan's thesis. The existence of Levi factors was announced by Cartan but only proven (publicly) by E. E. Levi in 1905. Weyl's Theorem (1925) was originally proven using integration on compact Lie groups. An algebraic proof of Weyl's theorem was found in 1935 by Casimir and van der Waerden. This and J. H. C. Whitehead's two lemmas (1936–1937) provided the first clues that enabled Chevalley and Eilenberg (1948 [ChE]) to construct the cohomology $H^*(\mathfrak{g}, M)$. The cohomological proofs in this section are close parallels of the treatment by Chevalley and Eilenberg.

Exercise 7.8.4 If $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field of characteristic 0, show that $\mathfrak{g}$ is semisimple iff $H^1(\mathfrak{g}, M) = 0$ for all finite-dimensional $\mathfrak{g}$-modules M.

Exercise 7.8.5 (Reductive Lie algebras) A Lie algebra $\mathfrak{g}$ is called *reductive* if $\mathfrak{g}$ is a completely reducible $\mathfrak{g}$-module (via the adjoint representation). That is, $\mathfrak{g}$ is reductive if $\mathfrak{g}$ is a direct sum of simple $\mathfrak{g}$-modules. Now assume that $\mathfrak{g}$ is finite-dimensional over a field of characteristic 0, so that $\mathfrak{g} \cong (\text{rad } \mathfrak{g}) \rtimes \mathcal{L}$ for some semisimple Lie algebra $\mathcal{L}$ by Levi's theorem. Show that the following are equivalent:

1. $\mathfrak{g}$ is reductive
2. $[\mathfrak{g}, \mathfrak{g}] = \mathcal{L}$
3. $\text{rad}(\mathfrak{g})$ is abelian and equals the center of $\mathfrak{g}$
4. $\mathfrak{g} \cong \mathfrak{a} \times \mathcal{L}$ where $\mathfrak{a}$ is abelian and $\mathcal{L}$ is semisimple

Then show that $\mathfrak{gl}_m$ is a reductive Lie algebra, and in fact that $\mathfrak{gl}_m \cong k \times \mathfrak{sl}_m$.

7.9 Universal Central Extensions

A *central extension* $\mathfrak{e}$ of a Lie algebra $\mathfrak{g}$ is an extension $0 \to M \to \mathfrak{e} \xrightarrow{\pi} \mathfrak{g} \to 0$ of Lie algebras such that M is in the center of $\mathfrak{e}$ (i.e., it is just an extension of Lie algebras of $\mathfrak{g}$ by a trivial $\mathfrak{g}$-module M in the sense of 7.6.1). A homomorphism over $\mathfrak{g}$ from $\mathfrak{e}$ to another central extension $0 \to M' \to \mathfrak{e}' \xrightarrow{\pi'} \mathfrak{g} \to 0$ is a map $f: \mathfrak{e} \to \mathfrak{e}'$ such that $\pi = \pi' f$. $\mathfrak{e}$ is called a *universal central extension* of $\mathfrak{g}$ if for every central extension $\mathfrak{e}'$ of $\mathfrak{g}$ there is a unique homomorphism $f: \mathfrak{e} \to \mathfrak{e}'$ over $\mathfrak{g}$. Clearly, a universal central extension of $\mathfrak{g}$ is unique up to isomorphism over $\mathfrak{g}$, provided it exists. As with groups (6.9.2), if $\mathfrak{g}$ has a universal central extension, then $\mathfrak{g}$ must be perfect, that is, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Construction of a Universal Central Extension 7.9.1 We may copy the construction 6.9.3 for groups. Choose a free Lie algebra $\mathfrak{f}$ mapping onto $\mathfrak{g}$ and let $\mathfrak{r} \subset \mathfrak{f}$ denote the kernel, so that $\mathfrak{g} \cong \mathfrak{f}/\mathfrak{r}$. This yields a central extension

$$0 \to \mathfrak{r}/[\mathfrak{r}, \mathfrak{f}] \to \mathfrak{f}/[\mathfrak{r}, \mathfrak{f}] \to \mathfrak{g} \to 0.$$

If $\mathfrak{g}$ is perfect, $[\mathfrak{f}, \mathfrak{f}]$ maps onto $\mathfrak{g}$, and we claim that

$$0 \to (\mathfrak{r} \cap [\mathfrak{f}, \mathfrak{f}])/[\mathfrak{r}, \mathfrak{f}] \to [\mathfrak{f}, \mathfrak{f}]/[\mathfrak{r}, \mathfrak{f}] \to \mathfrak{g} \to 0$$

is a universal central extension of $\mathfrak{g}$. Note that $H_2(\mathfrak{g}, k) = (\mathfrak{r} \cap [\mathfrak{f}, \mathfrak{f}])/[\mathfrak{r}, \mathfrak{f}]$ by exercise 7.5.2.

Theorem 7.9.2 *A Lie algebra $\mathfrak{g}$ has a universal central extension iff $\mathfrak{g}$ is perfect. In this case, the universal central extension is*

$$(*) \qquad 0 \to H_2(\mathfrak{g}, k) \to [\mathfrak{f}, \mathfrak{f}]/[\mathfrak{r}, \mathfrak{f}] \to \mathfrak{g} \to 0.$$

Proof We have seen that $(*)$ is a central extension. Set $\mathfrak{e} = [\mathfrak{f}, \mathfrak{f}]/[\mathfrak{r}, \mathfrak{f}]$. Since $[\mathfrak{f}, \mathfrak{f}]$ maps onto $\mathfrak{g}$, any $x, y \in \mathfrak{f}$ may be written as $x = x' + r$, $y = y' + s$ with $x', y' \in [\mathfrak{f}, \mathfrak{f}]$ and $r, s \in \mathfrak{r}$. Thus in $\mathfrak{f}/[\mathfrak{r}, \mathfrak{f}]$

$$[x, y] = [x', y'] + [x', s] + [r, y'] + [r, s] = [x', y'].$$

This shows that $\mathfrak{e}$ is also a perfect Lie algebra. If $0 \to M \to \mathfrak{e}' \xrightarrow{\pi} \mathfrak{g} \to 0$ is another central extension, lift $\mathfrak{f} \to \mathfrak{g}$ to a map $\phi: \mathfrak{f} \to \mathfrak{e}'$. Since $\pi\phi(\mathfrak{r}) = 0$, $\phi(\mathfrak{r}) \subseteq M$. This implies that $\phi([\mathfrak{r}, \mathfrak{f}]) = 1$. As in 6.9.5, ϕ induces a map $f: \mathfrak{e} \to \mathfrak{e}'$ over $\mathfrak{g}$. If f_1 is another such map, the difference $\delta = f_1 - f: \mathfrak{e} \to M$ is zero because $\mathfrak{e} = [\mathfrak{e}, \mathfrak{e}]$ and

$$f_1([xy]) = [f(x) + \delta(x), f(y) + \delta(y)] = [f(x), f(y)] = f([x, y]).$$

Hence $f_1 = f$, that is, f is unique. ◇

By copying the proofs of 6.9.6 and 6.9.7, we also have the following two results.

Lemma 7.9.3 *If $0 \to M \to \mathfrak{e} \to \mathfrak{g} \to 0$ and $0 \to M' \to \mathfrak{e}' \to \mathfrak{g} \to 0$ are central extensions, and $\mathfrak{e}$ is perfect, there is at most one homomorphism from $\mathfrak{e}$ to $\mathfrak{e}'$ over $\mathfrak{g}$.*

Recognition Criterion 7.9.4 *Call a Lie algebra $\mathfrak{g}$* simply connected *if every central extension $0 \to M \to \mathfrak{e} \to \mathfrak{g} \to 0$ splits in a unique way as a product Lie algebra $\mathfrak{e} = \mathfrak{g} \times M$. A central extension $0 \to M \to \mathfrak{e} \to \mathfrak{g} \to 0$ is universal iff $\mathfrak{e}$ is perfect and simply connected. Moreover, $H_1(\mathfrak{e}, k) = H_2(\mathfrak{e}, k) = 0$. In particular, if $\mathfrak{g}$ is perfect and $H_2(\mathfrak{e}, k) = 0$, then $\mathfrak{g}$ is simply connected.*

Corollary 7.9.5 *Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over a field of characteristic 0. Then $H_2(\mathfrak{e}, k) = 0$ and $\mathfrak{g}$ is simply connected.*

Proof $M = H_2(\mathfrak{e}, k)$ is a finite-dimensional $\mathfrak{g}$-module because it is a subquotient of $\Lambda^2\mathfrak{g}$ in the Chevalley-Eilenberg complex. By Whitehead's second lemma 7.8.12, $H^2(\mathfrak{g}, M) = 0$, so the universal central extension is $\mathfrak{e} = M \times \mathfrak{g}$. By universality, we must have $M = 0$. ◇

Exercise 7.9.1 Show that simply connected Lie algebras are perfect.

Exercise 7.9.2 If $0 \to M_i \to \mathfrak{e}_i \to \mathfrak{g}_i \to 0$ are universal central extensions, show that $0 \to M_1 \times M_2 \to \mathfrak{e}_1 \times \mathfrak{e}_2 \to \mathfrak{g}_1 \times \mathfrak{g}_2 \to 0$ is also a universal central extension.

In the rest of this section, we shall use the above ideas in the construction of Affine Lie algebras $\hat{\mathfrak{g}}$ corresponding to simple Lie algebras.

Let $\mathfrak{g}$ be a fixed finite-dimensional simple Lie algebra over a field k of characteristic 0. Write $\mathfrak{g}[t, t^{-1}]$ for the Lie algebra $\mathfrak{g} \otimes_k k[t, t^{-1}]$ over $k[t, t^{-1}]$. Elements of $\mathfrak{g}[t, t^{-1}]$ are Laurent polynomials $\sum x_i t^i$ with $x_i \in \mathfrak{g}$ and $i \in \mathbb{Z}$. Since the Chevalley-Eilenberg complex $V_*(\mathfrak{g}[t, t^{-1}])$ is $V_*(\mathfrak{g}) \otimes_k k[t, t^{-1}]$, we have

$$H_*(\mathfrak{g}[t, t^{-1}], k[t, t^{-1}]) = H_*(\mathfrak{g}, k) \otimes_k k[t, t^{-1}].$$

In particular, $H_1 = H_2 = 0$ (7.8.6, 7.8.12) so $\mathfrak{g}[t, t^{-1}]$ is perfect and simply connected as a Lie algebra over the ground ring $k[t, t^{-1}]$.

Now we wish to consider $\mathfrak{g}[t, t^{-1}]$ as an infinite-dimensional Lie algebra over k. Since $\mathfrak{g}[t, t^{-1}]$ is perfect, we still have $H_1(\mathfrak{g}[t, t^{-1}], k) = 0$, but we will no longer have $H_2(\mathfrak{g}[t, t^{-1}], k) = 0$. We now construct an example of a nontrivial central extension of $\mathfrak{g}[t, t^{-1}]$ over k.

Affine Lie Algebras 7.9.6 If $\kappa : \mathfrak{g} \otimes \mathfrak{g} \to k$ is the Killing form (7.8.3), set

$$\beta\left(\sum x_i t^i, \sum y_j t^j\right) = \sum i\kappa(x_i, y_{-i}).$$

Since β is alternating bilinear, it is a 2-cochain (7.7.3). Because k is a trivial $\mathfrak{g}[t, t^{-1}]$-module, β is a 2-cocycle: if $x = \sum x_i t^i$, $y = \sum y_j t^j$, and $z = \sum z_k t^k$, then the $\mathfrak{g}$-invariance of the Killing form gives

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$$\begin{aligned}\delta\beta(x,y,z) &= -\beta([xy],z)+\beta([xz],y)-\beta([yz],x)\\ &= \sum_{i,j,k} -\beta([x_iy_j]t^{i+j}, z_kt^k)+\beta([x_iz_k]t^{i+k}, y_jt^j)-\beta([y_jz_k]t^{j+k}, x_it^i)\\ &= \sum_{i+j+k=0} -(i+j)\kappa([x_iy_j],z_k)+(i+k)\kappa([x_iz_k],y_j)-(j+k)\kappa([y_jz_k],x_i)\\ &= \sum_{i+j+k=0} \{-(i+j)-(i+k)-(j+k)\}\kappa(x_i,[y_j,z_k])\\ &= 0.\end{aligned}$$

The class $[\beta]\in H^2(\mathfrak{g}[t,t^{-1}],k)$ corresponds to a central extension of Lie algebras over k:

$$0\to k\to \hat{\mathfrak{g}}\to \mathfrak{g}[t,t^{-1}]\to 0.$$

The Lie algebra $\hat{\mathfrak{g}}$ is called the *Affine Lie algebra* corresponding to $\mathfrak{g}$. It is a special type of Kac-Moody Lie algebra. We are going to prove that $\hat{\mathfrak{g}}$ is the universal central extension of $\mathfrak{g}[t,t^{-1}]$ following the proof in [Wil].

Lemma 7.9.7 *$\hat{\mathfrak{g}}$ is perfect.*

Proof Let $\rho\colon \mathfrak{g}[t,t^{-1}]\to\hat{\mathfrak{g}}$ be the vector space splitting corresponding to the 2-cocycle β. If $x,y\in\mathfrak{g}$ then $[\rho(xt^i),\rho(yt^{-i})]=\rho([xy])+i\,\kappa(x,y)$ for $i=0,1$ so $k\subseteq[\hat{\mathfrak{g}},\hat{\mathfrak{g}}]$. Since $[\hat{\mathfrak{g}},\hat{\mathfrak{g}}]$ maps onto the perfect $\mathfrak{g}[t,t^{-1}]$, we must have $\hat{\mathfrak{g}}=[\hat{\mathfrak{g}},\hat{\mathfrak{g}}]$. ◇

Now fix an arbitrary central extension $0\to M\to\mathfrak{e}\xrightarrow{\pi}\mathfrak{g}[t,t^{-1}]\to 0$. If $\sigma\colon\mathfrak{g}[t,t^{-1}]\to\mathfrak{e}$ is a vector space splitting of π, recall (exercise 7.7.5) that the corresponding 2-cocycle $f_\sigma\colon\Lambda^2(\mathfrak{g}[t,t^{-1}])\to M$ is defined by

$$[\sigma(x),\sigma(y)]=\sigma([xy])+f_\sigma(x,y),$$

and that conversely every 2-cocycle f determines a σ such that $f=f_\sigma$. Let S denote the set of all splittings σ of π such that

$$f_\sigma\Big(\sum x_it^i,y\Big)=0\ \text{ for all }\ x_i,y\in\mathfrak{g}\ \text{ and }\ i\in\mathbb{Z}.$$

Lemma 7.9.8 *S is nonempty for every central extension of $\mathfrak{g}[t,t^{-1}]$.*

Proof Given any splitting σ, write $f^i_\sigma(x,y)$ for $f_\sigma(xt^i,y)$. Each $f^i_\sigma(-,y)$ is an element of $\mathrm{Hom}_k(\mathfrak{g},M)$, so we may think of f^i_σ as a 1-cochain, that is, a map from $\mathfrak{g}$ to $\mathrm{Hom}_k(\mathfrak{g}, M)$. In fact, f_σ^i is a cocycle (exercise!). But $\mathrm{Hom}_k(\mathfrak{g}, M)$ is finite-dimensional, so by Whitehead's first lemma (7.8.10) there exists $\varphi^i \in \mathrm{Hom}_k(\mathfrak{g}, M)$ such that $f_\sigma^i(x, y) = \varphi^i([xy])$. Assembling the φ^i into a k-linear map $\varphi\colon \mathfrak{g}[t, t^{-1}] \to M$ by the rule $\varphi(\sum x_i t^i) = \sum \varphi^i(x_i)$, we see that the 2-cocycle $\delta\varphi\colon \Lambda^2\mathfrak{g}[t, t^{-1}] \to M$ satisfies

$$(\delta\varphi)(\sum x_i t^i, y) = -\sum \varphi^i([x_i y]) = -\sum f_\sigma^i(x_i, y) = -f(\sum x_i t^i, y).$$

Hence the splitting τ corresponding to the 2-cocycle $f + \delta\varphi$ is an element of S. $\diamond$

Exercise 7.9.3 Show that S contains exactly one element.

Lemma 7.9.9 *If $k = \mathbb{C}$ and $\sigma \in S$, then there exist $c_{ij} \in M$ such that*

$$f_\sigma(\sum x_i t^i, \sum y_j t^j) = \sum \kappa(x_i, y_j) c_{ij},$$

where κ is the Killing form on $\mathfrak{g}$.

Proof Because $\sigma \in S$, we have

$$0 = \delta f(x_i t^i, y_j t^j, z) = f_\sigma([x_i z] t^i, y_j t^j) - f_\sigma([x_i t^i, [z, y_j] t^j).$$

Therefore each $f_\sigma^{ij}(x, y) = f_\sigma(x t^i, y t^j)$ is a $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$:

$$f_\sigma^{ij}([xz], y) = f_\sigma^{ij}(x, [zy]).$$

On the other hand the Killing form is a nondegenerate $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$. Since $k = \mathbb{C}$, any $\mathfrak{g}$-invariant symmetric bilinear form must therefore be a multiple of κ (exercise 7.8.1). Thus $f_\sigma^{ij} = \kappa c_{ij}$ for some $c_{ij} \in M$. $\diamond$

Corollary 7.9.10 *If $k = \mathbb{C}$ and $\sigma \in S$, then there is a $c \in M$ such that for $x = \sum x_i t^i$, $y = \sum y_j t^j$ in $\mathfrak{g}[t, t^{-1}]$ we have*

$$f_\sigma(x, y) = \beta(x, y) c = \sum i\kappa(x_i, y_{-i}) c.$$

Proof Setting $c = c_{1,-1}$, it suffices to prove that $c_{i,-i} = ic$ and that $c_{ij} = 0$ if $i \neq -j$. As $\sigma \in S$, $c_{i0} = 0$ for all i; since f_σ is skew-symmetric, we have $c_{ij} = -c_{ji}$. Since κ is $\mathfrak{g}$-invariant and symmetric,

$$0 = \delta f_\sigma(x t^i, y t^j, z t^k) = -\kappa(x, [yz])(c_{i+j,k} + c_{i+k,j} + c_{j+k,i})$$

which yields $0 = c_{i+j,k} + c_{i+k,j} + c_{j+k,i}$. Taking $i + j = 1$ and $k = -1$, so that $j + k = -i$, we get

$$c_{i,-i} = -c_{-i,i} = c + c_{i-1,1-i}.$$

By induction on $|i| \geq 0$, this yields $c_{i,-i} = ic$ for all $i \in \mathbb{Z}$. Taking $i + j + k = s$ and $k = 1$, we get

$$c_{s-1,1} = c_{i,j+1} - c_{i+1,j}.$$

Summing from $i = 0$ to $s - 1$ if $s > 0$ (or from $i = s$ to -1 if $s < 0$) yields $sc_{s-1,1} = 0$, so $c_{t,1} = 0$ unless $t = -1$. This yields $c_{i,j+1} = c_{i+1,j}$ unless $i + j = -1$. Fixing $s \neq 0$, induction on $|i|$ shows that $c_{i,s-i} = 0$ for all $i \in \mathbb{Z}$. ◇

Theorem 7.9.11 (H. Garland) *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $k = \mathbb{C}$. Then the corresponding Affine Lie algebra $\hat{\mathfrak{g}}$ (7.9.6) is the universal central extension of $\mathfrak{g}[t, t^{-1}]$.*

Proof Let $0 \to M \to \mathfrak{e} \xrightarrow{\pi} \mathfrak{g}[t, t^{-1}] \to 0$ be a central extension. Choose a splitting σ in S (7.9.8), and let $c_{ij} \in M$ be the elements constructed in lemma 7.9.9. Recall that there is a vector space splitting $\iota\colon \mathfrak{g}[t, t^{-1}] \to \hat{\mathfrak{g}}$ corresponding to the 2-cocycle β, which yields a vector space decomposition $\hat{\mathfrak{g}} \cong k \times \mathfrak{g}[t, t^{-1}]$. Define $F\colon k \to M$ by $F(\alpha) = \alpha c_{1,-1}$ and extend this to a vector space map from $\hat{\mathfrak{g}}$ to $\mathfrak{e}$ by setting $F(\iota(x)) = \sigma(x)$ for $x \in \mathfrak{g}[t, t^{-1}]$. Since

$$\begin{aligned} F([\iota(x), \iota(y)]) &= F(\iota[x, y]) + F(\beta(x, y)) \\ &= \sigma([x, y]) + \sum i\kappa(x_i, y_{-i})c_{1,-1} \\ &= \sigma([x, y]) + f_\sigma(x, y) \\ &= [F(\iota(x)), F(\iota(y))], \end{aligned}$$

and k is in the center of $\hat{\mathfrak{g}}$, F is a Lie algebra homomorphism $\hat{\mathfrak{g}} \to \mathfrak{e}$ over $\mathfrak{g}[t, t^{-1}]$. Since $\hat{\mathfrak{g}}$ is perfect, there is at most one such map, so F is unique. ◇

Remark 7.9.12 If $\mathfrak{g}$ is semisimple over $\mathbb{C}$, then $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$ for simple Lie algebras $\mathfrak{g}_i$. Consequently the universal central extension of $\mathfrak{g}[t, t^{-1}]$ is the product

$$0 \to k^r \to \hat{\mathfrak{g}}_1 \times \cdots \times \hat{\mathfrak{g}}_r \to \mathfrak{g}[t, t^{-1}] \to 0.$$

If k is a subfield of $\mathbb{C}$ and $\mathfrak{g}$ is simple over k, $\mathfrak{g} \otimes \mathbb{C}$ is semisimple over $\mathbb{C}$. If $\mathfrak{g} \otimes \mathbb{C}$ is simple then since $H_2(\mathfrak{g}, k) \otimes_k \mathbb{C} = H_2(\mathfrak{g} \otimes_k \mathbb{C}, \mathbb{C}) = \mathbb{C}$ it follows that $\hat{\mathfrak{g}}$ is still the universal central extension of $\mathfrak{g}[t, t^{-1}]$. However, this fails if $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_r$ because then $H_2(\mathfrak{g}, k) = k^r$.

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