C2.7: Category Theory

Sheet 3 — MT23

Section B of this problem sheet should be turned in for marking. Each problem sheet has the following structure:

- **Section A**: one or more introductory questions, with solutions provided for students. The solutions will be provided at the end of the week to allow the students to try to solve the questions independently.
- **Section B**: core questions to be handed in for marking (only sheets 1 and 3), usually not with solutions for students, and to be discussed in classes
- **Section C**: one or more optional extension questions (which might, but need not, be harder than those in Section B), with sketches of solutions/references provided for students.

Section A

1. Show that $f\colon X \to Y$ is a monomorphism iff the square

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is Cartesian (a pullback square). Similarly, show that it is an epimorphism iff the square

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "\mathrm{id}"] \\ Y \arrow[r, "\mathrm{id}"'] \arrow[ur, phantom, "\ulcorner", very near end] & Y \end{tikzcd}$$

is coCartesian (a pushout square).

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Section B

2. Prove that inductive (direct) limits commute with binary products in Set; i.e. for infinite sequences of sets $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ with maps $X_n \to X_{n+1}$ and $Y_n \to Y_{n+1}$ construct a natural map

$$\operatorname{colim}_n(X_n \times Y_n) \to (\operatorname{colim}_n X_n) \times (\operatorname{colim}_n Y_n)$$

and show it is an isomorphism.

3. Let k be a field. Construct a coproduct in the categories of unital and non-unital commutative k-algebras.

4. Observe that in the category of sets every morphism into the initial object is an isomorphism. Deduce that the category of sets is not equivalent to its opposite.

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Section C

5. (a) Observe that any non-zero vector space V has a monomorphism $k \to V$ from a one-dimensional vector space.

   (b) Deduce that under an equivalence $\mathrm{Vect} \cong \mathrm{Vect}^{op}$ the one-dimensional vector space k would be sent to itself.

   (c) Using the fact that an infinite-dimensional vector space has a smaller dimension (in the sense of cardinal arithmetic) than its dual, deduce that there is no equivalence $\mathrm{Vect} \cong \mathrm{Vect}^{op}$.

6. Prove that the category Top of topological spaces and continuous maps is both complete and cocomplete.

7. Prove that a faithfully full functor reflects any limits and colimits that exist in its codomain.

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