# Homological Algebra

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# 1 Abelian Categories

**Definition 1.** Let $\mathcal{C}$ be a category, and let $x \in \mathcal{C}$. We say that $x$ is **terminal** if for every $c \in \mathcal{C}$, there is exactly one morphism $c \to x$. Dually, we say that $x$ is **initial** if for every $c \in \mathcal{C}$, there is exactly one morphism $x \to c$.

**Definition 2.** A **zero object** in a category is an object that is both initial and terminal.

## 1.1 Ab-enriched Categories

**Definition 3.** A **pre-additive** or **Ab-enriched** category is a category in which every hom-set is equipped with the structure of an abelian group, such that composition

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

is $\mathbb{Z}$-bilinear.

**Proposition 1.** In an **Ab**-enriched category, any initial object is also terminal.

*Proof.* Let $*$ be initial. Then $1_*$ is the unique element of $\mathrm{Hom}(*, *)$, so $1_*$ is zero in this group. Then since composition respects the group structures, we have for any map $f : A \to *$,

$$f = 1_* \circ f = 0 \circ f = 0$$

so $*$ is terminal. $\square$

**Proposition 2.** If $\mathcal{C}$ is an **Ab**-enriched category, then so is its opposite category $\mathcal{C}^{\mathrm{op}}$.

*Proof.* For $X, Y \in \mathcal{C}^{\mathrm{op}}$, the sets

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

are already endowed with the structure of an abelian group. Thus, we only have to prove that composition is bilinear. Let $X, Y, Z \in \mathcal{C}$ and let

$$f, f' \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y), \quad g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y, Z).$$

Then

$$g \circ_{\mathrm{op}} (f + f') = (f + f') \circ g = f \circ g + f' \circ g = g \circ_{\mathrm{op}} f + g \circ_{\mathrm{op}} f'.$$

Similarly, composition is linear in the other argument as well. $\square$

**Proposition 3.** In an **Ab**-enriched category $\mathcal{C}$, a binary product is also a binary coproduct.

*Proof.* Let $X_1, X_2$ be elments of an **Ab**-enriched category $\mathcal{C}$. Suppose that $X_1$ and $X_2$ have a product $X_1 \times X_2$ in $\mathcal{C}$, with projections $p_k : X_1 \times X_2 \to X_k$. By definition of products, there are unique morphisms $i_k : X_k \to X_1 \times X_2$ such that the following diagrams commute.

$$\begin{tikzcd}
X_1 \arrow[dr, dashed, "i_1"] \arrow[dd, "\mathrm{id}"'] \arrow[ddrr, bend left, "0"] & & \\
& X_1 \times X_2 \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X_1 & & X_2
\end{tikzcd}
\qquad
\begin{tikzcd}
& & X_2 \arrow[dl, dashed, "i_2"'] \arrow[dd, "\mathrm{id}"] \arrow[ddll, bend right, "0"'] \\
& X_1 \times X_2 \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X_1 & & X_2
\end{tikzcd}$$

Then we have

$$p_1 \circ (i_1 p_1 + i_2 p_2) = p_1, \quad p_2 \circ (i_1 p_1 + i_2 p_2) = p_2.$$

By definition of products, $\mathrm{id} X_1 \times X_2 \times X_1 \times X_2$ is the unique morphisms with $p_k \circ \mathrm{id} = p_k$ for each $k$, so $i_1 p_1 + i_2 p_2 = \mathrm{id}_{X_1 \times X_2}$. We claim that

$$\begin{tikzcd}
X_1 \arrow[dr, "i_1"] & & X_2 \arrow[dl, "i_2"'] \\
& X_1 \times X_2 &
\end{tikzcd}$$

is a universal cocone, so that $X_1 \times X_2 = X_1 \coprod X_2$. Suppose that

$$\begin{tikzcd} X_1 \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & A & \end{tikzcd}$$

is another cocone. Then we have a map

$$\phi = f_1 \circ p_1 + f_2 \circ p_2 : X_1 \times X_2 \to A,$$

which is easily seen to give a commutative diagram

$$\begin{tikzcd} X_1 \arrow[dr, "i_1"] \arrow[ddr, "f_1"'] & & X_2 \arrow[dl, "i_2"'] \arrow[ddl, "f_2"] \\ & X_1 \times X_2 \arrow[d, dashed, "\phi"] & \\ & A. & \end{tikzcd}$$

It remains to show that $\phi$ is unique. To see this, note that for any such $\phi$ we have

$$\begin{aligned} \phi &= \phi \circ \mathrm{id}_{X_1 \times X_2} \\ &= \phi \circ (i_1 p_1 + i_2 p_2) \\ &= \phi i_1 \circ p_1 + \phi i_2 \circ p_2 \\ &= f_1 \circ p_1 + f_2 \circ p_2. \end{aligned}$$

□

**Proposition 4.** In an **Ab**-enriched category, all binary coproducts are also binary products.

*Proof.* This is dual to the previous proposition. □

**Definition 4.** Let $\mathcal{C}$ be an **Ab**-enriched category, and let $x, y \in \mathcal{C}$. If $x$ and $y$ have a product in $\mathcal{C}$, then it is called the biproduct of $x$ and $y$, which we denote by $x \oplus y$.

**Definition 5.** Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between **Ab**-enriched categories. Then $F$ is said to be **additive** if it preserves finite biproducts.

**Lemma 1.** For any ring $R$, the category $R$-**mod** is **Ab**-enriched.

## 1.2 Additive Categories

**Definition 6.** A category is **additive** if it is **Ab**-enriched and admits finite coproducts.

**Lemma 2.** Let $\mathcal{A}$ be an additive category. Suppose that $i : a \to b$ is a monomorphism in $\mathcal{A}$ and $\alpha \in \mathrm{Hom}(a, b)$ is the zero morphism. Then $a = 0$.

*Proof.* Let $x \in \mathcal{A}$. Since $\mathrm{Hom}(a, x)$ is an abelian group, it contains at least one morphism (zero). Let $f : a \to x$ be any morphism. Then

$$\alpha \circ 0 = 0 = \alpha \circ f.$$

Since $\alpha$ is a monomorphism, we have $f = 0$. Therefore $a$ is initial, hence it is the zero object. □

**Lemma 3.** Let $\mathcal{A}$ be an additive category. Suppose that $q : a \to b$ is an epimorphism in $\mathcal{A}$. If $q = 0$, then $b = 0$.

*Proof.* Since $\mathcal{A}$ is additive, the opposite category $\mathcal{A}^{\mathrm{op}}$ is too. The map $q$ is a monomorphism $q : b \to a$ in $\mathcal{A}^{\mathrm{op}}$, and it is still the zero morphism. By the previous lemma we must therefore have that $b$ is the zero object in $\mathcal{A}^{\mathrm{op}}$, hence in $\mathcal{A}$. □

**Lemma 4.** For any ring $R$, the category $R$-**mod** is additive.

*Proof.* We know that the direct sum exists and is a coproduct in $R$-mod. □

## 1.3 Pre-abelian Categories

**Definition 7.** An additive category is **pre-abelian** if every morphism has a kernel and cokernel.

**Lemma 5.** Let $\mathcal{A}$ be a pre-abelian category. Every monomorphism has kernel 0, and every epimorphism has cokernel 0.

*Proof.* Let $i : a \to b$ be a monomorphism in $\mathcal{A}$. Let

$$\mathrm{Ker} i \xrightarrow{\ker i} a$$

be the kernel of $i$. Then $i \circ \ker i = 0 = i \circ 0$, so $\ker i$ is the zero morphism (since $i$ is a monomorphism). Since $\ker i$ is monomorphism, we have $\mathrm{Ker} i = 0$. □

**Lemma 6.** For any ring $R$, the category $R$-**mod** is pre-abelian.

## 1.4 Abelian Categories

**Definition 8.** A pre-abelian category is **abelian** if every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

**Lemma 7.** The category of left $R$-modules is an abelian category.

*Proof.* Let $i : A \to B$ be a monomorphism of $R$-modules. Then $\mathrm{Coker} i = B/i(A)$ and the cokernel map is the quotient $q : B \to B/i(A)$ with $q(b) = b + i(A)$. It is clear that $i(A) = \mathrm{Ker} q$ in the set-theoretic sense, so $i$ exhibits $A$ as the kernel of $q$.

Let $q : A \to B$ be an epimorphism of $R$-modules. Let $i : \mathrm{Ker} q \to A$ be the inclusion. Then $\mathrm{Coker} i = A/\mathrm{Ker} q \cong B$, so $q$ exhibits $B$ as the cokernel of $i$. □

**Lemma 8.** If $\mathcal{A}$ is abelian, then so is $\mathcal{A}^{\mathrm{op}}$.

*Proof.* Duality. □

**Lemma 9.** If $\mathcal{A}$ is an abelian category and $\mathcal{C}$ is any category, then $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ is abelian.