1
Chain Complexes

1.1 Complexes of R-Modules

Homological algebra is a tool used in several branches of mathematics: algebraic topology, group theory, commutative ring theory, and algebraic geometry come to mind. It arose in the late 1800s in the following manner. Let f and g be matrices whose product is zero. If $g \cdot v = 0$ for some column vector v, say, of length n, we cannot always write $v = f \cdot u$. This failure is measured by the *defect*

$$d = n - \operatorname{rank}(f) - \operatorname{rank}(g).$$

In modern language, f and g represent linear maps

$$U \xrightarrow{f} V \xrightarrow{g} W$$

with $gf = 0$, and d is the dimension of the *homology module*

$$H = \ker(g)/f(U).$$

In the first part of this century, Poincaré and other algebraic topologists utilized these concepts in their attempts to describe "n-dimensional holes" in simplicial complexes. Gradually people noticed that "vector space" could be replaced by "R-module" for any ring R.

This being said, we fix an associative ring R and begin again in the category **mod**–R of right R-modules. Given an R-module homomorphism $f: A \to B$, one is immediately led to study the kernel $\ker(f)$, cokernel $\operatorname{coker}(f)$, and image $\operatorname{im}(f)$ of f. Given another map $g: B \to C$, we can form the sequence

$$(*) \qquad A \xrightarrow{f} B \xrightarrow{g} C.$$

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We say that such a sequence is *exact* (at B) if $\ker(g) = \mathrm{im}(f)$. This implies in particular that the composite $gf: A \to C$ is zero, and finally brings our attention to sequences $(*)$ such that $gf = 0$.

Definition 1.1.1 A *chain complex* $C_.$ of R-modules is a family $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules, together with R-module maps $d = d_n: C_n \to C_{n-1}$ such that each composite $d \circ d: C_n \to C_{n-2}$ is zero. The maps d_n are called the *differentials* of $C_.$. The kernel of d_n is the module of *n-cycles* of $C_.$, denoted $Z_n = Z_n(C_.)$. The image of $d_{n+1}: C_{n+1} \to C_n$ is the module of *n-boundaries* of $C_.$, denoted $B_n = B_n(C_.)$. Because $d \circ d = 0$, we have

$$0 \subseteq B_n \subseteq Z_n \subseteq C_n$$

for all n. The n^{th} *homology module* of $C_.$ is the subquotient $H_n(C_.) = Z_n/B_n$ of C_n. Because the dot in $C_.$ is annoying, we will often write C for $C_.$.

Exercise 1.1.1 Set $C_n = \mathbb{Z}/8$ for $n \geq 0$ and $C_n = 0$ for $n < 0$; for $n > 0$ let d_n send $x \pmod 8$ to $4x \pmod 8$. Show that $C_.$ is a chain complex of $\mathbb{Z}/8$—modules and compute its homology modules.

There is a category **Ch(mod–R)** of chain complexes of (right) R-modules. The objects are, of course, chain complexes. A *morphism* $u: C_. \to D_.$ is a chain complex map, that is, a family of R-module homomorphisms $u_n: C_n \to D_n$ commuting with d in the sense that $u_{n-1}d_n = d_{n-1}u_n$. That is, such that the following diagram commutes

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{d} & C_{n+1} & \xrightarrow{d} & C_n & \xrightarrow{d} & C_{n-1} & \xrightarrow{d} & \cdots \\
 & & \downarrow u & & \downarrow u & & \downarrow u & & \\
\cdots & \xrightarrow{d} & D_{n+1} & \xrightarrow{d} & D_n & \xrightarrow{d} & D_{n-1} & \xrightarrow{d} & \cdots .
\end{array}$$

Exercise 1.1.2 Show that a morphism $u: C_. \to D_.$ of chain complexes sends boundaries to boundaries and cycles to cycles, hence maps $H_n(C_.) \to H_n(D_.)$. Prove that each H_n is a functor from **Ch(mod–R)** to **mod–R**.

Exercise 1.1.3 (Split exact sequences of vector spaces) Choose vector spaces $\{B_n, H_n\}_{n\in\mathbb{Z}}$ over a field, and set $C_n = B_n \oplus H_n \oplus B_{n-1}$. Show that the projection-inclusions $C_n \to B_{n-1} \subset C_{n-1}$ make $\{C_n\}$ into a chain complex, and that every chain complex of vector spaces is isomorphic to a complex of this form.

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Exercise 1.1.4 Show that $\{\mathrm{Hom}_R(A, C_n)\}$ forms a chain complex of abelian groups for every R-module A and every R-module chain complex C. Taking $A = Z_n$, show that if $H_n(\mathrm{Hom}_R(Z_n, C)) = 0$, then $H_n(C) = 0$. Is the converse true?

Definition 1.1.2 A morphism $C_. \to D_.$ of chain complexes is called a *quasi-isomorphism* (Bourbaki uses *homologism*) if the maps $H_n(C_.) \to H_n(D_.)$ are all isomorphisms.

Exercise 1.1.5 Show that the following are equivalent for every $C_.$:

1. $C_.$ is *exact*, that is, exact at every C_n.
2. $C_.$ is *acyclic*, that is, $H_n(C_.) = 0$ for all n.
3. The map $0 \to C_.$ is a quasi-isomorphism, where "0" is the complex of zero modules and zero maps.

The following variant notation is obtained by reindexing with superscripts: $C^n = C_{-n}$. A *cochain complex* $C^\cdot$ of R-modules is a family $\{C^n\}$ of R-modules, together with maps $d^n\colon C^n \to C^{n+1}$ such that $d \circ d = 0$. $Z^n(C^\cdot) = \ker(d^n)$ is the module of *n-cocycles*, $B^n(C^\cdot) = \mathrm{im}(d^{n-1}) \subseteq C^n$ is the module of *n-coboundaries*, and the subquotient $H^n(C^\cdot) = Z^n/B^n$ of C^n is the n^{th} *cohomology module* of $C^\cdot$. Morphisms and quasi-isomorphisms of cochain complexes are defined exactly as for chain complexes.

A chain complex $C_.$ is called *bounded* if almost all the C_n are zero; if $C_n = 0$ unless $a \le n \le b$, we say that the complex has *amplitude* in $[a, b]$. A complex $C_.$ is *bounded above* (resp. *bounded below*) if there is a bound b (resp. a) such that $C_n = 0$ for all $n > b$ (resp. $n < a$). The bounded (resp. bounded above, resp. bounded below) chain complexes form full subcategories of **Ch** = **Ch**(R–**mod**) that are denoted $\mathbf{Ch}_b$, $\mathbf{Ch}_-$ and $\mathbf{Ch}_+$, respectively. The subcategory $\mathbf{Ch}_{\ge 0}$ of non-negative complexes $C_.$ ($C_n = 0$ for all $n < 0$) will be important in Chapter 8.

Similarly, a cochain complex $C^\cdot$ is called *bounded above* if the chain complex $C_.$ ($C_n = C^{-n}$) is bounded below, that is, if $C^n = 0$ for all large n; $C^\cdot$ is *bounded below* if $C_.$ is bounded above, and *bounded* if $C_.$ is bounded. The categories of bounded (resp. bounded above, resp. bounded below, resp. non-negative) cochain complexes are denoted $\mathbf{Ch}^b$, $\mathbf{Ch}^-$, $\mathbf{Ch}^+$, and $\mathbf{Ch}^{\ge 0}$, respectively.

Exercise 1.1.6 (Homology of a graph) Let Γ be a finite graph with V vertices $(v_1, \cdots, v_V)$ and E edges $(e_1, \cdots, e_E)$. If we orient the edges, we can form the *incidence matrix* of the graph. This is a $V \times E$ matrix whose (ij) entry is $+1$

if the edge e_j starts at v_i, -1 if e_j ends at v_i, and 0 otherwise. Let C_0 be the free R-module on the vertices, C_1 the free R-module on the edges, $C_n = 0$ if $n \neq 0, 1$, and $d\colon C_1 \to C_0$ be the incidence matrix. If Γ is connected (i.e., we can get from v_0 to every other vertex by tracing a path with edges), show that $H_0(C)$ and $H_1(C)$ are free R-modules of dimensions 1 and $V - E - 1$ respectively. (The number $V - E - 1$ is the number of *circuits* of the graph.) *Hint:* Choose basis $\{v_0, v_1 - v_0, \cdots, v_V - v_0\}$ for C_0, and use a path from v_0 to v_i to find an element of C_1 mapping to $v_i - v_0$.

Application 1.1.3 (Simplicial homology) Here is a topological application we shall discuss more in Chapter 8. Let K be a geometric simplicial complex, such as a triangulated polyhedron, and let K_k $(0 \leq k \leq n)$ denote the set of k-dimensional simplices of K. Each k-simplex has $k+1$ faces, which are ordered if the set K_0 of vertices is ordered (do so!), so we obtain $k+1$ set maps $\partial_i\colon K_k \to K_{k-1} (0 \leq i \leq k)$. The *simplicial chain complex* of K with coefficients in R is the chain complex $C_.$, formed as follows. Let C_k be the free R-module on the set K_k; set $C_k = 0$ unless $0 \leq k \leq n$. The set maps ∂_i yield $k+1$ module maps $C_k \to C_{k-1}$, which we also call ∂_i; their alternating sum $d = \sum(-1)^i \partial_i$ is the map $C_k \to C_{k-1}$ in the chain complex $C_.$. To see that $C_.$ is a chain complex, we need to prove the algebraic assertion that $d \circ d = 0$. This translates into the geometric fact that each $(k-2)$-dimensional simplex contained in a fixed k-simplex σ of K lies on exactly two faces of σ. The homology of the chain complex $C_.$ is called the *simplicial homology* of K with coefficients in R. This simplicial approach to homology was used in the first part of this century, before the advent of singular homology.

Exercise 1.1.7 (Tetrahedron) The tetrahedron T is a surface with 4 vertices, 6 edges, and 4 2-dimensional faces. Thus its homology is the homology of a chain complex $0 \to R^4 \to R^6 \to R^4 \to 0$. Write down the matrices in this complex and verify computationally that $H_2(T) \cong H_0(T) \cong R$ and $H_1(T) = 0$.

Application 1.1.4 (Singular homology) Let X be a topological space, and let $S_k = S_k(X)$ be the free R-module on the set of continuous maps from the standard k-simplex Δ_k to X. Restriction to the i^{th} face of Δ_k $(0 \leq i \leq k)$ transforms a map $\Delta_k \to X$ into a map $\Delta_{k-1} \to X$, and induces an R-module homomorphism ∂_i from S_k to S_{k-1}. The alternating sums $d = \sum(-1)^i \partial_i$ (from S_k to S_{k-1}) assemble to form a chain complex

$$\cdots \xrightarrow{d} S_2 \xrightarrow{d} S_1 \xrightarrow{d} S_0 \longrightarrow 0,$$

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called the *singular chain complex* of X. The n^{th} homology module of $S_.(X)$ is called the n^{th} *singular homology* of X (with coefficients in R) and is written $H_n(X; R)$. If X is a geometric simplicial complex, then the obvious inclusion $C_.(X) \to S_.(X)$ is a quasi-isomorphism, so the simplicial and singular homology modules of X are isomorphic. The interested reader may find details in any standard book on algebraic topology.

1.2 Operations on Chain Complexes

The main point of this section will be that chain complexes form an abelian category. First we need to recall what an abelian category is. A reference for these definitions is [MacCW].

A category $\mathcal{A}$ is called an **Ab**-*category* if every hom-set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ in $\mathcal{A}$ is given the structure of an abelian group in such a way that composition distributes over addition. In particular, given a diagram in $\mathcal{A}$ of the form

$$A \xrightarrow{f} B \underset{g}{\overset{g'}{\rightrightarrows}} C \xrightarrow{h} D$$

we have $h(g+g')f = hgf + hg'f$ in $\mathrm{Hom}(A, D)$. The category **Ch** is an **Ab**-category because we can add chain maps degreewise; if $\{f_n\}$ and $\{g_n\}$ are chain maps from $C_.$ to $D_.$, their sum is the family of maps $\{f_n + g_n\}$.

An *additive functor* $F: \mathcal{B} \to \mathcal{A}$ between **Ab**-categories $\mathcal{B}$ and $\mathcal{A}$ is a functor such that each $\mathrm{Hom}_{\mathcal{B}}(B', B) \to \mathrm{Hom}_{\mathcal{A}}(FB', FB)$ is a group homomorphism.

An *additive category* is an **Ab**-category $\mathcal{A}$ with a zero object (i.e., an object that is initial and terminal) and a product $A \times B$ for every pair A, B of objects in $\mathcal{A}$. This structure is enough to make finite products the same as finite coproducts. The zero object in **Ch** is the complex "0" of zero modules and maps. Given a family $\{A_\alpha\}$ of complexes of R-modules, the product ΠA_α and coproduct (direct sum) $\oplus A_\alpha$ exist in **Ch** and are defined degreewise: the differentials are the maps

$$\prod d_\alpha : \prod_\alpha A_{\alpha,n} \to \prod_\alpha A_{\alpha,n-1} \quad \text{and} \quad \oplus\, d_\alpha : \oplus_\alpha A_{\alpha,n} \to \oplus_\alpha A_{\alpha,n-1},$$

respectively. These suffice to make **Ch** into an additive category.

Exercise 1.2.1 Show that direct sum and direct product commute with homology, that is, that $\oplus H_n(A_\alpha) \cong H_n(\oplus A_\alpha)$ and $\Pi H_n(A_\alpha) \cong H_n(\Pi A_\alpha)$ for all n.

Here are some important constructions on chain complexes. A chain complex B is called a *subcomplex* of C if each B_n is a submodule of C_n and the differential on B is the restriction of the differential on C, that is, when the inclusions $i_n : B_n \subseteq C_n$ constitute a chain map $B \to C$. In this case we can assemble the quotient modules C_n/B_n into a chain complex

$$\cdots \to C_{n+1}/B_{n+1} \xrightarrow{d} C_n/B_n \xrightarrow{d} C_{n-1}/B_{n-1} \xrightarrow{d} \cdots$$

denoted C/B and called the *quotient complex*. If $f\colon B \to C$ is a chain map, the kernels $\{\ker(f_n)\}$ assemble to form a subcomplex of B denoted $\ker(f)$, and the cokernels $\{\operatorname{coker}(f_n)\}$ assemble to form a quotient complex of C denoted $\operatorname{coker}(f)$.

Definition 1.2.1 In any additive category $\mathcal{A}$, a *kernel* of a morphism $f\colon B \to C$ is defined to be a map $i\colon A \to B$ such that $fi = 0$ and that is universal with respect to this property. Dually, a *cokernel* of f is a map $e\colon C \to D$, which is universal with respect to having $ef = 0$. In $\mathcal{A}$, a map $i\colon A \to B$ is *monic* if $ig = 0$ implies $g = 0$ for every map $g\colon A' \to A$, and a map $e\colon C \to D$ is an *epi* if $he = 0$ implies $h = 0$ for every map $h\colon D \to D'$. (The definition of monic and epi in a non-abelian category is slightly different; see A.1 in the Appendix.) It is easy to see that every kernel is monic and that every cokernel is an epi (exercise!).

Exercise 1.2.2 In the additive category $\mathcal{A} = R$–**mod**, show that:

1. The notions of kernels, monics, and monomorphisms are the same.
2. The notions of cokernels, epis, and epimorphisms are also the same.

Exercise 1.2.3 Suppose that $\mathcal{A} =$ **Ch** and f is a chain map. Show that the complex $\ker(f)$ is a kernel of f and that $\operatorname{coker}(f)$ is a cokernel of f.

Definition 1.2.2 An *abelian category* is an additive category $\mathcal{A}$ such that

1. every map in $\mathcal{A}$ has a kernel and cokernel.
2. every monic in $\mathcal{A}$ is the kernel of its cokernel.
3. every epi in $\mathcal{A}$ is the cokernel of its kernel.

The prototype abelian category is the category **mod**–R of R-modules. In any abelian category the *image* $\operatorname{im}(f)$ of a map $f\colon B \to C$ is the subobject $\ker(\operatorname{coker} f)$ of C; in the category of R-modules, $\operatorname{im}(f) = \{f(b) : b \in B\}$. Every map f factors as

$$B \xrightarrow{e} \operatorname{im}(f) \xrightarrow{m} C$$

with e an epimorphism and m a monomorphism. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of maps in $\mathcal{A}$ is called *exact* (at B) if $\ker(g) = \operatorname{im}(f)$.

A subcategory $\mathcal{B}$ of $\mathcal{A}$ is called an *abelian subcategory* if it is abelian, and an exact sequence in $\mathcal{B}$ is also exact in $\mathcal{A}$.

If $\mathcal{A}$ is any abelian category, we can repeat the discussion of section 1.1 to define chain complexes and chain maps in $\mathcal{A}$—just replace **mod**–R by $\mathcal{A}$! These form an additive category $\mathbf{Ch}(\mathcal{A})$, and homology becomes a functor from this category to $\mathcal{A}$. In the sequel we will merely write **Ch** for $\mathbf{Ch}(\mathcal{A})$ when $\mathcal{A}$ is understood.

Theorem 1.2.3 *The category $\mathbf{Ch} = \mathbf{Ch}(\mathcal{A})$ of chain complexes is an abelian category.*

Proof Condition 1 was exercise 1.2.3 above. If $f: B \to C$ is a chain map, I claim that f is monic iff each $B_n \to C_n$ is monic, that is, B is isomorphic to a subcomplex of C. This follows from the fact that the composite $\ker(f) \to C$ is zero, so if f is monic, then $\ker(f) = 0$. So if f is monic, it is isomorphic to the kernel of $C \to C/B$. Similarly, f is an epi iff each $B_n \to C_n$ is an epi, that is, C is isomorphic to the cokernel of the chain map $\ker(f) \to B_.$. ◇

Exercise 1.2.4 Show that a sequence $0 \to A_. \to B_. \to C_. \to 0$ of chain complexes is exact in **Ch** just in case each sequence $0 \to A_n \to B_n \to C_n \to 0$ is exact in $\mathcal{A}$.

Clearly we can iterate this construction and talk about chain complexes of chain complexes; these are usually called double complexes.

Example 1.2.4 A *double complex* (or *bicomplex*) in $\mathcal{A}$ is a family $\{C_{p,q}\}$ of objects of $\mathcal{A}$, together with maps

$$d^h: C_{p,q} \to C_{p-1,q} \quad \text{and} \quad d^v: C_{p,q} \to C_{p,q-1}$$

such that $d^h \circ d^h = d^v \circ d^v = d^v d^h + d^h d^v = 0$. It is useful to picture the bicomplex $C_{..}$ as a lattice

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$$
\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{p-1,q+1} & \xleftarrow{d^h} & C_{p,q+1} & \xleftarrow{d^h} & C_{p+1,p+1} & \longleftarrow & \cdots \\
 & & d^v\downarrow & & d^v\downarrow & & d^v\downarrow & & \\
\cdots & \longleftarrow & C_{p-1,q} & \xleftarrow{d^h} & C_{p,q} & \xleftarrow{d^h} & C_{p+1,q} & \longleftarrow & \cdots \\
 & & d^v\downarrow & & d^v\downarrow & & d^v\downarrow & & \\
\cdots & \longleftarrow & C_{p-1,q-1} & \xleftarrow{d^h} & C_{p,q-1} & \xleftarrow{d^h} & C_{p+1,q-1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}
$$

in which the maps d^h go horizontally, the maps d^v go vertically, and each square anticommutes. Each row C_{*q} and each column C_{p*} is a chain complex.

We say that a double complex C is *bounded* if C has only finitely many nonzero terms along each diagonal line $p + q = n$, for example, if C is concentrated in the first quadrant of the plane (a *first quadrant double complex*).

Sign Trick 1.2.5 Because of the anticommutivity, the maps d^v are not maps in **Ch**, but chain maps f_{*q} from C_{*q} to $C_{*,q-1}$ can be defined by introducing $\pm$ signs:

$$f_{p,q} = (-1)^p d^v_{p,q}\colon C_{p,q} \to C_{p,q-1}.$$

Using this sign trick, we can identify the category of double complexes with the category **Ch**(**Ch**) of chain complexes in the abelian category **Ch**.

Total Complexes 1.2.6 To see why the anticommutative condition $d^v d^h + d^h d^v = 0$ is useful, define the *total complexes* $\mathrm{Tot}(C) = \mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ by

$$\mathrm{Tot}^{\Pi}(C)_n = \prod_{p+q=n} C_{p,q} \quad \text{and} \quad \mathrm{Tot}^{\oplus}(C)_n = \bigoplus_{p+q=n} C_{p,q}.$$

The formula $d = d^h + d^v$ defines maps (check this!)

$$d : \mathrm{Tot}^{\Pi}(C)_n \to \mathrm{Tot}^{\Pi}(C)_{n-1} \quad \text{and} \quad d : \mathrm{Tot}^{\oplus}(C)_n \to \mathrm{Tot}^{\oplus}(C)_{n-1}$$

such that $d \circ d = 0$, making $\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ into chain complexes. Note that $\mathrm{Tot}^{\oplus}(C) = \mathrm{Tot}^{\Pi}(C)$ if C is bounded, and especially if C is a first quadrant double complex. The difference between $\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ will become apparent in Chapter 5 when we discuss spectral sequences.

Remark $\mathrm{Tot}^{\Pi}(C)$ and $\mathrm{Tot}^{\oplus}(C)$ do not exist in all abelian categories; they don't exist when $\mathcal{A}$ is the category of all finite abelian groups. We say that an abelian category is *complete* if all infinite direct products exist (and so Tot^{Π} exists) and that it is *cocomplete* if all infinite direct sums exist (and so $\mathrm{Tot}^{\oplus}$ exists). Both these axioms hold in R–**mod** and in the category of chain complexes of R-modules.

Exercise 1.2.5 Give an elementary proof that $\mathrm{Tot}(C)$ is acyclic whenever C is a bounded double complex with exact rows (or exact columns). We will see later that this result follows from the Acyclic Assembly Lemma 2.7.3. It also follows from a spectral sequence argument (see Definition 5.6.2 and exercise 5.6.4).

Exercise 1.2.6 Give examples of (1) a second quadrant double complex C with exact columns such that $\mathrm{Tot}^{\Pi}(C)$ is acyclic but $\mathrm{Tot}^{\oplus}(C)$ is not; (2) a second quadrant double complex C with exact rows such that $\mathrm{Tot}^{\oplus}(C)$ is acyclic but $\mathrm{Tot}^{\Pi}(C)$ is not; and (3) a double complex (in the entire plane) for which every row and every column is exact, yet neither $\mathrm{Tot}^{\Pi}(C)$ nor $\mathrm{Tot}^{\oplus}(C)$ is acyclic.

Truncations 1.2.7 If C is a chain complex and n is an integer, we let $\tau_{\geq n}C$ denote the subcomplex of C defined by

$$(\tau_{\geq n}C)_i = \begin{cases} 0 & \text{if } i < n \\ Z_n & \text{if } i = n \\ C_i & \text{if } i > n. \end{cases}$$

Clearly $H_i(\tau_{\geq n}C) = 0$ for $i < n$ and $H_i(\tau_{\geq n}C) = H_i(C)$ for $i \geq n$. The complex $\tau_{\geq n}C$ is called the (good) *truncation* of C below n, and the quotient complex $\tau_{<n}C = C/(\tau_{\geq n}C)$ is called the (good) truncation of C above n; $H_i(\tau_{<n}C)$ is $H_i(C)$ for $i < n$ and 0 for $i \geq n$.

Some less useful variants are the *brutal truncations* $\sigma_{<n}C$ and $\sigma_{\geq n}C = C/(\sigma_{<n}C)$. By definition, $(\sigma_{<n}C)_i$ is C_i if $i < n$ and 0 if $i \geq n$. These have the advantage of being easier to describe but the disadvantage of introducing the homology group $H_n(\sigma_{\geq n}C) = C_n/B_n$.

Translation 1.2.8 Shifting indices, or translation, is another useful operation we can perform on chain and cochain complexes. If C is a complex and p an integer, we form a new complex $C[p]$ as follows:

$$C[p]_n = C_{n+p} \quad (\text{resp. } C[p]^n = C^{n-p})$$

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with differential $(-1)^p d$. We call $C[p]$ the p^{th} *translate* of C. The way to remember the shift is that the degree 0 part of $C[p]$ is C_p. The sign convention is designed to simplify notation later on. Note that translation shifts homology:

$$H_n(C[p]) = H_{n+p}(C) \quad (\text{resp. } H^n(C[p]) = H^{n-p}(C)).$$

We make translation into a functor by shifting indices on chain maps. That is, if $f: C \to D$ is a chain map, then $f[p]$ is the chain map given by the formula

$$f[p]_n = f_{n+p} \quad (\text{resp. } f[p]^n = f^{n-p}).$$

Exercise 1.2.7 If C is a complex, show that there are exact sequences of complexes:

$$0 \longrightarrow Z(C) \longrightarrow C \xrightarrow{d} B(C)[-1] \longrightarrow 0;$$

$$0 \longrightarrow H(C) \longrightarrow C/B(C) \xrightarrow{d} Z(C)[-1] \longrightarrow H(C)[-1] \longrightarrow 0.$$

Exercise 1.2.8 (Mapping cone) Let $f: B \to C$ be a morphism of chain complexes. Form a double chain complex D out of f by thinking of f as a chain complex in **Ch** and using the sign trick, putting $B[-1]$ in the row $q = 1$ and C in the row $q = 0$. Thinking of C and $B[-1]$ as double complexes in the obvious way, show that there is a short exact sequence of double complexes

$$0 \longrightarrow C \longrightarrow D \xrightarrow{\delta} B[-1] \longrightarrow 0.$$

The total complex of D is cone(f'), the mapping cone (see section 1.5) of a map f', which differs from f only by some $\pm$ signs and is isomorphic to f.

1.3 Long Exact Sequences

It is time to unveil the feature that makes chain complexes so special from a computational viewpoint: the existence of long exact sequences.

Theorem 1.3.1 *Let* $0 \to A_. \xrightarrow{f} B_. \xrightarrow{g} C_. \to 0$ *be a short exact sequence of chain complexes. Then there are natural maps* $\partial: H_n(C) \to H_{n-1}(A)$, *called connecting homomorphisms, such that*

$$\cdots \xrightarrow{g} H_{n+1}(C) \xrightarrow{\partial} H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{f} \cdots$$

is an exact sequence.

Similarly, if $0 \to A^{\cdot} \xrightarrow{f} B^{\cdot} \xrightarrow{g} C^{\cdot} \to 0$ *is a short exact sequence of cochain complexes, there are natural maps* $\partial\colon H^n(C) \to H^{n+1}(A)$ *and a long exact sequence*

$$\cdots \xrightarrow{g} H^{n-1}(C) \xrightarrow{\partial} H^n(A) \xrightarrow{f} H^n(B) \xrightarrow{g} H^n(C) \xrightarrow{\partial} H^{n+1}(A) \xrightarrow{f} \cdots.$$

Exercise 1.3.1 Let $0 \to A \to B \to C \to 0$ be a short exact sequence of complexes. Show that if two of the three complexes A, B, C are exact, then so is the third.

Exercise 1.3.2 (3×3 lemma) Suppose given a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in an abelian category, such that every column is exact. Show the following:

1. If the bottom two rows are exact, so is the top row.
2. If the top two rows are exact, so is the bottom row.
3. If the top and bottom rows are exact, and the composite $A \to C$ is zero, the middle row is also exact.

Hint: Show the remaining row is a complex, and apply exercise 1.3.1.

The key tool in constructing the connecting homomorphism ∂ is our next result, the *Snake Lemma*. We will not print the proof in these notes, because it is best done visually. In fact, a clear proof is given by Jill Clayburgh at the beginning of the movie *It's My Turn* (Rastar-Martin Elfand Studios, 1980). As an exercise in "diagram chasing" of elements, the student should find a proof (but privately—keep the proof to yourself!).

Snake Lemma 1.3.2 *Consider a commutative diagram of* R*-modules of the form*

$$\begin{tikzcd} & A' \arrow[r] \arrow[d, "f"'] & B' \arrow[r] \arrow[d, "g"'] & C' \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A \arrow[r, "i"] & B \arrow[r] & C. \end{tikzcd}$$

If the rows are exact, there is an exact sequence

$$\ker(f) \to \ker(g) \to \ker(h) \xrightarrow{\partial} \operatorname{coker}(f) \to \operatorname{coker}(g) \to \operatorname{coker}(h)$$

with ∂ defined by the formula

$$\partial(c') = i^{-1} g p^{-1}(c'), \quad c' \in \ker(h).$$

Moreover, if $A' \to B'$ is monic, then so is $\ker(f) \to \ker(g)$, *and if $B \to C$ is onto, then so is* $\operatorname{coker}(f) \to \operatorname{coker}(g)$.

Etymology The term *snake* comes from the following visual mnemonic:

$$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d] & \ker(g) \arrow[r] \arrow[d] & \ker(h) \arrow[d] \arrow[dddll, dashed, rounded corners] \\ \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d, "h"] \\ \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \operatorname{coker}(f) \arrow[r] & \operatorname{coker}(g) \arrow[r] & \operatorname{coker}(h). \end{tikzcd}$$

Remark The Snake Lemma also holds in an arbitrary abelian category $\mathcal{C}$. To see this, let $\mathcal{A}$ be the smallest abelian subcategory of $\mathcal{C}$ containing the objects and morphisms of the diagram. Since $\mathcal{A}$ has a set of objects, the Freyd-Mitchell Embedding Theorem (see 1.6.1) gives an exact, fully faithful embedding of $\mathcal{A}$ into R–**mod** for some ring R. Since ∂ exists in R–**mod**, it exists in $\mathcal{A}$ and hence in $\mathcal{C}$. Similarly, exactness in R–**mod** implies exactness in $\mathcal{A}$ and hence in $\mathcal{C}$.

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Exercise 1.3.3 (5–Lemma) In any commutative diagram

$$\begin{array}{ccccccccc}
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\
a\downarrow\cong & & b\downarrow\cong & & c\downarrow & & d\downarrow\cong & & e\downarrow\cong \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E
\end{array}$$

with exact rows in any abelian category, show that if a, b, d, and e are isomorphisms, then c is also an isomorphism. More precisely, show that if b and d are monic and a is an epi, then c is monic. Dually, show that if b and d are epis and e is monic, then c is an epi.

We now proceed to the construction of the connecting homomorphism ∂ of Theorem 1.3.1 associated to a short exact sequence

$$0 \to A \to B \to C \to 0$$

of chain complexes. From the Snake Lemma and the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_nA & \longrightarrow & Z_nB & \longrightarrow & Z_nC & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n & \longrightarrow & 0 \\
 & & d\downarrow & & d\downarrow & & d\downarrow & & \\
0 & \longrightarrow & A_{n-1} & \longrightarrow & B_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \dfrac{A_{n-1}}{dA_n} & \longrightarrow & \dfrac{B_{n-1}}{dB_n} & \longrightarrow & \dfrac{C_{n-1}}{dC_n} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

we see that the rows are exact in the commutative diagram

$$\begin{array}{ccccccc}
 & & \dfrac{A_n}{dA_{n+1}} & \longrightarrow & \dfrac{B_n}{dB_{n+1}} & \longrightarrow & \dfrac{C_n}{dC_{n+1}} & \longrightarrow & 0 \\
 & & d\downarrow & & d\downarrow & & d\downarrow & & \\
0 & \longrightarrow & Z_{n-1}(A) & \xrightarrow{f} & Z_{n-1}(b) & \xrightarrow{g} & Z_{n-1}(C). & &
\end{array}$$

The kernel of the left vertical is $H_n(A)$, and its cokernel is $H_{n-1}(A)$. Therefore the Snake Lemma yields an exact sequence

$$H_n(A) \xrightarrow{f} H_n(B) \xrightarrow{g} H_n(C) \xrightarrow{\partial} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C).$$

The long exact sequence 1.3.1 is obtained by pasting these sequences together.

Addendum 1.3.3 When one computes with modules, it is useful to be able to push elements around. By decoding the above proof, we obtain the following formula for the connecting homomorphism: Let $z \in H_n(C)$, and represent it by a cycle $c \in C_n$. Lift the cycle to $b \in B_n$ and apply d. The element db of B_{n-1} actually belongs to the submodule $Z_{n-1}(A)$ and represents $\partial(z) \in H_{n-1}(A)$.

We shall now explain what we mean by the naturality of ∂. There is a category $\mathcal{S}$ whose objects are short exact sequences of chain complexes (say, in an abelian category $\mathcal{C}$). Commutative diagrams

$$(*) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

give the morphisms in $\mathcal{S}$ (from the top row to the bottom row). Similarly, there is a category $\mathcal{L}$ of long exact sequences in $\mathcal{C}$.

Proposition 1.3.4 *The long exact sequence is a functor from $\mathcal{S}$ to $\mathcal{L}$. That is, for every short exact sequence there is a long exact sequence, and for every map $(*)$ of short exact sequences there is a commutative ladder diagram*

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\partial} & H_n(A) & \longrightarrow & H_n(B) & \longrightarrow & H_n(C) & \xrightarrow{\partial} & H_{n-1}(A) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \xrightarrow{\partial} & H_n(A') & \longrightarrow & H_n(B') & \longrightarrow & H_n(C') & \xrightarrow{\partial} & H_{n-1}(A') & \longrightarrow & \cdots. \end{array}$$

Proof All we have to do is establish the ladder diagram. Since each H_n is a functor, the left two squares commute. Using the Embedding Theorem 1.6.1, we may assume $\mathcal{C} = \mathbf{mod}\text{–}R$ in order to prove that the right square commutes. Given $z \in H_n(C)$, represented by $c \in C_n$, its image $z' \in H_n(C')$ is represented by the image of c. If $b \in B_n$ lifts c, its image in B'_n lifts c'. Therefore by 1.3.3 $\partial(z') \in H_{n-1}(A')$ is represented by the image of db, that is, by the image of a representative of $\partial(z)$, so $\partial(z')$ is the image of $\partial(z)$. ◇

Remark 1.3.5 The data of the long exact sequence is sometimes organized into the mnemonic shape

$$\begin{tikzcd} H_*(A) \arrow[rr] & & H_*(B) \arrow[dl] \\ & H_*(C) \arrow[ul, "\partial"] & \end{tikzcd}$$

This is called an *exact triangle* for obvious reasons. This mnemonic shape is responsible for the term "triangulated category," which we will discuss in Chapter 10. The category **K** of chain equivalence classes of complexes and maps (see exercise 1.4.5 in the next section) is an example of a triangulated category.

Exercise 1.3.4 Consider the boundaries-cycles exact sequence $0 \to Z \to C \to B(-1) \to 0$ associated to a chain complex C (exercise 1.2.7). Show that the corresponding long exact sequence of homology breaks up into short exact sequences.

Exercise 1.3.5 Let f be a morphism of chain complexes. Show that if $\ker(f)$ and $\operatorname{coker}(f)$ are acyclic, then f is a quasi-isomorphism. Is the converse true?

Exercise 1.3.6 Let $0 \to A \to B \to C \to 0$ be a short exact sequence of double complexes of modules. Show that there is a short exact sequence of total complexes, and conclude that if $\operatorname{Tot}(C)$ is acyclic, then $\operatorname{Tot}(A) \to \operatorname{Tot}(B)$ is a quasi-isomorphism.

1.4 Chain Homotopies

The ideas in this section and the next are motivated by homotopy theory in topology. We begin with a discussion of a special case of historical importance. If C is any chain complex of vector spaces over a field, we can always choose vector space decompositions:

$$C_n = Z_n \oplus B'_n, \qquad B'_n \cong C_n/Z_n = d(C_n) = B_{n-1};$$
$$Z_n = B_n \oplus H'_n, \qquad H'_n \cong Z_n/B_n = H_n(C).$$

Therefore we can form the compositions

$$C_n \to Z_n \to B_n \cong B'_{n+1} \subseteq C_{n+1}$$

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to get splitting maps $s_n\colon C_n \to C_{n+1}$, such that $d = dsd$. The compositions ds and sd are projections from C_n onto B_n and B'_n, respectively, so the sum $ds + sd$ is an endomorphism of C_n whose kernel H'_n is isomorphic to the homology $H_n(C)$. The kernel (and cokernel!) of $ds + sd$ is the trivial homology complex $H_*(C)$. Evidently both chain maps $H_*(C) \to C$ and $C \to H_*(C)$ are quasi-isomorphisms. Moreover, C is an exact sequence if and only if $ds + sd$ is the identity map.

Over an arbitrary ring R, it is not always possible to split chain complexes like this, so we give a name to this notion.

Definition 1.4.1 *A complex C is called* split *if there are maps $s_n\colon C_n \to C_{n+1}$ such that $d = dsd$. The maps s_n are called the* splitting maps. *If in addition C is acyclic (exact as a sequence), we say that C is* split exact.

Example 1.4.2 Let $R = \mathbb{Z}$ or $\mathbb{Z}/4$, and let C be the complex

$$\cdots \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \cdots.$$

This complex is acyclic but not split exact. There is no map s such that $ds + sd$ is the identity map, nor is there any direct sum decomposition $C_n \cong Z_n \oplus B'_n$.

Exercise 1.4.1 The previous example shows that even an acyclic chain complex of free R-modules need not be split exact.

1. Show that acyclic *bounded below* chain complexes of free R-modules are always split exact.
2. Show that an acyclic chain complex of finitely generated free abelian groups is always split exact, even when it is not bounded below.

Exercise 1.4.2 Let C be a chain complex, with boundaries B_n and cycles Z_n in C_n. Show that C is split if and only if there are R-module decompositions $C_n \cong Z_n \oplus B'_n$ and $Z_n = B_n \oplus H'_n$. Show that C is split exact iff $H'_n = 0$.

Now suppose that we are given two chain complexes C and D, together with randomly chosen maps $s_n\colon C_n \to D_{n+1}$. Let f_n be the map from C_n to D_n defined by the formula $f_n = d_{n+1}s_n + s_{n-1}d_n$.

$$\begin{tikzcd} C_{n+1} \arrow[r, "d"] & C_n \arrow[r, "d"] \arrow[dl, "s"'] \arrow[d, "f"] & C_{n-1} \arrow[dl, "s"'] \\ D_{n+1} \arrow[r, "d"'] & D_n \arrow[r, "d"'] & D_{n-1} \end{tikzcd}$$

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Dropping the subscripts for clarity, we compute

$$df = d(ds + sd) = dsd = (ds + sd)d = fd.$$

Thus $f = ds + sd$ is a chain map from C to D.

Definition 1.4.3 We say that a chain map $f: C \to D$ is *null homotopic* if there are maps $s_n: C_n \to D_{n+1}$ such that $f = ds + sd$. The maps $\{s_n\}$ are called a *chain contraction* of f.

Exercise 1.4.3 Show that C is a split exact chain complex if and only if the identity map on C is null homotopic.

The chain contraction construction gives us an easy way to proliferate chain maps: if $g: C \to D$ is any chain map, so is $g + (sd + ds)$ for *any* choice of maps s_n. However, $g + (sd + ds)$ is not very different from g, in a sense that we shall now explain.

Definition 1.4.4 We say that two chain maps f and g from C to D are *chain homotopic* if their difference $f - g$ is null homotopic, that is, if

$$f - g = sd + ds.$$

The maps $\{s_n\}$ are called a *chain homotopy* from f to g. Finally, we say that $f: C \to D$ is a *chain homotopy equivalence* (Bourbaki uses *homotopism*) if there is a map $g: D \to C$ such that gf and fg are chain homotopic to the respective identity maps of C and D.

Remark This terminology comes from topology via the following observation. A map f between two topological spaces X and Y induces a map $f_*: S(X) \to S(Y)$ between the corresponding singular chain complexes. It turns out that if f is topologically null homotopic (resp. a homotopy equivalence), then the chain map f_* is null homotopic (resp. a chain homotopy equivalence), and if two maps f and g are topologically homotopic, then f_* and g_* are chain homotopic.

Lemma 1.4.5 *If $f: C \to D$ is null homotopic, then every map $f_*: H_n(C) \to H_n(D)$ is zero. If f and g are chain homotopic, then they induce the same maps $H_n(C) \to H_n(D)$.*

Proof It is enough to prove the first assertion, so suppose that $f = ds + sd$. Every element of $H_n(C)$ is represented by an n-cycle x. But then $f(x) = d(sx)$. That is, $f(x)$ is an n-boundary in D. As such, $f(x)$ represents 0 in $H_n(D)$. ◇

Exercise 1.4.4 Consider the homology $H_*(C)$ of C as a chain complex with zero differentials. Show that if the complex C is split, then there is a chain homotopy equivalence between C and $H_*(C)$. Give an example in which the converse fails.

Exercise 1.4.5 In this exercise we shall show that the chain homotopy classes of maps form a quotient category **K** of the category **Ch** of all chain complexes. The homology functors H_n on **Ch** will factor through the quotient functor **Ch** $\to$ **K**.

1. Show that chain homotopy equivalence is an equivalence relation on the set of all chain maps from C to D. Let $\mathrm{Hom}_{\mathbf{K}}(C, D)$ denote the equivalence classes of such maps. Show that $\mathrm{Hom}_{\mathbf{K}}(C, D)$ is an abelian group.
2. Let f and g be chain homotopic maps from C to D. If $u\colon B \to C$ and $v\colon D \to E$ are chain maps, show that vfu and vgu are chain homotopic. Deduce that there is a category **K** whose objects are chain complexes and whose morphisms are given in (1).
3. Let f_0, f_1, g_0, and g_1 be chain maps from C to D such that f_i is chain homotopic to g_i $(i = 1, 2)$. Show that $f_0 + f_1$ is chain homotopic to $g_0 + g_1$. Deduce that **K** is an additive category, and that **Ch** $\to$ **K** is an additive functor.
4. Is **K** an abelian category? Explain.

1.5 Mapping Cones and Cylinders

1.5.1 Let $f\colon B_. \to C_.$ be a map of chain complexes. The *mapping cone* of f is the chain complex $\mathrm{cone}(f)$ whose degree n part is $B_{n-1} \oplus C_n$. In order to match other sign conventions, the differential in $\mathrm{cone}(f)$ is given by the formula

$$d(b, c) = (-d(b), d(c) - f(b)), \quad (b \in B_{n-1}, c \in C_n).$$

That is, the differential is given by the matrix

$$\begin{bmatrix} -d_B & 0 \\ -f & +d_C \end{bmatrix} : \begin{array}{ccc} B_{n-1} & \xrightarrow{-} & B_{n-2} \\ \oplus & \searrow^{-} & \oplus \\ C_n & \xrightarrow[+]{} & C_{n-1} \end{array} .$$

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Here is the dual notion for a map $f: B^\cdot \to C^\cdot$ of cochain complexes. The mapping cone, cone(f), is a cochain complex whose degree n part is $B^{n+1} \oplus C^n$. The differential is given by the same formula as above with the same signs.

Exercise 1.5.1 Let cone(C) denote the mapping cone of the identity map id_C of C; it has $C_{n-1} \oplus C_n$ in degree n. Show that cone(C) is split exact, with $s(b, c) = (-c, 0)$ defining the splitting map.

Exercise 1.5.2 Let $f: C \to D$ be a map of complexes. Show that f is null homotopic if and only if f extends to a map $(-s, f): \mathrm{cone}(C) \to D$.

1.5.2 Any map $f_*: H_*(B) \to H_*(C)$ can be fit into a long exact sequence of homology groups by use of the following device. There is a short exact sequence

$$0 \to C \to \mathrm{cone}(f) \xrightarrow{\delta} B[-1] \to 0$$

of chain complexes, where the left map sends c to $(0, c)$, and the right map sends (b, c) to $-b$. Recalling (1.2.8) that $H_{n+1}(B[-1]) \cong H_n(B)$, the homology long exact sequence (with connecting homomorphism ∂) becomes

$$\cdots \to H_{n+1}(\mathrm{cone}(f)) \xrightarrow{\delta *} H_n(B) \xrightarrow{\partial} H_n(C) \to H_n(\mathrm{cone}(f)) \xrightarrow{\delta *} H_{n-1}(B) \xrightarrow{\partial} \cdots.$$

The following lemma shows that $\partial = f_*$, fitting f_* into a long exact sequence.

Lemma 1.5.3 *The map ∂ in the above sequence is f_*.*

Proof If $b \in B_n$ is a cycle, the element $(-b, 0)$ in the cone complex lifts b via δ. Applying the differential we get $(db, fb) = (0, fb)$. This shows that

$$\partial[b] = [fb] = f_*[b]. \qquad \diamond$$

Corollary 1.5.4 *A map $f: B \to C$ is a quasi-isomorphism if and only if the mapping cone complex cone(f) is exact. This device reduces questions about quasi-isomorphisms to the study of split complexes.*

Topological Remark Let K be a simplicial complex (or more generally a cell complex). The *topological cone* CK of K is obtained by adding a new vertex s to K and "coning off" the simplices (cells) to get a new $(n+1)$-simplex for every old n-simplex of K. (See Figure 1.1.) The simplicial (cellular) chain complex $C_.(s)$ of the one-point space $\{s\}$ is R in degree 0 and zero elsewhere. $C_.(s)$ is a subcomplex of the simplicial (cellular) chain complex $C_.(CK)$ of

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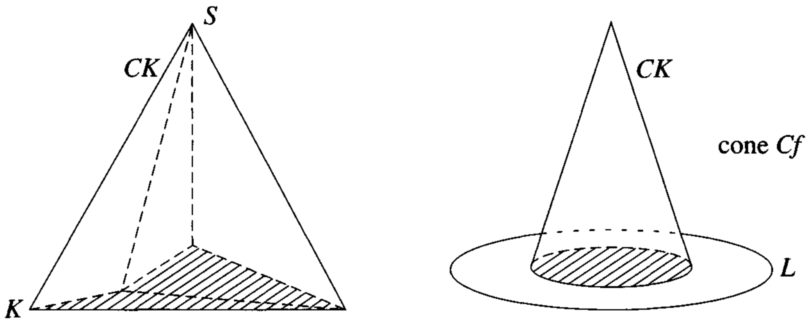

Figure 1.1. The topological cone CK and mapping cone Cf.

the topological cone CK. The quotient $C_.(CK)/C_.(s)$ is the chain complex cone$(C_.K)$ of the identity map of $C_.(K)$. The algebraic fact that cone$(C_.K)$ is split exact (null homotopic) reflects the fact that the topological cone CK is contractible.

More generally, if $f: K \to L$ is a simplicial map (or a cellular map), the *topological mapping cone* Cf of f is obtained by glueing CK and L together, identifying the subcomplex K of CK with its image in L (Figure 1.1). This is a cellular complex, which is simplicial if f is an inclusion of simplicial complexes. Write $C_.(Cf)$ for the cellular chain complex of the topological mapping cone Cf. The quotient chain complex $C_.(Cf)/C_.(s)$ may be identified with cone(f_*), the mapping cone of the chain map $f_*: C_.(K) \to C_.(L)$.

1.5.5 A related construction is that of the *mapping cylinder* cyl(f) of a chain complex map $f: B_. \to C_.$. The degree n part of cyl(f) is $B_n \oplus B_{n-1} \oplus C_n$, and the differential is

$$d(b, b', c) = (d(b) + b', -d(b'), d(c) - f(b')).$$

That is, the differential is given by the matrix

$$\begin{bmatrix} d_B & \mathrm{id}_B & 0 \\ 0 & -d_B & 0 \\ 0 & -f & d_C \end{bmatrix} \quad : \quad \begin{tikzcd} B_n \arrow[r, "+"] & B_{n-1} \\ \oplus & \oplus \\ B_{n-1} \arrow[uur, "+"] \arrow[r, "-"] \arrow[ddr, "-"] & B_{n-2} \\ \oplus & \oplus \\ C_n \arrow[r, "+"] & C_{n-1} \end{tikzcd}$$

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The cylinder is a chain complex because

$$d^2 = \begin{bmatrix} d_B^2 & d_B - d_B & 0 \\ 0 & d_B^2 & 0 \\ 0 & fd_B - d_C f & d_C^2 \end{bmatrix} = 0.$$

Exercise 1.5.3 Let $\mathrm{cyl}(C)$ denote the mapping cylinder of the identity map id_C of C; it has $C_n \oplus C_{n-1} \oplus C_n$ in degree n. Show that two chain maps $f, g\colon C \to D$ are chain homotopic if and only if they extend to a map (f, s, g): $\mathrm{cyl}(C) \to D$.

Lemma 1.5.6 *The subcomplex of elements* $(0, 0, c)$ *is isomorphic to* C*, and the corresponding inclusion* $\alpha\colon C \to \mathrm{cyl}(f)$ *is a quasi-isomorphism.*

Proof The quotient $\mathrm{cyl}(f)/\alpha(C)$ is the mapping cone of $-\mathrm{id}_B$, so it is null-homotopic (exercise 1.5.1). The lemma now follows from the long exact homology sequence for

$$0 \longrightarrow C \xrightarrow{\alpha} \mathrm{cyl}(f) \longrightarrow \mathrm{cone}(-\mathrm{id}_B) \longrightarrow 0. \qquad \diamond$$

Exercise 1.5.4 Show that $\beta(b, b', c) = f(b) + c$ defines a chain map from $\mathrm{cyl}(f)$ to C such that $\beta\alpha = \mathrm{id}_C$. Then show that the formula $s(b, b', c) = (0, b, 0)$ defines a chain homotopy from the identity of $\mathrm{cyl}(f)$ to $\alpha\beta$. Conclude that α is in fact a chain homotopy equivalence between C and $\mathrm{cyl}(f)$.

Topological Remark Let X be a cellular complex and let I denote the interval [0,1]. The space $I \times X$ is the topological cylinder of X. It is also a cell complex; every n-cell e^n in X gives rise to three cells in $I \times X$: the two n-cells, $0 \times e^n$ and $1 \times e^n$, and the $(n+1)$-cell $(0, 1) \times e^n$. If $C_.(X)$ is the cellular chain complex of X, then the cellular chain complex $C_.(I \times X)$ of $I \times X$ may be identified with $\mathrm{cyl}(\mathrm{id}_{C_.X})$, the mapping cylinder chain complex of the identity map on $C_.(X)$.

More generally, if $f\colon X \to Y$ is a cellular map, then the topological mapping cylinder $\mathrm{cyl}(f)$ is obtained by glueing $I \times X$ and Y together, identifying $0 \times X$ with the image of X under f (see Figure 1.2). This is also a cellular complex, whose cellular chain complex $C_.(\mathrm{cyl}(f))$ may be identified with the mapping cylinder of the chain map $C_.(X) \to C_.(Y)$.

The constructions in this section are the algebraic analogues of the usual topological constructions $I \times X \simeq X$, $\mathrm{cyl}(f) \simeq Y$, and so forth which were used by Dold and Puppe to get long exact sequences for any generalized homology theory on topological spaces.

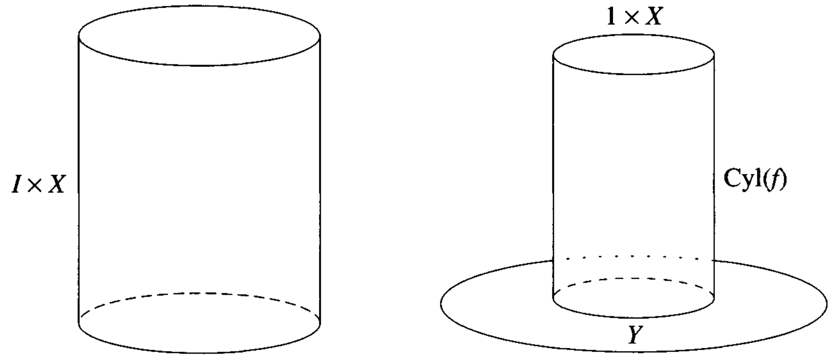

Figure 1.2. The topological cylinder of X and mapping cylinder $\mathrm{cyl}(f)$.

Here is how to use mapping cylinders to fit f_* into a long exact sequence of homology groups. The subcomplex of elements $(b, 0, 0)$ in $\mathrm{cyl}(f)$ is isomorphic to B, and the quotient $\mathrm{cyl}(f)/B$ is the mapping cone of f. The composite $B \to \mathrm{cyl}(f) \xrightarrow{\beta} C$ is the map f, where β is the equivalence of exercise 1.5.4, so on homology $f_*\colon H(B) \to H(C)$ factors through $H(B) \to H(\mathrm{cyl}(f))$. Therefore we may construct a commutative diagram of chain complexes with exact rows:

$$
\begin{tikzcd}
 & & C & & & \\
0 \arrow[r] & B \arrow[r] \arrow[ur, "f"] & \mathrm{cyl}(f) \arrow[r] \arrow[u, "\beta"'] & \mathrm{cone}(f) \arrow[r] & 0 & \\
 & 0 \arrow[r] & C \arrow[r] \arrow[u, "\alpha"'] & \mathrm{cone}(f) \arrow[r, "\delta"] \arrow[u, equal] & B[-1] \arrow[r] & 0.
\end{tikzcd}
$$

The homology long exact sequences fit into the following diagram:

$$
\begin{tikzcd}
\cdots \arrow[r, "-\partial"] & H_n(B) \arrow[r] \arrow[d, equal, "\wr"] \arrow[dr, "f"] & H_n(\mathrm{cyl}(f)) \arrow[r] \arrow[d, equal, "\wr"] & H_n(\mathrm{cone}(f)) \arrow[r, "-\partial"] \arrow[d, equal] & H_{n-1}(B) \arrow[r] \arrow[d, equal, "\wr"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(B[-1]) \arrow[r, "\partial"'] & H_n(C) \arrow[r] & H_n(\mathrm{cone}(f)) \arrow[r, "\delta"] & H_n(B[-1]) \arrow[r, "\partial"'] & \cdots
\end{tikzcd}
$$

Lemma 1.5.7 *This diagram is commutative, with exact rows.*

Proof It suffices to show that the right square (with $-\partial$ and δ) commutes.

Let (b, c) be an n-cycle in cone(f), so $d(b) = 0$ and $f(b) = d(c)$. Lift it to $(0, b, c)$ in cyl(f) and apply the differential:

$$d(0, b, c) = (0 + b, -db, dc - fb) = (b, 0, 0).$$

Therefore ∂ maps the class of (b, c) to the class of $b = -\delta(b, c)$ in $H_{n-1}(B)$.

◇

1.5.8 The cone and cylinder constructions provide a natural way to fit the homology of *every* chain map $f: B \to C$ into *some* long exact sequence (see 1.5.2 and 1.5.7). To show that the long exact sequence is well defined, we need to show that the usual long exact homology sequence attached to any short exact sequence of complexes

$$0 \to B \xrightarrow{f} C \xrightarrow{g} D \to 0$$

agrees both with the long exact sequence attached to f and with the long exact sequence attached to g.

We first consider the map f. There is a chain map φ: cone$(f) \to D$ defined by the formula $\varphi(b, c) = g(c)$. It fits into a commutative diagram with exact rows:

$$\begin{tikzcd}
 & 0 \arrow[r] & C \arrow[r] \arrow[d, "\alpha"] & \mathrm{cone}(f) \arrow[r, "\delta"] \arrow[d, equal] & B[-1] \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, equal] & \mathrm{cyl}(f) \arrow[r] \arrow[d, "\beta"] & \mathrm{cone}(f) \arrow[r] \arrow[d, "\varphi"] & 0 & \\
0 \arrow[r] & B \arrow[r, "f"] & C \arrow[r, "g"] & D \arrow[r] & 0. &
\end{tikzcd}$$

Since β is a quasi-isomorphism, it follows from the 5-lemma and 1.3.4 that φ is a quasi-isomorphism as well. The following exercise shows that φ need not be a chain homotopy equivalence.

Exercise 1.5.5 Suppose that the B and C of 1.5.8 are modules, considered as chain complexes concentrated in degree zero. Then cone(f) is the complex $0 \to B \xrightarrow{-f} C \to 0$. Show that φ is a chain homotopy equivalence iff $f : B \subset C$ is a split injection.

To continue, the naturality of the connecting homomorphism ∂ provides us with a natural isomorphism of long exact sequences:

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$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & H_n(B) \arrow[r] \arrow[d, equal] & H_n(\mathrm{cyl}(f)) \arrow[r] \arrow[d, "\cong"] & H_n(\mathrm{cone}(f)) \arrow[r, "\partial"] \arrow[d, "\cong"] & H_{n-1}(B) \arrow[r] \arrow[d, equal, "\wr"] & \cdots \\
\cdots \arrow[r, "\partial"] & H_n(B) \arrow[r] & H_n(C) \arrow[r] & H_n(D) \arrow[r, "\partial"] & H_{n-1}(B) \arrow[r] & \cdots .
\end{tikzcd}$$

Exercise 1.5.6 Show that the composite

$$H_n(D) \cong H_n(\mathrm{cone}(f)) \xrightarrow{-\delta_*} H_n(B[-1]) \cong H_{n-1}(B)$$

is the connecting homomorphism ∂ in the homology long exact sequence for

$$0 \to B \to C \to D \to 0.$$

Exercise 1.5.7 Show that there is a quasi-isomorphism $B[-1] \to \mathrm{cone}(g)$ dual to φ. Then dualize the preceding exercise, by showing that the composite

$$H_n(D) \xrightarrow{\partial} H_{n-1}(B) \xrightarrow{\simeq} H_n(\mathrm{cone}(g))$$

is the usual map induced by the inclusion of D in $\mathrm{cone}(g)$.

Exercise 1.5.8 Given a map $f: B \to C$ of complexes, let v denote the inclusion of C into $\mathrm{cone}(f)$. Show that there is a chain homotopy equivalence $\mathrm{cone}(v) \to B[-1]$. This equivalence is the algebraic analogue of the topological fact that for any map $f: K \to L$ of (topological) cell complexes the cone of the inclusion $L \subset Cf$ is homotopy equivalent to the suspension of K.

Exercise 1.5.9 Let $f: B \to C$ be a morphism of chain complexes. Show that the natural maps $\ker(f)[-1] \xrightarrow{\partial} \mathrm{cone}(f) \xrightarrow{\beta} \mathrm{coker}(f)$ give rise to a long exact sequence:

$$\cdots \xrightarrow{\partial} H_{n-1}(\ker(f)) \xrightarrow{\alpha} H_n(\mathrm{cone}(f)) \xrightarrow{\beta} H_n(\mathrm{coker}(f)) \xrightarrow{\partial} H_{n-2}(\ker(f)) \cdots .$$

Exercise 1.5.10 Let C and C' be split complexes, with splitting maps s, s'. If $f: C \to C'$ is a morphism, show that $\sigma(c, c') = (-s(c), s'(c') - s' f s(c))$ defines a splitting of $\mathrm{cone}(f)$ if and only if the map $f_*: H_*(C) \to H_*(C')$ is zero.

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1.6 More on Abelian Categories

We have already seen that R–**mod** is an abelian category for every associative ring R. In this section we expand our repertoire of abelian categories to include functor categories and sheaves. We also introduce the notions of left exact and right exact functors, which will form the heart of the next chapter. We give the Yoneda embedding of an additive category, which is exact and fully faithful, and use it to sketch a proof of the following result, which has already been used. Recall that a category is called *small* if its class of objects is in fact a set.

Freyd-Mitchell Embedding Theorem 1.6.1 (1964) *If $\mathcal{A}$ is a small abelian category, then there is a ring R and an exact, fully faithful functor from $\mathcal{A}$ into R–***mod***, which embeds $\mathcal{A}$ as a full subcategory in the sense that* $\mathrm{Hom}_{\mathcal{A}}(M, N) \cong \mathrm{Hom}_R(M, N)$.

We begin to prepare for this result by introducing some examples of abelian categories. The following criterion, whose proof we leave to the reader, is frequently useful:

Lemma 1.6.2 *Let $\mathcal{C} \subset \mathcal{A}$ be a full subcategory of an abelian category $\mathcal{A}$.*

1. *$\mathcal{C}$ is additive $\Leftrightarrow 0 \in \mathcal{C}$, and $\mathcal{C}$ is closed under $\oplus$.*
2. *$\mathcal{C}$ is abelian and $\mathcal{C} \subset \mathcal{A}$ is exact $\Leftrightarrow$ $\mathcal{C}$ is additive, and $\mathcal{C}$ is closed under* ker *and* coker.

Examples 1.6.3

1. Inside R–**mod**, the finitely generated R-modules form an additive category, which is abelian if and only if R is noetherian.
2. Inside **Ab**, the torsionfree groups form an additive category, while the p-groups form an abelian category. (A is a *p-group* if $(\forall a \in A)$ some $p^n a = 0$.) Finite p-groups also form an abelian category. The category $(\mathbb{Z}/p)$–**mod** of vector spaces over the field $\mathbb{Z}/p$ is also a full subcategory of **Ab**.

Functor Categories 1.6.4 Let C be any category, $\mathcal{A}$ an abelian category. The *functor category* $\mathcal{A}^C$ is the abelian category whose objects are functors $F: C \to \mathcal{A}$. The maps in $\mathcal{A}^C$ are natural transformations. Here are some relevant examples:

1. If C is the discrete category of integers, $\mathbf{Ab}^C$ contains the abelian category of *graded abelian groups* as a full subcategory.

2. If C is the poset category of integers $(\cdots \to n \to (n+1) \to \cdots)$ then the abelian category $\mathbf{Ch}(\mathcal{A})$ of cochain complexes is a full subcategory of $\mathcal{A}^C$.
3. If R is a ring considered as a one-object category, then R–**mod** is the full subcategory of all additive functors in $\mathbf{Ab}^R$.
4. Let X be a topological space, and $\mathcal{U}$ the poset of open subsets of X. A contravariant functor F from $\mathcal{U}$ to $\mathcal{A}$ such that $F(\emptyset) = \{0\}$ is called a *presheaf* on X with values in $\mathcal{A}$, and the presheaves are the objects of the abelian category $\mathcal{A}^{\mathcal{U}^{op}} = \text{Presheaves}(X)$.

A typical example of a presheaf with values in $\mathbb{R}$–**mod** is given by $C^0(U) = \{\text{continuous functions } f: U \to \mathbb{R}\}$. If $U \subset V$ the maps $C^0(V) \to C^0(U)$ are given by restricting the domain of a function from V to U. In fact, C^0 is a sheaf:

Definition 1.6.5 (Sheaves) A *sheaf* on X (with values in $\mathcal{A}$) is a presheaf F satisfying the

Sheaf Axiom. Let $\{U_i\}$ be an open covering of an open subset U of X. If $\{f_i \in F(U_i)\}$ are such that each f_i and f_j agree in $F(U_i \cap U_j)$, then there is a unique $f \in F(U)$ that maps to every f_i under $F(U) \to F(U_i)$.

Note that the uniqueness of f is equivalent to the assertion that if $f \in F(U)$ vanishes in every $F(U_i)$, then $f = 0$. In fancy (element-free) language, the sheaf axiom states that for every covering $\{U_i\}$ of every open U the following sequence is exact:

$$0 \to F(U) \longrightarrow \prod F(U_i) \xrightarrow{\text{diff}} \prod_{i<j} F(U_i \cap U_j).$$

Exercise 1.6.1 Let M be a smooth manifold. For each open U in M, let $C^\infty(M)$ be the set of smooth functions from U to $\mathbb{R}$. Show that $C^\infty(M)$ is a sheaf on M.

Exercise 1.6.2 (Constant sheaves) Let A be any abelian group. For every open subset U of X, let $A(U)$ denote the set of continuous maps from U to the discrete topological space A. Show that A is a sheaf on X.

The category Sheaves(X) of sheaves forms an abelian category contained in Presheaves(X), but it is not an abelian subcategory; cokernels in Sheaves(X) are different from cokernels in Presheaves(X). This difference gives rise to sheaf cohomology (Chapter 2, section 2.6). The following example lies at the heart of the subject. For any space X, let $\mathcal{O}$ (resp. $\mathcal{O}^*$) be the sheaf such that

$\mathcal{O}(U)$ (resp. $\mathcal{O}^*(U)$) is the group of continuous maps from U into $\mathbb{C}$ (resp. $\mathbb{C}^*$). Then there is a short exact sequence of sheaves:

$$0 \to \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0.$$

When X is the space $\mathbb{C}^*$, this sequence is not exact in Presheaves(X) because the exponential map from $\mathbb{C} = \mathcal{O}(X)$ to $\mathcal{O}^*(X)$ is not onto; the cokernel is $\mathbb{Z} = H^1(X, \mathbb{Z})$, generated by the global unit $1/z$. In effect, there is no global logarithm function on X, and the contour integral $\frac{1}{2\pi i} \oint f(z)\, dz$ gives the image of $f(z)$ in the cokernel.

Definition 1.6.6 Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. F is called *left exact* (resp. *right exact*) if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, the sequence $0 \to F(A) \to F(B) \to F(C)$ (resp. $F(A) \to F(B) \to F(C) \to 0$) is exact in $\mathcal{B}$. F is called *exact* if it is both left and right exact, that is, if it preserves exact sequences. A contravariant functor F is called left exact (resp. right exact, resp. exact) if the corresponding covariant functor $F'\colon \mathcal{A}^{op} \to \mathcal{B}$ is left exact (resp. . . .).

Example 1.6.7 The inclusion of Sheaves(X) into Presheaves(X) is a left exact functor. There is also an exact functor Presheaves(X) $\to$ Sheaves(X), called "sheafification." (See 2.6.5; the sheafification functor is left adjoint to the inclusion.)

Exercise 1.6.3 Show that the above definitions are equivalent to the following, which are often given as the definitions. (See [Rot], for example.) A (covariant) functor F is left exact (resp. right exact) if exactness of the sequence

$$0 \to A \to B \to C \quad (\text{resp. } A \to B \to C \to 0)$$

implies exactness of the sequence

$$0 \to FA \to FB \to FC \quad (\text{resp. } FA \to FB \to FC \to 0).$$

Proposition 1.6.8 *Let $\mathcal{A}$ be an abelian category. Then* $\mathrm{Hom}_{\mathcal{A}}(M, -)$ *is a left exact functor from $\mathcal{A}$ to* **Ab** *for every M in $\mathcal{A}$. That is, given an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$, the following sequence of abelian groups is also exact:*

$$0 \to \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C).$$

Proof If $\alpha \in \text{Hom}(M, A)$ then $f_*\alpha = f \circ \alpha$; if this is zero, then α must be zero since f is monic. Hence f_* is monic. Since $g \circ f = 0$, we have $g_* f_*(\alpha) = g \circ f \circ \alpha = 0$, so $g_* f_* = 0$. It remains to show that if $\beta \in \text{Hom}(M, B)$ is such that $g_*\beta = g \circ \beta$ is zero, then $\beta = f \circ \alpha$ for some α. But if $g \circ \beta = 0$, then $\beta(M) \subseteq f(A)$, so β factors through A. ◇

Corollary 1.6.9 $\text{Hom}_{\mathcal{A}}(-, M)$ *is a left exact contravariant functor.*

Proof $\text{Hom}_{\mathcal{A}}(A, M) = \text{Hom}_{\mathcal{A}^{op}}(M, A)$. ◇

Yoneda Embedding 1.6.10 Every additive category $\mathcal{A}$ can be embedded in the abelian category $\mathbf{Ab}^{\mathcal{A}^{op}}$ by the functor h sending A to $h_A = \text{Hom}_{\mathcal{A}}(-, A)$. Since each $\text{Hom}_{\mathcal{A}}(M, -)$ is left exact, h is a left exact functor. Since the functors h_A are left exact, the Yoneda embedding actually lands in the abelian subcategory $\mathcal{L}$ of all left exact contravariant functors from $\mathcal{A}$ to $\mathbf{Ab}$ whenever $\mathcal{A}$ is an abelian category.

Yoneda Lemma 1.6.11 *The Yoneda embedding h reflects exactness. That is, a sequence* $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ *in* $\mathcal{A}$ *is exact, provided that for every* M *in* $\mathcal{A}$ *the following sequence is exact:*

$$\text{Hom}_{\mathcal{A}}(M, A) \xrightarrow{\alpha *} \text{Hom}_{\mathcal{A}}(M, B) \xrightarrow{\beta *} \text{Hom}_{\mathcal{A}}(M, C).$$

Proof Taking $M = A$, we see that $\beta\alpha = \beta^*\alpha^*(id_A) = 0$. Taking $M = \ker(\beta)$, we see that the inclusion $\iota\colon \ker(\beta) \to B$ satisfies $\beta^*(\iota) = \beta\iota = 0$. Hence there is a $\sigma \in \text{Hom}(M, A)$ with $\iota = \alpha^*(\sigma) = \alpha\sigma$, so that $\ker(\beta) = \text{im}(\iota) \subseteq \text{im}(\alpha)$. ◇

We now sketch a proof of the Freyd-Mitchell Embedding Theorem 1.6.1; details may be found in [Freyd] or [Swan, pp. 14–22]. Consider the failure of the Yoneda embedding $h\colon \mathcal{A} \to \mathbf{Ab}^{\mathcal{A}^{op}}$ to be exact: if $0 \to A \to B \to C \to 0$ is exact in $\mathcal{A}$ and $M \in \mathcal{A}$, then define the abelian group $W(M)$ by exactness of

$$0 \to \text{Hom}_{\mathcal{A}}(M, A) \to \text{Hom}_{\mathcal{A}}(M, B) \to \text{Hom}_{\mathcal{A}}(M, C) \to W(M) \to 0.$$

In general $W(M) \neq 0$, and there is a short exact sequence of functors:

$$(*) \qquad 0 \to h_A \to h_B \to h_C \to W \to 0.$$

W is an example of a *weakly effaceable functor*, that is, a functor such that for all $M \in \mathcal{A}$ and $x \in W(M)$ there is a surjection $P \to M$ in $\mathcal{A}$ so that the map $W(M) \to W(P)$ sends x to zero. (To see this, take P to be the pullback $M \times_C B$, where $M \to C$ represents x, and note that $P \to C$ factors through B.) Next (see *loc. cit.*), one proves:

Proposition 1.6.12 *If $\mathcal{A}$ is small, the subcategory $\mathcal{W}$ of weakly effaceable functors is a localizing subcategory of $\mathbf{Ab}^{\mathcal{A}^{op}}$ whose quotient category is $\mathcal{L}$. That is, there is an exact "reflection" functor R from $\mathbf{Ab}^{\mathcal{A}^{op}}$ to $\mathcal{L}$ such that $R(L) = L$ for every left exact L and $R(W) \cong 0$ iff W is weakly effaceable.*

Remark Cokernels in $\mathcal{L}$ are different from cokernels in $\mathbf{Ab}^{\mathcal{A}^{op}}$, so the inclusion $\mathcal{L} \subset \mathbf{Ab}^{\mathcal{A}^{op}}$ is not exact, merely left exact. To see this, apply the reflection R to $(*)$. Since $R(h_A) = h_A$ and $R(W) \cong 0$, we see that

$$0 \to h_A \to h_B \to h_C \to 0$$

is an exact sequence in $\mathcal{L}$, but not in $\mathbf{Ab}^{\mathcal{A}^{op}}$.

Corollary 1.6.13 *The Yoneda embedding $h\colon \mathcal{A} \to \mathcal{L}$ is exact and fully faithful.*

Finally, one observes that the category $\mathcal{L}$ has arbitrary coproducts and has a faithfully projective object P. By a result of Gabriel and Mitchell [Freyd, p. 106], $\mathcal{L}$ is equivalent to the category R–**mod** of modules over the ring $R = \mathrm{Hom}_{\mathcal{L}}(P, P)$. This finishes the proof of the Embedding Theorem.

Example 1.6.14 The abelian category of graded R-modules may be thought of as the full subcategory of $(\prod_{i \in \mathbb{Z}} R)$-modules of the form $\oplus_{i \in \mathbb{Z}} M_i$. The abelian category of chain complexes of R-modules may be embedded in S–**mod**, where

$$S = (\prod_{i \in \mathbb{Z}} R)[d]/(d^2 = 0, \{dr = rd\}_{r \in R}, \{de_i = e_{i-1}d\}_{i \in \mathbb{Z}}).$$

Here $e_i\colon \prod R \to R \to \prod R$ is the i^{th} coordinate projection.