Homological Algebra

Sheet 2 — MT23

Section A

1. Show $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

   Solution: $\mathbb{Z}$ is a PID, thus projective $\mathbb{Z}$-mods are free. Divisible abelian groups can't be free.

   More explicit alternative: $\bigoplus_{i\in\mathbb{Q}} \mathbb{Z}e_i \to \mathbb{Q} \to 0$,

   $Hom(\mathbb{Q}, \bigoplus_{i\in\mathbb{Q}} \mathbb{Z}e_i) = 0$ thus $h^{\mathbb{Q}}$ not exact, i.e. $\mathbb{Q}$ not projective.

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Section B

2. Write an injective resolution for $\mathbb{Z}$ as a $\mathbb{Z}$-module.

3. Write free resolutions for:

    1. $\mathbb{Z}/2$ in $Mod_{\mathbb{Z}}$,
    2. $\mathbb{Z}/2$ in $Mod_{(\mathbb{Z}/2)[x]}$,
    3. $\mathbb{Z}/2$ in $Mod_{\mathbb{Z}[x]}$,
    4. $\mathbb{Z}/2$ in $Mod_{\mathbb{Z}[x]/2x}$.

4. R : commutative ring, $r \in R$, $M \in Mod_R$. $R[r^{-1}] := \frac{R[x]}{rx-1} = coker(R[x] \overset{rx-1}{\to} R[x])$, $M[r^{-1}] = coker(M[x] \overset{rx-1}{\to} M[x])$ where $M[x] = \{\sum_i m_i x^i\}$ is viewed naturally as an $R[x]$-module

    Show $M \otimes_R R[r^{-1}] \simeq M[r^{-1}]$.

5. Prove the general Frobenius reciprocity formula (Tensor-Hom adjunction):

    $Hom_S(A, Hom_R(B, C)) \cong Hom_R(A \otimes_S B, C)$. where A is a right S-module, B is an S-R-bimodule, and C is a right R-module.

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Section C

6. Show that every R-submodule of a free R-module M is free when R is a PID.

   Solution: Note that this would be a trivial statement if we were thinking about finitely-generated modules by the structure theorem (which implies free iff torsion-free. Generally for domain R, we only have free $\implies$ projective $\implies$ flat $\implies$ torsion-free. For a Dedekind domain, torsion-free $\implies$ flat! Note ideals in a Dedekind domain are projective, but may not be free).

   Let $N \leq M \cong \bigoplus_{i \in I} Re_i$. The well-ordering principle (equivalently, Zorn's lemma) equips I with an ordering $<$ rendering I well-ordered.

   For all $j \in I$, set $M_{<j} := \bigoplus_{i<j} Re_i$ and $M_j := M_{<j} \oplus Re_j$. Denote $\pi_j : N \cap M_j \to Re_j$

   Observe we have the following SES: $0 \to N \cap M_{<j} \to N \cap M_j \to im\pi_j \to 0$ since $\ker \pi_j \subseteq M_{<j}$ and $\subseteq N$ (and $N \cap M_{<j} \subseteq \ker \pi_j$).

   R is a PID, so $im\pi_j = a_j Re_j$ for some $a_j \in R$. Choose $n_j \in N \cap M_j$ such that $\pi_j(n_j) = a_j e_j$; in particular, set $n_j = 0$ iff $a_j = 0$. Let $J \subseteq I$ denote the j such that $a_j \neq 0$.

   (Linear Independence) Suppose $\sum_{i=1}^{k} r_{j_i} n_{j_i} = 0$ for some $r_{j_i} \in R$ with $j_1 < \cdots < j_k$. Thus $0 = \pi_{j_k}(\sum_{i=1}^{k} r_{j_i} n_{j_i}) = r_{j_k} a_{j_k}$; since R is a PID, this implies $r_{j_k} = 0$. Induction shows that $r_{j_{k-1}} = \cdots = r_{j_1} = 0$ as well. So we have $\sum Rn_j = \bigoplus Rn_j \subseteq N$.

   (Spanning) Assume $\bigoplus_{j \in J} Rn_j \subsetneq N$. Thus there exists a minimal $i \in J$ such that $\exists n \in M_i \cap N$ and $n \notin \bigoplus_{j \in J} Rn_j$. If $a_i = 0$, then $n \in M_i \cap N = M_{<i} \cap N$, which contradicts minimality of i; so $a_i \neq 0$, and there exists an $r \in R - \{0\}$ such that $\pi_i(n) = ra_i e_i$. Now $n - rn_i$ must also not be in $\bigoplus_{j \in J} Rn_j$ (since that would imply $n \in \bigoplus_{j \in J} Rn_j$). However, $\pi_i(n - rn_i) = 0$, thus $n - rn_i \in M_{<i} \cap N$, and this contradicts minimality of i.

   Thus, $\bigoplus_{j \in J} Rn_j \cong N$, and so N is a free R-module.

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