# Homological Algebra

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#### October 21, 2023

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# 1 Abelian Categories

**Definition 1.** Let  $\mathcal{C}$  be a category, and let  $x \in \mathcal{C}$ . We say that x is **terminal** if for every  $c \in \mathcal{C}$ , there is exactly one morphism  $c \to x$ . Dually, we say that x is **initial** if for every  $c \in \mathcal{C}$ , there is exactly one morphism  $x \to c$ .

**Definition 2.** A **zero object** in a category is an object that is both initial and terminal.

## 1.1 Ab-enriched Categories

**Definition 3.** A **pre-additive** or **Ab-enriched** category is a category in which every hom-set is equipped with the structure of an abelian group, such that composition

$$\operatorname{Hom}(X,Y)\times\operatorname{Hom}(Y,Z)\to\operatorname{Hom}(X,Z)$$

is  $\mathbb{Z}$ -bilinear.

**Proposition 1.** In an  ${\bf Ab}$ -enriched category, any initial object is also terminal.

*Proof.* Let \* be initial. Then  $1_*$  is the unique element of  $\operatorname{Hom}(*,*)$ , so  $1_*$  is zero in this group. Then since composition respects the group structures, we have for any map  $f: A \to *$ ,

$$f = 1_* \circ f = 0 \circ f = 0$$

so \* is terminal.

**Proposition 2.** If C is an **Ab**-enriched category, then so is its opposite category  $C^{op}$ .

*Proof.* For  $X, Y \in \mathcal{C}^{op}$ , the sets

$$\operatorname{Hom}_{\mathcal{C}^{\operatorname{op}}}(X,Y) = \operatorname{Hom}_{\mathcal{C}}(Y,X)$$

are already endowed with the structure of an abelian group. Thus, we only have to prove that composition is bilinear. Let  $X,Y,Z\in\mathcal{C}$  and let

$$f, f' \in \operatorname{Hom}_{\mathcal{C}^{\operatorname{op}}}(X, Y), \quad g \in \operatorname{Hom}_{\mathcal{C}^{\operatorname{op}}}(Y, Z).$$

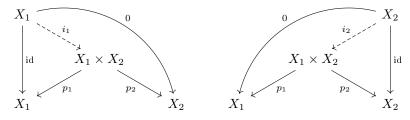
Then

$$g \circ_{\operatorname{op}} (f + f') = (f + f') \circ g = f \circ g + f' \circ g = g \circ_{\operatorname{op}} f + g \circ_{\operatorname{op}} f'.$$

Similarly, composition is linear in the other argument as well.

**Proposition 3.** In an Ab-enriched category C, a binary product is also a binary coproduct.

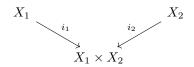
*Proof.* Let  $X_1, X_2$  be elments of an **Ab**-enriched category  $\mathcal{C}$ . Suppose that  $X_1$  and  $X_2$  have a product  $X_1 \times X_2$  in  $\mathcal{C}$ , with projections  $p_k : X_1 \times X_2 \to X_k$ . By definition of products, there are unique morphisms  $i_k : X_k \to X_1 \times X_2$  such that the following diagrams commute.



Then we have

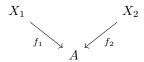
$$p_1 \circ (i_1p_1 + i_2p_2) = p_1, \quad p_2 \circ (i_1p_1 + i_2p_2) = p_2.$$

By definition of products,  $\mathrm{id} X_1 \times X_2 \times X_1 \times X_2$  is the unique morphisms with  $p_k \circ \mathrm{id} = p_k$  for each k, so  $i_1 p_1 + i_2 p_2 = \mathrm{id}_{X_1 \times X_2}$ . We claim that



 $\Box$ 

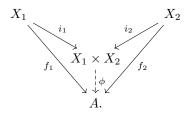
is a universal cocone, so that  $X_1 \times X_2 = X_1 \coprod X_2$ . Suppose that



is another cocone. Then we have a map

$$\phi = f_1 \circ p_1 + f_2 \circ p_2 : X_1 \times X_2 \to A,$$

which is easily seen to give a commutative diagram



It remains to show that  $\phi$  is unique. To see this, note that for any such  $\phi$  we have

$$\begin{split} \phi &= \phi \circ \operatorname{id}_{X_1 \times X_2} \\ &= \phi \circ (i_1 p_1 + i_2 p_2) \\ &= \phi i_1 \circ p_1 + \phi i_2 \circ p_2 \\ &= f_1 \circ p_1 + f_2 \circ p_2. \end{split}$$

**Proposition 4.** In an **Ab**-enriched category, all binary coproducts are also binary products.

*Proof.* This is dual to the previous proposition.

**Definition 4.** Let  $\mathcal{C}$  be an **Ab**-enriched category, and let  $x, y \in \mathcal{C}$ . If x and y have a product in  $\mathcal{C}$ , then it is called the biproduct of x and y, which we denote by  $x \oplus y$ .

**Definition 5.** Let  $F: A \to B$  be a functor between **Ab**-enriched categories. Then F is said to be **additive** if it preserves finite biproducts.

**Lemma 1.** For any ring R, the category R-mod is **Ab**-enriched.

#### 1.2 Additive Categories

**Definition 6.** A category is **additive** if it is **Ab**-enriched and admits finite coproducts.

**Lemma 2.** Let  $\mathcal{A}$  be an additive category. Suppose that  $i: a \to b$  is a monomorphism in  $\mathcal{A}$  and  $\alpha \in \operatorname{Hom}(a,b)$  is the zero morphism. Then a=0.

*Proof.* Let  $x \in \mathcal{A}$ . Since Hom(a, x) is an abelian group, it contains at least one morphism (zero). Let  $f: a \to x$  be any morphism. Then

$$\alpha \circ 0 = 0 = \alpha \circ f$$
.

Since  $\alpha$  is a monomorphism, we have f=0. Therefore a is initial, hence it is the zero object.

**Lemma 3.** Let  $\mathcal{A}$  be an additive category. Suppose that  $q:a\to b$  is an epimorphism in  $\mathcal{A}$ . If q=0, then b=0.

*Proof.* Since  $\mathcal{A}$  is additive, the opposite category  $\mathcal{A}^{\text{op}}$  is too. The map q is a monomorphism  $q:b\to a$  in  $\mathcal{A}^{\text{op}}$ , and it is still the zero morphism. By the previous lemma we must therefore have that b is the zero object in  $\mathcal{A}^{\text{op}}$ , hence in  $\mathcal{A}$ .

**Lemma 4.** For any ring R, the category R-mod is additive.

*Proof.* We know that the direct sum exists and is a coproduct in R-mod.  $\square$ 

#### 1.3 Pre-abelian Categories

**Definition 7.** An additive category is **pre-abelian** if every morphism has a kernel and cokernel.

**Lemma 5.** Let  $\mathcal{A}$  be a pre-abelian category. Every monomorphism has kernel 0, and every epimorphism has cokernel 0.

*Proof.* Let  $i: a \to b$  be a monomorphism in  $\mathcal{A}$ . Let

$$\operatorname{Ker} i \xrightarrow{\ker i} a$$

be the kernel of i. Then  $i \circ \ker i = 0 = i \circ 0$ , so  $\ker i$  is the zero morphism (since i is a monomorphism). Since  $\ker i$  is monomorphism, we have  $\operatorname{Ker} i = 0$ .

**Lemma 6.** For any ring R, the category R-mod is pre-abelian.

#### 1.4 Abelian Categories

**Definition 8.** A pre-abelian category is **abelian** if every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

**Lemma 7.** The category of left *R*-modules is an abelian category.

*Proof.* Let  $i:A\to B$  be a monomorphism of R-modules. Then  $\mathrm{Coker}i=B/i(A)$  and the cokernel map is the quotient  $q:B\to B/i(A)$  with q(b)=b+i(A). It is clear that  $i(A)=\mathrm{Ker}q$  in the set-theoretic sense, so i exhibits A as the kernel of q.

Let  $q:A\to B$  be an epimorphism of R-modules. Let  $i:\operatorname{Ker} q\to A$  be the inclusion. Then  $\operatorname{Coker} i=A/\operatorname{Ker} q\cong B$ , so q exhibits B as the cokernel of i.  $\square$ 

**Lemma 8.** If  $\mathcal{A}$  is abelian, then so is  $\mathcal{A}^{op}$ .

*Proof.* Duality.

**Lemma 9.** If  $\mathcal{A}$  is an abelian category and  $\mathcal{C}$  is any category, then  $Fun(\mathcal{C}, \mathcal{A})$  is abelian.

#### 1.5 Connection with R-mod

**Theorem 1** (Freyd-Mitchell Embedding Theorem). Let  $\mathcal{A}$  be a small abelian category. Then there is a ring R and an exact, fully faithful functor  $F: \mathcal{A} \to R\mathbf{mod}$ . This functor embeds  $\mathcal{A}$  as a full subcategory in  $R\mathbf{-mod}$ , by which we mean that for all  $M, N \in \mathcal{A}$ , we have

$$\operatorname{Hom}_{\mathcal{A}}(M,N) \cong \operatorname{Hom}_{R}(F(M),F(N)).s.$$

**Lemma 10.** The Freyd-Mitchell embedding preserves kernels and cokernels.

*Proof.* Let  $f: x \to y$  be a morphism in an abelian category  $\mathcal{A}$ , and let  $F: \mathcal{A} \to R$ **mod** be the Freyd-Mitchell embedding. Consider the sequence

$$0 \to \operatorname{Ker} f \xrightarrow{i} x \xrightarrow{f} y \xrightarrow{q} \operatorname{Coker} f \to 0.$$

**Lemma 11.** Let  $\mathcal{A}$  be an abelian category and let  $F: \mathcal{A} \to R$ -mod be the embedding from before. Then F(0) = 0.

1 ABELIAN CATEGORIES

#### 2 Exact Functors

### 2.1 Left- and Right-Exact Functors

**Definition 9.** A functor F is left-exact if for every short exact sequence  $0 \to A \to B \to C$ , the sequence

$$F(A) \to F(B) \to F(C) \to 0$$

is exact. Similarly, F is right-exact if instead

$$F(A) \to F(B) \to F(C) \to 0$$

is always exact.

**Lemma 12.** If  $F: A \to B$  is left-exact, and i is a monomorphism in A, then F(i) is a monomorphism in B.

*Proof.* If  $i: A \to B$  is a monomorphism, then we have a SES

$$0 \to A \to B \to \operatorname{coker} i \to 0$$
.

Therefore,  $0 \to F(A) \to F(B)$  is exact, so F(i) is a monomorphism.

**Lemma 13.** Let  $F: A \to \mathcal{B}$  be a functor between abelian categories. The following are equivalent:

- (i) F is left exact.
- (ii) For any exact sequence  $0 \to A \to B \to C$ , the corresponding sequence  $0 \to F(A) \to F(B) \to F(C)$  is also exact.

*Proof.* (ii)  $\Rightarrow$  (i): Trivial.

(i)  $\Rightarrow$  (ii): Let  $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C$  be exact. Then we have a short exact sequence  $0 \to A \to B \to \text{im} \pi \to 0$ , and therefor the sequence

$$0 \to F(A) \to F(B) \to F(\mathrm{im}\pi)$$

is exact. Now,  $\mathrm{im}\pi\to C$  is a monomorphism, so  $F(\mathrm{im}\pi)\to F(C)$  is too. Therefore

$$\ker(F(B) \to F(C)) = \ker(F(B) \to F(\operatorname{im}\pi) \to F(C))$$
$$= \ker(F(B) \to F(\operatorname{im}\pi))$$
$$= \operatorname{im}(F(A) \to F(B)).$$

By duality we have the dual result for right-exact functors.

**Lemma 14.** Let A be an abelian category, and consider maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in  $\mathcal{A}$ . Suppose that for all  $Z \in \mathcal{A}$ , the sequence

$$\operatorname{Hom}(A,Z) \xleftarrow{-\circ f} \operatorname{Hom}(B,Z) \xleftarrow{-\circ g} \operatorname{Hom}(C,Z) \leftarrow 0$$

is exact. Then  $A \to B \to C \to 0$  is exact.

*Proof.* We need to show that g exhibits C as the cokernel of f. Suppose that  $\alpha: B \to Z$  is some map with  $\alpha \circ f = 0$ . Then

$$\alpha \in \ker(-\circ f) = \operatorname{im}(g \circ -),$$

so  $\alpha = \phi \circ g$  for a unique map  $\phi : C \to Z$ . This is precisely the universal property of the cokernel.

#### Lemma 15. Suppose we have an adjunction



of additive functors between abelian categories, where F is the left adjoint. Then F is right-exact.

*Proof.* Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence in  $\mathcal{A}$ , and let  $Z \in \mathcal{B}$ . Then  $G(Z) \in \mathcal{A}$ , so

$$\operatorname{Hom}(A,G(Z)) \xleftarrow{-\circ i} \operatorname{Hom}(B,G(Z)) \xleftarrow{-\circ \pi} \operatorname{Hom}(C,G(Z)) \leftarrow 0$$

is exact by left-exactness of Hom. Therefore,

$$\operatorname{Hom}(F(A),Z) \xleftarrow{-\circ i} \operatorname{Hom}(F(B),Z) \xleftarrow{-\circ \pi} \operatorname{Hom}(F(C),Z) \leftarrow 0$$

is exact, so

$$F(A) \to F(B) \to F(C) \to 0$$

is exact by the previous lemma.

#### Corollary 1. If F, G are as in the previous lemma, then G is left exact.

*Proof.* This is just the dual statement. More explicitly, consider the opposite functor  $G: \mathcal{D}^{\text{op}} \to \mathcal{C}^{\text{op}}$  which is left adjoint (because the original G is right adjoint) and hence right exact. So  $G: \mathcal{D} \to \mathcal{C}$  is left exact.

#### 2.2 Exact Functors

**Definition 10.** A functor is **exact** if it is left-exact and right-exact.

Lemma 16. Suppose that we have a long exact sequence

$$\ldots \to A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \to \ldots$$

and an exact functor F. Then

$$\dots \to F(A_{n-1}) \to F(A_n) \to F(A_{n+1}) \to \dots$$

is also exact.

*Proof.* Since we only have to check exactness at each term, it suffices to prove that for an exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$
,

the sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is also exact. We prove this with a diagram-chase. Note that

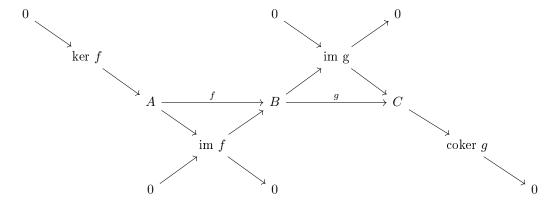
$$0 \to \ker\, f \to\!\! A \to \mathrm{im}\ f \to 0,$$

$$0 \to \ker g \to B \to \operatorname{im} g \to 0$$
,

and

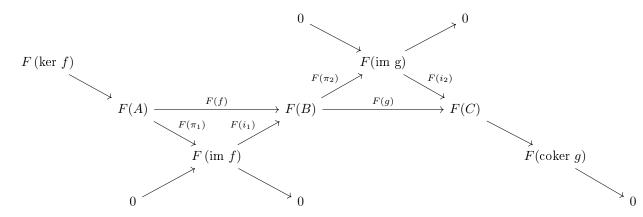
$$0 \to \mathrm{im}\ g \to C \to \mathrm{coker}\ g \to 0$$

are short exact sequences. We can fit these into a larger commutative diagram:



Note that the diagonals are exact. Applying F to the diagram (and removing

some redundant terms) gives a commutative diagram:



Again the diagonals are exact. Since  $F(\pi_1)$  is surjective, we have im  $F(f) = \lim F(i_1) = \ker F(\pi_2)$  by exactness at F(B). But  $F(i_2)$  is injective, so ker  $F(g) = \ker F(\pi_2)$ , and it follows that

im 
$$F(f) = \ker F(\pi_2) = \ker F(g)$$
.

### 2.3 Specific Functors

**Lemma 17.** Let  $\mathcal{A}$  be an abelian category, and let  $M \in \mathcal{A}$  be an object. Then the functor  $\operatorname{Hom}_{\mathcal{A}}(M,-): \mathcal{A} \to \mathbf{Ab}$  is left-exact.

*Proof.* Let  $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$  be a short exact sequence in  $\mathcal{A}$ . We have to show that the sequence

$$0 \to \operatorname{Hom}(M,A) \xrightarrow{i \circ -} \operatorname{Hom}(M,B) \xrightarrow{\pi \circ -} \operatorname{Hom}(M,C)$$

is exact. For exactness at  $\operatorname{Hom}(M,A)$ , suppose that  $i \circ \phi = 0$ , where  $\phi : M \to A$  is a map. Since i is a monomorphism and  $i \circ \phi = i \circ 0$ , we have  $\phi = 0$ . Therefore the sequence is exact at  $\operatorname{Hom}(M,A)$ .

Since  $\pi \circ i = 0$ , we have  $\operatorname{im}(i \circ -) \subset \ker(\pi \circ -)$ . Let  $\phi \in \ker(\pi \circ -)$ . Then  $\pi \circ \phi = 0$ . Since i exhibits A as the kernel of  $\pi$ , there is a unique map  $f: M \to A$  such that  $i \circ f = \phi$ . Therefore, the sequence is also exact at  $\operatorname{Hom}(M, B)$ .

Corollary 2. Let  $\mathcal{A}$  be an abelian category and let  $M \in \mathcal{A}$  be an object. Then the functor

$$\operatorname{Hom}_{\mathcal{A}}(-,M):\mathcal{A}^{\operatorname{op}}\to\mathbf{Ab}$$

is left-exact.

*Proof.* This follows from the definition of the opposite category.  $\Box$ 

Corollary 3. Let R be a ring and M be an R-module. Then the functors  $\operatorname{Hom}_R(M,-):R-\operatorname{mod}\to\operatorname{\mathbf{Ab}},\quad \operatorname{Hom}_R(-,M):R-\operatorname{\mathbf{mod}}^{\operatorname{op}}\to\operatorname{\mathbf{Ab}}$  are left-exact.

**Lemma 18.** For any ring R, the functor  $-\otimes_R N: R{\operatorname{-}\mathbf{mod}} \to R{\operatorname{-}\mathbf{mod}}$  is right-exact.

 ${\it Proof.}$  This follows from the adjunction

$$(-\otimes_R N)\dashv \operatorname{Hom}_R(N,-)$$

and Lemma 15.  $\Box$