The University of Oxford

MSc (Mathematics and Foundations of Computer Science)

Computational Game Theory

Michaelmas Term 2023

In this project, you are required to write an report on one of two topics in computational game theory, described below. The aim is to provide a clear mathematical exposition of the respective topic, precisely stating the key issues under consideration, the main results up to the state of the art, identifying any controversies or limitations and open problems, and drawing conclusions as you see fit.

For each of the topics below, some references are provided to start you off. You should not restrict your study to these, but should use them as a starting point, finding work which cites them, and related work, up to the present day.

Important: You should only choose *one* of the two topics. If you write on both, only the first will be marked.

What You Should Hand In

You should hand in a report that does not exceed 3500 words (approx. 10 pages maximum). Ideally your report will be typed (LaTeX or Word) but this is not a formal requirement.

Marking Scheme

- 20% for statement of the fundamental issue being addressed;
- 50% for exposition of the approaches proposed in the literature
- 20% for identifying open problems and controversies, and drawing conclusions as appropriate
- 10% for quality of exposition.

Topic 1: Computing Mixed Nash Equilibria

In the lectures, we saw an exponential time algorithm to compute mixed Nash equilibria in bimatrix games, by means of support enumeration and the indifference principle. At the moment this algorithm is essentially optimal: we have no other algorithms that are guaranteed to solve this problem in better than exponential time. This raises a natural question: is it possible to do better, or is worst case exponential running time the best we can do? Characterising the complexity of Nash equilibria in bimatrix games was one of the major problems in computational complexity at the turn of the century: it was finally settled in 2005, when the problem was shown to be *PPAD complete*. On the one hand this represented a major breakthrough – the first serious characterisation of the problem – but it left many questions open.

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> *Provide your own summary of the PPAD result and it's proof. State (with technical explanation as appropriate) what is currently known about how PPAD relates to other complexity classes (P, NP, EXPTIME. . .). What do these results tell us about algorithms for computing mixed Nash equilibria? What would the existence of a polynomial time algorithm imply?*

References:

- Christos Papadimitriou (1994). "On the complexity of the parity argument and other inefficient proofs of existence". Journal of Computer and System Sciences. 48(3):498–532.

  Paper that introduced PPAD and its motivation.

- https://blog.computationalcomplexity.org/2005/12/what-is-ppad.html

  Nice summary article about PPAD.

- https://people.cs.pitt.edu/~kirk/CS1699Fall2014/lect4.pdf

  Nice summary lecture on PPAD.

- Daskalakis, Constantinos.; Goldberg, Paul W.; Papadimitriou, Christos H. "The Complexity of Computing a Nash Equilibrium". SIAM Journal on Computing. 39 (1): 195–259.

  Journal version of paper that solved the problem.

Topic 2: Price of anarchy of simultaneous auctions

Imagine a scenario where a seller has a total of m different items available for sale. There are also n potential buyers that are interested in buying subsets of these items, and each one of them has a (submodular or subadditive) valuation for the subsets of items. In this situation, the seller faces a decision: how to effectively allocate these items to the buyers?

One approach is to directly ask the buyers to reveal their valuations. This would involve asking them to place bids on every possible combination of items they might be interested in. However, unless m is small, this approach becomes unmanageable because it necessitates each buyer to submit an overwhelming number of bids — specifically, 2^m bids per bidder, one for each possible combination of items.

So, how are multiple items typically auctioned in real-world scenarios? One of the most common methods is to sell each item separately. In this approach, each buyer submits a single bid for every item available (resulting in a total of m bids per buyer), and each item is awarded to the highest bidder for that item. The price can be determined by the highest bid or the second-highest bid, for example.

The key question is whether this simple method results in efficient item allocation. Do buyers bid in a way that the items end up with those who value them the most?

There are many papers that deal with this question. The following papers provide a good starting point:

- Christodoulou, G., Kovács, A., Schapira, M., 2016. Bayesian Combinatorial Auctions. J. ACM 63, 1–19.

- Syrgkanis, V., Tardos, E., 2013. Composable and efficient mechanisms, in: Proceedings of the Forty-Fifth Annual ACM Symposium on Theory of Computing, 211–220.

- Christodoulou, G., Kovács, A., Sgouritsa, A., Tang, B., 2016. Tight Bounds for the Price of Anarchy of Simultaneous First-Price Auctions. ACM Trans. Econ. Comput. 4, 1–33.

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Consider the price of anarchy of simultaneous first-price auctions. Define properly the problem for submodular and subadditive valuations. Explain the best known bounds on the price of anarchy for the full information setting and for the Bayesian setting, and provide their proofs. Provide a succinct overview of the current state of the art for second-price auctions.