Sheet 1

Mika Bohinen

November 16, 2023

Exercise 2

(a) Let $F \in \mathrm{Fun}(*/G_1, */G_2)$. We then have that $F(*_{G_1}) = *_{G_2}$. Moreover the functoriality of F means that if $g, g' \in G_1$ then $F(g' \cdot g) = F(g') \cdot F(g)$. Hence F is just a group homomorphism.

Now, if $\eta : F \Rightarrow H$ is a natural transformation for $F, H \in \mathrm{Fun}(*/G_1, */G_2)$ and $g_1 \in G_1$, then we get the following commutative diagram

$$\begin{tikzcd} *_{G_2} \arrow[r, "Fg_1"] \arrow[d, "\eta_*"'] & *_{G_2} \arrow[d, "\eta_*"] \\ *_{G_2} \arrow[r, "Hg_1"'] & *_{G_2} \end{tikzcd}$$

We then have that $\eta_* = g_2$ for some $g_2 \in G_2$. The commutativity of the diagram tells us that

$$Fg_1 = g_2^{-1} H(g_1) g_2.$$

Thus a natural transformation is just conjugation by an element of the group G_2.

(b) Let $F \in \mathrm{Fun}(*/\mathbb{Z}, */G)$. Then, by the previous exercise we know that $\mathrm{Fun}(*/\mathbb{Z}, */G) \cong \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G)$. Hence F corresponds to some $f \in \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G)$. Moreover, since $f(0) = e_G$ and $f(n) = f(1)^n$ we have that the image of f is simply the cyclic subgroup $\langle f(1) \rangle$. The most natural thing is therefore to associate $f \in \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G)$ with $f(1)$. As $f(1)$ can be any element in G we therefore have that $\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \cong G$ as a group isomorphism (the isomorphism being given by $f \mapsto f(1)$).

With $n \in \mathbb{Z}$ considered as a morphism and $a \in G$ which represents the natural transformation $\eta : F \Rightarrow H$, and $F(n) = g_1^n$, $H(n) = g_2^n$ we have the equation

$$g_1^n = a^{-1} g_2^n a.$$

This has to hold for all n but it suffices that it holds for $n = 1$ where we have the equation

$$g_1 = a^{-1} g_2 a.$$

Hence natural transformations $F \Rightarrow G$ are in one-to-one correspondence with conjugation relations of g_1 with g_2.

Exercise 3

As $\mathrm{Hom}_{\mathrm{Vect}_k^{fd}}(-,k)$ is contravariant we cannot technically have an equivalence of categories. However, we can speak of a duality of categories where we just require that both F and G are contravariant.

We first note that equivalence of categories is an equivalence relation. Considering the diagram

$$\begin{array}{ccc} \mathrm{Vect}_k^{fd} & \longrightarrow & (\mathrm{Vect}_k^{fd})^{\mathrm{op}} \\ \big\downarrow & & \big\downarrow \\ \mathrm{Mat}_k & \longrightarrow & (\mathrm{Mat}_k)^{\mathrm{op}} \end{array}$$

it suffices to show that $\mathrm{Mat}_k \to (\mathrm{Mat}_k)^{\mathrm{op}}$ is a duality of categories since we know that the two vertical arrows already are part of an equivalence of categories.

The contravariant functor $\mathrm{Mat}_k \to (\mathrm{Mat}_k)^{\mathrm{op}}$ is the identity on the objects and sends each matrix to its transpose. Hence if we create the contravariant functor $(\mathrm{Mat}_k)^{\mathrm{op}} \to \mathrm{Mat}_k$ which does exactly the same then their composition is the identity so that we have a contravariant isomorphism of categories.

Thus the map $\mathrm{Vect}_k^{fd} \to (\mathrm{Vect}_k^{fd})^{\mathrm{op}}$ is a duality of categories.

Exercise 4

Let $f, g \in \mathrm{Hom}(x,y)$. Then, since $\mathrm{End}(x)$ is the trivial group, we must have that $g^{-1} \circ f = 1_x$. This implies that $f = g$ showing that $|\mathrm{Hom}(x,y)| = 1$.

Next, let $F : \mathcal{C} \to *$ be the unique functor from $\mathcal{C}$ to $*$ (as $*$ is terminal in **Cat**). We then just have to show that F is fully-faithful and essentially surjective. As F is surjective on objects, it must be essentially surjective. We therefore only need to show that F is fully-faithful. This is quite straight-forward:

1. **Injectivity**. Let $f, g \in \mathrm{Hom}(x,y)$ such that $Ff = Fg$. Since $\mathrm{Hom}(x,y) = \{1\}$ we must have that $f = g$.
2. **Surjectivity**. For $x, y \in \mathcal{C}$ we have that $F(x) = * = F(y)$. Hence $\mathrm{Hom}(Fx, Fy) = \{1_*\}$. As $|\mathrm{Hom}(x,y)| = 1$ this unique isomorphism between x and y must get mapped to 1_* which shows that $F : \mathrm{Hom}(x,y) \to \mathrm{Hom}(Fx, Fy)$ is surjective.

Since F is both fully-faithful and essentially surjective we can conclude that there exists a $G : * \to \mathcal{C}$ such that $GF \cong \mathrm{id}_{\mathcal{C}}$ and $FG \cong \mathrm{id}_*$ which shows that $\mathcal{C}$ is equivalent to the discrete category with a single object.