Homological Algebra
Sheet 1

Mika Bohinen

November 7, 2023

Exercise 3

Since it wasn't specified we assume that the horizontal sequences are exact and that the diagram commutes.

The result we want to prove follows if we can show that $\ker(j) = 0 = \operatorname{coker}(j)$.

From the snake lemma we know that there exist an exact sequence of the form

$$0 \to \ker\, i \to \ker\, j \to \ker\, k \xrightarrow{\delta} \operatorname{coker}\, i \to \operatorname{coker}\, j \to \operatorname{coker}\, k \to 0.$$

Since i and k are isomorphisms, and the fact that we are in an abelian category, we have that this exact sequence turns into

$$0 \to 0 \to \ker\, j \to 0 \xrightarrow{\delta} 0 \to \operatorname{coker}\, j \to 0 \to 0.$$

Hence we must have that $\ker\, j = 0 = \operatorname{coker}\, j$ and so j is an isomorphism.

Exercise 4

Let the middle module be given by

$$M = \frac{k[x,y,z] \oplus k[x,y,z]}{\langle (x^2,0),(xz,0),(z^3,0)\rangle}.$$

Then the sequence

$$0 \to M_1 \to M \to M_2 \to 0$$

is exact. Moreover, there does not exist a section $s : M_2 \to M$ and hence the sequence does not split.

Exercise 5

This follows from showing that $\mathbb{Z}$ is a projective $\mathbb{Z}$-module and $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. More generally, it is true that any free $\mathbb{Z}$-module is projective (or for that matter any free R-module).

The case for just $\mathbb{Z}$ is straightforward. Since $p : B \to \mathbb{Z}$ is surjective there exist a $b \in B$ such that $p(b) = 1$. Let $s : \mathbb{Z} \to B$ be given by $s(n) = ns(1) = n \cdot b$. Then

$$(p \circ s)(n) = p(n \cdot b) = n \cdot p(b) = n.$$

Hence $p \circ s = \mathrm{id}_{\mathbb{Z}}$ so that the sequence must split by the splitting lemma.

To show that $0 \to \mathbb{Q} \to B \to C \to 0$ splits it's easiest to show that $\mathbb{Q}$ is an injective $\mathbb{Z}$-module because then the injectivity of $\mathbb{Q}$ tells us that $\mathrm{Hom}(-, \mathbb{Q})$ is exact and so we get that there is an $r \in \mathrm{Hom}(B, \mathbb{Q})$ such that $r \circ \iota = \mathrm{id}_{\mathbb{Q}}$ for $\iota : \mathbb{Q} \to B$ the specified map.

By Baer's we must show that for every ideal of $\mathbb{Z}$ and every map $f : n\mathbb{Z} \to \mathbb{Q}$ we can extend it to a map $\tilde{f} : \mathbb{Z} \to \mathbb{Q}$. This is easy enough. Suppose we have a map $f : n\mathbb{Z} \to \mathbb{Q}$ and let $\tilde{f}(1) = \frac{1}{n} f(n)$. Then, if $i : n\mathbb{Z} \to \mathbb{Z}$ is the inclusion map, we have that

$$(\tilde{f} \circ i)(n) = \frac{1}{n} f(n \cdot n) = \frac{n}{n} f(n) = f(n).$$

Hence $\mathbb{Q}$ is injective and there exists a retraction $r : B \to \mathbb{Q}$ so that $r \circ \iota = \mathrm{id}_{\mathbb{Q}}$. The splitting lemma then concludes the proof.

Exercise 6

For each prime p there is an embedding $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. These glue together to form a map $\alpha : \bigoplus_{p:\text{prime}} \mathbb{Z}[\frac{1}{p}]/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. We just have to show that this map is an isomorphism.

Surjectivity: Notice that for any rational number $\frac{m}{pq}$ with p and q relatively prime there exists a, b such that $\frac{m}{pq} = \frac{a}{p} + \frac{b}{q}$. Thus, for any rational number $\frac{m}{\prod_p p^{n_p}}$ with $n_p = 0$ for all but finitely many p's we can write

$$Q := \frac{m}{\prod_p p^{n_p}} = \sum_p \frac{a_p}{p^{n_p}}.$$

Then $Q \in \mathbb{Q}/\mathbb{Z}$ can be written as $Q = \alpha\left(\oplus_p (a_p/p^{n_p})\right)$ showing that α is surjective.

Injectivity: Take $\{a_p\}$ and $\{k_p\}$ with all but finitely many of the a_p equal to zero such that $\alpha\left(\oplus_p a_p/p^{k_p}\right) = 0$. This tells us that $\sum_p a_p/p^{k_p} = 0$ which turns into

$$\sum_p a_p \prod_{q \neq p} q^{k_q} = 0.$$

From this it follows that $p^{k_p} | a_p$ for all p and so we must have that $\left(\oplus_p a_p/p^{k_p}\right)$ is zero in $\bigoplus_{p:\text{prime}} \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

We thus see that α is an isomorphism.

Exercise 7

Let $(A)_{i \geq 1}$ be a strictly increasing chain of submodules such that their union is equal to A. Set $M = A$ and $N_i = A/A_i$. Then there is no map in no element in $\bigoplus_{i=1}^{\infty} \mathrm{Hom}(A, A/A_i)$ which can get sent to the projection map $p : A \to \bigoplus_{i=1}^{\infty} A/A_i$. This is because we can only big a finite amount of non-zero maps in $\bigoplus_{i=1}^{\infty} \mathrm{Hom}(A, A/A_i)$. Thus, if we pick any map $\alpha \in \bigoplus_{i=1}^{\infty} \mathrm{Hom}(A, A/A_i)$ there is some index j such that if $i > j$ then $\alpha_i = 0$. We can then pick some $a \in A - A_j$ and have that $\alpha_i(a) = 0$ while $p(a)_{j+1} \neq \alpha_{j+1}(a)$. Hence there is no isomorphism in general given that such an increasing sequence of submodules exists.

This is certainly the case as you can take $A = \bigoplus_{i=1}^{\infty} \mathbb{Z}$ and $A_k = \bigoplus_{i=1}^{k} \mathbb{Z}$.