B6.3 Integer Programming
Problem Sheet 2

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Section A

Problem A.1: Is the following matrix totally unimodular or not? Prove your answer, citing any results on total unimodularity you use from lectures.

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Problem A.2. Prove Lemma (Closure) from Lecture 7.

Problem A.3. Let $f : \mathscr{P}(\{1,\dots,n\}) \to \mathbb{R}$ be a submodular rank function, and $c \in \mathbb{R}^n$ a vector of weights. Consider the following variant of the greedy algorithm:

Algorithm 1 (Variant greedy)

1. *initialisation:*
   i) *re-index variables so that* $c_1 \geq \cdots \geq c_n$,
   ii) $S^0 := \emptyset$
2. *main loop:*

   for $k = 1,\dots,n$
   i) *set* $S^k = \{1,\dots,k\}$,
   ii) *set* $x_k = f(S^k) - f(S^{k-1})$

Define $\rho := \max\{|X| : X \in \mathcal{I}\}$, and let x be the output of Algorithm (Variant greedy). For $k = 1,\dots,\rho$, let $m(k)$ be the unique index such that $\sum_{i=1}^{m(k)} x_i = k$. Prove that for $k = 1,\dots,\rho$ the set $X_k := \{i : x_i > 0, i \leq m(k)\}$ is an optimal solution to the problem of finding a maximum weight independent subset of cardinality k,

$$(\text{MWISP}(k)) \quad \max_X \left\{ \sum_{j\in X} c_j : X \in \mathcal{I}, |X| = k \right\}.$$

Problem A.4.

i) Using the result of Problem A.3, derive a greedy algorithm that finds a maximum weight spanning tree in a graph $G = (V, E)$. [*A spanning tree in a graph* $G = (V, E)$ *is a tree* (V, E') *that is incident to all nodes* $v \in V$.]

ii) Find a maximum weight spanning tree in the graph $G = (V, E)$ with 9 nodes, edges $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 6\}$, $\{2, 3\}$, $\{2, 5\}$, $\{3, 6\}$, $\{3, 7\}$, $\{4, 7\}$, $\{4, 8\}$, $\{5, 6\}$, $\{5, 8\}$, $\{5, 9\}$, $\{6, 9\}$, $\{7, 9\}$, $\{8, 9\}$ and edge weights $c_{1,2} = 4$, $c_{1,3} = 9$, $c_{1,4} = 11$, $c_{1,6} = 7$, $c_{2,3} = 2$, $c_{2,5} = 3$, $c_{3,6} = 1$, $c_{3,7} = 6$, $c_{4,7} = 12$, $c_{4,8} = 8$, $c_{5,6} = 3$, $c_{5,8} = 6$, $c_{5,9} = 4$, $c_{6,9} = 4$, $c_{7,9} = 7$, $c_{8,9} = 8$.

Section B

Problem B.1. [Total unimodularity, short] Prove that matrices with the consecutive ones propetry are totally unimodular.

Problem B.2. A *matching* of a graph G is a set of edges meeting each node of G at most once. *König's Theorem* says that in a bipartite graph $G = (V_1, V_2, E)$ the number of edges in a matching of maximum cardinality is equal to the minimal cardinality needed for a set of vertices to be incident to all edges of E (covering by nodes). For any $v \in V$ let $E(v)$ be the set of edges incident to v. The maximum cardinality matching problem is thus given by

$$
\begin{aligned}
\text{(MaxMatch)} \quad \max \; & \sum_{e \in E} x_e \\
\text{s.t.} \; & \sum_{e \in E(v)} x_e \leq 1, \quad (v \in V_1) \\
& \sum_{e \in E(w)} x_e \leq 1, \quad (w \in V_2) \\
& x_e \geq 0, x_e \in \mathbb{Z}, \quad (e \in E).
\end{aligned}
$$

i) Introduce slack variables and show that the constraint matrix is totally unimodular.

ii) Set up the dual of the LP relaxation of (M) and interpret it as the LP relaxation of the minimum cardinality node covering problem.

iii) Using the Strong LP Duality Theorem, prove König's Theorem.

Problem B.3. Prove Theorem (Submodular rank function induced by matroid) from Lecture 7.

Problem B.4. Let $f : \mathscr{P} \to \mathbb{R}$ be a submodular rank function on $\mathscr{P} := \mathscr{P}(\{1, \ldots, n\})$, and

$$
P(f) = \left\{ x \in \mathbb{R}^n : x \geq 0, \quad \sum_{j \in S} x_j \leq f(S) \quad \forall S \in \mathscr{P} \setminus \emptyset \right\} \tag{1}
$$

the associated submodular polyhedron. A set $S \in \mathscr{P} \setminus \emptyset$ is called *inseparable* if $U \subseteq S$ such that $f(S) = f(U) + f(S \setminus U)$ implies that either $U = \emptyset$ or $U = S$. A set $S \in \mathscr{P} \setminus \emptyset$ is called *closed* if $\nexists j \in \{1, \ldots, n\} \setminus S$ for which $f(S \cup \{j\}) = f(S)$.

i) Show that if $S \in \mathscr{P}$ is not inseparable, then the rank inequality $\sum_{j \in S} x_j \leq f(S)$ is redundant in the polyhedral description (1) of $P(f)$.

ii) Prove that for every $S \in \mathscr{P} \setminus \emptyset$ there exists a unique maximal superset $\mathrm{cl}(S) \supseteq S$ in $\mathscr{P}$ such that $f(S) = f(\mathrm{cl}(S))$.

iii) Show that if $S \in \mathscr{P}$ is not closed, then the rank inequality $\sum_{j \in S} x_j \leq f(S)$ is redundant in the polyhedral description (1) of $P(f)$.

Section C

Problem C.1: Let $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \le b, x \ge 0\}$ be a polyhedron in $\mathbb{R}^n$ defined by m inequality constraints $\sum_{j=1}^n a_{ij}x_j \le b_i$, $(i = 1, \dots, m)$, and let us assume that the row vectors of A are linearly independent. Let $Ax + s = b$ be the system of equations obtained after introducing slack variables s_i for inequality i, $(i = 1, \dots, m)$.

Prove that $x \in \mathbb{R}^n$ is an extreme point of $\mathcal{P}$ if and only if there exists $s \in \mathbb{R}^m_+$ such that (x, s) is a basic feasible solution of the system $Ax + s = b$, $x, s \ge 0$.

Problem C.2. Consider the shortest path problem (SP) from the lecture slides.

i) Argue that capacity constraints on the arcs are unnecessary in the flow problem interpretation, and that the dual of its LP relaxation is equivalent to the following problem,

$$\begin{aligned} \text{(D)} \quad & \max \pi_t \\ & \text{s.t. } \pi_j - \pi_i \le c_{ij} \quad ((i,j) \in E), \\ & \quad \pi_s = 0. \end{aligned}$$

Give an interpretation of the dual variables π_i, and explain how an optimal solution of (D) can be used to find a shortest path from s to t.

ii) Show that the following algorithm solves the dual (D) in $\mathscr{O}(n \times m)$ time, where $n = |V|$ is the cardinality of the node set and $m = |E|$ the cardinality of the arc set:

1. Set $\pi_s = 0$ and $\pi_i = \infty$ for all $i \ne s$.
2. For $k = 1, \dots, n-1$
   for all $(i,j) \in E$ (using an arbitrary order)
      $\pi_j \leftarrow \min(\pi_j, \pi_i + c_{ij})$
   end
   end.

Problem C.3. A scheduling model in which a machine can be switched on at most $k < n$ times is modelled by the following constraints, where y_0 can be considered as zero,

$$\begin{aligned} \sum_{t=1}^n z_t &\le k, \\ z_t - y_t + y_{t-1} &\ge 0, \quad (t = 1, \dots, n), \\ z_t &\le y_t, \quad (t = 1, \dots, n), \\ 0 \le y_t, z_t &\le 1, \quad (t = 1, \dots, n), \\ y_t, z_t &\in \mathbb{Z}, \quad (t = 1, \dots, n). \end{aligned}$$

i) Give an economic interpretation of the decision variables y_t,z_t.

ii) It can be shown that the following are sufficient conditions for a matrix $A = (a_{ij})$ to be totally unimodular,

   a) $a_{ij} \in \{0, +1, -1\}$ for all i, j,

b) for any subset M of the rows of A, there exists a partition (M_1, M_2) of M such that each column j satisfies

$$\left| \sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} \right| \leq 1. \tag{2}$$

(M_1 and M_2 are the same for each column.)

Use this criterion to prove that the constraint matrix of the above described scheduling problem is totally unimodular.

Problem C.4. Let $f : \mathscr{P} \to \mathbb{R}$ be a non-decreasing function on $\mathscr{P} := \mathscr{P}(\{1, \dots, n\})$.

i) Prove that f is submodular if and only if it satisfies the *Law of Diminishing Returns*

$$A, T \in \mathscr{P},\, d \in T \setminus A \Rightarrow f(A \cup \{d\}) - f(A) \leq f\big((A \cup \{d\}) \cap T\big) - f(A \cap T).$$

ii) Let f be submodular and $h : \mathbb{R} \to \mathbb{R}$ a concave monotone non-decreasing function whose domain (the set of arguments for which $h(x)$ is well defined and finite) includes $\{f(A) : A \in \mathscr{P}\}$. Prove that $h \circ f : \mathscr{P} \to \mathbb{R}$ is a non-decreasing submodular function.

iii) Let $\{v_1, \dots, v_n\}$ be a list of training samples (images, audio files, texts, ...), $G_i \in \mathscr{P}$ for $(i = 1, \dots, k)$ a list of groups of samples $\{v_i : i \in G_i\}$. Prove that the entropy function

$$f : A \mapsto \sum_{i=1}^{k} \ln\left(1 + |G_i \cap A|\right)$$

is a submodular non-decreasing function.

Problem C.5. The rank of a subset $A \subseteq \mathbb{R}^n$ is defined as the dimension of the subspace spanned by the vectors in A, $\operatorname{rk}(A) := \dim(\operatorname{span}(A))$.

i) Show that $f : A \mapsto \operatorname{rk}(A)$ is a submodular increasing function on the set of subsets of $\mathbb{R}^n$.

ii) Show that for all $A, B \subseteq \mathbb{R}^n$,

$$\operatorname{rk}(A) \leq \operatorname{rk}(B) + \sum_{v_j \in A \setminus B} [\operatorname{rk}(B \cup \{v_j\}) - \operatorname{rk}(B)].$$

iii) Let $V = \{v_1, \dots, v_6\} \subset \mathbb{R}^3$ be given by

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix},\ v_3 = \begin{bmatrix} -9 \\ 1 \\ 0 \end{bmatrix},\ v_4 = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix},\ v_5 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix},\ v_6 = \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix}.$$

Solve the following mixed integer programming problem,

$$\begin{aligned} \text{(IP)}\ \ & \max\ x_1 - 2x_2 + 3x_3 - 4x_4 + 5x_5 + 6x_6 \\ & \text{s.t.} \sum_{\{j : v_j \in A\}} x_j \leq \operatorname{rk}(A) \quad \forall A \subseteq V, \\ & \quad 0 \leq x_i \leq i \quad (i = 1, \dots, 6), \\ & \quad x_i \in \mathbb{Z} \quad (i = 5, 6). \end{aligned}$$