B6.3 Integer Programming
Problem Sheet 1

Prof. Raphael Hauser

Section A

Problem A.1. A paper machine in a factory can only produce paper rolls of one standard width W. Orders arrive for q_i $(i = 1, \dots, m)$ paper rolls of m different narrower widths $w_1, \dots, w_m$, and the standard rolls are cut into smaller rolls to satisfy these orders. The sales team of the company have generated all possible patterns $(a_{1j}, \dots, a_{mj})$, $(j = 1, \dots, n)$, for cutting standard rolls into a_{ij} rolls of width w_i. That is, each pattern must satisfy

$$\sum_{i=1}^{m} a_{ij} w_i \leq W$$

and leaves a cut-off $s_j = W - \sum_{i=1}^{m} a_{ij} w_i$ that is too narrow to be sold, because $s_\ell < w_i$ for all i. The cut off has to be thrown away and causes a waste cost of c_j. Formulate an integer programming problem to help the factory determine how to cut standard rolls so as to satisfy all the orders whilst minimising the total cost of cut-offs.

Problem A.2. Consider the LP problem (P) $\max_{x \in \mathbb{R}^n} \{c^{\mathrm{T}} x : Ax = b,\ x \geq 0\}$, where $A \in \mathbb{R}^{m \times n}$ is of rank m, and let us assume that an initial basic feasible solution is available, that is, a set of m indices $B \subset \{1, \dots, n\}$ such that A_B is nonsingular and $x_B = A_B^{-1} b \geq 0$, where we use the notation of the lecture slides. Assume that i and j are the non-basic and basic indices chosen in Steps 2.i) and 2.ii) of the Simplex Algorithm, that is, the index set of basic variables is updated as $\tilde{B} = B \cup \{i\} \setminus \{j\}$. Prove that $A_{\tilde{B}}$ is nonsingular.

Problem A.3: Let $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $\mathcal{P} = \{x \in \mathbb{R}^n : Ax \leq b, x \geq 0\}$, and $\mathcal{P}' = \{z = (x, s) \in \mathbb{R}^{n+m} : Ax + s = b,\ x, s \geq 0\}$. Prove that if all extreme points $x \in \mathbb{R}^n$ of $\mathcal{P}$ are integer valued, then all extreme points z of $\mathcal{P}'$ are integer valued too.

Section B

Problem B.1.

i) Extend the big-M formulation of discrete alternatives to the union of two polyhedra $P_k = \{x \in \mathbb{R}^n : A^k x \leq b^k, 0 \leq x \leq u\}$ for $k = 1, 2$ where $\max_k \max_i \{a_i^k x - b_i^k : 0 \leq x \leq u\} \leq M$.

ii) Argue that the constraint $x \in P_1 \cup P_2$ is equivalent to

$$
\begin{aligned}
x &= z^1 + z^2 \\
A^k z^k &\leq b^k y^k \text{ for } k = 1, 2 \\
0 \leq z^k &\leq u y^k \text{ for } k = 1, 2 \\
y^1 + y^2 &= 1 \\
z^k \in \mathbb{R}^n, y^k &\in \mathbb{B}^1 \text{ for } k = 1, 2.
\end{aligned}
$$

(Hint: You need to show that $x \in P_1 \cup P_2$ if and only if there exists y, z such that (x, y, z) satisfy the second set of constraints (this is called an *extended formulation*, as it involves an augmentation by extra decision variables).

Problem B.2. Convert the following LP to standard primal form and set up its dual:

$$
\begin{aligned}
\min_{x \in \mathbb{R}^3} \ & \sum_{i=1}^{3} x_i \\
\text{s.t.} \ & -2x_1 + x_3 \leq 2, \\
& 3x_1 + x_2 + 2x_3 \leq 6, \\
& x_1 - x_2 + 3x_3 = 3, \\
& x_2 \geq x_3, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

Problem B.3. The generalisation of Gauss-Jordan Elimination to systems of linear inequalities is called Fourier-Motzkin Elimination. It works as follows: Consider a system of linear inequalities

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad (i = 1, \dots, m),$$

and let us select a variable x_k to eliminate. We partition the set $M = \{1, \dots, m\}$ into

$$
\begin{aligned}
M_+^k &:= \{i : a_{ik} > 0\}, \\
M_-^k &:= \{i : a_{ik} < 0\}, \\
M_0^k &:= \{i : a_{ik} = 0\}.
\end{aligned}
$$

The new system consists of the following inequalities,

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad (i \in M_0^k), \tag{1}$$

$$\sum_{j=1}^{n} (a_{ik} a_{\ell j} - a_{\ell k} a_{ij}) x_j \leq a_{ik} b_\ell - a_{\ell k} b_i, \quad \left((i, \ell) \in M_+^k \times M_-^k\right). \tag{2}$$

Prove that the new system of linear inequalities does not involve x_k and is equivalent to the original system in the following sense: $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ satisfies the new system if and only if there exists a value of x_k for which $(x_1, \ldots, x_k, \ldots, x_n)$ satisfies the original system. [*Hint: the set of values that x_k can take is an interval that you should determine. Note that the procedure can be applied repetitively, and if $M_+^k \cup M_0^k = \emptyset$ or $M_-^k \cup M_0^k = \emptyset$, then the new system is empty and is satisfied by all values of* $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$.]

Problem B.4. Prove part ii) of the strong duality theorem, using the following template, provided for one of the cases you need to consider: Suppose (P) is infeasible and (D) admits a feasible solution $\tilde{y} \in \mathbb{R}^m$. Show that the system

$$\begin{aligned} \sum_{i=1}^{m} u_i a_{ij} &= 0, \quad (j = 1, \ldots, n) \\ u_i &\geq 0, \quad (i = 1, \ldots, m) \\ \sum_{i=1}^{m} u_i b_i &< 0 \end{aligned}$$

has a solution $u \in \mathbb{R}^m$. Finally, using solutions of the form $y = \tilde{y} + \lambda u$, show that (D) is unbounded.

Section C

Problem C.1. A set of n jobs must be carried out on a single machine that can do only one job at a time. Each job j takes p_j hours to complete. Given job weights w_j for $j = 1, \dots, n$, in what order should the jobs be carried out so as to minimise the weighted sum of their starting times? Formulate this scheduling problem as a mixed integer programming problem. [*Hint: There exists a model that uses only* n^2 *binary and* n *real decision variables.*]

Problem C.2. Prove the Theorem of the Alternative for Linear Inequalities by breaking it down into the following steps:

i) Show that both systems cannot simultaneously have solutions.

ii) Suppose that the first system has no solution, and eliminate all of its n variables via Fourier-Motzkin Elimination. This yields an inconsistent system (a system with no solution) of the form

$$\sum_{j=1}^{n} 0 \cdot x_j \leq d_k, \quad (k = 1, \dots, p),$$

Show that there exists at least one index k^* for which $d_{k^*} < 0$, and values $y_1, \dots, y_m \geq 0$ such that the k^*-th inequality is obtained as

$$\sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) \leq \sum_{i=1}^{m} y_i b_i.$$

iii) Show that y is a solution of the alternative system.