

Category Theory

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1 Categories, Functors, Natural Transformations

1.1 Abstract and concrete categories

Definition 1. A *category* \mathcal{C} consists of the following data:

- (i) a collection $\text{ob}\mathcal{C}$ of object of \mathcal{C} ,
- (ii) for every two objects $x, y \in \text{ob}\mathcal{C}$ a collection $\text{Hom}_{\mathcal{C}}(x, y)$ of morphisms,
- (iii) the identity morphisms,
- (iv) the identity morphism $\text{id}_x \in \text{Hom}_{\mathcal{C}}(x, x)$ for every object $x \in \text{ob}\mathcal{C}$,
- (v) the composition map

$$\circ : \text{Hom}_{\mathcal{C}}(y, z) \times \text{Hom}_{\mathcal{C}}(x, y) \rightarrow \text{Hom}_{\mathcal{C}}(x, z)$$

for every triple of objects $x, y, z \in \text{ob}\mathcal{C}$

Definition 2. A category is **small** if it has only a set's worth of arrows.

Definition 3. A category is **locally small** if between any pair of objects there is only a set's worth of morphisms.

Definition 4. An **isomorphism** in a category is a morphism $f : X \rightarrow Y$ for which there exist a morphism $g : Y \rightarrow X$ so that $fg = 1_X$ and $gf = 1_Y$. The objects X and Y are **isomorphic** whenever there exist an isomorphism between X and Y , in which case one writes $X \cong Y$.

A **subcategory** \mathcal{D} of a category \mathcal{C} is defined by restricting to a subcollection of objects and subcollection of morphisms subject to the requirements that the subcategory \mathcal{D} contains the domain and codomain of any morphism in \mathcal{D} , the identity morphism of any object in \mathcal{D} , and the composite of any composable pair of morphisms in \mathcal{D} .

Lemma 1. Any category \mathcal{C} contains a maximal groupoid, the subcategory containing all of the objects and only those morphisms that are isomorphisms.

1.2 Duality

Definition 5. Let \mathcal{C} be any category. The **opposite category** \mathcal{C}^{op} has

1. the same objects as in \mathcal{C} , and
2. a morphism f^{op} in \mathcal{C}^{op} for each morphism f in \mathcal{C} so that the domain of f^{op} is defined to be the codomain of f and the codomain of f^{op} is defined to be the domain of f .
3. For each object X , the arrow 1_X^{op} serves as its identity in \mathcal{C}^{op} .
4. To define composition, observe that a pair of morphisms $f^{\text{op}}, g^{\text{op}}$ in \mathcal{C}^{op} is composable precisely when the pair g, f is composable in \mathcal{C} . We then define $g^{\text{op}} \circ f^{\text{op}}$ to be $(f \circ g)^{\text{op}}$.

Lemma 2. The following are equivalent

1. $f : x \rightarrow y$ is an isomorphism in \mathcal{C} .
2. For all objects $c \in \mathcal{C}$, post-composition with f defines a bijection

$$f_* : \mathcal{C}(c, x) \rightarrow \mathcal{C}(c, y).$$

3. For all objects $c \in \mathcal{C}$, pre-composition with f defines a bijection

$$f^* : \mathcal{C}(y, c) \rightarrow \mathcal{C}(x, c).$$

Definition 6. A morphism $f : x \rightarrow y$ in a category is

1. a **monomorphism** if for any parallel morphisms $h, k : w \rightrightarrows x$, $fh = fk$ implies that $h = k$; or
2. an **epimorphism** if for any parallel morphisms $h, k : y \rightrightarrows z$, $hf = kf$ implies that $h = k$.

Since the notions of monomorphism and epimorphism are dual, their abstract categorical properties are also dual, such as exhibited by the following lemma.

- Lemma 3.** 1. If $f : x \rightarrow y$ and $g : y \rightarrow z$ are monomorphisms, then so is $gf : x \rightarrow z$.
2. If $f : x \rightarrow y$ and $g : y \rightarrow z$ are morphisms so that gf is monic, then f is monic.

Dually.

1. If $f : x \rightarrow y$ and $g : y \rightarrow z$ are epimorphisms, then so is $gf : x \rightarrow z$.
2. If $f : x \rightarrow y$ and $g : y \rightarrow z$ are morphisms so that $gf : x \rightarrow z$ is epic, then g is epic.

1.3 Functors

Definition 7. A **functor** $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$, between categories \mathcal{C} and \mathcal{D} , consists of the following data:

- An object $\mathcal{F}c \in \mathcal{D}$, for each object $c \in \mathcal{C}$.
- A morphism $\mathcal{F}f : \mathcal{F}c \rightarrow \mathcal{F}c' \in \mathcal{D}$, for each morphism $f : c \rightarrow c' \in \mathcal{C}$, so that the domain and codomain of $\mathcal{F}f$ are, respectively, equal to \mathcal{F} applied to the domain and codomain of f .

The assignments are required to satisfy the following two functoriality axioms:

- For any composable pair f, g in \mathcal{C} , $\mathcal{F}g \circ \mathcal{F}f = \mathcal{F}(g \circ f)$.
- For each object $c \in \mathcal{C}$, $\mathcal{F}1_c = 1_{\mathcal{F}c}$.

There is also the dual notion of a **contravariant functor** which simply has as domain \mathcal{C}^{op} instead of \mathcal{C} .

Lemma 4. Functors preserve isomorphisms.

Proof. Straightforward. □

Corollary 1. When a group G acts functorially on an object X in a category \mathcal{C} , its elements g must act by automorphisms $g_* : X \rightarrow X$ and, moreover, $(g_*)^{-1} = (g^{-1})_*$.

Definition 8. If \mathcal{C} is locally small, then for any object $c \in \mathcal{C}$ we may define a pair of covariant and contravariant functors represented by c :

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{\mathcal{C}(c, -)} & \mathbf{Set} \\
 x & \mapsto & \mathcal{C}(c, x) \\
 \downarrow f & \mapsto & \downarrow f_* \\
 y & \mapsto & \mathcal{C}(c, y)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{C}^{\text{op}} & \xrightarrow{\mathcal{C}^{\text{op}}(-, c)} & \mathbf{Set} \\
 x & \mapsto & \mathcal{C}^{\text{op}}(x, c) \\
 \downarrow f & \mapsto & \uparrow f^* \\
 y & \mapsto & \mathcal{C}^{\text{op}}(y, c)
 \end{array}$$

Definition 9. If \mathcal{C} is locally small, then there is a **two-sided represented functor**

$$\mathcal{C}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \mathbf{Set}$$

defined in the evident manner. A pair of objects (x, y) is mapped to the hom-set $\mathcal{C}(x, y)$. A pair of morphisms $f : w \rightarrow x$ and $h : y \rightarrow z$ is sent to the function

$$\begin{aligned}
 \mathcal{C}(x, y) &\xrightarrow{(f^*, h_*)} \mathcal{C}(w, z) \\
 g &\mapsto hgf.
 \end{aligned}$$

1.4 Natural Transformations

Definition 10. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be two functors. A natural transformation $\eta : F \Rightarrow G$ consists of morphisms $\eta_x \in \text{Hom}_{\mathcal{D}}(F(x), G(x))$ for every object $x \in \mathcal{C}$ such that the diagram

$$\begin{array}{ccc}
 F(x) & \xrightarrow{F(f)} & F(y) \\
 \downarrow \eta_x & & \downarrow \eta_y \\
 G(x) & \xrightarrow{G(f)} & G(y)
 \end{array}$$

commutes for every morphism $f \in \text{Hom}_{\mathcal{C}}(x, y)$.

We say that a natural transformation $\eta : F \Rightarrow G$ is a natural isomorphism if the morphisms η_x are isomorphisms for any $x \in \mathcal{C}$.

Definition 11. An equivalence of categories \mathcal{C}, \mathcal{D} is a pair of functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ together with natural isomorphisms $e : \text{id}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \text{id}_{\mathcal{D}}$.

Definition 12. An adjoint equivalence of categories \mathcal{C}, \mathcal{D} is an equivalence (F, G, e, ε) satisfying the following axioms:

1. The composite natural transformation

$$F \cong F \circ \text{id}_{\mathcal{C}} \xrightarrow{\text{id}_F \circ e} FGF \xrightarrow{\varepsilon \circ \text{id}_F} \text{id}_{\mathcal{D}} \circ F \cong F$$

is the identity natural transformation on F .

2. The composite natural transformation

$$G \cong \text{id}_{\mathcal{D}} \circ G \xrightarrow{e \circ \text{id}_G} GFG \xrightarrow{\text{id}_G \circ \varepsilon} G \circ \text{id}_{\mathcal{C}} \cong G$$

is the identity natural transformation on G .