Tutorial 1: Preferences, Utilities, and Decisions

1. Recall the example of lexicographic preferences from the lecture. The attributes are colour, engine type, and nationality, with the ranking

$$\text{colour} > \text{engine} > \text{nationality}$$

The ordering for each of attribute is:

- *colour*: red $\succ$ blue $\succ$ green
- *engine type*: electric $\succ$ petrol $\succ$ diesel
- *nationality*: German $\succ$ French $\succ$ UK

Define a utility function that takes three inputs (colour, engine type, and nationality) and gives as output a real number, such that this utility function corresponds to the preference ordering defined above. *Don't* do this by enumerating all 27 cases! Argue for the correctness of your function.

2. To make things simple in the lectures, we assumed our set of outcomes Ω was finite. Let's now look at an example where this is not the case. Let the set of outcomes Ω be $\Omega = \mathbb{R}_+ \times \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of positive real numbers.

Now, define a preference relation $\succeq\ \subseteq \Omega \times \Omega$ by:

$$(x_1, x_2) \succeq (y_1, y_2) \qquad \text{iff} \qquad (x_1 > y_1 \quad \text{or} \quad [x_1 = y_1 \text{ and } x_2 \geq y_2])$$

(a) Prove that the relation $\succeq$ defined in this way is indeed a properly defined preference relation.

(b) Prove that there can exist no utility function $u : \Omega \to \mathbb{R}$ representing $\succeq$.

(It may be helpful to recall that between any two distinct real numbers there lies a rational number.)

3. You have three entertainment options: football, pub, or rowing. The utility you obtain from these will depend on whether it rains or not. The utilities you get from these outcomes are as follows:

Activity	Utility if rain	Utility if no rain
football	1	2
pub	3	0
rowing	0	1

You have a decision task ahead: to choose which activity to undertake. In what follows, let p_{rain} denote the probability of rain.

(a) Formulate this as a problem of decision making under uncertainty.

(b) Can you identify an alternative that you would never choose, irrespective of the value of p_{rain}?

(c) Can you write down of a rule that shows what the best thing to do is, as a function of p_{rain}? (The rule would say something like "if $p_{rain} \in [X, Y]$ then do Z, else if...")

4. Suppose $\Omega = \{A, B\}$ and $A \succ B$. Then I claim that, for all values $p \in (0, 1]$, we have:

$$[pA + (1-p)B] \succ B$$

Argue that if the relation $\succ$ satisfies the Von Neumann and Morgenstern axioms, then this property will indeed hold. You should make clear in your answer which of the Von Neumann and Morgenstern axioms you are appealing to.

5. Consider the following four lotteries with monetary rewards:

$$\ell_1 = \left[\frac{1}{2}£100 + \frac{1}{2}£0\right] \tag{1}$$

$$\ell_2 = £50 \tag{2}$$

$$\ell_3 = \left[\frac{1}{20}£100 + \frac{19}{20}£0\right] \tag{3}$$

$$\ell_4 = \left[\frac{1}{10}£50 + \frac{9}{10}£0\right] \tag{4}$$

Now suppose I claim that my preferences satisfy both $\ell_2 \succ \ell_1$ and $\ell_3 \succ \ell_4$. Show that my preferences in this case do not satisfy the Von Neumann and Morgenstern axioms.

6. Suppose $\Omega = \{A, B, C, D\}$, and consider the following four lotteries over Ω:

$$\ell_1 = \left[\frac{3}{5}A + \frac{2}{5}D\right] \tag{5}$$

$$\ell_2 = \left[\frac{3}{4}A + \frac{1}{4}C\right] \tag{6}$$

$$\ell_3 = \left[\frac{2}{5}A + \frac{1}{5}B + \frac{1}{5}C + \frac{1}{5}D\right] \tag{7}$$

$$\ell_4 = \left[\frac{2}{5}A + \frac{3}{5}C\right] \tag{8}$$

Suppose that a Von Neumann and Morgenstern preference relation $\succ \subseteq lott(\Omega) \times lott(\Omega)$ satisfies the following properties:

$$C \sim \ell_1 \tag{9}$$

$$B \sim \ell_2 \tag{10}$$

$$A \succ D \tag{11}$$

How are ℓ_3 and ℓ_4 ranked?

7. Suppose a person whose preferences satisfy the Von Neumann and Morgenstern axioms says that with respect to lotteries ℓ_1, ℓ_2, ℓ_3, and ℓ_4, her preferences are

$$\ell_1 \succeq \ell_2 \quad \text{and} \quad \ell_3 \succeq \ell_4.$$

Consider the following property. For all $p \in [0, 1]$ we have

$$p\ell_1 + (1-p)\ell_3 \;\succeq\; p\ell_2 + (1-p)\ell_4$$

Do you think this property should hold? If so, can you provide an argument that it does with respect to the von Neumann and Morgenstern axioms?

8. Consider the following scenario.

> You toss a fair coin repeatedly, until the coin shows heads for the first time. You are then paid $\pounds 2^k$, where k is the number of times the coin was tossed. For example, if the sequence of coin tosses was just H (i.e., heads appeared on the first toss) then $k = 1$ and you are paid $\pounds 2^1 = \pounds 2$. If the sequence was TTH then $k = 3$ and you are paid $\pounds 2^3 = \pounds 8$.

(a) Express this as a lottery (in which the set of outcomes is infinite).

(b) Assuming utility is expected monetary reward, compute the expected utility of this lottery.

(c) Now assume that the utility function u over monetary outcomes is such that $u(\pounds n) = \log_2 n$. Show that the agent's expected utility of this game is upward bounded.

(d) By giving examples, argue that the agent with the utility function as in the previous part (i.e., logarithmic) is *risk-averse*.

9. The following story, albeit slightly morbid, is nevertheless apparently true.

> In the second world war, a US bomber squadron was based 3000km from their target. The target was so far away that fighter plane escorts were impossible, making missions even more than usually dangerous. Planes could only carry a few bombs on each mission, so that they could carry enough fuel to return to base. Pilots were scheduled to fly 30 missions before returning to the USA, but on average only half of the pilots survived all 30 missions.
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> Logistics experts came up with the following proposal. Each plane would carry a much heavier bomb load – but only enough fuel to fly one way. Thus, each mission would be a suicide mission. However, the increased bomb load would mean that far fewer missions would be needed, allowing 75% of the pilots to return home. The other 25% of pilots, who had to fly the missions, would face certain death. Those to fly the missions would be selected randomly.
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> Every pilot who was presented with the new proposal rejected it in favour of the status quo.

(a) Formulate the above two scenarios as lotteries within the Von Neumann and Morgenstern framework, in which there are just two outcomes, $live$ and die, and such that $live \succ die$.

Show that, with preferences as expressed in the scenario above, the pilots violate Von Neumann and Morgenstern's axioms. Which of the axioms did the pilots violate when they made their choice?

(b) Now assume there are three outcomes, with associated preferences as follows:

$$\text{live with honour} \succ \text{die with honour} \succ \text{live without honour}$$

So: living with honour would mean flying a mission and surviving; to die with honour would mean flying a mission and being killed; and living without honour would mean living because somebody else had flown a mission to certain death.

Now reformulate the above scenarios as lotteries using these preferences. Do the pilots violate the Von Neumann and Morgenstern axioms?

(c) Are the preferences of airmen in this example lexicographic? If so can you give relevant attributes and explain how they are ranked?

(d) What factors do you think may have influenced the pilot's decisions?