Homological Algebra
Sheet 3

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Exercise 2

(i) Consider the following free resolution of $k[x]/(x-a)$

$$0 \to k[x] \xrightarrow{\cdot(x-a)} k[x] \to \frac{k[x]}{(x-a)} \to 0.$$

Tensoring with $k[x]/(x-b)$ over $k[x]$ gives us the following complex

$$0 \to \frac{k[x]}{(x-b)} \xrightarrow{\cdot(b-a)} k[x]/(x-b) \to 0.$$

Denoting the multiplication map $\cdot(x-a)$ by f there are two cases two consider.

$(a \neq b)$: In this case we have that

$$\ker(f) = \{c \in k \mid c(x-a) = c(b-a) = 0\}.$$

Hence, if $c \in \ker(f)$ we have that $cb = ca$ so we must have $c = 0$ as in the other case we get $b = a$ which is a contradiction.

$$\mathrm{Tor}_1^{k[x]}\left(\frac{k[x]}{(x-a)}, \frac{k[x]}{(x-b)}\right) = \ker(f) = 0$$
$$\mathrm{Tor}_0^{k[x]}\left(\frac{k[x]}{(x-a)}, \frac{k[x]}{(x-b)}\right) = \frac{k[x]}{(x-a)} \otimes_{k[x]} \frac{k[x]}{(x-b)} = \frac{k[x]}{(x-a, x-b)} = 0.$$

$(a = b)$: In this case we have that

$$\ker(f) = \{c \in k \mid c(x-a) = c(b-a) = 0\}.$$

However this requirement for being in $\ker(f)$ holds true for every $c \in k$ and hence $\ker(f) = k[x]/(x-b)$. Thus

$$\mathrm{Tor}_1^{k[x]}\left(\frac{k[x]}{(x-a)}, \frac{k[x]}{(x-b)}\right) = \ker(f) = \frac{k[x]}{(x-b)}$$
$$\mathrm{Tor}_0^{k[x]}\left(\frac{k[x]}{(x-a)}, \frac{k[x]}{(x-b)}\right) = \frac{k[x]}{(x-a)} \otimes_{k[x]} \frac{k[x]}{(x-b)} = \frac{k[x]}{(x-b, x-b)} = \frac{k[x]}{(x-b)}.$$

(ii) Take the free resolution

$$0 \to \mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z} \to \frac{\mathbb{Z}}{(a)} \to 0.$$

Tensoring with $\mathbb{Z}/b$ we get the complex

$$0 \to \frac{\mathbb{Z}}{(b)} \xrightarrow{f} \frac{\mathbb{Z}}{(b)} \to 0$$

with $f(1) = a \bmod b$. We then have that

$$\ker(f) = \{c \in \mathbb{Z}/(a) \mid b|ac\} = \mathbb{Z}/(d)$$

where $d = \gcd(a, b)$. Hence

$$\begin{aligned}\mathrm{Tor}_1^{\mathbb{Z}}\left(\frac{\mathbb{Z}}{(a)}, \frac{\mathbb{Z}}{(b)}\right) &= \ker(f) = \frac{\mathbb{Z}}{(d)}\\ \mathrm{Tor}_0^{\mathbb{Z}}\left(\frac{\mathbb{Z}}{(a)}, \frac{\mathbb{Z}}{(b)}\right) &= \mathrm{coker}(f) = \frac{\mathbb{Z}}{(a)} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{(b)} = \frac{\mathbb{Z}}{(d)}.\end{aligned}$$

(iii) Take the following free resolution of $\mathbb{Z}/(2)$ over $\mathbb{Z}/(4)$

$$\cdots \xrightarrow{\cdot 2} \frac{\mathbb{Z}}{(4)} \xrightarrow{\cdot 2} \frac{\mathbb{Z}}{(4)} \to \frac{\mathbb{Z}}{(2)} \to 0.$$

Applying the hom functor $\mathrm{Hom}_{\mathbb{Z}/(4)}(-, \mathbb{Z}/(2))$ we get the following complex

$$0 \to \mathrm{Hom}_{\mathbb{Z}/(4)}(\mathbb{Z}/(4), \mathbb{Z}/(2)) \xrightarrow{\cdot 2} \mathrm{Hom}_{\mathbb{Z}/(4)}(\mathbb{Z}/(4), \mathbb{Z}/(2)) \xrightarrow{\cdot 2} \cdots$$

which is equivalent to

$$0 \to \frac{\mathbb{Z}}{(2)} \xrightarrow{\cdot 0} \frac{\mathbb{Z}}{(2)} \xrightarrow{\cdot 0} \cdots$$

so that

$$\mathrm{Ext}^*_{\mathbb{Z}/(4)} = \bigoplus_{i=0}^{\infty} \frac{\mathbb{Z}}{(2)}.$$

(iv) Take the following free resolution of $\mathbb{Z}/(2^b)$

$$\cdots \xrightarrow{\cdot 2^{a-b}} \mathbb{Z}/(2^a) \xrightarrow{\cdot 2^b} \mathbb{Z}/(2^a) \xrightarrow{\cdot 2^{a-b}} \mathbb{Z}/(2^a) \xrightarrow{\cdot 2^b} \mathbb{Z}/(2^a) \to \mathbb{Z}/(2^b) \to 0.$$

Applying $\mathrm{Hom}_{\mathbb{Z}/(2^a)}(-, \mathbb{Z}/(2^c))$ we get the following complex

$$0 \to \mathbb{Z}/(2^c) \xrightarrow{\cdot 0} \mathbb{Z}/(2^c) \xrightarrow{\cdot 2^{a-b}} \mathbb{Z}/(2^c) \xrightarrow{\cdot 0} \cdots.$$

If $a - b \geq c$ then $\cdot 2^{a-b} = \cdot 0$ so that

$$\mathrm{Ext}^*_{\mathbb{Z}/(2^a)}(\mathbb{Z}/(2^b), \mathbb{Z}/(2^c)) = \bigoplus_{i=0}^{\infty} \mathbb{Z}/(2^c).$$

On the other hand, if $c > a - b$ then we have that