Homological Algebra

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October 21, 2023

Contents

1 Abelian Categories

Definition 1. Let $\mathcal{C}$ be a category, and let $x \in \mathcal{C}$. We say that x is **terminal** if for every $c \in \mathcal{C}$, there is exactly one morphism $c \to x$. Dually, we say that x is **initial** if for every $c \in \mathcal{C}$, there is exactly one morphism $x \to c$.

Definition 2. A **zero object** in a category is an object that is both initial and terminal.

1.1 Ab-enriched Categories

Definition 3. A **pre-additive** or **Ab-enriched** category is a category in which every hom-set is equipped with the structure of an abelian group, such that composition

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$$

is $\mathbb{Z}$-bilinear.

Proposition 1. In an **Ab**-enriched category, any initial object is also terminal.

Proof. Let $*$ be initial. Then 1_* is the unique element of $\mathrm{Hom}(*,*)$, so 1_* is zero in this group. Then since composition respects the group structures, we have for any map $f : A \to *$,

$$f = 1_* \circ f = 0 \circ f = 0$$

so $*$ is terminal. $\square$

> **Proposition 2.** If $\mathcal{C}$ is an **Ab**-enriched category, then so is its opposite category $\mathcal{C}^{\mathrm{op}}$.

Proof. For $X, Y \in \mathcal{C}^{\mathrm{op}}$, the sets

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$$

are already endowed with the structure of an abelian group. Thus, we only have to prove that composition is bilinear. Let $X, Y, Z \in \mathcal{C}$ and let

$$f, f' \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y), \quad g \in \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(Y,Z).$$

Then

$$g \circ_{\mathrm{op}} (f + f') = (f + f') \circ g = f \circ g + f' \circ g = g \circ_{\mathrm{op}} f + g \circ_{\mathrm{op}} f'.$$

Similarly, composition is linear in the other argument as well. $\square$

> **Proposition 3.** In an **Ab**-enriched category $\mathcal{C}$, a binary product is also a binary coproduct.

Proof. Let X_1, X_2 be elments of an **Ab**-enriched category $\mathcal{C}$. Suppose that X_1 and X_2 have a product $X_1 \times X_2$ in $\mathcal{C}$, with projections $p_k : X_1 \times X_2 \to X_k$. By definition of products, there are unique morphisms $i_k : X_k \to X_1 \times X_2$ such that the following diagrams commute.

$$\begin{tikzcd}
X_1 \arrow[d, "\mathrm{id}"'] \arrow[dr, dashed, "i_1"] \arrow[ddrr, bend left, "0"] & & \\
& X_1 \times X_2 \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X_1 & & X_2
\end{tikzcd}
\qquad
\begin{tikzcd}
& & X_2 \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "i_2"'] \arrow[ddll, bend right, "0"'] \\
& X_1 \times X_2 \arrow[dl, "p_1"] \arrow[dr, "p_2"'] & \\
X_1 & & X_2
\end{tikzcd}$$

Then we have

$$p_1 \circ (i_1 p_1 + i_2 p_2) = p_1, \quad p_2 \circ (i_1 p_1 + i_2 p_2) = p_2.$$

By definition of products, $\mathrm{id} X_1 \times X_2 \times X_1 \times X_2$ is the unique morphisms with $p_k \circ \mathrm{id} = p_k$ for each k, so $i_1 p_1 + i_2 p_2 = \mathrm{id}_{X_1 \times X_2}$. We claim that

$$\begin{tikzcd}
X_1 \arrow[dr, "i_1"] & & X_2 \arrow[dl, "i_2"'] \\
& X_1 \times X_2 &
\end{tikzcd}$$

is a universal cocone, so that $X_1 \times X_2 = X_1 \coprod X_2$. Suppose that

$$\begin{tikzcd} X_1 \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & A & \end{tikzcd}$$

is another cocone. Then we have a map

$$\phi = f_1 \circ p_1 + f_2 \circ p_2 : X_1 \times X_2 \to A,$$

which is easily seen to give a commutative diagram

$$\begin{tikzcd} X_1 \arrow[dr, "i_1"] \arrow[ddr, "f_1"'] & & X_2 \arrow[dl, "i_2"'] \arrow[ddl, "f_2"] \\ & X_1 \times X_2 \arrow[d, dashed, "\phi"] & \\ & A. & \end{tikzcd}$$

It remains to show that ϕ is unique. To see this, note that for any such ϕ we have

$$\begin{aligned} \phi &= \phi \circ \mathrm{id}_{X_1 \times X_2} \\ &= \phi \circ (i_1 p_1 + i_2 p_2) \\ &= \phi i_1 \circ p_1 + \phi i_2 \circ p_2 \\ &= f_1 \circ p_1 + f_2 \circ p_2. \end{aligned}$$

□

Proposition 4. In an **Ab**-enriched category, all binary coproducts are also binary products.

Proof. This is dual to the previous proposition. □

Definition 4. Let $\mathcal{C}$ be an **Ab**-enriched category, and let $x, y \in \mathcal{C}$. If x and y have a product in $\mathcal{C}$, then it is called the biproduct of x and y, which we denote by $x \oplus y$.

Definition 5. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between **Ab**-enriched categories. Then F is said to be **additive** if it preserves finite biproducts.

Lemma 1. For any ring R, the category R-**mod** is **Ab**-enriched.

1.2 Additive Categories

Definition 6. A category is **additive** if it is **Ab**-enriched and admits finite coproducts.

Lemma 2. Let $\mathcal{A}$ be an additive category. Suppose that $i : a \to b$ is a monomorphism in $\mathcal{A}$ and $\alpha \in \mathrm{Hom}(a, b)$ is the zero morphism. Then $a = 0$.

Proof. Let $x \in \mathcal{A}$. Since $\mathrm{Hom}(a, x)$ is an abelian group, it contains at least one morphism (zero). Let $f : a \to x$ be any morphism. Then

$$\alpha \circ 0 = 0 = \alpha \circ f.$$

Since α is a monomorphism, we have $f = 0$. Therefore a is initial, hence it is the zero object. $\square$

Lemma 3. Let $\mathcal{A}$ be an additive category. Suppose that $q : a \to b$ is an epimorphism in $\mathcal{A}$. If $q = 0$, then $b = 0$.

Proof. Since $\mathcal{A}$ is additive, the opposite category $\mathcal{A}^{\mathrm{op}}$ is too. The map q is a monomorphism $q : b \to a$ in $\mathcal{A}^{\mathrm{op}}$, and it is still the zero morphism. By the previous lemma we must therefore have that b is the zero object in $\mathcal{A}^{\mathrm{op}}$, hence in $\mathcal{A}$. $\square$

Lemma 4. For any ring R, the category R-**mod** is additive.

Proof. We know that the direct sum exists and is a coproduct in R-mod. $\square$

1.3 Pre-abelian Categories

Definition 7. An additive category is **pre-abelian** if every morphism has a kernel and cokernel.

Lemma 5. Let $\mathcal{A}$ be a pre-abelian category. Every monomorphism has kernel 0, and every epimorphism has cokernel 0.

Proof. Let $i : a \to b$ be a monomorphism in $\mathcal{A}$. Let

$$\mathrm{Ker} i \xrightarrow{\mathrm{ker} i} a$$

be the kernel of i. Then $i \circ \mathrm{ker} i = 0 = i \circ 0$, so $\mathrm{ker} i$ is the zero morphism (since i is a monomorphism). Since $\mathrm{ker} i$ is monomorphism, we have $\mathrm{Ker} i = 0$. $\square$

Lemma 6. For any ring R, the category R-**mod** is pre-abelian.

1.4 Abelian Categories

Definition 8. A pre-abelian category is **abelian** if every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

Lemma 7. The category of left R-modules is an abelian category.

Proof. Let $i : A \to B$ be a monomorphism of R-modules. Then $\mathrm{Coker} i = B/i(A)$ and the cokernel map is the quotient $q : B \to B/i(A)$ with $q(b) = b + i(A)$. It is clear that $i(A) = \mathrm{Ker} q$ in the set-theoretic sense, so i exhibits A as the kernel of q.

Let $q : A \to B$ be an epimorphism of R-modules. Let $i : \mathrm{Ker} q \to A$ be the inclusion. Then $\mathrm{Coker} i = A/\mathrm{Ker} q \cong B$, so q exhibits B as the cokernel of i. □

Lemma 8. If $\mathcal{A}$ is abelian, then so is $\mathcal{A}^{\mathrm{op}}$.

Proof. Duality. □

Lemma 9. If $\mathcal{A}$ is an abelian category and $\mathcal{C}$ is any category, then $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ is abelian.

1.5 Connection with R-mod

Theorem 1 (Freyd-Mitchell Embedding Theorem)**.** Let $\mathcal{A}$ be a small abelian category. Then there is a ring R and an exact, fully faithful functor $F : \mathcal{A} \to R\mathbf{mod}$. This functor embeds $\mathcal{A}$ as a full subcategory in R-**mod**, by which we mean that for all $M, N \in \mathcal{A}$, we have

$$\mathrm{Hom}_{\mathcal{A}}(M, N) \cong \mathrm{Hom}_R(F(M), F(N)).s.$$

Lemma 10. The Freyd-Mitchell embedding preserves kernels and cokernels.

Proof. Let $f : x \to y$ be a morphism in an abelian category $\mathcal{A}$, and let $F : \mathcal{A} \to R\mathbf{mod}$ be the Freyd-Mitchell embedding. Consider the sequence

$$0 \to \mathrm{Ker} f \xrightarrow{i} x \xrightarrow{f} y \xrightarrow{q} \mathrm{Coker} f \to 0.$$

□

Lemma 11. Let $\mathcal{A}$ be an abelian category and let $F : \mathcal{A} \to R$-**mod** be the embedding from before. Then $F(0) = 0$.

2 Exact Functors

2.1 Left- and Right-Exact Functors

Definition 9. A functor F is left-exact if for every short exact sequence $0 \to A \to B \to C$, the sequence

$$F(A) \to F(B) \to F(C) \to 0$$

is exact. Similarly, F is right-exact if instead

$$F(A) \to F(B) \to F(C) \to 0$$

is always exact.

Lemma 12. If $F : \mathcal{A} \to \mathcal{B}$ is left-exact, and i is a monomorphism in $\mathcal{A}$, then $F(i)$ is a monomorphism in $\mathcal{B}$.

Proof. If $i : A \to B$ is a monomorphism, then we have a SES

$$0 \to A \to B \to \operatorname{coker} i \to 0.$$

Therefore, $0 \to F(A) \to F(B)$ is exact, so $F(i)$ is a monomorphism. □

Lemma 13. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. The following are equivalent:

(i) F is left exact.

(ii) For any exact sequence $0 \to A \to B \to C$, the corresponding sequence $0 \to F(A) \to F(B) \to F(C)$ is also exact.

Proof. (ii) $\Rightarrow$ (i): Trivial.

(i) $\Rightarrow$ (ii): Let $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C$ be exact. Then we have a short exact sequence $0 \to A \to B \to \mathrm{im}\pi \to 0$, and therefor the sequence

$$0 \to F(A) \to F(B) \to F(\mathrm{im}\pi)$$

is exact. Now, $\mathrm{im}\pi \to C$ is a monomorphism, so $F(\mathrm{im}\pi) \to F(C)$ is too. Therefore

$$\begin{aligned}\ker(F(B) \to F(C)) &= \ker(F(B) \to F(\mathrm{im}\pi) \to F(C)) \\ &= \ker(F(B) \to F(\mathrm{im}\pi)) \\ &= \mathrm{im}(F(A) \to F(B)).\end{aligned}$$

□

By duality we have the dual result for right-exact functors.

Lemma 14. Let $\mathcal{A}$ be an abelian category, and consider maps

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathcal{A}$. Suppose that for all $Z \in \mathcal{A}$, the sequence

$$\mathrm{Hom}(A, Z) \xleftarrow{-\circ f} \mathrm{Hom}(B, Z) \xleftarrow{-\circ g} \mathrm{Hom}(C, Z) \leftarrow 0$$

is exact. Then $A \to B \to C \to 0$ is exact.

Proof. We need to show that g exhibits C as the cokernel of f. Suppose that $\alpha : B \to Z$ is some map with $\alpha \circ f = 0$. Then

$$\alpha \in \ker(- \circ f) = \mathrm{im}(g \circ -),$$

so $\alpha = \phi \circ g$ for a unique map $\phi : C \to Z$. This is precisely the universal property of the cokernel. □

Lemma 15. Suppose we have an adjunction

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] & \mathcal{B} \arrow[l, bend left, "G"] \end{tikzcd}$$

of additive functors between abelian categories, where F is the left adjoint. Then F is right-exact.

Proof. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence in $\mathcal{A}$, and let $Z \in \mathcal{B}$. Then $G(Z) \in \mathcal{A}$, so

$$\mathrm{Hom}(A, G(Z)) \xleftarrow{-\circ i} \mathrm{Hom}(B, G(Z)) \xleftarrow{-\circ \pi} \mathrm{Hom}(C, G(Z)) \leftarrow 0$$

is exact by left-exactness of Hom. Therefore,

$$\mathrm{Hom}(F(A), Z) \xleftarrow{-\circ i} \mathrm{Hom}(F(B), Z) \xleftarrow{-\circ \pi} \mathrm{Hom}(F(C), Z) \leftarrow 0$$

is exact, so

$$F(A) \to F(B) \to F(C) \to 0$$

is exact by the previous lemma. □

Corollary 1. If F, G are as in the previous lemma, then G is left exact.

Proof. This is just the dual statement. More explicitly, consider the opposite functor $G : \mathcal{D}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ which is left adjoint (because the original G is right adjoint) and hence right exact. So $G : \mathcal{D} \to \mathcal{C}$ is left exact. □

2.2 Exact Functors

Definition 10. A functor is **exact** if it is left-exact and right-exact.

Lemma 16. Suppose that we have a long exact sequence

$$\ldots \to A_{n-1} \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \to \ldots$$

and an exact functor F. Then

$$\ldots \to F(A_{n-1}) \to F(A_n) \to F(A_{n+1}) \to \ldots$$

is also exact.

Proof. Since we only have to check exactness at each term, it suffices to prove that for an exact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

the sequence

$$F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$$

is also exact. We prove this with a diagram-chase. Note that

$$0 \to \text{ker } f \to A \to \text{im } f \to 0,$$
$$0 \to \text{ker } g \to B \to \text{im } g \to 0,$$

and

$$0 \to \text{im } g \to C \to \text{coker } g \to 0$$

are short exact sequences. We can fit these into a larger commutative diagram:

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dr] & & 0 & & \\
& \text{ker } f \arrow[dr] & & & & \text{im g} \arrow[ur] \arrow[dr] & & & \\
& & A \arrow[rr, "f"] \arrow[dr] & & B \arrow[rr, "g"] \arrow[ur] & & C \arrow[dr] & & \\
& & & \text{im } f \arrow[ur] \arrow[dr] & & & & \text{coker } g \arrow[dr] & \\
& & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

Note that the diagonals are exact. Applying F to the diagram (and removing some redundant terms) gives a commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[dr] & & 0 & & \\
F(\ker\, f) \arrow[dr] & & & & F(\text{im g}) \arrow[ur] \arrow[dr, "F(i_2)"] & & & \\
 & F(A) \arrow[rr, "F(f)"] \arrow[dr, "F(\pi_1)"'] & & F(B) \arrow[rr, "F(g)"] \arrow[ur, "F(\pi_2)"] & & F(C) \arrow[dr] & & \\
 & & F(\text{im } f) \arrow[ur, "F(i_1)"'] \arrow[dr] & & & & F(\text{coker } g) \arrow[dr] & \\
 & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

Again the diagonals are exact. Since $F(\pi_1)$ is surjective, we have $\text{im } F(f) = \text{im } F(i_1) = \ker\, F(\pi_2)$ by exactness at $F(B)$. But $F(i_2)$ is injective, so $\ker\, F(g) = \ker\, F(\pi_2)$, and it follows that

$$\text{im } F(f) = \ker\, F(\pi_2) = \ker\, F(g).$$

□

2.3 Specific Functors

Lemma 17. Let $\mathcal{A}$ be an abelian category, and let $M \in \mathcal{A}$ be an object. Then the functor $\mathrm{Hom}_{\mathcal{A}}(M, -) : \mathcal{A} \to \mathbf{Ab}$ is left-exact.

Proof. Let $0 \to A \xrightarrow{i} B \xrightarrow{\pi} C \to 0$ be a short exact sequence in $\mathcal{A}$. We have to show that the sequence

$$0 \to \mathrm{Hom}(M, A) \xrightarrow{i\circ -} \mathrm{Hom}(M, B) \xrightarrow{\pi\circ -} \mathrm{Hom}(M, C)$$

is exact. For exactness at $\mathrm{Hom}(M, A)$, suppose that $i \circ \phi = 0$, where $\phi : M \to A$ is a map. Since i is a monomorphism and $i \circ \phi = i \circ 0$, we have $\phi = 0$. Therefore the sequence is exact at $\mathrm{Hom}(M, A)$.

Since $\pi \circ i = 0$, we have $\text{im}(i\circ -) \subset \ker(\pi\circ -)$. Let $\phi \in \ker(\pi\circ -)$. Then $\pi\circ\phi = 0$. Since i exhibits A as the kernel of π, there is a unique map $f : M \to A$ such that $i \circ f = \phi$. Therefore, the sequence is also exact at $\mathrm{Hom}(M, B)$. □

Corollary 2. Let $\mathcal{A}$ be an abelian category and let $M \in \mathcal{A}$ be an object. Then the functor

$$\mathrm{Hom}_{\mathcal{A}}(-, M) : \mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$$

is left-exact.

Proof. This follows from the definition of the opposite category. □

Corollary 3. Let R be a ring and M be an R-module. Then the functors

$$\mathrm{Hom}_R(M,-) : R\text{−}\mathbf{mod} \to \mathbf{Ab}, \quad \mathrm{Hom}_R(-,M) : R\text{−}\mathbf{mod}^{\mathrm{op}} \to \mathbf{Ab}$$

are left-exact.

Lemma 18. For any ring R, the functor $- \otimes_R N : R\text{−}\mathbf{mod} \to R\text{−}\mathbf{mod}$ is right-exact.

Proof. This follows from the adjunction

$$(- \otimes_R N) \dashv \mathrm{Hom}_R(N,-)$$

and Lemma 15. $\square$