The University of Oxford

MSc (Mathematics and Foundations of Computer Science)

Category Theory C2.7

Michaelmas Term 2023

The steps of (each) mini project are for your guidance; if you wish to take an alternative route to the desired goal, you are free to do so. But, if you follow the suggested route and find yourself unable to carry out any particular step, you may simply assume it so that you can continue with the mini project, but should make this assumption clear in your presentation.

The purpose of this miniproject is to study the concept of **reflective subcategory**. By definition, a reflective subcategory of a category $\mathcal{C}$ is a full subcategory $\mathcal{D}$ such that the inclusion $i : \mathcal{D} \hookrightarrow \mathcal{C}$ admits a left adjoint $L : \mathcal{C} \to \mathcal{D}$, called the *reflector.* A starting reference for this concept is §4.6 in E. Riehl's "Category Theory in Context", available online via Bodleian Library (you can find the link in the Moodle section of our course).

1. Begin by reviewing the notion of *adjoint functors* $F \dashv G$, with unit $e : 1_{\mathcal{C}} \Longrightarrow GF$ and counit $\epsilon : FG \Longrightarrow 1_{\mathcal{D}}$. In particular prove that:

(i) G is faithful if and only if each component of ϵ is an epimorphism.

(ii) G is full if and only if each component of ϵ is a split monomorphism.

(iii) G is fully faithful if and only if ϵ is an isomorphism.

(iv) F is faithful if and only if each component of e is a monomorphism.

(v) F is full if and only if each component of e is a split epimorphism.

(vi) F is fully faithful if and only if e is an isomorphism.

2.

(i) Show that abelian groups define a reflective subcategory of groups, where the reflector carries a group G to its abelianization $G/[G, G]$.

(ii) Show that the subcategory of commutative rings is a reflective subcategory of the category of rings.

(iii) Discuss some other example(s) of reflective subcategories, see for example Riehl's "Category Theory in Context".

3. For a reflective subcategory $i : \mathcal{D} \hookrightarrow \mathcal{C}$ with reflector $L : \mathcal{C} \to \mathcal{D}$:

(i) Show that an object $x \in \mathcal{C}$ is in the essential image of the inclusion i if and only if e_x is an isomorphism.

(ii) Show that the essential image of i consists of the objects $x \in \mathcal{C}$ that are *local* in the following sense: the pre-composition functions $\mathrm{Hom}_{\mathcal{C}}(y, z) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$ are isomorphisms for all maps $f : x \to y$ in $\mathcal{C}$ for which $L(f) : L(x) \to L(y)$ is an isomorphism in $\mathcal{D}$.

4. Show that the inclusion $i : \mathcal{D} \hookrightarrow \mathcal{C}$ for a reflective subcategory creates all limits that $\mathcal{C}$ admits and that $\mathcal{D}$ has all colimits that $\mathcal{C}$ admits.

Find an example which shows that the inclusion of a reflective subcategory does not create all colimits.

5. Explain how a reflective subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$ defines an *idempotent monad* on $\mathcal{C}$ via the endofunctor $Li : \mathcal{C} \to \mathcal{C}$. Prove that the inclusion $\mathcal{D} \hookrightarrow \mathcal{C}$ is monadic.

Can you show a converse of this: if there exists an idempotent monad on $\mathcal{C}$, can you construct a reflective subcategory of $\mathcal{C}$?