Homological Algebra

Sheet 1 — MT23

Section A

1. $A, B, C \in Mod_R$. Show that there exist canonical R-module isomorphisms

   $Hom(A \oplus B, C) \simeq Hom(A, C) \oplus Hom(B, C)$ and

   $Hom(A, B \oplus C) \simeq Hom(A, B) \oplus Hom(A, C)$.

   More generally, prove

   $Hom(\bigoplus_{i \in I} M_i, N) \simeq \prod_{i \in I} Hom(M_i, N)$

   and

   $Hom(M, \prod_{i \in I} N_i) \simeq \prod_{i \in I} Hom(M, N_i)$.

   Solution: We reduce to the general case by noting the equivalence between finite products and finite coproducts in abelian categories (in particular, in Mod_R). Now the canonical isomorphisms are precisely the content of the universal properties of products and coproducts.

   Explicitly, the natural maps $p_j : \prod_{i \in I} N_i \to N_j$ and $q_j : M_j \to \bigoplus_{i \in I} M_I$ yield

   $Hom(M, \prod_{i \in} N_i) \to \prod_{i \in I} Hom(M, N_i)$ via $\phi \mapsto (p_j \circ \phi)_{j \in I}$ and

   $Hom(\bigoplus_{i \in I} M_i, N) \to \prod_{i \in I} Hom(M_i, N)$ via $\phi \mapsto (\phi \circ q_j)_{j \in I}$. The inverse maps are provided by the universal properties.

   In the setting of Mod_R, these canonical bijections clearly preserve the R-module structures, e.g. because the explicit natural maps above do. We've implicitly used that the product in Mod_R is the product in Set (which of course follows from the right-adjointness of the forgetful functor (a left adjoint is the free functor).

2. A monomorphism is a morphism f satisfying $[f \circ g_1 = f \circ g_2] \implies [g_1 = g_2]$. An epimorphism is a morphism satisfying $[g_1 \circ f = g_2 \circ f] \implies [g_1 = g_2]$.

   Given $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, show (using the language of category theory) that f is a monomorphism and g is an epimorphism.

   Solution: We know that $f = \ker g$ and so is an equalizer.

   Claim: Any equalizer is a monomorphism. Let $X \xrightarrow{f} Y$ be the equalizer of $Y \overset{g_1}{\underset{g_2}{\rightrightarrows}} Z$, i.e. for any W, $Hom(W, X) \cong \{\phi \in Hom(W, Y) : g_1 \circ \phi = g_2 \circ \phi\}$, i.e. X represents

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the functor mapping W to $Hom(W, Y) \times_{g_1 \circ -, g_2 \circ -} Hom(W, Y)$. The bijection is induced by $\psi \mapsto f \circ \psi$. Thus, $f \circ - : Hom(W, X) \to Hom(W, Y)$ is injective for all W, i.e. f is injective. By applying $(-)^{op}$, we get the dual result that any co-equalizer is an epimorphism, and note that $coker(f) = g$.

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Section B

3. $$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "i"] & B \arrow[r, "g"] \arrow[d, "j"] & C \arrow[r] \arrow[d, "k"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd}.$$

   Suppose i, k are isomorphisms. Show that j must then be an isomorphism.

4. Let $R := k[x, y]$ where k is a field. Let $M_1 := R^2/\langle (x, 0), (y^2, -x), (0, y)\rangle$ and $M_2 := R/\langle x^2, xy, y^3\rangle$. Provide examples of non-split short exact sequences of R-modules

   $0 \to M_1 \to ??? \to M_2 \to 0.$

5. Prove that every $Mod_{\mathbb{Z}}$-SES of the form $0 \to A \to B \to \mathbb{Z} \to 0$ splits.

   Prove that every $Mod_{\mathbb{Z}}$-SES of the form $0 \to \mathbb{Q} \to B \to C \to 0$ splits.

6. Prove that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p:\text{prime}} \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$.

7. Prove that in general, $Hom(M, \bigoplus_{i\in I} N_i) \ncong \bigoplus_{i\in I} Hom(M, N_i)$.

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Section C

8. Prove that the natural inclusion $\bigoplus_{i\in\mathbb{N}}\mathbb{Z} \to Hom(\prod_{i\in\mathbb{N}}\mathbb{Z},\mathbb{Z})$ is an isomorphism.

   Solution: Note that as abelian groups, $\bigoplus_{i\in\mathbb{N}}\mathbb{Z} \simeq \mathbb{Z}[t]$ and $\prod_{i\in\mathbb{N}}\mathbb{Z} \simeq \mathbb{Z}[[t]]$. Note further that $\prod_{i\in\mathbb{N}}\mathbb{Z} \simeq Hom_{Mod_{\mathbb{Z}}}(\bigoplus_{i\in\mathbb{N}}\mathbb{Z},\mathbb{Z})$, so this result can be seen as verifying a double dual isomorphism.

   Following Kevin Buzzard and Richard Stanley.

   https://mathoverflow.net/questions/10239/is-it-true-that-as-bbb-z-modules-the-polynomial-ring-and-the-power-series-r which in turn seem influenced by some work of Kaplansky.

   Consider $f:\prod_{i\in\mathbb{N}}\mathbb{Z}\to\mathbb{Z}$. Let e_n be the n^{th} basis element of $\prod_{i\in\mathbb{N}}\mathbb{Z}$. Suppose there exist infinitely many n such that $f(e_n)\neq 0$. Set $\tau_j := \sum_{i\geq j} 2^{\sum_{k<i}(\lceil\log_2|f(e_k)|\rceil+1)}|f(e_i)|$ and note τ_0 is in $\mathbb{Z}_2-\mathbb{Z}$; it is the bitstring of the concatenation of $\{f(e_i)\}_i$ with some bonus spacing – the point is that if $f(e_n)\neq 0$ for infinitely many n, then this bitstring never stabilizes into a sequence of 0's as would befit an element of $\mathbb{Z}$.

   Now consider $b_j := \sum_{i\geq j} 2^{\sum_{k<i}(\lceil\log_2|f(e_k)|\rceil+1)}(f(e_i))e_i \in \prod_{i\in\mathbb{N}}\mathbb{Z}$. By assumption, we have $f(b_0)\in\mathbb{Z}\subseteq\mathbb{Z}_2$. Note $b_N = \sum_{i\geq N} 2^{\sum_{k<i}(\lceil\log_2|f(e_k)|\rceil+1)}(f(e_i))e_i = 2^N\cdot c_N$ for some $c_N\in\prod_{i\in\mathbb{N}}\mathbb{Z}$. Thus, $f(b_N)\in 2^N\mathbb{Z}$. Note $\tau_N\in 2^N\mathbb{Z}_2$.

   Note that b_0-b_N is a finite sum and $\tau_0-\tau_N = f(b_0-b_N) = f(b_0)-f(b_N)$. Thus we see that $\tau_0 - f(b_0) = \tau_N - f(b_N)\in 2^N\mathbb{Z}_2$ for all N. Thus, $\tau_0 = f(b_0)$, but this violates $f(b_0)\in\mathbb{Z}$.

   So $f(e_i)=0$ for all but finitely-many $i\in\mathbb{N}$. Let $g\in\bigoplus_{i\in\mathbb{N}}\mathbb{Z}$ be $g=\sum_n f(e_n)e_n$.

   Note that $f-g:\prod_{i\in\mathbb{N}}\mathbb{Z}\to\mathbb{Z}$ vanishes on $\bigoplus_{i\in\mathbb{N}}\mathbb{Z}$. So let's consider $\phi:\prod_{i\in\mathbb{N}}\mathbb{Z}/\bigoplus_{i\in\mathbb{N}}\mathbb{Z}\to\mathbb{Z}$

   Let $x=(x_0,\dots)\in\prod_{i\in\mathbb{N}}\mathbb{Z}$. Note $x_i = a_i2^i + b_i3^i$ for some $a_i,b_i\in\mathbb{Z}$. Thus $\phi(x) = \phi((a_i2^i)_i)+\phi((b_i3^i)_i)$.

   Note that

   $\phi((a_i2^i)_i) = \phi((a_i2^i)_{\text{with zero for } i<N}) + \phi((a_i2^i)_{\text{with zero for } i\geq N}) = 2^N\phi((a_i2^{i-N})_{\text{with zero for } i<N}) + 0$ must be divisible by 2^N for all N. Thus, $\phi((a_i2^i)_i)=0$, and $\phi((b_i3^i)_i)=0$ similarly. And so $\phi=0$. Thus, $f=g$.

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