

Sheet 1

Mika Bohinen

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Exercise 2

- (a) Let $F \in \text{Fun}(* / G_1, * / G_2)$. We then have that $F(*_{G_1}) = *_{G_2}$. Moreover the functoriality of F means that if $g, g' \in G_1$ then $F(g' \cdot g) = F(g') \cdot F(g)$. Hence F is just a group homomorphism.

Now, if $\eta : F \Rightarrow H$ is a natural transformation for $F, H \in \text{Fun}(* / G_1, * / G_2)$ and $g_1 \in G_1$, then we get the following commutative diagram

$$\begin{array}{ccc} *_{G_2} & \xrightarrow{Fg_1} & *_{G_2} \\ \eta_* \downarrow & & \downarrow \eta_* \\ *_{G_2} & \xrightarrow{Hg_1} & *_{G_2} \end{array}$$

We then have that $\eta_* = g_2$ for some $g_2 \in G_2$. The commutativity of the diagram tells us that

$$Fg_1 = g_2^{-1} H(g_1) g_2.$$

Thus a natural transformation is just conjugation by an element of the group G_2 .

- (b) Let $F \in \text{Fun}(* / \mathbb{Z}, * / G)$. Then, by the previous exercise we know that $\text{Fun}(* / \mathbb{Z}, * / G) \cong \text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G)$. Hence F corresponds to some $f \in \text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G)$. Moreover, since $f(0) = e_G$ and $f(n) = f(1)^n$ we have that the image of f is simply the cyclic subgroup $\langle f(1) \rangle$. The most natural thing is therefore to associate $f \in \text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G)$ with $f(1)$. As $f(1)$ can be any element in G we therefore have that $\text{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) \cong G$ as a group isomorphism (the isomorphism being given by $f \mapsto f(1)$).

With $n \in \mathbb{Z}$ considered as a morphism and $a \in G$ which represents the natural transformation $\eta : F \Rightarrow H$, and $F(n) = g_1^n$, $H(n) = g_2^n$ we have the equation

$$g_1^n = a^{-1} g_2^n a.$$

This has to hold for all n but it suffices that it holds for $n = 1$ where we have the equation

$$g_1 = a^{-1} g_2 a.$$

Hence natural transformations $F \Rightarrow G$ are in one-to-one correspondence with conjugation relations of g_1 with g_2 .

Exercise 3

As $\text{Hom}_{\text{Vect}_k^{fd}}(-, k)$ is contravariant we cannot technically have an equivalence of categories. However, we can speak of a duality of categories where we just require that both F and G are contravariant.

We first note that equivalence of categories is an equivalence relation. Considering the diagram

$$\begin{array}{ccc} \text{Vect}_k^{fd} & \longrightarrow & (\text{Vect}_k^{fd})^{\text{op}} \\ \downarrow & & \downarrow \\ \text{Mat}_k & \longrightarrow & (\text{Mat}_k)^{\text{op}} \end{array}$$

it suffices to show that $\text{Mat}_k \rightarrow (\text{Mat}_k)^{\text{op}}$ is a duality of categories since we know that the two vertical arrows already are part of an equivalence of categories.

The contravariant functor $\text{Mat}_k \rightarrow (\text{Mat}_k)^{\text{op}}$ is the identity on the objects and sends each matrix to its transpose. Hence if we create the contravariant functor $(\text{Mat}_k)^{\text{op}} \rightarrow \text{Mat}_k$ which does exactly the same then their composition is the identity so that we have a contravariant isomorphism of categories.

Thus the map $\text{Vect}_k^{fd} \rightarrow (\text{Vect}_k^{fd})^{\text{op}}$ is a duality of categories.

Exercise 4

Let $f, g \in \text{Hom}(x, y)$. Then, since $\text{End}(x)$ is the trivial group, we must have that $g^{-1} \circ f = 1_x$. This implies that $f = g$ showing that $|\text{Hom}(x, y)| = 1$.

Next, let $F : \mathcal{C} \rightarrow *$ be the unique functor from \mathcal{C} to $*$ (as $*$ is terminal in **Cat**). We then just have to show that F is fully-faithful and essentially surjective. As F is surjective on objects, it must be essentially surjective. We therefore only need to show that F is fully-faithful. This is quite straight-forward:

1. **Injectivity.** Let $f, g \in \text{Hom}(x, y)$ such that $Ff = Fg$. Since $\text{Hom}(x, y) = \{1\}$ we must have that $f = g$.
2. **Surjectivity.** For $x, y \in \mathcal{C}$ we have that $F(x) = * = F(y)$. Hence $\text{Hom}(Fx, Fy) = \{1_*\}$. As $|\text{Hom}(x, y)| = 1$ this unique isomorphism between x and y must get mapped to 1_* which shows that $F : \text{Hom}(x, y) \rightarrow \text{Hom}(Fx, Fy)$ is surjective.

Since F is both fully-faithful and essentially surjective we can conclude that there exists a $G : * \rightarrow \mathcal{C}$ such that $GF \cong \text{id}_{\mathcal{C}}$ and $FG \cong \text{id}_*$ which shows that \mathcal{C} is equivalent to the discrete category with a single object.