10
The Derived Category

There are many formal similarities between homological algebra and algebraic topology. The Dold-Kan correspondence, for example, provides a dictionary between positive complexes and simplicial theory. The algebraic notions of chain homotopy, mapping cones, and mapping cylinders have their historical origins in simplicial topology.

The *derived category* $\mathbf{D}(\mathcal{A})$ of an abelian category is the algebraic analogue of the homotopy category of topological spaces. $\mathbf{D}(\mathcal{A})$ is obtained from the category $\mathbf{Ch}(\mathcal{A})$ of (cochain) complexes in two stages. First one constructs a quotient $\mathbf{K}(\mathcal{A})$ of $\mathbf{Ch}(\mathcal{A})$ by equating chain homotopy equivalent maps between complexes. Then one "localizes" $\mathbf{K}(\mathcal{A})$ by inverting quasi-isomorphisms via a calculus of fractions. These steps will be explained below in sections 10.1 and 10.3. The topological analogue is given in section 10.9.

10.1 The Category $\mathbf{K}(\mathcal{A})$

Let $\mathcal{A}$ be an abelian category, and consider the category $\mathbf{Ch} = \mathbf{Ch}(\mathcal{A})$ of cochain complexes in $\mathcal{A}$. The quotient category $\mathbf{K} = \mathbf{K}(\mathcal{A})$ of $\mathbf{Ch}$ is defined as follows: The objects of $\mathbf{K}$ are cochain complexes (the objects of $\mathbf{Ch}$) and the morphisms of $\mathbf{K}$ are the chain homotopy equivalence classes of maps in $\mathbf{Ch}$. That is, $\mathrm{Hom}_{\mathbf{K}}(A, B)$ is the set $\mathrm{Hom}_{\mathbf{Ch}}(A, B)/\sim$ of equivalence classes of maps in $\mathbf{Ch}$. We saw in exercise 1.4.5 that $\mathbf{K}$ is well defined as a category and that $\mathbf{K}$ is an additive category in such a way that the quotient $\mathbf{Ch}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ is an additive functor.

It is useful to consider categories of complexes having special properties. If $\mathcal{C}$ is any full subcategory of $\mathbf{Ch}(\mathcal{A})$, let $\mathcal{K}$ denote the full subcategory of $\mathbf{K}(\mathcal{A})$ whose objects are the cochain complexes in $\mathcal{C}$. $\mathcal{K}$ is a "quotient category" of $\mathcal{C}$

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in the sense that

$$\mathrm{Hom}_{\mathcal{K}}(A, B) = \mathrm{Hom}_{\mathbf{K}}(A, B) = \mathrm{Hom}_{\mathbf{Ch}}(A, B)/\sim\, = \mathrm{Hom}_{\mathcal{C}}(A, B)/\sim .$$

If $\mathcal{C}$ is closed under $\oplus$ and contains the zero object, then by 1.6.2 both $\mathcal{C}$ and $\mathcal{K}$ are additive categories and $\mathcal{C} \to \mathcal{K}$ is also an additive functor.

We write $\mathbf{K}^b(\mathcal{A})$, $\mathbf{K}^-(\mathcal{A})$, and $\mathbf{K}^+(\mathcal{A})$ for the full subcategories of $\mathbf{K}(\mathcal{A})$ corresponding to the full subcategories $\mathbf{Ch}^b$, $\mathbf{Ch}^-$, and $\mathbf{Ch}^+$ of bounded, bounded above, and bounded below cochain complexes described in section 1.1. These will be useful in section 5 below.

Of course, we could have equally well considered chain complexes instead of cochain complexes when constructing $\mathbf{K}$. However, the historical origins of derived categories were in Grothendieck's study of sheaf cohomology [HartRD], and the choice to use cochains is fixed in the literature.

Having introduced the cast of categories, we turn to their properties.

Lemma 10.1.1 *The cohomology $H^*(C)$ of a cochain complex C induces a family of well-defined functors H^i from the category $\mathbf{K}(\mathcal{A})$ to $\mathcal{A}$.*

Proof As we saw in 1.4.5, the map $u^*\colon H^i(A) \to H^i(B)$ induced by $u\colon A \to B$ is independent of the chain homotopy equivalence class of u. ◇

Proposition 10.1.2 (Universal property) *Let $F\colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{D}$ be any functor that sends chain homotopy equivalences to isomorphisms. Then F factors uniquely through $\mathbf{K}(\mathcal{A})$.*

$$\begin{array}{ccc} \mathbf{Ch}(\mathcal{A}) & \xrightarrow{F} & \mathcal{D} \\ \downarrow & \nearrow \exists! & \\ \mathbf{K}(\mathcal{A}) & & \end{array}$$

Proof Let $\mathrm{cyl}(B)$ denote the mapping cylinder of the identity map of B; it has $B^n \oplus B^{n+1} \oplus B^n$ in degree n. We saw in exercise 1.5.4 that the inclusion $\alpha(b) = (0, 0, b)$ of B into $\mathrm{cyl}(B)$ is a chain homotopy equivalence with homopy inverse $\beta(b', b,'' b) = b' + b$; $\beta\alpha = \mathrm{id}_B$ and $\alpha\beta \sim \mathrm{id}_{\mathrm{cyl}(B)}$. By assumption, $F(\alpha)\colon F(B) \to F(\mathrm{cyl}(B))$ is an isomorphism with inverse $F(\beta)$. Now the map $\alpha'\colon B \to \mathrm{cyl}(B)$ defined by $\alpha'(b) = (b, 0, 0)$ has $\beta\alpha' = \mathrm{id}_B$, so

$$F(\alpha') = F(\alpha)F(\beta)F(\alpha') = F(\alpha)F(\beta\alpha') = F(\alpha).$$

Now suppose there is a chain homotopy s between two maps $f, g\colon B \to C$. Then $\gamma = (f, s, g)\colon \mathrm{cyl}(B) \to C$ is a chain complex map (exercise 1.5.3).

Moreover, $\gamma\alpha' = f$ and $\gamma\alpha = g$. Hence in $\mathcal{D}$ we have

$$F(f) = F(\gamma)F(\alpha') = F(\gamma)F(\alpha) = F(g).$$

It follows that F factors through the quotient $\mathbf{K}(\mathcal{A})$ of $\mathbf{Ch}(\mathcal{A})$. $\diamond$

Exercise 10.1.1 Taking F to be $\mathbf{Ch}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$, the proof shows that $\alpha' \colon B \to \text{cyl}(B)$ is a chain homotopy equivalence. Use an involution on $\text{cyl}(B)$ to produce an explicit chain homotopy $\beta\alpha' \sim \text{id}_{\text{cyl}(B)}$.

Definition 10.1.3 (Triangles in $\mathbf{K}(\mathcal{A})$) Let $u \colon A \to B$ be a morphism in $\mathbf{Ch}$. Recall from 1.5.2 that the mapping cone of u fits into an exact sequence

$$0 \to B \xrightarrow{v} \text{cone}(u) \xrightarrow{\delta} A[-1] \to 0$$

in $\mathbf{Ch}$. (The degree n part of $\text{cone}(u)$ is $A^{n+1} \oplus B^n$ and A^{n+1} is the degree n part of $A[-1]$; see 1.2.8.) The *strict triangle* on u is the triple (u, v, δ) of maps in $\mathbf{K}$; this data is usually written in the form

$$\begin{tikzcd} & \text{cone}(u) \arrow[dl, "\delta"'] & \\ A \arrow[rr, "u"] & & B. \arrow[ul, "v"'] \end{tikzcd}$$

Now consider three fixed cochain complexes A, B and C. Suppose we are given three maps $u \colon A \to B$, $v \colon B \to C$, and $w \colon C \to A[-1]$ in $\mathbf{K}$. We say that (u, v, w) is an *exact triangle* on (A, B, C) if it is "isomorphic" to a strict triangle (u', v', δ) on $u' \colon A' \to B'$ in the sense that there is a diagram of chain complexes,

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & A[-1] \arrow[d, "f[-1]"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & \text{cone}(u') \arrow[r, "\delta"] & A'[-1], \end{tikzcd}$$

commuting in $\mathbf{K}$ (i.e., commuting in $\mathbf{Ch}$ up to chain homotopy equivalences) and such that the maps f, g, h are isomorphisms in $\mathbf{K}$ (i.e., chain homotopy equivalences). If we replace u, v, and w by chain homotopy equivalent maps, we get the same diagram in $\mathbf{K}$. This allows us to think of (u, v, w) as a triangle

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in the category **K**. A triangle is usually written as follows:

$$\begin{array}{ccc} & C & \\ {}^{w}\swarrow & & \nwarrow{}^{v} \\ A & \xrightarrow{u} & B. \end{array}$$

Corollary 10.1.4 *Given an exact triangle (u, v, w) on (A, B, C), the cohomology sequence*

$$\cdots \xrightarrow{w^*} H^i(A) \xrightarrow{u^*} H^i(B) \xrightarrow{v^*} H^i(C) \xrightarrow{w^*} H^{i+1}(A) \xrightarrow{u^*} \cdots$$

is exact. Here we have identified $H^i(A[-1])$ and $H^{i+1}(A)$.

Proof For a strict triangle, this is precisely the long exact cohomology sequence of 1.5.2. Exactness for any exact triangle follows from this by the definition of a triangle and the fact that each H^i is a functor on **K**. ◇

Example 10.1.5 The endomorphisms 0 and 1 of A fit into the exact triangles

$$\begin{array}{ccc} & A \oplus A[-1] & \\ \swarrow & & \nwarrow \\ A & \xrightarrow{\quad 0 \quad} & A \end{array} \qquad \begin{array}{ccc} & 0 & \\ \swarrow & & \nwarrow \\ A & \xrightarrow{1} & A. \end{array}$$

Indeed, $\operatorname{cone}(0) = A \oplus A[-1]$ and we saw in exercise 1.5.1 that cone(1) is a split exact complex, that is, cone(1) is isomorphic to zero in **K**.

Example 10.1.6 (Rotation) If (u, v, w) is an exact triangle, then so are its "rotates"

$$\begin{array}{ccc} & A[-1] & \\ {}^{-u[-1]}\swarrow & & \nwarrow{}^{w} \\ B & \xrightarrow{v} & C \end{array} \quad \text{and} \quad \begin{array}{ccc} & B & \\ {}^{v}\swarrow & & \nwarrow{}^{u} \\ C[+1] & \xrightarrow{-w[1]} & A. \end{array}$$

To see this, we may suppose that $C = \operatorname{cone}(u)$. In this case, the assertions amount to saying that the maps $\operatorname{cone}(v) \to A[-1]$ and $B[-1] \to \operatorname{cone}(\delta)$ are chain homotopy equivalences. The first was verified in exercises 1.5.6 and 1.5.8, and the second assertion follows from the observation that $\operatorname{cone}(\delta) = \operatorname{cyl}(-u)[-1]$.

Remark 10.1.7 Given a short exact sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ of complexes, there may be no map $C \xrightarrow{w} A[-1]$ making (u, v, w) into an exact triangle in $\mathbf{K}(\mathcal{A})$, even though there is a long exact cohomology sequence begging to be seen as coming from an exact triangle (but see 10.4.9 below). This cohomology sequence does arise from the mapping cylinder triangle

$$\begin{tikzcd}[column sep=small] & \operatorname{cone}(u) \arrow[dl, "w"'] & \\ A \arrow[rr] & & \operatorname{cyl}(u) \arrow[ul] \end{tikzcd}$$

and the quasi-isomorphisms $\beta\colon \operatorname{cyl}(u) \to B$ and $\varphi\colon \operatorname{cone}(u) \to C$ of exercises 1.5.4 and 1.5.8.

Exercise 10.1.2 Regard the abelian groups $\mathbb{Z}/2$ and $\mathbb{Z}/4$ as cochain complexes concentrated in degree zero, and show that the short exact sequence $0 \to \mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{1} \mathbb{Z}/2 \to 0$ cannot be made into an exact triangle (2, 1, w) on $(\mathbb{Z}/2, \mathbb{Z}/4, \mathbb{Z}/2)$ in the category $\mathbf{K}(\mathcal{A})$.

10.2 Triangulated Categories

The notion of triangulated category generalizes the structure that exact triangles give to $\mathbf{K}(\mathcal{A})$. One should think of exact triangles as substitutes for short exact sequences.

Suppose given a category $\mathbf{K}$ equipped with an automorphism T. A *triangle* on an ordered triple (A, B, C) of objects of $\mathbf{K}$ is a triple (u, v, w) of morphisms, where $u\colon A \to B$, $v\colon B \to C$, and $w\colon C \to T(A)$. A triangle is usually displayed as follows:

$$\begin{tikzcd}[column sep=small] & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"] & & B \arrow[ul, "v"'] \end{tikzcd}$$

A *morphism of triangles* is a triple (f, g, h) forming a commutative diagram in $\mathbf{K}$:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & TA \arrow[d, "Tf"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & TA'. \end{tikzcd}$$

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Definition 10.2.1 (Verdier) An additive category $\mathbf{K}$ is called a *triangulated category* if it is equipped with an automorphism $T: \mathbf{K} \to \mathbf{K}$ (called the *translation functor*) and with a distinguished family of triangles (u, v, w) (called the *exact triangles* in $\mathbf{K}$), which are subject to the following four axioms:

(TR1) Every morphism $u: A \to B$ can be embedded in an exact triangle (u, v, w). If $A = B$ and $C = 0$, then the triangle $(\mathrm{id}_A, 0, 0)$ is exact. If (u, v, w) is a triangle on (A, B, C), isomorphic to an exact triangle (u', v', w') on (A', B', C'), then (u, v, w) is also exact.

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & TA' \end{array}$$

(TR2) (Rotation). If (u, v, w) is an exact triangle on (A, B, C), then both its "rotates" $(v, w, -Tu)$ and $(-T^{-1}w, u, v)$ are exact triangles on (B, C, TA) and $(T^{-1}C, A, B)$, respectively.

(TR3) (Morphisms). Given two exact triangles

$$\begin{array}{ccc} & C & \\ w \swarrow & & \nwarrow v \\ A & \xrightarrow{u} & B \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & C' & \\ w' \swarrow & & \nwarrow v' \\ A & \xrightarrow{u'} & B' \end{array}$$

with morphisms $f: A \to A'$, $g: B \to B'$ such that $gu = u'f$, there exists a morphism $h: C \to C'$ so that (f, g, h) is a morphism of triangles.

$$\begin{array}{ccccccc} A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \\ \downarrow f & & \downarrow g & & \exists\downarrow h & & \downarrow Tf \\ A' & \xrightarrow{u'} & B' & \xrightarrow{v'} & C' & \xrightarrow{w'} & TA' \end{array}$$

(TR4) (The octahedral axiom). Given objects A, B, C, A', B', C' in $\mathbf{K}$, suppose there are three exact triangles: (u, j, ∂) on (A, B, C'); (v, x, i) on (B, C, A'); (vu, y, δ) on (A, C, B'). Then there is a fourth exact triangle $(f, g, (Tj)i)$ on (C', B', A')

$$\begin{array}{ccc} & A' & \\ (Tj)i \swarrow & & \nwarrow g \\ C' & \xrightarrow[f]{} & B' \end{array}$$

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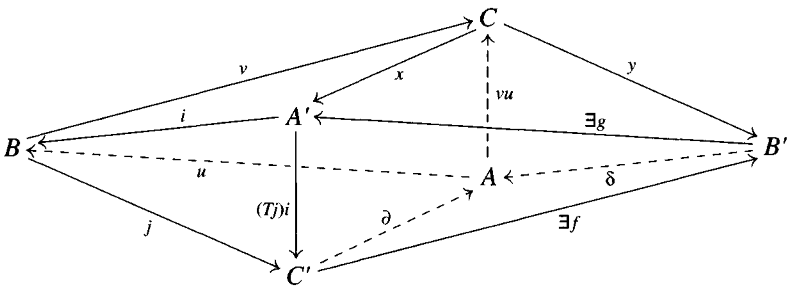
such that in the following octahedron we have (1) the four exact triangles form four of the faces; (2) the remaining four faces commute (that is, $\partial = \delta f : C' \to B' \to TA$ and $x = gy : C \to B' \to A'$); (3) $yv = fj : B \to B'$; and (4) $u\delta = ig : B' \to B$.



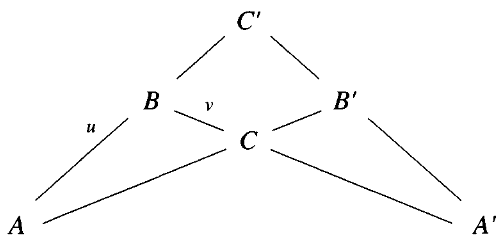
Exercise 10.2.1 If (u, v, w) is an exact triangle, show that the compositions vu, wv, and $(Tu)w$ are zero in $\mathbf{K}$. *Hint*: Compare the triangles $(\mathrm{id}_A, 0, 0)$ and (u, v, w).

Exercise 10.2.2 (5-lemma) If (f, g, h) is a morphism of exact triangles, and both f and g are isomorphisms, show that h is also an isomorphism.

$$\begin{array}{ccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & TA \\
\downarrow f & & \downarrow g & & \exists\downarrow h & & \downarrow Tf \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & TA'
\end{array}$$

Remark 10.2.2 Every exact triangle is determined up to isomorphism by any one of its maps. Indeed, (TR3) gives a morphism between any two exact triangles (u, v, w) on (A, B, C) and (u, v', w') on (A, B, C'), and the 5-lemma shows that it is an isomorphism. In particular, the data of the octahedral axiom are completely determined by the two maps $A \xrightarrow{u} B \xrightarrow{v} C$.

Exegesis 10.2.3 The octahehral axiom (TR4) is sufficiently confusing that it is worth giving another visualization of this axiom, following [BBD]. Write the triangles as straight lines (ignoring the morphism $C \to T(A)$), and form the diagram



The octahedral axiom states that the three lines through A, B, and C determine the fourth line through (C', B', A'). This visualization omits the identity $\partial = \delta f$.

Proposition 10.2.4 $\mathbf{K}(\mathcal{A})$ *is a triangulated category.*

Proof The translation $TA = A[-1]$ is defined in 1.2.8. We have already seen that axioms (TR1) and (TR2) hold. For (TR3) we may suppose that $C = \text{cone}(u)$ and $C' = \text{cone}(u')$; the map h is given by the naturality of the mapping cone construction.

It remains to check the octahedral axiom (TR4). For this we may assume that the given triangles are strict, that is, that $C' = \text{cone}(u)$, $A' = \text{cone}(v)$, and $B' = \text{cone}(vu)$. Define f^n from $(C')^n = B^n \oplus A^{n+1}$ to $(B')^n = C^n \oplus A^{n+1}$ by $f^n(b, a) = (v(b), a)$, and define g^n from $(B')^n = C^n \oplus A^{n+1}$ to $(A')^n = C^n \oplus B^{n+1}$ by $g^n(c, a) = (c, u(a))$. Manifestly, these are chain maps, $\partial = \delta f$ and $x = gy$. Since the degree n part of cone(f) is $(C^n \oplus A^{n+1}) \oplus (B^{n+1} \oplus A^{n+2})$, there is a natural inclusion γ of A' into cone(f) such that the following diagram of chain complexes commutes.

$$\begin{tikzcd}
C' \arrow[r, "f"] \arrow[d, equal] & B' \arrow[r, "g"] \arrow[d, equal] & A' \arrow[r, "(Tj)i"] \arrow[d, "\gamma"] & C'[-1] \arrow[d, equal] \\
C' \arrow[r, "f"] & B' \arrow[r] & \text{cone}(f) \arrow[r] & C'[-1]
\end{tikzcd}$$

To see that γ is a chain homotopy equivalence, define $\varphi\colon \text{cone}(f) \to A'$ by $\varphi(c, a_{n+1}, b, a_{n+2}) = (c, b + u(a_{n+1}))$. We leave it to the reader to check that φ is a chain map, that $\varphi\gamma = \text{id}_{A'}$ and that $\gamma\varphi$ is chain homotopic to the identity map on cone(f). (Exercise!) This shows that $(f, g, (Tj)i)$ is an exact triangle, because it is isomorphic to the strict triangle of f. ◇

Corollary 10.2.5 *Let* $\mathcal{C}$ *be a full subcategory of* $\mathbf{Ch}(\mathcal{A})$ *and* $\mathcal{K}$ *its corresponding quotient category. Suppose that* $\mathcal{C}$ *is an additive category and is closed* *under translation and the formation of mapping cones. Then $\mathcal{K}$ is a triangulated category.*

In particular, $\mathbf{K}^b(\mathcal{A})$, $\mathbf{K}^-(\mathcal{A})$, *and* $\mathbf{K}^+(\mathcal{A})$ *are triangulated categories.*

Definition 10.2.6 A *morphism* $F:\mathbf{K}' \to \mathbf{K}$ of triangulated categories is an additive functor that commutes with the translation functor T and sends exact triangles to exact triangles. There is a category of triangulated categories and their morphisms. We say that $\mathbf{K}'$ is a *triangulated subcategory* of $\mathbf{K}$ if $\mathbf{K}'$ is a full subcategory of $\mathbf{K}$, the inclusion is a morphism of triangulated categories, and if every exact triangle in $\mathbf{K}$ is exact in $\mathbf{K}'$.

For example, $\mathbf{K}^b$, $\mathbf{K}^+$, and $\mathbf{K}^-$ are triangulated subcategories of $\mathbf{K}(\mathcal{A})$. More generally, $\mathcal{K}$ is a triangulated subcategory of $\mathbf{K}$ in the above corollary.

Definition 10.2.7 Let $\mathbf{K}$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $H:\mathbf{K} \to \mathcal{A}$ is called a (covariant) *cohomological functor* if whenever (u, v, w) is an exact triangle on (A, B, C) the long sequence

$$\cdots \xrightarrow{w^*} H(T^iA) \xrightarrow{u^*} H(T^iB) \xrightarrow{v^*} H(T^iC) \xrightarrow{w^*} H(T^{i+1}A) \xrightarrow{u^*} \cdots$$

is exact in $\mathcal{A}$. We often write $H^i(A)$ for $H(T^iA)$ and $H^0(A)$ for $H(A)$ because, as we saw in 10.1.1, the zeroth cohomology $H^0:\mathbf{K}(\mathcal{A}) \to \mathcal{A}$ is the eponymous example of a cohomological functor. Here is another important cohomological functor:

Example 10.2.8 (Hom) If X is an object of a triangulated category $\mathbf{K}$, then $\mathrm{Hom}_{\mathbf{K}}(X, -)$ is a cohomological functor from $\mathbf{K}$ to $\mathbf{Ab}$. To see this, we have to see that for every exact triangle (u, v, w) on (A, B, C) that the sequence

$$\mathrm{Hom}_{\mathbf{K}}(X, A) \xrightarrow{u} \mathrm{Hom}_{\mathbf{K}}(X, B) \xrightarrow{v} \mathrm{Hom}_{\mathbf{K}}(X, C)$$

is exact; exactness elsewhere will follow from (TR2). The composition is zero since $vu = 0$. Given $g \in \mathrm{Hom}_{\mathbf{K}}(X, B)$ such that $vg = 0$ we apply (TR3) and (TR2) to

$$\begin{array}{ccccccc} X & = & X & \longrightarrow & 0 & \longrightarrow & TX \\ \exists\downarrow f & & \downarrow g & & \downarrow 0 & & \exists\downarrow Tf \\ A & \xrightarrow{u} & B & \xrightarrow{v} & C & \xrightarrow{w} & TA \end{array}$$

and conclude that there exists an $f \in \mathrm{Hom}_{\mathbf{K}}(X, A)$ so that $uf = g$.

Exercise 10.2.3 If $\mathbf{K}$ is triangulated, show that the opposite category $\mathbf{K}^{\mathrm{op}}$ is also triangulated. A covariant cohomological functor H from $\mathbf{K}^{\mathrm{op}}$ to $\mathcal{A}$ is sometimes called a *contravariant cohomological functor* on $\mathbf{K}$. If Y is any object of $\mathbf{K}$, show that $\mathrm{Hom}_{\mathbf{K}}(-, Y)$ is a contravariant cohomological functor on $\mathbf{K}$.

Exercise 10.2.4 Let $\mathcal{A}^{\mathbb{Z}}$ be the category of graded objects in $\mathcal{A}$, a morphism from $A = \{A_n\}$ to $B = \{B_n\}$ being a family of morphisms $f_n \colon A_n \to B_n$. Define TA to be the translated graded object $A[-1]$, and call (u, v, w) an exact triangle on (A, B, C) if for all n the sequence

$$A_n \xrightarrow{u} B_n \xrightarrow{v} C_n \xrightarrow{w} A_{n-1} \xrightarrow{u} B_{n-1}$$

is exact. Show that axioms (TR1) and (TR2) hold, but that (TR3) fails for $\mathcal{A} = \mathbf{Ab}$. If $\mathcal{A}$ is the category of vector spaces over a field, show that $\mathcal{A}^{\mathbb{Z}}$ is a triangulated category, and that cohomology $H^* \colon \mathbf{K}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$ is a morphism of triangulated categories.

Exercise 10.2.5 Let H be a cohomological functor on a triangulated category $\mathbf{K}$, and let $\mathbf{K}_H$ denote the full subcategory of $\mathbf{K}$ consisting of those objects A such that $H^i(A) = 0$ for all i. Show that $\mathbf{K}_H$ is a triangulated subcategory of $\mathbf{K}$.

Exercise 10.2.6 (Verdier) Show that every commutative square on the left in the diagram below can be completed to the diagram on the right, in which all the rows and columns are exact triangles and all the squares commute, except the one marked "–" which anticommutes. *Hint:* Use (TR1) to construct everything except the third column, and construct an exact triangle on (A, B', D). Then use the octahedral axiom to construct exact triangles on (C, D, B''), $(A,'' D, C')$, and finally (C', C'', C).

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ {\scriptstyle u}\downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \qquad \begin{array}{ccccccc} A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & T(A) \\ {\scriptstyle u}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Tu} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & T(A') \\ {\scriptstyle v}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle Tv} \\ A'' & \longrightarrow & B'' & \longrightarrow & C'' & \longrightarrow & T(A'') \\ {\scriptstyle w}\downarrow & & \downarrow & & \downarrow & - & \downarrow{\scriptstyle Tw} \\ T(A) & \xrightarrow{Ti} & T(B) & \xrightarrow{Tj} & T(C) & \xrightarrow{Tk} & T^2(A) \end{array}$$

10.3 Localization and the Calculus of Fractions

The derived category $\mathbf{D}(\mathcal{A})$ is defined to be the localization $Q^{-1}\mathbf{K}(\mathcal{A})$ of category $\mathbf{K}(\mathcal{A})$ at the collection Q of quasi-isomorphisms, in the sense of the following definition.

Definition 10.3.1 Let S be a collection of morphisms in a category $\mathcal{C}$. A *localization of $\mathcal{C}$ with respect to S* is a category $S^{-1}\mathcal{C}$, together with a functor $q: \mathcal{C} \to S^{-1}\mathcal{C}$ such that

1. $q(s)$ is a isomorphism in $S^{-1}\mathcal{C}$ for every $s \in S$.
2. Any functor $F: \mathcal{C} \to \mathcal{D}$ such that $F(s)$ is an isomorphism for all $s \in S$ factors in a unique way through q. (It follows that $S^{-1}\mathcal{C}$ is unique up to equivalence.)

Examples 10.3.2

1. Let S be the collection of chain homotopy equivalences in $\mathbf{Ch}(\mathcal{A})$. The universal property 10.1.2 for $\mathbf{Ch}(\mathcal{A}) \to \mathbf{K}(\mathcal{A})$ shows that $\mathbf{K}(\mathcal{A})$ is the localization $S^{-1}\mathbf{Ch}(\mathcal{A})$.
2. Let $\widetilde{Q}$ be the collection of all quasi-isomorphisms in $\mathbf{Ch}(\mathcal{A})$. Since $\widetilde{Q}$ contains the S of part (1), it follows that

$$\widetilde{Q}^{-1}\mathbf{Ch}(\mathcal{A}) = Q^{-1}(S^{-1}\mathbf{Ch}(\mathcal{A})) = Q^{-1}\mathbf{K}(\mathcal{A}) = \mathbf{D}(\mathcal{A}).$$

Therefore we could have defined the derived category to be the localization $\widetilde{Q}^{-1}\mathbf{Ch}(\mathcal{A})$. However, in order to prove that $\widetilde{Q}^{-1}\mathbf{Ch}(\mathcal{A})$ exists we must first prove that $Q^{-1}\mathbf{K}(\mathcal{A})$ exists, by giving an explicit description of the morphisms.

Set-Theoretic Remark 10.3.3 If $\mathcal{C}$ is a small category, every localization $S^{-1}\mathcal{C}$ of $\mathcal{C}$ exists. (Add inverses to the presentation of $\mathcal{C}$ by generators and relations; see [MacH, II.8].) It is also not hard to see that $S^{-1}\mathcal{C}$ exists when the class S is a set. However, when the class S is not a set, the existence of localizations is a delicate set-theoretic question.

The standard references [Verd], [HarRD], [GZ] all ignore these set-theoretic problems. Some adherents of the Grothendieck school avoid these difficulties by imagining the existence of a larger universe in which $\mathcal{C}$ is small and constructing the localization in that universe. Nevertheless, the issue of whether or not $S^{-1}\mathcal{C}$ exists in our universe is important to other schools of thought, and in particular to topologists who need to localize with respect to homology theories; see [A, III.14].

In this section we shall consider a special case in which localizations $S^{-1}\mathcal{C}$ may be constructed within our universe, the case in which S is a "locally small multiplicative system." This is due to the presence of a kind of calculus of fractions.

In section 10.4 we will see that the multiplicative system Q of quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$ is locally small when $\mathcal{A}$ is either **mod**–R or **Sheaves**(X). This will prove that $\mathbf{D}(\mathcal{A})$ exists within our universe. We will also see that if $\mathcal{A}$ has enough injectives (resp. projectives), the existence of Cartan-Eilenberg resolutions 5.7.1 allows us to forget about the set-theoretical difficulties in asserting that $\mathbf{D}^+(\mathcal{A})$ exists (resp. that $\mathbf{D}^-(\mathcal{A})$ exists).

Definition 10.3.4 A collection S of morphisms in a category $\mathcal{C}$ is called a *multiplicative system* in $\mathcal{C}$ if it satisfies the following three self-dual axioms:

1. S is closed under composition (if $s, t \in S$ are composable, then $st \in S$) and contains all identity morphisms ($\mathrm{id}_X \in S$ for all objects X in $\mathcal{C}$).
2. (Ore condition) If $t\colon Z \to Y$ is in S, then for every $g\colon X \to Y$ in $\mathcal{C}$ there is a commutative diagram "$gs = tf$" in $\mathcal{C}$ with s in S.

$$\begin{array}{ccc} W & \xrightarrow{f} & Z \\ s\downarrow & & \downarrow t \\ X & \xrightarrow{g} & Y \end{array}$$

   (The slogan is "$t^{-1}g = fs^{-1}$ for some f and s.") Moreover, the symmetric statement (whose slogan is "$fs^{-1} = t^{-1}g$ for some t and g") is also valid.
3. (Cancellation) If $f, g\colon X \to Y$ are parallel morphisms in $\mathcal{C}$, then the following two conditions are equivalent:
   (a) $sf = sg$ for some $s \in S$ with source Y.
   (b) $ft = gt$ for some $t \in S$ with target X.

Prototype 10.3.5 (Localizations of rings) An associative ring R with unit may be considered as an additive category $\mathcal{R}$ with one object $\cdot$ via $R = \mathrm{End}_{\mathcal{R}}(\cdot)$. Let S be a subset of R closed under multiplication and containing 1. If R is commutative, or more generally if S is in the center of R, then S is always a multiplicative system in $\mathcal{R}$; the usual ring of fractions $S^{-1}R$ is also the localization $S^{-1}\mathcal{R}$ of the category $\mathcal{R}$.

If S is not central, then S is a multiplicative system in $\mathcal{R}$ if and only if S is a "2-sided denominator set" in R in the sense of [Faith]. The classical ring of fractions $S^{-1}R$ is easy to construct in this case, each element being

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represented as either fs^{-1} or $t^{-1}g$ ($f, g \in R$ and $s, t \in S$), and again $S^{-1}R$ is the localization of the category $\mathcal{R}$.

The construction of the ring of fractions $S^{-1}R$ serves as the prototype for the construction of the localization $S^{-1}\mathcal{C}$. We call a chain in $\mathcal{C}$ of the form

$$fs^{-1}: X \xleftarrow{s} X_1 \xrightarrow{f} Y$$

a (left) "fraction" if s is in S. Call fs^{-1} *equivalent* to $X \xleftarrow{t} X_2 \xrightarrow{g} Y$ just in case there is a fraction $X \leftarrow X_3 \rightarrow Y$ fitting into a commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & X_3 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y. \\ & X_2 \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

It is easy to see that this is an equivalence relation. Write $\mathrm{Hom}_S(X, Y)$ for the family of equivalence classes of such fractions. Unfortunately, there is no *a priori* reason for this to be a set, unless S is "locally small" in the following sense.

Set-Theoretic Considerations 10.3.6 A multiplicative system S is called *locally small* (on the left) if for each X there exists a set S_X of morphisms in S, all having target X, such that for every $X_1 \rightarrow X$ in S there is a map $X_2 \rightarrow X_1$ in $\mathcal{C}$ so that the composite $X_2 \rightarrow X_1 \rightarrow X$ is in S_X.

If S is locally small, then $\mathrm{Hom}_S(X, Y)$ is a set for every X and Y. To see this, we make S_X the objects of a small category, a morphism from $X_1 \xrightarrow{s} X$ to $X_2 \xrightarrow{t} X$ being a map $X_2 \rightarrow X_1$ in $\mathcal{C}$ so that t is $X_2 \rightarrow X_1 \xrightarrow{s} X$. The Øre condition says that by enlarging S_X slightly we can make it a filtered category (2.6.13). There is a functor $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ from S_X to **Sets** sending s to the set of all fractions fs^{-1}, and $\mathrm{Hom}_S(X, Y)$ is the colimit of this functor.

Composition of fractions is defined as follows. To compose $X \leftarrow X' \xrightarrow{g} Y$ with $Y \xleftarrow{t} Y' \rightarrow Z$ we use the Ore condition to find a diagram

$$\begin{tikzcd} & W \arrow[r, "f"] \arrow[d, "s"] & Y' \arrow[r] \arrow[d, "t"] & Z \\ X & X' \arrow[l] \arrow[r, "g"] & Y & \end{tikzcd}$$

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with s in S; the composite is the class of the fraction $X \leftarrow W \rightarrow Z$ in $\mathrm{Hom}_S(X, Z)$. The slogan for the Ore condition, $t^{-1}g = fs^{-1}$, is a symbolic description of composition. It is not hard to see that the equivalence class of the composite is independent of the choice of X' and Y', so that we have defined a pairing

$$\mathrm{Hom}_S(X, Y) \times \mathrm{Hom}_S(Y, Z) \rightarrow \mathrm{Hom}_S(X, Z).$$

(Check this!) It is clear from the construction that composition is associative, and that $X = X = X$ is a 2-sided identity element. Hence the $\mathrm{Hom}_S(X, Y)$ (if they are sets) form the morphisms of a category having the same objects as $\mathcal{C}$; it will be our localization $S^{-1}\mathcal{C}$.

Gabriel-Zisman Theorem 10.3.7 ([GZ]) *Let S be a locally small multiplicative system of morphisms in a category $\mathcal{C}$. Then the category $S^{-1}\mathcal{C}$ constructed above exists and is a localization of $\mathcal{C}$ with respect to S. The universal functor $q: \mathcal{C} \rightarrow S^{-1}\mathcal{C}$ sends $f: X \rightarrow Y$ to the sequence $X = X \xrightarrow{f} Y$.*

Proof To see that $q: \mathcal{C} \rightarrow S^{-1}\mathcal{C}$ is a functor, observe that the composition of $X = X \xrightarrow{f} Y$ and $Y = Y \xrightarrow{h} Z$ is $X = X \xrightarrow{hf} Z$ since we can choose $t = \mathrm{id}_X$ and $f = g$. If s is in S, then $q(s)$ is an isomorphism because the composition of $X = X \xrightarrow{s} Y$ and $Y \xleftarrow{s} X = X$ is $X = X = X$ (take $W = X$). Finally, suppose that $F: \mathcal{C} \rightarrow \mathcal{D}$ is another functor sending S to isomorphisms. Define $S^{-1}F: S^{-1}\mathcal{C} \rightarrow \mathcal{D}$ by sending the fraction fs^{-1} to $F(f)F(s)^{-1}$. Given g and t, the equality $gs = tf$ in $\mathcal{C}$ shows that $F(g)F(s) = F(t)F(f)$, or $F(t^{-1}g) = F(fs^{-1})$; it follows that $S^{-1}F$ respects composition and is a functor. It is clear that $F = (S^{-1}F) \circ q$ and that this factorization is unique. ◇

Corollary 10.3.8 *$S^{-1}\mathcal{C}$ can be constructed using equivalence classes of "right fractions" $t^{-1}g: X \xrightarrow{g} Y' \xleftarrow{t} Y$, provided that S is "locally small on the right" (the dual notion to locally small, involving maps $Y \rightarrow Y'$ in S).*

Proof S^{op} is a multiplicative system in $\mathcal{C}^{\mathrm{op}}$. Since $\mathcal{C}^{\mathrm{op}} \rightarrow (S^{\mathrm{op}})^{-1}\mathcal{C}^{\mathrm{op}}$ is a localization, so is its dual $\mathcal{C} \rightarrow [(S^{\mathrm{op}})^{-1}(\mathcal{C}^{\mathrm{op}})]^{\mathrm{op}}$. But this is constructed using the fractions $t^{-1}g$. ◇

Corollary 10.3.9 *Two parallel maps $f, g: X \rightarrow Y$ in $\mathcal{C}$ become identified in $S^{-1}\mathcal{C}$ if and only if $sf = sg$ for some $s: X_3 \rightarrow X$ in S.*

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Exercise 10.3.1

1. If Z is a zero object (resp. an initial object, a terminal object) in $\mathcal{C}$, show that $q(Z)$ is a zero object (resp. an initial object, a terminal object) in $S^{-1}\mathcal{C}$.
2. If the product $X \times Y$ exists in $\mathcal{C}$, show that $q(X \times Y) \cong q(X) \times q(Y)$ in $S^{-1}\mathcal{C}$.

Corollary 10.3.10 *Suppose that $\mathcal{C}$ has a zero object. Then for every X in $\mathcal{C}$:*

$$q(X) \cong 0 \text{ in } S^{-1}\mathcal{C} \Leftrightarrow S \text{ contains the zero map } X \xrightarrow{0} X.$$

Proof Since $q(0)$ is a zero object in $S^{-1}\mathcal{C}$, $q(X) \cong 0$ if and only if the parallel maps $0, \mathrm{id}_X \colon X \to X$ become identified in $S^{-1}\mathcal{C}$, that is, iff $0 = s0 = s$ for some s. ◇

Corollary 10.3.11 *If $\mathcal{C}$ is an additive category, then so is $S^{-1}\mathcal{C}$, and q is an additive functor.*

Proof If $\mathcal{C}$ is an additive category, we can add fractions from X to Y as follows. Given fractions $f_1 s_1^{-1}$ and $f_2 s_2^{-1}$, we use the Ore condition to find an $s \colon X_2 \to X$ in S and $f_1', f_2' \colon X_2 \to Y$ so that $f_1 s_1^{-1} \sim f_1' s^{-1}$ and $f_2 s_2^{-1} \sim f_2' s^{-1}$; the sum $(f_1' + f_2')s^{-1}$ is well defined up to equivalence. (Check this!) Since $q(X \times Y) \cong q(X) \times q(Y)$ in $S^{-1}\mathcal{C}$ (exercise 10.3.1), it follows that $S^{-1}\mathcal{C}$ is an additive category (A.4.1) and that q is an additive functor. ◇

It is often useful to compare the localizations of subcategories with $S^{-1}\mathcal{C}$. For this we introduce the following definition.

Definition 10.3.12 (Localizing subcategories) Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$, and let S be a locally small multiplicative system in $\mathcal{C}$ whose restriction $S \cap \mathcal{B}$ to $\mathcal{B}$ is also a multiplicative system. For legibility, we will write $S^{-1}\mathcal{B}$ for $(S \cap \mathcal{B})^{-1}\mathcal{B}$. $\mathcal{B}$ is called a *localizing subcategory* of $\mathcal{C}$ (for S) if the natural functor $S^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$ is fully faithful. That is, if it identifies $S^{-1}\mathcal{B}$ with the full subcategory of $S^{-1}\mathcal{C}$ on the objects of $\mathcal{B}$.

Lemma 10.3.13 *A full subcategory $\mathcal{B}$ of $\mathcal{C}$ is localizing for S iff (1) holds. Condition (2) implies that $\mathcal{B}$ is localizing if S is locally small on the left, and condition (3) implies that $\mathcal{B}$ is localizing if S is locally small on the right.*

1. *For each B and B' in $\mathcal{B}$, the colimit $Hom_{S\cap\mathcal{B}}(B, B')$ (taken in $\mathcal{B}$) maps bijectively to the colimit $Hom_S(B, B')$ (taken in $\mathcal{C}$).*

2. *Whenever $C \to B$ is a morphism in S with B in $\mathcal{B}$, there is a morphism $B' \to C$ in $\mathcal{C}$ with B' in $\mathcal{B}$ such that the composite $B' \to B$ is in S.*
3. *Whenever $B \to C$ is a morphism in S with B in $\mathcal{B}$, there is a morphism $C \to B'$ in $\mathcal{C}$ with B' in $\mathcal{B}$ such that the composite $B \to B'$ is in S.*

Proof The statement that $S^{-1}\mathcal{B} \to S^{-1}\mathcal{C}$ is fully faithful means that the morphisms coincide (A.2.3), which by the Gabriel-Zisman Theorem 10.3.7 is assertion (1). Part (2) states that every left fraction $B \leftarrow C \to B''$ is equivalent to a fraction $B \leftarrow B' \to B''$, which must lie in the full subcategory $\mathcal{B}$. In particular, if two left fractions $\mathcal{B}$ are equivalent via a fraction $B \leftarrow C \to B''$ with C in $\mathcal{C}$, they are equivalent via a fraction with C in $\mathcal{B}$. Thus (2) implies (1) when S is locally small on the left. Replacing 'left' by 'right' and citing 10.3.8 proves that (3) implies (1) when S is locally small on the right. ◇

Corollary 10.3.14 *If $\mathcal{B}$ is a localizing subcategory of $\mathcal{C}$, and for every object C in $\mathcal{C}$ there is a morphism $C \to B$ in S with B in $\mathcal{B}$, then $S^{-1}\mathcal{B} \cong S^{-1}\mathcal{C}$.*

Suppose in addition that $S \cap \mathcal{B}$ consists of isomorphisms. Then

$$\mathcal{B} \cong S^{-1}\mathcal{B} \cong S^{-1}\mathcal{C}.$$

Example 10.3.15 Assume $\mathbf{D}(\mathcal{A})$ exists. The subcategories $\mathbf{K}^b(\mathcal{A})$, $\mathbf{K}^+(\mathcal{A})$, and $\mathbf{K}^-(\mathcal{A})$ of $\mathbf{K}(\mathcal{A})$ are localizing for Q (check this). Thus their localizations exist and are the full subcategories $\mathbf{D}^b(\mathcal{A})$, $\mathbf{D}^+(\mathcal{A})$, and $\mathbf{D}^-(\mathcal{A})$ of $\mathbf{D}(\mathcal{A})$ whose objects are the cochain complexes which are bounded, bounded below, and bounded above, respectively.

Example 10.3.16 Let S be a multiplicative system in a ring, and let Σ be the collection of all morphisms $A \to B$ in **mod**–R such that $S^{-1}A \to S^{-1}B$ is an isomorphism. It is not hard to see that Σ is a multiplicative system in **mod**–R. The subcategory **mod**–$S^{-1}R$ is localizing, because the natural map $A \to S^{-1}A$ is in Σ for every R-module A. Since $\Sigma \cap$ **mod**–$S^{-1}R$ consists of isomorphisms, we therefore have

$$\mathbf{mod}\text{–}S^{-1}R \cong \Sigma^{-1}\mathbf{mod}\text{–}R.$$

Exercise 10.3.2 (Serre subcategories) Let $\mathcal{A}$ be an abelian category. An abelian subcategory $\mathcal{B}$ is called a *Serre subcategory* if it is closed under subobjects, quotients, and extensions. Suppose that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$, and let Σ be the family of all morphisms f in $\mathcal{A}$ with $\ker(f)$ and $\operatorname{coker}(f)$ in $\mathcal{B}$.

1. Show that Σ is a multiplicative system in $\mathcal{A}$. We write $\mathcal{A}/\mathcal{B}$ for the localization $\Sigma^{-1}\mathcal{A}$ (provided that it exists).
2. Show that $q(X) \cong 0$ in $\mathcal{A}/\mathcal{B}$ if and only if X is in $\mathcal{B}$.

3. Assume that $\mathcal{B}$ is a small category, and show that Σ is locally small. This is one case in which $\mathcal{A}/\mathcal{B} = \Sigma^{-1}\mathcal{A}$ exists. More generally, $\mathcal{A}/\mathcal{B}$ exists whenever $\mathcal{A}$ is *well-powered*, that is, whenever the family of subobjects of any object of $\mathcal{A}$ is a set; see [Swan, pp.44ff].
4. Show that $\mathcal{A}/\mathcal{B}$ is an abelian category, and that $q: \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is an exact functor.
5. Let S be a multiplicative system in a ring R, and let $\mathbf{mod}_S R$ denote the full subcategory of R-modules A such that $S^{-1}A \cong 0$. Show that $\mathbf{mod}_S R$ is a Serre subcategory of $\mathbf{mod}\text{–}R$. Conclude that $\mathbf{mod}\text{–}S^{-1}R \cong \mathbf{mod}\text{–}R/\mathbf{mod}_S R$.

10.4 The Derived Category

In this section we show that $\mathbf{D}(\mathcal{A})$ is a triangulated category and that $\mathbf{D}^+(\mathcal{A})$ is determined by maps between bounded below complexes of injectives. We also show that $\mathbf{D}(\mathcal{A})$ exists within our universe, at least if $\mathcal{A}$ is $\mathbf{mod}\text{–}R$ or $\mathbf{Sheaves}(X)$.

For this we generalize slightly. Let $\mathbf{K}$ be a triangulated category. The system S *arising from* a cohomological functor $H: \mathbf{K} \to \mathcal{A}$ is the collection of all morphisms s in $\mathbf{K}$ such that $H^i(s)$ is an isomorphism for all integers i. For example, the quasi-isomorphisms Q arise from the cohomological functor H^0.

Proposition 10.4.1 *If S arises from a cohomological functor, then*

1. *S is a multiplicative system.*
2. *$S^{-1}\mathbf{K}$ is a triangulated category, and $\mathbf{K} \to S^{-1}\mathbf{K}$ is a morphism of triangulated categories (in any universe containing $S^{-1}\mathbf{K}$).*

Proof We first show that the system S is multiplicative (10.3.4). Axiom (1) is trivial. To prove (2), let $f: X \to Y$ and $s: Z \to Y$ be given. Embed s in an exact triangle (s, u, δ) on (Z, Y, C) using (TR1). Complete $uf: X \to C$ into an exact triangle (t, uf, v) on (W, X, C). By axiom (TR3) there is a morphism g such that

$$\begin{array}{ccccccc}
W & \xrightarrow{t} & X & \xrightarrow{uf} & C & \xrightarrow{v} & W[-1] \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle Tg} \\
Z & \xrightarrow{s} & Y & \xrightarrow{u} & C & \xrightarrow{\delta} & Z[-1]
\end{array}$$

is a morphism of triangles. If $H^*(s)$ is a isomorphism, then $H^*(C) = 0$. Applying this to the long exact sequence of the other triangle, we see that

$H^*(t)$ is also an isomorphism. The symmetric assertion may be proven similarly, or by appeal to $\mathbf{K}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$.

To verify axiom (3), we consider the difference $h = f - g$. Given $s: Y \to Y'$ in S with $sf = sg$, embed s in an exact triangle (u, s, δ) on (Z, Y, Y'). Note that $H^*(Z) = 0$. Since $\mathrm{Hom}_{\mathbf{K}}(X, -)$ is a cohomological functor (by 10.2.8),

$$\mathrm{Hom}_{\mathbf{K}}(X, Z) \xrightarrow{u} \mathrm{Hom}_{\mathbf{K}}(X, Y) \xrightarrow{s} \mathrm{Hom}_{\mathbf{K}}(X, Y')$$

is exact. Since $s(f - g) = 0$, there is a $g: X \to Z$ in $\mathbf{K}$ such that $f - g = ug$. Embed g in an exact triangle (t, g, w) on (X', X, Z). Since $gt = 0$, $(f - g)t = ugt = 0$, whence $ft = gt$. And since $H^*(Z) = 0$, the long exact sequence for H shows that $H^*(X') \cong H^*(X)$, that is, $t \in S$. The other implication of axiom (3) is analogous and may be deduced from the above by appeal to $\mathbf{K}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$.

Now suppose that $S^{-1}\mathbf{K}$ exists. The formula $T(fs^{-1}) = T(f)T(s)^{-1}$ defines a translation functor T on $S^{-1}\mathbf{K}$. To show that $S^{-1}\mathbf{K}$ is triangulated, we need to define exact triangles. Given $us_1^{-1}: A \to B$, $vs_2^{-1}: B \to C$, and $ws_3^{-1}: C \leftarrow C' \to T(A)$, the Ore condition for S yields morphisms $t_1: A' \to A$ and $t_2: B' \to B$ in S and $u': A' \to B'$, $v': B' \to C'$ in $\mathcal{C}$ so that $us_1^{-1} \cong t_2u't_1^{-1}$ and $vs_2^{-1} \cong s_3v't_2^{-1}$. We say that $(us_1^{-1}, vs_2^{-1}, ws_3^{-1})$ is an *exact triangle* in $S^{-1}\mathbf{K}$ just in case (u', v', w) is an exact triangle in $\mathbf{K}$. The verification that $S^{-1}\mathbf{K}$ is triangulated is left to the reader as an exercise, being straightforward but lengthy; one uses the fact that $\mathrm{Hom}_S(X, Y)$ may also be calculated using fractions of the form $t^{-1}g$. $\diamond$

Corollary 10.4.2 (Universal property) *Let $F: \mathbf{K} \to \mathbf{L}$ be a morphism of triangulated categories such that $F(s)$ is an isomorphism for all s in S, where S arises from a cohomological functor. Since $q: \mathbf{K} \to S^{-1}\mathbf{K}$ is a localization, there is a unique functor $F': S^{-1}\mathbf{K} \to \mathbf{L}$ such that $F = F' \circ q$. In fact, F' is a morphism of triangulated categories.*

Corollary 10.4.3 $\mathbf{D}(\mathcal{A})$, $\mathbf{D}^b(\mathcal{A})$, $\mathbf{D}^+(\mathcal{A})$ *and* $\mathbf{D}^-(\mathcal{A})$ *are triangulated categories (in any universe containing them).*

Proposition 10.4.4 *Let R be a ring. Then $\mathbf{D}(\mathcal{A})$ exists and is a triangulated category if $\mathcal{A}$ is* **mod**–*R, or either of*

- *Presheaves(X), presheaves of R-modules on a topological space X, or*
- *Sheaves(X), sheaves of R-modules on a topological space X.*

Proof We have to prove that the multiplicative system Q is locally small (10.3.6). Given a fixed cochain complex of R-modules A, choose an infinite cardinal number κ larger than the cardinality of the sets underlying the A^i and R. Call a cochain complex B *petite* if its underlying sets have cardinality $< \kappa$; there is a set of isomorphism classes of petite cochain complexes, hence a set S_X of isomorphism classes of quasi-isomorphisms $A' \to A$ with A' petite.

Given a quasi-isomorphism $B \to A$, it suffices to show that B contains a petite subcomplex B' quasi-isomorphic to A. Since $H^*(A)$ has cardinality $< \kappa$, there is a petite subcomplex B_0 of B such that the map $f_0^*\colon H^*(B_0) \to H^*(A)$ is onto. Since $\ker(f_0^*)$ has cardinality $< \kappa$, we can enlarge B_0 to a petite subcomplex B_1 such that $\ker(f_0^*)$ vanishes in $H^*(B_1)$. Inductively, we can construct an increasing sequence of petite subcomplexes B_n of B such that the kernel of $H^*(B_n) \to H^*(A)$ vanishes in $H^*(B_{n+1})$. But then their union $B' = \cup B_n$ is a petite subcomplex of B with

$$H^*(B') \cong \varinjlim H^*(B_n) \cong H^*(A).$$

The proof for presheaves is identical, except that κ must bound the number of open subsets U as well as the cardinality of $A(U)$ for every open subset U of X. The proof for sheaves is similar, using the following three additional facts, which may be found in [Hart] or [Gode]: (1) if κ bounds card $A(U)$ for all U and the number of such U, then κ also bounds the cardinality of the stalks A_x for $x \in X$ (2.3.12); (2) a map $B \to A$ is a quasi-isomorphism in **Sheaves**(X) iff every map of stalks $B_x \to A_x$ is a quasi-isomorphism; and (3) for every directed system of sheaves we have $H^*(\varinjlim B_n) = \varinjlim H^*(B_n)$. ◇

Remark 10.4.5 (Gabber) The proof shows that $\mathbf{D}(\mathcal{A})$ exists within our universe for every well-powered abelian category $\mathcal{A}$ that satisfies (AB5) and has a set of generators.

We conclude with a discussion of the derived category $\mathbf{D}^+(\mathcal{A})$. Assuming that $\mathcal{A}$ has enough injectives and we are willing to always pass to complexes of injectives, there is no need to leave the homotopy category $\mathbf{K}^+(\mathcal{A})$. In particular, $\mathbf{D}^+(\mathcal{A})$ will exist in our universe even if $\mathbf{D}(\mathcal{A})$ may not.

Lemma 10.4.6 *Let Y be a bounded below cochain complex of injectives. Every quasi-isomorphism $t\colon Y \to Z$ of complexes is a split injection in* $\mathbf{K}(\mathcal{A})$.

Proof The mapping cone $\text{cone}(t) = T(Y) \oplus Z$ is exact (1.5.4), and there is a natural map $\varphi\colon \text{cone}(t) \to T(Y)$. The Comparison Theorem of 2.3.7 (or rather its proof; see 2.2.6) shows that φ is null-homotopic, say, by a chain homotopy $v = (k, s)$ from $T(Y) \oplus Z$ to Y. The first coordinate of the equation $-y =$

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$\varphi(y,z) = (vd + dv)(y,z)$ yields the equation

$$y = (kdy + sty - dky) + (dsz - sdz).$$

Thus $ds = sd$ (i.e., s is a morphism of complexes) and $st = \mathrm{id}_Y + dk - kd$, that is, k is a chain homotopy equivalence $st \simeq \mathrm{id}_Y$. Hence $st = \mathrm{id}_Y$ in $\mathbf{K}^+(\mathcal{A})$. ◇

Corollary 10.4.7 *If I is a bounded below cochain complex of injectives, then*

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(X, I) \cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(X, I)$$

for every X. Dually, if P is a bounded above cochain complex of projectives, then

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(P, X) \cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(P, X).$$

Proof We prove the assertion for $Y = I$, using the notation of the lemma. Every right fraction $t^{-1}g\colon X \xrightarrow{g} Z \xleftarrow{t} Y$ is equivalent to $sg = (st)t^{-1}g\colon X \to Y$. Conversely, if two parallel arrows $f, g\colon X \to Y$ in $\mathbf{K}(\mathcal{A})$ become identified in $\mathbf{D}(\mathcal{A}) = Q^{-1}\mathbf{K}(\mathcal{A})$, then $tf = tg$ for some quasi-isomorphism $t\colon Y \to Z$ by 10.3.9, which implies that $f = stf = stg = g$ in $\mathbf{K}(\mathcal{A})$. ◇

Exercise 10.4.1 In the situation of the lemma, show that $(tk, 1)\colon \mathrm{cone}(t) \to Z$ induces an isomorphism $Z \cong Y \oplus \mathrm{cone}(t)$ in $\mathbf{K}(\mathcal{A})$.

Theorem 10.4.8 *Suppose that $\mathcal{A}$ has enough injectives. Then $\mathbf{D}^+(\mathcal{A})$ exists in our universe because it is equivalent to the full subcategory $\mathbf{K}^+(\mathcal{I})$ of $\mathbf{K}^+(\mathcal{A})$ whose objects are bounded below cochain complexes of injectives*

$$\mathbf{D}^+(\mathcal{A}) \cong \mathbf{K}^+(\mathcal{I}).$$

Dually, if $\mathcal{A}$ has enough projectives, then the localization $\mathbf{D}^-(\mathcal{A})$ of $\mathbf{K}^-(\mathcal{A})$ exists and is equivalent to the full subcategory $\mathbf{K}^-(\mathcal{P})$ of bounded above cochain complexes of projectives in $\mathbf{K}^-(\mathcal{A})$:

$$\mathbf{D}^-(\mathcal{A}) \cong \mathbf{K}^-(\mathcal{P}).$$

Proof Recall from 5.7.2 that every X in $\mathbf{Ch}^+(\mathcal{A})$ has a Cartan-Eilenberg resolution $X \to I$ with $\mathrm{Tot}(I)$ in $\mathbf{K}^+(\mathcal{I})$; since X is bounded below, this is a quasi-isomorphism (exercise 5.7.1). If $Y \to X$ is a quasi-isomorphism, then so is

$Y \to \mathrm{Tot}(I)$; by 10.3.13(3), $\mathbf{K}^+(\mathcal{I})$ is a localizing subcategory of $\mathbf{K}^+(\mathcal{A})$. This proves that $\mathbf{D}^+(\mathcal{A}) \cong S^{-1}\mathbf{K}^+(\mathcal{I})$, and by 10.3.14 it suffices to show that every quasi-isomorphism in $\mathbf{K}^+(\mathcal{I})$ is an isomorphism. Let Y and X be bounded below cochain complexes of injectives and $t\colon Y \to X$ a quasi-isomorphism. By lemma 10.4.6, there is a map $s\colon X \to Y$ so that $st = \mathrm{id}_Y$ in $\mathbf{K}^+(\mathcal{A})$. Interchanging the roles of X and Y, s and t, we see that $us = \mathrm{id}_X$ for some u. Hence t is an isomorphism in $\mathbf{K}^+(\mathcal{I})$ with $t^{-1} = s$.

Dually, if $\mathcal{A}$ has enough projectives, then $\mathcal{A}^{\mathrm{op}}$ has enough injectives and $\mathbf{D}^-(\mathcal{A}) \cong \mathbf{D}^+(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \cong \mathbf{K}^+(\mathcal{P}^{\mathrm{op}})^{\mathrm{op}} \cong \mathbf{K}^-(\mathcal{P})$. $\diamond$

Example 10.4.9 Every short exact sequence $0 \to A \xrightarrow{u} B \xrightarrow{v} C \to 0$ of cochain complexes fits into an exact triangle in $\mathbf{D}(\mathcal{A})$, isomorphic to the strict triangle on u. Indeed, the quasi-isomorphism $\varphi\colon \mathrm{cone}(u) \to C$ of 1.5.8 allows us to form the exact triangle $(u, v, \delta\varphi^{-1})$ on (A, B, C). This construction should be contrasted with the observation in 10.1.7 that there may be no similar exact triangle (u, v, w) in $\mathbf{K}(\mathcal{A})$.

Note that the construction of $\mathbf{D}(\mathcal{A})$ implies the following two useful criteria. A chain complex X is isomorphic to 0 in $\mathbf{D}(\mathcal{A})$ iff it is exact. A morphism $f\colon X \to Y$ in $\mathbf{Ch}(\mathcal{A})$ becomes the zero map in $\mathbf{D}(\mathcal{A})$ iff there is a quasi-isomorphism $s\colon Y \to Y'$ such that sf is null homotopic (chain homotopic to zero). The following exercise shows the subtlety of being zero.

Exercise 10.4.2 Give examples of maps f, g in $\mathbf{Ch}(\mathcal{A})$ such that (1) $f = 0$ in $\mathbf{D}(\mathcal{A})$, but f is not null homotopic, and (2) g induces the zero map on cohomology, but $g \neq 0$ in $\mathbf{D}(\mathcal{A})$. *Hint:* For (2) try $X\colon 0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0$, $Y\colon 0 \to \mathbb{Z} \xrightarrow{1} \mathbb{Z}/3 \to 0$, $g = (1, 2)$.

Exercise 10.4.3 ($\mathbf{K}_{\mathcal{B}}(\mathcal{A})$ and $\mathbf{D}_{\mathcal{B}}(\mathcal{A})$) Let $\mathcal{B}$ be a Serre subcategory of $\mathcal{A}$, and let $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ be the quotient map constructed in exercise 10.3.2.

1. Show that $H = \pi H^0\colon \mathbf{K}(\mathcal{A}) \to \mathcal{A} \to \mathcal{A}/\mathcal{B}$ is a cohomological functor, so that $\mathbf{K}_H(\mathcal{A})$ is a triangulated category by exercise 10.2.5. The notation $\mathbf{K}_{\mathcal{B}}(\mathcal{A})$ is often used for $\mathbf{K}_H(\mathcal{A})$, because of the description in part (2).
2. Show that X is in $\mathbf{K}_{\mathcal{B}}(\mathcal{A})$ iff the cohomology $H^i(X)$ is in $\mathcal{B}$ for all i.
3. Show that $\mathbf{K}_{\mathcal{B}}(\mathcal{A})$ is a localizing subcategory of $\mathbf{K}(\mathcal{A})$, and conclude that its localization $\mathbf{D}_{\mathcal{B}}(\mathcal{A})$ is a triangulated subcategory of $\mathbf{D}(\mathcal{A})$ (10.2.6).
4. Suppose that $\mathcal{B}$ has enough injectives and that every injective object of $\mathcal{B}$ is also injective in $\mathcal{A}$. Show that there is an equivalence $\mathbf{D}^+(\mathcal{B}) \cong \mathbf{D}^+_{\mathcal{B}}(\mathcal{A})$.

Exercise 10.4.4 (Change of Universe) This is a continuation of the previous exercise. Suppose that our universe is contained in a larger universe $\mathcal{U}$, and that **mod**–R and **Sheaves**(X) are small categories in $\mathcal{U}$. Let **MOD**–R and **SHEAVES**(X) denote the categories of modules and sheaves in $\mathcal{U}$, respectively. Show that **mod**–R and **Sheaves**(X) are Serre subcategories of **MOD**–R and **SHEAVES**(X), respectively. Conclude that $\mathbf{D}(\mathbf{mod}\text{–}R) \cong \mathbf{D}_{\mathbf{mod}\text{–}R}(\mathbf{MOD}\text{–}R)$ and $\mathbf{D}(\mathbf{Sheaves}(X)) \cong \mathbf{D}_{\mathbf{Sheaves}(X)}(\mathbf{SHEAVES}(X))$.

Exercise 10.4.5 Here is a construction of $\mathbf{D}(\mathcal{A})$ when $\mathcal{A}$ is **mod**–R, valid whenever $\mathcal{A}$ has enough projectives and satisfies (AB5). It is based on the construction of CW spectra in algebraic topology [LMS]. Call a chain complex C *cellular* if it is the increasing union of subcomplexes C_n, with $C_0 = 0$, such that each quotient C_n/C_{n-1} is a complex of projectives with all differentials zero. Let $\mathbf{K}_{\text{cell}}$ denote the full subcategory of $\mathbf{K}(\mathcal{A})$ consisting of cellular complexes. Show that

1. For every X there is a quasi-isomorphism $C \to X$ with C cellular.
2. If C is cellular and X is acyclic, then every map $C \to X$ is null-homotopic.
3. If C is cellular and $f: X \to Y$ is a quasi-isomorphism, then

$$f_* : \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(C, X) \cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(C, Y).$$

4. (Whitehead's Theorem) If $f: C \to D$ is a quasi-isomorphism of cellular complexes, then f is a homotopy equivalence, that is, $C \cong D$ in $\mathbf{K}(\mathcal{A})$.
5. $\mathbf{K}_{\text{cell}}$ is a localizing triangulated subcategory of $\mathbf{K}(\mathcal{A})$.
6. The natural map is an equivalence: $\mathbf{K}_{\text{cell}} \cong \mathbf{D}(\mathcal{A})$.

Exercise 10.4.6 Let R be a noetherian ring, and let $\mathbf{M}(R)$ denote the category of all finitely generated R-modules. Let $\mathbf{D}_{\text{fg}}(R)$ denote the full subcategory of $\mathbf{D}(\mathbf{mod}\text{–}R)$ consisting of complexes A whose cohomology modules $H^i(A)$ are all finitely generated, that is, the category $\mathbf{D}_{\mathbf{M}(R)}(\mathbf{mod}\text{–}R)$ of exercise 10.4.3. Show that $\mathbf{D}_{\text{fg}}(R)$ is a triangulated category and that there is an equivalence $\mathbf{D}^-(\mathbf{M}(R)) \cong \mathbf{D}^-_{\text{fg}}(R)$. *Hint:* $\mathbf{M}(R)$ is a Serre subcategory of **mod**–R (exercise 10.3.2).

10.5 Derived Functors

There is a category of triangulated categories; a *morphism* $F: \mathbf{K} \to \mathbf{K}'$ of triangulated categories is a (covariant) additive functor that commutes with the translation functor T and sends exact triangles to exact triangles. Morphisms are sometimes called *covariant ∂-functors*; a morphism $\mathbf{K}^{\mathrm{op}} \to \mathbf{K}'$ is of course a contravariant ∂-functor.

For example, suppose given an additive functor $F: \mathcal{A} \to \mathcal{B}$ between two abelian categories. Since F preserves chain homotopy equivalences, it extends to additive functors $\mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{B})$ and $\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$. Since F commutes with translation of chain complexes, it even preserves mapping cones and exact triangles. Thus $F: \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ is a morphism of triangulated categories.

We would like to extend F to a functor $\mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$. If $F: \mathcal{A} \to \mathcal{B}$ is exact, this is easy. However, if F is not exact, then the functor $\mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ will not preserve quasi-isomorphisms, and this may not be possible. The thing to expect is that if F is left or right exact, then the derived functors of F will be needed to extend something like the hyper-derived functors of F.

Our experience in Chapter 5, section 7 tells us that the right hyper-derived functors $\mathbb{R}^i F$ work best if we restrict attention to bounded below cochain complexes. With this in mind, let $\mathbf{K}$ denote $\mathbf{K}^+(\mathcal{A})$ or any other localizing triangulated subcategory of $\mathbf{K}(\mathcal{A})$, and let $\mathbf{D}$ denote the full subcategory of the derived category $\mathbf{D}(\mathcal{A})$ corresponding to $\mathbf{K}$.

Definition 10.5.1 Let $F: \mathbf{K} \to \mathbf{K}(\mathcal{B})$ be a morphism of triangulated categories. A *(total) right derived functor* of F on $\mathbf{K}$ is a morphism $\mathbf{R}F: \mathbf{D} \to \mathbf{D}(\mathcal{B})$ of triangulated categories, together with a natural transformation ξ from $qF: \mathbf{K} \to \mathbf{K}(\mathcal{B}) \to \mathbf{D}(\mathcal{B})$ to $(\mathbf{R}F)q: \mathbf{K} \to \mathbf{D} \to \mathbf{D}(\mathcal{B})$ which is universal in the sense that if $G: \mathbf{D} \to \mathbf{D}(\mathcal{B})$ is another morphism equipped with a natural transformation $\zeta: qF \Rightarrow Gq$, then there exists a unique natural transformation $\eta: \mathbf{R}F \Rightarrow G$ so that $\zeta_A = \eta_{qA} \circ \xi_A$ for every A in $\mathbf{D}$.

This universal property guarantees that if **RF** exists, then it is unique up to natural isomorphism, and that if $\mathbf{K}' \subset \mathbf{K}$, then there is a natural transformation from the right derived functor $\mathbf{R}'F$ on $\mathbf{D}'$ to the restriction of **RF** to $\mathbf{D}'$. If there is a chance of confusion, we will write $\mathbf{R}^b F$, $\mathbf{R}^+ F$, $\mathbf{R}_{\mathcal{B}} F$, and so on for the derived functors of F on $\mathbf{K}^b(\mathcal{A})$, $\mathbf{K}^+(\mathcal{A})$, $\mathbf{K}_{\mathcal{B}}(\mathcal{A})$, etc.

Similarly, a *(total) left derived functor* of F is a morphism $\mathbf{L}F: \mathbf{D} \to \mathbf{D}(\mathcal{B})$ together with a natural transformation $\xi: (\mathbf{L}F)q \Rightarrow qF$ satisfying the dual universal property (G factors through $\eta: G \Rightarrow \mathbf{L}F$). Since $\mathbf{L}F$ is $\mathbf{R}(F^{\mathrm{op}})^{\mathrm{op}}$, where $F^{\mathrm{op}}: \mathbf{K}^{\mathrm{op}} \to \mathbf{K}(\mathcal{B}^{\mathrm{op}})$, we can translate any statement about **RF** into a dual statement about $\mathbf{L}F$.

Exact Functors 10.5.2 If $F: \mathcal{A} \to \mathcal{B}$ is an exact functor, F preserves quasi-isomorphisms. Hence F extends trivially to $F: \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{B})$. In effect, F is its own left and derived functor. The following two examples generalize this observation.

Example 10.5.3 Let $\mathbf{K}^+(\mathcal{I})$ denote the triangulated category of bounded below complexes of injectives. We saw in 10.4.8 that every quasi-isomorphism in $\mathbf{K}^+(\mathcal{I})$ is an isomorphism, so $\mathbf{K}^+(\mathcal{I})$ is isomorphic to its derived category $\mathbf{D}^+(\mathcal{I})$. The functor $qFq^{-1}\colon \mathbf{D}^+(\mathcal{I}) \cong \mathbf{K}^+(\mathcal{I}) \xrightarrow{F} \mathbf{K}^+(\mathcal{B}) \to \mathbf{D}^+(\mathcal{B})$ satisfies $qF \cong (qFq^{-1})q$, so it is both the left and right total derived functor of F.

Similarly, for the category $\mathbf{K}^-(\mathcal{P})$ of bounded above cochain complexes of projectives, we have $\mathbf{K}^-(\mathcal{P}) \cong \mathbf{D}^-(\mathcal{P})$. Again, qFq^{-1} is both the left and right derived functor of F.

Definition 10.5.4 Let $F\colon \mathbf{K} \to \mathbf{K}(\mathcal{B})$ be a morphism of triangulated categories. A complex X in $\mathbf{K}$ is called *F-acyclic* if $F(X)$ is acyclic, that is, if $H^i(FX) \cong 0$ for all i. (Compare with 2.4.3.)

Example 10.5.5 (F-acyclic complexes) Suppose that $\mathbf{K}$ is a triangulated subcategory of $\mathbf{K}(\mathcal{A})$ such that every acyclic complex in $\mathbf{K}$ is F-acyclic. If $s\colon X \to Y$ is a quasi-isomorphism in $\mathbf{K}$, then $\operatorname{cone}(s)$ and hence $F(\operatorname{cone}(s))$ is acyclic. Since F preserves exact triangles, the cohomology sequence shows that $F(s)^*: H^*(FX) \cong H^*(FY)$, that is, that $F(s)$ is a quasi-isomorphism. By the universal property of the localization $\mathbf{D} = Q^{-1}\mathbf{K}$ there is a unique functor $Q^{-1}F$ from $\mathbf{D}$ to $\mathbf{D}(\mathcal{B})$ such that $qF = (Q^{-1}F)q$. Once again, $Q^{-1}F$ is both the left and right derived functor of F.

Existence Theorem 10.5.6 *Let $F\colon \mathbf{K}^+(\mathcal{A}) \to \mathbf{K}(\mathcal{B})$ be a morphism of triangulated categories. If $\mathcal{A}$ has enough injectives, then the right derived functor $\mathbf{R}^+F$ exists on $\mathbf{D}^+(\mathcal{A})$, and if I is a bounded below complex of injectives, then*

$$\mathbf{R}^+F(I) \cong qF(I).$$

Dually, if $\mathcal{A}$ has enough projectives, then the left derived functor $\mathbf{L}^-F$ exists on $\mathbf{D}^-(\mathcal{A})$, and if P is a bounded above cochain complex of projectives, then

$$\mathbf{L}^-F(P) \cong qF(P).$$

Proof Choose an equivalence $U\colon \mathbf{D}^+(\mathcal{A}) \xrightarrow{\cong} \mathbf{K}^+(\mathcal{I})$ inverse to the natural map $T\colon \mathbf{K}^+(\mathcal{I}) \xrightarrow{\cong} \mathbf{D}^+(\mathcal{A})$ of 10.4.8, and define $\mathbf{R}F$ to be the composite qFU:

$$\mathbf{D}^+(\mathcal{A}) \xrightarrow{\cong} \mathbf{K}^+(\mathcal{I}) \xrightarrow{F} \mathbf{K}^+(\mathcal{B}) \xrightarrow{q} \mathbf{D}^+(\mathcal{B}).$$

To construct ξ we use the natural isomorphism of 10.4.7

$$\operatorname{Hom}_{\mathbf{D}^+(\mathcal{A})}(qX, TUqX) \cong \operatorname{Hom}_{\mathbf{K}^+(\mathcal{A})}(X, UqX).$$

Under this isomorphism there is a natural map $f_X: X \to UqX$ in $\mathbf{K}^+(\mathcal{A})$ corresponding to the augmentation $\eta: qX \to TUqX$ in $\mathbf{D}^+(\mathcal{A})$. We define ξ_X to be the natural transformation $qF(f_X): qF(X) \to qF(UqX) \cong (qFU)(qX)$. It is not hard to see that ξ has the required universal property, making $(\mathbf{R}F, \xi)$ into a right derived functor of F. As usual, the dual assertion that the composite

$$\mathbf{D}^-(\mathcal{A}) \xrightarrow{\cong} \mathbf{K}^-(\mathcal{P}) \xrightarrow{F} \mathbf{K}^-(\mathcal{B}) \xrightarrow{q} \mathbf{D}^-(\mathcal{B})$$

is a left derived functor of F follows by passage to F^{op}. $\diamond$

Corollary 10.5.7 *Let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.*

1. *If $\mathcal{A}$ has enough injectives, the hyper-derived functors $\mathbb{R}^i F(X)$ are the cohomology of $\mathbf{R}F(X)$: $\mathbb{R}^i F(X) \cong H^i \mathbf{R}^+ F(X)$ for all i.*
2. *If $\mathcal{A}$ has enough projectives, the hyper-derived functors $\mathbb{L}_i F(X)$ are the cohomology of $\mathbf{L}F(X)$: $\mathbb{L}_i F(X) \cong H^{-i} \mathbf{L}^- F(X)$ for all i.*

Remark 10.5.8 The assumption in 5.7.4 that F be left or right exact was not necessary to define $\mathbb{R}^i F$ or $\mathbb{L}_i F$; it was made to retain the connection with F. Suppose that we consider an object A of $\mathcal{A}$ as a complex concentrated in degree zero. The assumption that F be left exact is needed to ensure that the $\mathbb{R}^i F(A)$ are the ordinary derived functors $R^i F(A)$ and in particular that $\mathbb{R}^0 F(A) = F(A)$. Similarly, the assumption that F be right exact is needed to ensure that $\mathbb{L}_i F(A)$ is the ordinary derived functor $L_i F(A)$, and that $\mathbb{L}_0 F(A) = F(A)$.

Exercise 10.5.1 Suppose that $F: \mathbf{K}^+(\mathcal{A}) \to \mathbf{K}(\mathcal{C})$ is a morphism of triangulated categories and that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$. If $\mathcal{A}$ has enough injectives, show that the restriction of $\mathbf{R}^+ F$ to $\mathbf{D}^+_{\mathcal{B}}(\mathcal{A})$ is the derived functor $\mathbf{R}^+_{\mathcal{B}} F$. If in addition $\mathcal{B}$ has enough injectives, which are also injective in $\mathcal{A}$, this proves that the composition $\mathbf{D}^+(\mathcal{B}) \to \mathbf{D}^+(\mathcal{A}) \xrightarrow{\mathbf{R}F} \mathbf{D}^+(\mathcal{C})$ is the derived functor $\mathbf{R}^+ F|\mathcal{B}$ of the restriction $F|\mathcal{B}$ of F to $\mathcal{B}$, since we saw in exercise 10.4.3 that in this case $\mathbf{D}^+(\mathcal{B}) \cong \mathbf{D}^+_{\mathcal{B}}(\mathcal{A})$.

Generalized Existence Theorem 10.5.9 ([HartRD, I.5.1]) *Suppose that $\mathbf{K}'$ is a triangulated subcategory of $\mathbf{K}$ such that*

1. *Every X in $\mathbf{K}$ has a quasi-isomorphism $X \to X'$ to an object of $\mathbf{K}'$.*

2. *Every exact complex in* $\mathbf{K}'$ *is* F*-acyclic (10.5.4).*

Then $\mathbf{D}' \xrightarrow{\cong} \mathbf{D}$ *and* $\mathbf{R}F\colon \mathbf{D} \cong \mathbf{D}' \xrightarrow{\mathbf{R}'F} \mathbf{D}(\mathcal{B})$ *is a right derived functor of* F.

Proof By (1) and 10.3.14, $\mathbf{K}'$ is localizing and $\mathbf{D}' \xrightarrow{\cong} \mathbf{D}$. Now modify the proof of the Existence Theorem 10.5.6, using F-acyclic complexes. ◇

Definition 10.5.10 Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. When $\mathcal{A}$ has enough injectives, so that the usual derived functors $R^i F$ (of Chapter 2) exist, we say that F has *cohomological dimension n* if $R^n F = 0$ for all $i > n$, yet $R^n F \neq 0$. Dually, when $\mathcal{A}$ has enough projectives, so that the usual derived functors $L_i F$ exist, we say that F has *homological dimension n* if $L_i F = 0$ for all $i > n$, yet $L_n F \neq 0$.

Exercise 10.5.2 If F has finite cohomological dimension, show that every exact complex of F-acyclic objects (2.4.3) is an F-acyclic complex in the sense of 10.5.4.

Corollary 10.5.11 *Let* $F\colon \mathcal{A} \to \mathcal{B}$ *be an additive functor. If* F *has finite cohomological dimension n, then* $\mathbf{R}F$ *exists on* $\mathbf{D}(\mathcal{A})$*, and its restriction to* $\mathbf{D}^+(\mathcal{A})$ *is* $\mathbf{R}^+F$*. Dually, if* F *has finite homological dimension n, then* $\mathbf{L}F$ *exists on* $\mathbf{D}(\mathcal{A})$*, and its restriction to* $\mathbf{D}^-(\mathcal{A})$ *is* $\mathbf{L}^-F$.

Proof Let $\mathbf{K}'$ be the full subcategory of $\mathbf{K}(\mathcal{A})$ consisting of complexes of F-acyclic objects in $\mathcal{A}$ (2.4.3). We need to show that every complex X has a quasi-isomorphism $X \to X'$ with X' a complex of F-acyclic objects. To see this, choose a Cartan-Eilenberg resolution $X^{\cdot} \to I^{\cdot\cdot}$ and let τI be the double subcomplex of I obtained by taking the good truncation $\tau_{\leq n}(I^{p\cdot})$ of each column (1.2.7). Since each $X^p \to I^{p\cdot}$ is an injective resolution, each $\tau_{\leq n}(I^{p\cdot})$ is a finite resolution of X^p by F-acyclic objects. Therefore $X' = \mathrm{Tot}(\tau I)$ is a chain complex of F-acyclic objects. The bounded spectral sequence $H^p H^q(\tau I) \Rightarrow H^{p+q}(X')$ degenerates to yield $H^*(X) \xrightarrow{\cong} H^*(X')$, that is, $X \to X'$ is a quasi-isomorphism. ◇

10.6 The Total Tensor Product

Let R be a ring. In order to avoid notational problems, we shall use the letters A, B, and so on to denote cochain complexes of R-modules. For each cochain complex A of right R-modules the total tensor product complex 2.7.1 is a

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functor $F(B) = \mathrm{Tot}^{\oplus}(A \otimes_R B)$ from $\mathbf{K}(R\text{–}\mathbf{mod})$ to $\mathbf{K}(\mathbf{Ab})$. Since R–**mod** has enough projectives, its derived functor $\mathbf{L}^{-}F\colon \mathbf{D}^{-}(R\text{–}\mathbf{mod}) \to \mathbf{D}(\mathbf{Ab})$ exists by 10.5.6.

Definition 10.6.1 The *total tensor product* of A and B is

$$A \otimes_R^{\mathbf{L}} B = \mathbf{L}^{-}\,\mathrm{Tot}^{\oplus}(A \otimes_R -)B.$$

Lemma 10.6.2 *If A, A', and B are bounded above cochain complexes and $A \to A'$ is a quasi-isomorphism, then $A \otimes_R^{\mathbf{L}} B \cong A' \otimes_R^{\mathbf{L}} B$.*

Proof We may change B up to quasi-isomorphism to suppose that B is a complex of flat modules. In this case $A \otimes_R^{\mathbf{L}} B$ is $\mathrm{Tot}^{\oplus}(A \otimes_R B)$ and $A' \otimes_R^{\mathbf{L}} B$ is $\mathrm{Tot}^{\oplus}(A' \otimes_R B)$ by 10.5.5. Now apply the Comparison Theorem 5.2.12 to $E_1^{pq}(A) \to E_1^{pq}(A')$, where

$$E_1^{pq}(A) = H^q(A) \otimes_R B^p \Rightarrow H^{p+q}(A \otimes_R^{\mathbf{L}} B).$$

The spectral sequences converge when A, A', and B are bounded above 5.6.2. ◇

Theorem 10.6.3 *The total tensor product is a functor*

$$\otimes_R^{\mathbf{L}}\colon \mathbf{D}^{-}(\mathbf{mod}\text{–}R) \times \mathbf{D}^{-}(R\text{–}\mathbf{mod}) \to D^{-}(\mathbf{Ab}).$$

Its cohomology is the hypertor of 5.7.8:

$$\mathbf{Tor}_i^R(A, B) \cong H^{-i}(A \otimes_R^{\mathbf{L}} B).$$

Proof For each fixed B, the functor $F(A) = A \otimes_R^{\mathbf{L}} B$ from $\mathbf{K}^{-}(\mathbf{mod}\text{–}R)$ to $\mathbf{D}^{-}(\mathbf{Ab})$ sends quasi-isomorphisms to isomorphisms, so F factors through the localization $\mathbf{D}^{-}(\mathbf{mod}\text{–}R)$ of $\mathbf{K}^{-}(\mathbf{mod}\text{–}R)$. If P and Q are chain complexes of flat modules, then by definition the hypertor groups $\mathbf{Tor}_i^R(P, Q)$ are the homology of $\mathrm{Tot}^{\oplus} P \otimes_R Q$. Reindexing the chain complexes as cochain complexes, the cochain complex $\mathrm{Tot}^{\oplus}(P \otimes_R Q)$ is isomorphic to $P \otimes_R^{\mathbf{L}} Q$. ◇

Corollary 10.6.4 *If A and B are R-modules, the usual* Tor*-group* $\mathrm{Tor}_i^R(A,B)$ *of Chapter 3 is $H^{-i}(A \otimes_R^{\mathbf{L}} B)$, where A and B are considered as cochain complexes in degree zero.*

Exercise 10.6.1 Form the derived functor $\mathbf{L}\,\mathrm{Tot}^{\oplus}(- \otimes_R B)$ and show that $A \otimes_R^{\mathbf{L}} B$ is naturally isomorphic to $\mathbf{L}^{-}\,\mathrm{Tot}^{\oplus}(- \otimes_R B)A$ in $\mathbf{D}(\mathbf{Ab})$. This isomorphism underlies the fact that hypertor is a balanced functor (2.7.7).

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Exercise 10.6.2 If A is a complex of R_1–R bimodules, and B is a complex of R–R_2 bimodules, $A \otimes_R B$ is a double complex of R_1–R_2 bimodules. Show that the total tensor product may be refined to a functor

$$\otimes_R^{\mathbf{L}} : \mathbf{D}^-(R_1\text{–}\mathbf{mod}\text{–}R) \times \mathbf{D}^-(R\text{–}\mathbf{mod}\text{–}R_2) \to \mathbf{D}^-(R_1\text{–}\mathbf{mod}\text{–}R_2).$$

By "refine" we mean that the composition to $\mathbf{D}(\mathbf{Ab})$ induced by the usual forgetful functor is the total tensor product in $\mathbf{D}(\mathbf{Ab})$. Then show that if R is a commutative ring, we may refine it to a functor

$$\otimes_R^{\mathbf{L}} : \mathbf{D}^-(R\text{–}\mathbf{mod}) \times \mathbf{D}^-(R\text{–}\mathbf{mod}) \to \mathbf{D}^-(R\text{–}\mathbf{mod}),$$

and that there is a natural isomorphism $A \otimes_R^{\mathbf{L}} B \cong B \otimes_R^{\mathbf{L}} A$.

Remark 10.6.5 (see [HartRD, II.4]) If X is a topological space with a sheaf $\mathcal{O}_X$ of rings, there is a category of $\mathcal{O}_X$-modules [Hart]. This category has enough flat modules (see [Hart, exercise III.6.4]), even though it may not have enough projectives, and this suffices to construct the total tensor product $\mathcal{E} \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{F}$ of complexes of $\mathcal{O}_X$-modules.

10.6.1 Ring Homomorphisms and $\mathbf{L}f^*$

10.6.6 Let $f: R \to S$ be a ring homomorphism. By the Existence Theorem 10.5.6, the functor $f^* = -\otimes_R S$ from R-modules to S-modules has a left-derived functor

$$\mathbf{L}f^* = \mathbf{L}(-\otimes_R S) : \mathbf{D}^-(\mathbf{mod}\text{–}R) \to \mathbf{D}^-(\mathbf{mod}\text{–}S).$$

The discussion in 5.7.8 shows that the hypertor groups are

$$\mathbf{Tor}_i^R(A, S) = \mathbb{L}_i f^*(A) \cong H^{-i}(\mathbf{L}f^* A).$$

If S has finite flat dimension n (4.1.1), then f^* has homological dimension n, and we may extend the derived functor $\mathbf{L}f^*$ using 10.5.11 to $\mathbf{L}f^*$: $\mathbf{D}(\mathbf{mod}\text{–}R) \to \mathbf{D}(\mathbf{mod}\text{–}S)$.

The forgetful functor f_*: $\mathbf{mod}\text{–}S \to \mathbf{mod}\text{–}R$ is exact, so it "is" its own derived functor f_*: $\mathbf{D}(\mathbf{mod}\text{–}S) \to \mathbf{D}(\mathbf{mod}\text{–}R)$. The composite $f_*(\mathbf{L}f^*)A$ is the total tensor product $A \otimes_R^{\mathbf{L}} S$ because, when A is a bounded above complex of flat modules, both objects of the derived category are represented by $A \otimes_R S$. We will see in the next section that f_* $(= \mathbf{R}f_*)$ and $\mathbf{L}f^*$ are adjoint functors in a suitable sense.

Remark If we pass from rings to schemes, the map f reverses direction, going from Spec(S) to Spec(R). This explains the use of the notation f_*, which suggests a covariant functor on Spec(R). Of course f_* is not always exact when we pass to more general schemes, and one needs to replace f_* by $\mathbf{R}f_*$; see [HartRD, II.5.5].

Lemma 10.6.7 *If $f: R \to S$ is a commutative ring homomorphism, there is a natural isomorphism in $\mathbf{D}^-(\mathbf{mod}\text{–}S)$ for every A, B in $\mathbf{D}^-(\mathbf{mod}\text{–}R)$:*

$$\mathbf{L}f^*(A) \otimes_S^{\mathbf{L}} \mathbf{L}f^*(B) \xrightarrow{\cong} \mathbf{L}f^*(A \otimes_R^{\mathbf{L}} B).$$

Proof Replacing A and B by complexes of flat R-modules, this is just the natural isomorphism $(A \otimes_R S) \otimes_S (S \otimes_R B) \cong (A \otimes_R B) \otimes_R S$. ◇

Exercise 10.6.3 (finite Tor-dimension) The *Tor-dimension* of a bounded complex A of right R-modules is the smallest n such that the hypertor $\mathbf{Tor}_i^R(A, B)$ vanish for all modules B when $i > n$. If A is a module, the Tor-dimension is just the flat dimension of 4.1.1.

1. Show that A has finite Tor-dimension if and only if there is a quasi-isomorphism $P \to A$ with P a bounded complex of flat R-modules.
2. If A has finite Tor-dimension, show that the derived functor $A \otimes_R^{\mathbf{L}} -$ on $\mathbf{D}^-(R\text{–}\mathbf{mod})$ extends to a functor

$$\mathbf{L}(A \otimes_R): \mathbf{D}(R\text{–}\mathbf{mod}) \to \mathbf{D}(\mathbf{Ab}).$$

3. Let $f: R \to S$ be a ring map, with S of finite flat dimension over R. Show that the forgetful functor $f_*: \mathbf{D}^b(\mathbf{mod}\text{–}S) \to \mathbf{D}^b(\mathbf{mod}\text{–}R)$ sends complexes of finite Tor-dimension over S to complexes of finite Tor-dimension over R.

10.6.2 The Derived Functors of Γ *and* f_*

10.6.8 Let X be a topological space, and Γ the global sections functor from $\mathbf{Sheaves}(X)$ (sheaves of abelian groups) to $\mathbf{Ab}$; see 2.5.4. For simplicity, we shall write $\mathbf{D}(X)$, $\mathbf{D}^+(X)$, and so on for the derived categories $\mathbf{D}(\mathbf{Sheaves}(X))$, $\mathbf{D}^+(\mathbf{Sheaves}(X))$, and so on. By 2.3.12 the category $\mathbf{Sheaves}(X)$ has enough injectives. Therefore Γ has a right-derived functor $\mathbf{R}^+\Gamma: \mathbf{D}^+(X) \to \mathbf{D}^+(\mathbf{Ab})$, and for every sheaf $\mathcal{F}$ the usual cohomology functors $H^i(X, \mathcal{F})$ of 2.5.4 are the groups $H^i(\mathbf{R}^+\Gamma(\mathcal{F}))$. More generally, if $\mathcal{F}^*$ is a bounded below complex of sheaves on X, then the hypercohomology groups of 5.7.10 are given by:

$$\mathbb{H}^i(X, \mathcal{F}^*) \cong H^i \mathbf{R}^+\Gamma(\mathcal{F}^*).$$

In algebraic geometry, one usually works with topological spaces that are noetherian (the closed subspaces satisfy the descending chain condition) and have finite Krull dimension n (the longest chain of irreducible closed subsets has length n). Grothendieck proved in [Tohuku, 3.6.5] (see [Hart, III.2.7]) that for such a space the functors $H^i(X, -)$ vanish for $i > n$, that is, that Γ has cohomological dimension n. As we have seen in 10.5.11, this permits us to extend $\mathbf{R}^+\Gamma$ to a functor

$$\mathbf{R}\Gamma : \mathbf{D}(X) \to \mathbf{D}(\mathbf{Ab}).$$

Now let $f: X \to Y$ be a continuous map of topological spaces. Just as for Γ, the direct image sheaf functor f_* (2.6.6) has a derived functor

$$\mathbf{R} f_* : \mathbf{D}^+(X) \to \mathbf{D}^+(Y).$$

If $\mathcal{F}$ is a sheaf on X, its higher direct image sheaves (2.6.6) are the sheaves

$$R^i f_*(\mathcal{F}) = H^i \mathbf{R} f_*(\mathcal{F}).$$

When X is noetherian of finite Krull dimension, the functor f_* has finite cohomological dimension because, by [Hart, III.8.1], $R^i f_*(\mathcal{F})$ is the sheaf on Y associated to the presheaf sending U to $H^i(f^{-1}(U), \mathcal{F})$. Once again, we can extend $\mathbf{R} f_*$ from $\mathbf{D}^+(X)$ to a functor $\mathbf{R} f_* : \mathbf{D}(X) \to \mathbf{D}(Y)$.

$\mathbf{R}\Gamma$ is just a special case of $\mathbf{R} f_*$. Indeed, if Y is a point, then $\mathbf{Sheaves}(Y) = \mathbf{Ab}$ and Γ is f_*; it follows that $\mathbf{R}\Gamma$ is $\mathbf{R} f_*$.

10.7 Ext and RHom

Let A and B be cochain complexes. In 2.7.4 we constructed the total Hom cochain complex $\mathrm{Hom}^{\cdot}(A, B)$, and observed that $H^n \mathrm{Hom}^{\cdot}(A, B)$ is the group of chain homotopy equivalence classes of morphisms $A \to B[-n]$. That is,

$$\mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, T^n B) = H^n(\mathrm{Hom}^{\cdot}(A, B)).$$

Both $\mathrm{Hom}^{\cdot}(A, -)$ and $\mathrm{Hom}^{\cdot}(-, B)$ are morphisms of triangulated functors, from $\mathbf{K}(\mathcal{A})$ and $\mathbf{K}(\mathcal{A})^{\mathrm{op}}$ to $\mathbf{K}(\mathbf{Ab})$, respectively. In fact, $\mathrm{Hom}^{\cdot}$ is a bimorphism

$$\mathrm{Hom}^{\cdot} : \mathbf{K}(\mathcal{A})^{\mathrm{op}} \times \mathbf{K}(\mathcal{A}) \to \mathbf{K}(\mathbf{Ab}).$$

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(Exercise!) In this section we construct an object $\mathbf{R}\mathrm{Hom}(A, B)$ in the derived category $\mathbf{D}(\mathcal{A})$ and prove that if A and B are bounded below, then

$$\mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A, T^n B) = H^n(\mathbf{R}\mathrm{Hom}(A, B)).$$

Since $\mathbf{D}^+(\mathcal{A})$ is a full subcategory of $\mathbf{D}(\mathcal{A})$, this motivates the following.

Definition 10.7.1 Let A and B be cochain complexes in an abelian category $\mathcal{A}$. The n^{th} *hyperext* of A and B is the abelian group

$$\mathrm{Ext}^n(A, B) = \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A, T^n B).$$

Note that since $\mathbf{D}(\mathcal{A})$ is a triangulated category, its Hom-functors $\mathrm{Ext}^n(A, -)$ and $\mathrm{Ext}^n(-, B)$ are cohomological functors, that is, they convert exact triangles into long exact sequences (10.2.8). Since $\mathbf{K}(\mathcal{A})$ is a triangulated category, its Hom-functors $H^n \mathrm{Hom}^{\cdot}(A, -)$ and $H^n \mathrm{Hom}^{\cdot}(-, B)$ are also cohomological functors, and there are canonical morphisms

$$H^n \mathrm{Hom}^{\cdot}(A, B) = \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, T^n B) \to \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A, T^n B) = \mathrm{Ext}^n(A, B).$$

Definition 10.7.2 Suppose that $\mathcal{A}$ has enough injectives, so that the derived functor $\mathbf{R}^+ \mathrm{Hom}^{\cdot}(A, -)\colon \mathbf{D}^+(\mathcal{A}) \to \mathbf{D}(\mathbf{Ab})$ exists for every cochain complex A. We write $\mathbf{R}\mathrm{Hom}(A, B)$ for the object $\mathbf{R}^+ \mathrm{Hom}^{\cdot}(A, -)B$ of $\mathbf{D}(\mathbf{Ab})$.

Lemma 10.7.3 *If $A \to A'$ is a quasi-isomorphism, then* $\mathbf{R}\mathrm{Hom}(A', B) \xrightarrow{\cong} \mathbf{R}\mathrm{Hom}(A, B)$.

Proof We may change B up to quasi-isomorphism to suppose that B is a bounded below cochain complex of injectives. But then $\mathbf{R}\mathrm{Hom}(A', B) \cong \mathrm{Hom}^{\cdot}(A', B)$ is quasi-isomorphic to $\mathbf{R}\mathrm{Hom}(A, B) \cong \mathrm{Hom}^{\cdot}(A, B)$, because we saw in 10.4.7 that

$$\begin{aligned} H^n \mathrm{Hom}^{\cdot}(A', B) &= \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A', T^n B) \\ &\cong \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A', T^n B) \cong \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A, T^n B) \\ &\cong \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, T^n B) = H^n \mathrm{Hom}^{\cdot}(A, B). \end{aligned}$$ ◇

Theorem 10.7.4 *If $\mathcal{A}$ has enough injectives, then* $\mathbf{R}$*Hom is a bifunctor*

$$\mathbf{R}\mathrm{Hom}\colon \mathbf{D}(\mathcal{A})^{\mathrm{op}} \times \mathbf{D}^+(\mathcal{A}) \to \mathbf{D}(\mathbf{Ab}).$$

Dually, if $\mathcal{A}$ has enough projectives, then $\mathbf{R}$Hom *is a bifunctor*

$$\mathbf{R}\mathrm{Hom}\colon \mathbf{D}^-(\mathcal{A})^{\mathrm{op}} \times \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathbf{Ab}).$$

In both cases, we have $\mathrm{Ext}^n(A, B) \cong H^n(\mathbf{R}\mathrm{Hom}(A, B))$.

Proof The lemma shows that, for each fixed B, the functor $F(A) = \mathbf{R}\mathrm{Hom}(A, B)$ from $\mathbf{K}(\mathcal{A})^{\mathrm{op}}$ to $\mathbf{D}(\mathbf{Ab})$ sends quasi-isomorphisms to isomorphisms, so F factors through the localization $\mathbf{D}(\mathcal{A})^{\mathrm{op}}$ of $\mathbf{K}(\mathcal{A})^{\mathrm{op}}$. Therefore, to compute $H^n(\mathbf{R}\mathrm{Hom}(A, B))$ we may suppose that B is a bounded below cochain complex of injectives. But then by the construction of $\mathbf{R}\mathrm{Hom}(A, B)$ as $\mathrm{Hom}^{\cdot}(A, B)$ we have

$$H^n\,\mathbf{R}\mathrm{Hom}(A, B) = H^n\,\mathrm{Hom}^{\cdot}(A, B) = \mathrm{Hom}_{\mathbf{K}(\mathcal{A})}(A, B) = \mathrm{Hom}_{\mathbf{D}(\mathcal{A})}(A, B). \qquad \diamond$$

Corollary 10.7.5 *If $\mathcal{A}$ has enough injectives, or enough projectives, then for any A and B in $\mathcal{A}$ the group* $\mathrm{Ext}^n(A, B)$ *is the usual Ext-group of Chapter 3.*

Proof If $B \to I$ is an injective resolution, then the usual definition of $\mathrm{Ext}^n(A, B)$ is $H^n\,\mathrm{Hom}(A, I) = H^n\,\mathrm{Tot}\,\mathrm{Hom}(A, I) \cong H^n\,\mathbf{R}\mathrm{Hom}(A, B)$. Similarly, if $P \to A$ is a projective resolution, the usual $\mathrm{Ext}^n(A, B)$ is $H^n\,\mathrm{Hom}(P, B) = H^n\,\mathbf{R}\mathrm{Hom}(A, B)$. $\diamond$

Exercise 10.7.1 (balancing $\mathbf{R}\mathrm{Hom}$) Suppose that $\mathcal{A}$ has both enough injectives and enough projectives. Show that the two ways of defining the functor $\mathbf{R}\mathrm{Hom}\colon \mathbf{D}^-(\mathcal{A})^{\mathrm{op}} \times \mathbf{D}^+(\mathcal{A}) \to \mathbf{D}^+(\mathbf{Ab})$ are canonically isomorphic.

Exercise 10.7.2 Suppose that $\mathcal{A}$ has enough injectives. We say that a bounded below complex B has *injective dimension n* if $\mathrm{Ext}^i(A, B) = 0$ for all $i > n$ and all A in $\mathcal{A}$, and $\mathrm{Ext}^n(A, B) \neq 0$ for some A.

1. Show that B has finite injective dimension $\Leftrightarrow$ there is a quasi-isomorphism $B \to I$ into a bounded complex I of injectives.
2. If B has finite injective dimension, show that $\mathbf{R}\mathrm{Hom}(-, B)\colon \mathbf{D}(\mathcal{A})^{\mathrm{op}} \to \mathbf{D}(\mathbf{Ab})$ of 10.7.4 is the derived functor 10.5.1 of $\mathrm{Hom}(-, B)$.

10.7.1 Adjointness of $\mathbf{L}f^*$ and f_*

We can refine the above construction slightly when $\mathcal{A}$ is the category R–**mod** of modules over a commutative ring R. For simplicity we shall write $\mathbf{D}(R)$, $\mathbf{D}^+(R)$, and so on for the derived categories $\mathbf{D}(R\text{–}\mathbf{mod})$, $\mathbf{D}^+(R\text{–}\mathbf{mod})$, and so on. Write $\mathrm{Hom}^{\cdot}_R(A, B)$ for $\mathrm{Hom}^{\cdot}(A, B)$, considered as a complex of R-modules. If we replace $\mathbf{D}(\mathbf{Ab})$ by $\mathbf{D}(R)$ in the above construction, we obtain an object $\mathbf{R}\mathrm{Hom}_R(A, B)$ in $\mathbf{D}(R)$ whose image under $\mathbf{D}(R) \to \mathbf{D}(\mathbf{Ab})$ is the unrefined $\mathbf{R}\mathrm{Hom}(A, B)$ of 10.7.2.

Suppose now that $f\colon R \to S$ is a map of commutative rings. The forgetful functor $f_*\colon \mathbf{mod}\text{–}S \to \mathbf{mod}\text{–}R$ is exact, so it is its own derived functor $f_*\colon \mathbf{D}(S) \to \mathbf{D}(R)$. If A is in $\mathbf{D}(S)$, the functor $f_*\,\mathbf{R}\mathrm{Hom}_S(A, -)\colon \mathbf{D}^+(S) \to \mathbf{D}(R)$ is the right derived functor of $f_*\,\mathrm{Hom}_S(A, -)$ because if I is a complex of injectives, then $f_*\,\mathbf{R}\mathrm{Hom}_S(A, I) = f_*\,\mathrm{Hom}_S(A, I)$. The universal property of derived functors yields a natural map:

$$(\dagger) \qquad \zeta\colon f_*\,\mathbf{R}\mathrm{Hom}_S(A, B) \to \mathbf{R}\mathrm{Hom}_R(f_*A, f_*B).$$

Theorem 10.7.6 *If $f\colon R \to S$ is a map of commutative rings, then the functor $\mathbf{L}f^*\colon \mathbf{D}^-(R) \to \mathbf{D}^-(S)$ is left adjoint to $f_*\colon \mathbf{D}^+(S) \to \mathbf{D}^+(R)$. That is, for A in $\mathbf{D}^-(R)$ and B in $\mathbf{D}^+(S)$ there is a natural isomorphism*

$$(*) \qquad \mathrm{Hom}_{\mathbf{D}(S)}(\mathbf{L}f^*A, B) \xrightarrow{\cong} \mathrm{Hom}_{\mathbf{D}(R)}(A, f_*B).$$

The adjunction morphisms are $\eta_A\colon A \to f_*\mathbf{L}f^*A$ and $\varepsilon_B\colon \mathbf{L}f^*(f_*B) \to B$, respectively. Moreover, the isomorphism $(*)$ comes from a natural isomorphism

$$\tau : f_*\,\mathbf{R}\mathrm{Hom}_S(\mathbf{L}f^*A, B) \xrightarrow{\cong} \mathbf{R}\mathrm{Hom}_R(A, f_*B).$$

Proof Since f_* is exact, $f_*\mathbf{L}f^*$ is the left derived functor of f_*f^*; the universal property gives a map $\eta_A\colon A \to \mathbf{L}(f_*f^*)A = f_*\mathbf{L}f^*A$. Using $(\dagger)$, this gives the map

$$\tau : f_*\,\mathbf{R}\mathrm{Hom}_S(\mathbf{L}f^*A, B) \xrightarrow{\zeta} \mathbf{R}\mathrm{Hom}_R(f_*\mathbf{L}f^*A, f_*B) \xrightarrow{\eta^*} \mathbf{R}\mathrm{Hom}_R(A, f_*B).$$

To evaluate this map, we suppose that A is a bounded above complex of projective R-modules. In this case the map τ is the isomorphism

$$\begin{aligned}\mathrm{Tot}(f_*\,\mathrm{Hom}_S(A \otimes_R S, B)) &\cong \mathrm{Tot}(\mathrm{Hom}_R(A, \mathrm{Hom}_S(S, B)))\\ &= \mathrm{Tot}(\mathrm{Hom}_R(A, f_*B)).\end{aligned}$$

Passing to cohomology, τ induces the adjoint isomorphism $(*)$. ◇

Remark For schemes one needs to be able to localize the above data to form the $\mathcal{O}_X$-module analogue of $\mathbf{R}\mathrm{Hom}_R$. By 3.3.8 one needs A to be finitely presented in order to have an isomorphism $S^{-1}\,\mathrm{Hom}_R(A, B) \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}A, S^{-1}B)$. Thus one needs to restrict A to a subcategory of $\mathbf{D}(X)$ which is locally the $\mathbf{D}_{\mathrm{fg}}(R)$ of exercise 10.4.6; see [HartRD, II.5.10] for details.

Exercise 10.7.3 Let X be a topological space. Given two sheaves $\mathcal{E}, \mathcal{F}$ on X, the *sheaf hom* is the sheaf $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is the sheaf on X associated to the presheaf sending U to $\mathrm{Hom}(\mathcal{E}|U, \mathcal{F}|U)$; see [Hart, exercise II.1.15]. Mimic the construction of **R**Hom to obtain a functor

$$\mathbf{R}\mathcal{H}om: \mathbf{D}(X)^{\mathrm{op}} \times \mathbf{D}^+(X) \to \mathbf{D}(X).$$

Now suppose that $f: X \to Y$ is a continuous map, and that X is noetherian of finite Krull dimension. Generalize (†) for $\mathcal{E}$ in $\mathbf{D}^-(X)$, $\mathcal{F}$ in $\mathbf{D}^+(X)$ to obtain a natural map in $\mathbf{D}^+(Y)$:

$$\zeta: \mathbf{R}f_* (\mathbf{R}\mathcal{H}om_X(\mathcal{E}, \mathcal{F})) \to \mathbf{R}\mathcal{H}om_Y(\mathbf{R}f_*\mathcal{E}, \mathbf{R}f_*\mathcal{F}).$$

10.8 Replacing Spectral Sequences

We have seen that the objects **R**F(A) in the derived category are more flexible than their cohomology groups, the hyper-derived functors $\mathbb{R}^i F(A) = H^i\mathbf{R}F(A)$. Of course, if we are interested in the groups themselves, we can use the spectral sequence $E_2^{pq} = (R^pF)(H^qA) \Rightarrow \mathbb{R}^{p+q}F(A)$ of 5.7.9. Things get more complicated when we compose two or more functors, because then we need spectral sequences to compute the E_2-terms of other spectral sequences.

Example 10.8.1 Consider the problem of comparing the two ways of forming the total tensor product of three bounded below cochain complexes $A \in \mathbf{D}^-(\mathbf{mod}\text{–}R)$, $B \in \mathbf{D}^-(R\text{–}\mathbf{mod}\text{–}S)$, and $C \in \mathbf{D}^-(S\text{–}\mathbf{mod})$. Replacing A and C by complexes of projectives, we immediately see that there is a natural isomorphism

$$(*) \qquad A \otimes_R^{\mathbf{L}} (B \otimes_S^{\mathbf{L}} C) \cong (A \otimes_R^{\mathbf{L}} B) \otimes_S^{\mathbf{L}} C.$$

However, it is quite a different matter to try to establish this quasi-isomorphism by studying the two hypertor modules $\mathbf{Tor}_i^R(A, B)$ and $\mathbf{Tor}_j^S(B, C)$! Cf. [EGA, III.6.8.3]. Another way to establish the isomorphism $(*)$ is to set $F = \mathrm{Tot}(A \otimes_R)$ and $G = \mathrm{Tot}(\otimes_S C)$. Since $FG \cong GF$, $(*)$ follows immediately from the following result.

Composition Theorem 10.8.2 *Let $\mathbf{K} \subset \mathbf{K}(\mathcal{A})$ and $\mathbf{K}' \subset \mathbf{K}(\mathcal{B})$ be localizing triangulated subcategories, and suppose given two morphisms of triangulated categories $G: \mathbf{K} \to \mathbf{K}'$, $F: \mathbf{K}' \to \mathbf{K}(\mathcal{C})$. Assume that* **R***F,* **R***G, and* $\mathbf{R}(FG)$ *exist, with* $\mathbf{R}F(\mathbf{D}) \subseteq \mathbf{D}'$*. Then:*

1. *There is a unique natural transformation* $\zeta = \zeta_{F,G} : \mathbf{R}(FG) \Rightarrow \mathbf{R}F \circ \mathbf{R}G$*, such that the following diagram commutes in* $\mathbf{D}(\mathcal{C})$ *for each* A *in* $\mathbf{K}$.

$$\begin{array}{ccc} qFG(A) & \xrightarrow{\xi_F} & (\mathbf{R}F)(qGA) \\ \downarrow \xi_{FG} & & \downarrow \xi_G \\ \mathbf{R}(FG)(qA) & \xrightarrow{\zeta_{qA}} & (\mathbf{R}F)(\mathbf{R}G)(qA) \end{array}$$

2. *Suppose that there are triangulated subcategories* $\mathbf{K}_0 \subseteq \mathbf{K}$, $\mathbf{K}'_0 \subseteq \mathbf{K}'$ *satisfying the hypotheses of the Generalized Existence Theorem 10.5.9 for* G *and* F*, and suppose that* G *sends* $\mathbf{K}_0$ *to* $\mathbf{K}'_0$*. Then* ζ *is an isomorphism*

$$\zeta : \mathbf{R}(FG) \cong (\mathbf{R}F) \circ (\mathbf{R}G).$$

Proof Part (1) follows from the universal property 10.5.1 of $\mathbf{R}(FG)$. For (2) it suffices to observe that if A is in $\mathbf{K}_0$, then

$$\mathbf{R}(FG)(qA) = qFG(A) \cong \mathbf{R}F(q(GA)) \cong \mathbf{R}F(\mathbf{R}G(qA)). \qquad \diamond$$

Corollary 10.8.3 (Grothendieck spectral sequences) *Let* $\mathcal{A}$, $\mathcal{B}$, *and* $\mathcal{C}$ *be abelian categories such that both* $\mathcal{A}$ *and* $\mathcal{B}$ *have enough injectives, and suppose given left exact functors* $G : \mathcal{A} \to \mathcal{B}$ *and* $F : \mathcal{B} \to \mathcal{C}$.

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "G"] \arrow[dr, "FG"'] & & \mathcal{B} \arrow[dl, "F"] \\ & \mathcal{C} & \end{tikzcd}$$

If G *sends injective objects of* $\mathcal{A}$ *to* F*-acyclic objects of* $\mathcal{B}$*, then*

$$\zeta : \mathbf{R}^+(FG) \cong (\mathbf{R}^+F) \circ (\mathbf{R}^+G).$$

If in addition G *sends acyclic complexes to* F*-acyclic complexes, and both* F *and* G *have finite cohomological dimension, then* $\mathbf{R}(FG) : \mathbf{D}(\mathcal{A}) \to \mathbf{D}(\mathcal{C})$ *exists, and*

$$\zeta : \mathbf{R}(FG) \cong (\mathbf{R}F) \circ (\mathbf{R}G).$$

In both cases, there is a convergent spectral sequence for all A:

$$E_2^{pq} = (R^pF)(\mathbb{R}^qG)(A) \Rightarrow \mathbb{R}^{p+q}(FG)(A).$$

If A *is an object of* $\mathcal{A}$*, this is the Grothendieck spectral sequence of 5.8.3.*

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Proof The hypercohomology spectral sequence 5.7.9 converging to $(\mathbb{R}^{p+q}F)(\mathbf{R}G(A))$ has E_2^{pq} term $(R^pF)H^q(\mathbf{R}G(A)) = (R^pF)(\mathbb{R}^qG(A))$. $\diamond$

Remark 10.8.4 Conceptually, the composition of functors $\mathbf{R}(FG) \cong (\mathbf{R}F) \circ (\mathbf{R}G)$ is much simpler than the original spectral sequence. The reader having some familiarity with algebraic geometry may wish to glance at [EGA, III.6], and especially at the "six spectral sequences" of III.6.6 or III.6.7.3, to appreciate the convenience of the derived category.

Exercise 10.8.1 If F, G, H are three consecutive morphisms, show that as natural transformations from $\mathbf{R}(FGH)$ to $\mathbf{R}F \circ \mathbf{R}G \circ \mathbf{R}H$ we have

$$\zeta_{G,H} \circ \zeta_{F,GH} = \zeta_{F,G} \circ \zeta_{FG,H}.$$

In the rest of this section, we shall enumerate three consequences of the Composition Theorem 10.8.2, usually replacing a spectral sequence with an isomorphism in the derived category. We will implicitly use the dual formulation $LF \circ LG \cong L(FG)$ of the Composition Theorem without comment.

10.8.1 The Projection Formula

10.8.5 Let $f: R \to S$ be a ring homomorphism, A a bounded above complex of right R-modules, and B a complex of left S-modules. The functor f^*: **mod**–R → **mod**–S sends A to $A \otimes_R S$, so it preserves projectives. Since $f^*(A) \otimes_S B = (A \otimes_R S) \otimes_S B \cong A \otimes_R f_*B$, the Composition Theorem 10.8.2 yields

$$(*) \qquad \mathbf{L}f^*(A) \otimes_S^{\mathbf{L}} B \xrightarrow{\cong} A \otimes_R^{\mathbf{L}} (f_*B)$$

in $\mathbf{D}(\mathbf{Ab})$. If S is commutative, we may regard B as an S–S bimodule and f_*B as an R–S bimodule. As we saw in exercise 10.6.2, this allows us to interpret $(*)$ as an isomorphism in $\mathbf{D}(S)$. From the standpoint of algebraic geometry, however, it is better to apply f_* to obtain the following isomorphism in $\mathbf{D}(R)$:

$$f_*(\mathbf{L}f^*(A) \otimes_S^{\mathbf{L}} B) \cong A \otimes_R^{\mathbf{L}} (f_*B).$$

This is sometimes called the "projection formula"; see [HartRD, II.5.6] for the generalization to schemes. The projection formula underlies the "Base change for Tor" spectral sequence 5.6.6.

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Exercise 10.8.2 Use the universal property of $\otimes^{\mathbf{L}}_R$ to construct the natural map $\mathbf{L}f^*(A) \otimes^{\mathbf{L}}_S B \to A \otimes^{\mathbf{L}}_R (f_* B)$.

10.8.6 Similarly, if $g: S \to T$ is another ring homomorphism, we have $(gf)^* \cong g^* f^*$. The Composition Theorem 10.8.2 yields a natural isomorphism

$$(\mathbf{L}g^*)(\mathbf{L}f^*)A \cong \mathbf{L}(gf)^* A.$$

This underlies the spectral sequence $\mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A, S), T) \Rightarrow \mathrm{Tor}^R_{p+q}(A, T)$.

10.8.2 *Adjointness of* $\otimes^{\mathbf{L}}$ *and* $\mathbf{R}\mathrm{Hom}$

Theorem 10.8.7 *If R is a commutative ring and B is a bounded above complex of R-modules, then $\otimes^{\mathbf{L}}_R B: \mathbf{D}^-(R) \to \mathbf{D}^-(R)$ is left adjoint to the functor $\mathbf{R}\mathrm{Hom}_R(B, -): \mathbf{D}^+(R) \to \mathbf{D}^+(R)$. That is, for A in $\mathbf{D}^-(R)$ and C in $\mathbf{D}^+(R)$ there is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{D}(R)}(A, \mathbf{R}\mathrm{Hom}_R(B, C)) \cong \mathrm{Hom}_{\mathbf{D}(R)}(A \otimes^{\mathbf{L}}_R B, C).$$

This isomorphism arises by applying H^0 to the isomorphism

$$(\dagger) \qquad \mathbf{R}\mathrm{Hom}_R(A, \mathbf{R}\mathrm{Hom}_R(B, C)) \xrightarrow{\cong} \mathbf{R}\mathrm{Hom}_R(A \otimes^{\mathbf{L}}_R B, C)$$

in $\mathbf{D}^+(R)$. The adjunction morphisms are $\eta_A: A \to \mathbf{R}\mathrm{Hom}_R(B, A \otimes^{\mathbf{L}}_R B)$ and $\varepsilon_C: \mathbf{R}\mathrm{Hom}_R(B, C) \otimes^{\mathbf{L}}_R B \to C$.

Proof Fix a projective complex A and an injective complex C. The functor $A \otimes^{\mathbf{L}}_R -$ preserves projectives, while the functor $\mathrm{Hom}_R(-, C)$ sends projectives to injectives. By the Composition Theorem 10.8.2, the two sides of $(\dagger)$ are both isomorphic to the derived functors of the composite functor $\mathrm{Hom}(A, \mathrm{Hom}(B, C)) \cong \mathrm{Hom}(A \otimes_R B, C)$. $\diamond$

Exercise 10.8.3 Let R be a commutative ring and C a bounded complex of finite Tor dimension over R (exercise 10.6.3). Show that there is a natural isomorphism in $\mathbf{D}(R)$:

$$\mathbf{R}\mathrm{Hom}_R(A, B) \otimes^{\mathbf{L}}_R C \xrightarrow{\cong} \mathbf{R}\mathrm{Hom}_R(A, B \otimes^{\mathbf{L}}_R C).$$

Here A is in $\mathbf{D}(R)$ and B is in $\mathbf{D}^+(R)$. For the scheme version of this result, see [HartRD, II.5.14].

We now consider the effect of a ring homomorphism $f: R \to S$ upon $\mathbf{R}$Hom. We saw in 2.3.10 that $\mathrm{Hom}_R(S, -): \mathbf{mod}\text{–}R \to \mathbf{mod}\text{–}S$ preserves injectives. Therefore for every S-module complex A, and every bounded below R-module complex B, we have

$$\mathbf{R}\mathrm{Hom}_S(A, \mathbf{R}\mathrm{Hom}_R(S, B)) \cong \mathbf{R}\mathrm{Hom}_R(f_*A, B).$$

This isomorphism underlies the "Base change for Ext" spectral sequence of exercise 5.6.3.

Exercise 10.8.4 Suppose that S is a flat R-module, so that f^* is exact and $\mathbf{L}f^* \cong f^*$. Suppose that A is quasi-isomorphic to a bounded above complex of finitely generated projective modules. Show that we have a natural isomorphism for every B in $\mathbf{D}^+(R)$:

$$\mathbf{L}f^*\,\mathbf{R}\mathrm{Hom}_R(A, B) \to \mathbf{R}\mathrm{Hom}_S(\mathbf{L}f^*A, \mathbf{L}f^*B).$$

Exercise 10.8.5 (Lyndon/Hochschild-Serre) Let H be a normal subgroup of a group G. Show that the functors $A_H = A \otimes_{\mathbb{Z}H} \mathbb{Z}$ and $A^H = \mathrm{Hom}_H(\mathbb{Z}, A)$ of Chapter 6 have derived functors $A \otimes^{\mathbf{L}}_H \mathbb{Z}: \mathbf{D}(G\text{–}\mathbf{mod}) \to \mathbf{D}(G/H\text{–}\mathbf{mod})$ and $\mathbf{R}\mathrm{Hom}_H(\mathbb{Z}, A): \mathbf{D}(G\text{–}\mathbf{mod}) \to \mathbf{D}(G/H\text{–}\mathbf{mod})$ such that

$$A \otimes^{\mathbf{L}}_G \mathbb{Z} \cong (A \otimes^{\mathbf{L}}_H \mathbb{Z}) \otimes^{\mathbf{L}}_{G/H} \mathbb{Z} \quad \text{and}$$

$$\mathbf{R}\mathrm{Hom}_G(\mathbb{Z}, A) \cong \mathbf{R}\mathrm{Hom}_{G/H}(\mathbb{Z}, \mathbf{R}\mathrm{Hom}_H(\mathbb{Z}, A)).$$

Use these to obtain the Lyndon/Hochschild-Serre spectral sequences 6.8.2.

10.8.3 Leray Spectral Sequences

10.8.8 Suppose that $f: X \to Y$ is a continuous map of topological spaces. We saw in 5.8.6 that f_* preserves injectives and that the Leray spectral sequence

$$E_2^{pq} = H^p(Y; R^q f_*\mathcal{F}) \Rightarrow H^{p+q}(X; \mathcal{F})$$

arose from the fact that $\Gamma(X, \mathcal{F})$ is the composite $\Gamma(Y, f_*\mathcal{F})$. The Composition Theorem 10.8.2 promotes this into an isomorphism for every $\mathcal{F}$ in $\mathbf{D}^+(X)$:

$$\mathbf{R}\Gamma(X, \mathcal{F}) \cong \mathbf{R}\Gamma(Y, \mathbf{R}f_*\mathcal{F}).$$

Of course, if X and Y are noetherian spaces of finite Krull dimension, then this isomorphism is valid for every $\mathcal{F}$ in $\mathbf{D}(X)$.

We can generalize this by replacing $\Gamma(Y, -)$ by g_*, where $g: Y \to Z$ is another continuous map. For this, we need the following standard identity.

Lemma 10.8.9 *$(gf)_*\mathcal{F} = g_*(f_*\mathcal{F})$ for every sheaf $\mathcal{F}$ on X.*

Proof By its very definition (2.6.6), for every open subset U of X we have

$$\begin{aligned}(gf)_*\mathcal{F}(U) &= \mathcal{F}((gf)^{-1}U)\\ &= \mathcal{F}(f^{-1}g^{-1}U) = (f_*\mathcal{F})(g^{-1}U) = g_*(f_*\mathcal{F})(U).\end{aligned} \qquad \diamond$$

Corollary 10.8.10 *For every $\mathcal{F}$ in $\mathbf{D}^+(X)$ there is a natural isomorphism*

$$\mathbf{R}(gf)_*(\mathcal{F}) \cong \mathbf{R}g_*(\mathbf{R}f_*(\mathcal{F}))$$

in $\mathbf{D}(Z)$. If moreover X and Y are noetherian of finite Krull dimension, then this isomorphism holds for every $\mathcal{F}$ in $\mathbf{D}(X)$.

Exercise 10.8.6 If $\mathcal{F}$ is an injective sheaf, the sheaf hom $\mathcal{H}om(\mathcal{E}, \mathcal{F})$ is Γ-acyclic ("flasque") by [Gode, II.7.3.2]. For any two sheaves $\mathcal{E}$ and $\mathcal{F}$, show that $\mathcal{H}om_X(\mathcal{E}, \mathcal{F}) \cong \Gamma(X, \mathcal{H}om(\mathcal{E}, \mathcal{F}))$. Then use the Composition Theorem 10.8.2 to conclude that there is a natural isomorphism

$$\mathbf{R}\mathrm{Hom}(\mathcal{E}, \mathcal{F}) \cong (\mathbf{R}\Gamma) \circ \mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{F})$$

of bifunctors from $\mathbf{D}^-(X)^{\mathrm{op}} \times \mathbf{D}^+(X)$ to $\mathbf{D}(\mathbf{Ab})$.

10.9 The Topological Derived Category

At the same time (1962–1963) as Verdier was inventing the algebraic notion of the derived category [Verd], topologists (e.g., D. Puppe) were discovering that the stable homotopy category $\mathbf{D}(\mathcal{S})$ was indeed a triangulated category. In this last section we show how to construct this structure with a minimum of topology, mimicking the passage from chain complexes to the homotopy category $\mathbf{K}(\mathbf{Ab})$ in section 10.1 and the localization from $\mathbf{K}(\mathbf{Ab})$ to the derived category $\mathbf{D}(\mathbf{Ab})$. This provides a rich analogy between derived categories and stable homotopy theory, which has only recently been exploited (see [Th] and [Rob], for example).

Our first task is to define the category of spectra $\mathcal{S}$. Here is the "modern" (coordinatized) definition, following [LMS].

Definition 10.9.1 A *spectrum* E is a sequence of based topological spaces E_n and based homeomorphisms $\alpha_n\colon E_n \xrightarrow{\cong} \Omega E_{n+1}$. A *map of spectra* $f\colon E \to F$ is a sequence of based continuous maps $f_n\colon E_n \to F_n$ strictly compatible with the given structural homeomorphisms. As these maps are closed under composition, the spectra and their maps form a category $\mathcal{S}$. The sequence of 1-point spaces forms a spectrum $*$, which is the zero object in $\mathcal{S}$, because $\mathrm{Hom}_{\mathcal{S}}(*, E) = \mathrm{Hom}_{\mathcal{S}}(E, *) = \{\text{point}\}$ for all E. The product $E \times F$ of two spectra is the spectrum whose n^{th} space is $E_n \times F_n$.

Historically, spectra arose from the study of "infinite loop spaces;" E_0 is an infinite loop space, because we have described it as the p-fold loop space $E_0 \cong \Omega^p E_p$ for all p. The most readable reference for this is part III of Adams' book [A], although it is far from optimal on the foundations, which had not yet been worked out in 1974.

Looping and Delooping 10.9.2 If E is a spectrum, we can form its loop spectrum ΩE by setting $(\Omega E)_n = \Omega(E_n)$, the structural maps being the $\Omega(\alpha_n)$. More subtly, we can form the delooping $\Omega^{-1}E$ by reindexing and forgetting E_0: $(\Omega^{-1}E)_n = E_{n+1}$. Clearly $\Omega\Omega^{-1}E = \Omega^{-1}\Omega E \cong E$, so Ω is a automorphism of the category $\mathcal{S}$. When we construct a triangulated structure on the stable homotopy category, Ω^{-1} will become our "translation functor."

Example 10.9.3 (Sphere spectra) There is a standard map from the m-sphere S^m to the ΩS^{m+1} (put S^m at the equator of S^{m+1} and use the longitudes). The *n-sphere spectrum* $\mathbf{S}^n$ is obtained by applying Ω^i and taking the colimit

$$(\mathbf{S}^n)_p = \operatorname*{colim}_{i\to\infty} \Omega^i S^{n+p+i}.$$

Of course, to define the negative sphere spectrum $\mathbf{S}^n$ we only use $i \geq -n$. The zero-th space of the sphere spectrum $\mathbf{S}^0$ is often written as $\Omega^\infty S^\infty$. Note that our notational conventions are such that for all integers n and p we have $\Omega^p\mathbf{S}^n = \mathbf{S}^{n-p}$.

Definition 10.9.4 (The stable category) The homotopy groups of a spectrum E are:

$$\pi_n E = \pi_{n+i}(E_i) \quad \text{for } i \geq 0, n+i \geq 0.$$

These groups are independent of the choice of i, because for all m $\pi_{i+1}E_m \cong \pi_i(\Omega E_m)$. We say that $f\colon E \to F$ is a *weak homotopy equivalence* if f induces an isomorphism on homotopy groups. Let $\widetilde{W}$ denote the family of all weak homotopy equivalences in $\mathcal{S}$. The *stable homotopy category*, or *topological derived category* $\mathbf{D}(\mathcal{S})$, is the localization $\widetilde{W}^{-1}\mathcal{S}$ of $\mathcal{S}$ at $\widetilde{W}$.

Of course, in order to see that the stable category exists within our universe we need to prove something. Mimicking the procedure of section 1 and section 3, we shall first construct a homotopy category $\mathbf{K}(\mathcal{S})$ and prove that the system W of weak homotopy equivalences form a locally small multiplicative system in $\mathbf{K}(\mathcal{S})$ (10.3.6). Then we shall show that the homotopy category of "CW spectra" forms a localizing subcategory $\mathbf{K}(\mathcal{S}_{CW})$ of $\mathbf{K}(\mathcal{S})$ (10.3.12), and that we may take the topological derived category to be $\mathbf{K}(\mathcal{S}_{CW})$. This parallels theorem 10.4.8, that the category $\mathbf{D}^+(\mathbf{Ab})$ is equivalent to the homotopy category of bounded below complexes of injective abelian groups.

For this program, we need the notion of homotopy in $\mathcal{S}$ and the notion of a CW spectrum, both of which are constructed using prespectra and the "spectrification" functor Ω^∞. Let SX denote the usual based suspension of a topological space X, and recall that maps $SX \to Y$ are in 1–1 correspondence with maps $X \to \Omega Y$.

Definition 10.9.5 A *prespectrum* D is a sequence of based topological spaces D_n and based continuous maps $S(\mathrm{D}_n) \to \mathrm{D}_{n+1}$, or equivalently, maps $\mathrm{D}_n \to \Omega \mathrm{D}_{n+1}$. If C and D are prespectra, a *function* $f\colon C \to \mathrm{D}$ is a sequence of based continuous maps $f_n\colon C_n \to \mathrm{D}_n$ which are strictly compatible with the given structural maps. There is a category $\mathcal{P}$ of prespectra and functions, as well as a forgetful functor $\mathcal{S} \to \mathcal{P}$. A *CW prespectrum* is a prespectrum D in which all the spaces D_n are CW complexes and all the structure maps $SD_n \to \mathrm{D}_{n+1}$ are cellular inclusions.

Warning: Terminology has changed considerably over the years, even since the 1970s. A prespectrum used to be called a "suspension spectrum," and the present notion of spectrum is slightly stronger than the notion of "Ω-spectrum," in which the structural maps were only required to be weak equivalences. Our use of "function" agrees with [A], but the category of CW

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prespectra in [A] has more morphisms than just the functions; see [A, p.140] or [LMS, p.2] for details.

10.9.6 There is a functor $\Omega^\infty : \mathcal{P} \to \mathcal{S}$, called "spectrification." It sends a CW prespectrum D to the spectrum Ω^∞D whose n^{th} space is

$$(\Omega^\infty \mathrm{D})_n = \operatorname*{colim}_{i\to\infty} \Omega^i D_{n+i},$$

where the colimit is taken with respect to the iterated loops on the maps $D_j \to \Omega D_{j+1}$. The structure maps $(\Omega^\infty D)_n \to (\Omega^\infty D)_{n+1}$ are obtained by shifting the indices, using the fact that Ω commutes with colimits. The effect of Ω^∞ on functions should be clear.

A *CW spectrum* is a spectrum of the form $E = \Omega^\infty D$ for some CW prespectrum D. The full subcategory of $\mathcal{S}$ consisting of CW spectra is written as $\mathcal{S}_{CW}$. Although the topological spaces E_n of a CW spectrum are obviously not CW complexes themselves, they do have the homotopy type of CW complexes.

Exercise 10.9.1 Show that $\Omega^\infty E \cong E$ in $\mathcal{S}$ for every spectrum E.

Topology Exercise 10.9.2 If D is a CW prespectrum, show that the structure maps $D_n \to \Omega D_{n+1}$ are closed embeddings. Use this to show that

$$\pi_n(\Omega^\infty D) = \operatorname*{colim}_{i\to\infty} \pi_{n+i}(D_{n+i}).$$

Analogy 10.9.7 There is a formal analogy between the theory of spectra and the theory of (chain complexes of) sheaves. The analogue of a presheaf is a prespectrum. Just as the forgetful functor from sheaves to presheaves has a left adjoint (sheafification), the forgetful functor from spectra to prespectra has Ω^∞ as its left adjoint. The reader is referred to the Appendix of [LMS] for the extension of Ω^∞ to general spectra, as well as the verification that Ω^∞ is indeed the left adjoint of the forgetful functor.

Just as many standard operations on sheaves (inverse image, direct sum, cokernels) are defined by sheafification, many standard operations on spectra (cylinders, wedges, mapping cones) are defined on spectra by applying Ω^∞ to the corresponding operation on prespectra. This is not surprising, since both are right adjoint functors and therefore must preserve coproducts and colimits by 2.6.10.

Example 10.9.8 (Coproduct) Recall that the coproduct in the category of based topological spaces is the *wedge* $\vee_\alpha X_\alpha$, obtained from the disjoint union by identifying the basepoints. If $\{D_\alpha\}$ is a family of prespectra, their wedge is the prespectrum whose n^{th} space is $(\vee D_\alpha)_n = \vee(D_\alpha)_n$; it is the coproduct in the category of prespectra. (Why?) Since Ω^∞ preserves coproducts, $\vee D_\alpha = \Omega^\infty\{\vee(D_\alpha)_n\}$ is the coproduct in the category of spectra.

Example 10.9.9 (Suspension) The *suspension* SE of a spectrum E is Ω^∞ applied to the prespectrum whose n^{th} space is SE_n and whose structure maps are the suspensions of the structure maps $SE_n \to E_{n+1}$. Adams proves in [A, III.3.7] that the natural maps $E_n \to \Omega S(E_n)$ induce a weak homotopy equivalence $E \to \Omega SE$, and hence a weak homotopy equivalence

$$\Omega^{-1}E \xrightarrow{\sim} SE.$$

Definition 10.9.10 (Homotopy category) The *cylinder spectrum* $\mathrm{cyl}(E)$ of a spectrum E is obtained by applying Ω^∞ to the prespectrum $(I_+ \wedge E)_n = [0, 1] \times E_n/[0, 1] \times \{*\}$. Just as in ordinary topology, we say that two maps of spectra $f_0, f_1 \colon E \to F$ are *homotopic* if there is a map $h \colon \mathrm{cyl}(E) \to F$ such that the f_i are the composites $E \cong \{i\} \times E \hookrightarrow \mathrm{cyl}(E) \to F$. It is not hard to see that this is an equivalence relation (exercise!).

We write $[E, F]$ for the set of homotopy classes of maps of spectra; these form the morphisms of the homotopy category $\mathbf{K}(\mathcal{S})$ of spectra. The full subcategory of $\mathbf{K}(\mathcal{S})$ consisting of the CW spectra is written as $\mathbf{K}(\mathcal{S}_{CW})$.

Exercise 10.9.3 Show that $E \times F$ and $E \vee F$ are also the product and coproduct in $\mathbf{K}(\mathcal{S})$.

Proposition 10.9.11 $\mathbf{K}(\mathcal{S})$ *is an additive category.*

Proof Since $\mathbf{K}(\mathcal{S})$ has a zero object $*$ and a product $E \times F$, we need only show that it is an **Ab**-category (Appendix, A.4.1), that is, that every Hom-set $[E, F]$ has the structure of an abelian group in such a way that composition distributes over addition. The standard proof in topology that homotopy classes of maps into any loop space form an abelian group proves this; one splits $\mathrm{cyl}(F)$ into $[0, \frac{1}{2}] \times F/\sim$ and $[\frac{1}{2}, 1] \times F/\sim$ and concatenates loops. We leave the verification of this to readers familiar with the standard proof. $\diamond$

Corollary 10.9.12 *The natural map* $E \vee F \to E \times F$ *is an isomorphism in* $\mathbf{K}(\mathcal{S})$.

The role of CW spectra is based primarily upon the two following fundamental results.

Proposition 10.9.13 *For each spectrum E there is a natural weak homotopy equivalence $C \to E$, with C a CW spectrum. In particular, $\mathbf{K}(\mathcal{S}_{CW})$ is a localizing subcategory of $\mathbf{K}(\mathcal{S})$ in the sense of 10.3.12.*

Proof Let $\mathrm{Sing}(X)$ denote the singular simplicial set (8.2.4) of a topological space X, and $|\mathrm{Sing}(X)| \to X$ the natural map. Since $|\mathrm{Sing}(X)|$ is a CW complex, the cellular inclusions $S|\mathrm{Sing}(E_n)| \hookrightarrow |\mathrm{Sing}(SE_n)| \hookrightarrow |\mathrm{Sing}(E_{n+1})|$ make $|\mathrm{Sing}(E)|$ into a CW prespectrum and give us a function of prespectra $|\mathrm{Sing}(E)| \to E$. Taking adjoints gives a map of spectra $C \to E$, where $C = \Omega^\infty|\mathrm{Sing}(E)|$. Since $\pi_*|\mathrm{Sing}(X)| \cong \pi_*(X)$ for every topological space X, we have

$$\pi_i|\mathrm{Sing}(E_m)| \cong \pi_i(E_m) \cong \pi_{i+1}(E_{m+1}) \cong \pi_{i+1}|\mathrm{Sing}(E_{m+1})|$$

for all m and i. Since $\pi_n(C) \cong \mathrm{colim}_{i\to\infty}\, \pi_{n+i}(|\mathrm{Sing}(E_{n+i})|$ by the topology exercise 10.9.2, it follows that $C \to E$ is a weak homotopy equivalence. $\diamond$

Whitehead's Theorem 10.9.14

1. *If C is a CW spectrum, then for every weak homotopy equivalence $f\colon E \to F$ of spectra (10.9.4) we have $f_*: [C, E] \cong [C, F]$.*
2. *Every weak homotopy equivalence of CW spectra is a homotopy equivalence (10.9.10), that is, an isomorphism in $\mathbf{K}(\mathcal{S})$.*

Proof See [A, pp.149–150] or [LMS, p.30]. Note that (1) implies (2), by setting $C = F$. $\diamond$

Corollary 10.9.15 *The stable homotopy category $\mathbf{D}(\mathcal{S})$ exists and is equivalent to the homotopy category of CW spectra*

$$\mathbf{D}(\mathcal{S}) \cong \mathbf{K}(\mathcal{S}_{CW}).$$

Proof The generalities on localizing subcategories in section 3 show that $\mathbf{D}(\mathcal{S}) \cong W^{-1}\mathbf{K}(\mathcal{S}_{CW})$. But by Whitehead's Theorem we have $\mathbf{K}(\mathcal{S}_{CW}) = W^{-1}\mathbf{K}(\mathcal{S}_{CW})$. $\diamond$

We are going to show in 10.9.18 that the topological derived category $\mathbf{D}(\mathcal{S}) \cong \mathbf{K}(\mathcal{S}_{CW})$ is a triangulated category in the sense of 10.2.1. For this we need to define exact triangles. The exact triangles will be the cofibration sequences, a term that we must now define. In order to avoid explaining a technical hypothesis ("cofibrant") we shall restrict our attention to CW spectra.

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Mapping Cones 10.9.16 Suppose that $u: E \to F$ is a map of spectra. The sequence of topological mapping cones $\mathrm{cone}(u_n) = \mathrm{cone}(E_n) \cup_u F_n$ form a prespectrum (why?), and the *mapping cone of* f is defined to be the spectrum $\Omega^\infty\{\mathrm{cone}(f_n)\}$. Applying Ω^∞ to the prespectrum functions $i_n: F_n \to \mathrm{cone}(f_n)$ and $\mathrm{cone}(f_n) \to SE_n$ give maps of spectra $i: F \to \mathrm{cone}(f)$ and $j:\mathrm{cone}(f) \to SE$. The triangle determined by this data is called the *Puppe sequence* associated to f:

$$E \xrightarrow{u} F \xrightarrow{i} \mathrm{cone}(u) \xrightarrow{j} SE.$$

A *cofibration sequence* in $\mathbf{K}(\mathcal{S}_{CW})$ is any triangle isomorphic to a Puppe sequence. Since $* \to E \xrightarrow{\mathrm{id}} E \to *$ is a Puppe sequence, the following elementary exercise shows that cofibration sequences satisfy axioms (TR1) and (TR2).

Exercise 10.9.4 (Rotation) Use the fact that SE_n is homotopy equivalent to the cone of $i_n: F_n \to \mathrm{cone}(f_n)$ to show that $SE \cong \mathrm{cone}(i)$. Then show that

$$F \xrightarrow{i} \mathrm{cone}(u) \xrightarrow{j} SE \xrightarrow{-Su} SF$$

is a cofibration sequence.

We say that a diagram of spectra is *homotopy commutative* if it commutes in the homotopy category $\mathbf{K}(\mathcal{S})$.

Proposition 10.9.17 *Every homotopy commutative square of spectra*

$$\begin{array}{ccc} E & \xrightarrow{u} & F \\ \downarrow f & & \downarrow g \\ E' & \xrightarrow{u'} & F' \end{array}$$

can be made to commute. That is, there is a homotopy commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{u} & F \\ \| & & \downarrow \simeq \\ E & \longrightarrow & \mathrm{cyl}(u) \\ \downarrow f & & \downarrow g' \\ E' & \xrightarrow{u'} & F' \end{array}$$

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in which the bottom square strictly commutes in $\mathcal{S}$ and the map $\simeq$ is a homotopy equivalence.

Proof Let $\mathrm{cyl}(u_n)$ denote the topological mapping cylinder of u_n (Chapter 1, section 5). The mapping cylinder spectrum $\mathrm{cyl}(u)$ is Ω^∞ of the prespectrum $\{\mathrm{cyl}(u_n)\}$. It is homotopy equivalent to F because the homotopy equivalences $F_n \xrightarrow{\simeq} \mathrm{cyl}(u_n)$ are canonical. The map $\mathrm{cyl}(E) \to F'$ expressing the homotopy commutativity of the square corresponds to a prespectrum function from $\{\mathrm{cyl}(E_n)\}$ to F'; together with g they define a prespectrum function from $\{\mathrm{cyl}(u_n)\}$ to F' and hence a spectrum map $g'\colon \mathrm{cyl}(u) \to F'$. The inclusions of E_n into the top of $\mathrm{cyl}(u_n)$ give the middle row after applying Ω^∞. It is now a straightforward exercise to check that the diagram homotopy commutes and that the bottom square commutes. $\diamond$

Theorem 10.9.18 $\mathbf{K}(\mathcal{S}_{CW})$ *is a triangulated category.*

Proof We have already seen that axioms (TR1) and (TR2) hold. For (TR3) we may suppose that $C = \mathrm{cone}(u)$ and $C' = \mathrm{cone}(u')$ and that $gu = u'f$ in $\mathcal{S}$; the map h is given by the naturality of the mapping cone construction.

It remains to check the octahedral axiom (TR4). For this we may assume that the given triangles are Puppe sequences, that is, that $C' = \mathrm{cone}(u)$, $A' = \mathrm{cone}(v)$, and $B' = \mathrm{cone}(vu)$. We shall mimic the proof in 10.2.4 that the octahedral axiom holds in $\mathbf{K}(\mathcal{A})$. Define a prespectrum function $\{f_n\}$ from $\{\mathrm{cone}(u_n)\}$ to $\{\mathrm{cone}(v_n u_n)\}$ by letting f_n be the identity on $\mathrm{cone}(A_n)$ and v_n on B_n. Define a prespectrum function $\{g_n\}$ from $\{\mathrm{cone}(v_n u_n)\}$ to $\{\mathrm{cone}(v_n)\}$ by letting g_n be $\mathrm{cone}(u_n)\colon \mathrm{cone}(A_n) \to \mathrm{cone}(B_n)$ and the identity on C. Manifestly, these are prespectrum functions; we define f and g by applying Ω^∞ to $\{f_n\}$ and $\{g_n\}$. Since it is true at the prespectrum level, ∂ is the composite $\mathrm{cone}(u) \xrightarrow{f} \mathrm{cone}(vu) \xrightarrow{\delta} SA$ and x is the composite $C \xrightarrow{y} \mathrm{cone}(vu) \xrightarrow{g} \mathrm{cone}(v)$. (Check this!)

Since $\mathrm{cone}(f_n)$ is a quotient of the disjoint union of $\mathrm{cone}(\mathrm{cone}(A_n))$, $\mathrm{cone}(B_n)$, and C_n, the natural maps from $\mathrm{cone}(B_n)$ and C_n to $\mathrm{cone}(f_n)$ induce an injection $\mathrm{cone}(v_n) \hookrightarrow \mathrm{cone}(f_n)$. As n varies, this forms a function of prespectra. Applying Ω^∞ gives a natural map of spectra $\gamma\colon \mathrm{cone}(v) \to \mathrm{cone}(f)$ such that the following diagram of spectra commutes in $\mathcal{S}$:

$$\begin{array}{ccccccc}
C' & \xrightarrow{f} & B' & \xrightarrow{g} & \mathrm{cone}(v) & \xrightarrow{(Tj)i} & SC' \\
\| & & \| & & \downarrow\gamma & & \| \\
C' & \xrightarrow{f} & B' & \longrightarrow & \mathrm{cone}(f) & \longrightarrow & SC'.
\end{array}$$

To see that γ is a homotopy equivalence, define $\varphi_n\colon \mathrm{cone}(f_n) \to \mathrm{cone}(v_n)$ by sending $\mathrm{cone}(B_n)$ and C_n to themselves via the identity, and composing the natural retract $\mathrm{cone}(\mathrm{cone}(A_n)) \to \mathrm{cone}(0 \times A_n)$ with $\mathrm{cone}(u_n)$: $\mathrm{cone}(A_n) \to \mathrm{cone}(B_n)$. Since the φ_n are natural, they form a function of prespectra; applying Ω^∞ gives a map of spectra φ: $\mathrm{cone}(f) \to \mathrm{cone}(v)$. We leave it to the reader to check that $\varphi\gamma$ is the identity on $\mathrm{cone}(v)$ and that $\gamma\varphi$ is homotopic to the identity map on $\mathrm{cone}(f)$. (Exercise!). This shows that $(f, g, (Tj)i)$ is a cofibration sequence (exact triangle), because it is isomorphic to the Puppe sequence of f. $\diamond$

Geometric Realization 10.9.19 By the Dold-Kan correspondence (8.4.1), there is a geometric realization functor from $\mathbf{Ch}(\mathbf{Ab})$ to $\mathcal{S}_{CW}$. Indeed, if A is a chain complex of abelian groups, then the good truncation $\tau A = \tau_{\geq 0}(A)$ corresponds to a simplicial abelian group, and its realization $|\tau A|$ is a CW complex. In the sequence

$$\tau A \longrightarrow \tau\,\mathrm{cone}(A) \xrightarrow{\delta} \tau(A[-1]),$$

the map δ is a Kan fibration (8.2.9, exercise 8.2.5). Since the mapping cone is contractible (exercise 1.5.1), there is a weak homotopy equivalence $|\tau A| \to \Omega|\tau A[-1]|$, and its adjoint $S|\tau A| \to |\tau A[-1]|$ is a cellular inclusion. (Check this!) Thus the sequence of spaces $|\tau A[-n]|$ form a CW prespectrum; applying Ω^∞ gives a spectrum. This construction makes it clear that the functor $|\tau|\colon \mathbf{Ch}(\mathbf{Ab}) \to \mathcal{S}_{CW}$ sends quasi-isomorphisms to weak equivalences and sends the translated chain complex $A[n]$ to $\Omega^n|\tau A|$. In particular, it induces a functor on the localized categories $|\tau|\colon \mathbf{D}(\mathbf{Ab}) \to \mathbf{D}(\mathcal{S})$.

Vista 10.9.20 Let $H\mathbb{Z}$ denote the geometric realization $|\tau\mathbb{Z}|$ of the abelian group $\mathbb{Z}$, regarded as a chain complex concentrated in degree zero. It turns out that $H\mathbb{Z}$ is a "ring spectrum" and that $\mathbf{D}(\mathbf{Ab})$ is equivalent to the stable category of "module spectra" over $H\mathbb{Z}$. This equivalence takes the total tensor product $\otimes^{\mathbf{L}}_{\mathbb{Z}}$ in $\mathbf{D}(\mathbf{Ab})$ to smash products of module spectra over $H\mathbb{Z}$. See [Rob] and {A. Elmendorf, I. Kriz, and J. P. May, "E_∞ Modules Over E_∞ Ring Spectra," preprint (1993)}.