C2.7: Category Theory

Sheet 1 — MT23

Sheet 1 Section B will be marked. Each problem sheet has the following structure:

- **Section A**: one or more introductory questions, with solutions provided for students. The solutions will be provided at the end of the week to allow the students to try to solve the questions independently.
- **Section B**: core questions to be handed in for marking (Sheets 1 and 3), usually not with solutions for students, and to be discussed in classes
- **Section C**: one or more optional extension questions (which might, but need not, be harder than those in Section B), with sketches of solutions/references provided for students.

Section A

1. (a) Let $(X, \leq)$ be a poset (i.e. a partially ordered set). Show that one can make it into a category such that for any $x_1, x_2 \in X$ the set $\mathrm{Hom}(x_1, x_2)$ has a single element if $x_1 \leq x_2$ and is empty otherwise.

   (b) Let $[n]$ be the category associated to the poset of integers $\{0, 1, ..., n\}$ with the usual ordering. Given another category $\mathcal{C}$ describe the functor category $\mathrm{Fun}([n], \mathcal{C})$.

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Section B

2. Given a group G consider the category $*/G$ with one object $*$ and $\mathrm{Hom}(*, *) = G$.
   (a) How can we describe functors $*/G_1 \to */G_2$ and natural transformations between such functors in group-theoretic terms?
   (b) Describe functors $*/\mathbb{Z} \to */G$. What are natural transformations between two such functors (for fixed G)?

3. Recall that an equivalence between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms relating their compositions to identity functors on $\mathcal{C}$ and $\mathcal{D}$.
   Show that the linear duality functor $\mathrm{Hom}_{\mathrm{Vect}_k^{fd}}(-, k)$ defines an equivalence
   $$\mathrm{Vect}_k^{fd} \to (\mathrm{Vect}_k^{fd})^{op}$$
   from the category Vect_k^{fd} of finite-dimensional vector spaces over a field k to its opposite.

4. Recall that a groupoid is a category in which every morphism is invertible. A contractible category is a groupoid $\mathcal{C}$ such that any two objects of $\mathcal{C}$ are isomorphic and $\mathrm{End}(c)$ is the trivial group for some $c \in \mathcal{C}$. Show that for any two objects x, y of a contractible category $\mathcal{C}$ the set $\mathrm{Hom}(x, y)$ consists of a single element, and that $\mathcal{C}$ is equivalent to a discrete category with a single object.

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Section C

5. Given a set of groups $\{G_i\}_{i\in I}$ indexed by I one has the category $\sqcup_{i\in I} */G_i$ with I its set of objects, endomorphisms of $i \in I$ given by G_i, and no other morphisms. Show that any (small) groupoid is equivalent to a category of this form.

6. Show that associating to a group its centre cannot be made into a functor Grp $\to$ Ab.

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