



Bayesian Combinatorial Auctions

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We study the following simple Bayesian auction setting: m items are sold to n selfish bidders in m independent second-price auctions. Each bidder has a *private* valuation function that specifies his or her complex preferences over *all* subsets of items. Bidders only have *beliefs* about the valuation functions of the other bidders, in the form of probability distributions. The objective is to allocate the items to the bidders in a way that provides a good approximation to the optimal social welfare value. We show that if bidders have submodular or, more generally, fractionally subadditive (aka XOS) valuation functions, every Bayes-Nash equilibrium of the resulting game provides a 2-approximation to the optimal social welfare. Moreover, we show that in the full-information game, a pure Nash always exists and can be found in time that is polynomial in both m and n .

CCS Concepts: • **Theory of computation** → **Quality of equilibria**; **Computational pricing and auctions**; Design and analysis of algorithms

Additional Key Words and Phrases: Mechanism design, combinatorial auctions, game theory, simultaneous item-bidding auctions

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1. INTRODUCTION

1.1. (Bayesian) Combinatorial Auctions with Item Bidding

Combinatorial auctions are fundamental to economic theory and have also been the subject of extensive research in computer science. In a *combinatorial auction*, m items $M = \{1, \dots, m\}$ are sold to n bidders $N = \{1, \dots, n\}$. Each bidder i has a *valuation function* (or valuation, in short) v_i that assigns a nonnegative real number to *every* subset of the items. v_i expresses i 's preferences over *bundles* of items. The value $v_i(S)$ can be thought of as specifying i 's maximum willingness to pay for S . Two standard assumptions are made regarding v_i : $v_i(\emptyset) = 0$ (normalization), and $v_i(S) \leq v_i(T)$ for every two bundles $S \subseteq T$ (monotonicity). The objective is to find a partition of the

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items among the bidders S_1, \dots, S_n (where $S_i \cap S_j = \emptyset$ for all $i \neq j$) such that the social welfare $\sum_i v_i(S_i)$ is maximized.

The interplay between selfishness and computation in combinatorial auctions is well studied. In particular, much work has been devoted to the design of mechanisms that are both truthful (i.e., incentivize bidders to report their private information) and computationally efficient. Unfortunately, in general, truthfulness and computational efficiency are at odds. The celebrated VCG mechanisms [Vickrey 1961; Clarke 1971; Groves 1973] optimize the social welfare and motivate agents to truthfully report their private information. The caveat is that this may take exponential time [Nisan and Ronen 1999, 2000] (in the natural parameters of the problem m and n). If we disregard incentives, nontruthful algorithms obtain good approximations to the optimal social welfare in polynomial time for restricted yet very expressive special cases of combinatorial auctions (e.g., combinatorial auctions in which bidders have *submodular* valuations, also called “combinatorial auctions with *submodular bidders*” [Lehmann et al. 2001; Dobzinski et al. 2005; Dobzinski and Schapira 2006; Feige 2006; Feige and Vondrak 2006; Vondrak 2008]). Unfortunately, no truthful mechanism can approximate the optimum social welfare in combinatorial auctions with submodular bidders within any constant factor [Dobzinski 2011], so it is natural to relax the equilibrium condition. Also, as seen in other contexts, for example, the famous Generalized Second-Price auction (GSP) for selling adwords, practitioners typically resort to nontruthful auction mechanisms.

We approach the central issue of reconciling selfishness and computation in auctions from an old-new perspective: Harsanyi [1967] introduced *Bayesian games* as an elegant way of modeling selfishness in partial-information settings. In a Bayesian game, players do not know exactly the private information of the other players, but only have *beliefs*, expressed by probability distributions over the different possible realizations of this private information. In combinatorial auctions, this translates to probability distributions over the possible valuation functions of the other bidders. We are interested in maximizing the social welfare in a way that is aligned with the interests of the different bidders. We ask the following question: can we design an auction for which *any* Bayesian Nash equilibrium provides a good approximation to the optimal social welfare? We thus seek a Bayesian analog of the *price of anarchy* [Koutsoupias and Papadimitriou 1999; Roughgarden and Tardos 2002] in this context.¹

Of much theoretical and practical interest are combinatorial auctions with item bidding, where the auctioneer sells the items by simultaneously running m independent single-item auctions. Inspired by auctions on eBay, we investigate the simple auction setting in which m items are sold in m independent second-price auctions, and each bidder can participate in any number of these single-item auctions. This auction setting induces a game in which a bidder’s strategy is the m -dimensional vector of bids he or she submits in the different single-item auctions, and his or her payoff is his or her value for the set of items he or she is allocated minus his or her payments. Unfortunately, some unnatural complications arise: consider the scenario that $m = 1$, $n = 2$, and the two bidders have complete information about each other. Let $v_1(1) = 1$ and $v_2(1) = 0$. Observe that the optimal social welfare is 1 (assign item 1 to bidder 1). Also observe that if bidder 1 bids 0 and bidder 2 bids 1, then this is a pure Nash equilibrium with a social welfare value of 0. Hence, the price of anarchy of this full-information game, that is, the ratio between the optimal solution and worst-case social welfare in equilibrium, is unbounded.

¹See Garg and Narahari [2005] and Gairing et al. [2005] for a similar approach to ours in selfish routing problems.

Observe that in the previous scenario, bidder 2 bid for (and got) an item he or she was not interested in possessing. We argue that such situations are unlikely to occur in practice, especially if bidders are only partially informed and are thus more inclined to avoid risks. We therefore make the assumption that bidders “play it safe,” in the sense that a bidder will not submit bids that might (in some scenario) result in getting a negative payoff. We call this the *no-overbidding* assumption. A strategy of bidder i is a *bid-vector* $b_i = (b_i(1), b_i(2), \dots, b_i(m))$, where $b_i(j)$ represents i ’s bid for item j . A bid-vector is called *nonoverbidding* if the sum of bids over any set of items S does not exceed the value $v_i(S)$ of this set (see Definition 2.1).

1.2. Our Contributions

(Bayesian) Price of Anarchy in Auctions. We initiate the study of (Bayesian) price of anarchy in auctions and, in particular, of combinatorial auctions with item bidding. Our main result establishes that when bidders’ valuations are *submodular*, that is, exhibit decreasing marginal utilities, *any* (mixed) Bayes-Nash equilibrium of this auction game provides a good approximation to the optimal social welfare. A bidder i is said to have a *submodular* valuation function if for all $S, T \subseteq M$ $v_i(S \cup T) + v_i(S \cap T) \leq v_i(S) + v_i(T)$. Equivalently (see, e.g., Lehmann et al. [2001]), v_i is submodular if for every two bundles $S \subseteq T$ that do not contain an item j , it holds that

$$v_i(S \cup \{j\}) - v_i(S) \geq v_i(T \cup \{j\}) - v_i(T).$$

We present the following result:

THEOREM: Under no-overbidding, and when all valuations are submodular, any (mixed) Bayes-Nash equilibrium of the auction game approximates the optimum social welfare within a factor of 2.

Our proof of the previous result combines several new ideas and is interesting in its own right. In particular, the proof involves the first application of *smoothness* [Roughgarden 2012a] to games with incomplete information [Roughgarden 2012b].

Moreover, this result holds for the strictly broader class “fractionally subadditive valuations” [Feige 2006] (defined, and termed XOS, in Nisan [2000]). Importantly, our result is independent of the bidders’ beliefs; that is, the 2-approximation ratio is guaranteed for *any* common probability distribution (“common prior”) over the valuation functions (we do require the common prior to be the product of independent probability distributions). Our approach thus suggests a middle ground between the classical economic and the standard computer science approaches: works in economics typically assume that the “input” is drawn from some *specific* probability distribution and prove results that apply to that specific distribution. In contrast, computer scientists typically prefer worst-case analysis.

Existence and computability of pure Nash equilibria. We study the scenario of interest in which the valuation function of each bidder is known to all other bidders (or, equivalently, the common prior selects a single valuation profile with probability 1). We show that such full-information games always possess a *pure* Nash equilibrium even if the no-overbidding assumption does not hold. In fact, a simple argument establishes that the socially optimal allocation of items to bidders is achievable in pure equilibrium. Hence, while the price of anarchy in these full-information games is unbounded (without the no-overbidding assumption), the *price of stability* (the social welfare of the *best* Nash equilibrium relative to the optimum [Anshelevich et al. 2004]) is 1.

Optimizing the social welfare in combinatorial auctions with submodular bidders is NP-hard [Lehmann et al. 2001]. Can a pure Nash equilibrium that provides a good

approximation to the optimal social welfare be computed in polynomial time? We give the following answer for submodular bidders:

THEOREM: *When bidders have submodular valuations, a pure Nash equilibrium of a full-information game that approximates the optimal social welfare within a ratio of 2 can be computed in polynomial time.*

The proof of this theorem shows that the 2-approximation algorithm for maximizing social welfare in combinatorial auctions with submodular bidders, due to Lehmann et al. [2001], can be used to compute the bids in a pure Nash equilibrium. We note that similar questions have been studied by Vetta in [2002].²

We provide, for the wider class of fractionally subadditive valuation functions, a constructive way of finding a pure Nash that yields a 2-approximation via a simple and natural myopic procedure (inspired by the greedy approximation algorithm in Dobzinski et al. [2005]). Unfortunately, this myopic procedure does not, in general, compute a pure Nash equilibrium in polynomial time for fractionally subadditive valuations. Indeed, we show that the myopic procedure may take exponential time by exhibiting a nontrivial construction of an instance on which this can occur.

1.3. Follow-Up Work

Since the publication of the conference version of this article [Christodoulou et al. 2008], there has been a fair amount of follow-up work on studying the price of anarchy of Bayes-Nash equilibria, and on combinatorial auctions with item bidding. Next, we briefly discuss the most relevant references.

Bhawalkar and Roughgarden [2011] and Feldman et al. [2013] study the price of anarchy of *simultaneous second-price auctions* for bidders with subadditive valuation functions. In Bhawalkar and Roughgarden [2011], it is shown that the price of anarchy for pure Nash equilibria is 2, but it is strictly greater than 2 for Bayes-Nash equilibria. Fu et al. [2012] extended this bound for general valuation functions. Bhawalkar and Roughgarden [2011] coined the term “combinatorial auctions with item bidding,” which refers to settings like ours in which multiple items are sold concurrently in single-item auctions. Feldman et al. [2013] prove upper and lower bounds on the price of anarchy for Bayes-Nash equilibria and present the best upper and lower bounds to date: 2.061 and 4, respectively.

The Bayesian price of anarchy of *simultaneous first-price auctions* was studied in Hassidim et al. [2011], Syrgkanis and Tardos [2013], Feldman et al. [2013], and Christodoulou et al. [2013]. Hassidim et al. [2011] investigated the existence of pure and mixed equilibria in such auctions and also studied the Bayesian price of anarchy in this context. They showed that pure Nash equilibria are always efficient (when they exist), and they also are upper bounds of 4, $O(\log m)$, and $O(m)$ for the price of anarchy of Bayes-Nash equilibria with XOS, subadditive, and general valuations, respectively. They also presented superconstant lower bounds for auctions with superadditive valuations. Syrgkanis and Tardos [2013] and Feldman et al. [2013] established upper bounds of $e/(e - 1)$ for XOS valuations and 2 for subadditive valuations, respectively. Recently, Christodoulou et al. [2013] showed tight lower bounds for both cases.

Paes Leme and Tardos [2010] initiated the study of the price of anarchy of *generalized second-price auctions*. This work was followed in Caragiannis et al. [2011, 2012]

²Vetta [2002] considers a general setting in which decisions are made by noncooperative agents, and the utility functions are submodular. He proves (among others) that in this setting, the price of anarchy is at most 2. The framework discussed there assumes players whose pure strategies are modeled by certain *subsets* of a ground set (e.g., the items). Apparently, even in the full-information and submodular case, this framework is not applicable to our auction, where the bids on the items play a crucial role.

for the full information and for the Bayesian setting and even for correlated distributions [Lucier and Paes Leme 2011]. Borodin and Lucier [2010] study the Bayesian price of anarchy of greedy mechanisms, and Roughgarden [2012b] explores methods to bound the price of anarchy in games with incomplete information.

Markakis and Telelis [2012] studied the inefficiency of uniform price *multiunit* auctions. De Keijzer et al. [2013] showed bounds for the price of anarchy of Bayesian equilibria for several formats of multiunit auctions with first or second pricing schemes. Bhawalkar and Roughgarden [2012] study the effect of the price rule of a single item in the equilibrium performance of simultaneous auctions. Alon et al. [2012] investigate the price of anarchy in network *cost sharing* games in a Bayesian context.

2. BAYESIAN PRICE OF ANARCHY

In Section 2.1, we present the setting explored in this article. Section 2.2 presents our Bayesian price of anarchy result for pure and mixed Bayes-Nash equilibria. The general proof for mixed equilibria can be found in Appendix 3.2.

2.1. Bayesian Combinatorial Auctions with Item Bidding

The auction. m items are sold to n bidders in m independent second-price auctions (with some tie-breaking rule). A bidder's strategy is a *bid-vector* $b_i \in \mathbb{R}_{\geq 0}^m$ ($b_i(j)$ represents i 's bid for item j). A (*pure*) *strategy profile* of all players is an n -tuple $b = (b_1, \dots, b_n)$. We will use the notation $b = (b_i, b_{-i})$ to denote a strategy profile in which bidder i bids b_i and other bidders bid as in $b_{-i} = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_n)$. We use $[j]$ to denote the set of integers from 1 to j .

Given a strategy profile b , the items are allocated according to the second-price rule; that is, every object is sold to the highest bidder at a price equal to the second-highest bid.

For any fixed b , we denote by $X_i(b)$ the set of items obtained by player i in the auction. For a set $S \subseteq M$, let the sum of the highest bids be denoted by

$$Bids(S, b) = \sum_{j \in S} \max_k b_k(j),$$

$$Bids(S, b_{-i}) = \sum_{j \in S} \max_{k \neq i} b_k(j),$$

and

$$Bids(S, b_i) = \sum_{j \in S} b_i(j).$$

The *utility* (payoff) of player i is then given by

$$u_i(b) = v_i(X_i(b)) - Bids(X_i(b), b_{-i}).$$

We make two assumptions about the bidders: no overbidding, and that the v_i s are *fractionally subadditive*. A valuation is fractionally subadditive if it is the pointwise maximum of a set of additive valuations: a valuation a is additive if for every $S \subseteq M$, $a(S) = \sum_{j \in S} a(\{j\})$. A valuation v_i is fractionally subadditive if there are additive valuations $A_i = \{a_1, \dots, a_l\}$ such that for every $S \subseteq M$, $v_i(S) = \max_{a \in A} a(S)$. (We will call $a_k \in A$ a *maximizing additive valuation* for the set S if $v_i(S) = a_k(S)$.)³

³In order to have a finite (discrete) model, we can restrict acceptable bids to multiples of an arbitrary fixed ϵ , and not to exceed some maximum value B_{max} . In this case, either we have to make the additional assumption that all $a(j)$ in the additive valuations of the sets A_i adhere to the same restrictions, or admit implicit ϵ errors in the derived bounds on the price of anarchy.

The class of fractionally subadditive valuations is known to be strictly contained in the class of subadditive valuations and to strictly contain the class of submodular valuations [Nisan 2000; Lehmann et al. 2001].

Definition 2.1. A bid vector b_i is said to be *nonoverbidding* given a valuation v_i if for all $S \subseteq M$, $v_i(S) \geq \sum_{j \in S} b_i(j)$.

Bayes-Nash equilibria. For all i , let V_i denote the finite set of possible valuations of player i . The set of possible valuation profiles of the players is then $V = V_1 \times \dots \times V_n$. There is a *known* probability distribution D over the valuations V (a *common prior*). D can be regarded as some market statistics that are known to all bidders (and to the auctioneer) and that specify their beliefs. We assume that $D = D_1 \times \dots \times D_n$ is the Cartesian product of independent probability distributions D_i : any valuation profile $v = (v_1, \dots, v_n)$ occurs with probability $D(v) = \prod_{i=1}^n D_i(v_i)$, where $D_i(v_i)$ is the probability that bidder i has the valuation function v_i . We will use the short notation $V_{-i} = \times_{k \neq i} V_k$, $D_{-i} = \times_{k \neq i} D_k$, and $v_{-i} = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n)$.

A *bidding-function* B_i for player i is a function that assigns a bid-vector $b_i = B_i(v_i)$ to every valuation function $v_i \in V_i$. The reader may find it helpful to think of B_i as a suggestion made to player i by the auctioneer as to which bid to submit. An n -tuple of bidding-functions $B = (B_1, \dots, B_n)$ is a *Bayes-Nash equilibrium* if for every $i \in N$ and for every valuation function v_i , the bid $B_i(v_i)$ maximizes i 's expected utility given that his or her valuation function is v_i , and that the bid of every other bidder j is $B_j(v_j)$, where v_j is drawn from D_j . That is, a Bayes-Nash maximizes i 's expected payoff for any valuation function he or she may have, given his or her beliefs about the other bidders.

Bayesian price of anarchy. For a fixed valuation profile of the bidders $v = (v_1, \dots, v_n)$, the optimal social welfare is $OPT(v) = \max_{S_1, \dots, S_n} \sum_i v_i(S_i)$, where the maximum is taken over all partitions of M into disjoint bundles S_1, \dots, S_n . For given D , the (expected) optimal social welfare $SW(OPT)$ is the expectation $E[OPT(v)]$, where v is drawn from D . That is,

$$SW(OPT) = \sum_{v \in V} D(v) OPT(v).$$

Given a valuation profile v , every pure strategy profile b induces a social welfare value $SW(b) = \sum_{i \in N} v_i(X_i(b))$. For an n -tuple of bidding-functions $B = (B_1, \dots, B_n)$, we denote by $SW(B)$ the expected social welfare $E[SW(B_1(v_1), \dots, B_n(v_n))]$, where the $v = (v_1, \dots, v_n)$ is drawn from D :

$$SW(B) = \sum_{v \in V} D(v) SW(B(v)).$$

We are interested in Bayes-Nash equilibria B for which the ratio $\frac{SW(OPT)}{SW(B)}$ is small. The *Bayesian price of anarchy* of a game is

$$PoA = \max_{D, B \text{ Bayes-Nash}} \frac{SW(OPT)}{SW(B)},$$

that is, the maximum of the expression $\frac{SW(OPT)}{SW(B)}$, taken over *all* probability distributions D , and *all* Bayes-Nash equilibria B (for these probability distributions). Intuitively, Bayesian price of anarchy of α means that no matter what the bidders' beliefs are, every Bayes-Nash equilibrium provides an α -approximation to the optimal social welfare.

2.2. Bayesian Price of Anarchy of 2

This subsection exhibits our main result. For the sake of readability, we prove the theorem regarding the Bayesian price of anarchy for *pure* Bayes-Nash equilibria. The

extension of the theorem to *mixed* Bayes-Nash equilibria is presented at the end of the section. The proof exploits the fractional subadditivity of the valuations via the following lemma:

LEMMA 2.2. *Let S be a set of items, and a be a maximizing additive valuation of player i for this set. If i bids according to a for the elements of S (and 0 for other elements) while all the others bid according to any pure profile b_{-i} , then*

$$u_i(a, b_{-i}) \geq v_i(S) - \text{Bids}(S, b_{-i}).$$

PROOF. Let $X_i := X_i(a, b_{-i})$ be the set of items that player i is going to get. Note that if i wins any item $j \notin S$, then the maximum bid on this j was 0. Thus, we can assume w.l.o.g. that $X_i \subseteq S$ (otherwise, the proof holds with $X_i \cap S$ instead of X_i). Moreover, $a(j) - \text{Bids}(\{j\}, b_{-i}) \leq 0$ holds for every nonobtained item $j \in S - X_i$. Therefore, we have

$$\begin{aligned} u_i(a, b_{-i}) &= v_i(X_i) - \text{Bids}(X_i, b_{-i}) \\ &\geq \sum_{j \in X_i} a(j) - \text{Bids}(X_i, b_{-i}) \\ &\geq \sum_{j \in S} a(j) - \text{Bids}(S, b_{-i}) \\ &= v_i(S) - \text{Bids}(S, b_{-i}). \end{aligned}$$

The first inequality uses the definition of $v_i(X_i)$ as the maximizing additive valuation for X_i , whereas a is one of the additive valuations. The second inequality follows from the inequality for nonobtained items. \square

THEOREM 2.3. *Let D be a distribution over fractionally subadditive valuations of the bidders. If $B = (B_1, \dots, B_n)$ is a Bayes-Nash, such that each B_i maps every valuation function v_i to a nonoverbidding bid (given v_i), then $\frac{\text{SW}(\text{OPT})}{\text{SW}(B)} \leq 2$.*

PROOF. Let $v = (v_1, \dots, v_n)$ be a fixed valuation profile. We denote by $O^v = (O_1^v, \dots, O_n^v)$ the optimum allocation with respect to profile v .

Now for every player i , let a_i denote the maximizing additive valuation for the set O_i^v , (we set $a_i(j) = 0$ if $j \notin O_i^v$). For all i , we consider a_i as an alternative strategy to $B_i(v_i)$.

Let us fix a bidder i . Let w_{-i} be an arbitrary valuation profile of all bidders except for i . We introduce the short notation

$$X_i^{w_{-i}} \stackrel{\text{def}}{=} X_i(B_i(v_i), B_{-i}(w_{-i})).$$

Furthermore, for any $S \subseteq M$, we will use

$$\text{Bids}^{w_{-i}}(S) \stackrel{\text{def}}{=} \text{Bids}(S, B_{-i}(w_{-i})),$$

respectively

$$\text{Bids}^w(S) \stackrel{\text{def}}{=} \text{Bids}(S, B(w)),$$

where $w = (w_i, w_{-i})$ is a complete valuation profile.

Since B is a Bayes-Nash, the strategy $B_i(v_i)$ provides higher expected utility to player i than the strategy a_i :

$$\sum_{w_{-i} \in V_{-i}} D(w_{-i}) u_i(B_i(v_i), B_{-i}(w_{-i})) \geq \sum_{w_{-i} \in V_{-i}} D(w_{-i}) u_i(a_i, B_{-i}(w_{-i})).$$

The utility values on the left-hand side are

$$u_i(B_i(v_i), B_{-i}(w_{-i})) = v_i(X_i^{w_{-i}}) - \text{Bids}^{w_{-i}}(X_i^{w_{-i}}) \leq v_i(X_i^{w_{-i}}).$$

On the right-hand side, applying Lemma 2.2 yields

$$u_i(a_i, B_{-i}(w_{-i})) \geq v_i(O_i^v) - Bids^{w_{-i}}(O_i^v).$$

By merging the previous inequalities, we get

$$\begin{aligned} \sum_{w_{-i} \in V_{-i}} D(w_{-i}) v_i(X_i^{w_{-i}}) &\geq \sum_{w_{-i} \in V_{-i}} D(w_{-i}) [v_i(O_i^v) - Bids^{w_{-i}}(O_i^v)] \\ &= v_i n(O_i^v) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) - \sum_{w_{-i} \in V_{-i}} D(w_{-i}) Bids^{w_{-i}}(O_i^v) \\ &= v_i(O_i^v) \cdot 1 - \sum_{w \in V} D(w) Bids^{w_{-i}}(O_i^v) \\ &\geq v_i(O_i^v) - \sum_{w \in V} D(w) Bids^w(O_i^v). \end{aligned}$$

Here the expected highest bids $\sum_{w_{-i} \in V_{-i}} D(w_{-i}) Bids^{w_{-i}}(O_i^v)$ and $\sum_{w \in V} D(w) Bids^{w_{-i}}(O_i^v)$ are equal, because D is independent for all bidders. Finally, $Bids^{w_{-i}}(O_i^v) \leq Bids^w(O_i^v)$ obviously holds, since in the latter case we consider maximum bids over a larger set of players. We obtained

$$v_i(O_i^v) \leq \sum_{w_{-i} \in V_{-i}} D(w_{-i}) v_i(X_i^{w_{-i}}) + \sum_{w \in V} D(w) Bids^w(O_i^v).$$

We sum over all i and then take the expectation over all valuations $v = (v_1, \dots, v_n)$ on both sides:

$$\begin{aligned} \sum_{v \in V} D(v) \sum_{i \in N} v_i(O_i^v) &\leq \sum_{v \in V} D(v) \sum_{i \in N} \sum_{w_{-i} \in V_{-i}} D(w_{-i}) v_i(X_i^{w_{-i}}) \\ &\quad + \sum_{v \in V} D(v) \sum_{i \in N} \sum_{w \in V} D(w) Bids^w(O_i^v). \end{aligned}$$

Note that $\sum_{v \in V} D(v) \sum_{i \in N} v_i(O_i^v) = SW(OPT)$. Furthermore, we claim that both summands on the right-hand side are at most $SW(B)$, so that $SW(OPT) \leq 2SW(B)$, which will conclude the proof. The first summand solves to

$$\begin{aligned} &\sum_{i \in N} \sum_{v_i \in V_i} D(v_i) \sum_{v_{-i} \in V_{-i}} D(v_{-i}) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) v_i(X_i^{w_{-i}}) \\ &= \sum_{i \in N} \sum_{v_i \in V_i} D(v_i) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) v_i(X_i^{w_{-i}}) \sum_{v_{-i} \in V_{-i}} D(v_{-i}) \\ &= \sum_{i \in N} \sum_{v_i \in V_i} \sum_{w_{-i} \in V_{-i}} D(v_i) D(w_{-i}) v_i(X_i^{w_{-i}}) \cdot 1 \\ &= \sum_{i \in N} \sum_{v \in V} D(v) v_i(X_i^{v_{-i}}) \\ &= \sum_{v \in V} D(v) \sum_{i \in N} v_i(X_i(B(v))) = SW(B). \end{aligned}$$

Finally, the second summand is

$$\begin{aligned}
 \sum_{v \in V} D(v) \sum_{w \in V} D(w) \sum_{i \in N} Bids^w(O_i^v) &= \sum_{v \in V} D(v) \sum_{w \in V} D(w) Bids^w(M) \\
 &= \sum_{w \in V} D(w) Bids^w(M) \sum_{v \in V} D(v) \\
 &= \sum_{w \in V} D(w) Bids^w(M) \cdot 1 \\
 &= \sum_{w \in V} D(w) \sum_{i \in N} Bids(X_i(B(w)), B_i(w_i)) \\
 &\leq \sum_{w \in V} D(w) \sum_{i \in N} w_i(X_i(B(w))) = SW(B).
 \end{aligned}$$

The last inequality holds since for all i , the $B_i(w_i)$ contains nonoverbidding bids for any set of items including the obtained set $X_i(B(w))$. \square

The bound on the price of anarchy holds in general for *mixed* Bayes-Nash equilibria:

THEOREM 2.4. *Let D be a distribution on the valuations of the bidders. If $p = (p_1, \dots, p_n)$ is a mixed Bayesian Nash, such that for every i and every valuation v_i , the $p^{v_i} (= p_i^{v_i})$ is a probability distribution over nonoverbidding bids w.r.t. v_i ,⁴ then $\frac{SW(OPT)}{SW(p)} \leq 2$.*

The proof of the general case consists of a straightforward extension of all (in)equalities to expectations over the mixed bidding strategies. Appendix 3.2 contains the proof.

A simple example shows that even in the full-information setting, this Bayesian price of anarchy result is tight:

Example 2.5. Consider the following example, with two items and two players. The first player values the objects $v_1(1) = v_1(\{1, 2\}) = 2$, and $v_1(2) = 1$, and symmetrically for the second player $v_2(2) = v_2(\{1, 2\}) = 2$, and $v_2(1) = 1$. In the optimum partition, the first player gets the first object and the second player gets the second object. This results in a social welfare of 4.

If the first player bids $b_1(2) = 1$, $b_1(1) = 0$, and the second $b_2(1) = 1$, $b_2(2) = 0$, then the first player will get the second object, while the second player will get the first object. This results in a social welfare of 2. Observe that b is a pure Nash equilibrium. Consequently, the price of anarchy is at least $\frac{4}{2} = 2$.

3. COMPUTING PURE NASH EQUILIBRIA

In this section, we consider the following full-information game: the m items are sold to n bidders with fractionally subadditive valuation functions in m independent second-price auctions. The players' valuation functions are assumed to be common knowledge.

In Section 3.1, we show that a pure Nash (with nonoverbidding bids) that provides a good approximation to the social welfare always exists in such games and provide a constructive way of finding one. We also prove that the price of stability [Anshelevich et al. 2004] is 1; that is, the optimum can always be achieved in a Nash equilibrium. Finally, we describe an example to demonstrate that with this procedure, it might take exponential time to find an equilibrium.

In Section 3.2, we show that if bidders have submodular valuation functions, then a good pure Nash can be reached in polynomial time.

⁴Technically, this condition can be realized so that $p^{v_i}(b_i) = 0$ whenever b_i is an *overbidding* bid.

3.1. Fractionally Subadditive Valuation Functions

Despite the fact that (as shown in the Introduction) some Nash equilibria may fail to provide good approximation to the social welfare, we present a constructive way for finding a pure Nash that yields a 2-approximation. We introduce a natural procedure we call **POTENTIAL PROCEDURE**, which always reaches such an equilibrium. The **POTENTIAL PROCEDURE** is a simple myopic procedure for fractionally subadditive bidders.⁵

For every i , let $A_i = \{a_1^i, \dots, a_m^i\}$ be a set of additive valuations such that for every $S \subseteq M$, $v_i(S) = \max_{a \in A_i} a(S)$. Since v_i is fractionally subadditive, such A_i must exist. The procedure simply starts with zero bids $b_i^*(j)$, zero per-item prices r_j , and an empty set of items $S_i = \emptyset$ allocated to each player i . Then it lets players best reply one by one to the bids of other players by switching to new nonoverbidding bids. After every round, the sets S_i of all agents form a partition (w.l.o.g.) of the item-set, and for every player i , the bids $b_i^*(j)$ on his or her “own” items from S_i are determined by a maximizing additive valuation from A_i for S_i and are zero for items not in S_i . The bids of *other* players we regard as current prices r_j on each item j for choosing a next best response.

In each round, an arbitrary dissatisfied player k selects a new set T that maximizes his or her utility given the current prices. Since prices are 0 over his or her own set S_k , it can be assumed w.l.o.g. that $T \supseteq S_k$ (otherwise, $T \cup S_k$ could be chosen), and also that the new bids of player k are never lower than the current price (otherwise, it would be better for him or her to choose a subset of T). This T becomes his or her new set; that is, he or she robs items from other players, which results in a new item-partition; subsequently, the bids and prices get updated as described earlier. We will prove that if after such a round all players are satisfied with their current sets, then the bids $(b_i^*)_{i \in N}$ correspond to a Nash equilibrium (even in the case when overbidding is allowed).

Note that after every round, the bids of *all* players need to be adjusted to sum up to the exact value of the current set of the player, and be zero elsewhere. In this sense, the procedure slightly differs from a pure best-response procedure but can nevertheless be understood as a natural and realistic distributed process that noncolluding agents could use to converge to an equilibrium.

Remark 3.1. This procedure requires bidder i to be able to determine which bundle he or she would prefer most, given a vector of per-item prices $r = (r_1, \dots, r_m)$, that is, to declare a bundle S for which $v_i(S) - \sum_{j \in S} r_j$ is maximized. This type of query is called a *demand query* and is very common in the combinatorial auctions literature.⁶ In addition, the agents must be capable of responding to XOS queries, in that for given S , they can choose their maximizing additive valuation over S .

POTENTIAL PROCEDURE:

- (1) Initialize $b_i^*(j) \leftarrow 0$, $S_i \leftarrow \emptyset$, $r_j \leftarrow 0$, for $i = 1, \dots, n$ and $j = 1, \dots, m$.
- (2) While there is a bidder k such that $S_k \neq \arg \max_{S \subseteq M} (v_k(S) - \sum_{j \in (S \setminus S_k)} r_j)$:
 - (a) Let $T = \arg \max_{S \subseteq M} (v_k(S) - \sum_{j \in (S \setminus S_k)} r_j)$; {w.l.o.g. $S_k \subseteq T$ }.
 - (b) Set $S_k \leftarrow T$.
 - (c) For all $i \neq k$, let $S_i \leftarrow S_i \setminus S_k$.
 - (d) For all $i \in N$, do
 - i. Let $a^i \in A_i$ be such that $v_i(S_i) = a^i(S_i)$.
 - ii. Set $b_i^*(j) \leftarrow a^i(j)$ and $r_j \leftarrow a^i(j)$ for all $j \in S_i$ and $b_i^*(j) \leftarrow 0$ for all $j \notin S_i$.
- (3) Output $b^* = (b_1^*, \dots, b_n^*)$.

We use a potential function argument and the fractional subadditivity of the bidders to show that the **POTENTIAL PROCEDURE** eventually converges to a “good” pure Nash.

⁵It can also be understood as a centralized local search procedure.

⁶See, for example, Blumrosen and Nisan [2005], Dobzinski et al. [2005], Feige [2006], and Feige and Vondrak [2006].

THEOREM 3.2. *If the valuation functions are fractionally subadditive, then the POTENTIAL PROCEDURE ends in a pure Nash equilibrium that provides a 2-approximation to the optimal social welfare.*

PROOF. We divide the proof into two claims: first, we show that the procedure terminates, and second, that the last bids b^* correspond to a Nash equilibrium.

CLAIM 1. *If the valuation functions are fractionally subadditive, then the POTENTIAL PROCEDURE terminates after a finite number of rounds.*

PROOF. We use a potential function argument. For this purpose, we use as a potential function the function that assigns to a partition of items $X = (X_1, \dots, X_n)$ the social welfare

$$SW(X) = \sum_{i \in N} v_i(X_i).$$

In particular, we are going to show that whenever there is a strict improvement to a player's utility, this translates into a strict improvement of the social welfare. This will prove that this procedure will terminate, as it converges to a local maximum of the social welfare. Later we will prove that any local maximum of the social welfare corresponds to a pure Nash equilibrium.

Let $S^t = (S_1^t, \dots, S_n^t)$ be the partition of the items and r_j^t the maximum bid for item j , after round $t = 1, 2, \dots$ of the POTENTIAL PROCEDURE. W.l.o.g. assume that at time step t , player k is chosen by the procedure to improve his or her utility, that is,

$$v_k(S_k^t) - \sum_{j \in S_k^t \setminus S_k^{t-1}} r_j^{t-1} > v_k(S_k^{t-1}),$$

where S_k^t and S_k^{t-1} play the roles of T and S_k , respectively, in step (2) of the procedure. For $i \neq k$, the prices r_j^{t-1} for items in S_i^{t-1} are determined by a maximizing additive valuation $a \in A_i$ for S_i^{t-1} . Therefore,

$$v_i(S_i^t) = v_i(S_i^{t-1} \setminus S_k^t) \geq \sum_{j \in S_i^{t-1} \setminus S_k^t} r_j^{t-1} = v_i(S_i^{t-1}) - \sum_{j \in S_k^t \cap S_i^{t-1}} r_j^{t-1}.$$

The inequality holds by fractional subadditivity, since the r_j^{t-1} are defined by some additive valuation from A_i , whereas $v_i(S_i^{t-1} \setminus S_k^t)$ is determined by the *maximizing* additive valuation for this set. Since the $(S_i^{t-1})_{i \in N}$ forms a partition, clearly $S_k^t \setminus S_k^{t-1} = \bigcup_{i \neq k} S_k^t \cap S_i^{t-1}$. Therefore, after summing up the previous inequalities over all players i (also for $i = k$), the prices cancel out and we obtain

$$\begin{aligned} \sum_{i \in N} v_i(S_i^t) &> v_k(S_k^{t-1}) + \sum_{j \in S_k^t \setminus S_k^{t-1}} r_j^{t-1} \\ &\quad + \sum_{i \neq k} \left(v_i(S_i^{t-1}) - \sum_{j \in S_k^t \cap S_i^{t-1}} r_j^{t-1} \right) \\ &= \sum_{i \in N} v_i(S_i^{t-1}) + \sum_{j \in S_k^t \setminus S_k^{t-1}} r_j^{t-1} - \sum_{i \neq k} \sum_{j \in S_k^t \cap S_i^{t-1}} r_j^{t-1} \\ &= \sum_{k \in N} v_k(S_k^{t-1}). \end{aligned}$$

Hence, the procedure will terminate, giving a partition that corresponds to a (local) maximum of the social welfare function. \square

Now we will prove that the strategy profile b^* resulting from the last partition of the POTENTIAL PROCEDURE is a Nash equilibrium.

CLAIM 2. *If the valuation functions are fractionally subadditive, then the POTENTIAL PROCEDURE terminates with an n -tuple of bids that is a Nash equilibrium.*

PROOF. By contradiction, let's suppose that b^* is not a pure Nash equilibrium; that is, there is a player i and a strategy b_i , such that $u_i(b_i, b_{-i}^*) > u_i(b^*) = v_i(S_i)$. Let $X_i = X_i(b_i, b_{-i}^*)$ be the corresponding bundle of items that player i gets if he or she bids b_i . Based on b_i , we are going to construct a nonoverbidding bid vector b'_i , which increases the utility of i at least as much as b_i does. We will show that b'_i corresponds to a set X'_i that would have been chosen by POTENTIAL PROCEDURE (X'_i or an even better set $T = \arg \max_{S \subseteq M} (v_i(S) - \sum_{j \in (S \setminus S_i)} r_j)$).

Let a be the maximizing additive valuation that corresponds to the set X_i . For every $j \notin X_i$, we assume that $b_i(j) = 0$ (we can decrease $b_i(j)$ to zero, without affecting player i 's utility).

Let O be the set of all items $j \in X_i$ for which $a(j) < r_j$, and let a' be the maximizing additive utility of the set $X_i \setminus O$. The utility that player i would get if he or she bid 0 for all items in O and a' on $X_i \setminus O$ would be

$$\begin{aligned}
 v_i(X_i \setminus O) - \sum_{j \in X_i \setminus O} r_j &= \sum_{j \in X_i \setminus O} (a'(j) - r_j) \\
 &\geq \sum_{j \in X_i \setminus O} (a(j) - r_j) \\
 &\geq \sum_{j \in X_i \setminus O} (a(j) - r_j) + \sum_{j \in O} (a(j) - r_j) \\
 &= \sum_{j \in X_i} (a(j) - r_j) \\
 &= v_i(X_i) - \sum_{j \in X_i} r_j \\
 &= u_i(b_i, b_{-i}^*) \\
 &> u_i(b^*) \\
 &= v_i(S_i).
 \end{aligned}$$

Therefore, the POTENTIAL PROCEDURE could choose the set $X'_i = X_i \setminus O$ or a set that increases even more the utility of player i , that is, $T = \arg \max_{S \subseteq M} v_i(S) - \sum_{j \in (S \setminus S_i)} r_j$. This contradicts the fact that the stop condition (line 2) of the POTENTIAL PROCEDURE has been satisfied. \square

The two claims and Theorem 2.3 imply that the POTENTIAL PROCEDURE results in a pure Nash equilibrium that is a 2-approximate solution.

Remark 3.3. Notice that by the second claim, the global optimum of the social welfare potential function is also a pure Nash. This means that for any optimum partition $S^* = (S_1^*, \dots, S_n^*)$, there always exist nonoverbidding bids that form a Nash equilibrium (every player bids according to the maximizing additive valuation with respect to his or her set S_i^*), and the price of stability is 1.

Table I. The Additive Valuations of Bidder 1, in the Order in Which This Bidder Responds in Even Steps of the POTENTIAL PROCEDURE

1	1'	2	2'	3	3'
1		1		1	
1		1			$1 + \epsilon$
1			$1 + \epsilon$		$1 + \epsilon$
1			$1 + \epsilon$	$1 + 2\epsilon$	
	$1 + \epsilon$		$1 + \epsilon$	$1 + 2\epsilon$	
	$1 + \epsilon$		$1 + \epsilon$		$1 + 3\epsilon$
	$1 + \epsilon$	$1 + 2\epsilon$			$1 + 3\epsilon$
	$1 + \epsilon$	$1 + 2\epsilon$		$1 + 4\epsilon$	

Thus, we have a natural procedure, which is essentially a myopic best-response sequence of the players, that leads to a pure Nash equilibrium. But how long will it take the POTENTIAL PROCEDURE to converge? A nontrivial construction shows that unfortunately, the worst-case running time is exponential in n and m .

THEOREM 3.4. *There is an instance with two bidders, each with a fractionally subadditive valuation function, on which the POTENTIAL PROCEDURE halts after $\Omega(2^m)$ steps.*

PROOF. Let $m = 2k$, and consider k pairs of items with the notation $M = \{1, 1', 2, 2', \dots, k, k'\}$. We call $P \subset M$ a *proper* set if ($|P| = k$ and) P contains exactly one item from each pair (e.g., the set $P = \{1, 2, \dots, k\}$ is proper). The set of all proper sets is denoted by Π .

We define fractionally subadditive valuations v_1 and v_2 for the two bidders, so that the POTENTIAL PROCEDURE should proceed as follows. Assume first that it starts from the partition $S_1 = \{1, 2, 3, \dots, k\}$ and $S_2 = \{1', 2', 3', \dots, k'\}$. In every two rounds, the bidders exchange a pair of items: after any even round, S_1 and S_2 are complementary proper sets. In the next (odd) round, bidder 2 “robs” an item $y \in S_1$, by changing to $T = S_2 \cup y$. In return, in the coming even round, bidder 1 takes the *pair* of the robbed item from bidder 2.

Bidder 1 (and also bidder 2) will possess each proper set exactly once. Since $|\Pi| = 2^k$, it takes $2 \cdot 2^k$ steps for the POTENTIAL PROCEDURE to converge. In order to go over every proper set systematically, the exchange of pairs follows the pattern of the Gray-code. This is a complete sequence of the k -long 0-1 vectors, such that every two consecutive vectors differ in exactly one bid. For example, the Gray-code for $k = 3$ is (000, 001, 011, 010, 110, 111, 101, 100).

The j th bid being 0 in the vector means that $j \in S_1$ and $j' \in S_2$; the j th bid being 1 means the opposite. When the bid in the code is changed, the items j and j' should be exchanged in the corresponding two steps of the procedure (observe in the code that item j is exchanged 2^{j-1} times).

Next, we define fractionally subadditive valuations v_1 and v_2 , for which the aforementioned sequence of exchanges is realized by the POTENTIAL PROCEDURE. The valuation v_1 consists of 2^k additive valuations $\{a_P\}_{P \in \Pi}$, one for each proper set. The rows in Table I show the additive valuations for $k = 3$, in the order in which these valuations and the respective proper sets appear as $a()$ and S_1 in even steps of the POTENTIAL PROCEDURE. Whenever an item x is in a proper set P , its value in the additive valuation a_P is $a_P(x) = 1 + s \cdot \epsilon$, where $\epsilon \ll 2^{-k}$, and the integer s indicates how many times this item was exchanged between the bidders so far.

Similarly, the valuation v_2 is the maximum of $2^k - 1$ additive valuations, as shown by Table II. The rows in the table follow the order of how bidder 2 responds in odd steps of the POTENTIAL PROCEDURE. Here the additive valuations take nonzero values

Table II. The Additive Valuations of Bidder 2, in the Order in Which This Bidder Responds in Odd Steps of the POTENTIAL PROCEDURE

1	1'	2	2'	3	3'
	1		1	$1 + \epsilon$	1
	1	$1 + \epsilon$	1	$1 + \epsilon$	
	1	$1 + \epsilon$		$1 + \epsilon$	$1 + 2\epsilon$
$1 + \epsilon$	1	$1 + \epsilon$			$1 + 2\epsilon$
$1 + \epsilon$		$1 + \epsilon$		$1 + 3\epsilon$	$1 + 2\epsilon$
$1 + \epsilon$		$1 + \epsilon$	$1 + 2\epsilon$	$1 + 3\epsilon$	
$1 + \epsilon$			$1 + 2\epsilon$	$1 + 3\epsilon$	$1 + 4\epsilon$

on the current proper set $M \setminus S_1$ of bidder 2 and on an additional item y , which is just being “robbed” from bidder 1 in the procedure. Again, the additive value of an item x is $1 + s \cdot \epsilon$, where s shows how often x was exchanged so far.

It remains to prove that the procedure selects the given sets in the aforementioned order to be the set T (see the POTENTIAL PROCEDURE). Consider first bidder 2. After an even round, he or she owns some proper set S_2 . His or her payoff on every item of this S_2 is at least 1. On any new item, the additional profit can be at most $s \cdot \epsilon < 1$. Therefore, the new set T will contain the current proper set S_2 , and at most one more item, since every additive valuation $a()$ of bidder 2 takes nonzero values on a set of $k + 1$ items. The proper set S_2 appears (with all members nonzero) in exactly two consecutive additive valuations, say, in the sets $S_2 \cup \{y\}$ and $S_2 \cup \{z\}$. These correspond to the odd steps *before* (*after*, respectively) the current even step when bidder 1 took $S_1 = M \setminus S_2$. (For example, the set $\{1, 2, 3\}$ appears in the sets $\{1, 2, 3, 3'\}$ and $\{1, 2, 2', 3\}$ in Table II.) Setting $T = S_2 \cup \{y\}$ would be a step back, and would mean a negative profit of $-\epsilon$, since the previous response of bidder 1 increased r_y by ϵ (e.g., $r_{3'} = 1 + 3\epsilon$ in our example). On the other hand, setting $T = S_2 \cup \{z\}$ means an additional profit of ϵ to the value of S_2 , so this is the best response for bidder 2.

The proof that the responses of bidder 1 follow the given order is analogous: at any step responding to bidder 2, bidder 1 has a value of $k - 1$ from items that currently only he or she bids for; giving up any of these would be a loss of about 1. Furthermore, he or she has the choice to bid for one of the two items in the remaining pair (one of which was just robbed by bidder 2), but he or she can only get the nonrobbed item, because of the ϵ increase in the price for the robbed item.

Finally, we show how to modify this instance so that the procedure can start with the empty allocation $S_1 = S_2 = \emptyset$. We add two new items to the instance, named 0 and 0'. For player one, item 0 appears with value 1 in the first row of the table, and with value 0 in every other row; item 0' has value $1 + \epsilon$ in every row. Similarly, for player two, item 0' has value 1 in the first row, and value 0 in other rows; item 0 adds value $1 + \epsilon$ to every row. Now, no matter which player responds first, the procedure follows the same order as shown earlier: bidding the additive valuation of the first row offers a player an additional value of about 1 compared to any other row. In the next step, he or she loses 1 by *any* response of the other player, and from then on all rows are equivalent concerning the items 0 and 0'. \square

Theorem 3.4 leaves us with two interesting open questions: First, will the POTENTIAL PROCEDURE converge in polynomial time for submodular valuations? Second, does the POTENTIAL PROCEDURE always run in time polynomial in the size of the sets A_i of additive valuations that underlie every fractionally subadditive valuation (i.e., the number of the different additive valuations l_i constituting A_i)? An affirmative answer to this question would imply that the POTENTIAL PROCEDURE runs in polynomial time

if the bidders have fractionally subadditive valuations encoded in a bidding language (see Nisan [2000], Lehmann et al. [2001], Dobzinski et al. [2005], and Dobzinski and Schapira [2006]). Observe that in our example, the size of these sets $|A_1|$ and $|A_2|$ is exponential in n and m .

3.2. Submodular Valuation Functions

In this subsection, we focus on submodular valuation functions. We present a polynomial time procedure that we call the **MARGINAL-VALUE PROCEDURE**, based on the algorithm due to Lehmann et al. [2001]. We show that it results in a pure Nash equilibrium that also satisfies the premises of Theorem 2.3. Therefore, it provides a 2-approximation to the optimal social welfare.

MARGINAL-VALUE PROCEDURE:

- (1) Fix an arbitrary order on the items. W.l.o.g. let this order be $1, \dots, m$.
- (2) Initialize $S_i \leftarrow \emptyset$, and $r_j \leftarrow 0$, for $i = 1, \dots, n$, and $j = 1, \dots, m$.
- (3) For each item $j = 1, \dots, m$:
 - (a) Let $i = \arg \max_{t \in N} v_t(S_t \cup \{j\}) - v_t(S_t)$. Set $S_i \leftarrow S_i \cup \{j\}$.
 - (b) Set $r_j \leftarrow \max_{t \in N} v_t(S_t \cup \{j\}) - v_t(S_t)$.
- (4) For every bidder i , set $b_i^*(j) \leftarrow r_j$ for all $j \in S_i$ and $b_i^*(j) \leftarrow 0$ for all $j \notin S_i$.
- (5) Output $b^* = (b_1^*, \dots, b_n^*)$.

Observe, that the resulting n -tuple of bid-vectors b^* is such that for each item, only one bidder offers a nonzero bid for that item. Also, notice that the **MARGINAL-VALUE PROCEDURE** only requires m rounds and so ends in polynomial time.

THEOREM 3.5. *If the valuation functions are submodular, then a pure Nash equilibrium that provides a 2-approximation to the optimal social welfare can be computed in polynomial time.*

PROOF. We show that the execution of the **MARGINAL-VALUE PROCEDURE** results in nonoverbidding bids of the players that correspond to a pure Nash equilibrium.

For any $j \in [m]$ and an arbitrary set of items Y , let $Y^j \stackrel{\text{def}}{=} Y \cap [j]$.

CLAIM 3. *For every i , b_i^* is nonoverbidding, given v_i .*

PROOF. Fix a player i . Clearly, it is enough to prove that $\sum_{j \in U} r_j \leq v_i(U)$ for every $U \subseteq S_i$. For an arbitrary set of items T , the claim then follows from

$$\sum_{j \in T} b_i^*(j) = \sum_{j \in S_i \cap T} r_j \leq v_i(S_i \cap T) \leq v_i(T).$$

We fix a subset U . For each item $j \in U \subseteq S_i$, we define $a(j)$ to be the marginal value of the item, according to the given ordering, restricted to U (i.e., $a(j)$ depends also on the subset U). That is, for $j \in U$,

$$a(j) \stackrel{\text{def}}{=} v_i(U^j) - v_i(U^{j-1}).$$

It is easy to see that $\sum_{j \in U} a(j) = v_i(U)$.

We claim that $r_j \leq a(j)$ for all $j \in U$. This will prove $\sum_{j \in U} r_j \leq v_i(U)$. The claim holds by submodularity, as

$$v_i(S_i^{j-1} \cup \{j\}) - v_i(S_i^{j-1}) \leq v_i(U^{j-1} \cup \{j\}) - v_i(U^{j-1}).$$

It follows from the **MARGINAL-VALUE PROCEDURE** that on the left-hand side, we have r_j . The right-hand side is obviously $a(j)$. \square

CLAIM 4. *The output b^* of the POTENTIAL PROCEDURE is a pure Nash equilibrium.*

PROOF. For the sake of contradiction, let's assume that b^* is not a Nash equilibrium. This means that player i , instead of keeping his or her set $S_i = X_i(b^*)$, would rather bid for another set $T_i = X_i(b'_i, b_{-i}^*)$ that he or she could get if he or she bid b'_i . Assume that T_i is maximal; that is, bidding for any superset of T_i would only decrease the utility of i . Player i is improving his or her utility by switching to b'_i , $u_i(b'_i, b_{-i}^*) > u_i(b^*)$. By construction of the MARGINAL-VALUE PROCEDURE, the items of S_i are free for player i (since the other players bid 0 for those items), and therefore we obtain

$$v_i(S_i) < v_i(T_i) - \sum_{j \in T_i \setminus S_i} r_j. \quad (1)$$

Note that the maximality of T_i implies $S_i \subseteq T_i$; otherwise, adding an item of $S_i \setminus T_i$ to T_i would not decrease $v_i(T_i)$ and would not increase the price, since the other players bid zero for this item.

As in the proof of Claim 3, we define $a(j)$ on the items of the set T_i : if $j \in T_i$, then $a(j) \stackrel{\text{def}}{=} v_i(T_i^j) - v_i(T_i^{j-1})$. We show that $a(j) \leq r_j$ for all $j \in T_i$. Using the definition of submodularity, $v_i(T_i^{j-1} \cup \{j\}) - v_i(T_i^{j-1}) \leq v_i(S_i^{j-1} \cup \{j\}) - v_i(S_i^{j-1})$. Note that on the left-hand side, we have $a(j)$. Moreover, it follows from the definition of the MARGINAL-VALUE PROCEDURE that for $j \in S_i$, the right-hand side equals r_j , whereas for $j \notin S_i$ it is at most r_j . Finally,

$$v_i(T_i) = \sum_{j \in T_i} a(j) \leq \sum_{j \in T_i} r_j = \sum_{j \in S_i} r_j + \sum_{j \in T_i \setminus S_i} r_j = v_i(S_i) + \sum_{j \in T_i \setminus S_i} r_j,$$

and this contradicts Equation (1). \square

The combination of these two claims together with Theorem 2.3 concludes the proof of Theorem 3.5.

APPENDIX

In this appendix, we prove Theorem 2.4, asserting the bound 2 for the Bayesian Price of Anarchy for mixed Nash equilibria.

THEOREM: *Let D be a distribution on the valuations of the bidders. If $p = (p_1, \dots, p_n)$ is a mixed Bayesian Nash, such that for every i and every valuation v_i , the $p^{v_i} (= p_i^{v_i})$ is a probability distribution over nonoverbidding bids w.r.t. v_i , then $\frac{SW(OPT)}{SW(p)} \leq 2$.*

PROOF. Let $v = (v_1, \dots, v_n)$ be a fixed valuation profile. We denote by $O^v = (O_1^v, \dots, O_n^v)$ the optimum allocation with respect to profile v . Now, for every player i , let a_i denote the maximizing additive valuation for the set O_i^v . For all i , we consider a_i as an alternative strategy to p^{v_i} .

Let us fix a bidder i . Let w_{-i} be an arbitrary valuation profile of all bidders except for i , and $w = (w_i, w_{-i})$ be a corresponding complete valuation profile.

Since p is a Bayes-Nash, the mixed strategy p^{v_i} provides better expected utility to player i than the (pure) strategy a_i :

$$\begin{aligned} & \sum_{b_i} p^{v_i}(b_i) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) u_i(b_i, b_{-i}) \\ & \geq \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) u_i(a_i, b_{-i}). \end{aligned}$$

The utility values on the left-hand side are

$$u_i(b) = v_i(X_i(b)) - Bids(X_i(b), b_{-i}) \leq v_i(X_i(b)).$$

On the right-hand side, applying Lemma 2.2 yields

$$u_i(a_i, b_{-i}) \geq v_i(O_i^v) - Bids(O_i^v, b_{-i}).$$

By merging the previous inequalities, we get

$$\begin{aligned} & \sum_{b_i} p^{v_i}(b_i) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) v_i(X_i(b)) \\ & \geq \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) [v_i(O_i^v) - Bids(O_i^v, b_{-i})] \\ & = v_i(O_i^v) \cdot 1 - \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) Bids(O_i^v, b_{-i}) \\ & = v_i(O_i^v) - \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) Bids(O_i^v, b_{-i}) \sum_{w_i \in V_i} D(w_i) \sum_{b_i} p^{w_i}(b_i) \\ & = v_i(O_i^v) - \sum_{w \in V} D(w) \sum_b p^w(b) Bids(O_i^v, b_{-i}) \geq v_i(O_i^v) - \sum_{w \in V} D(w) \sum_b p^w(b) Bids(O_i^v, b). \end{aligned}$$

Here we used that $\sum_{w_i \in V_i} D(w_i) \sum_{b_i} p^{w_i}(b_i) = 1$. Furthermore, note that $Bids(O_i^v, b_{-i}) \leq Bids(O_i^v, b)$ obviously holds, since in the latter case we consider maximum bids over a larger set of players. We obtained

$$\begin{aligned} v_i(O_i^v) & \leq \sum_{b_i} p^{v_i}(b_i) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) v_i(X_i(b)) \\ & \quad + \sum_{w \in V} D(w) \sum_b p^w(b) Bids(O_i^v, b). \end{aligned}$$

We sum over all i , and then take the expectation over all valuations $v = (v_1, \dots, v_n)$ on both sides:

$$\begin{aligned} & \sum_{v \in V} D(v) \sum_{i \in N} v_i(O_i^v) \leq \sum_{v \in V} D(v) \sum_{i \in N} \sum_{b_i} p^{v_i}(b_i) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_{b_{-i}} p^{w_{-i}}(b_{-i}) v_i(X_i(b)) \\ & \quad + \sum_{v \in V} D(v) \sum_{i \in N} \sum_{w \in V} D(w) \sum_b p^w(b) Bids(O_i^v, b). \end{aligned}$$

On the left-hand side, we have $\sum_{v \in V} D(v) \sum_{i \in N} v_i(O_i^v) = SW(OPT)$. Moreover, we show next that both summands on the right-hand side are at most $SW(p)$, so that $SW(OPT) \leq 2SW(p)$, which will conclude the proof. The first summand solves to

$$\begin{aligned} & \sum_{i \in N} \sum_{v_i \in V_i} D(v_i) \sum_{v_{-i} \in V_{-i}} D(v_{-i}) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_b p^{v_i, w_{-i}}(b) v_i(X_i(b)) \\ & = \sum_{i \in N} \sum_{v_i \in V_i} D(v_i) \sum_{w_{-i} \in V_{-i}} D(w_{-i}) \sum_b p^{v_i, w_{-i}}(b) v_i(X_i(b)) \\ & = \sum_{i \in N} \sum_{v \in V} D(v) \sum_b p^v(b) v_i(X_i(b)) \\ & = \sum_{v \in V} D(v) \sum_b p^v(b) \sum_{i \in N} v_i(X_i(b)) = SW(p). \end{aligned}$$

In the previous deduction, we obtained the third line from the second line by simply renaming w_{-i} by v_{-i} . Finally, the second summand is

$$\begin{aligned}
& \sum_{v \in V} D(v) \sum_{w \in V} D(w) \sum_b p^w(b) \sum_{i \in N} Bids(O_i^v, b) \\
&= \sum_{v \in V} D(v) \sum_{w \in V} D(w) \sum_b p^w(b) Bids(M, b) \\
&= \sum_{w \in V} D(w) \sum_b p^w(b) Bids(M, b) \\
&= \sum_{w \in V} D(w) \sum_b p^w(b) \sum_{i \in N} Bids(X_i(b), b_i) \\
&\leq \sum_{w \in V} D(w) \sum_b p^w(b) \sum_{i \in N} w_i(X_i(b)) = SW(p).
\end{aligned}$$

The last inequality holds since $p^w(b)$ can be nonzero only if for all i , the bid b_i is nonoverbidding w.r.t. w_i . \square

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REFERENCES

- Noga Alon, Yuval Emek, Michal Feldman, and Moshe Tennenholtz. 2012. Bayesian ignorance. *Theoretical Computer Science* 452 (Sept. 2012), 1–11. DOI : <http://dx.doi.org/10.1016/j.tcs.2012.05.017>
- Elliot Anshelevich, Anirban Dasgupta, Jon M. Kleinberg, Éva Tardos, Tom Wexler, and Tim Roughgarden. 2004. The price of stability for network design with fair cost allocation. In *Proceedings of the 45th Annual IEEE Symposium on Foundations of Computer Science (FOCS'04)*. 295–304.
- Kshipra Bhawalkar and Tim Roughgarden. 2011. Welfare guarantees for combinatorial auctions with item bidding. In *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms (SODA'11)*. 700–709.
- Kshipra Bhawalkar and Tim Roughgarden. 2012. Simultaneous single-item auctions. In *WINE'12*. 337–349.
- Liad Blumrosen and Noam Nisan. 2005. On the computational power of iterative auctions I: Demand queries. Discussion paper no. 381, The Center for the Study of Rationality, The Hebrew University. (2005). An extended abstract in EC'05 contained preliminary results.
- Ioannis Caragiannis, Christos Kaklamani, Panagiotis Kanellopoulos, and Maria Kyropoulou. 2011. On the efficiency of equilibria in generalized second price auctions. In *Proceedings of the ACM Conference on Electronic Commerce (EC'11)*. 81–90.
- Ioannis Caragiannis, Christos Kaklamani, Panagiotis Kanellopoulos, Maria Kyropoulou, Brendan Lucier, Renato Paes Leme, and Éva Tardos. 2012. On the efficiency of equilibria in generalized second price auctions. *CoRR* abs/1201.6429 (2012).
- George Christodoulou, Annamária Kovács, and Michael Schapira. 2008. Bayesian combinatorial auctions. In *Proceedings of the 35th International Colloquium Automata, Languages and Programming, (ICALP'08)*. 820–832.
- George Christodoulou, Annamária Kovács, Alkmini Sgouritsa, and Bo Tang. 2013. Tight bounds for the price of anarchy of simultaneous first price auctions. *arXiv:1312.2371* (2013).
- Edward H. Clarke. 1971. Multipart pricing of public goods. *Public Choice* 11, 1 (1971), 17–33.
- Bart de Keijzer, Evangelos Markakis, Guido Schäfer, and Orestis Telelis. 2013. Inefficiency of standard multi-unit auctions. In *ESA (Lecture Notes in Computer Science)*, Hans L. Bodlaender and Giuseppe F. Italiano (Eds.), Vol. 8125. Springer, 385–396.
- Shahar Dobzinski. 2011. An impossibility result for truthful combinatorial auctions with submodular valuations. In *Proceedings of the 43rd ACM Symposium on Theory of Computing (STOC'11)*. 139–148.
- Shahar Dobzinski, Noam Nisan, and Michael Schapira. 2005. Approximation algorithms for combinatorial auctions with complement-free bidders. In *Proceedings of the 37th ACM Symposium on Theory of Computing (STOC'05)*.

- Shahar Dobzinski and Michael Schapira. 2006. An improved approximation algorithm for combinatorial auctions with submodular bidders. (2006). In *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms (SODA'06)*.
- Uriel Feige. 2006. On maximizing welfare where the utility functions are subadditive. In *Proceedings of the Symposium on Theory of Computing (STOC'06)*.
- Uriel Feige and Jan Vondrak. 2006. The Allocation Problem with Submodular Utility Functions. Manuscript.
- Michal Feldman, Hu Fu, Nick Gravin, and Brendan Lucier. 2013. Simultaneous auctions are (almost) efficient. In *Proceedings of the Symposium on Theory of Computing Conference (STOC'13)*. 201–210.
- Hu Fu, Robert Kleinberg, and Ron Lavi. 2012. Conditional equilibrium outcomes via ascending price processes with applications to combinatorial auctions with item bidding. In *Proceedings of the ACM Conference on Electronic Commerce*. 586.
- Martin Gairing, Burkhard Monien, and Karsten Tiemann. 2005. Selfish routing with incomplete information. In *Proceedings of the 17th Annual ACM Symposium on Parallel Algorithms (SPAA'05)*. 203–212.
- Dinesh Garg and Yadati Narahari. 2005. Price of anarchy of network routing games with incomplete information. In *Proceedings of the 1st International Workshop on Internet and Network Economics (WINE'05)*. 1066–1075.
- Theodore Groves. 1973. Incentives in teams. *Econometrica* 41, 4 (1973), 617–631.
- John C. Harsanyi. 1967. Games with incomplete information played by “bayesian” players, parts i ii and iii. *Management Science* 14 (1967), 159–182.
- Avinatan Hassidim, Haim Kaplan, Yishay Mansour, and Noam Nisan. 2011. Non-price equilibria in markets of discrete goods. In *Proceedings of the ACM Conference on Electronic Commerce*. 295–296.
- Elias Koutsoupias and Christos Papadimitriou. 1999. Worst-case equilibria. In *Proceedings of the 16th Annual Symposium on Theoretical Aspects of Computer Science*. 404–413.
- Benny Lehmann, Daniel Lehmann, and Noam Nisan. 2001. Combinatorial auctions with decreasing marginal utilities. In *Proceedings of the ACM Conference on Electronic Commerce*.
- Renato Paes Leme and Éva Tardos. 2010. Pure and bayes-nash price of anarchy for generalized second price auction. In *Proceedings of the 51st Annual IEEE Symposium on Foundations of Computer Science (FOCS'10)*. 735–744.
- Brendan Lucier and Allan Borodin. 2010. Price of anarchy for greedy auctions. In *Proceedings of the 21st Annual ACM-SIAM Symposium on Discrete Algorithms (SODA'10)*. 537–553.
- Brendan Lucier and Renato Paes Leme. 2011. GSP auctions with correlated types. In *Proceedings of the 12th ACM Conference on Electronic Commerce (EC'11)*. San Jose, CA, USA, 71–80.
- Evangelos Markakis and Orestis Telelis. 2012. Uniform price auctions: Equilibria and efficiency. In *Proceedings of the 5th International Symposium on Algorithmic Game Theory (SAGT'12)*. 227–238.
- Noam Nisan. 2000. Bidding and allocation in combinatorial auctions. In *Proceedings of the ACM Conference on Electronic Commerce*.
- Noam Nisan and Amir Ronen. 1999. Algorithmic mechanism design. In *Proceedings of the Symposium on Theory of Computing Conference (STOC'99)*.
- Noam Nisan and Amir Ronen. 2000. Computationally feasible vcg-based mechanisms. In *Proceedings of the ACM Conference on Electronic Commerce*.
- Tim Roughgarden. 2012a. Intrinsic robustness of the price of anarchy. *Communications of the ACM* 55, 7 (2012), 116–123.
- Tim Roughgarden. 2012b. The price of anarchy in games of incomplete information. In *Proceedings of the ACM Conference on Electronic Commerce*. 862–879.
- Tim Roughgarden and Éva Tardos. 2002. How bad is selfish routing? *Journal of the ACM* 49, 2 (2002), 236–259.
- Vasilis Syrgkanis and Éva Tardos. 2013. Composable and efficient mechanisms. In *Proceedings of the Symposium on Theory of Computing Conference (STOC'13)*. 211–220.
- Adrian Vetta. 2002. Nash equilibria in competitive societies, with applications to facility location, traffic routing and auctions. In *FOCS*. IEEE Computer Society.
- William Vickrey. 1961. Counterspeculation, auctions and competitive sealed tenders. *Journal of Finance* 16, 1 (1961), 8–37.
- Jan Vondrak. 2008. Optimal approximation for the Submodular Welfare Problem in the value oracle model. (2008). In *Proceedings of the Symposium on Theory of Computing Conference (STOC'08)*.

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