

Homological Algebra

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1 Abelian Categories

Definition 1. Let \mathcal{C} be a category, and let $x \in \mathcal{C}$. We say that x is **terminal** if for every $c \in \mathcal{C}$, there is exactly one morphism $c \rightarrow x$. Dually, we say that x is **initial** if for every $c \in \mathcal{C}$, there is exactly one morphism $x \rightarrow c$.

Definition 2. A **zero object** in a category is an object that is both initial and terminal.

1.1 Ab-enriched Categories

Definition 3. A **pre-additive** or **Ab-enriched** category is a category in which every hom-set is equipped with the structure of an abelian group, such that composition

$$\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \rightarrow \mathrm{Hom}(X, Z)$$

is \mathbb{Z} -bilinear.

Proposition 1. In an **Ab-enriched** category, any initial object is also terminal.

Proof. Let $*$ be initial. Then 1_* is the unique element of $\mathrm{Hom}(*, *)$, so 1_* is zero in this group. Then since composition respects the group structures, we have for any map $f : A \rightarrow *$,

$$f = 1_* \circ f = 0 \circ f = 0$$

so $*$ is terminal. □

Proposition 2. If \mathcal{C} is an **Ab**-enriched category, then so is its opposite category \mathcal{C}^{op} .

Proof. For $X, Y \in \mathcal{C}^{\text{op}}$, the sets

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) = \text{Hom}_{\mathcal{C}}(Y, X)$$

are already endowed with the structure of an abelian group. Thus, we only have to prove that composition is bilinear. Let $X, Y, Z \in \mathcal{C}$ and let

$$f, f' \in \text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y), \quad g \in \text{Hom}_{\mathcal{C}^{\text{op}}}(Y, Z).$$

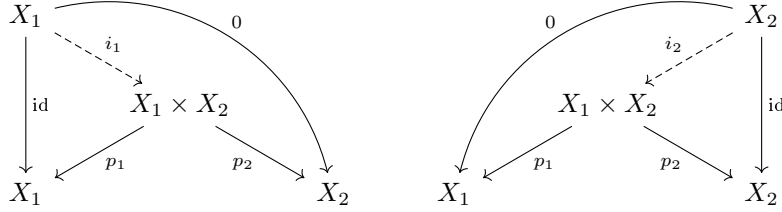
Then

$$g \circ_{\text{op}} (f + f') = (f + f') \circ g = f \circ g + f' \circ g = g \circ_{\text{op}} f + g \circ_{\text{op}} f'.$$

Similarly, composition is linear in the other argument as well. □

Proposition 3. In an **Ab**-enriched category \mathcal{C} , a binary product is also a binary coproduct.

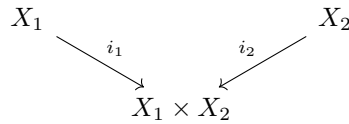
Proof. Let X_1, X_2 be elements of an **Ab**-enriched category \mathcal{C} . Suppose that X_1 and X_2 have a product $X_1 \times X_2$ in \mathcal{C} , with projections $p_k : X_1 \times X_2 \rightarrow X_k$. By definition of products, there are unique morphisms $i_k : X_k \rightarrow X_1 \times X_2$ such that the following diagrams commute.



Then we have

$$p_1 \circ (i_1 p_1 + i_2 p_2) = p_1, \quad p_2 \circ (i_1 p_1 + i_2 p_2) = p_2.$$

By definition of products, $\text{id}_{X_1 \times X_2} : X_1 \times X_2 \rightarrow X_1 \times X_2$ is the unique morphism with $p_k \circ \text{id} = p_k$ for each k , so $i_1 p_1 + i_2 p_2 = \text{id}_{X_1 \times X_2}$. We claim that



is a universal cocone, so that $X_1 \times X_2 = X_1 \coprod X_2$. Suppose that

$$\begin{array}{ccc} X_1 & & X_2 \\ & \searrow f_1 & \swarrow f_2 \\ & A & \end{array}$$

is another cocone. Then we have a map

$$\phi = f_1 \circ p_1 + f_2 \circ p_2 : X_1 \times X_2 \rightarrow A,$$

which is easily seen to give a commutative diagram

$$\begin{array}{ccccc} X_1 & & & & X_2 \\ & \searrow i_1 & & \swarrow i_2 & \\ & & X_1 \times X_2 & & \\ & \searrow f_1 & \downarrow \phi & \swarrow f_2 & \\ & & A & & \end{array}$$

It remains to show that ϕ is unique. To see this, note that for any such ϕ we have

$$\begin{aligned} \phi &= \phi \circ \text{id}_{X_1 \times X_2} \\ &= \phi \circ (i_1 p_1 + i_2 p_2) \\ &= \phi i_1 \circ p_1 + \phi i_2 \circ p_2 \\ &= f_1 \circ p_1 + f_2 \circ p_2. \end{aligned}$$

□

Proposition 4. In an **Ab**-enriched category, all binary coproducts are also binary products.

Proof. This is dual to the previous proposition. □

Definition 4. Let \mathcal{C} be an **Ab**-enriched category, and let $x, y \in \mathcal{C}$. If x and y have a product in \mathcal{C} , then it is called the biproduct of x and y , which we denote by $x \oplus y$.

Definition 5. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be a functor between **Ab**-enriched categories. Then F is said to be **additive** if it preserves finite biproducts.

Lemma 1. For any ring R , the category $R\text{-mod}$ is **Ab**-enriched.

1.2 Additive Categories

Definition 6. A category is **additive** if it is **Ab**-enriched and admits finite coproducts.

Lemma 2. Let \mathcal{A} be an additive category. Suppose that $i : a \rightarrow b$ is a monomorphism in \mathcal{A} and $\alpha \in \text{Hom}(a, b)$ is the zero morphism. Then $a = 0$.

Proof. Let $x \in \mathcal{A}$. Since $\text{Hom}(a, x)$ is an abelian group, it contains at least one morphism (zero). Let $f : a \rightarrow x$ be any morphism. Then

$$\alpha \circ 0 = 0 = \alpha \circ f.$$

Since α is a monomorphism, we have $f = 0$. Therefore a is initial, hence it is the zero object. \square

Lemma 3. Let \mathcal{A} be an additive category. Suppose that $q : a \rightarrow b$ is an epimorphism in \mathcal{A} . If $q = 0$, then $b = 0$.

Proof. Since \mathcal{A} is additive, the opposite category \mathcal{A}^{op} is too. The map q is a monomorphism $q : b \rightarrow a$ in \mathcal{A}^{op} , and it is still the zero morphism. By the previous lemma we must therefore have that b is the zero object in \mathcal{A}^{op} , hence in \mathcal{A} . \square

Lemma 4. For any ring R , the category $R\text{-mod}$ is additive.

Proof. We know that the direct sum exists and is a coproduct in $R\text{-mod}$. \square

1.3 Pre-abelian Categories

Definition 7. An additive category is **pre-abelian** if every morphism has a kernel and cokernel.

Lemma 5. Let \mathcal{A} be a pre-abelian category. Every monomorphism has kernel 0, and every epimorphism has cokernel 0.

Proof. Let $i : a \rightarrow b$ be a monomorphism in \mathcal{A} . Let

$$\text{Ker } i \xrightarrow{\text{ker } i} a$$

be the kernel of i . Then $i \circ \text{ker } i = 0 = i \circ 0$, so $\text{ker } i$ is the zero morphism (since i is a monomorphism). Since $\text{ker } i$ is monomorphism, we have $\text{Ker } i = 0$. \square

Lemma 6. For any ring R , the category $R\text{-mod}$ is pre-abelian.

1.4 Abelian Categories

Definition 8. A pre-abelian category is **abelian** if every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.

Lemma 7. The category of left R -modules is an abelian category.

Proof. Let $i : A \rightarrow B$ be a monomorphism of R -modules. Then $\text{Coker } i = B/i(A)$ and the cokernel map is the quotient $q : B \rightarrow B/i(A)$ with $q(b) = b + i(A)$. It is clear that $i(A) = \text{Ker } q$ in the set-theoretic sense, so i exhibits A as the kernel of q .

Let $q : A \rightarrow B$ be an epimorphism of R -modules. Let $i : \text{Ker } q \rightarrow A$ be the inclusion. Then $\text{Coker } i = A/\text{Ker } q \cong B$, so q exhibits B as the cokernel of i . \square

Lemma 8. If \mathcal{A} is abelian, then so is \mathcal{A}^{op} .

Proof. Duality. \square

Lemma 9. If \mathcal{A} is an abelian category and \mathcal{C} is any category, then $\text{Fun}(\mathcal{C}, \mathcal{A})$ is abelian.