Category Theory

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1 Categories, Functors, Natural Transformations

1.1 Abstract and concrete categories

Definition 1. A *category* $\mathcal{C}$ consists of the following data:

(i) a collection ob$\mathcal{C}$ of object of $\mathcal{C}$,

(ii) for every two objects $x, y \in \mathrm{ob}\mathcal{C}$ a collection $\mathrm{Hom}_{\mathcal{C}}(x, y)$ of morphisms,

(iii) the identity morphisms,

(iv) the identity morphism $\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)$ for every object $x \in \mathrm{ob}\mathcal{C}$,

(v) the composition map

$$\circ : \mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$$

for every triple of objects $x, y, z \in \mathrm{ob}\mathcal{C}$

Definition 2. A category is **small** if it has only a set's worth of arrows.

Definition 3. A category is **locally small** if between any pair of objects there is only a set's worth of morphisms.

Definition 4. An **isomorphism** in a category is a morphism $f : X \to Y$ for which there exist a morphism $g : Y \to X$ so that $fg = 1_X$ and $gf = 1_Y$. The objects X and Y are **isomorphic** whenever there exist an isomorphism between X and Y, inn which case one writes $X \cong Y$.

A **subcategory** $\mathcal{D}$ of a category $\mathcal{C}$ is defined by restricting to a subcollection of objects and subcollection of morphisms subject to the requirements that the subcategory $\mathcal{D}$ contains the domain and codomain of any morphism in $\mathcal{D}$, the identity morphism of any object in $\mathcal{D}$, and the composite of any composable pair of morphisms in $\mathcal{D}$.

Lemma 1. Any category $\mathcal{C}$ contains a maximal groupoid, the subcategory containing all of the objects and only those morphisms that are isomorphisms.

1.2 Duality

Definition 5. Let $\mathcal{C}$ be any category. The **opposite category** $\mathcal{C}^{\mathrm{op}}$ has

1. the same objects as in $\mathcal{C}$, and
2. a morphism f^{op} in $\mathcal{C}^{\mathrm{op}}$ for each morphism f in $\mathcal{C}$ so that the domain of f^{op} is defined to be the codomain of f and the codomain of f^{op} is defined to be the domain of f.
3. For each object X, the arrow 1_X^{op} serves as its identity in $\mathcal{C}^{\mathrm{op}}$.
4. To define composition, observe that a pair of morphisms $f^{\mathrm{op}}, g^{\mathrm{op}}$ in $\mathcal{C}^{\mathrm{op}}$ is composable precisely when the pair g, f is composable in $\mathcal{C}$. We then define $g^{\mathrm{op}} \circ f^{\mathrm{op}}$ to be $(f \circ g)^{\mathrm{op}}$.

Lemma 2. The following are equivalent

1. $f : x \to y$ is an isomorphism in $\mathcal{C}$.
2. For all objects $c \in \mathcal{C}$, post-composition with f defines a bijection
$$f_* : \mathcal{C}(c, x) \to \mathcal{C}(c, y).$$
3. For all objects $c \in \mathcal{C}$, pre-composition with f defines a bijection
$$f^* : \mathcal{C}(y, c) \to \mathcal{C}(x, c).$$

Definition 6. A morphism $fx \to y$ in a category is

1. a **monomorphism** if for any parallel morphisms $h, k : w \rightrightarrows x, fh = fk$ implies that $h = k$; or
2. an **epimorphism** if for any parallel morphisms $h, k : y \rightrightarrows z, hf = kf$ implies that $h = k$.

Since the notions of monomorphism and epimorphism are dual, their abstract categorical properties are also dual, such as exhibited by the following lemma.

Lemma 3. 1. If $f : x \to y$ and $g : y \to z$ are monomorphisms, then so is $gf : x \to z$.

2. If $f : x \to y$ and $g : y \to z$ are morphisms so that gf is monic, then f is monic.

Dually.

1. If $f : x \to y$ and $g : y \to z$ are epimorphisms, then so is $gf : x \to z$.
2. If $f : x \to y$ and $g : y \to z$ are morphisms so that $gf : x \to z$ is epic, then g is epic.

1.3 Functors

Definition 7. A **functor** $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, between categories $\mathcal{C}$ and $\mathcal{D}$, consists of the following data:

- An object $\mathcal{F}c \in \mathcal{D}$, for each object $c \in \mathcal{C}$.
- A morphism $\mathcal{F}f : \mathcal{F}c \to \mathcal{F}c' \in \mathcal{D}$, for each morphism $f : c \to c' \in \mathcal{C}$, so that the domain and codomain of $\mathcal{F}f$ are, respectively, equal to $\mathcal{F}$ applied to the domain and codomain of f.

The assignments are required to satisfy the following two functoriality axioms:

- For any composable pair f, g in $\mathcal{C}$, $\mathcal{F}g \circ \mathcal{F}f = \mathcal{F}(g \circ f)$.
- For each object $c \in \mathcal{C}$, $\mathcal{F}1_c = 1_{\mathcal{F}c}$.

There is also the dual notion of a **contravariant functor** which simply has as domain $\mathcal{C}^{\mathrm{op}}$ instead of $\mathcal{C}$.

Lemma 4. Functors preserve isomorphisms.

Proof. Straightforward. □

Corollary 1. When a group G acts functorially on an object X in a category $\mathcal{C}$, its elements g must act by automorphisms $g_* : X \to X$ and, moreover, $(g_*)^{-1} = (g^{-1})_*$.

Definition 8. If $\mathcal{C}$ is locally small, then for any object $c \in \mathcal{C}$ we may define a pair of covariant and contravariant functors represented by c:

$$\begin{array}{ccccccc}
\mathcal{C} & \xrightarrow{\mathcal{C}(c,-)} & \mathrm{Set} & \qquad & \mathcal{C}^{\mathrm{op}} & \xrightarrow{\mathcal{C}^{\mathrm{op}}(-,c)} & \mathrm{Set} \\
x & \mapsto & \mathcal{C}(c,x) & & x & \mapsto & \mathcal{C}^{\mathrm{op}}(x,c) \\
\Big\downarrow f & \mapsto & \Big\downarrow f_* & & \Big\downarrow f & \mapsto & \Big\uparrow f^* \\
y & \mapsto & \mathcal{C}(c,y) & & y & \mapsto & \mathcal{C}^{\mathrm{op}}(y,c)
\end{array}$$

Definition 9. If $\mathcal{C}$ is locally small, then there is a **two-sided represented functor**

$$\mathcal{C}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$$

defined in the evident manner. A pair of objects (x, y) is mapped to the hom-set $\mathcal{C}(x, y)$. A pair of morphisms $f : w \to x$ and $h : y \to z$ is sent to the function

$$\mathcal{C}(x,y) \xrightarrow{(f^*,h_*)} \mathcal{C}(w,z)$$
$$g \mapsto hgf.$$

1.4 Natural Transformations

Definition 10. Let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\eta : F \Rightarrow G$ consists of morphisms $\eta_x \in \mathrm{Hom}_{\mathcal{D}}(F(x), G(x))$ for every object $x \in \mathcal{C}$ such that the diagram

$$\begin{tikzcd}
F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"] & F(y) \arrow[d, "\eta_y"] \\
G(x) \arrow[r, "G(f)"] & G(y)
\end{tikzcd}$$

commutes for every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$.

We say that a natural transformation $\eta : F \Rightarrow G$ is a natural isomorphism if the morphisms η_x are isomorphisms for any $x \in \mathcal{C}$.

Definition 11. An equivalence of categories $\mathcal{C}, \mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $e : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\varepsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.

Definition 12. An adjoint equivalence of categories $\mathcal{C}, \mathcal{D}$ is an equivalence (F, G, e, ε) satisfying the following axioms:

1. The composite natural transformation

$$F \cong F \circ \mathrm{id}_{\mathcal{C}} \xRightarrow{\mathrm{id}_F \circ e} FGF \xRightarrow{\varepsilon \circ \mathrm{id}_F} \mathrm{id}_{\mathcal{D}} \circ F \cong F$$

is the identity natural transformation on F.

2. The composite natural transformation

$$G \cong \mathrm{id}_{\mathcal{C}} \circ G \xRightarrow{e \circ \mathrm{id}_G} GFG \xRightarrow{\mathrm{id}_G \circ \varepsilon} G \circ \mathrm{id}_{\mathcal{D}} \cong G$$

is the identity natural transformation on G.

Fix a functor $F : \mathcal{C} \to \mathcal{D}$ and consider the category Equiv_F. Its objects are functors $G : \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms e and ε such that (F, G, e, ε) is an adjoint equivalence. A morphism $(G_1, e_1, \varepsilon_1) \to (G_2, e_2, \varepsilon_2)$ consists of a natural transformation $f : G_1 \Rightarrow G_2$ making the diagrams

$$\begin{tikzcd}
 & \mathrm{id}_{\mathcal{C}} \arrow[dl, Rightarrow, "e_1"'] \arrow[dr, Rightarrow, "e_2"] & \\
G_1 F \arrow[rr, Rightarrow, "f \circ \mathrm{id}_F"] & & G_2 F
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & \mathrm{id}_{\mathcal{D}} & \\
F G_1 \arrow[ur, Rightarrow, "\varepsilon_1"] \arrow[rr, Rightarrow, "\mathrm{id}_F \circ f"] & & F G_2 \arrow[ul, Rightarrow, "\varepsilon_2"']
\end{tikzcd}$$

commute.

Proposition 1. Any two objects $(G_1, e_1, \varepsilon_1)$ and $(G_2, e_2, \varepsilon_2)$ of Equiv_F are isomorphic and this isomorphism is unique.

Proof. To prove the first statement, consider the composite natural isomorphism $f : G_1 \Rightarrow G_2$ given by

$$G_1 \cong \mathrm{id}_{\mathcal{C}} \circ G_1 \xRightarrow{e_2 \circ \mathrm{id}_{G_1}} G_2 F G_1 \xRightarrow{\mathrm{id}_{G_2} \circ \varepsilon_1} G_2.$$

To show that the first diagram commutes, consider the composite natural transformation

$$\mathrm{id} \xRightarrow{e_1} G_1 F \xRightarrow{e_2 \circ \mathrm{id}_{G_1} \circ \mathrm{id}_F} G_2 F G_1 F \xRightarrow{\mathrm{id}_{G_2} \circ \varepsilon_1 \circ \mathrm{id}_F} G_2 F.$$

Since e_1 and e_2 are natural transformations, we can alternatively write this as

$$\mathrm{id} \xRightarrow{e_2} G_2 F \xRightarrow{\mathrm{id}_{G_2} \circ \mathrm{id}_F \circ e_1} G_2 F G_1 F \Rightarrow G_2 F.$$

$\square$