The University of Oxford

MSc (Mathematics and Foundations of Computer Science)

Homological Algebra

Michaelmas Term 2023

The steps of (each) mini project are for your guidance; if you wish to take an alternative route to the desired goal, you are free to do so. But, if you follow the suggested route and find yourself unable to carry out any particular step, you may simply assume it so that you can continue with the mini project, but should make this assumption clear in your presentation.

The purpose of this mini-project is to define and study Lie algebra cohomology and homology. Let k be a field. A *k-Lie algebra* is a k-vector space $\mathfrak{g}$ with a bilinear map $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[x, x] = 0 \tag{1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. \tag{2}$$

Morphisms of Lie algebras are linear maps compatible with the bracket. A Lie algebra is called *abelian* if $[x, y] = 0$ for all x, y. A left $\mathfrak{g}$-*module* M is a k-module equipped with a bilinear action map $\mathfrak{g} \otimes_k M \to M$ (where $x \otimes m$ goes to xm) such that

$$[x, y]m = x(ym) - y(xm) \tag{3}$$

for all $x, y \in \mathfrak{g}$ and all $m \in M$. Morphisms of $\mathfrak{g}$-modules are linear maps which commute with the actions.

1. Show that the category of $\mathfrak{g}$-modules is abelian.

2. Show that the category of $\mathfrak{g}$-modules is closed symmetric monoidal where the action map on $M \otimes_k N$ is given by $x(m \otimes n) = xm \otimes n + m \otimes xn$.

3. For any associative k-algebra A let $Lie(A)$ be the Lie algebra with underlying k-module A and bracket given by $[a, b] = ab - ba$. Show that this defines a functor from the category of associative k-algebras to the category of Lie algebras. Show that this functor has a left adjoint (the universal enveloping algebra denoted by U).

4. Show that the category of $\mathfrak{g}$-modules is naturally isomorphic to the category of $U(\mathfrak{g})$-modules.

5. Show that the category of $\mathfrak{g}$-modules has enough projectives and enough injectives.

6. Show that the functor from k-modules to $\mathfrak{g}$-modules sending V to the trivial $\mathfrak{g}$-module with underlying k-module V ($xv = 0$ for all $x \in \mathfrak{g}$ and all $v \in V$) has a left adjoint (coinvariants denoted by $M_{\mathfrak{g}}$) and a right adjoint (invariants denoted by $M^{\mathfrak{g}}$). Denote by $H_*(\mathfrak{g}, M)$ the left derived functor of coinvariants, and by $H^*(\mathfrak{g}, M)$ the right derived functor of invariants.

7. Show that if M is a trivial $\mathfrak{g}$-module then $H_1(\mathfrak{g}, M) \cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \otimes_k M$.

8. Define the *Chevalley-Eilenberg complex*. Show that it gives a projective resolution of the trivial $\mathfrak{g}$-module k. You are allowed to use the PBW theorem as long as you state it precisely.

9. Let M be a $\mathfrak{g}$-module. Define what is a *Lie algebra extension* of $\mathfrak{g}$ by M (viewed as an abelian Lie algebra). Define when two extensions are equivalent. Show that $H^2(\mathfrak{g}, M)$ is in a natural one to one correspondence with the set of equivalence classes of Lie algebra extensions of $\mathfrak{g}$ by M.

10. Discuss more examples of Lie algebra cohomology computations. One possibility is the BGG resolution and how it can be used to compute Lie algebra cohomology of Borel subalgebras of semisimple Lie algebras (Bott's theorem).