Integer Programming
Sheet 1

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Exercise B.1

(i) My interpretation of this exercise is to take this extension to mean switching out $a_k^x x \leq b_k$ in the original formulation for $A^k x \leq b^k$. The extended alternative disjunction model then becomes

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} \; c^T x \\
&\text{s.t. } 0 \leq x \leq u \\
&A^1 x \leq b^1 \\
&A^2 x \leq b^2
\end{aligned}
$$

where we require that at least one of $A^1 x \leq b^1$ or $A^2 x \leq b^2$ is satisfied.

Letting $M \geq \max_k \max_i \{a_i^k x - b_i^k \mid 0 \leq x \leq u\}$ we can extend the big-M formulation to become

$$
\begin{aligned}
\min_{(x,y^1,y^2) \in \mathbb{R}^{n+1+1}} \quad & c^T x \\
& A^k x - b^k \leq M(\mathbf{1} - \mathbf{1} y^k), \quad (k = 1, 2) \\
& y^1 + y^2 = 1 \\
& 0 \leq x \leq u \\
& y^k \in \mathbb{B}^1 \text{ for } k = 1, 2
\end{aligned}
$$

where $\mathbf{1} = (\underbrace{1, \ldots, 1}_{m^k \text{ times}})$ and m^k is the number of rows in A^k.

(ii) There are two directions to prove.

($\Rightarrow$): Assume that $x \in P_1 \cup P_2$. We can without loss of generality assume that $x \in P_1$. We then let

$$
\begin{aligned}
y^1 &= 1 \\
y^2 &= 0 \\
z^1 &= x \\
z^2 &= 0.
\end{aligned}
$$

The constraints then become

$$\begin{aligned} x &= x \\ A^1 x &\leq b^1 \\ 0 \leq x &\leq u \\ 1 &= 1 \end{aligned}$$

which are obviously all true.

($\Leftarrow$): Assume that the there exists x, z^1, z^2, y^1, y^2 so that the constraints are satisfied. We want to show that $x \in P_1 \cup P_2$. Hence, we must show that both $0 \leq x \leq u$ and $A^1 x \leq b^1$ or $A^2 x \leq b^2$.

For the first requirement, notice that

$$0 \leq x = z^1 + z^2 \leq u(y^1 + y^2) = u.$$

It thus remains to show either $A^1 x \leq b^1$ or $A^2 x \leq b^2$. Since $y^1 + y^2 = 1$ we have either $y^1 = 1 \wedge y^2 = 0$ or $y^1 = 0 \wedge y^2 = 1$. We can without loss of generality assume that $y^1 = 1$ and $y^2 = 0$. The constraints then say that $A^1 z^1 \leq b^1$ and $0 \leq z^2 \leq u \cdot 0 = 0$ so that

$$A^1 x = A^1 z^1 + A^1 0 \leq b^1$$

as desired. This concludes the proof.

Exercise B.2

The standard form is given by

$$\begin{aligned} \max_{x \in \mathbb{R}^5} & -x_{11} + x_{12} - x_{21} + x_{22} - x_3 \\ \text{s.t} & -2x_{11} + 2x_{12} + x_3 \leq 2, \\ & 3x_{11} - 3x_{12} + x_{21} - x_{22} + 2x_3 \leq 6 \\ & x_{11} - x_{12} - x_{21} + x_{22} + 3x_3 \leq 3, \\ & -x_{11} + x_{12} + x_{21} - x_{22} - 3x_3 \leq -3 \\ & -x_{21} + x_{22} + x_3 \leq 0 \end{aligned}$$

where $x_1 = x_{11} - x_{12}$ and $x_2 = x_{21} - x_{22}$. This has corresponding dual

$$\begin{aligned} \min_{y \in \mathbb{R}^5} \ & 2y_1 + 6y_2 + 3y_3 - 3y_4 \\ \text{s.t} & -2y_1 + 3y_2 + y_3 - y_4 = -1 \\ & y_2 - y_3 + y_4 - y_5 = -1 \\ & y_1 + 2y_2 + 3y_3 - 3y_4 + y_5 = -1 \\ & y_1, y_2, y_3, y_4, y_5 \geq 0. \end{aligned}$$

Exercise B.3

First, to see that the new system doesn't involve x_k is fairly straightforward. As $a_{ij} = 0$ for $i \in M_0^k$ we have that

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad (i \in M_0^k)$$

clearly doesn't involve x_k. Moreover, if we look at the kth term of the sum

$$\sum_{j=1}^{n} (a_{ik} a_{lj} - a_{lk} a_{ij}) x_j$$

we see that it equates to

$$a_{ik} a_{lk} - a_{lk} a_{ik} = 0$$

so that the coefficient of x_k is zero. Hence this sum also does not involve x_k and so the new system does not involve x_k.

For the second statement there are two directions to be proven.

($\Rightarrow$): Assume that $(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ satisfies the new system. We then want to find a value of x_k so that $(x_1, \ldots, x_k, \ldots, x_n)$ satisfies the original system.

Since all the x_j's are given, with the exception of x_k of course, this places restrictions on what values x_k can have. Solving the original system with respect to x_k we get the following constraints:

$$\begin{aligned} x_k &\leq \frac{b_i - \sum_{j \neq k} a_{ij} x_j}{a_{ik}} \quad & a_{ik} \in M_+^k \\ x_k &\geq \frac{b_i - \sum_{j \neq k} a_{ij} x_j}{a_{ik}} \quad & a_{ik} \in M_-^k. \end{aligned}$$

Putting this information together tells us that a value x_k exists if and only if

$$\max_{i \in M_-^k} \left(\frac{b_i - \sum_{j \neq k} a_{ij} x_j}{a_{ik}} \right) \leq \min_{i \in M_+^k} \left(\frac{b_i - \sum_{j \neq k} a_{ij} x_j}{a_{ik}} \right).$$

Letting i^* and l^* be the corresponding maximizing and minimizing indices we get, after rearranging terms, the requirement

$$\sum_{j \neq k} (a_{i^*k} a_{l^*j} - a_{l^*k} a_{i^*j}) x_j \leq a_{i^*k} b_{l^*} - a_{l^*k} b_{i^*}$$

which holds by assumption.

($\Leftarrow$): Suppose $(x_1, \ldots, x_k, \ldots x_n)$ satisfies the original system. By assumption we then have

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \quad (i \in M_0^k).$$

We also have for all $(i,l) \in M_+^k \times M_-^k$ that

$$\begin{aligned}\sum_{j=1}^n (a_{ik}a_{lj} - a_{lk}a_{ij})x_j &= a_{ik}\sum_{j=1}^n a_{lj}x_j - a_{lk}\sum_{j=1}^n a_{ij}x_j \\ &\leq a_{ik}b_l - a_{lk}b_i\end{aligned}$$

which concludes the proof.

Exercise B.4

If

$$\sum_{j=1}^n a_{ij}x_j \leq b_i, \quad (i = 1, \ldots, m)$$

is infeasible, then by the theorem of alternatives for linear inequalities (Farkas' lemma) we get that there exist a $u \in \mathbb{R}^m$ such that

$$\begin{aligned}\sum_{i=1}^m u_i a_{ij} &= 0, \quad (j = 1, \ldots, n) \\ u_i &\geq 0, \quad (i = 1, \ldots, m) \\ \sum_{i=1}^m u_i b_i &< 0.\end{aligned}$$

The objective function of (D) is $\sum_{i=1}^m b_i y_i$ which we want to minimize. We know by assumption that there exists at least one feasible solution $\tilde{y} \in \mathbb{R}^m$ for (D). Setting $y = \tilde{y} + \lambda u$ with $\lambda \in \mathbb{R}_+$ we have that

$$\begin{aligned}\sum_{i=1}^m y_i a_{ij} &= \sum_{i=1}^m \tilde{y}_i a_{ij} + \lambda \sum_{i=1}^m u_i a_{ij} \\ &= \sum_{i=1}^m y_i a_{ij} \\ &= c_j, \quad (j = 1, \ldots, n)\end{aligned}$$

so that y is also a feasible solution of (D). Moreover, letting $\tilde{d} \coloneqq \sum_{i=1}^m b_i \tilde{y}_i$ we have that

$$\begin{aligned}d_\lambda &\coloneqq \sum_{i=1}^m b_i y_i \\ &= \sum_{i=1}^m b_i \tilde{y}_i + \lambda \sum_{i=1}^m u_i b_i \\ &= \tilde{d} + \lambda d'\end{aligned}$$

where $d' = \sum_{i=1}^m u_i b_i < 0$. Hence by letting $\lambda \to \infty$ we get that $d_\lambda \to -\infty$ showing that (D) is unbounded.