Tutorial 2: Strategic Form Games & Mixed Strategies

1. Solution concepts and social welfare concepts:

   Either prove or disprove each of the following statements in the context of 2×2 games (disproving is usually best done with a counterexample):

   (a) If a player i has a dominant strategy in a game, then in every Nash equilibrium of that game player i will choose a dominant strategy.

   (b) If a game has a dominant strategy equilibrium, then it is unique: the game has no other dominant strategy equilibria.

   (c) Every dominant strategy equilibrium of a game is a Nash equilibrium.

   (d) Every Nash equilibrium of a game is a dominant strategy equilibrium.

   (e) If a game outcome ω maximises utilitarian social welfare, then ω is Pareto efficient.

   (f) If a game outcome ω is Pareto efficient, then it maximises utilitarian social welfare.

   (g) If all utilities in a game are positive, then any outcome that maximises the product of utilities of players is Pareto efficient.

   (h) If all utilities in a game are positive, then any Pareto efficient outcome of the game will maximise the product of utilities of players.

2. Nash equilibria in Mixed Strategies:

   If we use mixed strategies in a game, then we are in the domain of expected utility.

   (a) Write down an expression for the expected utility of each player in a generic 2×2 game, when a mixed strategy is given as a pair $(p, q) \in [0, 1]^2$. That is, define the expressions $EU_1(p, q)$ and $EU_2(p, q)$.

   (b) Generalise the expression you obtained in the first part to n player games, where each player $i \in N$ has pure strategy set Σ_i. Denote a mixed strategy profile by $\vec{\mu} = (\mu_1, \ldots, \mu_n)$, where $\mu_i \in \Delta\Sigma_i$ is a mixed strategy for i, i.e., a probability distribution over Σ_i. Use $\mu_i(\sigma)$ to denote the probability of $\sigma \in \Sigma_i$ being played in the mixed strategy μ_i.

   Hint: First define an expression $P(\vec{\sigma}, \vec{\mu})$, meaning the probability that the pure strategies $\vec{\sigma}$ are chosen given the mixed strategy profile $\vec{\mu}$. Then define expected utility using this expression.

3. Nash equilibria:

   For each of the following games:

(i) identify any dominant strategies, dominant strategy equilibria, and pure Nash equilibria;

(ii) identify outcomes that are Pareto efficient, that maximise utilitarian social welfare, and that maximise egalitarian social welfare

(iii) apply the principle of indifference to identify any fully mixed Nash equilibria

(iv) compute the expected utility of each player for each fully mixed strategy equilibrium you identify

(v) sketch the best response curves of the players in each game (no need to make it fancy – a sketch will do)

The games are:

(a) This game is called “Bach or Stravinsky”.

		2: L	2: R
1	T	1 2	0 0
	B	0 0	2 1

(b)

		2: L	2: R
1	T	0 0	5 3
	B	4 4	3 0

4. Mixed Nash Equilibria and the Indifference Principle:

This question refers to the “generic” 2×2 game that we discussed in the lecture, where the row players has pure strategies T and B, and the column player has strategies L and R. Prove that a pair of probabilities $(p, q) \in (0,1)^2$ is a fully mixed strategy Nash equilibrium in the generic 2×2 game iff:

$$\begin{aligned} EU_1(T, q) &= EU_1(B, q) \quad \text{and} \\ EU_2(L, p) &= EU_2(R, p) \end{aligned}$$

5. Consider the following scenaruio:

Two firms, X and Y, provide a service based on machine learning. The market share of each firm is directly proportional to the quality of its service, which, in turn, is directly proportional to the size of the data set it has: thus, if X has x data points and Y has y data points, the profit of X (in £) is $\frac{x}{x+y} \cdot M$ and the profit of Y (in £) is $\frac{y}{x+y} \cdot M$, where M is the overall value of the market. Initially, firm X possesses 1 million data points and firm Y possesses 2 million data points. They are both presented with an opportunity to buy a new corpus of data, consisting of n million data points, where $n > 0$, at price P (in £), $P > 0$. If both firms express the desire to buy, each of them gets one half of the new data, i.e., $\frac{n}{2}$ points, and pays $\frac{P}{2}$.

(a) Describe this setting as a two-player game, where each player's choice of actions is Buy (buy) and $NotBuy$ (do not buy). For simplicity, assume $M = 1$ in this and subsequent parts.

(b) Suppose that n is fixed. Under what conditions on the price P is Buy a weakly dominant strategy for firm X? Under what conditions on P is Buy a weakly dominant strategy for firm Y? NB: note that we are after a *weakly* dominant strategy here, so must always be at least as good and in one case strictly better.

(c) Suppose that n is fixed. Characterise the range of values of P such that (Buy, Buy) is a Nash equilibrium.

(d) Suppose that n is fixed. Characterise the range of values of P such that $(NotBuy, NotBuy)$ is a Nash equilibrium.

(e) Suppose that n is fixed. Are there values of P such that $(Buy, NotBuy)$ or $(NotBuy, Buy)$ is a Nash equilibrium?