# Composable and Efficient Mechanisms

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## ABSTRACT

We initiate the study of efficient mechanism design with guaranteed good properties even when players participate in multiple mechanisms simultaneously or sequentially. We define the class of smooth mechanisms, related to smooth games defined by Roughgarden, that can be thought of as mechanisms that generate approximately market clearing prices. We show that smooth mechanisms result in high quality outcome both in equilibrium and in learning outcomes in the full information setting, as well as in Bayesian equilibrium with uncertainty about participants. Our main result is to show that smooth mechanisms compose well: smoothness locally at each mechanism implies global efficiency.

For mechanisms where good performance requires that bidders do not bid above their value, we identify the notion of a weakly smooth mechanism. Weakly smooth mechanisms, such as the Vickrey auction, are approximately efficient under the no-overbidding assumption, and the weak smoothness property is also maintained by composition.

In most of the paper we assume participants have quasi-linear valuations. We also extend some of our results to settings where participants have budget constraints.

## Categories and Subject Descriptors

J.4 [**Social and Behavioral Sciences**]: Economics; F.2 [**Theory of Computation**]: Analysis of Algorithms and Problem Complexity

## Keywords

Efficiency, Mechanisms, Composition, Smoothness

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## 1. INTRODUCTION

The goal of our paper is to initiate the study of efficient mechanism design with guaranteed good properties even when players participate in multiple different mechanisms either simultaneously or sequentially. In most markets, (e.g. online markets) people participate in various mechanisms and the value of each player overall is a complex function of their outcomes. Predominantly, these mechanisms are run by different principals (e.g. different sellers on eBay or different ad-exchange platforms) and coordinating them to run a single combined mechanism is infeasible or impractical. The goal of this paper is to develop a theory of how to design mechanisms so that the efficiency guarantees for a single mechanism (when studied in isolation) carry over to the same or approximately the same guarantees for a market composed of such mechanisms. The key question considered in this paper can be summarized as follows:

> *What properties of local mechanisms guarantee global efficiency in a market composed of such mechanisms?*

Mechanism design is a subject with a long and distinguished history aiming to design games that produce a certain desired outcome (such as revenue or social welfare maximization) in equilibrium. However, traditional mechanism design considered such mechanisms only in isolation, an assumption not so realistic in online markets, where players can cover their needs through multiple different mechanisms. Mechanism design has mostly focused on truthful mechanisms, where players participate by revealing their true preferences to the mechanism. In an environment with several auctions running simultaneously or sequentially, truthfulness of each individual auction loses its appeal, as the global mechanism is no longer truthful, even if each individual part is. The literature's focus on truthful mechanisms is based on the revelation principle, showing that if there are better non-truthful solutions, the mechanism designer can run this alternate solution on the players' behalf. However, the revelation principle is limited to mechanisms running in isolation: with multiple mechanisms run by different parties, there is no global coordinator to implement the solution. Requiring global coordination between mechanisms is not viable and could lead to complicated coordination problems, such as agreeing on ways to divide up the global revenue.

The online market setting introduces new desiderata for designing mechanisms. Typical mechanisms used in practice are extremely simple, and not truthful. The Internet environment allows for running millions of auctions, which necessitates the use of very simple and intuitive auction schemes.

Second, participants of such a dynamic and complex setting are bound to use learning strategies. Therefore, a mechanism should have good properties even under learning behavior. Third, we cannot assume that the designer knows all parameters of the environment at the design phase. Most mechanisms in online markets run in a dynamic environment and constantly adapting the mechanism is infeasible. Last, we cannot expect the participants to know all the parameters of the game (e.g. valuations of opponents). Therefore, the mechanism should also be robust with respect to informational assumptions and should be approximately efficient, independent of the distribution of valuations.

### *Our Results.*

We define the notion of a $(\lambda, \mu)$-*smooth mechanism* and show that *smooth mechanisms* possess all the aforementioned desired properties of composability and robustness under learning behavior and incomplete information. If a mechanism has the property that in any outcome, any participant can change her bid to receive her allocation of choice by paying the price paid at the current outcome, then the equilibrium outcome and prices are *market clearing*, implying that the outcome is socially optimal. $(\lambda, \mu)$-smooth mechanisms satisfy an approximate analog of this, requiring the property only in aggregate and only approximately (both in the value of the outcome achieved by the deviating bid and in the price paid, roughly speaking achieving at least a $\lambda$ fraction of the value, for at most $\mu$ times the price), but not allowing the deviating bid to depend on the current actions of other players; a property crucial for the efficiency results described next. Our notion of smoothness is focused on mechanisms where players have quasilinear utilities and is closely related to the notion of smooth games introduced by Roughgarden [26].

**Efficiency of Smooth Mechanisms.** We show that a $(\lambda, \mu)$-smooth mechanism achieves at least a $\frac{\lambda}{\max(1,\mu)}$ fraction of the maximum possible social welfare in the full information setting. This is true in all correlated equilibria and thereby at all no-internal-regret learning outcomes [4]. We show that this result extends to the Bayesian setting with uncertainty about participants. This extension theorem generalizes to smooth games, and strengthens the results of Roughgarden [27] and Syrgkanis [29] who showed a similar extension theorem requiring a complex smoothness condition involving multiple types which additionally couldn't capture sequential games. Our proof uses a bluffing technique to handle the fact that we allow the deviating action to depend on the previous action of the deviating player (needed for sequential composition).

**Complement-Free Valuations.** For our composability results, we need to assume that user's valuations have no complements across the different mechanisms. We develop a hierarchy of valuations on outcomes that have no complements across mechanisms. Existing valuation hierarchies consider only valuations on sets of items. We identify analogs of complement-free valuations across mechanisms, without making any assumption about the valuations of players' for outcomes within a mechanism. We define a natural generalization of fractionally subadditive valuations. In the context of valuations on sets of items, fractionally subadditive is a generalization of submodular valuations, and is known to be equivalent to the class of XOS valuations. We extend the notions of fractionally subadditive and XOS valuations to our more general setting, and show that these two classes are equivalent, extending the result of Feige [10].

**Composability of Smooth Mechanisms.** We show that smooth mechanisms compose well in parallel: if we run any number of $(\lambda, \mu)$-smooth mechanisms simultaneously and players have fractionally subadditive valuations over outcomes of different mechanisms, then the global mechanism is also a $(\lambda, \mu)$-market clearing, and hence achieves a $\lambda/\max(1, \mu)$ fraction of the maximum social welfare in all correlated equilibria of the full information setting and in all mixed Bayes-Nash equilibria in the Bayesian setting.

We also show that smooth mechanisms compose well sequentially: if we run any number of $(\lambda, \mu)$-smooth mechanisms sequentially and a player's value is the maximum valued allocation she got among all mechanisms then the global game is also $(\lambda, \mu + 1)$-smooth and thereby achieves a $\lambda/(\mu + 1)$ fraction of the optimal social welfare.

**Applications.** We show that many well-known auctions are *smooth* and can be analyzed in our framework. We list a few representative examples below, and note that our composition result applies when running any set of such auctions simultaneously or sequentially.

- We show that the first price auction is $(1 - 1/e, 1)$-smooth implying an efficiency bound of approximately 1.58 for simultaneous first price item auctions, improving the bound of Hassidim et al [15] and matching [29].
- All-pay auctions, and a simple first price position auction are $(1/2, 1)$-smooth, implying a bound of 2.
- The first price greedy combinatorial auction of Lucier and Borodin [19] based on a $c$-approximation algorithm is $(1 - e^{-1/c}, c)$-smooth, improving the efficiency bound of [19] from $c + O(\log(c))$ to $c + 0.58$.
- The bandwidth allocation game of Johari and Tsitsiklis [17] is $(2 - \sqrt{3}, 1)$-smooth, proving a somewhat weaker efficiency bound than [17], but extending the bound also to Bayesian games and learning outcomes.

**No-overbidding.** For some mechanisms, such as the second price auction, good performance requires that participants do not bid above their value. For such mechanisms, we identify the notion of a weakly $(\lambda, \mu)$-smooth mechanism. Roughly speaking, we will require that bidders' declared maximum willingness to pay doesn't exceed their valuation, and add a term to the smoothness definition using the participants maximum willingness to pay. As in the case of smooth mechanisms, weakly smooth mechanisms remain weakly smooth when composed, and have high quality outcome in equilibrium (assuming no overbidding) both in the full information setting, in learning outcomes, and in the Bayesian setting.

**Budget Constraints.** The results discussed so far, assume that participants have quasi-linear valuations. The most common non-quasi-linear valuation is when players have budget constraints. We extend our results to settings where participants have budget constraints. With budget constraints, maximizing welfare is not an achievable goal, as we cannot expect a low budget participant to be effective at maximizing her contribution to welfare. We define a new benchmark in this setting, which we call the optimal "effective welfare"; capping the contribution of each player to the welfare by their budget. We show that all our results about efficiency for the case of simultaneous mechanisms carry over to bounds for this benchmark when players have budget constraints.

### Related Work.

There has been a long line of research on quantifying inefficiency of equilibria starting from Koutsoupias and Papadimitriou [18] who introduced the notion of the price of anarchy. More recently, this analysis technique has also been used to quantify the inefficiency of auction games, including games of incomplete information. A series of papers, Bikhchandani [3], Christodoulou et al [6], Bhawalkar and Roughgarden [2], Hassidim et al [15], Paes Leme et al [24], Syrgkanis and Tardos [30] studied the efficiency of equilibria of non-truthful combinatorial auctions that are based on running separate item auctions (simultaneously or sequentially) for each item. Lucier and Borodin [19] studied Bayes-Nash Equilibria of non-truthful auctions based on greedy allocation algorithms. Caragiannis et al [5] studied the inefficiency of Bayes-Nash equilibria of the generalized second price auction. All this literature can be thought of as special cases of our framework and all the proofs can be understood as smoothness proofs giving the same or even tighter results. A recent exception is the paper by Feldman et al. [11] giving a tighter bound for simultaneous item-auctions with subadditive bidders, than what would follow from our framework.

Roughgarden [26] proposed a framework, which he calls smoothness in games, and showed that a number of classical price of anarchy results (such as routing and valid utility games) can be proved using this framework. Further, he showed that such efficiency proofs carry over to efficiency of coarse correlated equilibria (no-regret learning outcomes). Nadav and Roughgarden [23] give the broadest solution concept for which smoothness proofs apply. Schoppmann and Roughgarden [28] extend the framework to games with continuous strategy spaces, providing tighter results. However, these papers consider only the full information setting and do not capture several of the auctions described previously. Our definition of a smooth mechanism is closely related to the notion of a smooth game. If utilities of the game were always non-negative (which we only assume in expectation) then a $(\lambda,\mu)$-smooth mechanism can be thought of as a $(\lambda,\mu-1)$-smooth game, but with much weaker requirements, allowing us to capture all the auctions above, as well as sequential composition.

Recent papers offer extensions of the smoothness framework to incomplete information games. Lucier and Paes Leme [20] introduced the concept of semi-smoothness (inspired by their GSP analysis), and showed that efficiency results shown via semi-smoothness extend to the incomplete information version of the game, even if the types of the players are arbitrarily correlated. Semi-smoothness is a much more restrictive property (for instance, not satisfied by the item-bidding auctions) than just requiring that every complete information instance of the game is smooth. Recently Roughgarden [27] (and independently Syrgkanis [29]) offered a more general such extension theorem. They show that one can prove bounds on the price of anarchy of an incomplete information game (assuming type distributions of players are independent) by restricting attention to induced complete information instances and proving a stronger version of the smoothness property, which [29] calls universal smoothness. Our extension theorem is based on simply assuming that for any choice of valuations, the induced full information game is smooth. In contrast, the stronger universal smoothness property relates utilities of players with different types in a single inequality. While many of the known examples satisfy this stronger notion of smoothness, our extension theorem is more natural, requiring only smoothness to hold for each instance of the player types, and does not mix over different type profiles. In addition, our smoothness is a weaker property that allows us to capture efficiency in sequential games of incomplete information in a unified framework.

A recent survey by Pai [25] highlights settings where different sellers compete by announcing mechanisms, starting from the seminal work of McAfee [22] and focusing on revenue maximization. Our work is in the same spirit, and aims to analyze the effect of such competition on social welfare.

## 2. COMPOSITION FRAMEWORK

We will consider a setting where players participate in a set of mechanisms. We assume that players' preferences are quasilinear in money. In this section we introduce our framework and set up the notation we need for defining mechanisms and compositions of mechanisms.

**Mechanism with Quasilinear Preferences.** A mechanism design setting consists of a set of $n$ players and a set of outcomes $\mathcal{X} \subseteq \times_i \mathcal{X}_i$, where $\mathcal{X}_i$ is the set of allocations for player $i$. Each player has a valuation $v_i : \mathcal{X}_i \to \mathbb{R}_+$ over allocations. Let $\mathcal{V}_i$ be the set of possible valuations of player $i$. Given an allocation $x_i \in \mathcal{X}_i$ and a payment $p_i \in \mathbb{R}_+$, we assume that the utility of player $i$ is:

$$u_i^{v_i}(x_i, p_i) = v_i(x_i) - p_i \tag{1}$$

Observe that although we assume that the outcome space is in the form of a subset of a product space, this doesn't restrict at all the space of mechanism design settings we can model, since we don't put any restriction on the structure of the subset of the product space. Hence, our model captures a range of problems, including games where players have externalities or share a single outcome. A few of the special cases are: 1) *combinatorial auctions* where $\mathcal{X}_i$ is the power set of items and $\mathcal{X}$ is the subset of this product space such that no item is assigned to more than one player, 2) *combinatorial public projects* where $\mathcal{X}_i$ is the power set of projects and $\mathcal{X}$ is the subset of the product space such that every coordinate is the same, 3) *position auctions* where $\mathcal{X}_i$ is the set of positions and $\mathcal{X}$ is the subset of the product space where no two coordinates are assigned the same position, 4) *bandwidth allocation mechanisms* where $\mathcal{X}_i$ is the portion of the bandwidth assigned to player $i$ and $\mathcal{X}$ is the subset such that the sum of the coordinates is at most the bandwidth capacity. Using this product space formulation allows us to encode which part of the outcome the valuation of a player is affected by and it facilitates the formulation of valuation classes on outcome spaces as we will see in the next section.

Given a valuation space $\mathcal{V} = \times_i \mathcal{V}_i$ and an outcome space $\mathcal{X}$, a mechanism $\mathcal{M}$ is a triple $(\mathcal{A}, X, P)$, where $\mathcal{A} = \times_i \mathcal{A}_i$ is a set of actions $\mathcal{A}_i$ for each player $i$, $X : \mathcal{A} \to \mathcal{X}$ is an allocation function that maps each action profile $a$ to an outcome $x$ and $P : \mathcal{A} \to \mathbb{R}^n_+$ is a payment function that maps each action profile to a payment $P_i$ for each player.

We will only consider settings where each player has the option to not participate, and hence at any rational outcome gets non-negative utility in expectation over the information she doesn't have and over the randomness of the other players and the mechanisms.

**The Composition Framework.** Mechanisms rarely run in isolation but rather, several mechanisms take place simultaneously and/or sequentially, and players typically have valuations that are complex functions of the outcomes of different mechanisms.

We consider the following general setting: there are $n$ bidders and $m$ mechanisms. Each mechanism $\mathcal{M}_j$ has its own outcome space $\mathcal{X}^j$ and consists of a triple $(\mathcal{A}^j, X^j, P^j)$ as described previously, i.e. $\mathcal{A}^j = \times_i \mathcal{A}_i^j$ is the action space, $X^j : \mathcal{A}^j \to \mathcal{X}^j$ is the allocation function and $P^j : \mathcal{A}^j \to \mathbb{R}_+^n$ the payment function.

We assume that a player has a valuation over vectors of outcomes from the different mechanisms: $v_i : \mathcal{X}_i \to \mathbb{R}^+$ where $\mathcal{X}_i = \times_j \mathcal{X}_i^j$. A player's utility is still quasi-linear in this extended setting in the sense that his utility from an allocation vector $x_i = (x_i^1, \ldots, x_i^m)$ and payment vector $p_i = (p_i^1, \ldots, p_i^m)$ is given by:

$$u_i^{v_i}(x_i, p_i) = v_i(x_i^1, \ldots, x_i^m) - \textstyle\sum_{j=1}^m p_i^j \tag{2}$$

We will consider both simultaneous and sequential composition of mechanisms. In the case of simultaneous composition, a player's strategy space is to report an action $a_i^j$ at each mechanism $j$. In the case of sequential composition a player can base the action she submits at mechanism $j$ on the history of the submitted action profiles in previous mechanisms (alternatively we could assume that bidders observe only allocations and payments in previous mechanisms; our results are robust to such information assumptions).

The simultaneous composition of $m$ mechanisms can be viewed as a global mechanism $\mathcal{M} = (\mathcal{A}, X, P)$, where $\mathcal{A}_i = \times_j \mathcal{A}_i^j, \mathcal{X} = \times_j \mathcal{X}^j, X(a) = (X^j(a^j))_j$ and $P(a) = \sum_j P^j(a^j)$. Sequential composition can also be viewed as a global mechanism with a more complex action space, where actions are functions of the observed history of play in earlier mechanisms. Our goal is to give properties of the individual mechanisms that guarantee efficiency of the global mechanism.

**Efficiency Measure.** We will measure efficiency of an action profile $a$ in terms of social welfare

$$SW^v(a) = \textstyle\sum_i v_i(X_i(a)) \tag{3}$$

which is the sum of the utilities of all players and the revenue of all the mechanisms. For any valuation profile $v \in \times_i \mathcal{V}_i$ there exists an optimal allocation $x^*(v)$ that maximizes $\sum_i v_i(x_i)$ over all allocations $x \in \mathcal{X}$ and we will denote with

$$\textsc{Opt}(v) = \textstyle\sum_i v_i(x_i^*(v)) \tag{4}$$

# 3. COMPLEMENT-FREE VALUATIONS

In order to infer good properties of the global mechanism from properties of individual mechanisms, we will need to assume that player valuations have no complements across outcomes of different mechanisms. Roughly speaking this means that winning more in one mechanism does not increase a player's marginal valuation in a different mechanism. When each mechanism is a single item auction, this is captured by well-understood assumptions on valuations, such as subadditive, fractionally subadditive, submodular, etc. In this work we extend these definitions to the case when the component mechanisms have arbitrary possible outcome sets, without making any assumptions on valuations within each mechanism. Since we focus on the valuation of a specific player $i$, for notational simplicity we will drop the index $i$ in the current section. We define classes of complement free valuations $v : \mathcal{X} \to \mathbb{R}_+$ on a product space of allocations $\mathcal{X} = \times_j \mathcal{X}_j$. In a composition setting, $\mathcal{X}_j$ is the set of possible allocations to player $i$ from mechanism $\mathcal{M}_j$.

In the full version of this paper we give generalizations of several classes of complement-free valuations across mechanisms, such as submodular, subadditive and monotone, together with valuation hierarchy theorems. Here we focus on the broad class of fractionally subadditive valuations (including submodular valuations), that will be important in our composability theorems.

**Definition 3.1 (Fractionally Subadditive)** *A valuation is fractionally subadditive across mechanisms if*

$$v(x) \leq \textstyle\sum_{\hat{x} \in \mathcal{X}} \alpha_{\hat{x}} v(\hat{x}),$$

*whenever each coordinate $x_j$ is covered by coefficients $\mathbf{a} = (a_{\hat{x}})_{\hat{x} \in \mathcal{X}}$, that is $\sum_{\hat{x}: x_j = \hat{x}_j} \alpha_{\hat{x}} \geq 1$.*

The above is the natural extension of the class of fractionally subadditive valuations that has been defined only for valuations defined on sets (i.e. special case of $\mathcal{X}_j \in \{0, 1\}$). In the context of set valuations it has been shown that fractionally subadditive valuations is equivalent to the class of XOS valuations. We give here the natural generalization of XOS valuations in our setting and then we show that the analogous equivalence theorem still holds, thereby extending the result of Feige [10]. We defer the details of the proof to the full version of the paper.

**Definition 3.2 (XOS)** *A valuation is XOS if there exist a set $\mathcal{L}$ of additive valuations $v_j^\ell(x_j)$, such that: $v(x) = \max_{\ell \in \mathcal{L}} \sum_j v_j^\ell(x_j)$.*

**Theorem 3.3 (XOS≡ Fractionally Subadditive)** *A valuation is fractionally subadditive over the outcomes of different mechanisms if and only if it is XOS.*

Proof sketch. It is easy to see that XOS valuations are fractionally subadditive. The interesting direction is showing that any fractionally subadditive valuation can be expressed as an XOS valuation. To achieve this we need to construct the set of additive valuations $\mathcal{L}$ such that for any allocation $x$ we will have: $v(x) = \max_{\ell \in \mathcal{L}} \sum_j v_j^\ell(x_j)$. Each additive valuation in the set $\mathcal{L}$ that we construct in the proof, will be associated with an allocation $x$ and hence we can denote it's induced valuation at mechanism $j$ as $v_j^x(\cdot)$.

Each of these induced valuations will be a single minded valuation where $v_j^x(\hat{x}_j) = t_j^x$ if $\hat{x}_j = x_j$, for some value $t_j^x$, and 0 otherwise. To compute the value $t_j^x$ associated with each outcome $x$ and mechanism $j$ we construct a linear program associated with outcome $x$, minimizing $\sum_{\hat{x} \in \mathcal{X}} \alpha_{\hat{x}} v(\hat{x})$ over all possible fractional covers $\mathbf{a} = (a_x)_{x \in \mathcal{X}}$, as given in Definition 3.1. Since the valuation is fractionally subadditive, the value of the program is equal to $v(x)$. The value $t_j^x$ will simply be the dual variable associated with the covering constraint for mechanism $j$ of the primal program. ■

In the full version, we give an analogous definition of the class of $\beta$-fractionally subadditive valuations, and show that similarly it is equivalent to a generalized version of $\beta$-XOS valuations. We also give natural definitions of monotonicity, submodularity and subadditivity *across mechanisms* and we show that monotone submodular valuations are a subclass of XOS valuations and that subadditive valuations are a subclass of $H_m$-XOS valuations, thus providing a generalized hierarchy that extends the existing one that has been shown only for valuations defined on sets of items.

**Restricted Classes of Induced Valuations.** In some applications we want to consider restricted subclasses of valuations that are natural in the context. For such applications, we want to prove a strengthening of Theorem 3.3 where if the valuation $v(x)$ comes from some class, the component valuations $v_j^\ell : \mathcal{X}_j \to \mathbb{R}^+$ used in the equivalent XOS valuation also come from this class, which may not contain the single-minded valuations used in the above proof.

We examine general classes of valuations (that may not contain single minded valuations) when the outcome space of a component mechanism is ordered or is a lattice. In this case, it is appropriate to consider only induced valuations that are monotone with respect to the predefined order, or submodular on the lattice. In the full version of the paper we provide two such strengthenings of Theorem 3.3.

# 4. SMOOTH MECHANISMS

In this section we introduce the notion of a smooth mechanism for settings where agents have quasi-linear preferences. Our notion is similar to the smoothness of games of Roughgarden [26], but is tailored to the setting of mechanisms where participants have quasilinear preferences. We denote by $u_i^{v_i}(\mathbf{a})$ the expected utility of a player $i$ with valuation $v_i$ when the randomized strategy vector $\mathbf{a}$ is played.

**Definition 4.1 (Smooth Mechanism)** *A mechanism $\mathcal{M}$ is $(\lambda, \mu)$-smooth for some $\lambda, \mu \geq 0$, if there exists a randomized action $\mathbf{a}_i^*(v, a_i)$ that is dependent on the whole valuation profile $v$ and the player's action $a_i$, such that for any valuation profile $v \in \times_i \mathcal{V}_i$ and for any action profile $a$:*

$$\textstyle\sum_i u_i^{v_i}(\mathbf{a}_i^*(v, a_i), a_{-i}) \geq \lambda \textsc{Opt}(v) - \mu \sum_i P_i(a) \qquad (5)$$

The definition of a smooth mechanism has a very natural interpretation as guaranteeing an approximate analog of market cleaning prices. Bikhchandani [3] showed that pure Nash equilibria of a simultaneous first price auction define market clearing prices, and this implies that the outcome is efficient. Aggregate market clearing prices are guaranteed when each participant can modify her bid to claim her optimal bundle at the price paid for this bundle in the current solution. A mechanism is $(1, 1)$-smooth, in essence if the above property is satisfied only in aggregate, but for any outcome of the mechanism, not only at equilibrium. While a $(\lambda, \mu)$-smooth mechanism satisfies this only approximately, both in terms of the bundle claimed, as well as the price paid for it. In addition, unlike the pure equilibrium analysis, it requires the modified bid to be ignorant of the actions of the rest of the players.

We show that smooth mechanisms have low price of anarchy and that this result extends to all correlated equilibria (and hence learning outcomes) in the complete information setting and to all Bayes-Nash equilibria in the incomplete information setting without any change in the assumption.

**Theorem 4.2** *If a mechanism is $(\lambda, \mu)$-smooth and players have the possibility to withdraw from the mechanism then the expected social welfare at any Correlated Equilibrium of the game is at least $\frac{\lambda}{\max\{\mu,1\}}$ of the optimal social welfare.*

Proof sketch. We prove the theorem for the case of a Pure Nash Equilibrium $a$. Since players have quasi-linear utilities we have: $v_i(X_i(a)) = u_i^{v_i}(a) + P_i(a)$. Using that no player wants to deviate to $\mathbf{a}_i^*(v, a_i)$ we get:

$$\begin{aligned} \textstyle\sum_i v_i(X_i(a)) \geq\ & \textstyle\sum_i u_i^{v_i}(\mathbf{a}_i^*(v, a_i), a_{-i}) + \sum_i P_i(a) \\ \geq\ & \lambda \textsc{Opt}(v) + (1-\mu) \textstyle\sum_i P_i(a) \end{aligned}$$

The result follows if $\mu \leq 1$. When $\mu > 1$, to get the result, we note that $v_i(X_i(a)) \geq P_i(a)$, as players have the possibility to withdraw from the mechanism and get 0 utility. ∎

The smoothness property of a mechanism has several differences with Roughgarden's notion of smoothness of games. To think of a mechanism as a game, we will consider the mechanism also as a player, with utility $\sum_i P_i(a)$ and no strategic decision to make. Our definition of a $(\lambda, \mu)$-smooth mechanism, is closely related to the game being $(\lambda, \mu - 1)$-smooth in the sense of [26], with two differences. We dropped the term $-(\mu-1)\sum_i u_i^{v_i}(a)$ on the right hand side, to make the definition more natural in the context of mechanisms. Note that this change makes the definitions incomparable, as with an arbitrary action profile $a$, the player utilities $u_i^{v_i}(a)$ can be negative. Second, we allow the deviating strategy $a_i^*(v, a_i)$ to depend both on the valuation vector $v$ and the strategy of the deviating player $i$. This difference causes our Theorem 4.2 to only hold for correlated equilibria, and not coarse correlated equilibria. Allowing the deviating strategy to depend on $a_i$ makes it possible to prove a composability theorem for sequential mechanisms, where it is important to allow the deviating player to "wait for the right moment" to deviate. In games where the deviation required by smoothness does not depend on $a_i$, our results extend to coarse correlated equilibria. We focus on the version that allows this dependence so as to capture sequential composition. Simultaneous composition works well with either version of the definition.

**Incomplete Information Setting.** Next we consider the case where the valuation of each player is drawn from a distribution $F_i$ over his valuation space $\mathcal{V}_i$. These distributions are independent and are common knowledge. A mechanism $\mathcal{M} = (\mathcal{A}, X, P)$ now defines a game of incomplete information. The strategy of each player is a function $s_i : \mathcal{V}_i \to \mathcal{A}_i$. We will use $s(v) = (s_i(v_i))_{i \in N}$ to denote the vector of actions given a valuation profile $v$ and $s_{-i}(v_{-i}) = (s_j(v_j))_{j \neq i}$ to denote the vector of actions for all players except $i$.

The dominant solution concept in incomplete information games is the Bayes-Nash Equilibrium (BNE). A Bayes-Nash Equilibrium is a strategy profile (possibly randomized) such that each player maximizes his expected utility conditional on his private information.

$$\forall a_i \in \mathcal{A}_i : \mathbb{E}_{v_{-i}|v_i}\left[u_i^{v_i}(s(v))\right] \geq \mathbb{E}_{v_{-i}|v_i}\left[u_i^{v_i}(a_i, s_{-i}(v_{-i}))\right]$$

Given a strategy profile $s : \times_i \mathcal{V}_i \to \times_i \mathcal{A}_i$, our measure of efficiency will be the *expected social welfare over the valuations* of the players: $\mathbb{E}_v\left[SW^v(s(v))\right]$. We will compare the efficiency of our solution concepts with respect to the *expected optimal social welfare*: $\mathbb{E}_v\left[\textsc{Opt}(v)\right]$.

**Extension Theorem.** The main result of this section is to show that if a mechanism is smooth according to definition 4.1 then it achieves a good fraction of the expected optimal social welfare at every Bayes-Nash equilibrium of the incomplete information game, irrespective of the distributions of valuations.

Note that the deviating strategy $\mathbf{a}_i^*(v, a_i)$ of player $i$ required by the smoothness property depends on the whole valuation profile $v$ and not only on the valuation of player $i$. As a result $\mathbf{a}_i^*(v, a_i)$ cannot be directly used as deviation for the player in the incomplete information game, as she is not aware of the valuations $v_{-i}$. We use random sampling to handle the dependence on the values of other players, and a bluffing technique to handle the dependence on the action of the deviating player.

**Theorem 4.3** *If a mechanism $\mathcal{M}$ is $(\lambda, \mu)$-smooth and players have the possibility to withdraw, then for any set of independent distributions $F_i$, every mixed Bayes-Nash Equilibrium of the game induced by $\mathcal{M}$ has expected social welfare at least $\frac{\lambda}{\max\{\mu,1\}}$ of the expected optimal social welfare.*

Proof. We will prove it for the case of a pure Bayes-Nash equilibrium $s(v)$ (the generalization to mixed equilibria is straightforward). Consider the following randomized deviation for each player $i$ that depends only on the information that he has which is his own value $v_i$ and the equilibrium strategies $s(\cdot)$: He random samples a valuation profile $w \sim \times_i F_i$. Then he plays $\mathbf{a}_i^*((v_i, w_{-i}), s_i(w_i))$, i.e., the player considers the equilibrium actions $s(w)$, using the randomly sampled type (including the random sample of his own type), and deviates from this action profile using the action given by the smoothness property for his true type $v_i$, the random sample of the types of the others $w_{-i}$, and the equilibrium action $s_i(w_i)$ of his randomly sampled type $w_i$. Using the action $s_i(w_i)$ as the base, corresponds to a bluffing technique that was introduced in [29] in the context of sequential first price auctions, where player $i$ "pretends" that his valuation was $w_i$ until he deviates.
Since this is not a profitable deviation for player $i$:

$$\begin{aligned}\mathbb{E}_v\left[u_i^{v_i}(s(v))\right] \geq\ & \mathbb{E}_{v,w}\left[u_i^{v_i}(\mathbf{a}_i^*((v_i, w_{-i}), s_i(w_i)), s_{-i}(v_{-i})\right]\\ =\ & \mathbb{E}_{v,w}\left[u_i^{w_i}(\mathbf{a}_i^*((w_i, w_{-i}), s_i(v_i)), s_{-i}(v_{-i}))\right]\\ =\ & \mathbb{E}_{v,w}\left[u_i^{w_i}(\mathbf{a}_i^*(w, s_i(v_i)), s_{-i}(v_{-i}))\right],\end{aligned}$$

where the first equation is an exchange of variable names, using independence. Summing over players and using smoothness:

$$\begin{aligned}\mathbb{E}_v\left[\textstyle\sum_i u_i^{v_i}(s(v))\right] \geq\ & \mathbb{E}_{v,w}\left[\textstyle\sum_i u_i^{w_i}(\mathbf{a}_i^*(w, s_i(v_i)), s_{-i}(v_{-i}))\right]\\ \geq\ & \mathbb{E}_{v,w}\left[\lambda \mathrm{OPT}(w) - \mu \textstyle\sum_i P_i(s(v))\right]\\ =\ & \lambda \mathbb{E}_w\left[\mathrm{OPT}(w)\right] - \mu \mathbb{E}_v\left[\textstyle\sum_i P_i(s(v))\right]\end{aligned}$$

By quasi-linearity of utility and using the fact that players have the possibility to withdraw from the mechanism, we have the result. ∎

**Extension Theorem for General Games.** In the full version of the paper we generalize our extension theorem to general normal form games. Specifically, we define a notion of smoothness that is the analog of our definition of a smooth mechanism replacing the total revenue on the right hand side of the smoothness property with the total social welfare in the current action profile. We still allow for the deviation to depend on the whole valuation profile and on the previous action of the player. We simply assume that the complete information game that corresponds to each instance of the valuation profile is smooth. Our notion of smoothness is more relaxed than what has been previously used in Roughgarden [27] and Syrgkanis [29] where a strengthened notion of smoothness (universal smoothness) was used to establish efficiency results in the incomplete information setting. In addition, the previous definitions of smoothness in normal form games did not allow the deviating strategy to depend on the previous action of the deviating player.

# 5. COMPOSITION THEOREMS

**Simultaneous Composition of Mechanisms.** For simultaneous composability of mechanisms we require that each mechanism is $(\lambda, \mu)$-smooth, and that the valuation is fractionally subadditive over outcomes of mechanisms. To state the result more generally, recall that Theorem 3.3 implies that the valuation is also XOS.

**Theorem 5.1 (Simultaneous Composition)** *Consider a simultaneous composition of $m$ mechanisms. Suppose that each mechanism $\mathcal{M}_j$ is $(\lambda, \mu)$-smooth when the mechanism restricted valuations of the players come from a set $(\mathcal{V}_i^j)_{i\in[n]}$. If the valuation $v_i : \mathcal{X}_i \to \mathbb{R}^+$ of each player across mechanisms is fractionally subadditive, and can be expressed as an XOS valuation by component valuations $v_{ij}^\ell \in \mathcal{V}_i^j$ then the global mechanism is also $(\lambda, \mu)$-smooth.*

Proof. Consider a valuation profile $v$ and an action profile $a$. Let $x^*$ be the optimal allocation for type profile $v$. Let $v_{ij}^*$ be the representative additive valuation for player $i$ for $x_i^*$ as implied by the definition of XOS valuations, i.e. $v_i(x_i^*) = \sum_j v_{ij}^*(x_{ij}^*)$ and for all $x_i \in \mathcal{X}_i$: $v_i(x_i) \geq \sum_j v_{ij}^*(x_i^j)$.
To prove the theorem we will show that there exists a deviation $\mathbf{a}_i^* = \mathbf{a}_i^*(v, a_i)$ of the global mechanism such that:

$$\textstyle\sum_i u_i^{v_i}(\mathbf{a}_i^*, a_{-i}) \geq \lambda \sum_i v_i(x_i^*) - \mu \sum_i P_i(a)$$

To define such a deviation we use the fact that each mechanism $\mathcal{M}_j$ is $(\lambda, \mu)$-smooth. Suppose that we run mechanism $\mathcal{M}_j$ and each player has valuation $v_{ij}^*$ on $\mathcal{X}_i^j$ and let $v_j^*$ be this valuation profile. Since, by assumption those valuations fall in the valuation space $\mathcal{V}_i^j$ for which smoothness of $\mathcal{M}_j$ holds, for any action profile $a^j$ there exists a randomized action $\mathbf{a}_{ij}^* = \mathbf{a}_{ij}^*(v_j^*, a_i^j)$ for each player, such that the sum of the utilities of the agents when each agent unilaterally deviates to it, is at least $\lambda \sum_i v_{ij}^*(x_{ij}^*) - \mu \sum_i P_i^j(a^j)$.

For the global mechanism, we consider a randomized deviation $\mathbf{a}_i^* = \mathbf{a}_i^*(v, a_i)$ of player $i$ that consists of independent randomized deviations $\mathbf{a}_{ij}^* = \mathbf{a}_{ij}^*(v_j^*, a_i^j)$ for each mechanism $j$ as described in the previous paragraph. For each action $a_i^*$ in the support of $\mathbf{a}_i^*$ we denote with $X_i(a_i^*, a_{-i})$ the outcome vector in that action profile. By the properties of the representative additive valuation, we have that $v_i(X_i(a_i^*, a_{-i})) \geq \sum_j v_{ij}^*(X_i^j(a_{ij}^*, a_{-i}^j))$. Thus the expected utility of player $i$ from the deviation will be at least:

$$u_i^{v_i}(\mathbf{a}_i^*, a_{-i}) \geq\ \mathbb{E}_{\mathbf{a}_i^*}\left[\textstyle\sum_j v_{ij}^*\left(X_i^j(\mathbf{a}_{ij}^*, a_{-i}^j)\right) - P_i^j(\mathbf{a}_{ij}^*, a_{-i}^j)\right]$$

Now adding over all players $i$ we have:

$$\textstyle\sum_i u_i^{v_i}(\mathbf{a}_i^*, a_{-i}) \geq \sum_{j,i} \mathbb{E}_{\mathbf{a}_{ij}^*}\left[v_{ij}^*(X_i^j(\mathbf{a}_{ij}^*, a_{-i}^j)) - P_i^j(\mathbf{a}_{ij}^*, a_{-i}^j)\right]$$

The key argument is that

$$\textstyle\sum_i \mathbb{E}_{\mathbf{a}_{ij}^*}\left[v_{ij}^*(X_i^j(\mathbf{a}_{ij}^*, a_{-i}^j)) - P_i^j(\mathbf{a}_{ij}^*, a_{-i}^j)\right]$$

is the sum of the expected utilities where starting from strategy profile $a^j$ each player $i$ unilaterally deviates to a randomized bid $\mathbf{a}_{ij}^* = \mathbf{a}_{ij}^*(v_j^*, a_i^j)$ in mechanism $\mathcal{M}_j$ and when each player $i$ has valuation $v_{ij}^*$ for the different outcomes $x_i^j$ of mechanism $\mathcal{M}_j$. By the $(\lambda,\mu)$-smoothness of each mechanism $\mathcal{M}_j$:

$$\begin{aligned}\sum_i u_i^{v_i}(\mathbf{a}_i^*, a_{-i}) \geq\ & \sum_j \left(\lambda \sum_i v_{ij}^*(x_{ij}^*) - \mu \sum_i P_i^j(a^j)\right)\\ =\ & \lambda \sum_i \sum_j v_{ij}^*(x_{ij}^*) - \mu \sum_i \sum_j P_i^j(a^j)\\ =\ & \lambda \sum_i v_i(x_i^*) - \mu \sum_i P_i(a)\end{aligned}$$

where we used that by the definition of the representative additive valuation $v_i(x_i^*) = \sum_{j\in[m]} v_{ij}^*(x_{ij}^*)$. ■

In Sections 6 and 9 we will give a number of applications of this result. Note that if the classes $\mathcal{V}_i^j$ contain single-minded valuations (e.g. in the case of combinatorial auctions), then our composability theorem holds for any fractionally subadditive valuation. For classes of valuations that do not contain single-minded valuations (such as ad-auctions), we can apply the strengthenings of Theorem 3.3 for monotone and lattice-submodular valuations that we refer to in Section 3 and prove in the full version of the paper.

**Sequential Composition of Mechanisms.** In many scenarios, mechanisms might not all take place simultaneously. Sequentiality however can lead to inefficiencies as was shown by recent works on sequential auctions [24, 29]. Here, we show that the positive results of [24, 29] on unit-demand sequential first price auctions are a special case of a more general property of smooth mechanisms. For the sequential composition of $m$ mechanisms we prove that if each mechanism $\mathcal{M}_j$ is $(\lambda,\mu)$-smooth, then the resulting mechanism is $(\lambda,\mu+1)$-smooth (for the normal form representation of the extensive form of game) if an agents valuation is the best of her valuation over the different mechanisms: $v_i(x_i) = \max_{j\in[m]} v_i^j(x_i^j)$.

An interesting aspect of the sequential composition is that the strategy of a player is no longer just an action $a_i^j \in \mathcal{A}_i^j$ for each mechanism but rather a whole contingency plan of what action she will submit to mechanism $\mathcal{M}_j$ conditional on any observed history of play. Our result doesn't depend on what part of the history is observed by the players, whether players just observe their own allocation, or all allocations, or also all prices, or bids. We don't even need that all players observe the same things. However, we do assume that the information structure is common knowledge.

**Theorem 5.2 (Sequential Composition)** *Consider a sequential composition of $m$, $(\lambda,\mu)$-smooth, mechanisms defined on valuation spaces $\mathcal{V}_i^j$. If each valuation $v_i : \mathcal{X}_i \to \mathbb{R}^+$ of is of the form $v_i(x_i) = \max_{j\in[m]} v_i^j(x_i^j)$, with $v_i^j \in \mathcal{V}_i^j$, then the global mechanism is $(\lambda,\mu+1)$-smooth, independent of the information released to players during the sequential rounds.*

We defer the proof of this theorem to the full version of the paper. There we also combine these two theorems to prove efficiency guarantees when mechanisms are run in a sequence of rounds and at each round several mechanisms are run simultaneously.

## 6. FIRST APPLICATION: ITEM AUCTIONS

In this section we present a simple, yet rich, application of our framework to the case where each component mechanism is a single-item auction. We consider the three main single-item auctions: first-price, all-pay and second-price.

**First Price Auction.** The first price auction is a $(1-1/e, 1)$-smooth mechanism. To see why smoothness holds, note that under any valuation profile $v$ (observe that we only need to argue about the full information setting), the highest value player with value $v_h$ can deviate to submitting a randomized bid $b_h'$ drawn from a distribution with density function $f(x) = \frac{1}{v_h - x}$ and support $[0, (1-1/e)v_h]$, while all non-highest value players should just deviate to bidding 0. No matter what the rest of the players are bidding, the utility of the highest bidder from the deviation is:

$$\begin{aligned}u_h^{v_h}(b_h', b_{-h}) \geq\ & \int_{\max_{i\neq h} b_i}^{(1-\frac{1}{e})v_h} (v_h - x) f(x)dx\\ \geq\ & \left(1-\tfrac{1}{e}\right) v_h - \max_i b_i\end{aligned}$$

Theorem 5.1 now implies that if we run $m$ simultaneous first price auctions and bidders have fractionally subadditive valuations then any Correlated Equilibrium in the full information setting and any mixed Bayes-Nash Equilibrium in the incomplete information setting has social welfare at least $(1-\frac{1}{e})$ of the optimal. A looser result of $1/4$ for this setting and only for mixed Nash and Bayes-Nash appeared in [15]. The tighter result of $(1-\frac{1}{e})$ appeared in [29]. Theorem 5.2 implies that if we run $m$ first price auctions sequentially and bidders have unit-demand valuations then any Correlated Equilibrium in the full information setting and any Bayes-Nash Equilibrium in the incomplete information setting has social welfare at least $\frac{1}{2}(1-\frac{1}{e})$ of the optimal. The latter result was given in a sequence of two papers [24, 29].

**All-Pay Auction.** The all-pay auction is a $(1/2, 1)$-smooth mechanism. The smoothness proof is similar to the first price auction with the only alteration that we make the highest value player submit a bid drawn uniformly at random from $[0, v_h]$. The utility from such a deviation is:

$$\begin{aligned}u_h^{v_h}(b_h', b_{-h}) \geq\ & \int_{\max_{i\neq h} b_i}^{v_h} v_h f(x)dx - \mathbb{E}[b_h']\\ \geq\ & \tfrac{1}{2}v_h - \max_i b_i \geq \tfrac{1}{2}v_h - \sum_i b_i\end{aligned}$$

Therefore we get an efficiency guarantee of $1/2$ for the simultaneous composition of $m$ all-pay auctions and an efficiency guarantee of $1/4$ for the sequential composition both in the Bayesian setting and in learning outcomes. Simultaneous and sequential all-pay auctions have not been studied in the literature and could prove useful in capturing simultaneous or sequential all-pay contests, which is a natural model for several online crowd-sourcing environments.

**Second-Price Auction.** The second price auction is not a smooth mechanism. In fact, the second price auction is not as robust as the previous auctions. Second price auctions have arbitrary bad equilibria when players bid above their value. Goeree [14] shows that signaling is bound to arise in a second price auction when bidders are strategising about future opportunities, and Paes Leme et al [24] show an example with unbounded inefficiency when running second price auctions sequentially and bidders are unit-demand. The main difference of the second price auction and the previous two auctions is that it makes very loose connection between the bid a player needs to make to win and the price that was previously paid to the auctioneer. Several papers [6, 2, 21, 5] have used an assumption that players will not bid above their valuations to give good efficiency guarantees for second-price type of auctions. Next, we extend our results to mechanisms that require such no-overbidding assumptions.

# 7. WEAK SMOOTHNESS

In this section we give a generalization of our framework to capture mechanisms that produce high efficiency under a no-overbidding refinement. First, we give a definition of no-overbidding that generalizes the no-overbidding assumption. In a single-item second-price auction the bid of a player is his maximum willingness to pay when he wins. The following defines maximum willingness to pay in the general mechanism design setting.

**Definition 7.1 (Declared Willingness to Pay)** *Given a mechanism $(\mathcal{A}, X, P)$, a player's declared maximum willingness to pay for an allocation $x_i$ when using strategy $a_i$, is defined as the maximum he could ever pay conditional on allocation $x_i$:*

$$B_i(a_i, x_i) = \max_{a_{-i}:\ X_i(a)=x_i} P_i(a) \tag{6}$$

**Definition 7.2 (Weakly Smooth Mechanism)** *A mechanism is weakly $(\lambda, \mu_1, \mu_2)$-smooth for $\lambda, \mu_1, \mu_2 \geq 0$, if there exists a randomized action $\mathbf{a}_i^*(v, a_i)$ for each player $i$, that is dependent on the whole valuation profile $v$ and on his previous actions $a_i$, such that for any valuation profile $v \in \times_i \mathcal{V}_i$ and for any action profile $a$:*

$$\sum_i u_i^{v_i}(\mathbf{a}_i^*(v, a_i), a_{-i}) \geq \lambda \textsc{Opt}(v) - \mu_1 \sum_i P_i(a) - \mu_2 \sum_i B_i(a_i, X_i(a))$$

**Definition 7.3 (No-overbidding)** *A randomized strategy profile* $\mathbf{a}$ *satisfies the no-overbidding assumption if:*

$$\mathbb{E}_{\mathbf{a}}[B_i(\mathbf{a}_i, X_i(\mathbf{a}))] \leq \mathbb{E}_{\mathbf{a}}[v_i(X_i(\mathbf{a}))] \tag{7}$$

*i.e., at this strategy profile no player is bidding in a way that she could potentially pay more than her value subject to her expected allocation remaining the same.*

Note that the smoothness property must be satisfied for any action profile and not only for non-overbidding action profiles. This is essential for the smoothness property to be composable. Note that in a composition setting, local no-overbidding is not meaningful, since there is no clear induced valuation of a player at each local mechanism and is only meaningful on the overall mechanism. As an example, note that the single item second price auction is weakly $(1, 0, 1)$-smooth, but is not $(1, 1)$-smooth (which would be true if smoothness was required only for non-overbidding strategies), and while a single item second price auction is optimal, this property is not maintained in composition.

**Theorem 7.4** *If a mechanism is weakly $(\lambda, \mu_1, \mu_2)$-smooth then any Correlated Equilibrium in the full information setting and any mixed Bayes-Nash Equilibrium in the Bayesian setting that satisfies the no-overbidding assumption achieves efficiency at least $\frac{\lambda}{\mu_2 + \max\{\mu_1, 1\}}$ of the expected optimal.*

In the full version, we show, analogously to the results in Section 5, that the simultaneous composition of weakly $(\lambda, \mu_1, \mu_2)$-smooth mechanisms is weakly $(\lambda, \mu_1, \mu_2)$-smooth and the sequential composition is weakly $(\lambda, \mu_1 + 1, \mu_2)$-smooth.

**Remark 1.** In contrast to the smoothness used in [27] our definition of weakly smooth mechanisms allows us to prove efficiency under the weaker assumption of no-overbidding in expectation, rather than point-wise no-overbidding. The main difference is that we incorporate the willingness-to-pay inside the smoothness definition, while previous smoothness approaches would relate to value directly. The latter approach would require to use point-wise no-overbidding to relate bids to welfare in second-price auctions.

**Remark 2.** We use the non-overbidding assumption as an equilibrium refinement rather than as a strategy-space restriction. Several papers in the literature have used non-overbidding as a strategy space restriction (rather than as an equilibrium refinement). The two uses are equivalent in settings where the restricted strategy space always contains best-responses. Note that while overbidding is a dominated strategy in a single item auction, global no-overbidding is not dominated when running second price auctions simultaneously or sequentially. Overbidding equilibria that survive elimination of dominated strategies and that have non-constant inefficiency have been given both for the case of sequential [24] and simultaneous [11] second price auctions, even in the simplest scenario when bidders are unit-demand. Restricting the strategy space to non-overbidding strategies, could potentially create artificial equilibria that were not equilibria of the original game, since this restricted strategy space does not always contain best-responses (see [11] for an example). On the other hand, the refined set of non-overbidding equilibria might be empty. Some of our results carry over to the strategy-space restriction version. A detailed exposition is deferred to the full version.

# 8. BUDGET CONSTRAINTS

An important class of non-quasilinear preferences is when players have hard budget constraints on the payments they make. Studying the effect of budgets on efficiency has received great attention in recent algorithmic game theory literature [7, 12, 13, 8] mostly in the realm of truthful mechanism design and assuming that the budgets are common knowledge. Little is known about the effect of budgets in the case of non-truthful mechanisms. For instance, only recently Huang et al. [16] analyzed efficiency in a two-player sequential first price auction game with budget constraints in the complete information setting.

Most of the literature has focused on producing pareto-optimal outcomes, i.e. a pair of allocation and prices such that there is no other pair that respects feasibility and budget constraints and such that all players have strictly higher utility and the auctioneer receives strictly higher revenue.

We introduce an orthogonal benchmark, which we call *Effective Welfare*, by capping a player's value by his budget:

$$EW(x) = \sum_i \min\{v_i(x_i), B_i\} \tag{8}$$

We compare the social welfare resulting in our mechanism to the maximum possible effective welfare. This benchmark reflects that we cannot expect players with low budgets to be effective at maximizing their own value.

We show that a lot of our results carry over to the effective welfare benchmark, by introducing a strengthening of the smoothness property of mechanisms (a strengthening that is satisfied by almost all the applications we consider) and assuming that the valuation spaces $\mathcal{V}$ considered is closed under capping, i.e., for any valuation $v \in \mathcal{V}$ and any budget $B$, we also have $\min(v(x), B) \in \mathcal{V}$. We focus on smooth mechanisms, but all the results in this section extend to weakly smooth mechanisms assuming no-overbidding.

**Definition 8.1 (Conservatively Smooth Mechanism)** *A mechanism is conservatively $(\lambda,\mu)$-smooth if it is $(\lambda,\mu)$-smooth in the quasilinear utility setting and the actions in the support of the smoothness deviations satisfy:*

$$\max_{a_{-i}} P_i(a_i^*(v,a_i),a_{-i}) \leq \max_{x_i\in\mathcal{X}_i} v_i(x_i) \qquad (9)$$

The next theorem shows that the expected social welfare at Correlated Equilibria and at Bayes-Nash equilibria of conservatively smooth mechanisms is a good fraction of the optimal effective welfare. Note that in the incomplete information setting, the private information of a player is his valuation and his budget. We will denote the valuation and budget pair as the type $t_i = (v_i, B_i)$ of player $i$ and we will assume that it is distributed independently according to some distribution $F_i$ on $\mathcal{V}_i \times \mathbb{R}^+$. Note that we allow the budget of a player to be correlated with his valuation.

**Theorem 8.2** *If a mechanism is conservatively $(\lambda,\mu)$-smooth and its valuation space is closed under capping, then the social welfare at any correlated equilibrium and at any Bayes-Nash equilibrium is at least $\frac{\lambda}{\max\{1,\mu\}}$ of the expected maximum effective welfare.*

Last we show that efficiency guarantees for bidders with budget-constraints are simultaneously composable under the conservative smoothness property Unfortunately, sequential composition doesn't carry over. In sequential mechanisms a good deviation may require that the player waits and plays according to equilibrium until his optimal mechanism arrives. While "waiting" he might exhaust his budget.

**Theorem 8.3** *Consider the simultaneous composition of $m$ conservatively $(\lambda,\mu)$-smooth mechanisms defined on valuation spaces $\mathcal{V}_i^j$ that are closed under capping. If players have XOS valuations and can be expressed by valuations $v_{ij}^\ell \in \mathcal{V}_i^j$ then the social welfare at any correlated equilibrium and at any Bayes-Nash equilibrium of the global mechanism is at least $\frac{\lambda}{\max\{1,\mu\}}$ of the expected maximum effective welfare.*

The composability result is proved in a sequence of two lemmas: first we prove that the conservative smoothness property of a mechanism composes under XOS valuations and second we show that if the valuation space of each component mechanism is closed under capping then the corresponding valuation space of the composition mechanism is also closed under capping. The latter is shown by proving a structural property of XOS valuations: a valuation produced by capping an XOS valuation is also XOS and can be described by component valuations that are cappings of the component valuations of the XOS representation of the initial valuation. Using these two lemmas we can invoke Theorem 8.2 to get efficiency guarantees for budget constrained bidders in the global mechanism.

# 9. APPLICATIONS

In this section we give several applications of our framework. Some are new smoothness proofs implying new bounds on efficiency, others are reinterpretations of existing literature as smoothness proofs. In each case adding budget constraints gives new results on efficiency of mechanisms, and our results show that the efficiency is preserved by composition of mechanisms. The efficiency guarantees hold for correlated equilibria in the full information setting and for mixed Bayes-Nash equilibria in the incomplete information setting. Our guarantees are with respect to the optimal effective welfare when the players have budget constraints.

**Single Item Auctions.** Extending the results of Section 6 we show that the first price single item auction is conservatively $\left(1-\frac{1}{e},1\right)$-smooth, the all-pay auction is conservatively $\left(\frac{1}{2},1\right)$-smooth and the second price auction is weakly and conservatively $(1,0,1)$-smooth. We also give a smoothness proof for the hybrid auction in which the winner pays a convex combination of her own bid and the second highest bid. Our framework implies that running $m$ simultaneous first price auctions and bidders have fractionally subadditive valuations and budget constraints achieves efficiency at least $1-\frac{1}{e}$ of the optimal effective welfare. All-pay auctions achieve a guarantee of $\frac{1}{2}$. Second price auctions achieve a guarantee of $\frac{1}{2}$ under the no-overbidding assumption. For sequential auctions with unit-demand bidders and no budget constraints the first price, all-pay and second price auctions give guarantees of $\frac{1}{2}(1-\frac{1}{e})$, $\frac{1}{4}$ and $\frac{1}{2}$ respectively.

**Greedy Direct Auctions.** Lucier and Borodin [19] considers combinatorial auctions, whose allocation function is based on a greedy $c$-approximation algorithm. When a first price payment is used, they show that such a greedy auction has a $c + O(\log(c))$ efficiency guarantee. We improve this bound, by showing that this mechanism is conservatively $(1-e^{-1/c}, c)$-smooth implying an efficiency guarantee of at least $\frac{1}{c+0.58}$. This bound extends to the simultaneous composition of such mechanisms when bidders have fractionally subadditive valuations across auctions and budget constraints. For example, when each auctions sells only a small number of items, greedy algorithms can do quite well (giving a $\sqrt{k}$-approximation for arbitrary valuations, if each auction sells at most $k$ items). Observe, that fractionally subadditive valuations across auctions allow for complements within the items of a single greedy auction, hence is more general than just assuming that players have fractionally subadditive valuations over the whole universe of items. In the full version of this paper, we show that the above analysis is a special case of a more general class of *direct auctions.*

**Position Auctions.** We analyze position auctions for more general valuation spaces than what has been typically considered [9, 5]. We use the model of Abrams et al [1], where each player $i$ has an arbitrary valuation $v_{ij}$ for appearing at position $j$, that is monotone in the position. Most of the literature in position auctions has considered valuations of the form $v_{ij} = a_j\gamma_i v_i$, i.e. players have only value per click $v_i$ and their click-through-rate is dependent in a separable way on their quality and on the position. The more general class of valuations can capture settings where players have value both for click and for the impression itself, and settings where the click-through-rates are not separable. We show that the following very simple first price analog of the auction of [1] is conservatively $(\frac{1}{2},1)$-smooth: solicit bids from the players, allocate positions in order of bids and charge each player his bid. The implied guarantee of $\frac{1}{2}$ holds for simultaneous composition when players have monotone fractionally subadditive valuations and budget constraints. Such valuations capture, for instance, settings where bidders have value $v_i$ only for the first $k$ clicks, or settings where the marginal value per-click of a player decreases with the number of clicks he gets. In addition, a bound of $\frac{1}{4}$ is implied for the sequential composition when bidders value is the maximum value among all impressions he got. In contrast, [1] consider the second price analog of this auction, and show that it always has an efficient Nash equilibrium, but do not consider the price of anarchy. We show that the second price version is conservatively weakly $(\frac{1}{2}, 0, 1)$-smooth, implying an efficiency guarantee of $\frac{1}{4}$ for simultaneous and sequential composition of such auctions under the no-overbidding assumption. In the full version of this paper we also consider other variations of the well-studied GFP and GSP mechanisms for the case when players have only values per click.

**Bandwidth Allocation Mechanisms.** We consider the setting studied by Johari and Tsitsiklis [17] where a set of players want to share a resource: an edge with bandwidth $C$. Each player has a concave valuation $v_i(x_i)$ for getting $x_i$ units of bandwidth. The mechanism studied in [17] is the following: solicit bids $b_i$, allocate to each player bandwidth proportional to his bid $x_i = \frac{b_i}{\sum_j b_j}$, charge each player $b_i$. We show that this mechanism is conservatively $(2-\sqrt{3}, 1)$-smooth, implying an efficiency guarantee of approximately 1/4 for correlated equilibria and Bayes-Nash equilibria. The same efficiency guarantee extends to the case when we run such mechanisms simultaneously and players have budget constraints and monotone, lattice-submodular valuations on the lattice defined on $\mathbb{R}^m$ by the coordinate-wise ordering. If the valuations are twice differentiable, being monotone and lattice-submodular translates to: every partial derivative is non-negative and every cross-derivative is non-positive.

**Multi-Unit Auctions.** For the setting of multi-unit auctions where players have concave utilities in the amount of units they get, we give two smooth mechanisms. Recently, Markakis et al. [21] studied a greedy mechanism and showed a $O(\log(m))$ approximation for the case of mixed Bayes-Nash equilibria under a no-overbidding assumption. In the full version of this paper, we show that a first price version of their mechanism where each player is charged his declared marginal bids for the items he acquired is conservatively $\left(\frac{1}{2}\left(1-\frac{1}{e}\right), 1\right)$-smooth, while the uniform price version of [21] is weakly $\left(\frac{1}{2}\left(1-\frac{1}{e}\right), 0, 1\right)$-smooth. Therefore our smooth analysis improves the $O(\log(m))$ bound of [21] to a constant $\frac{1}{4}\left(1-\frac{1}{e}\right)$ and to $\frac{1}{2}\left(1-\frac{1}{e}\right)$ when a first price payment rule is used.

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