

Computational Game Theory

Sheet 2

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October 27, 2023

Exercise 1

- (a) Consider the following payoff matrix for a 2 player game:

		Player 2	
		L	R
Player 1	T	$(1, 1)$	$(1, 0)$
	B	$(1, 1)$	$(0, -1)$

Here both (T, L) and (B, L) are Nash equilibria, and while T is a dominant strategy for player 1, B is not. Hence there is a Nash equilibrium in which a player has not chosen a dominant strategy.

- (b) This is false. Simply consider the following payoff matrix:

		Player 2	
		L	R
Player 1	T	$(1, 1)$	$(1, 1)$
	B	$(1, 1)$	$(1, 1)$

Here every combination of strategies is a dominant strategy equilibrium. Hence there is no reason for a dominant strategy equilibrium to be unique.

- (c) If $\vec{\sigma}$ is a dominant strategy equilibrium then we must show that there is no $i \in N$ and $\sigma'_i \in \Sigma_i$ such that

$$u_i(\vec{\sigma}_{-i}, \sigma'_i) > u_i(\vec{\sigma}).$$

This follows directly from the definition of a dominant strategy equilibrium. As if such σ'_i were to exist then it would contradict the assumption that

$$u_i(\vec{\sigma}_{-i}, \sigma_i) \geq u_i(\vec{\sigma}_{-i}, \sigma'_i).$$

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- (d) This is not true. To see why, consider the following payoff matrix

		Player 2	
		L	R
Player 1	T	$(-2, -2)$	$(-3, -3)$
	B	$(-3, 0)$	$(-1, -1)$

We see that the strategy $\vec{\sigma} = (T, T)$ is a Nash equilibrium. However, it is not a dominant strategy equilibrium. To see why, suppose player Y choose R . Then the best response for X is to choose T . Hence X 's best strategy is dependent upon the strategy Y chooses which goes against the defining property of a dominant strategy.

- (e) Suppose ω is not Pareto efficient. Then there exist a player $i \in N$ such that we can keep the utility for each player $j \neq i$ the same (or increase it) while increasing the utility for player i . This new outcome ω' will then have higher utilitarian social welfare compared to the previous outcome ω . Hence ω does not maximize utilitarian social welfare. This proves the contrapositive of the claim which is equivalent to proving the claim itself.
- (f) This is not true. To see why, consider the payoff matrix:

		Player 2	
		L	R
Player 1	T	$(2, 2)$	$(0, 1)$
	B	$(1, 0)$	$(1, 1)$

Here (B, B) is Pareto efficient, but does not maximize utilitarian social welfare.

- (g) Suppose all utilities are positive. Let $\omega^* \in \Omega$ be such that

$$\omega^* = \operatorname{argmax}_{\omega \in \Omega} \prod_{i=1}^n u_i(\omega).$$

Suppose there was a player j and a different outcome $\omega' \neq \omega$ such that $u_j(\omega') > u_j(\omega^*)$ and $u_i(\omega') \geq u_i(\omega^*)$ for all other $i \neq j$. Then, since $u_i(\omega^*) > 0$ for $i = 1, \dots, n$, we must have that

$$\prod_{i=1}^n u_i(\omega') > \prod_{i=1}^n u_i(\omega^*)$$

which is a contradiction. Hence it must be the case that ω^* is Pareto efficient.

- (h) This is not true. To see why, consider the following payoff matrix:

		Player 2	
		L	R
Player 1	T	$(3, 3)$	$(1, 1)$
	B	$(1, 1)$	$(2, 2)$

We see that (B, B) is Pareto efficient, but it does not maximize the product of utilities of players.

Exercise 2

- (a) As expected utility is defined to be the sum over the utility of all outcomes multiplied with the probability of that outcome occurring, we must have that

$$EU_1(p, q) = v_1^1 pq + v_1^2 p(1 - q) + v_1^3 (1 - p)q + v_1^4 (1 - p)(1 - q)$$

$$EU_2(q, p) = v_2^1 pq + v_2^2 p(1 - q) + v_2^3 (1 - p)q + v_2^4 (1 - p)(1 - q)$$

where v_j^i is the utility player i gets for the j th outcome.

- (b) Let $P(\vec{\sigma}, \vec{\mu})$ be the probability that the pure strategy combinations $\vec{\sigma}$ is played. We must then have that

$$P(\vec{\sigma}, \vec{\mu}) = \prod_{i=1}^n \mu_i(\sigma_i)$$

assuming that all players make choices independently. We want to define $EU(\vec{\mu})$. Having defined $P(\vec{\sigma}, \vec{\mu})$ this is easy enough as there is only one way to define expected value:

$$EU_i(\mu) = \sum_{\vec{\sigma} \in \Sigma} u_i(\vec{\sigma}) P(\vec{\sigma}, \vec{\mu}).$$

This sum is well-defined as we assume the set of outcomes Ω is finite.

Exercise 3

- (a) (i) There are no dominant strategies, hence no dominant strategy equilibria. However, there are two pure Nash equilibria given by (T, L) and (B, R) .
- (ii) The two Nash equilibria mentioned above both maximize utilitarian social welfare. Thus they must also both be Pareto efficient by Exercise 1 (e). In addition, both Nash equilibria also maximize egalitarian social welfare as the worst off player in both scenarios has utility 1.
- (iii) Using the Indifference Principle we get two equations

$$EU_1(T, q) = EU_1(B, q)$$

$$EU_2(L, p) = EU_2(R, p).$$

Concentrating on the first equation we get

$$\begin{aligned} 2 \cdot q + 0 \cdot (1 - q) &= 0 \cdot q + 1 \cdot (1 - q) \\ 2q &= 1 - q \\ q &= \frac{1}{3}. \end{aligned}$$

For the second equation we have

$$\begin{aligned} 1 \cdot p + 0 \cdot (1 - p) &= 0 \cdot p + 2 \cdot (1 - p) \\ p &= 2 - 2p \\ p &= \frac{2}{3}. \end{aligned}$$

Hence there is a fully mixed Nash equilibrium given by $(p, q) = (\frac{2}{3}, \frac{1}{3})$.

(iv) We have that

$$\begin{aligned} EU_1\left(\frac{2}{3}, \frac{1}{3}\right) &= 2 \cdot \frac{2}{3} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{3} \\ EU_2\left(\frac{1}{3}, \frac{2}{3}\right) &= 1 \cdot \frac{2}{3} \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{3}. \end{aligned}$$

(v) Knowing that the Nash equilibrium is $(p, q) = (\frac{2}{3}, \frac{1}{3})$ we get that the best response for player 1 given q is given by

$$BR_1(q) = \begin{cases} \{0\}, & \text{if } q < \frac{1}{3} \\ [0, 1], & \text{if } q = \frac{1}{3} \\ \{1\}, & \text{if } q > \frac{1}{3} \end{cases}.$$

Similarly, for player 2 we have

$$BR_2(p) = \begin{cases} \{0\}, & \text{if } p < \frac{2}{3} \\ [0, 1], & \text{if } p = \frac{2}{3} \\ \{1\}, & \text{if } p > \frac{2}{3} \end{cases}.$$

From this we get the following response curve plot

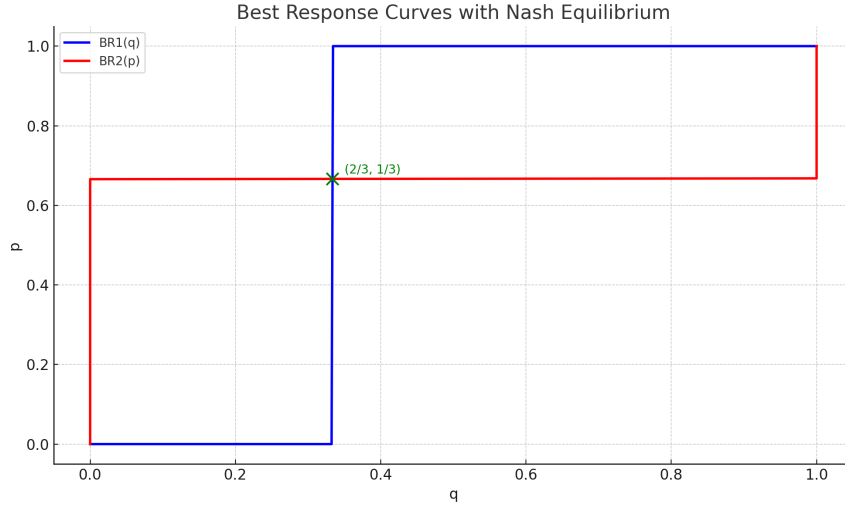


Figure 1: The best response for player 1 and player 2 given the other player's strategy. The point in the middle is the Nash equilibrium

- (b) (i) There are no dominant strategies and hence no dominant strategy equilibria. There are, however, two pure Nash equilibria given by (B, L) and (T, R) .
- (ii) Both Nash equilibria maximize utilitarian social welfare and so are both Pareto optimal. However, only (B, L) maximizes egalitarian social welfare.
- (iii) Using the same two equations as before we get that the first equation turns into

$$\begin{aligned} 0 \cdot q + 3 \cdot (1 - q) &= 4 \cdot q + 0 \cdot (1 - q) \\ 3 - 3q &= 4q \\ q &= \frac{3}{7} \end{aligned}$$

while the second equation gives

$$\begin{aligned} 0 \cdot p + 4 \cdot (1 - p) &= 5 \cdot p + 3 \cdot (1 - p) \\ 4 - 4p &= 5p + 3 - 3p \\ p &= \frac{1}{6}. \end{aligned}$$

We thus have a fully mixed Nash equilibrium given by $(\frac{1}{6}, \frac{3}{7})$.

- (iv) Computing the utilities, we get

$$\begin{aligned} EU_1\left(\frac{1}{6}, \frac{3}{7}\right) &= 3 \cdot \frac{1}{6} \cdot \frac{4}{7} + 4 \cdot \frac{5}{6} \cdot \frac{3}{7} = \frac{12}{7} \\ EU_2\left(\frac{3}{7}, \frac{1}{6}\right) &= 5 \cdot \frac{1}{6} \cdot \frac{4}{7} + 4 \cdot \frac{5}{6} \cdot \frac{3}{7} + 3 \cdot \frac{5}{6} \cdot \frac{4}{7} = \frac{7}{3}. \end{aligned}$$

- (v) Since the Nash equilibrium is $(p, q) = (\frac{1}{6}, \frac{3}{7})$ we have the following best response functions

$$BR_1(q) = \begin{cases} \{0\}, & \text{if } q < \frac{3}{7} \\ [0, 1], & \text{if } q = \frac{3}{7} \\ \{1\}, & \text{if } q > \frac{3}{7} \end{cases}$$

$$BR_2(p) = \begin{cases} \{0\}, & \text{if } p < \frac{1}{6} \\ [0, 1], & \text{if } p = \frac{1}{6} \\ \{1\}, & \text{if } p > \frac{1}{6} \end{cases}$$

from which we get the plot:

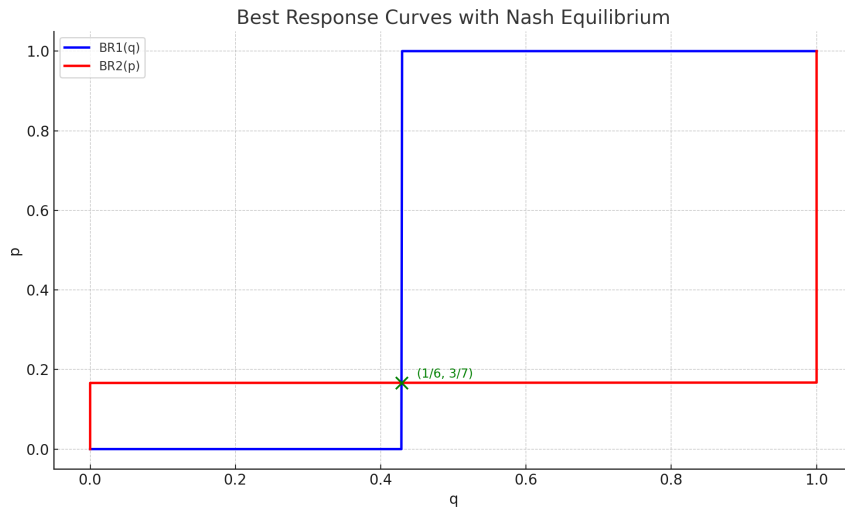


Figure 2: The best response for player 1 and player 2 given the other player's strategy. The point in the middle is the Nash equilibrium

Exercise 4

There are two directions to prove.

(\Rightarrow): Assume that $(p, q) \in (0, 1)^2$ is a fully mixed strategy Nash equilibrium in the generic 2×2 game. We can without loss of generality focus on the first equation. To see why it must hold, consider the fact that

$$EU_1(p, q) = pEU_1(T, q) + (1 - p)EU_1(B, q).$$

If either $EU_1(T, q) > EU_1(B, q)$ or $EU_1(T, q) < EU_1(B, q)$ then player 1 would be better off choosing a pure strategy T or U . Hence to satisfy the requirement that (p, q) is fully mixed Nash equilibrium we must have that $EU_1(T, q) = EU_1(B, q)$.

(\Leftarrow): Assume the two equations hold. Considering again the equation

$$EU_1(p, q) = pEU_1(T, q) + (1 - p)EU_1(B, q)$$

we see that this must imply (p, q) is a Nash equilibrium since there is no way for player 1 to alter his strategy so that the expected utility gets strictly higher. No matter what value he chooses for p the expected utility will always be the same and so he has no reason to deviate from his current strategy.

Exercise 5

- (a) If we suppose that the utility function for both firms is simply the profit then we can describe this setting by the following 2-player payoff matrix:

		Player Y	
		Buy	Not Buy
Player X	Buy	$(\frac{2+n}{6+2n} - \frac{P}{2}, \frac{4+n}{6+2n} - \frac{P}{2})$	$(\frac{1+n}{3+n} - P, \frac{2}{3+n})$
	Not Buy	$(\frac{1}{3+n}, \frac{2+n}{3+n} - P)$	$(\frac{1}{3}, \frac{2}{3})$

- (b) If *Buy* is to be a weakly dominant strategy for firm X then we need that

$$\begin{aligned} u_X(\text{Buy}, \text{Buy}) &\geq u_X(\text{Not Buy}, \text{Buy}) \\ u_X(\text{Buy}, \text{Not Buy}) &\geq u_X(\text{Not Buy}, \text{Not Buy}) \end{aligned}$$

and in one of these cases the inequality needs to be strict. The two inequalities gives us the conditions

$$\begin{aligned} \frac{n}{3+n} &\geq P \\ \frac{2n}{9+3n} &\geq P. \end{aligned}$$

Hence, *Buy* is a weakly dominant strategy for X if $P \leq 2n/(9+3n)$ since in this case we also get one strict inequality.

Doing the same for player Y we have the two inequalities

$$\begin{aligned} u_Y(\text{Buy}, \text{Buy}) &\geq u_Y(\text{Buy}, \text{Not Buy}) \\ u_Y(\text{Not Buy}, \text{Buy}) &\geq u_Y(\text{Not Buy}, \text{Not Buy}) \end{aligned}$$

from which we get the requirements

$$\begin{aligned} \frac{n}{3+n} &\geq P \\ \frac{n}{9+3n} &\geq P. \end{aligned}$$

Thus, *Buy* is a weakly dominant strategy for Y if $P \leq \frac{n}{9+3n}$.

- (c) For (Buy, Buy) to be a Nash equilibrium we need that

$$\begin{aligned} u_X(\text{Buy}, \text{Buy}) &\geq u_X(\text{Not Buy}, \text{Buy}) \\ u_Y(\text{Buy}, \text{Buy}) &\geq u_Y(\text{Buy}, \text{Not Buy}) \end{aligned}$$

which gives

$$\frac{n}{3+n} \geq P$$

$$\frac{n}{3+n} \geq P.$$

Hence (Buy, Buy) is a Nash equilibrium iff $P \leq n/(3+n)$.

- (d) Similarly, for $(Not\ Buy, Not\ Buy)$ to be a Nash equilibrium. We have the two inequalities

$$u_X(Not\ Buy, Not\ Buy) \geq u_X(Buy, Not\ Buy)$$

$$u_Y(Not\ Buy, Not\ Buy) \geq u_Y(Not\ Buy, Buy)$$

which gives

$$P \geq \frac{2n}{9+3n}$$

$$P \geq \frac{n}{9+3n}$$

and so we have a Nash equilibrium for $(Not\ Buy, Not\ Buy)$ when $P \geq 2n/(9+3n)$.

- (e) For $(Buy, Not\ Buy)$ we have the inequalities

$$u_X(Buy, Not\ Buy) \geq u_X(Not\ Buy, Not\ Buy)$$

$$u_Y(Buy, Not\ Buy) \geq u_Y(Buy, Buy)$$

which gives

$$\frac{2n}{9+3n} \geq P$$

$$\frac{n}{3+n} \leq P.$$

Thus there can be no Nash equilibrium for $(Buy, Not\ Buy)$. Similarly, for $(Not\ Buy, Buy)$ there are inequalities

$$u_X(Not\ Buy, Buy) \geq u_X(Buy, Buy)$$

$$u_Y(Not\ Buy, Buy) \geq u_Y(Not\ Buy, Not\ Buy)$$

which gives

$$\frac{n}{3+n} \leq P$$

$$\frac{n}{9+3n} \geq P.$$

Hence there is also no Nash equilibrium for $(Not\ Buy, Buy)$.