Homological Algebra

Sheet 3 — MT23

Section A

1. Prove for $A, B \in \mathrm{Mod}_{\mathbb{Z}}$, $\forall i > 1, \mathrm{Ext}^i_{\mathbb{Z}}(A, B) = 0 = \mathrm{Tor}_i^{\mathbb{Z}}(A, B)$.

   Solution: Note any submodule of a free module over a PID is free (previous sheet). So $\ker(\bigoplus_{a \in A} \mathbb{Z}e_a \twoheadrightarrow A) =: K$ is free and $\cdots \to 0 \to K \to \bigoplus_{a \in A} \mathbb{Z}e_a$ is a projective resolution of A concentrated in degrees 0 and 1. Thus, applying $Hom_{\mathbb{Z}}(-, B)$ (or $- \otimes_{\mathbb{Z}} B$) and taking cohomology (homology, resp.) we see the claim.

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Section B

2. Compute the following Ext, Tor groups:

   - $\mathrm{Tor}_*^{k[x]}(\frac{k[x]}{x-a}, \frac{k[x]}{x-b})$ for $a, b \in k$ a field,
   - $\mathrm{Tor}_*^{\mathbb{Z}}(\frac{\mathbb{Z}}{a}, \frac{\mathbb{Z}}{b})$ for $a, b \in \mathbb{Z}$,
   - $\mathrm{Ext}^*_{\mathbb{Z}/4}(\frac{\mathbb{Z}}{2}, \frac{\mathbb{Z}}{2})$,
   - $\mathrm{Ext}^*_{\mathbb{Z}/2^a}(\frac{\mathbb{Z}}{2^b}, \frac{\mathbb{Z}}{2^c})$ for $a \geq b \geq c$,
   - $\mathrm{Ext}^*_{k[x,y]/(x^2,xy,y^2)}(k, k)$.

3. Compute all terms and maps in the following long exact sequences:

   i) $\cdots \to \mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) \to \mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/4, \mathbb{Z}) \to \mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) \to \mathrm{Ext}^{i+1}_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}) \to \cdots$

   ii) $\cdots \to \mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2) \to \mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/4) \to \mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2) \to \mathrm{Ext}^{i+1}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2) \to \cdots$

   associated to the SES $0 \to \mathbb{Z}/2 \xrightarrow{f} \mathbb{Z}/4 \xrightarrow{g} \mathbb{Z}/2 \to 0$ in $\mathrm{Mod}_{\mathbb{Z}}$.

   iii) $\cdots \to \mathrm{Ext}^i_{\mathbb{Z}/8}(\mathbb{Z}/2, \mathbb{Z}/4) \to \mathrm{Ext}^i_{\mathbb{Z}/8}(\mathbb{Z}/4, \mathbb{Z}/4) \to \mathrm{Ext}^i_{\mathbb{Z}/8}(\mathbb{Z}/2, \mathbb{Z}/4) \to \mathrm{Ext}^{i+1}_{\mathbb{Z}/8}(\mathbb{Z}/2, \mathbb{Z}/4) \to \cdots$

   associated to the SES $0 \to \mathbb{Z}/2 \xrightarrow{f} \mathbb{Z}/4 \xrightarrow{g} \mathbb{Z}/2 \to 0$ in $\mathrm{Mod}_{\mathbb{Z}/8}$.

   iv) $\cdots \to \mathrm{Ext}^i_{\mathbb{Z}/2^a}(\mathbb{Z}/2, \mathbb{Z}/2^b) \to \mathrm{Ext}^i_{\mathbb{Z}/2^a}(\mathbb{Z}/4, \mathbb{Z}/2^b) \to \mathrm{Ext}^i_{\mathbb{Z}/2^a}(\mathbb{Z}/2, \mathbb{Z}/2^b) \to \mathrm{Ext}^{i+1}_{\mathbb{Z}/2^a}(\mathbb{Z}/2, \mathbb{Z}/2^b) \to \cdots$

   associated to the SES $0 \to \mathbb{Z}/2 \xrightarrow{f} \mathbb{Z}/4 \xrightarrow{g} \mathbb{Z}/2 \to 0$ in $\mathrm{Mod}_{\mathbb{Z}/2^a}$ where $a > b \geq 2$.

4. Is $\prod_I : \mathrm{Mod}_R \to \mathrm{Mod}_R$ left exact? Right exact? What is the derived functor?

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Section C

5. Let k be a field, $R = k[x, y], M := R/(x, y)^2 \cong k \oplus kx \oplus ky$ as a k-module.

   Consider the following Mod_R-SES's and compute the associated Tor-LES's:

   $0 \to k \oplus k \to M \to k \to 0$:

   i) LES from $M \otimes_R -$

   $0 \to k \to \mathrm{Hom}_k(M, k) \to k \oplus k \to 0$:

   ii) LES from $M \otimes_R -$

   iii) LES from $k \otimes_R -$

   $0 \to k^{\oplus 3} \to \frac{M \oplus M}{\langle (y,-x) \rangle} \to k \oplus k \to 0$:

   iv) LES from $k \otimes_R -$.

   For the SES $0 \to A \to B \to C \to 0$ above, we name the morphisms $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$.

   Solution: First, let's take a second to appreciate that the SES in iv $0 \to k^{\oplus 3} \to \frac{M \oplus M}{(y,-x)R} \to k^{\oplus 2} \to 0$ might appear, at first glance, to be ambiguous. If we think of the coproduct being taken in the category of k-modules, then by mapping to the first factor and quotienting, we have a SES of R-modules: $0 \to M \xrightarrow{q \circ i_1} \frac{M \oplus M}{(y,-x)R} \to \frac{R}{(x,y^2)} = \frac{k[y]}{y^2} \to 0$. We also have $0 \to \frac{(kx \oplus ky) \oplus (kx \oplus ky)}{(y,-x)R} \to \frac{M \oplus M}{(y,-x)R} \to k^{\oplus 2} \to 0$; this is a non-isomorphic SES of R-modules (e.g. since the first module has 0-action by x, y). Thus, the only reasonable way to interpret the $\oplus$'s in this question is as the coproducts in Mod_R (maybe this was obvious to the ever-alert; however when starting with parts i and ii, one might be distracted since it isn't difficult to see that the only possible Mod_R objects of the correct $\dim_k$ on the left and right have 0-action by x, y – hence why my Frankensteined[2] answer awkwardly shows that). Note that $kx \oplus ky$ is allowed notation since kx in this usage refers to the R-submodule of M generated by x; in particular, this is Mod_R-isomorphic to k with 0-action by x, y, ie. $R/(x, y)$.

   For $a \in R$, we write $a = a_0 + a_1x + a_2y + O((x, y)^2)$ for $a_i \in k$ without always highlighting that we're doing this.

   Let's note another important thing: there is not a unique injective R-module homomorphism $k^{\oplus 3} \to \frac{M \oplus M}{(y,-x)R}$, even though the image is unique (since for $a_i, b_i \in k$, we have that $x(a_0 + a_1x + a_2y, b_0 + b_1x + b_2y) = (a_0x + b_0y, 0)$ and $y(a_0 + a_1x + a_2y, b_0 + b_1x + b_2y) = (0, a_0x + b_0y)$ are both 0 iff $a_0 = b_0 = 0$). For example, mapping $\alpha : e_1 \mapsto (x, 0), e_2 \mapsto (y, 0), e_3 \mapsto (0, y)$ is not really preferred, yet is distinct, from the same map except instead $e_1 \mapsto (-x, 0)$. This is slightly different from earlier where we were in situations involving $\mathrm{Hom}_{\mathrm{Mod}_{\mathbb{Z}}}(\mathbb{Z}/2, \mathbb{Z}/2) = \mathbb{Z}/2$ where there is an on-the-nose unique isomorphism.

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To write the maps in the induced LES explicitly for a general right exact functor requires using the horseshoe lemma on the SES to get a split SES of chain complexes. The image of this under our right exact functor is another SES of chain complexes which we then apply the snake lemma towards. However, since our right exact functor is $N \otimes_R -$, we have an alternative approach to computing the LES that goes as follows: we write a projective resolution $P_\bullet^N$ for N, and then apply $P_\bullet^N \otimes_R -$ to get a SES of chain complexes[3]. Applying snake lemma to this produces a canonically equivalent LES as the one produced via the first method!

We'll do part iv) via the first method and parts i), ii), and ii) via the second.

iv) A $\mathrm{Mod}_{k[x,y]}$-projective resolution of $k = R/(x,y)$ is $\cdots \to 0 \to R \overset{d_2}{\to} R^2 \overset{d_1}{\to} R = P_0$ given by $d_1(f,g) = xf + yg$, $d_2(h) = (yh, -xh)$. We know have resolutions for $k^{\oplus n}$, apply the horseshoe lemma to get a split SES of complexes, apply $k \otimes_R -$, and finally apply the snake lemma:

$$
\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & R^{\oplus 3} \arrow[r] \arrow[d, "\partial_2'"] & R^{\oplus 3} \oplus R^{\oplus 2} \arrow[r] \arrow[d, "\partial_2"] & R^{\oplus 2} \arrow[d, "\partial_2''"] & \\
 & (R^2)^{\oplus 3} \arrow[r] \arrow[d, "\partial_1'"] & (R^2)^{\oplus 3} \oplus (R^2)^{\oplus 2} \arrow[r] \arrow[d, "\partial_1"] & (R^2)^{\oplus 2} \arrow[d, "\partial_1''"] & \\
 & R^{\oplus 3} \arrow[r] \arrow[d, "\partial_0'"] & R^{\oplus 3} \oplus R^{\oplus 2} \arrow[r] \arrow[d, "\partial_0"] & R^{\oplus 2} \arrow[d, "\partial_0''"] & \\
0 \arrow[r] & k^{\oplus 3} \arrow[r, "\alpha"'] & \frac{M \oplus M}{(y,-x)R} \arrow[r, "\beta"'] & k^{\oplus 2} \arrow[r] & 0
\end{tikzcd}
\overset{k \otimes_R -}{\rightsquigarrow}
\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
k^{\oplus 3} \arrow[r] \arrow[d, "\partial_2'"] & k^{\oplus 3} \oplus k^{\oplus 2} \arrow[r] \arrow[d, "\partial_2"] & k^{\oplus 2} \arrow[d, "\partial_2''"] \\
(k^2)^{\oplus 3} \arrow[r] \arrow[d, "\partial_1'"] & (k^2)^{\oplus 3} \oplus (k^2)^{\oplus 2} \arrow[r] \arrow[d, "\partial_1"] & (k^2)^{\oplus 2} \arrow[d, "\partial_1''"] \\
k^{\oplus 3} \arrow[r] \arrow[d, "\partial_0'"] & k^{\oplus 3} \oplus k^{\oplus 2} \arrow[r] \arrow[d, "\partial_0"] & k^{\oplus 2} \arrow[d, "\partial_0''"] \\
0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
$$

Where wlog we take $\alpha : e_1 \mapsto (x,0), e_2 \mapsto (y,0), e_3 \mapsto (0,y)$ and $\beta : (m,n) \mapsto (m_0, n_0)$ for $m, n \in R$ representing an element in $\frac{M \oplus M}{(y,-x)R}$. Let's narrate the above horseshoe lemma: We lift ∂_0'' to get $R^{\oplus 2} \to \frac{M \oplus M}{(y,-x)R} : (m,n) \mapsto \overline{(m,n)}$. We subsequently won't bother with writing the overlines (which just indicates that the element is in an obvious quotient of $R^{\oplus 2}$). Thus we have $\partial_0(r,s,t,a,b) = (r_0 x + s_0 y + a, t_0 y + b)$. This is 0 iff $a_0 = b_0 = 0$ and $t_0 = -b_2, r_0 = -a_1, s_0 = -(a_2 + b_1)$. So $\ker \partial_0 = \{(-f + (x,y)R, -(g+m) + (x,y)R, -n + (x,y)R, xf + yg, xm + yn) : f, g, m, n \in R\}$ where by the $(x,y)R$ I mean any element in this ideal as opposed to a quotient – this naturally surjects onto $\ker \partial_0''$. We can then lift $\partial_1'' : (f,g,m,n) \mapsto (fx + gy, mx + ny)$ to get $\sigma_1 : (R^2)^{\oplus 2} \to R^{\oplus 3} \oplus R^{\oplus 2} : (f,g,m,n) \mapsto (-f, -g-m, -n, fx + gy, mx + ny)$. Thus we get $\partial_1(r,f,g,m,n) = (\partial_1'(r), 0, 0) + (-f, -g-m, -n, fx + gy, mx + ny)$ for $r \in (R^2)^{\oplus 3}$. Note $\ker \partial_1$ implies $x|g, y|f, x|n, y|m$, so g_0, f_0, m_0, n_0 are 0 for elements in this kernel. In fact, $f = yh, g = -xh, m = yp, n = -xp$. Set $r = (r,s,t,u,v,w) \in R^{\oplus 6}$. So we have

$rx + sy = hy, tx + uy = py - xh, vx + wy = xp$. So we may write a lift of ∂_2'' to this kernel via $\sigma_2 : (h, p) \mapsto (0, h, -h, p, p, 0, yh, -xh, yp, -xp)$. Now $\partial_2 = \partial_2' \oplus \sigma_2$, and our projective resolution terminates (since σ_2, ∂_2' are both injective).

Note that $\overline{\partial_1} = \overline{\partial_1' \oplus \sigma_1} : (r, s, t, u, v, w, f, g, m, n) \mapsto (-f_0, -g_0 - m_0, -n_0, 0, 0)$, and $\overline{\partial_2} = \overline{\partial_2' \oplus \sigma_2} : (a, b, c, h, p) \mapsto (0, h_0, -h_0, p_0, p_0, 0, 0, 0, 0, 0)$. The differentials in the left and right columns are all identically 0. So $\mathrm{Tor}_0^R(k, \frac{M \oplus M}{(y,-x)R}) = ke_4 \oplus ke_5$, $\mathrm{Tor}_1^R(k, \frac{M \oplus M}{(y,-x)R}) = \frac{\bigoplus_{i=1}^6 ke_i \bigoplus k(e_8 - e_9)}{k(e_2 - e_3) \oplus k(e_4 + e_5)} \cong k^{\oplus 5}$, $\mathrm{Tor}_2^R(k, \frac{M \oplus M}{(y,-x)R}) = \bigoplus_{i=1}^3 ke_i$.

So our LES looks like:

$$\begin{tikzcd}
 & \cdots \arrow[r] & 0 \arrow[dll] \\
k^{\oplus 3} \arrow[r, "\alpha_2"] & \bigoplus_{i=1}^3 ke_i \arrow[r, "\beta_2"] & k^{\oplus 2} \arrow[dll, "\delta_2"] \\
k^{\oplus 6} \arrow[r, "\alpha_1"] & \mathrm{Tor}_1^R(k, \frac{M \oplus M}{(y,-x)R}) \arrow[r, "\beta_1"] & k^{\oplus 4} \arrow[dll, "\delta_1"] \\
k^{\oplus 3} \arrow[r, "\alpha_0"] & ke_4 \oplus ke_5 \arrow[r, "\beta_0"] & k^{\oplus 2} \arrow[dll, "\delta_0"] \\
0 \arrow[r] & \cdots &
\end{tikzcd}$$

and finally, we write the explicit maps:

- $\beta_0 : e_4 \mapsto e_1, e_5 \mapsto e_2$
- $\alpha_0 = 0$
- $\delta_1 : e_1 \mapsto -e_1, e_2 \mapsto -e_2, e_3 \mapsto -e_2, e_4 \mapsto e_3$ (we look at pre-images of $\overline{\partial_1}(e_i)$ for $i = 7, 8, 9, 10$, resp.)
- $\beta_1 : e_8 - e_9 \mapsto e_2 - e_3$, all other $e_i \mapsto 0$
- $\alpha_1 : e_i \mapsto e_i$ for $i = 1, \ldots, 6$ (note $e_2 = e_3, e_4 = -e_5$)
- $\delta_2 : e_1 \mapsto e_2 - e_3, e_2 \mapsto e_4 + e_5$ (we look at pre-images of $\overline{\partial_2}(e_i)$ for $i = 4, 5$, resp.)
- $\beta_2 = 0$
- $\alpha_2 : e_i \mapsto e_i$.

Note that we could have deduced many things (e.g. α_2 being an isomorphism and $\alpha_0 = 0$) just from knowing the groups in the long exact sequence, however we wouldn't have been able to write explicit non-zero maps.

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Now, the above process was a little painful, but it clearly could be used to write the $\mathbb{L}F$-LES associated to a SES for any additive functor F. However, for $\mathrm{Tor}(D,-) = \mathbb{L}(D\otimes_R -)$ we (as mentioned earlier) have an alternative approach. Let $P^A_\bullet$ be a projective resolution for an R-module A. Recall that we have the following quasi-isomorphisms which are *functorial* in A and B: $P^A_\bullet \otimes_R B \xleftarrow{\sim} \mathrm{Tot}^{\bigoplus}_\bullet(P^A_\bullet \otimes P^B_\bullet) \xrightarrow{\sim} A\otimes_R P^B_\bullet$. Thus, instead of understanding our LES from the SES of chain complexes $D\otimes_R P^A_\bullet \to D\otimes_R P^B_\bullet \to D\otimes_R P^C_\bullet$,[4] we can consider the quasi-isomorphic SES $P^\bullet_D \otimes_R A \to P^\bullet_D \otimes_R B \to P^\bullet_D \otimes_R C$[5]; this will make life a substantially easier since we won't have the additional complexity of working with the horseshoe lemma before tensoring.

The following is a $\mathrm{Mod}_{k[x,y]}$-projective resolution of M: $\cdots \to 0 \to R^2 \xrightarrow{d_2} R^3 \xrightarrow{d_1} R \to 0$ with $d_1(f,g,h) = x^2f + xyg + y^2h$ and $d_2(a,b) = (ya, -xa - yb, xb)$.

To justify this, we check: $x^2f + xyg + y^2h = 0 \implies f = yf', h = xh' \implies x|g + yh', y|g + xf'$since $xf' + g + yh' = 0$.

If $x \nmid h'$, then $g = xg' - yh'$ and $f' = -g'$. Thus our triple looks like $(-yg', xg' - yh', xh')$.

If $y \nmid f'$, then $g = yg' - xf'$ and $h' = -g'$. Thus our triple looks like $(yf', yg' - xf', -xg')$.

Now suppose $h' = xh'', f' = yf''$. Thus $g = xyg''$ and we have $f'' + g'' + h'' = 0$. Our triple looks like $(y^2f'', -xy(f'' + h''), x^2h'')$.

We note that these are all in the image of $d_2(a,b) = (ya, -xa - yb, xb)$ which is also always in $\ker d_1$ and so we have our P_2. Further, $\ker d_2 = 0$ so our resolution terminates here.

i) We consider $0 \to k^{\oplus 2} \xrightarrow{\alpha} M \xrightarrow{\beta} k \to 0$. The submodule of M on which x and y act by 0 is $kx + ky$, and so α must be an isomorphism onto this. We set $\alpha : e_1 \mapsto x, e_2 \mapsto y$. β is thus $f \mapsto f_0$.

So applying $P^M_\bullet \otimes_R -$, we have

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & R^2 \otimes k^{\oplus 2} \arrow[r, "\mathrm{id}_{R^2}\otimes\alpha"] \arrow[d, "\partial_2'"] & R^2 \otimes M \arrow[r, "\mathrm{id}_{R^2}\otimes\beta"] \arrow[d, "\partial_2"] & R^2 \otimes k \arrow[d, "\partial_2''"] & \\
 & R^3 \otimes k^{\oplus 2} \arrow[r, "\mathrm{id}_{R^3}\otimes\alpha"] \arrow[d, "\partial_1'"] & R^3 \otimes M \arrow[r, "\mathrm{id}_{R^3}\otimes\beta"] \arrow[d, "\partial_1"] & R \otimes k \arrow[d, "\partial_1''"] & \\
 & R \otimes k^{\oplus 2} \arrow[r, "\mathrm{id}_{R}\otimes\alpha"] \arrow[d, "\partial_0'"] & R \otimes M \arrow[r, "\mathrm{id}_{R}\otimes\beta"] \arrow[d, "\partial_0"] & R \otimes k \arrow[d, "\partial_0''"] & \\
0 \arrow[r] & k^{\oplus 2} \arrow[r, "\alpha"'] & M \arrow[r, "\beta"'] & k \arrow[r] & 0
\end{tikzcd}$$

where the $\partial_i' = d_i \otimes \mathrm{id}_{k^{\oplus 2}}, \partial_i = d_i \otimes \mathrm{id}_M$, and $\partial_i'' = d_i \otimes \mathrm{id}_k$. Simplifying, we have

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$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
(k^{\oplus 2})^2 \arrow[r, "\alpha^2"] \arrow[d, "0"] & M^2 \arrow[r, "\beta^2"] \arrow[d, "\partial_2"] & k^2 \arrow[d, "0"] \\
(k^{\oplus 2})^3 \arrow[r, "\alpha^3"] \arrow[d, "0"] & M^3 \arrow[r, "\beta^3"] \arrow[d, "0"] & k^3 \arrow[d, "0"] \\
k^{\oplus 2} \arrow[r, "\alpha"] & M \arrow[r, "\beta"] & k
\end{tikzcd}
\overset{LES}{\to}
\begin{tikzcd}
 & \cdots \arrow[r] & 0 \arrow[dll] \\
(k^{\oplus 2})^2 \arrow[r, "\overline{\alpha^2}"] & \mathrm{Tor}_2^R(M,M) \arrow[r, "\overline{\beta^2}"] & k^2 \arrow[dll, "\delta_2"] \\
(k^{\oplus 2})^3 \arrow[r, "\overline{\alpha^3}"] & \frac{M^3}{\partial_2(M^2)} \arrow[r, "\overline{\beta^3}"] & k^3 \arrow[dll, "\delta_1"] \\
k^{\oplus 2} \arrow[r, "\overline{\alpha}"] & M \arrow[r, "\overline{\beta}"] & k \arrow[dll] \\
0 \arrow[r] & \cdots &
\end{tikzcd}
$$

Where $\mathrm{Tor}_2^R(M,M) = \ker \partial_2 = \{(a_1 x + a_2 y, b_1 x + b_2 y) \in M : a_i, b_i \in k\} \simeq k^{\oplus 4}$. $\mathrm{Tor}_1^R(M,M) = \frac{M^3}{k(y,-x,0) \oplus k(0,-y,x)}$ is 7-dimensional as a k-module. We can write the morphisms explicitly:

- $\overline{\alpha^2} : (a,b,c,d) \mapsto (ax+by, cx+dy)$ is an isomorphim,
- $\overline{\beta^2} = 0$ (alternatively, immediate since α^2 is surjective!),
- $\delta_2 : e_1 \mapsto (0,1,-1,0,0,0) = e_2 - e_3, e_2 \mapsto (0,0,0,-1,1,0)$ is an injection with image $k(e_2 - e_3) \oplus k(e_5 - e_4)$,
- $\overline{\alpha^3} : (a,b,c,d,e,f) \mapsto (ax+by, cx+dy, ex+fy) = (ax+(b+c)y, 0, (d+e)x+fy)$ has image $\frac{(kx \oplus ky)^3}{k(y,-x,0) \oplus k(0,-y,x)}$,
- $\overline{\beta^3} : (a,b,c) \mapsto (a_0, b_0, c_0)$ is surjective,
- $\delta_1 = 0$,
- $\overline{\alpha} = \alpha$ and $\overline{\beta} = \beta$ are the original maps from our SES.

ii) $\mathrm{Hom}_k(M,k) \cong ke_0^* + ke_1^* + ke_2^*$ where e_i^* represent the dual basis for $1, x, y$ resp. Note $x \cdot (f_0 e_0^* + f_1 e_1^* + f_2 e_2^*) = f_1 e_0^*$ and $y \cdot (f_0 e_0^* + f_1 e_1^* + f_2 e_2^*) = f_2 e_0^*$. Now $k\mathrm{Hom}_k(M,k)$, so we note that x^2 acts by 0 on k. Thus x can't act on this k by a non-zero scalar, ie. x acts by 0. Similarly, y acts by 0 on the submodule k. The submodule of $\mathrm{Hom}_k(M,k)$ on which x and y act by 0 is ke_0^* which is one-dimensional$_k$ and so this injection (and thus the SES) is canonically determined.

So for our SES $0 \to k \overset{\alpha}{\to} \mathrm{Hom}_k(M,k) \overset{\beta}{\to} k \oplus k \to 0$, we take $\alpha : e_1 \mapsto e_0^*, \beta : f_0 e_0^* + f_1 e_1^* + f_2 e_2^* \mapsto f_1 e_1 + f_2 e_2$ (for $f_i \in k$).

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So applying $P_\bullet^M \otimes_R -$, we have

$$\begin{tikzcd}
& 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
& R^2 \otimes_R k \arrow[r, "\mathrm{id}_{R^2}\otimes\alpha"] \arrow[d, "\partial_2'"] & R^2 \otimes_R \mathrm{Hom}_k(M,k) \arrow[r, "\mathrm{id}_{R^2}\otimes\beta"] \arrow[d, "\partial_2"] & R^2 \otimes_R k^{\oplus 2} \arrow[d, "\partial_2''"] & \\
& R^3 \otimes_R k \arrow[r, "\mathrm{id}_{R^3}\otimes\alpha"] \arrow[d, "\partial_1'"] & R^3 \otimes_R \mathrm{Hom}_k(M,k) \arrow[r, "\mathrm{id}_{R^3}\otimes\beta"] \arrow[d, "\partial_1"] & R^3 \otimes_R k^{\oplus 2} \arrow[d, "\partial_1''"] & \\
& R \otimes_R k \arrow[r, "\mathrm{id}_{R}\otimes\alpha"] \arrow[d, "\partial_0'"] & R \otimes_R \mathrm{Hom}_k(M,k) \arrow[r, "\mathrm{id}_{R}\otimes\beta"] \arrow[d, "\partial_0"] & R \otimes_R k^{\oplus 2} \arrow[d, "\partial_0''"] & \\
0 \arrow[r] & k \arrow[r, "\alpha"'] & \mathrm{Hom}_k(M,k) \arrow[r, "\beta"'] & k^{\oplus 2} \arrow[r] & 0
\end{tikzcd}$$

where the $\partial_i' = d_i \otimes \mathrm{id}_k, \partial_i = d_i \otimes \mathrm{id}_{\mathrm{Hom}_k(M,k)}$, and $\partial_i'' = d_i \otimes \mathrm{id}_{k^{\oplus 2}}$. Simplifying, we have

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
k^2 \arrow[r, "\alpha^2"] \arrow[d, "0"] & (\mathrm{Hom}_k(M,k))^2 \arrow[r, "\beta^2"] \arrow[d, "\partial_2"] & (k^{\oplus 2})^2 \arrow[d, "0"] \\
k^3 \arrow[r, "\alpha^3"] \arrow[d, "0"] & (\mathrm{Hom}_k(M,k))^3 \arrow[r, "\beta^3"] \arrow[d, "0"] & (k^{\oplus 2})^3 \arrow[d, "0"] \\
k \arrow[r, "\alpha"] & \mathrm{Hom}_k(M,k) \arrow[r, "\beta"] & k^{\oplus 2}
\end{tikzcd}
\overset{LES}{\to}
\begin{tikzcd}
& \cdots \arrow[r] & 0 \arrow[dll] \\
k^2 \arrow[r, "\overline{\alpha^2}"] & \mathrm{Tor}_2^R(M, \mathrm{Hom}_k(M,k)) \arrow[r, "\overline{\beta^2}"] & (k^{\oplus 2})^2 \arrow[dll, "\delta_2"] \\
k^3 \arrow[r, "\overline{\alpha^3}"] & \mathrm{Tor}_1^R(M, \mathrm{Hom}_k(M,k)) \arrow[r, "\overline{\beta^3}"] & (k^{\oplus 2})^3 \arrow[dll, "\delta_1"] \\
k \arrow[r, "\overline{\alpha}"] & \mathrm{Hom}_k(M,k) \arrow[r, "\overline{\beta}"] & k \arrow[dll] \\
0 \arrow[r] & \cdots &
\end{tikzcd}$$

where $\partial_2(a,b) = (a_2e_0^*, (-a_1 - b_2)e_0^*, b_1e_0^*)$. Thus$\mathrm{Tor}_2^R(M, \mathrm{Hom}_k(M,k)) = \ker \partial_2 = \{(a_0e_0^* + a_1e_1^*, b_0e_0^* - a_1e_2^*) : a_1, a_0, b_0 \in k\} = k(e_0^*, 0) \oplus k(e_1^*, -e_2^*) \oplus k(0, e_0^*)$ with the inherited x, y action from $\mathrm{Hom}_k(M,k)^2$. Note $\mathrm{im}\partial_2 = (ke_0^*) \oplus (ke_0^*) \oplus (ke_0^*) \subseteq M^3$, so $\mathrm{Tor}_1^R(M, \mathrm{Hom}_k(M,k)) = \frac{(\mathrm{Hom}_k(M,k))^3}{(ke_0^*)\oplus(ke_0^*)\oplus(ke_0^*)} \simeq k^6$ where x, y act by 0. We write the morphisms explicitly:

- $\overline{\alpha^2} : ae_1 + be_2 \mapsto (ae_0^*, be_0^*)$ which is injective with image $(ke_0^*)^2$,
- $\overline{\beta^2} : (e_0^*, 0) \mapsto 0, (0, e_0^*) \mapsto 0, (e_1^*, -e_2^*) \mapsto e_1 - e_4$ which has image $k(e_1 - e_4)$,
- $\delta_2 : e_1 \mapsto -e_2, e_2 \mapsto e_1, e_3 \mapsto e_3, e_4 \mapsto -e_2$ which is surjective,
- $\overline{\alpha^3} = 0$,
- $\overline{\beta^3} : e_i \mapsto e_i$ (where in the source: $e_1 = (e_1^*, 0, 0), e_2 = (e_2^*, 0, 0), e_3 = (0, e_1^*, 0)$, etc.) which is surjective,

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- $\delta_1 = 0$,
- $\overline{\alpha} = \alpha, \overline{\beta} = \beta$ are exactly as in our original SES.

iii) Now recall our resolution for $k = R/(x, y)$: $\cdots \to 0 \to R \xrightarrow{d_2} R^2 \xrightarrow{d_1} R =: P_0^k$ given by $d_1(f, g) = xf + yg$, $d_2(h) = (yh, -xh)$. We apply $P_\bullet^k \otimes -$ to the same SES as in part ii)

So for our SES $0 \to k \xrightarrow{\alpha} \mathrm{Hom}_k(M, k) \xrightarrow{\beta} k \oplus k \to 0$, we take $\alpha : e_1 \mapsto e_0^*, \beta : f_0e_0^* + f_1e_1^* + f_2e_2^* \mapsto f_1e_1 + f_2e_2$ (for $f_i \in k$).

So applying $P_\bullet^k \otimes_R -$, we have

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] & \\
 & R \otimes_R k \arrow[r, "\mathrm{id}_R \otimes \alpha"] \arrow[d, "\partial_2'"] & R \otimes_R \mathrm{Hom}_k(M, k) \arrow[r, "\mathrm{id}_R \otimes \beta"] \arrow[d, "\partial_2"] & R \otimes_R k^{\oplus 2} \arrow[d, "\partial_2''"] & \\
 & R^2 \otimes_R k \arrow[r, "\mathrm{id}_{R^2} \otimes \alpha"] \arrow[d, "\partial_1'"] & R^2 \otimes_R \mathrm{Hom}_k(M, k) \arrow[r, "\mathrm{id}_{R^2} \otimes \beta"] \arrow[d, "\partial_1"] & R^2 \otimes_R k^{\oplus 2} \arrow[d, "\partial_1''"] & \\
 & R \otimes_R k \arrow[r, "\mathrm{id}_R \otimes \alpha"] \arrow[d, "\partial_0'"] & R \otimes_R \mathrm{Hom}_k(M, k) \arrow[r, "\mathrm{id}_R \otimes \beta"] \arrow[d, "\partial_0"] & R \otimes_R k^{\oplus 2} \arrow[d, "\partial_0''"] & \\
0 \arrow[r] & k \arrow[r, "\alpha"'] & \mathrm{Hom}_k(M, k) \arrow[r, "\beta"'] & k^{\oplus 2} \arrow[r] & 0
\end{tikzcd}$$

where the $\partial_i' = d_i \otimes \mathrm{id}_k, \partial_i = d_i \otimes \mathrm{id}_{\mathrm{Hom}_k(M,k)}$, and $\partial_i'' = d_i \otimes \mathrm{id}_{k^{\oplus 2}}$. Simplifying, we have

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
k \arrow[r, "\alpha"] \arrow[d, "0"] & \mathrm{Hom}_k(M, k) \arrow[r, "\beta"] \arrow[d, "\partial_2"] & k^{\oplus 2} \arrow[d, "0"] \\
k^2 \arrow[r, "\alpha^2"] \arrow[d, "0"] & (\mathrm{Hom}_k(M, k))^2 \arrow[r, "\beta^2"] \arrow[d, "\partial_1"] & (k^{\oplus 2})^2 \arrow[d, "0"] \\
k \arrow[r, "\alpha"] & \mathrm{Hom}_k(M, k) \arrow[r, "\beta"] & k^{\oplus 2}
\end{tikzcd}
\quad \overset{LES}{\to} \quad
\begin{tikzcd}
 & \cdots \arrow[r] & 0 \arrow[dll] \\
k \arrow[r, "\alpha_2"] & \mathrm{Tor}_2^R(k, \mathrm{Hom}_k(M, k)) \arrow[r, "\beta_2"] & k^{\oplus 2} \arrow[dll, "\delta_2"] \\
k^2 \arrow[r, "\alpha_1"] & \mathrm{Tor}_1^R(k, \mathrm{Hom}_k(M, k)) \arrow[r, "\beta_1"] & (k^{\oplus 2})^2 \arrow[dll, "\delta_1"] \\
k \arrow[r, "\alpha_0"] & \mathrm{Tor}_0^R(k, \mathrm{Hom}_k(M, k)) \arrow[r, "\beta_0"] & k^{\oplus 2} \arrow[dll] \\
0 \arrow[r] & \cdots &
\end{tikzcd}$$

where $\alpha_i := \mathrm{Tor}_i^R(k, \alpha), \beta_i := \mathrm{Tor}_i^R(k, \beta)$. Recall from before: $\mathrm{Hom}_k(M, k) \cong ke_0^* + ke_1^* + ke_2^*$ where e_i^* represent the dual basis for $1, x, y$ resp. and $x \cdot (f_0e_0^* + f_1e_1^* + f_2e_2^*) = f_1e_0^*$, $y \cdot (f_0e_0^* + f_1e_1^* + f_2e_2^*) = f_2e_0^*$.

So $\partial_1(\sum f_ie_i^*, \sum g_ie_i^*) = (f_1 + g_2)e_0^*$ and $\partial_2(f_0e_0^* + f_1e_1^* + f_2e_2^*) = (f_2e_0^*, -f_1e_0^*)$. Thus, $\mathrm{Tor}_0^R(k, \mathrm{Hom}_k(M, k)) = \frac{ke_0^* + ke_1^* + ke_2^*}{ke_0^*} \simeq k^{\oplus 2}$ and $\mathrm{Tor}_2^R(k, \mathrm{Hom}_k(M, k)) = ke_0^*$. $\mathrm{Tor}_1^R(k, \mathrm{Hom}_k(M, k)) =$

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$\frac{k(e_1^*,-e_2^*)+k(e_0^*,0)+k(0,e_0^*)+k(e_2^*,0)+k(0,e_1^*)}{k(e_0^*,0)+k(0,e_0^*)} \simeq \bigoplus_{i=1}^3 ke_i$ with $(0, e_1^*) \mapsto e_1, (e_2^*, 0) \mapsto e_2, (e_1^*, -e_2^*) \mapsto e_3$.

We write the morphisms explicitly:

- $\alpha_2 : e_1 \mapsto e_0^*$ which is surjective,
- $\beta_2 = 0$,
- $\delta_2 : e_1 \mapsto -e_2, e_2 \mapsto e_1$ which is surjective,
- $\alpha_1 = 0$,
- $\beta_1 : (0, e_1^*) \mapsto e_3, (e_2^*, 0) \mapsto e_2, (e_1^*, -e_2^*) \mapsto e_1 - e_4$ which has image $k(e_1 - e_4) \oplus ke_2 \oplus ke_3$,
- $\delta_1 : e_1 \mapsto e_1, e_2 \mapsto 0, e_3 \mapsto 0, e_4 \mapsto e_1$ which is surjective,
- $\alpha_0 = 0$,
- $\beta_1 : e_1^* \mapsto e_1, e_2^* \mapsto e_2$ which is surjective.

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