Numerical analysis
Chapter 0: Fundamentals

Purpose of book

Understanding the way computer calculations work

Evaluating polynomials

Evaluate $P(x) = 2x^4 + 3x^3 - 3x^2 + 5x - 1$

$x = \frac{1}{2}$ is stored in memory

Method 1 (naive)

simply substitute x with $\frac{1}{2}$

this will require 10 mults. and 4 adds.

Method 2 (store powers)

First, calculate powers of x, and store them

$x = \frac{1}{2}$

$x^2 = x \cdot x = \frac{1}{4}$

$x^3 = x \cdot x^2 = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$ and so on ...

And then substitute

this will require 7 mults (?), still 4 adds

Method 3 (nesting/Horner's method)

Rewrite for inside out eval.

this will require 4 mults (!) and 4 adds

$\boxed{\text{A degree } n \text{ polynom. can be eval. in } n \text{ mults. and } n \text{ adds.}}$

Exercise solutions

Polynomial evaluation

1a) $x = \frac{1}{3}$ $\quad P(x) = 6x^4 + x^3 + 5x^2 + x + 1$

Eval. without:

$$P\left(\tfrac{1}{3}\right) = 6 \cdot \frac{1}{3^4} + \frac{1}{3^3} + 5 \cdot \frac{1}{3^2} + \frac{1}{3} + 1$$

$$= \frac{6}{81} + \frac{1}{27} + \frac{5}{9} + \frac{1}{3} + \frac{3}{3}$$

$$= \frac{6+3+45+27+81}{81} = \frac{162}{81} = 2$$

Retrieve nested form:

$$P(x) = 6x^4 + x^3 + 5x^2 + x + 1$$

$$= 1 + x(1 + 5x + x^2 + 6x^3)$$

$$= 1 + x(1 + x(5 + x + 6x^2))$$

$$= 1 + x(1 + x(5 + x(1 + 6x)))$$

Eval with

(1) $\quad 1 + 6x = 1 + 2 = 3$

(2) $\Rightarrow$ $5 + x(1+6x) = 5 + \frac{1}{3} \cdot 3 = 6$

(3) $\Rightarrow$ $1 + x(\ldots = 1 + \frac{1}{3} \cdot 6 = 3$

(4) $\Rightarrow$ $1 + x(\ldots = 1 + \frac{1}{3} \cdot 3 = 2$

b) <u>Eval without</u>

$$P(\tfrac{1}{3}) = -3 \cdot \frac{1}{3^4} + 4 \cdot \frac{1}{3^3} + 5\frac{1}{3^2} - 5\frac{1}{3} + 1$$

$$= -\frac{1}{27} + \frac{4}{27} + \frac{5}{9} - \frac{5}{3} + \frac{3}{3}$$

$$= \frac{-1+4+15-45+27}{27} = 0$$

<u>Horner's</u>

$$P(x) = 1 + x(-5 + x(5 + x(4 - 3x)))$$

<u>Eval with</u>

(1) $4 - 3x = 4 - 1 = 3$

(2) $5 + x(\ldots = 5 + \frac{1}{3}3 = 6$

(3) $-5 + x(\ldots = -5 + \frac{1}{3}6 = -3$

(4) $1 + x(\ldots = 1 + \frac{1}{3} \cdot -3 = 0$

c) <u>Eval without</u>

$$P(\tfrac{1}{3}) = 2\frac{1}{3^4} + \frac{1}{3^3} - \frac{1}{3^2} + 1$$

$$= \frac{2}{81} + \frac{1}{27} - \frac{1}{9} + \frac{3}{3}$$

$$= \frac{2+3-9+81}{81} = \frac{77}{81}$$

<u>Horner's</u>

$$P(x) = 1 + x(0 + x(-1 + x(1 + 2x)))$$

Eval with

(1) $1+2x = \frac{5}{3}$

(2) $-1+x(\ldots = -1+\frac{5}{9} = \frac{-4}{9}$

(3) $0+x(\ldots = 0-\frac{4}{27} = -\frac{4}{27}$

(4) $1+x(\ldots = 1-\frac{4}{81} = \frac{77}{81}$

2a) Eval w/o

$$P(-\tfrac{1}{2}) = -6\frac{1}{2^3} - 2\frac{1}{2^2} + 3\frac{1}{2} + 7$$

$$= -\frac{3}{4} - \frac{1}{2} + \frac{3}{2} + \frac{28}{4}$$

$$= \frac{-3-2+6+28}{4} = \frac{29}{4}$$

Horner's

$$P(x) = 7 + x(-3 + x(-2+6x))$$

Eval w/

(1) $-2+6x = -2-3 = -5$

(2) $-3+x(\ldots = -3-\frac{1}{2}(-5) = -\frac{1}{2}$

(3) $7+x(\ldots = 7-\frac{1}{2}(-\frac{1}{2}) = 7+\frac{1}{4} = \frac{29}{4}$

b) Eval w/o

$$P\left(-\frac{1}{2}\right) = -8\frac{1}{2^5} - \frac{1}{2^4} + 3\frac{1}{2^3} + \frac{1}{2^2} + \frac{3}{2} + 1$$

$$= -\frac{8}{32} - \frac{1}{16} + \frac{3}{8} + \frac{1}{4} + \frac{3}{2} + \frac{2}{2}$$

$$= \frac{-4 - 1 + 6 + 4 + 24 + 16}{16}$$

$$= \frac{45}{16}$$

Horner's

$$P(x) = 1 + x(-3 + x(1 + x(-3 + x(-1 + 8x))))$$

Eval w/

(1) $-1 + 8x = -1 - 4 = -5$

(2) $-3 + x(\ldots = -3 - \frac{1}{2}(-5) = -\frac{1}{2}$

(3) $1 + x(\ldots = 1 - \frac{1}{2}(-\frac{1}{2}) = \frac{5}{4}$

(4) $-3 + x(\ldots = -3 - \frac{1}{2}\frac{5}{4} = \frac{-29}{8}$

(5) $1 + x(\ldots = 1 - \frac{1}{2}(-\frac{29}{8}) = \frac{16 + 29}{16} = \frac{45}{16}$

c) Skipping, getting tedious

3) assume $t = x^2$

means that if $x = \frac{1}{2}$ then $t = \frac{1}{4}$

and $P(t) = t^3 - 4t^2 + 2t + 1$

<u>Eval w/o</u>

$$P(\tfrac{1}{4}) = \frac{1}{4^3} - 4\frac{1}{4^2} + 2\frac{1}{4} + 1$$

$$= \frac{1}{64} - \frac{1}{4} + \frac{2}{4} + \frac{4}{4}$$

$$= \frac{1 + 16 + 64}{64} = \frac{81}{64}$$

<u>Horner's</u>

$$P(t) = 1 + t(2 + t(-4 + t))$$

<u>Eval w/</u>

(1) $-4 + t = -4 + \frac{1}{4} = -\frac{15}{4}$

(2) $2 + t(\ldots = 2 + \frac{1}{4}(-\frac{15}{4}) = \frac{17}{16}$

(3) $1 + t(\ldots = 1 + \frac{1}{4}(\frac{17}{16}) = \frac{81}{64}$

4a) I'm a little confused what the intent of the question is...

am I supposed to convert it to nested and then evaluate, or just evaluate?

Since it says eval., I guess I'll just eval.

(1) $\frac{1}{2} + (5-3)(-\frac{1}{2}) = \frac{1}{2} - \frac{2}{2} = -\frac{1}{2}$

(2) $\frac{1}{2} + (5-2)(\ldots = \frac{1}{2} - \frac{3}{2} = -1$

(3) $1 + 5(\ldots = -4$

b) (1) $\frac{1}{2} + (-1-3)(-\frac{1}{2}) = \frac{1}{2} + \frac{4}{2} = \frac{5}{2}$

(2) $\frac{1}{2} + (-1-2)\ldots = \frac{1}{2} - \frac{15}{2} = -\frac{14}{2} = -7$

(3) $1 - (\ldots = 8$

5a) (1) $3 + (\frac{1}{2} - 3)(2) = -2$

(2) $1 + (\frac{1}{2} - 2)(\ldots = 4$

(3) $4 + (\frac{1}{2} - 1)(\ldots = 2$

(4) $4 + \frac{1}{2}(\ldots = 5$

b) (1) $3 + (-\frac{1}{2} - 3)(2) = -4$

(2) $1 + (-\frac{1}{2} - 2)(\ldots = 1 + \frac{20}{2} = 11$

(3) $4 + (-\frac{1}{2} - 1)(\ldots = 4 - \frac{33}{2}$

(4) $4 - \frac{1}{2}(\ldots = 4 - 2 + \frac{33}{4} = 2 + \frac{33}{4} = \frac{41}{4}$

6a) In this case we should first assert $t = x^5$, then apply Horner's

$$P(t) = a_0 + a_5 t + a_{10} t^2 + a_{15} t^3$$

$$= a_0 + t(a_5 + t(a_{10} + t(a_{15})))$$

Which would result in 3 mults & 3 adds

b) Here, the difference in powers follows the pattern $n + m \cdot 5$, where $n = 7$ therefore, we can rewrite as

$$P(x) = x^7(a_7 + x^5(a_{12} + x^5(a_{17} + x^5(a_{22} + x^5(a_{27})))))$$

This requires 5 mults and 4 adds

7) n mults, 2n adds

Computer problems

See sperate file for source code.

1) Input: nest (50, ones (50), 1.00001)

Output: 51,0128

Actual value: 51,012752082 7452

Home → Preferences → Command window → Format → long g
to disable rounding

Error = $4{,}76 \cdot 10^{-12}$

2) $\boxed{\text{Code for array: } (-1).\text{\^{}}(0:n)}$

Input: nest (99, (−1).^(0:99), 1.00001)

Output: −0,000500245079647632

$$1 - x(1 - x(\ldots \quad = \frac{1 - x^{100}}{1 + x}$$

Error = $1{,}713 \cdot 10^{-16}$

No way to check if this is correct, lol.