Numerical analysis

Chapter 3: Interpolation

Data and interpolating functions

A function interpolates a point it if passes through it

Lagrange interpolation

Helps us find a function/polynomial which runs through given points

$y = (y_1, y_2, \dots, y_n)$

Results in a degree $n-1$

See book for formula.

Newton's divided differences

A simpler way to compute the polynomial (same polynomial)

Also results in a degree $n-1$

Calculates each coefficient recursively

Approximating functions

We can approx. funcs. with polynomial interpolation

Interpolation error

Interpolation error formula

$f(x) - P(x)$

$\uparrow$ original $\uparrow$ interpolated

Runge phenomenon

A phenomenon where the interpolated function does not accurately describe the original, for example when points are equally spaced

$x_{n+1} = x_n + \Delta x$ with $f(x_n) = 0$

for all n

This introduces a "wiggle"

Chebyshev interpolation

An optimization of point spacing as to minimize the interpolation error

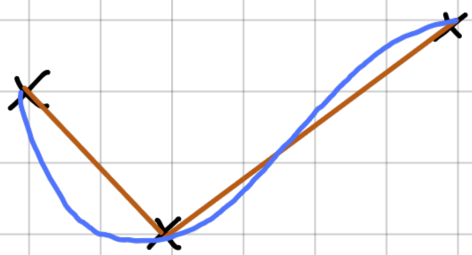
$$\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{n!} f^{(n)}(c)$$

A polynomial that uses Chebyshev roots is called a Chebyshev interpolating polynomial

Cubic splines

Intersect the points by using several polynomials, instead of one

Simplest being linear splines, but they aren't smooth like cubic ones:



Cubic splines

$$S_k(x) = y_k + b_k(x - x_k) + c_k(x - x_k)^2 + d_1(x - x_k)^3$$

on $[x_k, x_{k+1}]$

Properties

1. $S_i(x_i) = y_i$, $S_i(x_{i+1}) = y_{i+1}$
   for $i \in [1, n-1]$

2. $S'_{i-1}(x_i) = S'_i(x_i)$ for $i \in [2, n-1]$

3. S'' – // –

Additional props.

a. Natural spline:

$S_1''(x_1) = 0$ and $S_{n-1}''(x_n) = 0$

(one exist for every set of unique data points)

b. Curvature adjusted cubic spline:

$S_1''(x_1)$, $S_{n-1}''(x_n)$ set to arbitrary values $\neq 0$

c. Clamped cubic spline

$S_1'(x_1)$, $S_{n-1}'(x_n)$ arbitrary $\neq 0$

d. Parabolically terminated cubic spline

e. Not-a-knot cubic spline

Bézier curves

Arbitrary slopes around knots

Exercises

3.1 Data and interpolating functions

1a)

$$P_2(x) = \frac{(x-2)(x-3)}{(0-2)(0-3)} + 3\frac{(x-0)(x-3)}{(2-0)(2-3)}$$

b)

$$P_3(x) = 0 + \frac{(x+1)(x-3)(x-5)}{(2+1)(2-3)(2-5)}$$

$$+ \frac{(x+1)(x-2)(x-5)}{(3+1)(3-2)(3-5)}$$

$$+ 2\frac{(x+1)(x-2)(x-3)}{(5+1)(5-2)(5-3)}$$

c) $$P_2(x) = -2\frac{(x-2)(x-4)}{(0-2)(0-4)}$$

$$+ \frac{(x-0)(x-4)}{(2-0)(2-4)}$$

$$+ \frac{(x-0)(x-2)}{(4-0)(4-2)}$$

2a)

0	1	
		1
2	3	
		-3
3	0	

$$f[x_1 x_2] = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{2-0} = 1$$

$$f[x_2 x_3] = \frac{y_3 - y_2}{x_3 - x_2} = \frac{0-3}{3-2} = -3$$

$$f[x_1 x_2 x_3] = \frac{f[x_2 x_3] - f[x_1 x_2]}{x_3 - x_1}$$

$$= \frac{-3-1}{3-0} = -\frac{4}{3}$$

$$P(x) = f[x_1] + f[x_1 x_2](x - x_1) + f[x_1 x_2 x_3](x - x_1)(x - x_2)$$

$$= 1 + (x-0) - \frac{4}{3}(x-0)(x-2)$$

$$= 1 + x - \frac{4}{3}(x^2 - 2x)$$

$$= -\frac{4}{3}x^2 + \frac{11}{3}x + 1$$

In 1a, we got:

$$p(x) = \frac{(x-2)(x-3)}{(0-2)(0-3)} + 3\,\frac{(x-0)(x-3)}{(2-0)(2-3)}$$

which simplifies like:

$$= \frac{x^2-5x+6}{6} + 3\,\frac{x^2-3x}{-2}$$

$$= \frac{x^2-5x+6-9x^2+27x}{6}$$

$$= \frac{-8x^2+22x+6}{6} = \frac{-4x^2-11x+3}{3}$$

$$= -\frac{4}{3}x^2 - \frac{11}{3}x + 1$$

which matches!

b)

$$\begin{array}{r|llll}
-1 & 0 & & & \\
 & & \frac{1}{3} & & \\
2 & 1 & & -\frac{1}{12} & \\
 & & 0 & & \frac{1}{72} \\
3 & 1 & & \frac{1}{6} & \\
 & & \frac{1}{2} & & \\
5 & 2 & & &
\end{array}$$

$$f[x_1\, x_2\, x_3] = \frac{f[x_2\, x_3] - f[x_1\, x_2]}{x_3 - x_1}$$

$$= \frac{0 - \frac{1}{3}}{3+1} = -\frac{1}{12}$$

$$f[x_2\, x_3\, x_4] = \frac{f[x_3\, x_4] - f[x_2\, x_3]}{x_4 - x_2}$$

$$= \frac{\frac{1}{2} - 0}{5-2} = \frac{1}{6}$$

$$f[x_1\, x_2\, x_3\, x_4] = \frac{f[x_2\, x_3\, x_4] - f[x_1\, x_2\, x_3]}{x_4 - x_1}$$

$$= \frac{\frac{1}{6} - \frac{1}{12}}{5+1} = \frac{1}{72}$$

$$P(x) = f[x_1] + f[x_1\, x_2](x - x_1)$$

$$+ f[x_1\, x_2\, x_3](x - x_1)(x - x_2)$$

$$+ f[x_1\, x_2\, x_3\, x_4](x - x_1)(x - x_2)(x - x_3)$$

$$= 0 + \frac{1}{3}(x+1) - \frac{1}{12}(x+1)(x-2)$$

$$+ \frac{1}{72}(x+1)(x-2)(x-3)$$

$$= \frac{1}{3}x + \frac{1}{3} - \frac{1}{12}x^2 + \frac{1}{12}x + \frac{3}{12}$$

$$+ \frac{1}{72}(x+1)(x^2-5x+6)$$

$$= \frac{4x + 4 - x^2 + x + 3}{12}$$

$$+ \frac{1}{72}(x^3 - 5x^2 + 6x + x^2 - 5x + 6)$$

$$= \frac{-6x^2 + 30x + 42 + x^3 + 4x^2 - 10x + 6}{72}$$

$$= \frac{x^3 - 2x^2 + 20x + 48}{72}$$

In 1b, we got:

$$P(x) = 0 + \frac{(x+1)(x-3)(x-5)}{(2+1)(2-3)(2-5)}$$

$$+ \frac{(x+1)(x-2)(x-5)}{(3+1)(3-2)(3-5)}$$

$$+ 2\frac{(x+1)(x-2)(x-3)}{(5+1)(5-2)(5-3)}$$

which simplifies like:

$$(x+1)\left(\frac{x^2-8x+15}{9}-\frac{x^2-7x+10}{8}\right.$$

$$\left.+\frac{x^2-5x+6}{18}\right)$$

$$=(x+1)\left(\frac{2x^2-16x+30+x^2-5x+6}{18}\right.$$

$$\left.-\frac{x^2-7x+10}{8}\right)$$

$$=(x+1)\left(\frac{3x^2-21x+36}{18}-\frac{x^2-7x+10}{8}\right)$$

$$=(x+1)\left(\frac{8(x^2-7x+12)-6(x^2-7x+10)}{48}\right)$$

$$=(x+1)\left(\frac{2x^2-14x+36}{48}\right)$$

$$=(x+1)\left(\frac{x^2-7x+18}{24}\right)$$

$$=\frac{x^3-7x^2+18x+x^2-7x+18}{24}$$

$$= \frac{x^3 - 6x^2 + 11x + 18}{24}$$

Can't be bothered redoing, lol

3a)

$$\begin{array}{r|cccc}
-1 & 3 & & & \\
 & & -1 & & \\
1 & 1 & & 1 & \\
 & & 2 & & 0 \\
2 & 3 & & 1 & \\
 & & 4 & & \\
3 & 7 & & &
\end{array}$$

$$P_2(x) = 3 - (x+1) + (x+1)(x-1)$$

$$= 3 - x - 1 + x^2 - 1 = x^2 - x + 1$$

$P_2(-1) = 3$ ✓

$P_2(1) = 1$ ✓

$P_2(2) = 3$ ✓

$P_2(3) = 7$ ✓

$\Rightarrow$ yes, one solution exists ?

b) $P_3(x) = P_2(x) + f[x_1 x_2 x_3](x-x_1)(x-x_2)(x-x_3)$

$= x^2 - x + 1 + (x+1)(x-1)(x-2)$

$= x^2 - x + 1 + (x-2)(x^2-1)$

$= x^2 - x + 1 + x^3 - x - 2x^2 + 2$

$= x^3 - x^2 - 2x + 3$

$P_3(-1) = 1$ X

no, none such exist

c)

Infinite

Skipping rest !

3.2 Interpolation error

Skipping !

3.3 Chebyshev interpolation

1a) $x_i = \cos\frac{(2i-1)\pi}{2n}$ when $-1 \leq x_1, \ldots, x_n \leq 1$

$x_1 = \cos\frac{\pi}{12}$ $x_2 = \cos\frac{3\pi}{12} = \cos\frac{\pi}{4}$

$x_3 = \cos\frac{5\pi}{12}$ $x_4 = \cos\frac{7\pi}{12}$

$x_5 = \cos\frac{9\pi}{12} = \cos\frac{3\pi}{4}$

$x_6 = \cos\frac{11\pi}{12}$

b) For ranges $\neq [-1, 1]$,

$x_i = \frac{b+a}{2} + \frac{b-a}{2}\cos\frac{(2i-1)\pi}{2n}$

$\frac{b+a}{2} = 0$ $\frac{b-a}{2} = 2$

so:

$x_1 = 2\cos\frac{\pi}{8}$ $x_2 = 2\cos\frac{3\pi}{8}$

$x_3 = 2\cos\frac{5\pi}{8}$ $x_4 = 2\cos\frac{7\pi}{8}$

c) $\frac{b+a}{2} = \frac{16}{2} = 8 \qquad \frac{b-a}{2} = \frac{8}{2} = 4$

$x_1 = 8 + 4\cos\frac{\pi}{12} \qquad x_2 = 8 + 4\cos\frac{\pi}{4}$

$x_3 = 8 + 4\cos\frac{5\pi}{12} \qquad x_4 = 8 + 4\cos\frac{7\pi}{12}$

$x_5 = 8 + 4\cos\frac{3\pi}{4} \qquad x_6 = 8 + 4\cos\frac{11\pi}{12}$

d) $\frac{b+a}{2} = 0.2 \qquad \frac{b-a}{2} = 0.5$

$x_1 = 0.2 + 0.5\cos\frac{\pi}{10}$

$x_2 = 0.2 + 0.5\cos\frac{3\pi}{10}$

$x_3 = 0.2 + 0.5\cos\frac{\pi}{5}$

$x_4 = 0.2 + 0.5\cos\frac{7\pi}{10}$

$x_5 = 0.2 + 0.5\cos\frac{9\pi}{10}$

2a) $$|(x-x_1)\cdots(x-x_n)| \leq \frac{\left(\frac{b-a}{2}\right)^n}{2^{n-1}}$$

$$\frac{\left(\frac{2}{2}\right)^6}{2^5} = \frac{1}{2^5}$$

b) $$\frac{2^4}{2^5} = 2$$

c) $$\frac{4^6}{2^5} = \frac{(2^2)^6}{2^5} = \frac{2^{12}}{2^5} = 2^7 = 128$$

d) $$\frac{\left(\frac{1}{2}\right)^5}{2^4} = \frac{\frac{1}{2^5}}{2^4} = \frac{1}{2^9}$$

3a) $$|e^x - Q_5(x)| \leq \frac{(x-x_1)\cdots(x-x_n)}{6!} f^{(5)}(c)$$

Since $|(x-x_1)\cdots(x-x_n)| \leq \frac{1}{2^5}$
we have

$$|e^x - Q_5(x)| \leq \frac{f^{(5)}(c)}{2^5 \cdot 6!}$$

$$\iff |e^x - Q_5(x)| \leq \frac{e^c}{23040}$$

for $c \in [-1, 1]$, e^c ranges $[\frac{1}{e}, e]$

which means

$$|e^x - Q_5(x)| \leq \frac{e}{23040}$$

where $\frac{e}{23040} \approx 0.\underbrace{000}\,118$

3 decimal points will be correct

4) Upper bound:

$$\frac{\left(\frac{1-0.6}{2}\right)^6}{2^5} = \frac{\frac{1}{5^6}}{2^5} = \frac{1}{5^6 \cdot 2^5}$$

$$|e^x - Q_5(x)| \leq \frac{e}{5^6 \cdot 2^5 \cdot 6!}$$

$(\approx 7{,}55 \cdot 10^{-9})$ 8 decimal places correct

Skipping rest of exercises in section...

3.4 Cubic splines

1a) It should hold that:

1. $S_i(x_i) = y_i$, $S_i(x_{i+1}) = y_{i+1}$ for $i \in [1, n-1]$

2. $S_{i-1}'(x_i) = S_i'(x_i)$ for $i \in [2, n-1]$

3. S'' - '' -

1. $S_1(0) = -1$, $S_1(1) = 1$

$S_2(1) = 1$

It is connected

2. $S'(x) = \begin{cases} 3x^2 + 1 & \text{on } [0,1] \\ -3(x-1)^2 + 6(x-1) + 3 & \text{on } [1,2] \end{cases}$

$S_2'(1) = 3 \quad S_1'(1) = 4$

$3 \neq 4 \Rightarrow$ prop. 2 fails $\Rightarrow$ not a cubic spline

b) 1. $S_1(0) = 5 \quad S_1(1) = 12$ ✓

$S_2(1) = 12$

it connects

2. $$S'(x) = \begin{cases} 6x^2 + 2x + 4 & \text{on } [0, 1] \\ 3(x-1)^2 + 14(x-1) + 12 & \text{on } [1, 2] \end{cases}$$

$S_1'(1) = 12 \quad S_2'(1) = 12$ ✓

3. $$S''(x) = \begin{cases} 12x + 2 & \text{on } [0, 1] \\ 6(x-1) + 14 & \text{on } [1, 2] \end{cases}$$

$S_1''(1) = 14 \quad S_2''(1) = 14$ ✓

This is a cubic spline

2a) 1. $S_1(1) = 10 \quad S_2(1) = 10$ ✓

2.
$$S'(x) = \begin{cases} 12x^2 + 6x + 2 & \text{on } [0, 1] \\ 12(x-1)^2 + 30(x-1) + 20 & \text{on } [1, 2] \end{cases}$$

$S_1'(1) = 20 \quad S_2'(1) = 20$ ✓

3.
$$S''(x) = \begin{cases} 24x + 6 & \text{on } [0,1] \\ 24(x-1) + 30 & \text{on } [1,2] \end{cases}$$

$S_1''(1) = 30 \quad S_2''(1) = 30$ ✓

It is a cubic spline!

b)

$\boxed{\begin{array}{l} \text{4a (natural)} \\ S_1''(x_1) = 0 \text{ and } S_{n-1}''(x_n) = 0 \end{array}}$

4a

$S_1''(0) = 6 \neq 0 \quad X$

not natural

$\boxed{\begin{array}{l} \text{4b (parabolically terminated)} \\ S_1''(x_1) \text{ and } S_{n-1}''(x_n) = \text{arbitrary value} \end{array}}$

4b $S_1''(0) = S_2''(2)$

$\Leftrightarrow 6 = 54 \Rightarrow$ doesn't satisfy

4c $S_1'(0) = S_2'(2)$

$\Leftrightarrow 2 = 62 \Rightarrow$ doesn't satisfy

Skipping. don't know what I'm doing...

Skipping rest of exercises. Feels like a waste of time considering the lack of guidance

3.5 Bézier curves

Skipping to save on time