Numerical analysis

Chapter 5: Numerical Diff. and Int.

Numerical differentiation

Finite difference formulas

Two-point forward-difference formula

If f is twice cont. diff. then:

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(c)$$

where $x < c < x+h$ (not chosen)

The error will be proportional to h, so making h smaller will minimize the error $O(h^n)$, where n is the nth order approx.

error = approximation − real $f'(x)$

Three-point centered-difference formula

f is thrice diff.

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f''(c)$$

where $x - h < c < x + h$ (not chosen)

Rounding error

The given formulas will cause rounding errors on computers

If h is too small we run into loss of significance

Extrapolation

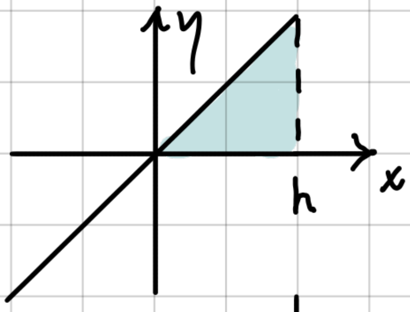
A higher order approx.

$$Q \approx \frac{2^n F(h/2) - F(h)}{2^n - 1}$$

Newton-Cotes formulas for num. int.

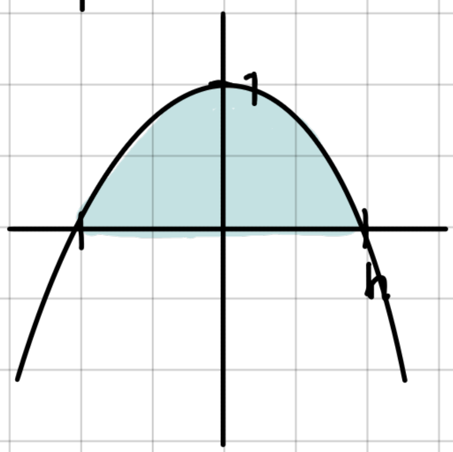
The three integrals

1.



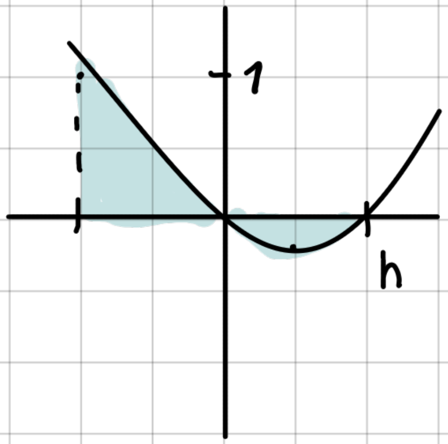
$$\int_0^h \frac{x}{h}\,dx = \frac{h}{2}$$

2.



$$\int_{-h}^{h} P(x)\,dx = x - \frac{x^3}{3h^2} = \frac{4}{3}h$$

3.



$$\int_{-h}^{h} P(x)\,dx = \frac{1}{3}h$$

Trapezoid rule

$$\int_{x_0}^{x_1} f(x)\,dx = \frac{h}{2}(y_0 + y_1) - \frac{h^3}{12} f''(c)$$

where $h = x_1 - x_0$ and $x_0 < c < x_1$

Simpson's rule

Uses three points rather than two, and replaces the degree 1 interpolant with a parabola

$$\int_{x_0}^{x_2} f(x)\,dx = \frac{h}{3}(y_0 + 4y_1 + y_2) - \frac{h^5}{90} f^{(4)}(c)$$

where $h = x_2 - x_1 = x_1 - x_0$ (same dist.)

and $x_0 < c < x_2$

Degree of precision

The greatest k for which all degree k or less polynomials are integrated exactly (without error)

Composite Newton-Cotes

Simply compute several adjacent subintegrals (panels) and sum them up

Comp. Trapezoid rule

$$\int_a^b f(x)\,dx = \frac{h}{2}\left(y_0 + y_m + 2\sum_{i=1}^{m-1} y_i\right) - \frac{(b-a)h^2}{12} f''(c)$$

where $h = \frac{b-a}{m}$ and $a < c < b$

Comp. Simpson's rule

See book.

Open Newton - Cotes Methods

(the others are considered closed)

Midpoint rule

Only requires one function eval rather than 2 as in Trapezoid

$$\int_{x_0}^{x_1} f(x)\,dx = h f(w) + \frac{h^3}{24} f''(c)$$

where $h = x_1 - x_0$, $w = x_0 + \frac{h}{2}$

and $x_0 < c < x_1$

Composite

$$\int_a^b f(x)\,dx = h \sum_{i=1}^{m} f(w_i) + \frac{(b-a)h^2}{24} f''(c)$$

where $h = \frac{b-a}{m}$, $a < c < b$

and w_i is the midpoint for sub interval i

Romberg integration

Not part of my course, skipping!

Exercises

5.1 Numerical differentiation

1) $$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(c)$$

$$f''(x) = -\frac{1}{x^2}$$

$$\Rightarrow f_a'(x) \approx \frac{\ln(x+h) - \ln x}{h} + \frac{h}{2c^2}$$

The real derivative is $f'(x) = \frac{1}{x}$

note that $\ln(1) = 0$, so:

$$f_a'(x) \approx \frac{\ln(x+h)}{h} + \frac{h}{2c^2}$$

a) $f_a'(1) \approx 0.953$ (we don't choose c)
making the error $|0.953 - 1| \approx 0.047$

b, c) skipping, not interesting

2a) $f'(x) = \dfrac{f(x+h) - f(x-h)}{2h} - \dfrac{h^2}{6} f'''(x)$

$f'''(x) = e^x$

The real $f'(x) = e^x$

$$f_a'(x) \approx \frac{e^{x+h} - e^{x-h}}{2h} - \frac{h^2 e^x}{6}$$

$$= \frac{e^x(3(e^h - e^{-h}) - h)}{6h}$$

which for $x = 0$:

$$f_a'(0) = \frac{3(e^h - e^{-h}) - h}{6h}$$

a) $f_a'(0) \approx 0.835$

giving the error $|0.835 - 1| \approx 0.16$

b, c) Shipping

Shipping rest in section to save on time!

5.2 Newton - Cotes

1a) The exact value should be $\frac{1}{3}$

$m = 1$:

$$h = \frac{b-a}{m} = 1$$

$$\frac{1}{2} \cdot 1 = \frac{1}{2}$$

error is $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$m = 2$:

$$h = \frac{1}{2}$$

$$\frac{1}{4} \cdot \left(0 + 1 + 2 \sum_{i=1}^{2} y_i\right) = \frac{1}{4}\left(1 + 2\left(\frac{1}{4}\right)\right)$$

where $y_i = f(x_0 + hi)$

$$= \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

error is $\frac{3}{8} - \frac{1}{3} = \frac{1}{24}$

$m = 4$:

$h = \frac{1}{4}$

$\frac{1}{8} \cdot \left(1 + 2\left(\frac{1}{16} + \frac{4}{16} + \frac{9}{16}\right)\right) = \frac{1}{8} \cdot \frac{44}{16} = \frac{44}{128}$

$= \frac{11}{32}$

error is $\frac{11}{32} - \frac{1}{3} = 0.0104$

b) Compute exact value:

$\int_0^{\frac{\pi}{2}} \cos x \, dx = \left[\sin x\right]_0^{\frac{\pi}{2}} = 1$

$m = 1$:

$h = \frac{\pi}{2}$

$\frac{\pi}{4}(1 + 0) = \frac{\pi}{4}$

error $= \left|\frac{\pi}{4} - 1\right| \approx 0.2146$

$m = 2$:

$h = \frac{\pi}{4}$

$\frac{\pi}{8}\left(1 + 2\left(\cos\frac{\pi}{4}\right)\right) \approx 0.948$

error ≈ 0.0519

$m = 4$:

$h = \frac{\pi}{8}$

$\frac{\pi}{16}\left(1 + 2\left(\cos\frac{\pi}{8} + \cos\frac{2\pi}{8} + \cos\frac{3\pi}{8}\right)\right)$

≈ 0.9871

error ≈ 0.0129

c) Skipping

2a) $m = 1$:

$h = 1$

$\sum_{i=1}^{1} f(u_i) = f\left(\frac{1}{2}\right) = \frac{1}{4}$

error $= \frac{1}{4} - \frac{1}{3} = \frac{1}{12}$

$m = 2$:

$h = \frac{1}{2}$

$$\frac{1}{2}\left(f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)\right) = \frac{1}{2}\left(\frac{1}{16} + \frac{9}{16}\right)$$

$$= \frac{10}{32} = \frac{5}{16}$$

$$\text{error} = \left|\frac{5}{16} - \frac{1}{3}\right| = \frac{1}{48}$$

$m = 4$:

$h = \frac{1}{4}$

$$\frac{1}{4}\left(f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right)$$

$$= \frac{1}{4}\left(\frac{1 + 9 + 25 + 49}{64}\right)$$

$$= \frac{84}{256} = \frac{21}{64}$$

$$\text{error} = \left|\frac{21}{64} - \frac{1}{3}\right| = \frac{1}{192}$$

6, c) Shipping

3a) $m = 1$:

$n = 1$

$\frac{1}{3}\left(0 + 0 + 4 \cdot 0 + 2 \cdot f(1)\right)$

$= \frac{1}{3}(1) = \frac{1}{3}$

error $= 0$

$m = 2$:

$h = \frac{1}{2} \quad x = \{0, \frac{1}{2}, 1\}$

$\frac{1}{6}\left(0 + 4(f(\frac{1}{2})) + 2(f(1))\right) = \frac{1}{6}(1 + 2) =$

error $= 0$

$m = 4$:

$h = \frac{1}{4} \qquad x = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$

$\frac{1}{12}\left(1 + 4(f(0) + f(\frac{3}{4})) + 2(f(\frac{2}{4}) + f(1))\right)$

$= \frac{1}{12}\left(1 + \frac{9}{4} + \frac{4}{8} + 2\right) = \frac{1}{12}\left(\frac{8 + 18 + 4 + 16}{8}\right)$

$$= \frac{1}{12}\left(\frac{46}{8}\right) = \frac{46}{48} = \frac{23}{24}$$