Numerical analysis

Chapter 12: Eigenvalues and singular values

Power iteration methods

Power iteration

def 12.1:

the dominant eigenvalue
is the largest of the eigenvalues

the eigenvector associated with it
is also considered dominant

Repeated multiplication with A will
grow towards the dominant eigenvalue

Rayleigh quotient

An approx. eigenvalue

$$\lambda = \frac{x^T A x}{x^T x}$$

Algerithm

For each iteration:

1. normalize the vector x_{j-1} from the previous step (or initial), save as u_{j-1}
2. multiply u_{j-1} with matrix A save as x_j
3. Compute Rayleigh quotient save as l_j

fin. normalize one last time save as u_j

Convergence

Converges at a rate $S = \left|\frac{l_2}{l_1}\right|$

Inverse

We can also find the smallest eigenvector/value in a similar fashion

1. normalize as before

2. solve $(A - s\underline{I})x_j = u_{j-1}$

3. compute $l_j = u_{j-1}^T x_j$

fin. normalize last time

Rayleigh quotient iteration

Can also be used to find the smallest eigenvalue

1. normalize as before

2. compute $\lambda_{j-1} = u_{j-1}^T A u_{j-1}$

3. solve $(A - \lambda_{j-1} I)x_j = u_{j-1}$

fin. normalize last time

QR algorithm

Simutaneous iteration

As power iteration iterates, the previously orthogonal vectors grow closer to the dominant eigenvector and hence cease to be orthogonal

As such, we need to re-orthogonalize at each step

init. set $\bar{Q}_0 = I$

for each column j:

$$A\bar{Q}_j = \bar{Q}_{j+1} R_{j+1}$$

Real Schur form & the QR algo.

Upper triangular, except for a bump (possibly)

$$\begin{bmatrix} x & x & x & x & x \\ & x & x & x & x \\ & & x & x & x \\ & & x & x & x \\ & & & & x \end{bmatrix}$$

if A is square then there exists orthogonal Q and schur T such that

$$A = Q^T T Q$$

Upper Hessenberg form

An "offset" upper triangular matrix

$$\begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ & x & x & x & x \\ & & x & x & x \\ & & & x & x \end{bmatrix}$$

$$A = QBQ^T$$

Singular value decomposition

$$A = USV^T$$

The matrix V is found by finding the eigen vectors/values for A^TA

and U is found by solving

$$\sqrt{\lambda} \cdot U = AV$$

or individually: $\sqrt{\lambda_n} \cdot u_n = Av_1$, e.t.c.

Special case: symmetric matrices

If the matrix is symmetric

$$s_n = \lambda_n \quad \text{and} \quad u_n = \begin{cases} +v_n & \text{if } \lambda_n \geq 0 \\ -v_n & \text{if } \lambda_n < 0 \end{cases}$$

where λ is ordered by size first

v is eigenvectors

Applications of the SVD

Properties of the SVD

1. the rank of A is the number of non-zero entries in S

2. $\det A = s_1 \cdots s_n$ (if A is $n \times n$)

3. if A is an invertible $m \times m$ matrix
$$A^{-1} = V S^{-1} U^T$$

4. A $(m \times m)$ can be written as the sum of rank 1 matrices

Dimension reduction

SVD can be used to reduce the number of dimensions in data

Compression

SVD can compress by using prop. 4

Exercises

5.1 Power iteration methods

1a) We need to satisfy $\det(A - \lambda I) = 0$

$$A - \lambda I = \begin{bmatrix} 3.5 - \lambda & -1.5 \\ -1.5 & 3.5 - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (3.5 - \lambda)(3.5 - \lambda) - 1.5^2$$

$$= 3.5^2 - 7\lambda + \lambda^2 - 1.5^2 = 0$$

$$\Leftrightarrow (\lambda - 3.5)^2 - 1.5^2 = 0$$

$$\Leftrightarrow \lambda - 3.5 = \pm 1.5 \Leftrightarrow \begin{matrix} \lambda_1 = 5 \\ \lambda_2 = 2 \end{matrix}$$

Solve for eigenvectors

$$(A - \lambda I)v = 0$$

$\lambda = 5$:

$$A - \lambda I = \begin{bmatrix} -1.5 & -1.5 \\ -1.5 & -1.5 \end{bmatrix}$$

$$\Rightarrow \quad -1.5(v_{11} + v_{12}) = 0$$

$$\Leftrightarrow \quad v_{11} = -v_{12} \quad \Leftrightarrow \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda = 2$:

$$A - \lambda I = \begin{bmatrix} 1.5 & -1.5 \\ -1.5 & 1.5 \end{bmatrix}$$

$$\Rightarrow \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

b)

$$\det(A - \lambda I) = \lambda^2 - 4 = 0$$

$$\Leftrightarrow \lambda = \pm 2$$

λ_1:

$$A - \lambda I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$

$$\Rightarrow \quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \dots \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

c) $\det(A - \lambda I)$

$$= (-0.2 - \lambda)(1.2 - \lambda) - 2.4^2$$

$$= -0.24 + 0.2\lambda - 1.2\lambda + \lambda^2 - 2.4^2$$

$$= \lambda^2 - \lambda - 6 = \left(\lambda - \frac{1}{2}\right)^2 - \frac{1}{4} - 6 = 0$$

$$\Leftrightarrow \lambda - \frac{1}{2} = \pm\sqrt{\frac{25}{4}} = \pm \frac{5}{2}$$

$$\Rightarrow \lambda_1 = 3$$
$$\lambda_2 = -2$$

λ_1:

$$A - \lambda I = \begin{bmatrix} -3.2 & -2.4 \\ -2.4 & -1.8 \end{bmatrix}$$

sub $\frac{2.4}{3.2}$ x row 1 from row 2 $\Rightarrow \begin{bmatrix} -3.2 & -2.4 \\ 0 & 0 \end{bmatrix}$

$$\Rightarrow \quad -3.2 v_{11} + 2.4 v_{12} = 0$$

$$\Leftrightarrow v_{11} = -\frac{2.4}{3.2} v_{12} = -0.75 v_{12}$$

$$\Rightarrow \quad v_1 = \begin{bmatrix} -0{,}75 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

λ_2:

$$A - \lambda I = \begin{bmatrix} 1.8 & -2.4 \\ -2.4 & 3.2 \end{bmatrix}$$

sub $-\frac{2.4}{1.8} \times$ row 1 from row 2

$$\Rightarrow \begin{bmatrix} 1.8 & -2.4 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad v_{22} = \frac{1.8}{2.4} v_{21} = 0.75 v_{21}$$

$$\Rightarrow \quad v_2 = \begin{bmatrix} 1 \\ 0.75 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

d) Shipping

2) -//-

3) -//-

4) -//-

5a) It converges to the dom./largest one, so

$\lambda_1 = 4$

$s = \left|\frac{\lambda_2}{\lambda_1}\right| = \frac{3}{4}$

the nth largest !

b) $\lambda_1 = -4 \quad (|-4| \geq 3)$

$\lambda_2 = 3$

$s = \frac{3}{4}$

c) $\lambda_1 = 4 \quad \lambda_2 = 2 \Rightarrow s = 2$

d) $\lambda_1 = 10 \quad \lambda_2 = 9 \Rightarrow s = \frac{9}{10}$

6) skipping

7a) The shifted eigenvalues are: $\{3, 1, 4\}$

It will converge to the number closest to s, so 1

$\lambda_1 = 1 \quad \lambda_2 = 3$

$$S = \left|\frac{\lambda_1 - s}{\lambda_2 - s}\right| = \left|\frac{1-0}{3-0}\right| = \frac{1}{3}$$

b) $\lambda_1 = 1 \quad \lambda_2 = 3 \Rightarrow S = \frac{1}{3}$

c) $\lambda_1 = -1 \quad \lambda_2 = 2 \Rightarrow S = \frac{1}{2}$

d) $\lambda_1 = 9 \quad \lambda_2 = 10 \Rightarrow S = \frac{3}{4}$

8) Skipping

9a) $\det(A-\lambda I) = (1-\lambda)(3-\lambda) - 8$

$= 3 - 4\lambda + \lambda^2 - 8 = 0$

$\Leftrightarrow (\lambda-2)^2 - 4 - 5 = 0$

$\Leftrightarrow \lambda - 2 = \pm 3 \Rightarrow \begin{array}{l}\lambda_1 = 5\\ \lambda_2 = -1\end{array}$

λ_1:

$A - \lambda I = \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$

$\Rightarrow -4v_{11} = 2v_{12}$

$\Rightarrow v_{11} = -\frac{1}{2} v_{12} \Rightarrow v_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

λ_2:

$A - \lambda I = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b) $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$u_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_1 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$\lambda_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = 1$

$$u_1 = \frac{1}{\sqrt{17}} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$x_2 = \frac{1}{\sqrt{17}} \begin{bmatrix} 9 \\ 16 \end{bmatrix}$$

$$\lambda_2 = \frac{73}{17}$$

$$u_2 = \frac{1}{\sqrt{337}} \begin{bmatrix} 9 \\ 16 \end{bmatrix}$$

$$x_2 = \frac{1}{\sqrt{337}} \begin{bmatrix} 41 \\ 84 \end{bmatrix}$$

$$\lambda_2 = \frac{1713}{337} \approx 5.08$$

c) The eigenvalues are $\{5, -1\}$

$$s = 0 \Rightarrow \lambda_1 = -1, \lambda_2 = 5$$

$$\Rightarrow S = \frac{1}{5}$$

d) $\lambda_1 = 5, \lambda_2 = -1 \Rightarrow S = \frac{2}{4} = \frac{1}{2}$

Skipping rest !

12.2 QR algorithm

Skipping, doesn't seem to appear on my exam, and I need to save on time!

12.3 Singular value decomposition

1a)

Since this is a symmetric matrix it suffices to find the eigenvectors and values, and use them as singular values

v_i are then eigenvectors, and u_i is determined like:

$$u_i = \begin{cases} v_i & \text{if } \lambda \geq 0 \\ -v_i & \text{if } \lambda < 0 \end{cases}$$

This matrix is already diagonal so $\lambda_1 = -3$ $\lambda_2 = 2$

and the vectors are just

the Standard basis vectors, essentially

$v_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

then

$$u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (\lambda_1 < 0)$$

$$u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad (\lambda_2 \geq 0)$$

this gives

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad S = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

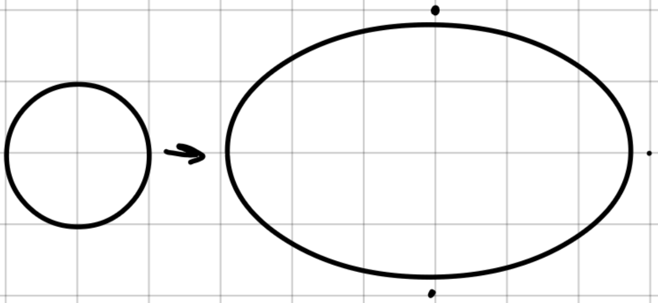
and $V = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

the singular values tell us:

the unit circle is stretched

in x-axis: 3

y-axis: 2



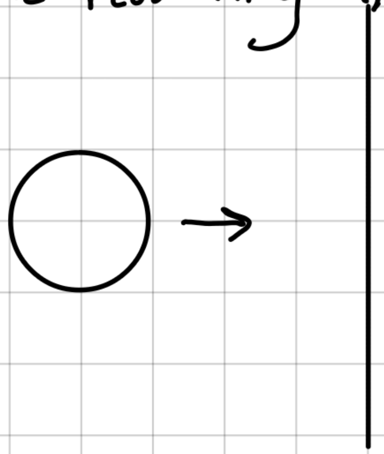
b) again, $\lambda_1 = 0$ $\lambda_2 = 3$

we have $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\Rightarrow u_1 = v_1 \quad u_2 = v_2$

$$USV^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This corresponds to flattening/projecting onto the y-axis, and stretching the resulting line by a factor of 3



c) Let's find the eigenvalues: (decreasing)

$$\det(A - \lambda I) = \left(\frac{3}{2} - \lambda\right)^2 - \frac{1}{4} = 0$$

$$\Leftrightarrow \frac{3}{2} - \lambda = \pm \frac{1}{2} \Leftrightarrow \lambda = \frac{3}{2} \pm \frac{1}{2} \Rightarrow \begin{matrix} \lambda_1 = 2 \\ \lambda_2 = 1 \end{matrix}$$

λ_1:

$$A - \lambda I = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

λ_2:

$$A - \lambda I = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

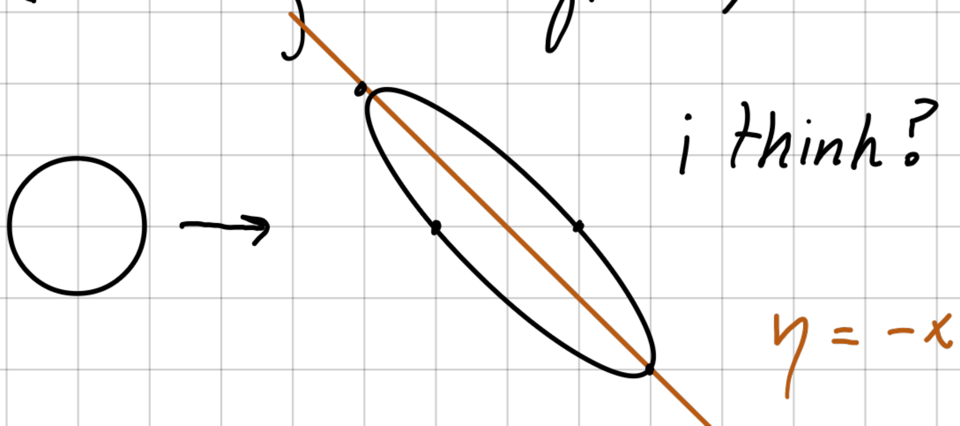
$$u_1 = v_1 = \frac{1}{\|x_1\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad u_2 = v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(they need to be normalized)

$$USV^T = \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

Stretched 2 along $\frac{1}{\sqrt{2}} [-1 \ \ 1]^T$

which corresponds to stretching along $y = -x$ by a factor of 2 (resulting in length 4)



i think?

e) (accidentally skipped d.)

$$\det(A - \lambda I) = \left(\frac{3}{4} - \lambda\right)^2 - \left(\frac{5}{4}\right)^2 = 0$$

$$\Leftrightarrow \frac{3}{4} - \lambda = \pm \frac{5}{4} \Leftrightarrow \lambda = \frac{3}{4} \pm \frac{5}{4}$$

$$\Rightarrow \begin{matrix} \lambda_1 = 2 \\ \lambda_2 = -\frac{1}{2} \end{matrix} \quad = \quad \begin{matrix} s_1 = 2 \\ s_2 = \frac{1}{2} \end{matrix}$$

λ_1

$$A - \lambda I = \begin{bmatrix} -1.25 & 1.25 \\ 1.25 & -1.25 \end{bmatrix}$$

$$\Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow u_1 = v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

λ_2

$$A - \lambda I = \begin{bmatrix} 1.25 & 1.25 \\ 1.25 & 1.25 \end{bmatrix}$$

$$\Rightarrow x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow u_2 = -v_2 = -\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

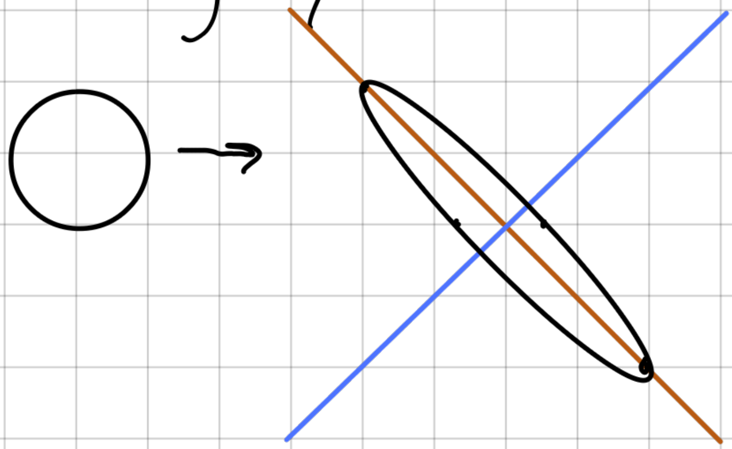
$$USV^T = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

again, expands 2 along $\frac{1}{\sqrt{2}} [1 \ 1]^T$

so along $y = -1$

contracts by 2 along $[1 \ -1]^T$

so along $y = x$



2a) We compute the eigenvalues of $A^T A$ instead

$$B = A^T A = \begin{bmatrix} 25 & 0 \\ 0 & 0 \end{bmatrix}$$

which gives eigenvalues $\lambda_1 = 25$ and $\lambda_2 = 0$

and gives $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

and $s_1 = \sqrt{\lambda_1} = 5$ $s_2 = 0$

this gives

$$Av_1 = su_1 \iff \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 5u_1$$

$$\iff \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 3u_1 \iff u_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$ (just needs to be orthogonal)

$$USV^T = \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

6)
$$B = A^TA = \begin{bmatrix} 6 & 8 \\ -2 & \frac{3}{2} \end{bmatrix}\begin{bmatrix} 6 & -2 \\ 8 & \frac{3}{2} \end{bmatrix} = \begin{bmatrix} 100 & 0 \\ 0 & \frac{25}{4} \end{bmatrix}$$

which gives $\lambda_1 = 100$ $\lambda_2 = \frac{25}{4}$

and $s_1 = 10$ $s_2 = \frac{5}{2}$

$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$Av_1 = s_1u_1 \iff \begin{bmatrix} 6 \\ 8 \end{bmatrix} = 10u_1 \iff u_1 = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$Av_2 = s_2u_2 \iff \begin{bmatrix} -2 \\ \frac{3}{2} \end{bmatrix} = \frac{5}{2}u_2 \iff u_2 = \frac{1}{5}\begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$USV^T = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & \frac{5}{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Shipping c, d, e

shipping rest !