Numerical analysis
Chapter 2: Systems of equations

Gaussian elimination

It can be helpful to visualize geometrically.

Naive

Three options:

- swap equations
- add/subtract one eq. to/from another
- multiply equation by non-zero constant

Not guaranteed to find solution or be accurate

Operation counts

Elimination Step

$$\frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n$$

in a system with n vars. and n eqs.

Back-substitution Step

n^2, n same as above

LU Factorization

$$\underbrace{\begin{bmatrix} x & x & x \\ 0 & x & x \\ 0 & 0 & x \end{bmatrix}}_{\text{Upper triangular}} \qquad \underbrace{\begin{bmatrix} x & 0 & 0 \\ x & x & 0 \\ x & x & x \end{bmatrix}}_{\text{Lower triangular}}$$

The U matrix is what you'd get after eliminating, and the L matrix what you used to eliminate

e.g subtract $m \times$ row i
from row n would be put
in column i, row n in L

The result is that: $LU = A$!

This makes the algorithm more efficient

1. Retrieve L matrix
2. Calculate corresponding new right hand side c by setting $Lc = b$
3. Retrieve U matrix
4. Using c values, set $Ux = c$ and solve for x

Classic gaussian elim. takes $2hn^3/3$ ops., while LU takes $2n^3/3 + 2hn^2$, which makes a difference if n is large

n is the dimension and h the size of the right hand vector

Sources of error

Maximum norm

The largest absolute value in the vector

Backward error

Here, we calculate the norm of the difference in the vectors
$\|b - Ax_a\|_\infty$

The relative backward error is then

$$\frac{\|b - Ax_a\|_\infty}{\|b\|_\infty} \quad \text{or} \quad \frac{\|r\|_\infty}{\|b\|_\infty}$$

Forward error

Same, but we calculate $\|x - x_a\|_\infty$

The relative forward error is then

$$\frac{\|x - x_a\|_\infty}{\|x\|_\infty}$$

Error magnification factor

$$\frac{\text{Relative forward error}}{\text{Relative backward error}}$$

Condition number

Maximum possible error magnification factor

For a square matrix this is

$$\text{cond}(A) = \|A\| \cdot \|A^{-1}\|$$

Operator norms

$$\|A\| = \max \frac{\|Ax\|}{\|x\|}$$

"A size of the linear operation"

Not sure I understand...

Swamping

Rounding on computers can cause errors during gaussian elimination causing equations to dissassocciate from eachother

PA = LU factorization

Partial pivoting

For each column n replace the pivot row (the nth row)
with the row that has the largest element in column n (if bigger)

Permutation matrices

Matrices where every row & column has a single 1
The rest are 0's

Fundamental theorem

If the permutation matrix is the identity matrix with replaced rows, then the same replacement will occur to a matrix multiplied with it

$PA = LU$ factorization

The P matrix is included to preserve the row exchange information

$$PAx = b$$
$$LUx = b$$

Solve

$Lc = Pb$ for c
$Ux = c$ for x

Iterative methods

Jacobi method

An iterative method (a form of FPI)

First, assume $x_0 = \begin{bmatrix} x_{01} \\ x_{02} \\ \vdots \\ x_{0n} \end{bmatrix} = 0$

then insert those values into the original equation and solve for $x_1 = \begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{bmatrix}$

repeat!

x_m will eventually converge to the approx. answer and, given enough iterations, the exact answer

does not always succeed!

only succeeds if strictly diagonally dominant, e.g.
the diagonal entry $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$
(which also means it is non-singular)

Gauss - Seidel method

Like Jacobi, but the current steps values are used in the equations

Successive Over-Relaxation (JOR)

Uses a relaxation parameter ω to overshoot the guesses

$\omega > 1$ - over-relaxation

$(1-\omega) x_{kn}$ is added to $x_{(k+1)n}$

Methods for symm. pos.-def. matrices

Symmetric matrices

A matrix is symmetric if

$A^T = A$

Positive-definitive

If $x^T A x > 0$ for all $x \neq 0$

If symm., A is also pos. def.
if all eigenvalues are positive

Cholesky factorization

If A is symm. pos. def and square
then an upper triangular matrix R
exists such that $A = R^T R$

$$\underset{(U)}{R} \quad \begin{matrix} \circ & \circ & \circ \\ & \circ & \circ \\ & & \circ \end{matrix} \qquad \underset{(L)}{R^T} \quad \begin{matrix} \circ & & \\ \circ & \circ & \\ \circ & \circ & \circ \end{matrix}$$

(a LU factorization!)

Conjugate gradient method

A-inner product

$(v, w)_A = v^T A w$

inherits symmetry, linearity, and pos. def

symmetry also means that

$(v, w)_A = (w, v)_A$

A-conjugate

if $(v, w)_A = 0$

Method

See book.

In some ways simpler than gaussian elim. but can fail on an ill-conditioned matrix

To fix this, preconditioning can be performed before running the algorithm

Preconditioning

Accelerates the convergence of iterative methods

The original matrix is multiplied with a "preconditioner" M

M is invertible, and has the same dimensions as A

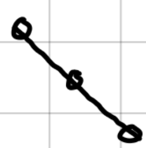
$M^{-1}Ax = M^{-1}b$

M should be:

- as close to A as possible
- simple to invert

Jacobi preconditioner

$M = D$, where D is the diagonal elements of A



Symmetric SOR preconditioner

$$M = (D + wL) D^{-1} (D + wU)$$

$w = 1 \Leftrightarrow$ Gauss-Seidel preconditioner

Nonlinear systems of equations

Multivariate Newton's Method

Use the derivatives of the matrix at each step

Skipping rest, as it is not part of my course

Exercises

2.1 Gaussian elimination

1a) $\begin{cases} 2x - 3y = 2 \\ 5x - 6y = 8 \end{cases} \Rightarrow$ sub $\frac{5}{2} \times$ row 1 from row 2

$\Rightarrow \begin{cases} 2x - 3y = 2 \\ 1{,}5y = 3 \end{cases}$

Backsolve $1{,}5y = 3 \Rightarrow y = 2$

$2x - 6 = 2 \Leftrightarrow 2x = 8 \Leftrightarrow x = 4$

b) $\begin{matrix} x + 2y = -1 \\ 2x + 3y = 1 \end{matrix} \Rightarrow$ sub $2 \times$ row 1 from row 2

$\Rightarrow \begin{matrix} x + 2y = -1 \\ -y = 3 \end{matrix}$

$y = -3$

$x - 6 = -1 \Leftrightarrow x = 5$

c) $$\begin{aligned} -x + y &= 2 \\ 3x + 4y &= 15 \end{aligned} \Rightarrow$$ add 3 x row 1 to row 2

$$\Rightarrow \begin{aligned} -x + y &= 2 \\ 7y &= 21 \end{aligned}$$

$$\Rightarrow y = 3$$

$$-x + 3 = 2 \iff x = 1$$

2a) Skipping, already familiar

3a) $z = 1$

$$3y - 4 = -1 \iff 3y = 3 \iff y = 1$$

$$3x - 4 + 5 = 2 \iff 3x = 1$$

$$\iff x = \frac{1}{3}$$

b) Skipping

4a) $\left[\begin{array}{ccc|c} 3 & -4 & -2 & 3 \\ 6 & -6 & 1 & 2 \\ -3 & 8 & 2 & -1 \end{array}\right]$ sub 2 x row 1 from row 2

$\left[\begin{array}{ccc|c} 3 & -4 & -2 & 3 \\ 0 & 6 & 5 & -4 \\ -3 & 8 & 2 & -1 \end{array}\right]$ add 1 x row 1 to row 3

$\left[\begin{array}{ccc|c} 3 & -4 & -2 & 3 \\ 0 & 6 & 5 & -4 \\ 0 & 4 & 0 & 2 \end{array}\right]$ sub $\frac{4}{6}$ x row 2 from row 3

$\left[\begin{array}{ccc|c} 3 & -4 & -2 & 3 \\ 0 & 6 & 5 & -4 \\ 0 & 0 & -\frac{20}{6} & \frac{28}{6} \end{array}\right]$

$$z = \frac{\frac{28}{6}}{-\frac{20}{6}} = -\frac{28}{20} = -\frac{14}{10} = \frac{-7}{5}$$

$$6y - 7 = -4 \Leftrightarrow 6y = 3 \Leftrightarrow y = \frac{1}{2}$$

$$3x - 2 + \frac{14}{5} = 3 \Leftrightarrow 3x = \frac{11}{5}$$

$$\Leftrightarrow x = \frac{11}{15}$$

b) Shipping

5) $m = 3n \Rightarrow \frac{2m^3}{3} = \frac{2 \cdot 27n^3}{3}$

as compared to $\frac{2n^3}{3}$

this is 27 times slower

6) $\frac{\left(\frac{2n^3}{3}\right)}{n^2} = \frac{2n}{3}$ times the time for

back sub., so $\frac{1000}{3} \cdot 0{,}005 = \frac{5}{3}$ s

for elim.

Total $= 0{,}005 + \frac{5}{3}$ seconds

7) The amount of ops. for $n = 4000$
is 4000^2

which means the ops. per sec is

$\frac{4000^2}{0{,}002}$ and thereby the time

for 9000 (complete) would be

$$\frac{\frac{2 \cdot 9000^3}{3} \cdot 0.002}{4000^2} = 60{,}75 \text{ secs.}$$

8) Skipping

2.2 LU factorization

1a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ sub $3 \times$ row 1 from row 2

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = U$

gives $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$

b) $U = \begin{bmatrix} 1 & 3 \\ 0 & -4 \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

c) $U = \begin{bmatrix} 3 & -4 \\ 0 & -\frac{14}{3} \end{bmatrix}$ $L = \begin{bmatrix} 1 & 0 \\ -\frac{5}{3} & 1 \end{bmatrix}$

2a) $\begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}$ sub 2 × row 1 from row 2
sub 1 × row 1 from row 3

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

b) $\begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}$ sub 1× row 1 from row 2
sub $\frac{1}{2}$ × row 1 from row 3

$\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ sub $\frac{1}{2}$ × row 2 from row 3

$$\begin{bmatrix} 4 & 2 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} = U \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

c) Shipping

3a) $\begin{bmatrix} 3 & 7 \\ 6 & 1 \end{bmatrix}$ sub 2 x row 1 from row 2

$$\begin{bmatrix} 3 & 7 \\ 0 & -13 \end{bmatrix} = U \quad L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

Solve $Lc = b$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \end{bmatrix}$$

$\Rightarrow c_1 = 1 \Rightarrow 2 + c_2 = -11$

$\Leftrightarrow c_2 = -13$

Solve $Ux = c$

$$\begin{bmatrix} 3 & 7 \\ 0 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -13 \end{bmatrix} \Rightarrow x_2 = 1$$

$\Rightarrow 3x_1 + 7 = 1 \Leftrightarrow 3x_1 = -6$

$\Leftrightarrow x_1 = -2$

b) $\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$ sub 2 row 1 from row 2

$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = U \quad L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

Solve $Lc = b$

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \Rightarrow c_1 = 1$

$2 + c_2 = 3 \Leftrightarrow c_2 = 1$

Solve $Ux = c$

$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow x_2 = 1$

$2x_1 + 3 = 1 \Leftrightarrow 2x_1 = -2$

$\Leftrightarrow x_1 = -1$

4) Shipping

5) Solve $Lc = b$

$c_1 = 1 \quad c_2 = 1$

$1 + 3 + c_3 = 2 \iff c_3 = -2$

$4 + 1 - 4 + c_4 = 0 \iff c_4 = -1$

Solve $Ux = c$

$x_4 = -1$

$-x_3 - 1 = -2 \iff x_3 = 1$

$x_2 + 2 = 1 \iff x_2 = -1$

$2x_1 - 1 = 1 \iff x_1 = 1$

Skipping rest in section !!

2.3 Sources of error

1a) $\|A\|_\infty = \max \begin{bmatrix} 3 \\ 7 \end{bmatrix} = 7$

b) $\max \begin{bmatrix} 7 \\ 6 \\ 8 \end{bmatrix} = 8$

2a) $A^{-1} = \frac{1}{\det A} \cdot A^T$

$= \frac{1}{4-6} \cdot \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$\|A^{-1}\| = \frac{1}{2} \max \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 3$

$\|A\| = 7$ (from 1a)

$\operatorname{cond}(A) = \|A\| \cdot \|A^{-1}\|$

$= 3 \cdot 7 = 21$

b) $A^{-1} = \frac{1}{6-6.03}\begin{bmatrix} 1 & 3 \\ 2.01 & 6 \end{bmatrix}$

$\|A^{-1}\| = \frac{1}{0.03} \cdot \max \begin{bmatrix} 4 \\ 8.01 \end{bmatrix}$

$= \frac{8.01}{0.03} = \frac{801}{3} = 267$

$\|A\| = \max \begin{bmatrix} 3.01 \\ 9 \end{bmatrix} = 9$

$\text{cond}(A) = 267 \cdot 9 = 801 \cdot 3$

$= 2403$

3a) <u>Backward error</u> <u>Forward error</u>

$\|b - Ax_a\|_\infty$ $\quad$ $\|x - x_a\|_\infty$

$A = \begin{bmatrix} 1 & 1 \\ 1.0001 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$FE = \left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right\| = \max \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2$

$$BE = \left\| \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 2 \\ 1,9999 \end{bmatrix} \right\|$$

$$= \max \begin{bmatrix} 0 \\ 0.0002 \end{bmatrix} = 0.0002$$

$$EMF = \frac{RFE}{RBE}$$

$$RFE = \frac{FE}{\|x\|} = 2$$

$$RBE = \frac{BE}{\|b\|} = \frac{0.0002}{2.0001}$$

$$EMF = \frac{2 \cdot 2.0001}{0.0002} = \frac{2.0001}{10^{-4}}$$

$$= 20001$$

b) $FE = \max \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1$

$$BE = \left\| \begin{bmatrix} 2 \\ 2.0001 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\|$$

$$= 0.0001$$

$$RFE = 1 \quad RBE \ \frac{0.0001}{2.0001}$$

$$EMF = \frac{2.0001}{0.0001} = 20001$$

c) $FE = 1 \quad BE = \left\| \begin{bmatrix} 2 & -4 \\ 2.0001 & -4.0002 \end{bmatrix} \right\|$

$$= 2.0001$$

$$EMF = \frac{2.0001}{2.0001} = 1$$

d) $FE = 3.0001 \quad BE = \left\| \begin{matrix} 2 & -2.0001 \\ 2.0001 & -1.9999 \end{matrix} \right\|$

$$= 0.0002$$

$$EMF = \frac{3.0001 \cdot 2.0001}{0.0002} \approx 30002.5$$

4a) The system in matrix form is

$$\begin{bmatrix} 1 & 2 \\ 2 & 4.01 \end{bmatrix} x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solve for x

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4.01 & 2 \end{array}\right]$$ sub 2 x row 1 from row 2

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0.01 & 0 \end{array}\right] \Rightarrow x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$FE = \max \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2$$

$$BE = \left\| \begin{bmatrix} 1 - 1 \\ 2 - 2.01 \end{bmatrix} \right\| = 0.01$$

$$RFE = \frac{2}{1} = 2 \quad RBE = \frac{0.01}{2}$$

$$EMF = \frac{2 \cdot 2}{0.01} = 400$$

b) $$FE = \max \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2$$

$$BE = \left\| \begin{bmatrix} 1 - 1 \\ 2 - 3.99 \end{bmatrix} \right\| = 1.99$$

$$EMF = \frac{2 \cdot 2}{1.99} \approx 2$$

c) Skipping

5a) $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$ $b = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

Solve for x

$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 3 & -4 & 7 \end{array}\right]$ sub 3 x row 1 from row 2

$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 2 & -2 \end{array}\right] \Rightarrow x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$FE = \max \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3$

$BE = \left\|\begin{bmatrix} 3 - 6 \\ 2 - 10 \end{bmatrix}\right\| = 3$

$RFE = \frac{3}{1} = 3$ $RBE = \frac{3}{7}$

$EMF = \frac{7 \cdot 3}{3} = 7$

b, c, d) Skip

e) $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 3 \\ -2 & -4 \end{bmatrix}$

$||A^{-1}|| = \frac{1}{2} \max \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 3$

$||A|| = \max \begin{bmatrix} 3 \\ 7 \end{bmatrix} = 7$

$\text{cond}(A) = 3 \cdot 7 = 21$

6) Skip

7) Skip

Skipping rest in section

2.4 PA = LU Factorization

1a) $\begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$ exchange row 1 w/ row 2 $\Rightarrow \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$

sub. $\frac{1}{2} \times$ row 1 from row 2 $\Rightarrow \begin{bmatrix} 2 & 3 \\ 0 & \frac{3}{2} \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

b) sub $\frac{1}{2} \times$ row 1 from row 2 $\Rightarrow \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c) exc. row 1 w/ row 2 $\Rightarrow \begin{bmatrix} 5 & 12 \\ 1 & 5 \end{bmatrix}$

sub $\frac{1}{5} \times$ row 1 from row 2 $\Rightarrow \begin{bmatrix} 5 & 12 \\ 0 & \frac{13}{5} \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{5} & 1 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

d) exc. row 1 w/ row 2 $\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2a) exc. row 1 w/ row 2 $\Rightarrow \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 0 \\ -1 & 1 & -1 \end{bmatrix}$

$\left.\begin{array}{l} \text{sub } \frac{1}{2} \times \text{ row 1} \\ \text{From row 2} \\ \text{sub } -\frac{1}{2} \times \text{ row 1} \\ \text{from row 3} \end{array}\right\} \Rightarrow \begin{bmatrix} 2 & 1 & -1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

exc. row 2 w/ row 3 $\Rightarrow \begin{bmatrix} 2 & 1 & -1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

sub $\frac{1}{3}$ x row 2 from row 3 $\Rightarrow \begin{bmatrix} 2 & 1 & -1 \\ -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Control:

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

✓ checks out!

b) exc. row 1 w/ row 2 $\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ -1 & -1 & 2 \end{bmatrix}$

sub $-\frac{1}{2} \times$ row 1 from row 3 $\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} \end{bmatrix}$

sub $-\frac{1}{2} \times$ row 2 from row 3 $\Rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ -\frac{1}{2} & -\frac{1}{2} & 4 \end{bmatrix} = U$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Skipping rest

3a) exc. row 1 w/ row 2 $\Rightarrow \begin{bmatrix} 6 & 1 \\ 3 & 7 \end{bmatrix}$

sub $\frac{1}{2}$ x row 1 from row 2 $\Rightarrow \begin{bmatrix} 6 & 1 \\ \frac{1}{2} & \frac{13}{2} \end{bmatrix} = U$

$$L = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$Lc = Pb \iff \begin{cases} c_1 = -11 \\ \frac{1}{2}c_1 + c_2 = 1 \end{cases}$$

$$\Rightarrow c_1 = -11 \Rightarrow -\frac{11}{2} + c_2 = 1$$

$$\iff c_2 = \frac{13}{2}$$

$$Ux = c \iff \begin{cases} 6x_1 + x_2 = -11 \\ \frac{13}{2}x_2 = \frac{13}{2} \end{cases}$$

$x_2 = 1 \Rightarrow 6x_1 + 1 = -11$

$\Leftrightarrow x_1 = -2$

b) exc. row 1 w/ row 2 $\Rightarrow \begin{bmatrix} 6 & 3 & 4 \\ 3 & 1 & 2 \\ 3 & 1 & 5 \end{bmatrix}$

exc. row 2 w/ row 3 $\Rightarrow \begin{bmatrix} 6 & 3 & 4 \\ 3 & 1 & 5 \\ 3 & 1 & 2 \end{bmatrix}$ because $3 \geq 3$!

$\left.\begin{array}{l} \text{sub } \frac{1}{2} \times \text{row 1} \\ \text{from row 2} \\ \text{- " - row 3} \end{array}\right\} \Rightarrow \begin{bmatrix} 6 & 3 & 4 \\ \frac{1}{2} & -\frac{1}{2} & 3 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$

exc. row 2 w/ row 3 $\Rightarrow \begin{bmatrix} 6 & 3 & 4 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 3 \end{bmatrix}$ $-\frac{1}{2} \geq -\frac{1}{2}$!

sub 1 × row 2 from row 3 $\Rightarrow \begin{bmatrix} 6 & 3 & 4 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & 1 & 3 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\underline{Lc = Pb}$

$$\begin{cases} c_1 & = 1 \\ \frac{1}{2}c_1 + c_2 & = 0 \\ \frac{1}{2}c_1 + c_2 + c_3 & = 3 \end{cases}$$

$\Rightarrow c_1 = 1$

$\frac{1}{2} + c_2 = 0 \Leftrightarrow c_2 = -\frac{1}{2}$

$\frac{1}{2} - \frac{1}{2} + c_3 = 3$

$\Leftrightarrow c_3 = 3$

$\underline{Ux = c}$

$$\begin{cases} 6x_1 + 3x_2 + 4x_3 & = 1 \\ -\frac{1}{2}x_2 & = -\frac{1}{2} \\ 3x_3 & = 3 \end{cases}$$

$\Rightarrow x_3 = 1 \quad x_2 = 1$

$6x_1 + 3 + 4 = 1 \Leftrightarrow x_1 = -1$

Skipping rest in section ∇

2.5 Iterative methods

1a) **Jacobi**

First assume $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = 0$

Which means

$$\begin{cases} 3u_1 - v_0 = 5 \\ -u_0 + 2v_1 = 4 \end{cases} \Leftrightarrow \begin{cases} u_1 = \dfrac{5+v_0}{3} = \dfrac{5}{3} \\ v_1 = \dfrac{4+u_0}{2} = 2 \end{cases}$$

Which then means that

$$\begin{cases} u_2 = \dfrac{5+2}{3} = \dfrac{7}{3} \\ v_2 = \dfrac{4+\frac{5}{3}}{2} = 2 + \dfrac{5}{6} = \dfrac{17}{6} \end{cases}$$

Gauss-Seidel

Assume $\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = 0$

then:

$$\begin{cases} u_1 = \dfrac{5+v_0}{3} \\ v_1 = \dfrac{4+u_1}{2} \end{cases} \Rightarrow \begin{cases} u_1 = \dfrac{5}{3} \\ v_1 = \dfrac{17}{6} \end{cases}$$

second Step:

$$\begin{cases} u_2 = \dfrac{5+v_1}{3} \\ v_2 = \dfrac{4+u_2}{2} \end{cases} \Rightarrow \begin{cases} u_2 = \dfrac{5}{3} + \dfrac{17}{18} = \dfrac{47}{18} \\ v_2 = 2 + \dfrac{47}{36} = \dfrac{119}{36} \end{cases}$$

6) Jacobi

The iteration is $\begin{cases} u_{n+1} = \dfrac{v_n}{2} \\ v_{n+1} = \dfrac{2+u_n+w_n}{2} \\ w_{n+1} = \dfrac{v_n}{2} \end{cases}$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

Gauss-seidel

Iteration: $\begin{cases} u_{n+1} = \dfrac{v_n}{2} \\ v_{n+1} = \dfrac{2+u_{n+1}+w_n}{2} \\ w_{n+1} = \dfrac{v_{n+1}}{2} \end{cases}$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix} \quad \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ \frac{3}{4} \end{bmatrix}$$

c) Jacobi

Iteration:

$$\begin{cases} u_{n+1} = \dfrac{6 - v_n - w_n}{3} \\ v_{n+1} = \dfrac{3 - u_n - w_n}{3} \\ w_{n+1} = \dfrac{5 - u_n - v_n}{3} \end{cases}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \frac{6}{3} \\ 1 \\ \frac{5}{3} \end{bmatrix} \qquad \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ -\frac{2}{9} \\ \frac{6}{9} \end{bmatrix}$$

Gauss-seidel

Iteration:

$$\begin{cases} u_{n+1} = \dfrac{6 - v_n - w_n}{3} \\ v_{n+1} = \dfrac{3 - u_{n+1} - w_n}{3} \\ w_{n+1} = \dfrac{5 - u_{n+1} - v_{n+1}}{3} \end{cases}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{3} \\ \frac{11}{9} \end{bmatrix} \qquad \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \rightsquigarrow \text{Skipping?}$$

2a) $A = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix}$

Jacobi

$$u_{n+1} = \frac{6 - 4v_n}{5}$$

$$v_{n+1} = \frac{-1 - u_n}{3}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ -\frac{1}{3} \end{bmatrix} \quad \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{22}{15} \\ -\frac{11}{15} \end{bmatrix}$$

Gauss-seidel

$$u_{n+1} = \frac{6 - 4v_n}{5}$$

$$v_{n+1} = \frac{-1 - u_{n+1}}{3}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ -\frac{11}{15} \end{bmatrix} \quad \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{62}{15} \\ -\frac{65}{45} \end{bmatrix}$$

b) $A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & -8 & -2 \\ 1 & 1 & 5 \end{bmatrix}$

Jacobi

$$u_{n+1} = \frac{-2 + v_n - w_n}{3}$$

$$v_{n+1} = \frac{1 - u_n + 2w_n}{-8}$$

$$w_{n+1} = \frac{4 - u_n - v_n}{5}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{8} \\ \frac{4}{5} \end{bmatrix} \qquad \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \frac{-2 - \frac{1}{8} - \frac{4}{5}}{3} \\ \frac{1 + \frac{2}{3} + \frac{8}{5}}{-8} \\ \frac{4 + \frac{2}{3} + \frac{1}{8}}{5} \end{bmatrix} = \begin{bmatrix} \frac{-2 - \frac{37}{40}}{3} \\ \frac{1 + \frac{34}{15}}{-8} \\ \frac{4 + \frac{19}{24}}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{117}{120} \\ -\frac{49}{120} \\ \frac{115}{120} \end{bmatrix}$$

c) $$A = \begin{bmatrix} 4 & 0 & 3 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Jacobi

$$u_{n+1} = \frac{-3w_n}{4}$$

$$v_{n+1} = \frac{5-u_n}{4}$$

$$w_{n+1} = \frac{2-v_n}{2}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{4} \\ 1 \end{bmatrix} \quad \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4} \\ \frac{5}{4} \\ \frac{3}{8} \end{bmatrix}$$

Gauss-seidel

$$u_{n+1} = \frac{-3w_n}{4}$$

$$v_{n+1} = \frac{5-u_{n+1}}{4}$$

$$w_{n+1} = \frac{2-v_{n+1}}{2}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{4} \\ \frac{3}{8} \end{bmatrix} \quad \begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} -\frac{9}{32} \\ \frac{151}{128} \\ \frac{105}{256} \end{bmatrix}$$

3a) $u_{n+1} = (1-w)u_n + w\left(\frac{5+v_n}{3}\right)$

$v_{n+1} = (1-w)v_n + w\left(\frac{4+u_{n+1}}{2}\right)$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5w}{3} \\ \frac{12w + 5w^2}{6} \end{bmatrix} = \begin{bmatrix} \frac{15}{6} \\ \frac{18 + \frac{45}{4}}{6} \end{bmatrix} = \begin{bmatrix} \frac{15}{6} \\ \frac{117}{24} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = -\frac{1}{2} \cdot \frac{15}{6} + \frac{3}{2} ($$

Skipping tedious asf

Skipping rest in section...

2.6 Methods for pos.-def. matrices

1a) Symmetry

$$A^T = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow A^T = A \checkmark$$

Pos. def.

$$x^T A x = [x_1 \ x_2] \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= [x_1 \ x_2] \begin{bmatrix} x_1 \\ 3x_2 \end{bmatrix} = x_1^2 + 3x_2^2$$

Which is non-negative for all $x \neq 0$

b) The matrix is symm.

$$x^T A x = x_1^2 + 3x_1x_2 + 3x_1x_2 + 10x_2^2$$

$$= x_1^2 + 6x_1x_2 + 10x_2^2$$

$$= (x_1 - 3x_2)^2 - 9x_2^2 + 10x_2^2$$

$$= (x_1 - 3x_2)^2 + x_2^2$$

which is always non-negative

c) The matrix is symm.

$x^T A x = x_1^2 + 2x_2^2 + 3x_3^2$

which is always positive

2a) Symmetric

$x^T A x = x_1^2 - 3x_2^2 = 0$

$\Leftrightarrow x_2 = \sqrt{\frac{x_1^2}{3}}$

Any vector where $x_2 \geq \frac{x_1}{\sqrt{3}}$

for ex $(1, 1)$

b) Symm.

$x^T A x = x_1^2 + 2x_1x_2 + 2x_1x_2 + 2x_2^2$

$= (x_1 + 2x_2)^2 - 4x_2^2 + 2x_2^2$

$= (x_1 + 2x_2)^2 - 2x_2^2 = 0$

occurs when

$x_2 \geq \frac{x_1 + 2x_2}{\sqrt{2}} \Leftrightarrow x_2 - \sqrt{2}x_2 \geq \frac{x_1}{\sqrt{2}}$

$\Leftrightarrow x_2(1-\sqrt{2}) \geq \frac{x_1}{\sqrt{2}}$

$\Leftrightarrow x_2 \geq \frac{x_1}{\sqrt{2}(1-\sqrt{2})} \Leftrightarrow x_2 \geq \frac{x_1}{\sqrt{2}-2}$

for example $(\sqrt{2}-2, 2)$

c) Symm.

$x^T A x = x_1^2 - x_1 x_2 - x_1 x_2 = x_1^2 - 2x_1 x_2$

$= (x_1 - x_2)^2 - x_2^2 = 0$

occurs when

$x_2 \geq x_1 - x_2 \Leftrightarrow x_2 \geq \frac{x_1}{2}$

ex. when $x = (2, 1)$

d) Symm

$x^T A x = x_1^2 - 2x_2^2 + 3x_3^2 = 0$

when

$x_2 \geq \sqrt{\frac{x_1^2 + 3x_3^2}{2}}$

for ex. $x = (1, \sqrt{2}, 1)$

3a) $R_{11} = \sqrt{a_{11}} = 1$

$R_{12} = \frac{a_{12}}{R_{11}} = 0$

$R_{21} = 0 \quad R_{22} = \sqrt{a_{22} - R_{12}^2} = \sqrt{a_{22}} = \sqrt{3}$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix}$$

b) $R_{11} = 1 \Rightarrow R_{12} = 3$

$R_{21} = 0 \quad R_{22} = \sqrt{10-9} = 1$

$$R = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

c) $R_{11} = 1 \quad R_{1,2:3} = [0 \; 0]$

$R_{2:3,1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$uu^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow R_{22} = \sqrt{a_{22}} = \sqrt{2}$

$R_{23} = 0$

$R_{32} = 0 \quad R_{33} = \sqrt{3-0} = \sqrt{3}$

4) ——

5a) $R = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ b) $R = \begin{bmatrix} 2 & -1 \\ 0 & \frac{1}{2} \end{bmatrix}$

c) $R = \begin{bmatrix} 5 & 1 \\ 0 & 5 \end{bmatrix}$ d) $R = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

6a) $R_1 = 2 \quad R_{1,2:3} = [-1 \; 0] = u$

$uu^T = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \Rightarrow \text{submatrix} = \begin{bmatrix} 2 & -3 \\ -3 & 10 \end{bmatrix}$

$R_{22} - \sqrt{2} \Rightarrow R_{23} = -\frac{3}{\sqrt{2}}$

$R_{33} = -\frac{3}{\sqrt{2}} \cdot -\frac{3}{\sqrt{2}} = \frac{9}{4}$

Skipping rest as I realized this is not part of the course ?

2.7 Nonlinear systems of equations

1a) $\begin{bmatrix} 3u^2 & 0 \\ 3v^3 & 3uv^2 \end{bmatrix}$ b)

c) $\begin{bmatrix} 2u+v^2 & 2v \\ 2(u-1) & 2v \end{bmatrix}$ d)

Skipping rest due to time constraints

!!