Numerical analysis

Chapter 1: Solving equations

Equation solving methods are important for effectivizing computer calculations.

The bisection method

Bracketing a root

def 1.1: $f(x)$ has a root r if $f(r)=0$

th. 1.2: if f is cont. in $[a,b]$ and $f(a)f(b)<0$ a root $r \in [a,b]$ exists

Finding the roots of a function can be made more efficient by repetitively narrowing our search interval (increasing certainty).

This is called bracketing.

Bisection method

if $c=$ middle, $f(c)=0$: end, c is a root

else narrow until the TOLerance is reached

How accurate & fast?

Error: $|x_c - r| < \frac{b-a}{2^{n+1}}$

No. evals: $n+2$

A solution is correct within p decimals if error $< 0{,}5 \times 10^{-p}$

Fixed point iteration

def 1.4: the real num. r is a fixed point of g if $g(r) = r$

Such values are found by repetively feeding the function the result of the previous iteration, until the value stops changing

Geometry

If we graphically represent these steps we get a cobweb diagram, spiralling around r in a spiral pattern

Linear convergence

The slope of the function determines the spiralling direction: whether it converges or diverges

Convergence means the error grows smaller whilst the opposite holds for divergence

def 1.5: the rate of convergence is

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i} = S$$ where e_n is error at step n

def 1.7: an iterative method is locally convergent within a neighborhood if all guesses within it converge to r

Stopping criteria

Knowing when to stop with FPI is hard
we need a stopping criterion

Absolute

$$|x_{i+1} - x_i| < TOL$$

Relative

If the solution is too near zero

$$\frac{|x_{i+1} - x_i|}{|x_{i+1}|} < TOL$$

Hybrid

$$\frac{|x_{i+1} - x_i|}{\max(|x_{i+1}|, \theta)}$$

FPI is only locally convergent, and even then linearly

Unlike bisection method, errors grow/shrink at different rates, so FPI may either be slower or faster than it depending on circumstance

Limits of accuracy

Some calculations cannot be made accurate on a computer, to the same degree as others.

Forward and backward error

def. 1.8: given $x_a \approx r$ where r is a root of f

the backward error is $|f(x_a)|$

and the forward error is $|r - x_a|$

Backward error

How much the function f would need to change to nullify the error

Forward error

How much the solution x_a would need to change to nullify the error

Root multiplicity

A root is a multiple root of multiplicity m if $0 = f(r) = f'(r) = \dots = f^{(m-1)}(r)$ but $f^{(m)}(r) \neq 0$

A root is simple if $m = 1$

Wilhinson polynomial

Hard to determine roots due to cancellation of nearly equal large numbers

Sensitivity of root finding

A problem is sensitive if small errors in the input lead to large errors in the output.

$$\text{sensitivity factor} = \Delta r = \frac{\epsilon g(r)}{f'(r)}$$

where r is root for f
and $r + \Delta r$ is root for $f + \epsilon g$

$$\text{error magnification factor} = \frac{\text{relative forward err.}}{\text{relative backward err.}}$$

Newton's method

This method takes into account the differential at the current estimate, and adjusts it by an inverse factor. It is a form of FPI

If the function is slowing down, eg. nearing a root, we hasten the movement.

Therefore, this method is even faster than previous ones

x_0 = initial guess

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Quadratic convergence

def 1.10: a method is quadratically convergent if $M = \lim_{i \to \infty} \frac{e_{i+1}}{e_i^2} < \infty$

Root finding w/o derivatives

Secant method

Converges almost as quickly as Newton's method

Replaces tangent w/ secant or (in variants) an approx. parabola

Original

Replace $f'(x_i)$ w/ $\dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$ in Newton's method

x_0, x_1 init. guesses

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

for $i \in \mathbb{N}^+$

The convergence is superlinear e.g. between linear and quad. convergent.

Regula Falsi

Like bisection, but midpoint is calc. with a secant-like approx

The new point is guaranteed to lie in [a, b] unlike the secant method

Muller's method

Use a parabola by using three prev. guesses

Check for intersections

Inverse quad. interpolation (IQI)

Like Muller's, but parabola is of form $x = p(y)$ (laying down)
so it only intersects at one point.

Three init guesses

Converges faster than og. secant

Brent's method

Combines IQI, secant method and bisection method

Uses backwards error as metric

1. apply IQI
2. bracketing interval should be halved if not, apply secant
3. if it still fails, apply bisection

Exercise solutions

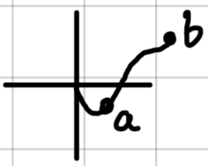
1.1 The bisection method

1a) $x^3 = 9$

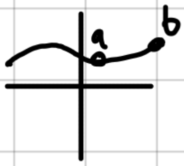
Can be restated as $f(x) = x^3 - 9$

We are looking for $f(c) = \eta = 0$
inbetween a and b
where $a < c < b$ and $f(a)f(b) < 0$

in other words $\text{sign} f(a) \neq \text{sign} f(b)$
because then there wouldn't be a root
inbetween



compared to



Negative

$f(a) < 0 \iff a^3 - 9 < 0$

$\iff a^3 < 9 \iff a < \sqrt[3]{9}$

such a value could be $2 \Rightarrow a = 2$

Positive

$b > \sqrt[3]{9}$ and $b = a + 1 \Rightarrow b = 3$

Result: $[2, 3]$

b) $3x^3 + x^2 = x + 5 \iff 3x^3 + x^2 - x - 5 = 0$

$f(x) = 3x^3 + x^2 - x - 5$

$f(c) = y = 0$

$f(a) < 0 \iff 3a^3 + a^2 - a - 5 < 0$

Such a value could simply be $a = 1$

$f(b) > 0 \iff 3b^3 + b^2 - b - 5 > 0$

Such a value would need to be 2
which does hold, so $b = 2$

Result: $[1, 2]$

c) $\cos^2 x + 6 = x \iff \cos^2 x - x + 6 = 0$

$f(x) = \cos^2 x - x + 6$

$f(b) > 0 \iff \cos^2 b - b + 6 > 0$

$\cos^2 b$ could at maximum be 1
which means the first value that satisfies
is around 7, $b = 7$
and then $a = 6 \Rightarrow [6, 7]$

2a) $x^5 + x = 1 \Rightarrow f(x) = x^5 + x - 1$

$a^5 + a - 1 < 0 \iff a^5 + a < 1$

$\iff a(a^4 + 1) < 1$

$\Rightarrow a = 0$ works and $b = 1$

$[0, 1]$

b) $\sin a - 6a - 5 < 0$

can't be bothered

c) $-//-$

3a) We want the error $|x_c - r|$ to be less than $\frac{1}{8}$

Starting interval $[2, 3]$

The real root $r = \sqrt[3]{9} \approx 2{,}080\ldots$

which means $x_c = 2{,}08 \pm 0{,}125$

1° $c = (2 + 3)/2 = 2{,}5$

I $f(2{,}5) \neq 0$

II $f(a)f(c) < 0$? yes $\Rightarrow b = c$

$[2, 2.5]$

2: $c = (2+2,5)/2 = 2,25$

I $f(2,25) \neq 0$

II yes $\Rightarrow b = c$

$[2, 2.25]$

fin: $x_c = (2+2,25)/2 = 2.125$

This is sufficiently close

b) Start $[1,2]$ $r =$

1. $c = (1+2)/2 = 1,5$

I $f(c) = 5,875 \neq 0$

II yes $\Rightarrow b = 1,5$

2. $c = (1+1,5)/2 = 1,25$

I $f(c) = 1,17 \neq 0$

II yes $\Rightarrow b = 1,25$

fin: $x_c = (1+1,25)/2 = 0,5625$

the real root should be in a range of this

$r \in [0,5625 - \frac{1}{8}, 0,5625]$

$r_{lower} \approx 0,5625 - \frac{1}{8} = 0,4375 \quad f(r)_{lower} = -4,99$

since the sign flipped we can conclude x_c is within the margin of error

(I can't be bothered solving the equation)

c) Start $[6,7]$

1. $c = (6+7)/2 = 6,5$

I $f(c) = 0,487 \neq 0$

II no $\Rightarrow a = 6,5$

2. $c = (6,5+7)/2 = 6,75$

I $f(6,75) = 0,236 \neq 0$

II no $\Rightarrow a = 6,75$

fin: $x_c = (6,75, 7)/2 = 6,875$

meaning $r \in [6,75, 7]$

$f(r_{lower}) = 0,236$

$f(r_{upper}) = -0,01$

sign flipped so x_c is within margin.

4) Shipping since 1 dnf. 2

5a) $f(x) = x^4 - x^3 + 10$

$a^4 - a^3 + 10 < 0 \iff a^3(a-1) < -10$

$a = 1$

$b^3(b-1) > -10$

$b = 2$

$[1, 2]$

b)

$$|x_c - r| < \frac{b-a}{2^{n+1}}$$

$$\Rightarrow 10^{-10} < \frac{2-1}{2^{n+1}} \iff 2^{n+1} > \frac{1}{10^{-10}}$$

$$\iff 2^{n+1} > 10^{10} \iff (n+1)\log 2 > 10 \log 10$$

$$\iff n > \frac{10}{\log 2} - 1 \iff n > \frac{10}{0{,}301} - 1$$

$\iff n > 32{,}21 \Rightarrow$ next int. $n = 33$ steps

6) Since $\frac{1}{x}$ can never be equal to 0, there is no root

The method may still converge toward something, but it cannot be the root. It will however give $x_c \in \mathbb{R}$

1.2 Fixed point iteration

1a) Solving by hand:

$$f(x) = \frac{3}{x} = x \iff 3 = x^2$$

yields $x = \pm\sqrt{3}$

b) $$x^2 - 2x + 2 = x$$

$$\iff x^2 - 3x + 2 = 0$$

$$\iff \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{8}{4} = 0$$

$$\iff \left(x - \frac{3}{2}\right)^2 = \frac{1}{4} \iff x - \frac{3}{2} = \pm\frac{1}{2}$$

$$\iff \begin{cases} x_1 = 2 \\ x_2 = 1 \end{cases}$$

c) $x^2 - 4x + 2 = x$

$$\Leftrightarrow \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{8}{4} = 0$$

$$\Leftrightarrow x - \frac{5}{2} = \pm \frac{\sqrt{17}}{2} \Leftrightarrow \begin{cases} x_1 = \frac{\sqrt{17} + 5}{2} \\ x_2 = \frac{5 - \sqrt{17}}{2} \end{cases}$$

2a) $$\frac{x + 6}{3x - 2} = x \Leftrightarrow 3x^2 - 2x = x + 6$$

$$\Leftrightarrow 3x^2 - 3x - 6 = 0$$

$$\Leftrightarrow x^2 - x - 2 = 0 \Leftrightarrow \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{8}{4} = 0$$

$$\Leftrightarrow x - \frac{1}{2} = \pm \frac{3}{2} \Leftrightarrow \begin{cases} x_1 = 2 \\ x_2 = -1 \end{cases}$$

b) $$\frac{8 + 2x}{2 + x^2} = x \Leftrightarrow 8 + 2x = 2x + x^3$$

$$\Leftrightarrow x^3 = 8 \Leftrightarrow x = 2$$

c) $$x^5 = x \Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 1 \end{cases}$$

3a) $\underline{x = 1}$

$$g(x) = \frac{1+1-6}{6-10} = \frac{-4}{-4} = 1 = x \quad \checkmark$$

$\underline{x = 2}$

$$g(x) = \frac{8+2-6}{12-10} = \frac{4}{2} = 2 = x \quad \checkmark$$

$\underline{x = 3}$

$$g(x) = \frac{27+3-6}{18-10} = \frac{24}{8} = 3 = x \quad \checkmark$$

b) $\underline{x = 1}$

$$g(x) = \frac{6+6-1}{11} = \frac{11}{11} = 1 = x \quad \checkmark$$

$\underline{x = 2}$

$$g(x) = \frac{6+24-8}{11} = \frac{22}{11} = 2 = x \quad \checkmark$$

$\underline{x = 3}$

$$g(x) = \frac{6+52-27}{11} = \frac{33}{11} = 3 = x \quad \checkmark$$

4a) tedious... Skipping

5) a: X

b: $\frac{2\sqrt{3}}{3} + \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$ ✓

c: X

d: $1 + \frac{2}{\sqrt{3}+1} = \sqrt{3} \iff \frac{\sqrt{3} + 3 - \sqrt{3}(\sqrt{3}+1)}{\sqrt{3}+1}$

$= 0 \iff \frac{\sqrt{3} + 3 - 3 - \sqrt{3}}{\sqrt{3}+1} = 0$

$\iff \frac{0}{\sqrt{3}+1} = 0$ ✓

6) a: $\frac{5 + 7\sqrt{5}}{\sqrt{5}+7} = \sqrt{5} \iff 5 + 7\sqrt{5} = \sqrt{5}(\sqrt{5}+7)$

$\iff \sqrt{5}(\sqrt{5}+7) = \sqrt{5}(\sqrt{5}+7)$ ✓

b: $\frac{10}{3\sqrt{5}} + \frac{\sqrt{5}}{3} = \frac{10 + \sqrt{5}\sqrt{5}}{3\sqrt{5}} = \frac{13}{3\sqrt{5}} = \sqrt{5}$

$\iff 13 = 15$ X

c: X

d: $1 + \frac{4}{\sqrt{5}+1} = \sqrt{5}$

$\iff \sqrt{5} + 1 + 4 = \sqrt{5}(\sqrt{5}+1)$

$\iff \sqrt{5} + 5 = 5 + \sqrt{5}$ ✓

7a) I $g(r) = (2-1)^{\frac{1}{3}} = \sqrt[3]{1} = 1 = r$ ✓

II $g'(x) = \frac{1}{3}(x-1)^{-\frac{2}{3}}$

$|g'(r)| < 1 \Rightarrow |\frac{1}{3} \cdot 0| < 1$ ✓

yes!

b) I $g(r) = (1+1)/2 = 1 = r$ ✓

II $g'(x) = \frac{3}{2}x^2$

$|g'(r)| < 1 \Rightarrow |\frac{3}{2}| < 1$ ✗

no!

c) I $g(r) = \sin 0 + 0 = 0 = r$ ✓

II $g'(x) = \cos x \quad |g'(r)| < 1 \Rightarrow |1| < 1$ ✗

no!

8a) $g(x) = \dfrac{2x-1}{x^2} = 2x^{-1} - x^{-2}$

$g'(x) = -2x^{-2} + 2x^{-3}$

$g(r) = \dfrac{2-1}{1} = 1 = r$ ✓

$g'(r) = -2 + 2 = 0 < 1$ ✓

b) $g'(x) = -\sin x$

$g(r) = -1 + \pi + 1 = \pi = r$ ✓

$g'(r) = -0 < 1$ ✓

c) $g'(x) = 2e^{2x}$

$g(r) = 1 - 1 = 0 = r$ ✓

$g'(r) = 2 > 1$ X

not convergent !

9a) $\frac{1}{2}x^2 + \frac{1}{2}x = x$

$\Leftrightarrow x^2 - x = 0 \Leftrightarrow x(x-1) = 0$

$\Leftrightarrow \begin{cases} x_1 = 0 \\ x_2 = 1 \end{cases}$

$g'(x) = x + \frac{1}{2}$

$g'(0) = \frac{1}{2} < 1 \Rightarrow x_1$ local conv.

$g'(1) = \frac{3}{2} > 1 \Rightarrow x_2$ not conv.

b) $x^2 - \frac{5}{4}x + \frac{3}{8} = 0$

$\Leftrightarrow (x - \frac{5}{8})^2 - \frac{25}{64} + \frac{24}{64} = 0$

$\Leftrightarrow x - \frac{5}{8} = \pm\frac{1}{8} \Leftrightarrow \begin{cases} x_1 = \frac{3}{4} \\ x_2 = \frac{1}{2} \end{cases}$

$g'(x) = 2x - \frac{1}{4}$

$g'(\frac{3}{4}) = \frac{5}{4} > 1$ X not conv.

$g'(\frac{1}{2}) = 1 - \frac{1}{4} = \frac{3}{4} < 1$ ✓ conv.

10a) $x^2 - \frac{5}{2}x + \frac{3}{2} = 0$

$<=> (x - \frac{5}{4})^2 - \frac{25}{16} + \frac{24}{16} = 0$

$<=> x - \frac{5}{4} = \pm \frac{1}{4} <=> \begin{cases} x_1 = \frac{3}{2} \\ x_2 = 1 \end{cases}$

$g'(x) = 2x - \frac{3}{2}$

$g'(\frac{3}{2}) = 3 - \frac{3}{2} = \frac{3}{2} > 1$ X not conv.

$g'(1) = 2 - \frac{3}{2} = \frac{1}{2} < 1$ ✓ conv.

b) $x^2 - \frac{1}{2}x - \frac{1}{2} = 0 <=> (x - \frac{1}{4})^2 - \frac{1}{16} - \frac{8}{16} = 0$

$<=> x - \frac{1}{4} = \pm \frac{3}{4} <=> \begin{cases} x_1 = 1 \\ x_2 = -\frac{1}{2} \end{cases}$

$g'(x) = 2x + \frac{1}{2}$

$g(1) = 2 + \frac{1}{2} > 1$ X not conv.

$g(-\frac{1}{2}) = -1 + \frac{1}{2} < 1$ ✓ conv.

11a) 1∘ $x^3 - x + e^x = 0 \iff x^3 + e^x = x$

$\Rightarrow \quad g_1(x) = x^3 + e^x = x$

2∘ $x^3 = x - e^x \iff x = \sqrt[3]{x - e^x}$

$g_2(x) = \sqrt[3]{x - e^x} = x$

3∘ $e^x = x - x^3 \iff x = \ln(x - x^3)$

$g_3(x) = \ln(x - x^3) = x$

b) 1∘ $3x^{-2} + 9x^3 = x^2 \iff x = \sqrt{3x^{-2} + 9x^3}$

2∘ $9x^3 = x^2 - 3x^{-2} \iff x = \sqrt[3]{\frac{x^2 - 3x^{-2}}{9}}$

3∘ $\frac{3}{x^2} = x^2 - 9x^3 \iff x = \sqrt{\frac{3}{x^2 - 9x^3}}$

12a) The step size S for bisection is $\frac{1}{2}$

whereas FPI is $S = |g'(r)|$

$g'(x) = 2x$

so $g'(-0{,}2) = -0{,}4 \Rightarrow S = 0{,}4$

so bisection would be faster

for this root

b)

$x^2 - x - 0{,}24 = 0$

$\Leftrightarrow (x - 0{,}5)^2 - 0{,}25 - 0{,}24 = 0$

$\Leftrightarrow x - 0{,}5 = \pm 0{,}7 \Rightarrow x_2 = 1{,}2$

$g'(1{,}2) = 2{,}4 \Rightarrow$ not conv.

13a) $x^2 + x - 0{,}39 = 0$

$\Leftrightarrow (x + 0{,}5)^2 - 0{,}25 - 0{,}39 = 0$

$\Leftrightarrow x + 0{,}5 = \pm 0{,}8 \Rightarrow \begin{cases} x_1 = 0{,}3 \\ x_2 = 1{,}3 \end{cases}$

b)

$g'(x) = -2x$

$|g'(0{,}3)| = |-0{,}6| < 1$ ✓ conv.

$|g'(1{,}3)| = |-2{,}6| > 1$ X not conv

c) Slower for both

14) $A'(x) = \frac{1}{2} - \frac{1}{x^2}$

$B'(x) = \frac{2}{3} - \frac{2}{3x^2}$

$C'(x) = \frac{3}{4} - \frac{1}{2x^2}$

wrong!

use caution

$A(\sqrt{2}) = \frac{\sqrt{2}}{2} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$ ✓

I $B(\sqrt{2}) = \frac{2\sqrt{2}}{3} - \frac{2}{3\sqrt{2}} = \frac{4-6}{3\sqrt{2}} = -\frac{\sqrt{2}}{3}$ X

$C(\sqrt{2}) = \frac{3\sqrt{2}}{4} - \frac{1}{2\sqrt{2}} = \frac{3\sqrt{2} - \sqrt{2}}{4} = \frac{2\sqrt{2}}{4}$ X

II $A'(-\sqrt{2}) = \frac{1}{2} - \frac{1}{2} = 0 < 7$ ✓

A is

idk which is fastest

i make mistakes here, use caution

15) $A'(x) = \frac{4}{5} - \frac{1}{x^2}$

$B'(x) = \frac{1}{2} - \frac{5}{2x^2}$

$c'(x) =$

$$A(\sqrt{5}) = \frac{4\sqrt{5}}{5} + \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5} \checkmark$$

$$B(-\sqrt{5}) = \frac{\sqrt{5}}{2} + \frac{5}{2\sqrt{5}} = \frac{5+5}{2\sqrt{5}} = \frac{5}{\sqrt{5}} = \sqrt{5} \checkmark$$

$$C(-\sqrt{5}) = \frac{\sqrt{5}+5}{\sqrt{5}+1} = 1 + \frac{4}{\sqrt{5}+1} = \sqrt{5}$$

$$\Leftrightarrow \sqrt{5}+1+4 = 5+\sqrt{5}$$

$$\Leftrightarrow 0 = 0 \checkmark$$

$$A'(\sqrt{5}) = \frac{4}{5} - \frac{1}{5} < 1 \checkmark$$

$$B'(\sqrt{3}) = \frac{1}{2} - \frac{5}{10} = 0 < 1 \checkmark$$

$$C'(\sqrt{3}) =$$

16) Shipping

Shipping rest in section, tedious!

1.3 Limits of accuracy

1a) Backward error $= |f(x_a)| = |-0{,}04|$

Forward error $= \left|\frac{3}{4} - 0{,}74\right| = 0{,}01$

b) $f(x_a) = -1{,}6 \cdot 10^{-3}$

c) $f(x_a) = -6{,}4 \cdot 10^{-5}$

d) $f(x_a) = -0{,}342$

2a) Forward error $= \left|\frac{1}{3} - 0{,}3333\right| = 3.33 \cdot 10^{-5}$

$f(x_a) = -1 \cdot 10^{-4}$

b) $f(x_a) = 1 \cdot 10^{-8}$

c) $f(x_a) = -1 \cdot 10^{-12}$

d) $f(x_a) = -0{,}046$

3a) $f(0) = 1 - 1 = 0$ ✓

$f'(x) = \sin x$

$f'(0) = 0$ ✓

$f''(x) = \cos x$

$f''(0) = 1$ ✗

multiplicity is 2

b) Forward err. $= 0{,}0001$

Backward err. $= f(0{,}0001) = 1{,}521 \cdot 10^{-12}$

does not match solutions manual
not sure why...

4a) $f(0) = 0$ ✓

$f'(x) = 2x \sin x^2 + x^2 \cdot 2x \cdot \cos x^2$

$f'(0) = 0$ ✓

$$f''(x) = 2 \sin x^2 + x^2 \cdot 2x \cdot \cos x^2 + 6x^2 \cdot \cos x^2 - 2x^3 \cdot 2x \cdot \sin x^2$$

c.t.c

i kind of don't care lol

b) Shipping

5) $f(x) = ax + b \Rightarrow f'(x) = a$

Since errors are differences in function values, the backward error grows in terms of $|a|$ times the forward error, e.g. $BE = |a| FE$

6a) $x^n - A = 0 \iff x^n = A$

$\iff x = \sqrt[n]{A} = A^{\frac{1}{n}}$

$f(x) = x^n - A$

$\Rightarrow f'(x) = nx^{n-1}$

$f'(A^{\frac{1}{n}}) = n(A^{\frac{1}{n}})^{n-1} = nA^{\frac{n-1}{n}}$

$= nA^{1-\frac{1}{n}} \neq 0$

The multiplicity is 1 (simple root)

b) With FE as a small error we get our approx. solution

$$x_a = A^{\frac{1}{n}} + FE$$

which means

$$x^n = (A^{\frac{1}{n}} + FE)^n$$

which w/ binom. approx. gives

$$x_a^n \approx A + nA^{\frac{n-1}{n}} FE$$

The backward error is

$$f(x_a) = x_a^n - A$$

$$= nA^{\frac{n-1}{n}} FE$$

where we see that

$$BE = \underbrace{nA^{\frac{n-1}{n}}}_{\text{the multiplier}} FE$$

7a) Shipping due to lack of guidance...

8) $f(x) = x^n - ax^{n-1}$

$g(x) = x^n$

$f_\epsilon(x) = x^n - ax^{n-1} + \epsilon x^n$

Sensitivity formula:

$$\Delta r \approx -\frac{f_\epsilon(r)}{f'(r)}$$

if $\epsilon \ll f'(r)$

so:

$$\Delta r \approx \frac{r^n - ar^{n-1} + \epsilon r^n}{nr^{n-1} - (n-1)r^{n-2}}$$

This is a waste of time...

theres no guidance!

1.4 Newton's method

1a) $x_0 = 0$

$$f(x) = x^3 + x - 2$$

$$f'(x) = 3x^2 + 1$$

Each iteration is:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_1 = 0 - \frac{-2}{1} = 2$$

$$x_2 = 2 - \frac{8}{13} = \frac{18}{13}$$

b) $f'(x) = 4x^3 - 2x + 1$

$$x_1 = 0 - \frac{-1}{1} = 1$$

$$x_2 = 1 - \frac{0}{3} = 1$$

converges immideately to 1 !

c) $f'(x) = 2x - 1$

$$x_1 = 0 - \frac{-1}{-1} = -1$$

$$x_2 = -1 - \frac{1}{-3} = -1 + \frac{1}{3} = -\frac{2}{3}$$

2a) $x_0 = 1$

$$f'(x) = 3x^2 + 2x$$

$$x_1 = 1 - \frac{1}{5} = \frac{4}{5}$$

$$x_2 = \frac{4}{5} - \frac{\left(\frac{64}{125} + \frac{16}{25} - \frac{125}{125}\right)}{\frac{48}{25} + \frac{8}{5}}$$

$$= \frac{4}{5} - \frac{-\frac{1}{125}}{\frac{88}{25}}$$

$$= \frac{4}{5} + \frac{25}{125 \cdot 88} = \frac{4}{5} + \frac{1}{25 \cdot 88}$$

$$= \frac{4 \cdot 5 \cdot 88 + 1}{2200} = \frac{1761}{2200}$$

b) $f'(x) = 2x - \frac{1}{(x+1)^2} - 3$

$$x_1 = 1 - \frac{1 + \frac{1}{2} - 3}{2 - \frac{1}{4} - 3} = 1 - \frac{\frac{-3}{2}}{-\frac{5}{4}}$$

$$= 1 - \frac{12}{10} = -\frac{1}{5}$$

$x_2 = -\frac{1}{5}$ no ! tedious :(

c) $f'(x) = 5$

$$x_1 = 1 - \frac{-5}{5} = 0$$

$$x_2 = 0 - \frac{-10}{5} = 2$$

3) Newton's method is quadratically convergent if the function is twice differentiable

$f(r) = 0$ and $f'(r) \neq 0$ are requirements too

The error at step $i+1$ can be described as $e_{i+1} \approx M e_i^2$

where $M = \left| \frac{f''(r)}{2f'(r)} \right|$

a)

$f'(x) = 5x^4 - 8x^3 + 4x - 1$

$f''(x) = 20x^3 - 24x^2 + 4$

for $r = -1$ this gives

$M = \left| \frac{-20 - 24 + 4}{2(5 + 8 - 4 - 1)} \right| = \left| \frac{-40}{16} \right|$

$= \frac{5}{2}$

which gives $e_{i+1} \approx \frac{5}{2} e_i^2$

(linearly convergent)

Skipping rest, tedious

4) Skip

5) $f'(x) = 32x^3 - 36x^2 + 12x - 1$

$f''(x) = 96x^2 - 72x + 12$

$f(0) = 0$ ✓

$f'(0) = -1 \neq 0$ ✓

Quadratic convergence, so Newton's is faster

$f(\frac{1}{2}) = 0$ ✓

$f'(\frac{1}{2}) = 4 - 9 + 6 - 1 = 0$ ✗

linear convergence, bisection is faster

6)

7) $f'(x) = 4x^3 - 21x^2 + 36x - 20$

$f''(x) = 12x^2 - 42x + 36$

$f'(2) = 32 - 84 + 72 - 20$

$= 104 - 104 = 0$

linear convergence ?

$$\lim_{i \to \infty} \frac{e_{i+1}}{e_i} = S$$

times differentiable

where $S = \frac{(m-1)}{m}$

$m = 4$, so $S = \frac{3}{4}$

8) $f'(x) = a \quad f''(x) = 0$

so we have linear convergence

the root is $x = -\frac{b}{a}$

Each iteration is

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{1}{a} \cdot (ax_i + b)$$

$$= x_i - x_i - \frac{b}{a} = -\frac{b}{a}$$

meaning that no matter initial guess x_i, it cancels out and we always get $-\frac{b}{a}$

9) $f'(x) = 2x$

$$x_{i+1} = x_i - \frac{x_i^2 - A}{2x_i}$$

$$= \frac{2x_i \cdot x_i - x_i^2 + A}{2x_i} = \frac{x_i^2 + A}{2x_i}$$

$$= \frac{x_i + \frac{A}{x_i}}{2}$$ which is equal if $A = 2$

10) $f'(x) = 3x^2$

$$x_{i+1} = x_i - \frac{x_i^3 - A}{3x_i^2}$$

$$= \frac{3x_i^3 - x_i^3 + A}{3x_i^2} = \frac{2x_i + \frac{A}{x_i^2}}{3}$$

11) $x^n = A \iff f(x) = x^n - A = 0$

$f'(x) = nx^{n-1} \quad f''(x) = n(n-1)x^{n-2}$

$r = \sqrt[n]{A} = A^{\frac{1}{n}}$

$f'(r) = n(A^{\frac{1}{n}})^{n-1} = nA^{\frac{n-1}{n}} \neq 0$

Which means quadratic convergence

12) $f'(x) = -x^{-2}$

$$x_{i+1} = x_i - \frac{\frac{1}{x_i}}{-\frac{1}{x_i^2}} = x_i + x_i = 2x_i$$

x doubles each time so $x_{50} = 2^{49} x_0$

$= 2^{49}$

13a) $e_4 = x_4 - r = x_4 - 2$

$\Leftrightarrow x_4 - 2 = 10^{-6} \Leftrightarrow x_4 = 2 + 10^{-6}$

$f'(x) = 3x^2 - 4 \Rightarrow x_{i+1} = x_i - \frac{x_i^3 - 4x_i}{3x_i^2 - 4}$

$$\Rightarrow x_5 = 2 + 10^{-6} - \frac{8{,}000006 \cdot 10^{-6}}{8{,}000012}$$

$$= 2 + 10^{-6} - 0{,}99999925 \cdot 10^{-6}$$

$$\approx 2 + 7{,}5 \cdot 10^{-7} \cdot 10^{-6} = 2 + 7{,}5 \cdot 10^{-13}$$

$$\Rightarrow e_5 = 7{,}5 \cdot 10^{-13} = 0{,}75 \cdot 10^{-12}$$

b) Shipping

14) Shipping...

1.5 Root-finding without derivatives

1a) $$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

$x_0 = 1 \quad x_1 = 2$

$f(x) = -x^3 + 2x + 2$

$$x_2 = 2 - \frac{-2(2-1)}{-2+1-2-2} = 2 - \frac{-2}{-5}$$

$$= 2 - \frac{2}{5} = \frac{8}{5}$$

x_3 = can't be bothered

b,c) shipping, tedious

2a) $$c = \frac{bf(a) - af(b)}{f(a) - f(b)}$$

1: $a = 1 \quad b = 2$

$$c = \frac{2 \cdot 3 + 2}{3 + 2} = \frac{8}{5}$$

$f(c) = 1.104$

$f(a)f(c) < 0$? no

$\Rightarrow a = c$

2: $a = \frac{8}{5} \quad b = 2$

$$c = \frac{2 \cdot 1{,}104 + \frac{8}{5} \cdot 2}{1{,}104 + 2} \approx 1{,}7423$$

$f(c) = 0{,}196$

$\Rightarrow a = c$

$a = 1{,}7423 \quad b = 2$

b,c) shipping

3) you know what? no...

I'd rather watch paint dry lol

4) $f(9) = 8$

$f(5) = 15$

$f(x) = 10$

Since water temp. increases toward surface, 10 should lie between 9 and 5 meters

thus we have starting interval

$[5, 9]$

$$1: \quad c = \frac{9 \cdot 15 - 5 \cdot 8}{15 - 8} = \frac{135 - 40}{7} = \frac{95}{7}$$

$\approx 13{,}6$ m

further than that I cannot proceed w/o f

5) Skip

6) Skip

7a) A: bisection is always $S = \frac{1}{2}$

B: lies between linear and quad

C: $g'(x) = \frac{1}{2} - \frac{3}{x^4} \approx \frac{1}{2}$

D: $g'(x) = \frac{1}{3} - \frac{1}{x^4} \approx \frac{1}{3}$

BDAC

b) Newton's