Trabalho 1 de Cálculo 2

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Ex 1: $\displaystyle\int \log(x)\ dx$

$f' = 1, \quad f = x, \quad g = \log(x), \quad g' = \dfrac{1}{x}$

$$\begin{aligned}
\int \log(x)dx &= \int 1 \cdot \log(x)dx \\
&= x \cdot \log x - \int x \cdot \frac{1}{x}dx \\
&= x \cdot \log x - x \\
&= x(\log x - 1).
\end{aligned}$$

Ex 2: $\displaystyle\int x \cdot \log(x)\ dx$

$f' = x, \quad f = \dfrac{x^2}{2}, \quad g = \log(x), \quad g' = \dfrac{1}{x}$

$$\begin{aligned}
\int x \cdot \log(x)\ dx &= \frac{x^2}{2} \cdot \log x - \int \frac{x^2}{2} \cdot \frac{1}{x}\ dx \\
&= \frac{x^2}{2} \cdot \log x - \frac{1}{2} \cdot \int x\ dx \\
&= \frac{x^2}{2} \cdot \log x - \frac{1}{2} \cdot \frac{x^2}{2} \\
&= \frac{x^2}{2} \cdot \left(\log x - \frac{1}{2}\right).
\end{aligned}$$

Ex 3: $\displaystyle\int x^n \cdot \log(x)\ dx$

$f' = x^n, \qquad f = \dfrac{x^{n+1}}{n+1}, \qquad g = \log x, \qquad g' = \dfrac{1}{x}$

$$\begin{aligned}
\int x^n \cdot \log(x)\ dx &= \frac{x^{n+1}}{n+1} \cdot \log x - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x}\ dx \\
&= \frac{x^{n+1}}{n+1} \cdot \log x - \int \frac{x^n}{n+1}\ dx \\
&= \frac{x^{n+1}}{n+1} \cdot \log x - \frac{1}{n+1} \cdot \int x^n\ dx \\
&= \frac{x^{n+1}}{n+1} \cdot \log x - \frac{1}{n+1} \cdot \frac{x^{n+1}}{n+1} \\
&= \frac{x^{n+1}}{n+1} \cdot \left(\log x - \frac{1}{n+1} \right).
\end{aligned}$$

Ex 4: $\displaystyle\int x \cdot \log(1+x)\ dx$

$f' = x, \qquad f = \dfrac{x^2}{2}, \qquad g = \log(1+x), \qquad g' = \dfrac{1}{x+1}$

$$\begin{aligned}
\int x \cdot \log(1+x)\ dx &= \frac{x^2}{2} \cdot \log(1+x) - \int \frac{x^2}{2} \cdot \frac{1}{x+1}\ dx \\
&= \frac{x^2}{2} \cdot \log(1+x) - \frac{1}{2} \cdot \int \frac{x^2}{1+x}\ dx \\
&= \frac{x^2}{2} \cdot \log(1+x) - \frac{1}{2} \cdot \left[\frac{(x-1)^2}{2} + \log(1+x) \right] \\
&= \frac{x^2}{2} \cdot \log(1+x) - \frac{(x-1)^2}{4} - \frac{1}{2} \log(1+x) \\
&= \frac{x^2-1}{2} \cdot \log(1+x) - \frac{(x-1)^2}{4}.
\end{aligned}$$

obs: $\displaystyle\int \frac{x^2}{x-1}$: Referência usada pelo Ex 6.

Ex 5: $\displaystyle\int x \cdot \log(1-x)\ dx$

$f' = x, \qquad f = \dfrac{x^2}{2}, \qquad g = \log(1-x), \qquad g' = -\dfrac{1}{1-x}$

$$\begin{aligned}
\int x \cdot \log(1-x)\ dx &= \frac{x^2}{2} \cdot \log(1-x) - \int \frac{x^2}{2} \cdot \left(-\frac{1}{x-1}\right)\ dx \\
&= \frac{x^2}{2} \cdot \log(1-x) + \frac{1}{2} \cdot \int \frac{x^2}{1-x}\ dx \\
&= \frac{x^2}{2} \cdot \log(1-x) + \frac{1}{2} \cdot \left[-\frac{(x+1)^2}{2} - \log(1-x)\right] \\
&= \frac{x^2}{2} \cdot \log(1-x) - \frac{(x+1)^2}{4} - \frac{1}{2}\log(1-x) \\
&= \frac{x^2-1}{2} \cdot \log(1-x) - \frac{(1+x)^2}{4}.
\end{aligned}$$

obs: $\displaystyle\int \frac{x^2}{x-1}$: Referência usada pelo Ex 7.

Ex 6: $\displaystyle\int \frac{x^2}{1+x}\ dx$

$u = 1+x, \qquad du = dx, \qquad x^2 = (u-1)^2,$

$$\begin{aligned}
\int \frac{x^2}{1+x}\ dx &= \int \frac{(u-1)^2}{u}\ du \\
&= \int \frac{u^2-2u+1}{u}\ du \\
&= \int u + \frac{1}{u} - 2\ du \\
&= \int u\ du + \int \frac{1}{u}\ du - 2 \cdot \int 1\ du \\
&= \frac{u^2}{2} + \log u - 2 \cdot u
\end{aligned}$$

$$= \frac{(1+x)^2}{2} + \log(1+x) - 2\cdot(1+x)$$

$$= \frac{1+2x+x^2}{2} - 2 - 2x + \log(1+x)$$

$$= \log(1+x) + \frac{x^2}{2} - x + C$$

$$= \log(1+x) + \frac{(x-1)^2}{2}.$$

Ex 7: $\displaystyle\int \frac{x^2}{1-x}\ dx$

$u = 1-x, \qquad du = -dx, \qquad x^2 = (1-u)^2,$

$$\int \frac{x^2}{1-x}\ dx = -\int \frac{(1-u)^2}{u}\ du$$

$$= -\int \frac{u^2-2u+1}{u}\ du$$

$$= -\left(\int u\ du - 2\cdot\int 1\ du + \int \frac{1}{u}\ du\right)$$

$$= -\frac{u^2}{2} - \log u + 2\cdot u$$

$$= -\log(1-x) - \frac{(1-x)^2}{2} + 2\cdot(1-x)$$

$$= -\log(1-x) - \left(\frac{x^2-2x+1}{2}\right) + (2-2x)$$

$$= -\log(1-x) - \frac{x^2}{2} + x - \frac{1}{2} + 2 - 2x$$

$$= -\log(1-x) - \frac{x^2}{2} - x + C$$

$$= -\log(1-x) - \frac{(x-1)^2}{2}.$$

Ex 8: $\displaystyle\int \log(1+x)\ dx$

$u = 1+x, \qquad du = dx, \qquad f = u, \qquad f' = 1, \qquad g = \log(u), \qquad g' = \dfrac{1}{u}$

$$\begin{aligned}
\int \log(1+x)\ dx &= \int \log(u)\ du \\
&= \int 1 \cdot \log u\ du \\
&= u \cdot \log u - \int u \cdot \frac{1}{u}\ du \\
&= u \cdot \log u - \int 1\ du \\
&= u \cdot \log u - u \\
&= (1+x) \cdot \log(1+x) - (1+x) \\
&= (1+x) \cdot \log(1+x) - x.
\end{aligned}$$

Ex 9: $\displaystyle\int \log(1-x)\ dx$

$u = 1-x, \qquad du = -dx, \qquad f = u, \qquad f' = 1, \qquad g = \log(u), \qquad g' = \dfrac{1}{u}$

$$\begin{aligned}
\int \log(1-x)\ dx &= -\int \log(u)\ du \\
&= -\int 1 \cdot \log u\ du \\
&= -\left(u \cdot \log u - \int u \cdot \frac{1}{u}\ du\right) \\
&= -\left(u \cdot \log u - \int 1\ du\right) \\
&= -u \cdot \log u + u \\
&= -(1+x) \cdot \log(1-x) + (1-x) \\
&= -(1+x) \cdot \log(1-x) - x.
\end{aligned}$$

Ex 10: $\displaystyle\int \log(1+x^2)\ dx$

$f = x, \qquad f' = 1, \qquad g = \log(1+x^2), \qquad g' = \dfrac{1}{1+x^2}\cdot 2x$

$$\begin{aligned}
\int \log(1+x^2)\ dx &= x\cdot\log(1+x^2) - \int x\cdot\frac{2x}{1+x^2}\ dx \\
&= x\cdot\log(1+x^2) - \int \frac{2x^2}{1+x^2}\ dx \\
&= x\cdot\log(1+x^2) - 2\cdot\int \frac{x^2}{1+x^2}\ dx \\
&= x\cdot\log(1+x^2) - 2\cdot\int \frac{x^2}{1+x^2} + \frac{1+x^2}{1+x^2} - \frac{1+x^2}{1+x^2}\ dx \\
&= x\cdot\log(1+x^2) - 2\cdot\left[\int \frac{1+x^2}{1+x^2}\ dx - \int \frac{1}{1+x^2}\ dx\right] \\
&= x\cdot\log(1+x^2) - 2\cdot(x - \operatorname{arc\,tg} x).
\end{aligned}$$

Ex 11: $\displaystyle\int \operatorname{arc\,tg}(x)\ dx$

$f = x, \qquad f' = 1, \qquad g = \operatorname{arc\,tg} x, \qquad g' = \dfrac{1}{1+x^2}$

$$\begin{aligned}
\int \operatorname{arc\,tg}(x)\ dx &= x\cdot\operatorname{arc\,tg} x - \int \frac{x}{x^2+1}\ dx \\
&= x\cdot\operatorname{arc\,tg} x - \int \frac{x}{x^2+1}\ dx
\end{aligned}$$

$u = x^2+1, \qquad du = 2x\ dx$

$$\begin{aligned}
&= x\cdot\operatorname{arc\,tg} x - \int \frac{x}{u}\cdot\frac{1}{2x}\ du \\
&= x\cdot\operatorname{arc\,tg} x - \frac{1}{2}\cdot\int \frac{1}{u}\ du \\
&= x\cdot\operatorname{arc\,tg} x - \frac{1}{2}\cdot\log u \\
&= x\cdot\operatorname{arc\,tg} x - \frac{1}{2}\cdot\log(x^2+1).
\end{aligned}$$

Ex 12: $2\displaystyle\int x \operatorname{arc\,tg}(x)\ dx$

$f = \dfrac{x^2}{2}, \qquad f' = x, \qquad g = \operatorname{arc\,tg} x, \qquad g' = \dfrac{1}{x^2+1}$

$$\int x \operatorname{arc\,tg}(x)\ dx = 2 \cdot \left[\frac{x^2 \operatorname{arc\,tg} x}{2} - \int \frac{x^2}{2\cdot(x^2+1)}\ dx\right]$$

$$= x^2 \operatorname{arc\,tg} x - \int \frac{x^2}{x^2+1}\ dx$$

$$= x^2 \operatorname{arc\,tg} x - \int \frac{x^2+1-1}{x^2+1}\ dx$$

$$= x^2 \operatorname{arc\,tg} x - \int 1 - \frac{1}{x^2+1}\ dx$$

$$= x^2 \operatorname{arc\,tg} x - \int 1 + \int \frac{1}{x^2+1}\ dx$$

$$= x^2 \operatorname{arc\,tg} x - x + \operatorname{arc\,tg} x.$$

Ex 13: $\displaystyle\int \sqrt{\operatorname{tg}(x)}\ dx$

Resolução 1:

$u = \sqrt{\operatorname{tg} x}, \quad u^2 = \operatorname{tg} x, \qquad 2\ du = \dfrac{1}{\cos^2 x}\ dx = (1+\operatorname{tg}^2 x) = (1+u^4), \qquad dx = \dfrac{2u}{1+u^4}\ du$

$$\int \sqrt{\operatorname{tg}(x)}\ dx = \int u \cdot \frac{2u}{1+u^4\ du}$$

$$= \int \frac{2u^2}{1+u^4}$$

obs*: $1+u^4 = (u^2+i)(u^2-i)$

$\dfrac{2u^2}{1+u^4} = \dfrac{1}{(u^2+i)} + \dfrac{1}{(u^2-i)}$

$u^2+i = (u+i\sqrt{i})(u-i\sqrt{i})\ u^2-i = (u+\sqrt{i})(u-\sqrt{i})$

$$\int \sqrt{\operatorname{tg}(x)}\ dx = \int \frac{1}{u^2+i} + \int \frac{1}{u^2-i}\ du$$

$$= \int \frac{1}{u^2+i}\ du + \int \frac{1}{u^2-i}\ du$$

$$\int \frac{1}{u^2+i}\ du = \int \frac{1}{(u+i\sqrt{i})(u-i\sqrt{i})}\ du$$

$$= \int \frac{1}{2i\sqrt{i}} \left(\frac{1}{u-i\sqrt{i}} - \frac{1}{u-i\sqrt{i}} \right)\ du$$

$$= \frac{1}{2i\sqrt{i}} \int \left(\frac{1}{u-i\sqrt{i}} - \frac{1}{u-i\sqrt{i}} \right)\ du$$

$$= \frac{1}{2i\sqrt{i}} \left(\int \frac{1}{u-i\sqrt{i}}\ du - \int \frac{1}{u-i\sqrt{i}}\ du \right)$$

$$= \frac{1}{2i\sqrt{i}} \left[\log(u-i\sqrt{i} - \log(u+i\sqrt{i}) \right]$$

$$= \frac{1}{2i\sqrt{i}} \cdot \log \frac{u-i\sqrt{i}}{u+i\sqrt{i}}$$

$$\int \frac{1}{u^2-i}\ du = \int \frac{1}{(u+\sqrt{i})(u-\sqrt{i})}\ du$$

$$= \int \frac{1}{2\sqrt{i}} \left(\frac{1}{u-\sqrt{i}} - \frac{1}{u+i\sqrt{i}} \right)\ du$$

$$= \frac{1}{2\sqrt{i}} \left(\int \frac{1}{u-\sqrt{i}}\ du - \int \frac{1}{u+i\sqrt{i}}\ du \right)$$

$$= \frac{1}{2\sqrt{i}} \left[\log(u-\sqrt{i} - \log(u+\sqrt{i}) \right]$$

$$= \frac{1}{2\sqrt{i}} \cdot \log \frac{u-\sqrt{i}}{u+\sqrt{i}}$$

$$\int \frac{1}{u^2+i}\ du + \int \frac{1}{u^2-i}\ du = \left(\frac{1}{2i\sqrt{i}} \cdot \log \frac{u-i\sqrt{i}}{u+i\sqrt{i}} \right) + \left(\frac{1}{2\sqrt{i}} \cdot \log \frac{u-\sqrt{i}}{u+\sqrt{i}} \right)$$

$$= \frac{1}{2i\sqrt{i}} \cdot \log \frac{\sqrt{\operatorname{tg} x}-i\sqrt{i}}{\sqrt{\operatorname{tg} x}+i\sqrt{i}} + \frac{1}{2\sqrt{i}} \cdot \log \frac{\sqrt{\operatorname{tg} x}-\sqrt{i}}{\sqrt{\operatorname{tg} x}+\sqrt{i}}.$$

Resolução 2:

$u = \sqrt{\operatorname{tg} x}, \quad u^2 = \operatorname{tg} x, \quad dx = \dfrac{2u}{1+u^4}\ du$

$$\int \sqrt{\operatorname{tg}(x)}\ dx = \int u \cdot \frac{2u}{1+u^4\ du}$$

$$= \int \frac{2u^2}{1+u^4}$$

$$= 2\int \frac{u^2}{1+u^4}$$

$$obs\ 1: 1+u^4 = (u^2-\sqrt{2}u+1)(u^2+\sqrt{2}u+1)$$

$$obs\ 2: \frac{1}{2\sqrt{2}}\left(\frac{u}{u^2-\sqrt{2}u+1} - \frac{u}{u^2+\sqrt{2}u+1}\right)$$

$$obs\ 3: \int \frac{u}{u^2-\sqrt{2}u+1}\ du = \frac{1}{2}\int \frac{2u-\sqrt{2}+\sqrt{2}}{u^2-\sqrt{2}u+1}du = \frac{1}{2}\left[\log(u^2-\sqrt{2}u+1)+\sqrt{2}\int \frac{1}{u^2-\sqrt{2}u+1}\ du\right]$$

$$\int \sqrt{\operatorname{tg}(x)}\ dx = \frac{2}{2\sqrt{2}}\int\left(\frac{u}{u^2-\sqrt{2}u+1} - \frac{u}{u^2+\sqrt{2}u+1}\right)\ du$$

$$= \frac{2}{2\sqrt{2}}\left[\frac{1}{2}\log(u^2-\sqrt{2}u+1)+\operatorname{arc\,tg}(\sqrt{2}u-1)\right]$$

$$= \frac{2}{2\sqrt{2}}\left[\frac{1}{2}\log(\sqrt{\operatorname{tg}x}^2-\sqrt{2}\cdot\sqrt{\operatorname{tg}x}+1)+\operatorname{arc\,tg}(\sqrt{2}\cdot\sqrt{\operatorname{tg}x}-1)\right]$$

$$= \frac{2}{2\sqrt{2}}\left[\frac{1}{2}\log(\operatorname{tg}x-\sqrt{2\operatorname{tg}x}+1)+\operatorname{arc\,tg}(\sqrt{2\operatorname{tg}x}-1)\right].$$

Resolução 3:
$u = \sqrt{\operatorname{tg}x}, \quad \frac{du}{dx} = \frac{1}{2\sqrt{\operatorname{tg}x}\cos^2 x}$

$$\int \sqrt{\operatorname{tg}(x)}\ dx = \int \frac{2u^2}{u^4+1}\ du$$

$u^2 = iy^2, \quad du = \sqrt{i}\ dy, \quad u^4 = -y^4$

$$= \int \frac{2iy^2}{-y^4+1}\cdot\sqrt{i}\ dy$$

$$= 2i\sqrt{i}\int \frac{y^2}{1-y^4}\ dy$$

$$= 2i\sqrt{i}\int \frac{1}{2}\left(\frac{y^2}{1-y^2}+\frac{y^2}{1+y^2}\right)\ dy$$

$$= i\sqrt{i}\int \left(\frac{y^2}{1-y^2}+\frac{y^2}{1+y^2}\right)\ dy$$

$$= i\sqrt{i}\int \frac{y^2}{1-y^2}\ dy+\int \frac{y^2}{1+y^2}\ dy$$

$$= i\sqrt{i}\left(-y+\frac{1}{2}\log\frac{1+y}{1-y}+y-\operatorname{arc\,tg} y\right)$$

$$= i\sqrt{i}\left(\frac{1}{2}\log\frac{1+y}{1-y}-\operatorname{arc\,tg} y\right)$$

$$y = \frac{\sqrt{i}\sqrt{\operatorname{tg} x}}{i}$$

$$= i\sqrt{i}\left(\frac{1}{2}\log\frac{1+\dfrac{\sqrt{i}\sqrt{\operatorname{tg} x}}{i}}{1-\dfrac{\sqrt{i}\sqrt{\operatorname{tg} x}}{i}}-\operatorname{arc\,tg}\frac{\sqrt{i}\sqrt{\operatorname{tg} x}}{i}\right).$$