On the Onset of Chaos in the Pythagorean Three-body Problem

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1 Setup of the Problem

The three-body problem (3BP) is the problem of solving for the motion of three gravitationally interacting point masses $\{m_i\}$ given their initial positions $\{\boldsymbol{r}_i\}$ and velocities $\{\boldsymbol{v}_i\}$. The governing equation may be expressed as:

$$m_i \frac{d^2 \boldsymbol{r}_i}{dt^2} = \frac{\partial U}{\partial \boldsymbol{r}_i} \tag{1.1}$$

where U is the force function in Newtonian gravity:

$$U = \sum_{i \neq j} \frac{G m_i m_j}{||\boldsymbol{r}_j - \boldsymbol{r}_i||}, \quad i = 1, 2, 3. \tag{1.2}$$

The Pythagorean configuration consists of masses $\{3, 4, 5\}$ initially at rest at the vertices of a right triangle with sides $\{3, 4, 5\}$ (Fig. 1.1). For the main project, we considered the equal-mass case in which all three $m_i = 0.5$ and set the gravitational constant $G = 1$, following Dejonghe and Hut [1].

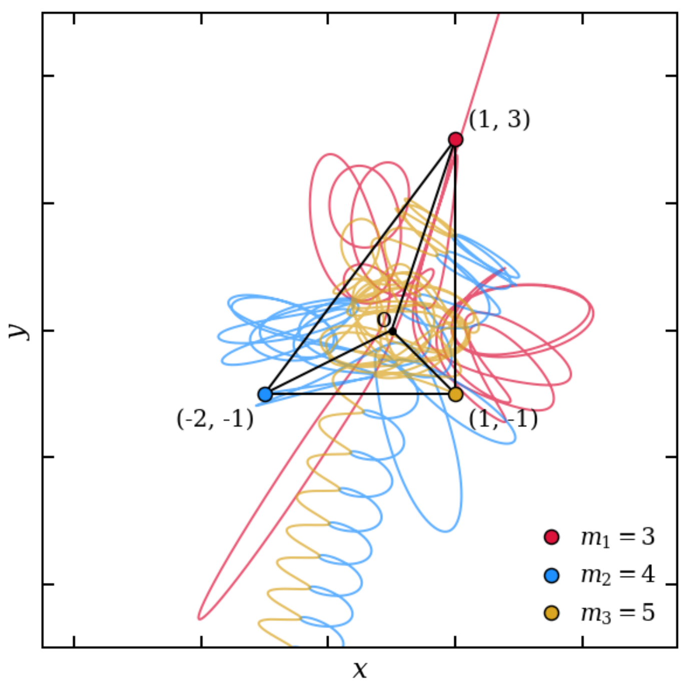

Figure 1.1: Trajectories of the Pythagorean problem in position space. After a series of close encounters and long excursions, the lightest body (red) is ejected while the other two form a tight binary.

1.1 Exponential Divergence

Two three-body trajectories offset by an initial phase space separation Δ_0 diverge from each other at an exponential rate (Fig. 5.3). Since the rate of divergence is dominated by the maximum Lyapunov exponent (MLE), the direction of the initial offset is not important. The total phase space separation between the two trajectories at a given time t is given by:

$$\Delta(t) = \sqrt{\sum_{i=1,2,3} \left(||\boldsymbol{r}_i - \boldsymbol{r}'_i||^2 + ||\boldsymbol{v}_i - \boldsymbol{v}'_i||^2\right)} \tag{1.3}$$

where $\{\boldsymbol{r}'_i\}$ and $\{\boldsymbol{v}'_i\}$ are the position and velocity vectors of the three bodies in the perturbed solution, respectively. Using these quantities, we may also define the amplification factor which represents the magnitude by which the initial separation is amplified over time:

$$\mathrm{AF} = \frac{\Delta(t)}{\Delta_0} \approx e^{\lambda t} \tag{1.4}$$

where the approximate relationship between the amplification factor and the maximum Lyapunov exponent is also shown.

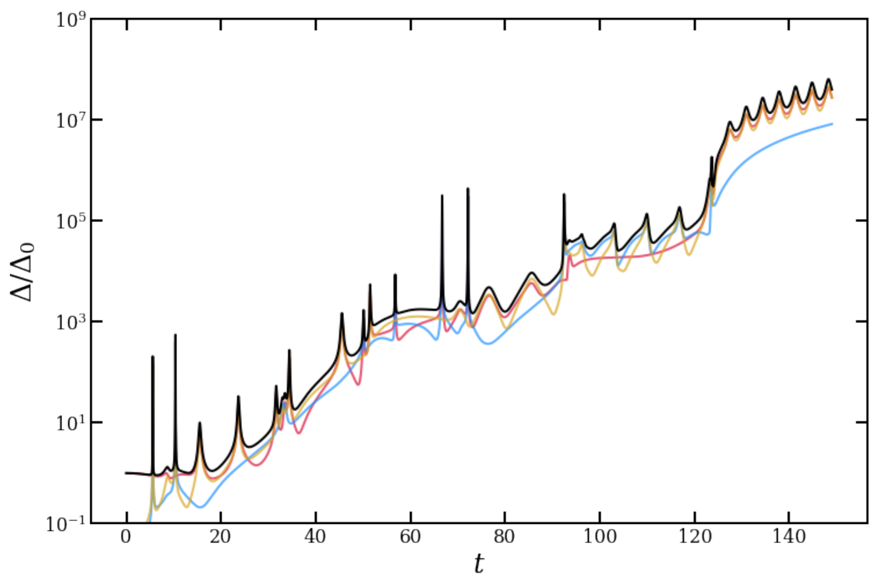

Figure 1.2: The divergence between two solutions to the equal-mass Pythagorean problem with an initial separation $\Delta_0 = 1 \times 10^{-10}$. The black line corresponds to the total phase space divergence, whereas the coloured ones (matching Fig. 1) represent the divergence between each body and its counterpart in the perturbed solution.

1.2 Integration Scheme

All solutions were obtained using the time-transformed leapfrog (TTL) algorithm introduced by Mikkola and Tanikawa [2]. In the N-body formulation, the update equations are:

$$\text{DRIFT} \qquad \boldsymbol{r}_{i,1/2} = \boldsymbol{r}_{i,0} + \frac{(h/2)(\boldsymbol{p}_{i,0}/m_i)}{T_0 + P_t} \tag{1.5}$$

$$t_{1/2} = t_0 + \frac{h/2}{T_0 + P_t} \tag{1.6}$$

$$\text{KICK} \qquad \boldsymbol{p}_{i,1} = \boldsymbol{p}_{i,0} + \frac{h}{U_{i,1/2}} \frac{\partial U_{i,1/2}}{\partial \boldsymbol{r}_{i,1/2}} \tag{1.7}$$

$$\text{DRIFT} \qquad \boldsymbol{r}_{i,1} = \boldsymbol{r}_{i,1/2} + \frac{(h/2)(\boldsymbol{p}_{ki1}/m_i)}{T_1 + P_t} \tag{1.8}$$

$$t_1 = t_{1/2} + \frac{h/2}{T_1 + P_t} \tag{1.9}$$

where $T_s = \sum_i \frac{|\boldsymbol{p}_{i,s}|^2}{2m_i}$ and $P_t = -E_0$. The method provides a form of algorithmic regularisation which allows for close encounters to be handled without excplicit regularisation of the problem.

2 Two-Body Treatment

2.1 Jacobi Coordinates

During its lifetime, a three-body system often consists of a temporary binary and an ejected body orbiting the centre of mass (CoM) of that binary. In this situation, the ejected body is said to be undergoing an excursion. The state of the system may then be represented in terms of the Jacobi coordinates $\{\boldsymbol{r}, \boldsymbol{R}\}$ where $\boldsymbol{R}$ is the separation between the two bodies forming the temporary binary and $\boldsymbol{r}$ is the separation between the ejected body and the centre of mass (CoM) of the binary.

When the ratio of the Jacobi coordinates $r/R \gg 1$, the three-body system may effectively be treated as a two-body one. This treatment may further be extended to the entire time evolution of the system by defining the ejected body at a given time as that for which the ratio r/R is maximised.

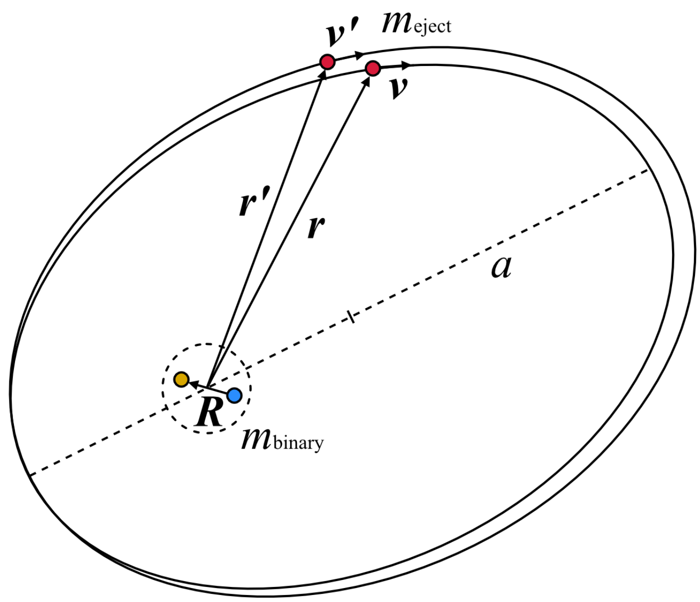

Figure 2.1: The setup for the two-body approximation.

2.2 Conserved Quantities

Two-body motion is characterised by two conserved quantities: the specific orbital energy E_{orb} and the specific relative angular momentum h. These and a number of other quantities that appear multiple times in the following sections are collected below:

Standard Gravitational Parameter
$$\mu = GM = G \sum_{i=1,2,3} m_i \tag{2.1}$$

Reduced Mass
$$m = \frac{m_{\text{eject}} m_{\text{binary}}}{m_{\text{eject}} + m_{\text{binary}}} \tag{2.2}$$

Specific Orbital Energy
$$E_{\text{orb}} = \frac{v^2}{2} - \frac{\mu}{r} \tag{2.3}$$

Specific Relative Angular Momentum
$$\boldsymbol{h} = \boldsymbol{r} \times \boldsymbol{v} \tag{2.4}$$

Eccentricity
$$e = \sqrt{1 + \frac{2E_{\text{orb}} h^2}{\mu^2}} \tag{2.5}$$ [1]

Semi-Major Axis
$$a = \frac{\mu}{2|E_{\text{orb}}|} \tag{2.6}$$

Coefficient of Potential Energy
$$\alpha = \mu m \tag{2.7}$$

Eccentric Anomaly
$$E \in [0, 2\pi] \tag{2.8}$$

Here the eccentric anomaly E is used as the independent variable and has no explicit expression. The initialisation of the Keplerian approximation introduced in the next section relies on translating deviations in the position and velocity coordinates, δr and δv, into deviations in the conserved quantities E_{orb} and h. These are given by:

$$\begin{aligned} \delta E_{\text{orb}} &= \frac{\partial E_{\text{orb}}}{\partial v} \delta v + \frac{\partial E_{\text{orb}}}{\partial r} \delta r \\ &= v \delta v + \frac{\mu}{r^2} \delta r \end{aligned} \tag{2.9}$$

$$\begin{aligned} \delta h &= \frac{\partial h}{\partial v} \delta v + \frac{\partial h}{\partial r} \delta r \\ &= r \sin\theta \delta v + v \sin\theta \delta r \end{aligned} \tag{2.10}$$

where the angle $\theta = \angle(\boldsymbol{r}, \boldsymbol{v})$ is assumed to remain unchanged. This assumption may be relaxed in further work.

2.3 Idea behind the Keplerian Approximation

[1] This expression comes from Bate, Mueller & White, *Fundamentals of Astrodynamics*, 1971.

3 Keplerian Approximation in 1D

A further simplification to the two-body approximation is to consider the motion of the ejected body w.r.t. the binary only along one dimension, i.e. away from and towards the binary. In this case, the system reduces to a radial Kepler problem, a special case of the Kepler problem effectively equivalent to setting $e = 1$.

3.1 Elliptic Case ($E_{\text{orb}} < 0$)

In the elliptic case, the parametric equations describing the motion of the ejected body are:

$$t = \sqrt{\frac{ma^3}{\alpha}}[E - \sin E] = \frac{a}{\sqrt{2|E_{\text{orb}}|}}[E - \sin E] \tag{3.1}$$

$$r = a[1 - \cos E] \tag{3.2}$$

where the variables are as stated in the previous section. The radial speed $\dot{r}$ may be obtained by first differentiating Eq. 3.1 w.r.t. E s.t.

$$\begin{aligned} \frac{dt}{dE} &= \frac{a}{\sqrt{2|E_{\text{orb}}|}}[1 - \cos E] \\ \Rightarrow \frac{dE}{dt} &= \frac{\sqrt{2|E_{\text{orb}}|}}{a}\frac{1}{1 - \cos E}. \end{aligned} \tag{3.3}$$

and then differentiating Eq. 3.2 w.r.t t:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{dE}\frac{dE}{dt} = \sqrt{2|E_{\text{orb}}|}\frac{\sin E}{1 - \cos E}. \tag{3.4}$$

3.2 Hyperbolic Case ($E_{\text{orb}} > 0$)

In the hyperbolic case, the parametric equations are:

$$t = \sqrt{\frac{ma^3}{\alpha}}[\sinh E - E] = \frac{a}{\sqrt{2|E_{\text{orb}}|}}[\sinh E - E] \tag{3.5}$$

$$r = a[\cosh E - 1] \tag{3.6}$$

Similarly to the elliptic case, the radial speed $\dot{r}$ may be obtained by first differentiating Eq.

3.5 w.r.t. E s.t.

$$
\begin{aligned}
\frac{dt}{dE} &= \frac{a}{\sqrt{2|E_{\text{orb}}|}}\left[\cosh E - 1\right] \\
\Rightarrow \frac{dE}{dt} &= \frac{\sqrt{2|E_{\text{orb}}|}}{a}\frac{1}{\cosh E - 1}.
\end{aligned} \tag{3.7}
$$

and then differentiating Eq. 3.6 w.r.t t:

$$
\dot{r} = \frac{dr}{dt} = \frac{dr}{dE}\frac{dE}{dt} = \sqrt{2|E_{\text{orb}}|}\frac{\sinh E}{\cosh E - 1}. \tag{3.8}
$$

4 Keplerian Approximation in 2D

The Keplerian approximation may be extended to two dimensions, which introduces a non-zero angular momentum $\boldsymbol{h}$ and thus eccentricity $e \neq 1$ to the problem.

4.1 Elliptic Case ($E_{\text{orb}} < 0$)

We start with the parametric equations from Landau & Lifshitz (Eq. 15.10, 15.11):

$$t = \sqrt{\frac{ma^3}{\alpha}}[E - e\sin E] = \frac{a}{\sqrt{2|E_{\text{orb}}|}}[E - e\sin E] \tag{4.1}$$

$$r = a[1 - e\cos E] \tag{4.2}$$

$$x = a[\cos E - e] \tag{4.3}$$

$$y = a\sqrt{1-e^2}\sin E \tag{4.4}$$

where the variables are as stated Section 2.2. The corresponding speeds may be obtained by first differentiating Eq. 4.1 w.r.t E s.t.

$$\begin{aligned} \frac{dt}{dE} &= \frac{a}{\sqrt{2|E_{\text{orb}}|}}[1 - e\cos E] \\ \Rightarrow \frac{dE}{dt} &= \frac{\sqrt{2|E_{\text{orb}}|}}{a}\frac{1}{1 - e\cos E} \end{aligned} \tag{4.5}$$

and then differentiating Eq. 4.2, 4.3 and 4.4 w.r.t t:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{dE}\frac{dE}{dt} = \sqrt{2|E_{\text{orb}}|}\frac{e\sin E}{1 - e\cos E} \tag{4.6}$$

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{dE}\frac{dE}{dt} = -\sqrt{2|E_{\text{orb}}|}\frac{\sin E}{1 - e\cos E} \tag{4.7}$$

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dE}\frac{dE}{dt} = \sqrt{2|E_{\text{orb}}|}\sqrt{1-e^2}\frac{\cos E}{1 - e\cos E} \tag{4.8}$$

We may further combine the last two equations to obtain the orbital speed v:

$$v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{2|E_{\text{orb}}|}\sqrt{\frac{1 + e\cos E}{1 - e\cos E}} \tag{4.9}$$

To obtain the growth of the deviation in orbital radius and speed, we differentiate Eq. 4.2

and 4.9 w.r.t the mutually independent parameters E_{orb} and h:

$$\begin{aligned}\delta r &= \frac{\partial r}{\partial E_{\text{orb}}}\delta E_{\text{orb}} + \frac{\partial r}{\partial h}\delta h \\ &= \left\{ \frac{da}{dE_{\text{orb}}}\left[1 - e\cos E\right] - \frac{de}{dE_{\text{orb}}}a\cos E + \frac{dE}{dE_{\text{orb}}}ae\sin E \right\}\delta E_{\text{orb}} \\ &\quad - a\left\{ \frac{de}{dh}\cos E - \frac{dE}{dh}\sin E \right\}\delta h\end{aligned} \tag{4.10}$$

$$\begin{aligned}\delta v &= \frac{\partial v}{\partial E_{\text{orb}}}\delta E_{\text{orb}} + \frac{\partial v}{\partial h}\delta h \\ &= \left\{ -\frac{1}{\sqrt{2|E_{\text{orb}}|}}\sqrt{\frac{1+e\cos E}{1-e\cos E}} + \frac{2\sqrt{2|E_{\text{orb}}|}}{[1-e\cos E]^2}\left[\frac{de}{dE_{\text{orb}}}\cos E - \frac{dE}{dE_{\text{orb}}}e\sin E\right] \right\}\delta E_{\text{orb}} \\ &\quad + \left\{ \frac{2\sqrt{2|E_{\text{orb}}|}}{[1-e\cos E]^2}\left[\frac{de}{dh}\cos E - \frac{dE}{dh}e\sin E\right] \right\}\delta h\end{aligned} \tag{4.11}$$

We may similarly obtain the growth of the deviation along the orthogonal coordinate axes $\{x, y, \dot{x}, \dot{y}\}$ by differentiating Eq. 4.3, 4.4, 4.7 and 4.8 w.r.t E_{orb} and h:

$$\begin{aligned}\delta x &= \frac{\partial x}{\partial E_{\text{orb}}}\delta E_{\text{orb}} + \frac{\partial x}{\partial h}\delta h \\ &= \left\{ \frac{da}{dE_{\text{orb}}}\left[\cos E - e\right] - a\left[\frac{de}{dE_{\text{orb}}} + \frac{dE}{dE_{\text{orb}}}\sin E\right] \right\}\delta E_{\text{orb}} \\ &\quad - a\left\{ \frac{de}{dh} + \frac{dE}{dh}\sin E \right\}\delta h\end{aligned} \tag{4.12}$$

$$\begin{aligned}\delta y &= \frac{\partial y}{\partial E_{\text{orb}}}\delta E_{\text{orb}} + \frac{\partial y}{\partial h}\delta h \\ &= \left\{ \frac{da}{dE_{\text{orb}}}\sqrt{1-e^2}\sin E - a\left[\frac{de}{dE_{\text{orb}}}\frac{e}{\sqrt{1-e^2}}\sin E + \frac{dE}{dE_{\text{orb}}}\sqrt{1-e^2}\cos E\right] \right\}\delta E_{\text{orb}} \\ &\quad - a\left\{ \frac{de}{dh}\frac{e}{\sqrt{1-e^2}}\sin E - \frac{dE}{dh}\sqrt{1-e^2}\cos E \right\}\delta h\end{aligned} \tag{4.13}$$

$$\begin{aligned}\delta \dot{x} &= \frac{\partial \dot{x}}{\partial E_{\text{orb}}}\delta E_{\text{orb}} + \frac{\partial \dot{x}}{\partial h}\delta h \\ &= \left\{ \frac{1}{\sqrt{2|E_{\text{orb}}|}}\frac{\sin E}{1-e\cos E} - \frac{\sqrt{2|E_{\text{orb}}|}}{(1-e\cos E)^2}\left[\frac{de}{dE_{\text{orb}}}\sin E\cos E - \frac{dE}{dE_{\text{orb}}}\left[e - \cos E\right]\right] \right\}\delta E_{\text{orb}} \\ &\quad - \left\{ \frac{\sqrt{2|E_{\text{orb}}|}}{(1-e\cos E)^2}\left[\frac{de}{dh}\sin E\cos E - \frac{dE}{dh}\left[e - \cos E\right]\right] \right\}\delta h\end{aligned} \tag{4.14}$$

$$
\begin{aligned}
\delta\dot{y} &= \frac{\partial \dot{y}}{\partial E_{\text{orb}}}\delta E_{\text{orb}} + \frac{\partial \dot{y}}{\partial h}\delta h \\
&= \left\{ -\left[\frac{\sqrt{1-e^2}}{\sqrt{2|E_{\text{orb}}|}} + \frac{de}{dE_{\text{orb}}}\frac{e\sqrt{2|E_{\text{orb}}|}}{\sqrt{1-e^2}} \right] \frac{\cos E}{1-e\cos E} + \frac{\sqrt{2|E_{\text{orb}}|}\sqrt{1-e^2}}{(1-e\cos E)^2}\left[\frac{de}{dE_{\text{orb}}}\cos^2 E - \frac{dE}{dE_{\text{orb}}}\sin E \right] \right\} \delta E_{\text{orb}} \\
&\quad - \left\{ \frac{de}{dh}\frac{e\sqrt{2|E_{\text{orb}}|}}{\sqrt{1-e^2}}\frac{\cos E}{1-e\cos E} - \frac{\sqrt{2|E_{\text{orb}}|}\sqrt{1-e^2}}{(1-e\cos E)^2}\left[\frac{de}{dh}\cos^2 E - \frac{dE}{dh}\sin E \right] \right\} \delta h.
\end{aligned}
\tag{4.15}
$$

The derivatives w.r.t E_{orb} may be expressed as:

$$
\frac{da}{dE_{\text{orb}}} = \frac{\mu}{2E_{\text{orb}}^2} \tag{4.16}
$$

$$
\frac{de}{dE_{\text{orb}}} = \frac{h^2}{\mu^2}\left(1 + \frac{2E_{\text{orb}}h^2}{\mu^2}\right)^{-\frac{1}{2}} \tag{4.17}
$$

$$
\frac{dE}{dE_{\text{orb}}} = \frac{3}{2E_{\text{orb}}}\frac{E - e\sin E}{1-e\cos E} + \frac{de}{dE_{\text{orb}}}\frac{\sin E}{1-e\cos E} \tag{4.18}
$$

and w.r.t. h as:

$$
\frac{de}{dh} = \frac{2E_{\text{orb}}h}{\mu^2}\left(1 + \frac{2E_{\text{orb}}h^2}{\mu^2}\right)^{-\frac{1}{2}} \tag{4.19}
$$

$$
\frac{dE}{dh} = \frac{de}{dh}\frac{\sin E}{1-e\cos E} \tag{4.20}
$$

where the last expression in both cases is obtained by setting $dt/dE_{\text{orb}} = 0$ and $dt/dh = 0$ and rearranging.

4.2 Hyperbolic Case ($E_{\text{orb}} > 0$)

Again from Landau & Lifshitz (Eq. 15.12):

$$
t = \sqrt{\frac{ma^3}{\alpha}}[e\sinh E - E] = \frac{a}{\sqrt{2|E_{\text{orb}}|}}[e\sinh E - E] \tag{4.21}
$$

$$
r = a[e\cosh E - 1] \tag{4.22}
$$

$$
x = a[e - \cosh E] \tag{4.23}
$$

$$
y = a\sqrt{e^2-1}\sinh E \tag{4.24}
$$

where all variables have the same meaning as in the elliptic case. We can evaluate the corresponding speeds by first differentiating Eq. 4.21 w.r.t E:

$$\begin{aligned} \frac{dt}{dE} &= \frac{a}{\sqrt{2|E_{\text{orb}}|}}[e\cosh E - 1] \\ \Rightarrow \frac{dE}{dt} &= \frac{\sqrt{2|E_{\text{orb}}|}}{a}\frac{1}{e\cosh E - 1} \end{aligned} \tag{4.25}$$

and then differentiating Eq. 4.22, 4.23 and 4.24 w.r.t t:

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{dE}\frac{dE}{dt} = \sqrt{2|E_{\text{orb}}|}\frac{e\sinh E}{e\cosh E - 1} \tag{4.26}$$

$$\dot{x} = \frac{dx}{dt} = \frac{dx}{dE}\frac{dE}{dt} = -\sqrt{2|E_{\text{orb}}|}\frac{\sinh E}{e\cosh E - 1} \tag{4.27}$$

$$\dot{y} = \frac{dy}{dt} = \frac{dy}{dE}\frac{dE}{dt} = \sqrt{2|E_{\text{orb}}|}\sqrt{e^2 - 1}\frac{\cosh E}{e\cosh E - 1} \tag{4.28}$$

We may further combine the last two equations to obtain the orbital speed v:

$$\boxed{v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{2|E_{\text{orb}}|}\sqrt{\frac{e\cosh E + 1}{e\cosh E - 1}}} \tag{4.29}$$

To obtain the growth of the deviation in orbital radius and speed, we differentiate Eq. 4.22 and 4.29 w.r.t the mutually independent parameters E_{orb} and h:

$$\begin{aligned} \delta r &= \frac{\partial r}{\partial E_{\text{orb}}}\delta E_{\text{orb}} + \frac{\partial r}{\partial h}\delta h \\ &= \left\{\frac{da}{dE_{\text{orb}}}[e\cosh E - 1] + a\left[\frac{de}{dE_{\text{orb}}}\cosh E + \frac{dE}{dE_{\text{orb}}}e\sinh E\right]\right\}\delta E_{\text{orb}} \\ &\quad + a\left\{\frac{de}{dh} + \frac{dE}{dh}e\sinh E\right\}\delta h \end{aligned} \tag{4.30}$$

$$\begin{aligned} \delta v &= \frac{\partial v}{\partial E_{\text{orb}}}\delta E_{\text{orb}} + \frac{\partial v}{\partial h}\delta h \\ &= \left\{\frac{1}{\sqrt{2|E_{\text{orb}}|}}\sqrt{\frac{e\cosh E + 1}{e\cosh E - 1}} - \frac{2\sqrt{2|E_{\text{orb}}|}}{[e\cosh E - 1]^2}\left[\frac{de}{dE_{\text{orb}}}\cosh E + \frac{dE}{dE_{\text{orb}}}e\sinh E\right]\right\}\delta E_{\text{orb}} \\ &\quad - \left\{\frac{2\sqrt{2|E_{\text{orb}}|}}{[e\cosh E - 1]^2}\left[\frac{de}{dh}\cosh E + \frac{dE}{dh}e\sinh E\right]\right\}\delta h \end{aligned} \tag{4.31}$$

We may similarly obtain the growth of the deviation along the orthogonal coordinate axes

$\{x, y, \dot{x}, \dot{y}\}$ by differentiating Eq. 4.23, 4.24, 4.27 and 4.28 w.r.t E_{orb} and h:

$$\begin{aligned}\delta x &= \frac{\partial x}{\partial E_{\text{orb}}}\delta E_{\text{orb}} + \frac{\partial x}{\partial h}\delta h \\ &= \left\{\frac{da}{dE_{\text{orb}}}\left[e - \cosh E\right] + a\left[\frac{de}{dE_{\text{orb}}} - \frac{dE}{dE_{\text{orb}}}\sinh E\right]\right\}\delta E_{\text{orb}} \\ &\quad + a\left\{\frac{de}{dh} - \frac{dE}{dh}\sinh E\right\}\delta h\end{aligned} \tag{4.32}$$

$$\begin{aligned}\delta y &= \frac{\partial y}{\partial E_{\text{orb}}}\delta E_{\text{orb}} + \frac{\partial y}{\partial h}\delta h \\ &= \left\{\frac{da}{dE_{\text{orb}}}\sqrt{e^2-1}\sinh E + a\left[\frac{de}{dE_{\text{orb}}}\frac{e}{\sqrt{e^2-1}}\sinh E + \frac{dE}{dE_{\text{orb}}}\sqrt{e^2-1}\cosh E\right]\right\}\delta E_{\text{orb}} \\ &\quad + a\left\{\frac{de}{dh}\frac{e}{\sqrt{e^2-1}}\sinh E + \frac{dE}{dh}\sqrt{e^2-1}\cosh E\right\}\delta h\end{aligned} \tag{4.33}$$

The derivatives w.r.t E_{orb} may be expressed as:

$$\frac{da}{dE_{\text{orb}}} = -\frac{\mu}{2E_{\text{orb}}^2} \tag{4.34}$$

$$\frac{de}{dE_{\text{orb}}} = \frac{h^2}{\mu^2}\left(1 + \frac{2E_{\text{orb}}h^2}{\mu^2}\right)^{-\frac{1}{2}} \tag{4.35}$$

$$\frac{dE}{dE_{\text{orb}}} = \frac{3}{2E_{\text{orb}}}\frac{E - e\sinh E}{1 - e\cosh E} + \frac{de}{dE_{\text{orb}}}\frac{\sinh E}{1 - e\cosh E} \tag{4.36}$$

and w.r.t. h as:

$$\frac{de}{dh} = \frac{2E_{\text{orb}}h}{\mu^2}\left(1 + \frac{2E_{\text{orb}}h^2}{\mu^2}\right)^{-\frac{1}{2}} \tag{4.37}$$

$$\frac{dE}{dh} = \frac{de}{dh}\frac{\sinh E}{1 - e\cosh E}. \tag{4.38}$$

where the last expression in both cases is again obtained by setting $dt/dE_{\text{orb}} = 0$ and $dt/dh = 0$ and rearranging.

5 Divergence Model

5.1 Parameter Evaluation

In the descriptive model, the procedure for obtaining the initial parameter values for an excursion is as follows:

1. Determine which of the three bodies is ejected throughout the system lifetime using r/R. Represent this by the index of the ejected body $i \in \{1, 2, 3\}$.
2. Define exchanges $\{t_{\text{ex}}\}$ as the points in time when the value of i changes.
3. Delete exchanges followed by a short excursion ($\Delta t <?$) from $\{t_{\text{ex}}\}$.
4. Define the approximation start times $\{t_{\text{start}}\}$ and end times $\{t_{\text{end}}\}$ as the points in time corresponding to a local maximum of $\dot{r}_i$ immediately preceding and following each exchange, i.e. when $d\dot{r}_i/dt = 0$ and $d^2\dot{r}_i/dt^2 < 0$.
5. Define the interval of strict two-body validity $[t^{\text{2BP}}_{\text{start}}, t^{\text{2BP}}_{\text{end}}] \subset [t_{\text{start}}, t_{\text{end}}]$ as the time interval during which the ejected body is the only one corresponding to $r/R > 1$.

Each divergence approximation is started at the corresponding time t_{start} and the initialisation parameters $\{r, v, \theta\}$ are evaluated as an average over the interval of strict two-body validity $[t^{\text{2BP}}_{\text{start}}, t^{\text{2BP}}_{\text{end}}]$.

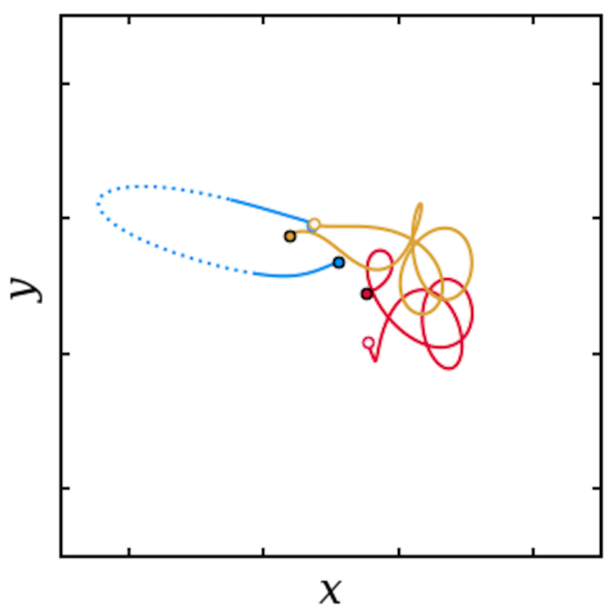

Figure 5.1: Single excursion of the blue body.

5.2 Approximation for a Single Excursion

For an excursion starting at time $t_0 \in \{t_{\text{start}}\}$ and with the initial parameter values $\{r_0, \delta r_0, v_0, \delta v_0, \theta\}$ s.t. $\Delta_0 = \sqrt{(\delta r_0)^2 + (\delta v_0)^2}$, the corresponding values of E_{orb} and h are evaluated using Eq. 2.3 and 2.4 and the deviations in them using Eq. 2.9 and 2.10. The initial value of the eccentric anomaly E_{start} is evaluated by inverting the equation for the orbital radius. The approximation of the divergence for the excursion is then given by:

$$\Delta(E) = \sqrt{\frac{(\delta r(E))^2 + (\delta v(E))^2}{(\delta r(E_{\text{start}}))^2 + (\delta v(E_{\text{start}}))^2}}\Delta_0 \tag{5.1}$$

where δr and δv are both functions of $\{E_{\text{orb}}, \delta E_{\text{orb}}, h, \delta h\}$. The sign of the initial value of E_{orb} determines whether the elliptic or the hyperbolic approximation is used. To relate this divergence to time, Eq. 4.1 and 4.21 are used s.t.

$$t(E) = \begin{cases} t_0 + \frac{a}{\sqrt{2|E_{\text{orb}}|}}[E - e\sin E] - \frac{a}{\sqrt{2|E_{\text{orb}}|}}[E_{\text{start}} - e\sin E_{\text{start}}], & E_{\text{orb}} < 0 \\ t_0 + \frac{a}{\sqrt{2|E_{\text{orb}}|}}[e\sinh E - E] - \frac{a}{\sqrt{2|E_{\text{orb}}|}}[e\sinh E_{\text{start}} - E_{\text{start}}], & E_{\text{orb}} > 0 \end{cases} \tag{5.2}$$

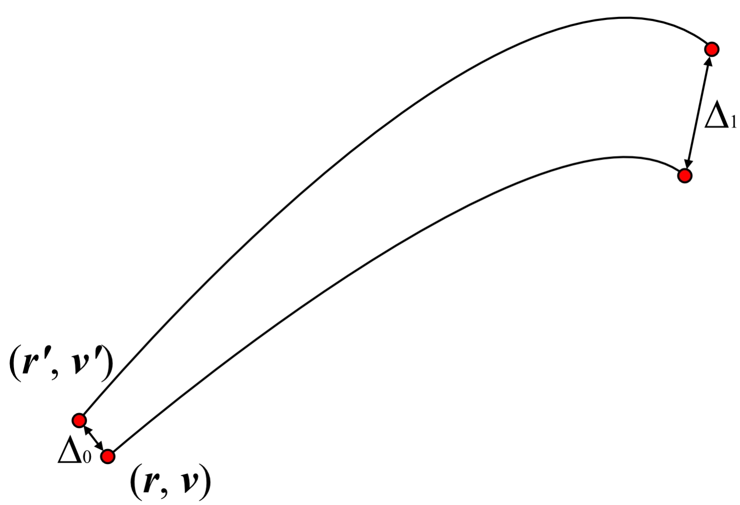

Figure 5.2: The growth of the initial deviation Δ_0 between the unperturbed $(\boldsymbol{r}, \boldsymbol{v})$ and perturbed solution $(\boldsymbol{r}', \boldsymbol{v}')$.

5.3 Application to the Pythagorean 3BP

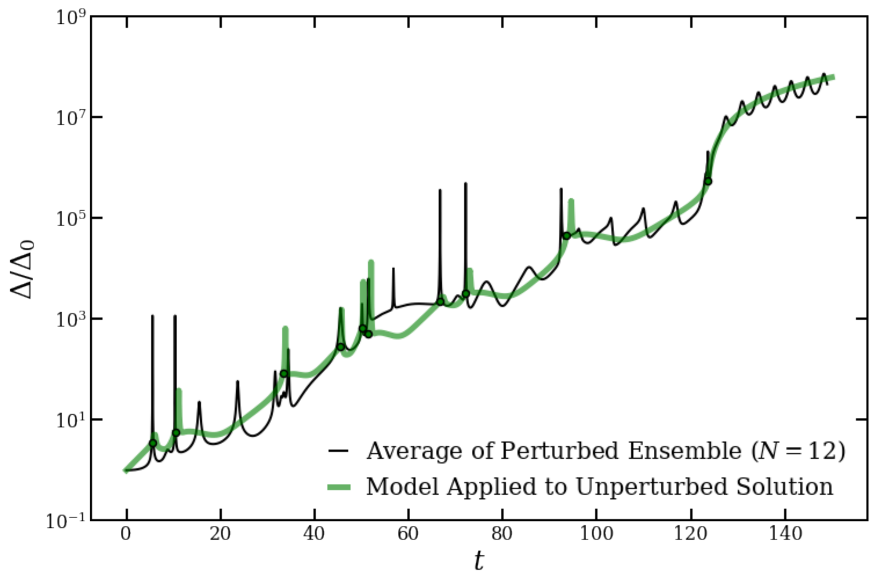

Figure 5.3: The average phase space divergence between the unperturbed Pythagorean solution and an ensemble of 12 perturbed solutions with $\Delta_0 = 1 \times 10^{-10}$ (black). Applying the Keplerian model to the unperturbed solution reproduces the overall shape and magnitude of the divergence (green).

References

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