Topological Chiral Field Synthesis

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Abstract

We present a compact mechanism to generate chiral second-order random fields on the cubic three-torus $\mathbb{T}^3 = [0, L)^3$ using a measured-bundle projection with thread weights. Information degrees of freedom live on a base space B and are transported to $\mathbb{T}^3$ by a kernel factoring into a spatial envelope and a $U(1)$ phase. A Hopf-type phase lift supplies nontrivial holonomy, which induces a helical part of the Fourier-space covariance. Under a finite, normalized fiber measure and a uniform pushforward of the thread density, the scalar power spectrum obeys the transfer law $P_\Phi(k) = \left|\widehat{G}_\sigma^{(T)}(k)\right|^2 P_\psi(k)$. For vector outputs, the covariance splits into scalar and helical pieces with the positivity bound $P_S(k) \geq |P_H(k)|$. We also record an optional anisotropic envelope that imprints angle dependence while preserving the same Gaussian falloff. Definitions and terminology for bundles follow standard sources [1, 2]; Hopf-type phases are motivated by recent observations and models of Hopf solitons [3, 4]. The "thread" terminology is deliberately analogous to statistical fiber-bundle models in materials physics [5, 6], while the helical projector formalism on $\mathbb{T}^3$ follows the vector analysis on tori [7]. Toroidal helical fields are classically discussed in [8].

1 Measured-bundle projection with threads

Definition 1.1 (Measured bundle and threads)**.** *Let $\pi : E \to B$ be a measurable bundle with typical fiber F.*[1] *Equip B with a base measure ν_B and F with a fiber measure ν_F. A thread density is a function $\lambda \in L^1(B, \nu_B)$; it induces a product measure on E:*

$$d\nu_E(b, f) := \lambda(b)\, d\nu_B(b)\, d\nu_F(f).$$

An information observable is any $\psi \in L^2(E, \nu_E)$.

Definition 1.2 (Finite fiber and normalization)**.** *Assume the fiber measure is finite and normalized,*

$$\nu_F(F) = 1.$$

Throughout we take $B = \mathbb{T}^3$ with the identity embedding $X : B \to \mathbb{T}^3$, $X(b) = b$, and a constant thread density $\lambda(b) = \Lambda/L^3$. We transport the observable ψ to a target field $\Phi : \mathbb{T}^3 \to \mathbb{C}$ (or $\Phi : \mathbb{T}^3 \to \mathbb{C}^3$) by

$$\Phi(x) = \int_B \int_F K(b, f; x)\, \psi(b, f)\, d\nu_F(f)\, \lambda(b)\, d\nu_B(b).$$

Definition 1.3 (Kernel structure)**.** *Let the kernel factor as*

$$K(b, f; x) = U(b, f; x)\, G_\sigma^{(T)}\left(x - X(b)\right),$$

where $G_\sigma^{(T)}$ is the periodized Gaussian on $\mathbb{T}^3$ with width $\sigma > 0$, and U is a $U(1)$ phase with $|U| = 1$. With Fourier wavevectors $k \in (2\pi/L)\mathbb{Z}^3$, the Fourier coefficients of the periodized Gaussian are

$$\widehat{G}_\sigma^{(T)}(k) = \exp\left(-\frac{1}{2}\sigma^2 \|k\|^2\right).$$

Definition 1.4 (Uniform pushforward of threads)**.** *With the choices above, the pushforward of $\lambda\, d\nu_B$ under X is spatially uniform:*

$$X_*(\lambda\, d\nu_B) = \frac{\Lambda}{L^3} d^3x$$

on $\mathbb{T}^3$.

[1] For background on fiber bundles and notation, see [1, 2].

Fourier convention. We use

$$\widehat{\Phi}(k) = \frac{1}{L^3} \int_{\mathbb{T}^3} e^{-ik\cdot x}\, \Phi(x)\, d^3x, \quad \Phi(x) = \sum_{k\in(2\pi/L)\mathbb{Z}^3} \widehat{\Phi}(k)\, e^{ik\cdot x}.$$

2 Stationarity and the scalar transfer law

Assume ψ is centered and second-order with covariance $C_\psi((b,f),(b',f')) := \mathbb{E}[\psi(b,f)\overline{\psi(b',f')}]$, and that U has unit modulus.

Proposition 2.1 (Stationarity and transfer)**.** *Under the uniform pushforward and $\nu_F(F) = 1$, the scalar output Φ defined above is stationary on $\mathbb{T}^3$. Its power spectrum satisfies*

$$P_\Phi(k) = \left|\widehat{G}_\sigma^{(T)}(k)\right|^2 P_\psi(k),$$

where $P_\psi(k)$ is the spectrum induced by the pushforward of C_ψ to $\mathbb{T}^3$. In particular, if the pushforward correlations are white, then $P_\Phi(k) = \Lambda \left|\widehat{G}_\sigma^{(T)}(k)\right|^2$.

Proof sketch. Translation of x acts only through $G_\sigma^{(T)}(x - X(b))$, so the covariance of Φ depends on $x - y$ once $\nu_F(F) = 1$ and the uniform pushforward are used. Fourier transforming pulls out $\left|\widehat{G}_\sigma^{(T)}(k)\right|^2$, yielding the transfer law.

3 Hopf lift and chirality

We now endow the kernel with a phase that encodes nontrivial holonomy.

Definition 3.1 (Hopf-type phase)**.** *Let $h : B \to S^2$ be smooth and let A denote a $U(1)$ connection on the Hopf fibration $S^3 \to S^2$; pull back A to B and push it forward to a 1-form a on $\mathbb{T}^3$ via X. For coupling $q \in \mathbb{R}$ define*

$$U(b,f;x) := \exp\left(iq \int_{\gamma_{X(b)\to x}} a\right),$$

where $\gamma_{X(b)\to x}$ is the straight geodesic segment on $\mathbb{T}^3$ (coordinate differences taken modulo L with $|\Delta x_i| \leq L/2$).

Gauge note. The open-path phase depends on the gauge choice for a; however, closed-loop Wilson holonomies are gauge-invariant. In the present construction, the two-point spectrum depends on a only through the curvature two-form $F = da$ pulled back from $h^*(dA)$, so P_S and P_H are gauge-invariant. See [3, 4] for physically realized Hopf textures.

For vector outputs $\Phi : \mathbb{T}^3 \to \mathbb{C}^3$, write the spectral covariance as

$$\langle \widehat{\Phi}_i(k)\overline{\widehat{\Phi}_j(k)}\rangle = P_S(k)\Pi_{ij}^{(S)}(k) + iP_H(k)\Pi_{ij}^{(H)}(k),$$

with $\hat{k} = k/\|k\|$, $\Pi_{ij}^{(S)} = \delta_{ij} - \hat{k}_i\hat{k}_j$ and $\Pi_{ij}^{(H)} = \epsilon_{ijl}\hat{k}_l$ (cf. helical projectors on $\mathbb{T}^3$ [7]).

Proposition 3.1 (Helical power and positivity)**.** *If $q \neq 0$ and the pulled curvature has nonzero Chern number on some 2-cycle in B,*

$$\frac{1}{2\pi}\int_\Sigma h^*(dA) = m \in \mathbb{Z} \setminus \{0\},$$

then the phase induces a nonzero helical spectrum $P_H(k)$ supported by the same Gaussian envelope as $P_S(k)$. Moreover, $P_S(k) \geq |P_H(k)|$ for all k, and the left/right helical powers are $P_\pm(k) = P_S(k) \pm P_H(k)$.

Remark 1 (Envelope and shape). The scalar part follows the transfer law $P_S(k) = \left|\widehat{G}_\sigma^{(T)}(k)\right|^2 P_\psi(k)$, and $P_H(k)$ shares the same envelope, with amplitude controlled by q and m.

4 Optional anisotropic envelope

To imprint angle dependence, replace the isotropic Gaussian by an anisotropic one. Let $n : B \to S^2$ be a director field and choose $\sigma_\parallel, \sigma_\perp > 0$. Set

$$G^{(T)}_{\Sigma(b)}(x - X(b)) \propto \exp\left(-\frac{1}{2}(x - X(b))^\top \Sigma(b)^{-1}(x - X(b))\right),$$

$$\Sigma(b) = \sigma_\perp^2 I + (\sigma_\parallel^2 - \sigma_\perp^2) n(b) n(b)^\top.$$

For slowly varying n (on scales $\gg \sigma_\perp, \sigma_\parallel$) one obtains

$$P_\Phi(k) \approx \exp\left(-\frac{1}{2}k^2 \left[\sigma_\perp^2 + (\sigma_\parallel^2 - \sigma_\perp^2)\mathbb{E}_B\left[(\hat{k} \cdot n)^2\right]\right]\right) P_\psi(k),$$

which is parity-even by itself, yet modulates the chirality fraction $\chi(k) := P_H(k)/P_S(k)$ when combined with the Hopf phase.

5 Diagnostics and minimal recipe

Spectral helicity. For any realization,

$$H(k) := \Im\left(\widehat{\Phi}(k)^* \cdot (\hat{k} \times \widehat{\Phi}(k))\right), \quad \mathbb{E}[H(k)] = 2P_H(k).$$

Minimal implementation. Choose $B = \mathbb{T}^3$, X, λ, and σ. Pick h with $(2\pi)^{-1} \int_\Sigma h^*(dA) = m \neq 0$ and a coupling q. Use the kernel above to synthesize Φ. In a helical basis $\{e_\pm(k)\}$, draw complex Gaussians $a_\pm(k)$ with variances $P_\pm(k)$ and set $\widehat{\Phi}(k) = a_+(k)e_+(k) + a_-(k)e_-(k)$. For a real-valued output field, the Fourier coefficients must satisfy the reality condition $\widehat{\Phi}(-k) = \overline{\widehat{\Phi}(k)}$.

A Periodized Gaussian on $\mathbb{T}^3$

Let $g_\sigma(x) = (2\pi\sigma^2)^{-3/2} \exp(-\|x\|^2/2\sigma^2)$ on $\mathbb{R}^3$. The periodization is

$$G^{(T)}_\sigma(x) = \sum_{n \in \mathbb{Z}^3} g_\sigma(x + Ln),$$

with Fourier coefficients $\exp\left(-\frac{1}{2}\sigma^2\|k\|^2\right)$ for $k \in (2\pi/L)\mathbb{Z}^3$.

B Assumptions at a glance

Finite fiber measure: $\nu_F(F) = 1$.

Uniform pushforward: $X_*(\lambda\, d\nu_B) = (\Lambda/L^3) d^3x$.

Unit-modulus phase U. These ensure stationarity on $\mathbb{T}^3$ and the Gaussian transfer envelope; adding the Hopf phase yields controlled, physically interpretable chirality.

References

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