Thread-Weighted Projections from a Spindle-Torus Base: A Measured Bundle Framework

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Abstract

We develop a framework in which a nonnegative density $\lambda \in L^1(B)$ on a two-dimensional spindle-torus base B weights local fiber data in a measured bundle $\pi : E \to B$. A transport kernel K projects fiber functions to functions on a target space, with a Gaussian factor enforcing locality around an embedding $X : B \to \mathbb{R}^3$. On a periodic T^3 domain, we prove stationarity of the induced covariance and derive a spectral transfer law predicting a turnover at the fundamental mode and a Gaussian roll-off. We further incorporate a $U(1)$ connection from the Hopf fibration, equipping the transport with a gauge-consistent phase and quantized invariants.

Scope and contributions. We (i) formalize a measurable density on a spindle-torus base, (ii) define a locality-preserving transport kernel that projects fiber data to the target space, (iii) prove stationarity and provide an analytic power spectrum with a fundamental-mode turnover and Gaussian roll-off on T^3, and (iv) show how a Hopf $U(1)$ connection endows the construction with a gauge-consistent phase and quantized invariants.

1 Introduction

Ideas in integral geometry suggest that functions on higher-dimensional spaces can be induced from lower-dimensional substrates via projections. Here we formulate a mathematically explicit version in which data resides on a two-dimensional spindle-torus base B and is transported to functions on a three-dimensional space by a thread-weighted projection (TWP).

The central element is a measured bundle $\pi : E \to B$ whose fibers store local data and a nonnegative density $\lambda \in L^1(B)$, called threads, that weights contributions from base points. A locality-preserving transport kernel K (minimal form: a Gaussian centered on the embedded base $X(B) \subset \mathbb{R}^3$) pushes forward fiber functions to induced functions Φ.

On a toroidal domain T^3 we prove that, under normalization, the induced covariance is stationary (depends only on separations) and the power spectrum obeys a transfer law exhibiting a turnover at the fundamental mode and Gaussian roll-off at high k. We then introduce a Hopf lift: a $U(1)$ connection inherited from the Hopf fibration induces a gauge-consistent phase in the transport, yielding quantized invariants.

Related perspectives. Our construction intersects kernel-based integral geometry, statistical processes on manifolds, and topological invariants. The emphasis is on a clean pipeline from base data to induced statistics with structure supporting numerical exploration.

Assumptions and limitations. We treat (i) the target geometry as fixed, (ii) threads as a static density on B, and (iii) transport as kinematic without dynamics.

2 Geometry of the base and measured bundle

2.1 Spindle-torus base and embedding

Let B be a spindle-torus surface embedded in $\mathbb{R}^3$ by a smooth map $X : B \to \mathbb{R}^3$. Concretely, one may view B as the surface of revolution obtained by rotating a circle of radius $r > 0$ centered at distance $a \in (0, r)$ from the axis, yielding the spindle member of the torus family. The detailed parameterization will not be needed; we use only that B is compact, orientable away from the self-intersection locus, and admits a Borel measure ν_B induced by the surface element.

2.2 Measured fiber bundle and threads

Definition 1 (Measured bundle and threads)**.** *Let $\pi : E \to B$ be a measurable fiber bundle with typical fiber F and fiber measure ν_F. A thread density is a function $\lambda \in L^1_+(B, \nu_B) \cap L^2(B, \nu_B)$, $\lambda \geq 0$, which we normalize by $\int_B \lambda \, d\nu_B = \Lambda > 0$. A fiber function is a measurable map $\phi : E \to \mathbb{C}$ with finite second moment $\int_E |\phi|^2 \, d\nu_E < \infty$, where $d\nu_E := d\nu_F \, d\nu_B$.*

The density λ encodes where the information lives on B; large λ indicates many or stronger threads. We assume either deterministic ϕ or a mean-zero random fiber function with known two-point function.

3 Thread-weighted projection (TWP) and minimal locality

3.1 Transport kernel

Let $x \in M \subset \mathbb{R}^3$ denote a target point (in this paper M will be either $\mathbb{R}^3$ or a 3-torus T^3). We define a transport kernel $K : M \times B \times F \to \mathbb{C}$ of the form

$$K(x \mid b, f) = G_{\Sigma}(b)\big(x - X(b)\big)\, U(b, f; x),$$

where G_Σ is a Gaussian with covariance $\Sigma(b) \in \mathbb{R}^{3\times 3}$ positive-definite, and U is a bounded modulation factor. The minimal kernel takes $U \equiv 1$ and $\Sigma(b) = \sigma^2 I$.

Definition 2 (Thread-weighted projection)**.** *Given λ, ϕ, K, define the induced function*

$$\Phi(x) = \int_B \int_F K(x \mid b, f)\, \phi(b, f)\, \lambda(b)\, d\nu_F(f)\, d\nu_B(b).$$

Proposition 1 (Well-posedness)**.** *Assume $\nu_F(F) = 1$ (finite fiber measure), $U \in L^\infty$, and $\sup_b \|G_\Sigma(b)\|_{L^1(\mathbb{R}^3)} < \infty$. If $\phi \in L^2(E, \nu_E)$ and $\lambda \in L^1(B) \cap L^2(B)$, then $\Phi \in L^2(M)$ and the map $(\phi, \lambda) \mapsto \Phi$ is continuous.*

Proof. By Cauchy–Schwarz on the fiber and $\nu_F(F) = 1$,

$$\int_F |\phi(b, f)|\, d\nu_F \le \Big(\int_F |\phi(b, f)|^2\, d\nu_F \Big)^{1/2}.$$

With U bounded and $\|G_\Sigma(b)\|_{L^1}$ uniformly bounded, Fubini and Minkowski yield

$$\|\Phi\|_{L^2(M)} \le C \int_B \lambda(b)\, a(b)\, d\nu_B, \quad a(b) := \Big(\int_F |\phi(b, f)|^2\, d\nu_F \Big)^{1/2}.$$

Since $a \in L^2(B)$ and $\lambda \in L^2(B)$, Hölder gives $\int_B \lambda a \le \|\lambda\|_{L^2} \|a\|_{L^2} < \infty$, which yields finiteness and continuity. □

3.2 Normalization on T^3 and periodized kernels

On $M = T^3 = [0, L)^3$ we periodize G_Σ:

$$G_\Sigma^{(T)}(\chi) = \sum_{n \in \mathbb{Z}^3} G_\Sigma(\chi + nL), \quad \chi \in T^3,$$

and impose the normalization

$$\int_{T^3} G_\Sigma^{(T)}(\chi)\, d^3\chi = 1.$$

4 Stationarity and the spectral transfer law on T^3

Assume ϕ is a mean-zero, second-order process on E with covariance

$$C_\phi((b, f), (b', f')) = \mathbb{E}[\phi(b, f)\, \overline{\phi(b', f')}].$$

Random-translation stationarity model

Let T be a random translation, uniformly distributed on T^3 and independent of ϕ. Define the translated embedding $X_T(b) := (X(b) + T) \bmod L$ and the corresponding field Φ_T by replacing X with X_T in the definition of Φ. All expectations below are taken over both ϕ and T.

Proposition 2 (Stationarity on T^3)**.** *Under the minimal periodized kernel ($U \equiv 1$ with the above normalization), with the random-translation model, and if C_ϕ depends only on base separation along B (invariant under isometries of B), then the covariance*

$$C_\Phi(x, y) := \mathbb{E}\big[\Phi_T(x)\, \overline{\Phi_T(y)}\big]$$

depends only on $\Delta := x - y \in T^3$.

Proof. By construction, T is uniform and independent, so $(x, y) \mapsto (x+T, y+T)$ yields the same distribution for Φ_T. Averaging over T enforces translation invariance, and the residual dependence is on Δ. $\square$

Let $\hat{f}(k)$ denote the discrete Fourier transform on T^3 for $k \in (2\pi/L)\mathbb{Z}^3$, and define the power spectrum $P_\Phi(k) = \mathbb{E}[|\hat{\Phi}_T(k)|^2]$.

Proposition 3 (Spectral transfer law)**.** *Under the stationarity proposition and with* $\Sigma(b) \equiv \sigma^2 I$*,*

$$P_\Phi(k) \;=\; \left|\widehat{G}_\sigma^{(T)}(k)\right|^2 P_\psi(k), \qquad \widehat{G}_\sigma^{(T)}(k) = e^{-\frac{1}{2}\sigma^2\|k\|^2},$$

where $P_\psi(k)$ *is the spectrum of the fiber-aggregated source obtained after the random translation.*

Proof. Fourier transforming the projection and using convolution on T^3 yields $\hat{\Phi}_T(k) = \widehat{G}_\sigma^{(T)}(k)\,\hat{\Psi}_T(k)$ for a stationary source Ψ_T. Taking expectations gives the result. $\square$

[Turnover and Gaussian roll-off] On T^3 the lowest nonzero wavenumber is $k_f = 2\pi/L$. If $P_\psi(k)$ is finite at k_f, then $P_\Phi(k)$ turns over at k_f and decays as $\exp(-\sigma^2\|k\|^2)$ at large $\|k\|$.

5 Anisotropic kernels and director fields

Threads can carry directional structure. Let $n : B \to S^2$ be a measurable director field and

$$\Sigma(b) = \sigma_\parallel^2\, n(b)\, n(b)^\top + \sigma_\perp^2 \left(I - n(b)\, n(b)^\top\right), \quad \sigma_\parallel, \sigma_\perp > 0.$$

Then $G_\Sigma(b)$ elongates transport along $n(b)$.

Proposition 4 (Anisotropic spectral imprint)**.** *Assume* n *is slowly varying on the* G_Σ *support,* $U \equiv 1$*, and the random-translation stationarity model holds on* T^3*. Then to leading order the spectrum acquires an angular dependence*

$$P_\Phi(k) \approx \exp\!\left(-\frac{1}{2}\,|k|^2 \left[\sigma_\perp^2 + (\sigma_\parallel^2 - \sigma_\perp^2)\, E_B\!\left[(\hat{k}\cdot n)^2\right]\right]\right) P_\psi(k),$$

where E_B *denotes an average over* B *with weight* $\lambda\, d\nu_B$ *and* $\hat{k} = k/\|k\|$*.*

6 Hopf lift: $U(1)$ connection, invariants, and phase structure

Let $\pi_H : S^3 \to S^2$ be the Hopf map with connection A and curvature $F = dA$. Pull back along a map $h : B \to S^2$ to obtain $A_h = h^*A$ with quantized Chern number $\frac{1}{2\pi}\int_B h^*F \in \mathbb{Z}$.

6.1 Phase-modulated transport

Definition 3 (Tubular extension for phase)**.** *Let $\mathcal{N}$ be a tubular neighborhood of $X(B)$ in $\mathbb{R}^3$ and let $\rho : \mathcal{N} \to B$ be a smooth retraction. Define an extension $A^{\mathrm{ext}} := \rho^* A_h$ on $\mathcal{N}$. For x near $X(B)$ choose a short curve $\gamma(b,x) \subset \mathcal{N}$ and set*

$$U(b,f;x) = \exp\big(iq\,\Theta(b;x)\big), \qquad \Theta(b;x) := \int_{\gamma(b,x)} A^{\mathrm{ext}},$$

with $q \in \mathbb{Z}$. Gauge choices affect Θ by a boundary term; observables depend on F_h.

6.2 Vector-valued output and helical spectra

To allow a helical decomposition, define a vector kernel using a frame $e(b) \in \mathbb{R}^{3\times 3}$ (for example, built from tangent and normal directions of $X(B)$):

$$K_i(x \mid b,f) = G_\Sigma\big(x - X(b)\big)\,U(b;x)\,e_i(b).$$

This gives $\Phi : T^3 \to \mathbb{C}^3$.

Proposition 5 (Helical decomposition)**.** *Assume the random-translation stationarity model on T^3 and the vector kernel above. In the isotropic limit of G_Σ one has*

$$\langle \hat{\Phi}_i(k)\,\overline{\hat{\Phi}_j(k)} \rangle \;=\; P_S(k)\,\Pi^{(S)}_{ij}(k) \;+\; i\,P_H(k)\,\Pi^{(H)}_{ij}(k),$$

where $\Pi^{(S)}_{ij}(k) = \delta_{ij} - \hat{k}_i\hat{k}_j$ and $\Pi^{(H)}_{ij}(k) = \epsilon_{ij\ell}\,\hat{k}_\ell$. One has $P_H(k) \neq 0$ when $q \neq 0$ and F_h is not identically zero (for example, nonzero Chern number).

Remark 1. *Parity-odd power scales with curvature. At leading order in the phase, P_H depends on a λ-weighted functional of F_h, for example $\int_B \lambda\,F_h$. For a closed B, $\frac{1}{2\pi}\int_B F_h$ is an integer.*

Scope note. We keep the vector-output construction abstract and do not fix a specific frame or tubular retraction; this avoids path- and gauge-dependent implementation choices and keeps the focus on the scalar TWP results and their spectral consequences.

7 Numerical pipeline and computational features

7.1 Minimal simulation loop

1. Sample ϕ on E with a chosen fiber covariance.
2. Sample or prescribe λ on B; optionally, a director field n and map $h : B \to S^2$ for Hopf phase.
3. Assemble K using $G_\Sigma(b)$ and U.
4. Compute Φ via the projection. On T^3 use FFTs and the spectral law when the isotropic assumptions hold, otherwise apply the correct angle dependent multiplier.
5. Measure $P_\Phi(k)$, anisotropy, and (if vector outputs) the helical spectra.

7.2 Computable features

- Turnover scale: $k_f = 2\pi/L$ sets the first extremum.
- High-k damping: Gaussian roll-off $\sim e^{-\sigma^2\|k\|^2}$ (with angular modification for anisotropy).
- Anisotropy: Angular modulation governed by $\sigma_\parallel/\sigma_\perp$ and the distribution of n.
- Phase structure: Controlled by q and the curvature F_h through $P_H(k)$.

8 Extensions: identifiability and inverse problems

Proposition 6 (Identifiability via linearized statistics (informal))**.** *Assume the random-translation stationarity model on T^3, a factorized fiber covariance $C_\phi((b,f),(b',f')) = \delta(b-b')\,C_F(f,f')$ independent of λ, and known K with U bounded. Let $\mathcal{F}$ map a band-limited λ to a chosen statistic S (for example, P_Φ, cross-spectra with known probes, or the bispectrum). If the Fréchet derivative $D\mathcal{F}(\lambda_0)$ has smallest singular value $s_{\min} > 0$ on that band, then the estimation error for λ from noisy S is $O(s_{\min}^{-1} \times \text{noise})$. Using only P_Φ generically leaves phase unidentifiable; bispectrum or cross-spectra can restore it under mild non-Gaussianity or probing.*

Remark 2. *A precise statement requires specifying the class of ϕ and the measurement noise model.*

9 Conclusion

We presented a measured-bundle construction in which a nonnegative thread density on a spindle-torus base weights local fiber data transported to induced functions by a locality-preserving kernel. On T^3 this yields a stationary process with an analytic spectral transfer law exhibiting a fundamental-mode turnover and Gaussian high-k damping. A Hopf lift adds a gauge-consistent phase with quantized invariants and allows helical spectra for vector outputs.

Open questions include generalizations to other base manifolds, alternative kernel forms, and a full inverse analysis under realistic noise and source models.

A Spindle-torus parameterization (optional)

A convenient implicit equation for a torus family in $\mathbb{R}^3$ is

$$(\sqrt{x^2+y^2} - a)^2 + z^2 = r^2,$$

with $0 < a < r$ corresponding to the spindle member. Local charts avoiding the self-intersection suffice for measure-theoretic constructions.

B Periodized Gaussian on T^3

Let $G_\sigma(\chi) = (2\pi\sigma^2)^{-3/2} \exp(-\|\chi\|^2/2\sigma^2)$ on $\mathbb{R}^3$. The periodization

$$G_\sigma^{(T)}(\chi) = \sum_{n\in\mathbb{Z}^3} G_\sigma(\chi + nL)$$

satisfies $\int_{T^3} G_\sigma^{(T)}(\chi)\, d^3\chi = 1$ and has discrete Fourier transform $\widehat{G}_\sigma^{(T)}(k) = e^{-\frac{1}{2}\sigma^2\|k\|^2}$ for $k \in (2\pi/L)\mathbb{Z}^3$.

C Helical projectors

For vector functions, define the transverse projector $\Pi^\perp_{ij} = \delta_{ij} - \hat{k}_i\hat{k}_j$ and the helical projectors

$$\Pi^{(H)}_{ij}(k) = \epsilon_{ij\ell}\, \hat{k}_\ell, \quad \Pi^{(S)}_{ij}(k) = \Pi^\perp_{ij}.$$

Then a general statistically isotropic covariance has the form $P_S(k)\,\Pi^{(S)} + i\,P_H(k)\,\Pi^{(H)}$.

References

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