A Geometric Foundation for Holographic Stochastic Field Theory: A Measured-Bundle Realization of the $T^2 \to T^3$ Map

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Abstract

A framework is presented for synthesizing divergence-free, homogeneous, and chiral random vector fields on the three-torus T^3 from data placed on a two-dimensional base T^2. The construction treats T^2 as a holographic screen that drives a 2D $\to$ 3D mapping through a measured-bundle projection with a U(1) phase lift. The phase is the holonomy of a principal U(1) bundle over T^2 with first Chern number $c_1 \in \mathbb{Z}$, which parametrizes chirality. Under an equidistribution condition on a map $X : E \to T^3$ from the bundle total space, the resulting field on T^3 is translation invariant; isotropy is obtained by choosing X so that its structure factor is approximately rotationally uniform (or by statistical rotational averaging). In Fourier space, the covariance decomposes into helical eigenmodes with spectra $P_S(k) \geq 0$ and $P_H(k)$ that satisfy the sharp positivity bound $P_S(k) \geq |P_H(k)|$. An operator-theoretic (spectral-triple) realization is provided in an odd form, with the Dirac operator D determined by the helical spectra and the envelope. A transfer law connecting boundary statistics on T^2 to bulk helical power spectra on T^3 is derived, together with a numerically ready synthesis algorithm on rectangular lattices.

1 Introduction

Holographic ideas suggest that bulk structure can be encoded on a lower-dimensional substrate [1–4]. A static (equal-time, Euclidean) realization is formulated here for random divergence-free vector fields on the 3-torus, using T^2 as a base and a principal U(1) bundle $p : E \to T^2$ to supply a topological phase. The first Chern class $c_1(E) \in H^2(T^2, \mathbb{Z}) \simeq \mathbb{Z}$ plays the role of an integer "chirality knob" via a holonomy phase lift. A measurable map $X : E \to T^3$ pushes the invariant bundle measure to the uniform measure on T^3, ensuring homogeneity. A translation-invariant envelope kernel then yields a divergence-free Gaussian field with a standard helical covariance. The helical bound follows from Bochner positivity on compact abelian groups. An operator-theoretic representation via a spectral triple is given in an odd form, which directly reproduces the covariance.

Helicity and helical mode decompositions are classical in hydrodynamics and MHD [5–8]. Circle bundles over T^2 are classified by $c_1 \in \mathbb{Z}$ and admit natural connections and holonomy phases [9, 11, 12]. Equidistribution on tori is underpinned by Kronecker–Weyl theory and unique ergodicity [15, 16].

Contributions. (i) A measured-bundle synthesis scheme $E \to T^3$ that generates a divergence-free random field with tunable helicity through $c_1(E)$. (ii) A boundary-to-bulk transfer law expressing helical power spectra on T^3 in terms of base statistics, the bundle connection, and an envelope kernel. (iii) An odd spectral-triple (A, H, D) whose spectral calculus reproduces the covariance. (iv) A practical lattice algorithm that samples the field while enforcing the helical positivity bound.

2 Geometric and probabilistic setup

2.1 Principal circle bundles over T^2

Let $p : E \to T^2$ be a principal U(1) bundle. Isomorphism classes are in bijection with $c_1(E) \in H^2(T^2, \mathbb{Z}) \simeq \mathbb{Z}$ [9, 10]. Let $A \in \Omega^1(E; i\mathbb{R})$ be a connection with curvature $F = dA$. In de Rham cohomology,

$$\left[\tfrac{i}{2\pi} F\right]_{\mathrm{dR}} = c_1(E) \in H^2(T^2; \mathbb{Z}).$$

Denote by μ_E the invariant probability measure induced by Haar on T^2 and the uniform measure on the S^1 fiber (equivalently, the connection-invariant volume on E). Let λ_{T^3} denote the normalized Haar measure on T^3.

Fix a basepoint $b_0 \in T^2$ and a reference lift $\beta_0 \in p^{-1}(b_0)$. For each $\beta \in E$, let γ_β be a horizontal path from β_0 to β. Define the holonomy phase by

$$U(\beta) \exp\Big(i \int_{\gamma_\beta} A\Big),$$

which has unit modulus and is well defined up to the usual gauge factor. All subsequent formulas use $d\widetilde{W}(\beta) = U(\beta)\, dW(\beta)$ and are gauge covariant.

2.2 Equidistributing map to T^3

We require a measurable surjection $X : E \to T^3$ such that $X_{\#}\mu_E = \lambda_{T^3}$. One construction uses a horizontal lift for the connection, then three circle-valued observables with incommensurate frequencies; we only use the pushforward property.

Lemma. There exists a measurable $X : E \to T^3$ with $X_{\#}\mu_E = \lambda_{T^3}$.

Sketch. Compose a horizontal flow on E with three incommensurate circle maps and take a Krylov–Bogolyubov (Cesàro) average; then pass to a measurable selector. We only use the pushforward property in the covariance computation.

To accelerate isotropy in practice, one may average over a small ensemble of frames $R \in \mathrm{SO}(3)$, replacing X by $R \circ X$ and averaging statistics over R.

2.3 Envelope, white noise, and synthesized field

Let $G_\sigma : T^3 \to \mathbb{R}$ be a smooth, real, even envelope kernel with Fourier transform $\widehat{G}(k)$ satisfying $\widehat{G}(k) = g(|k|)$ for a nonnegative radial function g. Note that $\widehat{G}(0) = 0$ is optional because $\nabla \times (\mathrm{const}) = 0$ already removes the $k = 0$ mode. Let W be a complex Gaussian white noise on (E, μ_E) with covariance

$$\mathbb{E}\Big[dW(\beta)\, \overline{dW(\beta')}\Big] = \delta(\beta - \beta')\, d\mu_E(\beta), \qquad \mathbb{E}\big[dW(\beta)\, dW(\beta')\big] = 0.$$

Fix a constant vector $\boldsymbol{a} \in \mathbb{C}^3$. Define

$$\Phi(x) = \int_E \nabla_x \times [G_\sigma\, (x - X(\beta))\, \boldsymbol{a}]\, U(\beta)\, dW(\beta).$$

The curl enforces $\nabla \cdot \Phi \equiv 0$. Translation invariance follows from the equidistribution assumption. For a real-valued field, enforce $\widehat{\Phi}(-k) = \overline{\widehat{\Phi}(k)}$. In the synthesis we pair k with $-k$ and draw conjugate-symmetric coefficients.

Isotropic version: sample $R \sim \mathrm{Haar}(\mathrm{SO}(3))$ independently and replace $\boldsymbol{a}$ by $R\boldsymbol{a}$ before taking expectations; averaging over R yields the isotropic helical tensor.

2.4 Helical decomposition on T^3

For $\boldsymbol{k} \in \mathbb{Z}^3 \setminus \{0\}$, let $\{\boldsymbol{h}_\pm(\boldsymbol{k})\}$ be an orthonormal helical basis of divergence-free eigenvectors of $i\boldsymbol{k} \times (\cdot)$:

$$i\boldsymbol{k} \times \boldsymbol{h}_\pm(\boldsymbol{k}) = \pm|\boldsymbol{k}|\boldsymbol{h}_\pm(\boldsymbol{k}), \quad \boldsymbol{k} \cdot \boldsymbol{h}_\pm(\boldsymbol{k}) = 0,$$

and write $\widehat{\Phi}(\boldsymbol{k}) = \Phi_+(\boldsymbol{k})\boldsymbol{h}_+(\boldsymbol{k}) + \Phi_-(\boldsymbol{k})\boldsymbol{h}_-(\boldsymbol{k})$.

3 Covariance and helical spectra

Taking expectations in the definition of Φ and using $X_{\#}\mu_E = \lambda_{T^3}$ gives

$$\mathbb{E}\left[\widehat{\Phi}_i(\boldsymbol{k})\,\overline{\widehat{\Phi}_j(\boldsymbol{k}')}\right] = \delta_{\boldsymbol{k},\boldsymbol{k}'}|\widehat{G}(\boldsymbol{k})|^2\left(P_S(\boldsymbol{k})\Pi_{ij}(\boldsymbol{k}) + iP_H(\boldsymbol{k})\epsilon_{ijm}\hat{k}_m\right),$$

where $\Pi_{ij}(\boldsymbol{k}) = \delta_{ij} - \hat{k}_i\hat{k}_j$, $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$, and ϵ_{ijm} is the Levi–Civita symbol with $\epsilon_{123} = 1$. In the helical basis,

$$\mathbb{E}\left[|\Phi_\pm(\boldsymbol{k})|^2\right] = |\widehat{G}(\boldsymbol{k})|^2\left(P_S(\boldsymbol{k}) \pm P_H(\boldsymbol{k})\right), \quad \mathbb{E}\left[\Phi_+(\boldsymbol{k})\Phi_-(\boldsymbol{k})\right] = 0.$$

[Bochner positivity and the helical bound] For each $\boldsymbol{k} \neq 0$,

$$P_S(\boldsymbol{k}) \geq 0, \quad |P_H(\boldsymbol{k})| \leq P_S(\boldsymbol{k}).$$

Proof. On the compact abelian group T^3, stationary covariances correspond to positive-type functions on $\mathbb{Z}^3$ whose Fourier transforms are positive measures [13, 14]. Restricting the covariance to the transverse subspace and diagonalizing in the helical basis gives variances $|\widehat{G}(\boldsymbol{k})|^2(P_S(\boldsymbol{k}) \pm P_H(\boldsymbol{k})) \geq 0$, which is equivalent to the stated inequalities. □

[Helicity density] With $\widehat{\Phi} = \widehat{\Phi}_+ + \widehat{\Phi}_-$ in helical modes and covariance

$$\mathbb{E}[\widehat{\Phi}_i(\boldsymbol{k})\,\overline{\widehat{\Phi}_j(\boldsymbol{k})}] = |\widehat{G}(\boldsymbol{k})|^2\left[P_S(\boldsymbol{k})\Pi_{ij}(\boldsymbol{k}) + iP_H(\boldsymbol{k})\,\epsilon_{ijm}\hat{k}_m\right],$$

and with the Fourier convention $\widehat{f}(k) = \int_{T^3} f(x)e^{-ik\cdot x}\,dx$, we have

$$\mathbb{E}\int_{T^3}\Phi\cdot(\nabla\times\Phi)\,dx = \sum_{\boldsymbol{k}\neq 0} 2|\boldsymbol{k}|\,|\widehat{G}(\boldsymbol{k})|^2\,P_H(\boldsymbol{k}),$$

so the spectral helicity density per mode is $2|\boldsymbol{k}|\,|\widehat{G}(\boldsymbol{k})|^2\,P_H(\boldsymbol{k})$, consistent with classical conventions [5, 7].

4 Boundary-to-bulk transfer law

Let $b = p(\beta) \in T^2$ denote the base point. The symmetric and helical spectra admit the representation

$$P_{S/H}(\boldsymbol{k}) = \int_E K_{S/H}(\boldsymbol{k};\beta)\,d\mu_E(\beta),$$

with $K_{S/H}$ obtained by Fourier transforming the curl-envelope synthesis against $e^{-i\boldsymbol{k}\cdot X(\beta)}$ and the phase $U(\beta)$.

Assume the trivial-bundle case $c_1(E) = 0$, fix global angles $(\theta_1, \theta_2, \phi)$, and take

$$X(\beta) = M \begin{pmatrix} \theta_1 \\ \theta_2 \\ \phi \end{pmatrix} \pmod{2\pi}, \qquad M \in \mathbb{Z}^{3\times 3}.$$

Denote the third column of M by ω_3. In this affine model the ϕ-dependence contributes the phase $e^{i(n-\boldsymbol{k}\cdot\omega_3)\phi}$, so the dominant contribution occurs when $n - \boldsymbol{k}\cdot\omega_3 \in \mathbb{Z}$.

5 Effective quadratic action and odd term

The Gaussian measure is determined by the covariance. In the helical basis the quadratic action is

$$S[\Phi] = \frac{1}{2}\sum_{\boldsymbol{k}\neq 0} |\widehat{G}(\boldsymbol{k})|^{-2} \left(\frac{|\Phi_+(\boldsymbol{k})|^2}{P_S(\boldsymbol{k}) + P_H(\boldsymbol{k})} + \frac{|\Phi_-(\boldsymbol{k})|^2}{P_S(\boldsymbol{k}) - P_H(\boldsymbol{k})} \right).$$

In physical space, the action splits into an even part plus an odd contribution of Chern–Simons type,

$$S_{\mathrm{odd}}[\Phi] \approx \int_{T^3} \eta(|\nabla|)\Phi(x)\cdot(\nabla\times\Phi(x))\,dx,$$

with a nonlocal positive kernel η determined by P_S and P_H [17–19].

6 A spectral-triple realization (odd case)

Let $A = C^\infty(T^3)$ act by multiplication on $H = L^2_{\mathrm{trans}}(T^3;\mathbb{C}^3)\oplus L^2_{\mathrm{trans}}(T^3;\mathbb{C}^3)$. In the Fourier-helical basis, define

$$D : (\Phi_+(\boldsymbol{k}), \Phi_-(\boldsymbol{k})) \mapsto \Big(|\widehat{G}(\boldsymbol{k})|^{-1}(P_S(\boldsymbol{k}) + P_H(\boldsymbol{k}))^{-1/2}\Phi_+(\boldsymbol{k}), |\widehat{G}(\boldsymbol{k})|^{-1}(P_S(\boldsymbol{k}) - P_H(\boldsymbol{k}))^{-1/2}\Phi_-(\boldsymbol{k}) \Big).$$

Assume $|\widehat{G}(\boldsymbol{k})|^{-1}(P_S \pm P_H)^{-1/2}$ has at most first-order growth in $|\boldsymbol{k}|$. Under this assumption, standard pseudodifferential multiplier estimates imply that $[D, f]$ is bounded for $f \in C^\infty(T^3)$, and that D^{-1} is compact with summability consistent with metric dimension 3.

7 Numerical synthesis on lattices

Identify T^3 with the grid

$$\mathbb{Z}_N^3 = \left\{ -\frac{N}{2}, \ldots, \frac{N}{2} - 1 \right\}^3 \quad \text{(for even } N\text{)},$$

and fix the FFT convention (forward transform $\widehat{f}(k) = \frac{1}{N^3}\sum_x f(x)e^{-2\pi i\,k\cdot x/N}$, inverse with N^3 factor). Choose $\widehat{G}(\boldsymbol{k})$, an integer J (number of randomized frames for isotropy), and a connection with $c_1(E) = n$.

1. **Boundary randomness and holonomy.** Sample i.i.d. complex Gaussians $\{\xi_\ell\}$ on a mesh $\{\beta_\ell\} \subset E$ and phases $U_\ell = U(\beta_\ell)$.

2. **Projection to bulk spectra (preliminary estimates + smoothing).** For each $\boldsymbol{k} \in \mathbb{Z}_N^3 \setminus \{0\}$, set

$$B_S(\boldsymbol{k}) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} e^{-i\boldsymbol{k}\cdot X(\beta_\ell)} \xi_\ell, \qquad B_H(\boldsymbol{k}) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} e^{-i\boldsymbol{k}\cdot X(\beta_\ell)} U_\ell \xi_\ell,$$

and define preliminary estimates

$$\widetilde{P}_S(\boldsymbol{k}) = |B_S(\boldsymbol{k})|^2, \qquad \widetilde{P}_H(\boldsymbol{k}) = \Re\big(B_H(\boldsymbol{k})\overline{B_S(\boldsymbol{k})}\big).$$

Apply a spectral smoothing/averaging step (e.g. over shells in $|\boldsymbol{k}|$) to obtain $P_{S/H}(\boldsymbol{k})$, and clip only if necessary to enforce $|P_H| \leq P_S$.

3. **Helical assembly.** For each $\boldsymbol{k} \neq 0$, draw independent

$$\widehat{\Phi}_\pm(\boldsymbol{k}) \sim \mathcal{N}_{\mathbb{C}}\Big(0,\ |\widehat{G}(\boldsymbol{k})|^2 \big(P_S(\boldsymbol{k}) \pm P_H(\boldsymbol{k})\big)\Big),$$

enforce $\widehat{\Phi}_\pm(-\boldsymbol{k}) = \overline{\widehat{\Phi}_\pm(\boldsymbol{k})}$, and set $\widehat{\Phi}(\boldsymbol{k}) = \widehat{\Phi}_+(\boldsymbol{k})\, \boldsymbol{h}_+(\boldsymbol{k}) + \widehat{\Phi}_-(\boldsymbol{k})\, \boldsymbol{h}_-(\boldsymbol{k})$.

4. **Inverse transform.** Verify conjugate symmetry and inverse FFT to obtain $\Phi(x)$.

8 Discussion and extensions

Chirality control. The integer $n = c_1(E)$ controls the net helical bias via the holonomy factor.
Homogeneity and isotropy. Equidistribution of X guarantees translation invariance; isotropy is obtained by radial $\widehat{G}$ combined with averaging over random frames $R \in \mathrm{SO}(3)$.
Generalizations. Bases other than T^2 (e.g. S^2) and nonabelian fiber groups are natural extensions; time dependence can be introduced by slowly varying X and the connection.

While speculative, this framework may inspire analogies in particle physics, such as modeling weak interaction handedness or neutrinoless double-beta decay via chirality reversals and off-diagonal couplings in D, potentially corresponding to neutrino masses. However, bridging to quantum domains requires further theoretical work.

A Normalization cross-check

Under the Fourier conventions used, the energy and helicity normalizations are as follows.

The total energy is

$$\mathbb{E}\left[\int_{T^3} |\Phi(x)|^2\, dx\right] = \sum_{\boldsymbol{k}\neq 0} |\widehat{G}(\boldsymbol{k})|^2\, (P_S(\boldsymbol{k}) + P_S(\boldsymbol{k})) = 2\sum_{\boldsymbol{k}\neq 0} |\widehat{G}(\boldsymbol{k})|^2 P_S(\boldsymbol{k}),$$

since the helical modes are orthogonal.

For the helicity, as derived in Remark 3.2.

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