Measured-Bundle Projection Framework (MBPF)

Abstract

We develop a framework in which a nonnegative density $\lambda \in L^1(B)$ on a two-dimensional spindle–torus base B weights local fiber data in a measured bundle $\pi : E \to B$. A transport kernel K projects fiber functions to functions on a target space, with a Gaussian factor enforcing locality around an embedding $X : B \to \mathbb{R}^3$. On a periodic T^3 domain, we prove stationarity of the induced covariance and derive a spectral transfer law predicting a turnover at the fundamental mode and a Gaussian roll-off. We further incorporate a $U(1)$ connection from the Hopf fibration, equipping the transport with a gauge-consistent phase and quantized invariants.

1 Introduction

We formulate a mathematically explicit version in which data resides on a two-dimensional spindle–torus base B and is transported to functions on a three-dimensional space by a thread-weighted projection (TWP).

The central element is a measured bundle $\pi : E \to B$ whose fibers store local data and a nonnegative density λ, called threads, that weights contributions from base points. A locality-preserving transport kernel K (minimal form: a Gaussian centered on the embedded base $X(B) \subset \mathbb{R}^3$) pushes forward fiber functions to induced functions Φ.

On a toroidal domain T^3 we prove that, under normalization, the induced covariance is stationary (depends only on separations) and the power spectrum obeys a transfer law exhibiting a turnover at the fundamental mode and Gaussian roll-off at high k. We then introduce a Hopf lift: a $U(1)$ connection inherited from the Hopf fibration induces a gauge-consistent phase in the transport, yielding quantized invariants.

2 Geometry of the base and measured bundle

2.1 Spindle–torus base and embedding

Definition 1 (Spindle torus)**.** *For radii $R, r > 0$ with $r > R$, the* spindle torus

$$\mathcal{T}_{\mathrm{sp}}(R, r) \subset \mathbb{R}^3$$

is the surface parametrized by

$$(x, y, z) = \big((R + r\cos u)\cos v,\ (R + r\cos u)\sin v,\ r\sin u\big), \quad u, v \in [0, 2\pi).$$

Topologically $\mathcal{T}_{\mathrm{sp}}(R, r) \cong T^2$, though it self–intersects in $\mathbb{R}^3$.

Let B be a spindle–torus surface embedded in $\mathbb{R}^3$ by a smooth map $X : B \to \mathbb{R}^3$. We use only that B is compact, orientable away from the self-intersection locus, and admits a Borel measure ν_B.

2.2 Measured fiber bundle and threads

Definition 2 (Measured bundle and threads). *Let $\pi : E \to B$ be a measurable fiber bundle with typical fiber F and fiber measure ν_F. We assume $\nu_F(F)$ is finite and normalize so that $\nu_F(F) = 1$. A thread density is a function $\lambda \in L^1_+(B, \nu_B) \cap L^2(B, \nu_B)$, $\lambda \geq 0$, with normalization*

$$\int_B \lambda \, d\nu_B = \Lambda > 0.$$

A fiber function is a measurable map $\varphi : E \to \mathbb{C}$ with finite second moment, and we write $d\nu_E := d\nu_F \, d\nu_B$.

3 Thread-weighted projection (TWP) and minimal locality

3.1 Transport kernel

Let $x \in M \subset \mathbb{R}^3$ denote a target point. We define a transport kernel

$$K(x|b, f) = G_{\Sigma(b)}(x - X(b))U(b, f; x),$$

where G_Σ is a Gaussian with covariance $\Sigma(b)$ positive-definite, and U is a bounded modulation factor.

Definition 3 (Thread-weighted projection). *Given λ, φ, K, define*

$$\Phi(x) = \int_B \int_F K(x|b, f)\varphi(b, f)\lambda(b) \, d\nu_F(f) d\nu_B(b).$$

Proposition 1 (Well-posedness). *Assume $U \in L^\infty$ and $\sup_b \|G_{\Sigma(b)}\|_{L^1(\mathbb{R}^3)} < \infty$. If $\varphi \in L^2(E, \nu_E)$ and $\lambda \in L^1(B) \cap L^2(B)$, then $\Phi \in L^2(M)$ and the map $(\varphi, \lambda) \mapsto \Phi$ is continuous.*

3.2 Normalization on T^3

On $M = T^3 = [0, L)^3$ we periodize G_Σ and impose

$$\int_{T^3} G^{(T)}_\Sigma(\chi) d^3\chi = 1.$$

This ensures stationarity of the induced covariance.

4 Stationarity and the spectral transfer law on T^3

Proposition 2 (Stationarity). *With the periodized kernel and base covariance invariant under isometries of B, the induced covariance $C_\Phi(x, y)$ depends only on $x - y \in T^3$.*

Proposition 3 (Spectral transfer law). *With isotropic $\Sigma = \sigma^2 I$, one has*

$$P_\Phi(k) = |\hat{G}^{(T)}_\sigma(k)|^2 P_\psi(k), \quad \hat{G}^{(T)}_\sigma(k) = e^{-\frac{1}{2}\sigma^2\|k\|^2},$$

where P_ψ is the spectrum of the fiber-aggregated source.

5 Anisotropic kernels and director fields

Proposition 4 (Anisotropic spectral imprint)**.** *Assume n is C^1 and $\|\nabla n\|$ is uniformly small on the support of G_Σ. Let $E_B[\cdot]$ denote averaging over B with weight $\lambda\, d\nu_B$. Then*

$$P_\Phi(k) \approx \exp\left(-\tfrac{1}{2}|k|^2[\sigma_\perp^2 + (\sigma_\parallel^2 - \sigma_\perp^2)E_B[(\hat{k}\cdot n)^2]]\right) P_\psi(k).$$

6 Hopf lift: $U(1)$ connection and invariants

Let $\pi_H : S^3 \to S^2$ be the Hopf map with connection A and curvature $F = dA$. Pull back along $h : B \to S^2$ to obtain A_h. Quantized invariants $\frac{1}{2\pi}\int_B h^*F \in \mathbb{Z}$ are defined only on orientable patches of B away from self-intersection.

Definition 4 (Phase-modulated transport)**.** *Let N be a tubular neighborhood of $X(B)$ and $\rho : N \to B$ a smooth retraction. Set $A_{\mathrm{ext}} := \rho^* A_h$. For x near $X(B)$ choose the radial retraction path $\gamma(b,x) \subset N$ from $X(b)$ to x and define*

$$\Theta(b;x) := \int_{\gamma(b,x)} A_{\mathrm{ext}}, \qquad U(b,f;x) := \exp\{iq\Theta(b;x)\},$$

with $q \in \mathbb{Z}$. Radial paths are contractible in N, so Θ is well defined up to gauge; observables depend only on F_h.

7 Vector-valued outputs and helicity

Proposition 5 (Helical decomposition)**.** *For isotropic kernels, the vector covariance decomposes as*

$$\langle \hat{\Phi}_i(k)\hat{\Phi}_j(k)\rangle = P_S(k)\Pi_{ij}^{(S)}(k) + iP_H(k)\Pi_{ij}^{(H)}(k).$$

When $q \neq 0$ and $F_h \not\equiv 0$, $P_H(k) \neq 0$. If anisotropy is present, mixing corrections occur; here we restrict to isotropic analysis.

8 Identifiability and inverse problems

Proposition 6 (Identifiability (informal))**.** *If $DF(\lambda_0)$ has a bounded inverse on the chosen statistic space (e.g. band-limited L^2), then identifiability holds up to noise. A rigorous statement requires fixing the function class and noise model.*