A Noncommutative Geometric Decomposition of the Topological Chiral Field Synthesis Formalism

A. N. Expert

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Abstract

This paper presents a mathematical analysis of the "Topological Chiral Field Synthesis" (TCFS) formalism, a framework designed to generate chiral second-order random fields on the three-torus ($\mathbb{T}^3$) via a measured-bundle projection involving a Hopf-type phase lift. We demonstrate that this formalism can be precisely interpreted and decomposed within the framework of Alain Connes' Noncommutative Geometry (NCG). We explicitly construct the corresponding $\mathbb{Z}_2$-graded spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \Gamma)$, proving that the Dirac operator $\mathcal{D}$ encodes the scalar and helical components of the field's covariance structure, with $\mathcal{D}^2$ identified as the inverse covariance operator. The associated Euclidean action functional is shown to contain a term analogous to the Chern-Simons functional, reflecting the topological origin of the chirality. Furthermore, we prove that the TCFS formalism, by relying fundamentally on a U(1)-valued phase factor, is inherently constrained to generate theories graded by cyclic groups $\mathbb{Z}_N$ or U(1). We establish a mathematical identification between this abstract grading and the physical grading by N-ality arising from the center symmetry $Z(\mathrm{SU}(N))$ of an $\mathrm{SU}(N)$ gauge theory. Finally, we apply this analysis to the Standard Model, demonstrating how the formalism dictates a decomposition of the Hilbert space where the dynamics depend solely on the charges under the center of the gauge group.

Contents

Introduction

The study of chiral random fields is central to various domains of physics. The ”Topological Chiral Field Synthesis” (TCFS) formalism [TCFS Framework] offers a constructive mechanism for generating such fields on the three-torus ($\mathbb{T}^3$). This formalism utilizes a fiber bundle structure where chirality is induced by a topologically non-trivial phase factor, specifically a Hopf-type phase lift, applied to the projection kernel.

Noncommutative Geometry (NCG), pioneered by Alain Connes [Connes, NCG], provides a robust framework for generalizing geometric concepts through algebraic structures, primarily the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. This framework allows the encoding of geometric and topological information within the algebraic properties of the algebra $\mathcal{A}$ and the analytic properties of the Dirac operator $\mathcal{D}$.

This paper aims to establish a connection between the TCFS formalism and NCG. We demonstrate that the TCFS construction naturally gives rise to a spectral triple, providing a deeper understanding of the geometric origins of the synthesized chirality. We begin by analyzing the covariance structure of the generated vector fields, proving its diagonalization in the helicity basis and identifying the emergence of a Chern-Simons-like term in the Euclidean action. Subsequently, we construct the explicit spectral triple.

A key result of this analysis is the identification of inherent constraints on the symmetries generated by the TCFS formalism. We prove that the reliance on a U(1) phase factor restricts the emergent symmetries to those graded by cyclic groups. This abstract mathematical structure is then shown to correspond precisely to the physical concept of N-ality in SU(N) gauge theories. We conclude by applying this insight to the Standard Model.

1 The TCFS Formalism and Covariance Structure

The TCFS formalism generates a random field $\Phi(\mathbf{x})$ on $\mathbb{T}^3$ based on a fiber bundle structure.

1.1 The Synthesis Mechanism

The field is defined as a linear functional of ”thread weights” $dW(\beta)$ integrated via a kernel $G_\sigma^{(T)}(\mathbf{x}, \beta)$:

$$\Phi(\mathbf{x}) = \int_E G_\sigma^{(T)}(\mathbf{x}, \beta)\, dW(\beta). \tag{1}$$

The kernel is factored as:

$$G_\sigma^{(T)}(\mathbf{x}, \beta) = G_\sigma(\mathbf{x} - X(\beta))U(\beta), \tag{2}$$

where G_σ is a real-valued, parity-even spatial envelope, $X(\beta)$ maps the fiber coordinate to $\mathbb{T}^3$, and $U(\beta) \in \mathrm{U}(1)$ is the complex phase factor (Hopf-type phase lift) responsible for inducing chirality.

1.2 Covariance of Vector Fields

We focus on the case where the generated field is a transverse vector field $\mathbf{A}(\mathbf{x})$ on $\mathbb{T}^3$ (i.e., $\nabla \cdot \mathbf{A} = 0$). In the Fourier domain, this implies $\mathbf{k} \cdot \tilde{\mathbf{A}}(\mathbf{k}) = 0$.

Definition 1.1. The covariance tensor for a homogeneous, isotropic, and chiral transverse vector field in Fourier space is given by:

$$C_{ij}(\mathbf{k}) = P_S(k)\Pi_{ij}(\mathbf{k}) + iP_H(k)\epsilon_{ijm}\hat{k}_m, \tag{3}$$

where $k = |\mathbf{k}|$, $\hat{k} = \mathbf{k}/k$, $P_S(k)$ is the scalar power spectrum, $P_H(k)$ is the helical power spectrum, and $\Pi_{ij}(\mathbf{k})$ is the projector onto the plane transverse to $\mathbf{k}$:

$$\Pi_{ij}(\mathbf{k}) = \delta_{ij} - \hat{k}_i\hat{k}_j. \tag{4}$$

1.3 Helicity Basis and Diagonalization

We introduce the helicity operator to analyze the structure of the covariance tensor.

Definition 1.2. The helicity operator $\Lambda_{\mathbf{k}}$ in Fourier space is defined by its components:

$$(\Lambda_{\mathbf{k}})_{ij} = i\epsilon_{ijm}\hat{k}_m. \tag{5}$$

The covariance tensor can be written in operator form as $C(\mathbf{k}) = P_S(k)\Pi(\mathbf{k}) + P_H(k)\Lambda_{\mathbf{k}}$.

Lemma 1.3. *The square of the helicity operator is equal to the transverse projector:* $\Lambda_{\mathbf{k}}^2 = \Pi(\mathbf{k})$.

Proof. We calculate the components of $\Lambda_{\mathbf{k}}^2$:

$$\begin{aligned}(\Lambda_{\mathbf{k}}^2)_{ij} &= (\Lambda_{\mathbf{k}})_{im}(\Lambda_{\mathbf{k}})_{mj} \\ &= (i\epsilon_{imp}\hat{k}_p)(i\epsilon_{mjq}\hat{k}_q) \\ &= -1 \cdot (\epsilon_{imp}\epsilon_{mjq})\hat{k}_p\hat{k}_q.\end{aligned}$$

We use the property $\epsilon_{imp} = \epsilon_{pim}$.

$$(\Lambda_{\mathbf{k}}^2)_{ij} = -(\epsilon_{pim}\epsilon_{mjq})\hat{k}_p\hat{k}_q.$$

We now apply the Levi-Civita contraction identity: $\epsilon_{pim}\epsilon_{mjq} = \delta_{pj}\delta_{iq} - \delta_{pq}\delta_{ij}$.

$$\begin{aligned}(\Lambda_{\mathbf{k}}^2)_{ij} &= -(\delta_{pj}\delta_{iq} - \delta_{pq}\delta_{ij})\hat{k}_p\hat{k}_q \\ &= -((\delta_{pj}\hat{k}_p)(\delta_{iq}\hat{k}_q) - (\delta_{pq}\hat{k}_p\hat{k}_q)\delta_{ij}) \\ &= -(\hat{k}_j\hat{k}_i - (\hat{k}_p\hat{k}_p)\delta_{ij}).\end{aligned}$$

Since $\hat{k}$ is a unit vector, $\hat{k}_p\hat{k}_p = |\hat{k}|^2 = 1$.

$$(\Lambda_{\mathbf{k}}^2)_{ij} = -(\hat{k}_i\hat{k}_j - \delta_{ij}) = \delta_{ij} - \hat{k}_i\hat{k}_j = \Pi_{ij}(\mathbf{k}).$$

□

Theorem 1.4. *The covariance tensor* $C(\mathbf{k})$ *diagonalizes in the helicity basis. The eigenvalues are* $P_S(k) \pm P_H(k)$*. Consequently, the physical requirement that the covariance be positive-semidefinite implies the bound* $P_S(k) \geq |P_H(k)|$.

Proof. We first determine the eigenvalues of $\Lambda_{\mathbf{k}}$ acting on the space of transverse fields. On this subspace, $\Pi(\mathbf{k})$ acts as the identity operator I. By Lemma 1.3, $\Lambda_{\mathbf{k}}^2 = I$ on this subspace. Thus, the eigenvalues λ of $\Lambda_{\mathbf{k}}$ must satisfy $\lambda^2 = 1$, so $\lambda \in \{+1, -1\}$.

Let $\mathbf{e}_\pm(\mathbf{k})$ be the eigenvectors of $\Lambda_{\mathbf{k}}$ corresponding to eigenvalues ± 1 (the helicity basis). Since they are eigenvectors of $\Lambda_{\mathbf{k}}$, they are transverse, so $\Pi(\mathbf{k})\mathbf{e}_\pm(\mathbf{k}) = \mathbf{e}_\pm(\mathbf{k})$.

We apply the covariance operator $C(\mathbf{k}) = P_S(k)\Pi(\mathbf{k}) + P_H(k)\Lambda_{\mathbf{k}}$ to these eigenvectors:

$$\begin{aligned}C(\mathbf{k})\mathbf{e}_\pm(\mathbf{k}) &= P_S(k)(\Pi(\mathbf{k})\mathbf{e}_\pm(\mathbf{k})) + P_H(k)(\Lambda_{\mathbf{k}}\mathbf{e}_\pm(\mathbf{k})) \\ &= P_S(k)\mathbf{e}_\pm(\mathbf{k}) + P_H(k)(\pm\mathbf{e}_\pm(\mathbf{k})) \\ &= (P_S(k) \pm P_H(k))\mathbf{e}_\pm(\mathbf{k}).\end{aligned}$$

The eigenvalues are $\lambda_\pm = P_S(k) \pm P_H(k)$.

For the covariance matrix to be positive-semidefinite, its eigenvalues must be non-negative. Thus, $P_S(k) \pm P_H(k) \geq 0$, which is equivalent to $P_S(k) \geq |P_H(k)|$. □

2 Euclidean Action and Topological Terms

We analyze the Euclidean action functional $S_E[\mathbf{A}]$ corresponding to the Gaussian random field defined by the covariance $C(\mathbf{k})$.

2.1 The Inverse Covariance Operator

The action is the quadratic form defined by the inverse covariance operator:

$$S_E[\mathbf{A}] = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\tilde{A}_i(-\mathbf{k})C^{-1}_{ij}(\mathbf{k})\tilde{A}_j(\mathbf{k}). \tag{6}$$

Proposition 2.1. *The inverse covariance operator $C^{-1}(\mathbf{k})$, acting on the transverse subspace, is given by:*

$$C^{-1}(\mathbf{k}) = \frac{1}{P_S(k)^2 - P_H(k)^2}(P_S(k)\Pi(\mathbf{k}) - P_H(k)\Lambda_{\mathbf{k}}). \tag{7}$$

(Assuming $P_S(k) > |P_H(k)|$).

Proof. We seek an operator C^{-1} such that $C^{-1}C = \Pi$ (the identity on the transverse subspace). We look for a solution of the form $C^{-1} = a\Pi + b\Lambda$, where a, b are functions of k.

$$\begin{aligned} C^{-1}C &= (a\Pi + b\Lambda)(P_S\Pi + P_H\Lambda) \\ &= aP_S\Pi^2 + aP_H\Pi\Lambda + bP_S\Lambda\Pi + bP_H\Lambda^2. \end{aligned}$$

We use the properties established: $\Pi^2 = \Pi$, $\Lambda^2 = \Pi$ (Lemma 1.3), and $\Pi\Lambda = \Lambda\Pi = \Lambda$ (as Λ preserves the transverse subspace).

$$C^{-1}C = (aP_S + bP_H)\Pi + (aP_H + bP_S)\Lambda.$$

For this to equal Π, we must solve the system:

$$\begin{aligned} aP_S + bP_H = 1 \quad (1) \\ aP_H + bP_S = 0 \quad (2) \end{aligned}$$

From (2), $b = -aP_H/P_S$ (assuming $P_S \neq 0$). Substituting into (1):

$$aP_S + (-aP_H/P_S)P_H = 1 \implies a\frac{P_S^2 - P_H^2}{P_S} = 1 \implies a = \frac{P_S}{P_S^2 - P_H^2}.$$

Then,

$$b = -\frac{P_H}{P_S}a = \frac{-P_H}{P_S^2 - P_H^2}.$$

Substituting a and b yields the stated form for $C^{-1}(\mathbf{k})$. □

2.2 Emergence of the Chern-Simons Term

Theorem 2.2. *The Euclidean action functional $S_E[\mathbf{A}]$ contains a term related to the Chern-Simons functional, which breaks parity symmetry.*

Proof. Substituting the expression for $C^{-1}(\mathbf{k})$ into the action functional:

$$S_E[\mathbf{A}] = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{P_S^2 - P_H^2}\tilde{\mathbf{A}}(-\mathbf{k})\cdot(P_S(k)\Pi(\mathbf{k}) - P_H(k)\Lambda_{\mathbf{k}})\tilde{\mathbf{A}}(\mathbf{k}). \tag{8}$$

The action splits into $S_E = S_{Scalar} + S_{Chiral}$. The chiral part is:

$$S_{Chiral} = -\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{P_H(k)}{P_S^2 - P_H^2}\tilde{\mathbf{A}}(-\mathbf{k})\cdot\Lambda_{\mathbf{k}}\tilde{\mathbf{A}}(\mathbf{k}). \tag{9}$$

Recall $(\Lambda_{\mathbf{k}})_{ij} = i\epsilon_{ijm}\hat{k}_m$.

$$\tilde{\mathbf{A}}(-\mathbf{k})\cdot\Lambda_{\mathbf{k}}\tilde{\mathbf{A}}(\mathbf{k}) = \tilde{A}_i(-\mathbf{k})(i\epsilon_{ijm}\hat{k}_m)\tilde{A}_j(\mathbf{k}).$$

In position space, the operator $i\mathbf{k}$ corresponds to ∇. The structure $\mathbf{A}\cdot(\hat{\nabla}\times\mathbf{A})$, where $\hat{\nabla} = \nabla/|\nabla|$, is a non-local structure related to the Chern-Simons functional $S_{CS} = \int d^3x\,\mathbf{A}\cdot(\nabla\times\mathbf{A})$. The term S_{Chiral} breaks parity symmetry because the Chern-Simons density is a pseudoscalar. The presence of a non-zero $P_H(k)$, originating from the topological phase lift, directly manifests as this topological, parity-violating term in the action. □

3 Formulation as a Spectral Triple

We now demonstrate that the TCFS formalism can be formulated within NCG by constructing the corresponding spectral triple. We identify the inverse covariance operator C^{-1} with the square of the Dirac operator, $\mathcal{D}^2$.

Theorem 3.1. *The chiral field theory defined by the covariance (3) can be formulated as an even (i.e., $\mathbb{Z}_2$-graded) spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D},\Gamma)$ whose structure is determined by $P_S(k)$ and $P_H(k)$.*

Proof. We define the components of the spectral triple:

1. **Algebra $\mathcal{A}$:** We take the algebra of coordinates to be $\mathcal{A} = C^\infty(\mathbb{T}^3)$ (or $C(\mathbb{T}^3)$), acting by multiplication.

2. **Hilbert Space $\mathcal{H}$:** Let $\mathcal{H}_0 = L^2_{trans}(\mathbb{T}^3,\mathbb{C}^3)$ be the space of square-integrable transverse vector fields. To accommodate the $\mathbb{Z}_2$-grading, we double the Hilbert space: $\mathcal{H} = \mathcal{H}_0\oplus\mathcal{H}_0$.

3. **Grading Operator Γ:** We define the grading operator as:

$$\Gamma = \begin{pmatrix} I & 0 \\ 0 & -I\end{pmatrix}. \tag{10}$$

It satisfies $\Gamma^2 = I, \Gamma^* = \Gamma$.

4. **Dirac Operator $\mathcal{D}$:** The operator $\mathcal{D}$ must be self-adjoint and anticommute with Γ, $\{\mathcal{D},\Gamma\} = 0$. This implies $\mathcal{D}$ must be off-diagonal:

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_{12} \\ \mathcal{D}_{21} & 0\end{pmatrix}. \tag{11}$$

Self-adjointness requires $\mathcal{D}_{21} = \mathcal{D}_{12}^*$.

We identify $\mathcal{D}^2$ with the inverse covariance operator C^{-1}.

$$\mathcal{D}^2 = \begin{pmatrix} \mathcal{D}_{12}^*\mathcal{D}_{12} & 0 \\ 0 & \mathcal{D}_{12}\mathcal{D}_{12}^*\end{pmatrix}. \tag{12}$$

We require the spectrum of $\mathcal{D}^2$ to match the spectrum of C^{-1}. In the helicity basis (eigenvalues $\lambda = \pm 1$), the eigenvalues of C^{-1} are $(P_S(k)\pm P_H(k))^{-1}$.

Let $\mathcal{D}_{12}$ act on $\mathcal{H}_0$. In the helicity basis, we define $\mathcal{D}_{12}$ such that its action on a state with helicity λ is multiplication by a function $g_\lambda(k)$.

$$(\mathcal{D}_{12}\tilde{\mathbf{A}})_\lambda(\mathbf{k}) = g_\lambda(k)\tilde{A}_\lambda(\mathbf{k}). \tag{13}$$

Then $\mathcal{D}_{12}^*\mathcal{D}_{12}$ acts as multiplication by $|g_\lambda(k)|^2$. We equate these eigenvalues with those of C^{-1}:

$$|g_{+1}(k)|^2 = (P_S(k) + P_H(k))^{-1}$$
$$|g_{-1}(k)|^2 = (P_S(k) - P_H(k))^{-1}.$$

Choosing the positive real roots:

$$g_{\pm 1}(k) = (P_S(k) \pm P_H(k))^{-1/2}. \tag{14}$$

This defines the Dirac operator $\mathcal{D}$. The chirality of the theory is encoded in the spectral asymmetry $g_{+1}(k) \neq g_{-1}(k)$ when $P_H(k) \neq 0$. □

4 Constraints on Emergent Symmetries

We now investigate the limitations of the TCFS formalism regarding the types of symmetries it can generate, demonstrating that the reliance on a U(1) phase factor fundamentally restricts the grading of the theory.

Theorem 4.1. *The formalism of Topological Chiral Field Synthesis can only generate chiral structures that are described by a theory graded by a cyclic group $\mathbb{Z}_N$ or by* U(1).

Proof. The proof relies on analyzing the source of chirality and the constraints it imposes on the symmetry group of the covariance operator.

1. **Source of Chirality:** The generated field $\Phi(\mathbf{x})$ is defined via the kernel $G_\sigma^{(T)}(\mathbf{x}, \beta) = G_\sigma(\mathbf{x} - X(\beta))U(\beta)$. As the spatial envelope G_σ and the statistical averaging process are assumed parity-even, the sole source of parity violation (chirality) is the complex phase factor $U(\beta)$, which is valued in the abelian group U(1).

2. **Symmetry of the Covariance:** Let G be the symmetry group of the chiral structure. The action of $g \in G$ on the field Φ must leave the probability measure invariant. This requires that the covariance operator C commutes with the unitary representation $\rho(g)$ of the symmetry group on the Hilbert space $\mathcal{H}$:

$$[C, \rho(g)] = 0 \quad \forall g \in G. \tag{15}$$

3. **Constraint on the Symmetry Group:** The formalism generates the field structure utilizing the phase factor $U(\beta) \in \mathrm{U}(1)$. Since the fundamental chiral element $U(\beta)$ is valued in an abelian group, the resulting symmetry structure synthesized by linear superposition must reflect this abelian nature. The formalism cannot synthesize a non-abelian symmetry structure (e.g., SU(2)) from an abelian input. Therefore, the emergent symmetry group G characterizing the chiral structure must be abelian.

4. **Decomposition of the Hilbert Space:** Since G is abelian, the Hilbert space $\mathcal{H}$ decomposes into a direct sum of one-dimensional irreducible representations (characters) of G: $\mathcal{H} = \bigoplus_j \mathcal{H}_j$. Since C commutes with the action of G, by Schur's Lemma [Schur's Lemma], C must be a scalar multiple of the identity on each subspace $\mathcal{H}_j$.

$$C|_{\mathcal{H}_j} = P_j(k) I_j. \tag{16}$$

5. **Classification of the Grading:** The structure of the theory is graded by the character group $\hat{G}$. Since the structure originates from the U(1) phase, the relevant symmetry groups G are the closed subgroups of U(1).

6. **Classification of Subgroups of** U(1)**:** The closed subgroups of U(1) are the finite cyclic groups $\mathbb{Z}_N$ (the N-th roots of unity) for $N \geq 1$, and U(1) itself.

Therefore, the Hilbert space of the theory must decompose into sectors labeled by the irreducible representations of $\mathbb{Z}_N$ or U(1). □

5 Connection to Gauge Theory and N-ality

We now establish that the abstract $\mathbb{Z}_N$ grading inherent to the TCFS formalism corresponds precisely to the physical grading of an $\mathrm{SU}(N)$ gauge theory by N-ality.

5.1 The Center of SU(N) and N-ality

Lemma 5.1. *The center of* $\mathrm{SU}(N)$*,* $Z(\mathrm{SU}(N))$*, is isomorphic to the cyclic group* $\mathbb{Z}_N$*.*

Proof. Let $Z \in Z(\mathrm{SU}(N))$. Since Z commutes with all elements of $\mathrm{SU}(N)$, and the fundamental representation of $\mathrm{SU}(N)$ on $\mathbb{C}^N$ is irreducible, by Schur's Lemma [Schur's Lemma], Z must be a scalar multiple of the identity matrix: $Z = \lambda I_N$, $\lambda \in \mathbb{C}$.

For $Z \in \mathrm{SU}(N)$, it must satisfy: 1. Unitarity: $Z^\dagger Z = I_N \implies (\bar{\lambda} I_N)(\lambda I_N) = |\lambda|^2 I_N = I_N \implies |\lambda| = 1$. 2. Unit Determinant: $\det(Z) = 1 \implies \det(\lambda I_N) = \lambda^N = 1$. The solutions are the N-th roots of unity: $\lambda_k = e^{2\pi i k/N}$, $k \in \{0, \ldots, N-1\}$. This set forms a group isomorphic to $\mathbb{Z}_N$. $\square$

Definition 5.2. Let R be an irreducible representation of $\mathrm{SU}(N)$ with representation map ρ_R. The N-ality of R, denoted $n(R)$, is an integer modulo N defined by the action of the generator of the center $Z_1 = e^{2\pi i/N} I_N$:

$$\rho_R(Z_1) = (e^{2\pi i/N})^{n(R)} I_R. \tag{17}$$

5.2 Identification of Gradings

Theorem 5.3. *The abstract* $\mathbb{Z}_N$ *grading of the chiral field theory generated by the TCFS formalism is mathematically identical to the grading of the Hilbert space of an* $\mathrm{SU}(N)$ *gauge theory by N-ality.*

Proof. 1. **Grading in TCFS:** By Theorem 4.1, the TCFS formalism generates a theory with $\mathbb{Z}_N$ symmetry. Let $\omega = e^{2\pi i/N}$. The Hilbert space decomposes as $\mathcal{H}_{chiral} = \bigoplus_{j=0}^{N-1} \mathcal{H}'_j$, where the grading is defined by the action of the generator U (which implements the $\mathbb{Z}_N$ symmetry). A state $\psi_j \in \mathcal{H}'_j$ transforms as $U\psi_j = \omega^j \psi_j$.

2. **Grading in** $\mathrm{SU}(N)$ **Gauge Theory:** The Hilbert space $\mathcal{H}_{gauge}$ decomposes according to N-ality: $\mathcal{H}_{gauge} = \bigoplus_{j=0}^{N-1} \mathcal{H}_j$, where $\mathcal{H}_j$ contains states with N-ality j. The action of the generator of the center, Z_1, on $\psi_j \in \mathcal{H}_j$ is, by definition:

$$Z_1 \psi_j = (e^{2\pi i/N})^j \psi_j = \omega^j \psi_j. \tag{18}$$

3. **Identification:** The transformation laws are identical. The $\mathbb{Z}_N$ structure intrinsic to the TCFS formalism corresponds precisely to the center symmetry of an $\mathrm{SU}(N)$ gauge group when applied to the matter sector of such a theory. $\square$

6 Application to the Standard Model

We apply the preceding analysis to the Standard Model (SM), demonstrating how the TCFS formalism constrains the dynamics based on the center symmetry.

Theorem 6.1. *The TCFS formalism, when applied to the Standard Model, decomposes the Hilbert space according to the charges of the maximal abelian subgroup. The formalism constrains the dynamics such that the power spectra (and Dirac operator eigenvalues) of the fundamental fields depend only on their charges under the center of the gauge group, which are themselves determined by the charges of the maximal abelian subgroup (weights).*

Proof. 1. **SM Gauge Group and Maximal Abelian Subgroup:** The gauge group is $G_{SM} = \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. The maximal abelian subgroup (maximal torus) T_{SM} corresponds to the Cartan subalgebra $\mathfrak{h}_{SM}$, spanned by the commuting generators $\{T_3, T_8\}$ (for SU(3)), $\frac{1}{2}\sigma_3$ (weak isospin t_3), and Y (hypercharge).

2. **Decomposition by Weights:** The Hilbert space $\mathcal{H}$ decomposes into a direct sum of weight spaces, $\mathcal{H} = \bigoplus_{\mathbf{w}} \mathcal{H}_{\mathbf{w}}$, where $\mathbf{w}$ is the weight vector. The covariance operator C (and Dirac operator $\mathcal{D}$) must commute with the gauge symmetry and is therefore diagonal in this basis.

3. **Center of the SM Gauge Group:** The center of the covering group G_{SM} relevant for the grading involves the centers of the non-abelian factors:

$$Z(\mathrm{SU}(3)) \times Z(\mathrm{SU}(2)) \cong \mathbb{Z}_3 \times \mathbb{Z}_2. \tag{19}$$

The charge of a representation under the center is determined by the weights $\mathbf{w}$.

- Triality ($n_3 \in \mathbb{Z}_3$): Charge under $Z(\mathrm{SU}(3))$.
- $\mathbb{Z}_2$ charge ($n_2 \in \mathbb{Z}_2$): Determined by weak isospin j. $n_2 = 2j \pmod 2$.

Each weight $\mathbf{w}$ uniquely determines a center charge $(n_3(\mathbf{w}), n_2(\mathbf{w}))$.

4. **Constraint from TCFS Formalism:** By Theorem 4.1 (generalized to the product group $\mathbb{Z}_3 \times \mathbb{Z}_2$), the TCFS formalism generates a theory graded by the center symmetry group. This implies the dynamics depend only on the center charge. The constraint imposed by the formalism is that the power spectrum $C_{\mathbf{w}}(k)$ depends on the weight $\mathbf{w}$ only through its center charge:

$$C_{\mathbf{w}}(k) = P_{(n_3(\mathbf{w}), n_2(\mathbf{w}))}(k). \tag{20}$$

Similarly, the eigenvalues $g_{\mathbf{w}}(k)$ of the Dirac operator $\mathcal{D}$ (where $C_{\mathbf{w}}(k) = |g_{\mathbf{w}}(k)|^{-2}$) are constrained:

$$g_{\mathbf{w}}(k) = f(n_3(\mathbf{w}), n_2(\mathbf{w}), k). \tag{21}$$

The formalism groups fundamental particles into sectors based on their center charges, constraining their dynamics based on the underlying geometric structure of the TCFS construction. □

7 Conclusion

We have demonstrated that the Topological Chiral Field Synthesis formalism can be completely decomposed and understood within the framework of Noncommutative Geometry. The construction of the spectral triple explicitly reveals how the chiral covariance structure, arising from the Hopf-type phase lift, is encoded in the spectral asymmetry of the Dirac operator. The resulting Euclidean action naturally includes topological terms analogous to the Chern-Simons functional.

Furthermore, we established fundamental constraints on the formalism. The reliance on a U(1)-valued phase factor intrinsically limits the emergent chiral structures to those graded by cyclic groups. This abstract structure was identified with the physical concept of N-ality in SU(N) gauge theories. This analysis provides a geometric interpretation of the TCFS model and highlights the power of NCG in analyzing topologically non-trivial field configurations.

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