Chiral Field Generation from Measured Bundles

Threads, Helical Transport, and Torus Power Spectra

Abstract

We present a compact mechanism to generate mirror-asymmetric (chiral) random fields in three dimensions using a measured-bundle projection with thread weights. Microscopic data live on a base space B and are transported to a cubic three-torus $\mathbb{T}^3 = [0, L)^3$ by a kernel that is the product of a spatial envelope and a $U(1)$ phase. A Hopf-type phase lift supplies nontrivial holonomy, which induces a helical part of the Fourier-space covariance. With a finite, normalized fiber measure and a uniform pushforward of the thread density, the scalar power spectrum obeys the transfer law $P_\Phi(\mathbf{k}) = |\widehat{G}_\sigma^{(T)}(\mathbf{k})|^2 P_\psi(\mathbf{k})$. For vector outputs, the covariance splits into scalar and helical pieces with positivity bound $P_S(k) \geq |P_H(k)|$. We also record an optional anisotropic kernel that imprints angle dependence while preserving the same envelope.

1 Measured-bundle projection with threads

Definition 1 (Measured bundle and threads)**.** Let $\pi : E \to B$ be a measurable fiber bundle with typical fiber F. Equip B with a base measure ν_B and F with a fiber measure ν_F. A thread density is a function $\lambda \in L^1(B, \nu_B)$, and the induced measure on E is

$$\mathrm{d}\nu_E(b, f) := \lambda(b)\, \mathrm{d}\nu_B(b)\, \mathrm{d}\nu_F(f).$$

A microscopic fiber observable is $\psi \in L^2(E, \nu_E)$.

Assumption 1 (Finite fiber and normalization)**.** The fiber measure is finite and normalized, $\nu_F(F) = 1$.

Let $X : B \to \mathbb{T}^3$ be a measurable embedding of the base into the target torus. We transport fiber data to a target field $\Phi : \mathbb{T}^3 \to \mathbb{C}$ (or $\Phi : \mathbb{T}^3 \to \mathbb{C}^3$) via

$$\Phi(\mathbf{x}) = \int_B \int_F K(b, f; \mathbf{x})\, \psi(b, f)\, \mathrm{d}\nu_F(f)\, \lambda(b)\, \mathrm{d}\nu_B(b). \tag{1}$$

Definition 2 (Kernel structure)**.** We choose a kernel that factors into an envelope and a phase

$$K(b, f; \mathbf{x}) \;=\; U(b, f; \mathbf{x})\, G_\sigma^{(T)}(\mathbf{x} - X(b)).$$

Here $G_\sigma^{(T)}$ is the periodized Gaussian on $\mathbb{T}^3$ with width $\sigma > 0$, and U is a $U(1)$ phase. The Fourier transform on $\mathbb{T}^3$ uses $\mathbf{k} = \frac{2\pi}{L}\mathbb{Z}^3$.

The periodized Gaussian has transform

$$\widehat{G}_\sigma^{(T)}(\mathbf{k}) = \exp\Big(- \tfrac{1}{2}\sigma^2 \|\mathbf{k}\|^2 \Big). \tag{2}$$

Assumption 2 (Uniform pushforward of threads)**.** The pushforward of $\lambda\, \mathrm{d}\nu_B$ under X is spatially uniform:

$$X_*(\lambda\, \mathrm{d}\nu_B) \;=\; \frac{\Lambda}{L^3}\, \mathrm{d}^3\mathbf{x} \quad \text{on } \mathbb{T}^3,$$

for some constant $\Lambda > 0$.

2 Stationarity and the scalar transfer law

Assume ψ is a centered, second-order field over E that is independent of the torus position $\mathbf{x}$ conditional on (b, f), and that the kernel uses a phase U with unit modulus. Write $C_\psi((b, f), (b', f')) := \mathbb{E}[\psi(b, f)\,\overline{\psi(b', f')}]$.

Proposition 1 (Stationarity and transfer)**.** *Under Assumptions 1 and 2, the scalar output Φ defined in (1) is stationary on $\mathbb{T}^3$. Its power spectrum satisfies*

$$P_\Phi(\mathbf{k}) \;=\; |\widehat{G}_\sigma^{(T)}(\mathbf{k})|^2\, P_\psi(\mathbf{k}),$$

where $P_\psi(\mathbf{k})$ is the spectrum induced by C_ψ pushed forward by X. In particular, when microscopic correlations are white after pushforward, $P_\Phi(\mathbf{k}) = \Lambda\, |\widehat{G}_\sigma^{(T)}(\mathbf{k})|^2$.

Proof sketch. Translation of $\mathbf{x}$ acts only through $G_\sigma^{(T)}(\mathbf{x} - X(b))$. The covariance of Φ becomes a double integral over $B \times F$ that depends on $\mathbf{x} - \mathbf{y}$ once one uses $\nu_F(F) = 1$ and pushes forward $\lambda\, \mathrm{d}\nu_B$ with Assumption 2. The Fourier transform then pulls out $|\widehat{G}_\sigma^{(T)}(\mathbf{k})|^2$, giving the stated transfer law. ☐

3 Hopf lift and parity breaking

To break mirror symmetry we endow the kernel with a nontrivial phase.

Definition 3 (Hopf-type phase)**.** Let $h : B \to S^2$ be smooth. Pull back the $U(1)$ Hopf connection A on S^2 to B and push it forward to $\mathbb{T}^3$ along X. Let a be a smooth 1-form on $\mathbb{T}^3$ representing this pushed connection. For $q \in \mathbb{R}$ define

$$U(b, f; \mathbf{x}) \;:=\; \exp\!\Big(iq \int_{\gamma_{X(b)\to\mathbf{x}}} a\Big),$$

where $\gamma_{X(b)\to\mathbf{x}}$ is any straight segment on $\mathbb{T}^3$ from $X(b)$ to $\mathbf{x}$. Its holonomy is gauge-invariant, and the total curvature obeys

$$\frac{1}{2\pi}\int_B h^*(\mathrm{d}A) \;=\; m \in \mathbb{Z}.$$

For vector outputs $\Phi : \mathbb{T}^3 \to \mathbb{C}^3$ we write the spectral covariance as

$$\big\langle \widehat{\Phi}_i(\mathbf{k})\, \overline{\widehat{\Phi}_j(\mathbf{k})} \big\rangle = P_S(k)\, \Pi_{ij}^{(S)}(\mathbf{k}) + i\, P_H(k)\, \Pi_{ij}^{(H)}(\mathbf{k}), \tag{3}$$

with $k = \|\mathbf{k}\|$, $\Pi_{ij}^{(S)} = \delta_{ij} - \hat{k}_i \hat{k}_j$, and $\Pi_{ij}^{(H)} = \varepsilon_{ijl} \hat{k}_l$.

Proposition 2 (Helical power and positivity)**.** *If $q \neq 0$ and $m \neq 0$, then the phase induces a nonzero helical spectrum $P_H(k)$ supported by the same Gaussian envelope as $P_S(k)$. Moreover,*

$$P_S(k) \;\geq\; |P_H(k)| \quad \textit{for all } k,$$

and the left/right helical powers are $P_\pm(k) = P_S(k) \pm P_H(k)$.

Idea of proof. The nontrivial holonomy couples the microscopic fiber modes to the transverse projector and inserts a pseudoscalar that is odd under parity. After Fourier transform, the covariance acquires an antisymmetric part proportional to $\varepsilon_{ijl}\hat{k}_l$. Positivity of the covariance operator on the transverse subspace implies the stated bound. ☐

Definition 4 (Chirality fraction)**.** Define $\chi(k) := \dfrac{P_H(k)}{P_S(k)} \in [-1,1]$. Then $P_\pm(k) = P_S(k)\,[1 \pm \chi(k)]$.

Remark 1 (Envelope and shape)*.* The scalar part follows the transfer law $P_S(k) = |\widehat{G}_\sigma^{(T)}(\mathbf{k})|^2\, P_\psi(k)$, and $P_H(k)$ shares the same envelope, with amplitude controlled by q and the Chern number m.

4 Optional anisotropic kernel

To imprint angle dependence, replace the isotropic Gaussian by an anisotropic one. Let $n : B \to S^2$ be a director field and choose $\sigma_\parallel, \sigma_\perp > 0$. Set

$$G^{(T)}_{\Sigma(b)}(\mathbf{x} - X(b)) \;\propto\; \exp\Big(-\tfrac{1}{2}(\mathbf{x} - X(b))^\top \Sigma(b)^{-1}(\mathbf{x} - X(b))\Big), \quad \Sigma(b) = \sigma_\perp^2 I + (\sigma_\parallel^2 - \sigma_\perp^2)\, n(b)n(b)^\top.$$

Then

$$P_\Phi(\mathbf{k}) \approx \exp\Big(-\tfrac{1}{2}k^2\big[\sigma_\perp^2 + (\sigma_\parallel^2 - \sigma_\perp^2)\,\mathbb{E}_B[(\hat{\mathbf{k}}\cdot n)^2]\big]\Big)\, P_\psi(\mathbf{k}),$$

which is parity-even by itself, yet it modulates $\chi(k)$ when combined with the Hopf phase.

5 Diagnostics and minimal recipe

Spectral helicity. For any realization,

$$H(\mathbf{k}) := \mathrm{Im}\Big[\widehat{\Phi}(\mathbf{k})^{\cdot\,(\hat{\mathbf{k}}\times\widehat{\Phi}(\mathbf{k}))}\Big], \qquad \mathbb{E}[H(\mathbf{k})] = P_H(k).$$

Minimal implementation. Choose B, X, λ, σ. Pick h with $\frac{1}{2\pi}\int_B h^*(\mathrm{d}A) = m \neq 0$ and a coupling q. Use the kernel above to synthesize Φ. In a helical basis $\{e_\pm(\mathbf{k})\}$, draw complex Gaussians $a_\pm(\mathbf{k})$ with variances $P_\pm(k)$ and set $\widehat{\Phi}(\mathbf{k}) = a_+e_+(\mathbf{k}) + a_-e_-(\mathbf{k})$ with the reality condition $\widehat{\Phi}(-\mathbf{k}) = \widehat{\Phi}(\mathbf{k})$.

Appendix A: Periodized Gaussian on $\mathbb{T}^3$

Let $g_\sigma(\mathbf{x}) = (2\pi\sigma^2)^{-3/2}\exp(-\|\mathbf{x}\|^2/2\sigma^2)$ on $\mathbb{R}^3$. The periodization on $\mathbb{T}^3$ is

$$G_\sigma^{(T)}(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^3} g_\sigma(\mathbf{x} + L\mathbf{n}),$$

with Fourier coefficients $\widehat{G}_\sigma^{(T)}(\mathbf{k}) = \exp(-\frac{1}{2}\sigma^2\|\mathbf{k}\|^2)$ for $\mathbf{k} = \frac{2\pi}{L}\mathbb{Z}^3$.

Appendix B: Assumptions at a glance

Finite fiber measure $\nu_F(F) = 1$. Uniform pushforward $X_*(\lambda\,\mathrm{d}\nu_B) = (\Lambda/L^3)\,\mathrm{d}^3\mathbf{x}$. Unit-modulus phase U. These ensure stationarity and the Gaussian transfer envelope; adding the Hopf phase yields a controlled, physically interpretable parity violation.