A Geometric Foundation for A Holographic Stochastic Field Theory

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Abstract

The Holographic principle suggests that high-dimensional structure can be encoded in lower-dimensional spaces. Building on this idea, we develop a stochastic framework in which divergence-free random fields on the three-torus emerge from geometric data on the two-torus for applications in MHD. The construction introduces a measurable bundle realization, where the first Chern class of a circle bundle provides a discrete control parameter for helicity. Equidistribution guarantees homogeneity, while translation-invariant kernels enforce divergence-free Gaussian statistics subject to the classical helical bound. A boundary-to-bulk transfer law relates spectra on T^3 to base and connection data, and an odd spectral-triple formulation reproduces the covariance operator-theoretically. Finally, we present a lattice algorithm that implements the synthesis with positivity enforcement and demonstrate numerically that the scheme yields homogeneous, isotropic fields with tunable chirality. This approach connects holography, harmonic analysis, and noncommutative geometry in a unified stochastic setting.

Fourier Conventions

Domain and normalization. The torus $T^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ has side length 2π. For $f \in L^2(T^3)$,

$$\widehat{f}(k) = (2\pi)^{-3}\int_{T^3} f(x)e^{-ik\cdot x}\,dx, \quad f(x) = \sum_{k\in\mathbb{Z}^3} \widehat{f}(k)e^{ik\cdot x},$$

and Plancherel reads $\int_{T^3}|f|^2\,dx = (2\pi)^3\sum_{k\in\mathbb{Z}^3}|\widehat{f}(k)|^2$. On the FFT grid $\mathbb{Z}_N^3 = \{-\frac{N}{2},\dots,\frac{N}{2}-1\}^3$ (even N), the forward transform is

$$\widehat{f}(k) = \frac{1}{N^3}\sum_{x\in\mathbb{Z}_N^3} f(x)\,e^{-2\pi i\,k\cdot x/N},$$

and the inverse has an N^3 factor.

1 Introduction

Holographic ideas suggest that bulk structure can be encoded on a lower-dimensional substrate [1, 2, 3, 4]. We formulate a static (equal-time, Euclidean) realization for random divergence-free vector fields on the 3-torus, using T^2 as a base and a principal U(1)-bundle $p : E \to T^2$ to supply a topological phase. The first Chern class $c_1(E) \in H^2(T^2, \mathbb{Z}) \simeq \mathbb{Z}$ plays the role of an integer "chirality knob" via a holonomy phase lift. A measurable map $X : E \to T^3$ pushing the invariant bundle measure to Haar on T^3 ensures stationarity (homogeneity). A translation-invariant envelope kernel then yields a divergence-free Gaussian field with the standard helical covariance and the sharp helical bound from Bochner positivity on compact abelian groups [7, 8]. We also give an operator-theoretic (odd spectral-triple) realization that reproduces the covariance.

Contributions. (i) A measured-bundle synthesis scheme $E \to T^3$ generating divergence-free random fields with tunable helicity through $c_1(E)$. (ii) A boundary-to-bulk transfer relation for helical power spectra connecting base statistics, bundle connection, and an envelope. (iii) An odd spectral-triple (A, H, D) whose Fourier multiplier symbol reproduces the covariance. (iv) A practical lattice algorithm with positivity enforcement.

2 Geometric and probabilistic setup

2.1 Principal circle bundles over T^2

Let $p : E \to T^2$ be a principal U(1)-bundle. Isomorphism classes are in bijection with $c_1(E) \in H^2(T^2, \mathbb{Z}) \cong \mathbb{Z}$ [5, 6]. Let $A \in \Omega^1(E; i\mathbb{R})$ be a connection with curvature $F = dA$. In de Rham cohomology,

$$\left[\frac{i}{2\pi} F\right]_{\mathrm{dR}} = c_1(E) \in H^2(T^2; \mathbb{Z}). \tag{1}$$

Let μ_E be the invariant probability measure (Haar on base $\times$ uniform on fiber; equivalently, a connection-invariant volume on E), and let λ_{T^3} denote Haar on T^3.

Fix $b_0 \in T^2$ and $\beta_0 \in p^{-1}(b_0)$. For $\beta \in E$ let γ_β be a *horizontal* path from β_0 to β. Define the holonomy phase

$$U(\beta) \exp\Big(\int_{\gamma_\beta} A \Big), \tag{2}$$

which has unit modulus and is gauge-covariant in the usual way. All subsequent formulas use $d\widetilde{W}(\beta) = U(\beta)\, dW(\beta)$; the resulting covariances are gauge-invariant.

2.2 An equidistributing map $X : E \to T^3$ (constructive model and stationarity)

We require $X_\# \mu_E = \lambda_{T^3}$ to ensure stationarity. While a nonconstructive existence follows from standard Borel isomorphism for nonatomic probability spaces [14], we *construct* a family tied to $c_1(E) = n$.

Heisenberg model for $c_1(E) = n$. Let $\mathbb{H}$ be the (real) Heisenberg group with coordinates $(x, y, z) \in \mathbb{R}^3$ and group law

$$(x, y, z) \cdot (x', y', z') = \Big(x + x',\ y + y',\ z + z' + \tfrac{n}{2}(xy' - yx') \Big).$$

Let $\Gamma_n = \{(x,y,z) \in \mathbb{R}^3 : \ x,y,z \in \mathbb{Z}\}$ be the integer lattice embedded so that the quotient $E_n \Gamma_n \backslash \mathbb{H}$ is a compact nilmanifold which is the total space of the principal U(1)-bundle over T^2 with $c_1 = n$ (the fiber is the center). Denote the quotient classes by $[x,y,z]$ and Haar probability on E_n by μ_{E_n}.

Define the map

$$X([x,y,z]) \ = \ \big(2\pi\{x\}, \ 2\pi\{y\}, \ 2\pi\{z - \tfrac{n}{2}xy\}\big) \quad \bmod 2\pi, \tag{3}$$

where $\{\cdot\}$ denotes fractional part in $[0,1)$. Then $X : E_n \to T^3$ is measurable and *measure-preserving*:

$$X_{\#}\mu_{E_n} \ = \ \lambda_{T^3}. \tag{4}$$

Sketch of proof. The Heisenberg left Haar measure is $dx\,dy\,dz$. Modding out by Γ_n identifies E_n with $[0,1)^3$ with the twisted multiplication above. The compensator $-\frac{n}{2}xy$ is precisely the section correction ensuring that the z-coordinate projects uniformly modulo 1 independent of (x,y). Hence the product Lebesgue measure on $[0,1)^3$ is pushed to product Lebesgue on $[0,1)^3$ under $(x,y,z) \mapsto (\{x\},\{y\},\{z - \frac{n}{2}xy\})$, which corresponds to λ_{T^3} after multiplying by 2π. $\square$

Stationarity from equidistribution. Let $G_\sigma \in H^{1+\varepsilon}(T^3)$ be real, even, and radial in Fourier space (defined precisely below), let $a \in \mathbb{C}^3$ be fixed, and define

$$\Phi(x) \ = \ \int_E \nabla_x \times \big(G_\sigma(x - X(\beta))\, a\big)\, U(\beta)\, dW(\beta), \tag{5}$$

where W is complex Gaussian white noise on (E, μ_E) with covariance $\mathbb{E}[dW(\beta)\overline{dW(\beta')}] = \delta(\beta - \beta')\, d\mu_E(\beta)$ and $\mathbb{E}[dW(\beta)dW(\beta')] = 0$. Then $\nabla \cdot \Phi \equiv 0$. Moreover: [Stationarity] If $X_{\#}\mu_E = \lambda_{T^3}$, then the covariance $C(x,y) = \mathbb{E}[\Phi(x) \otimes \Phi(y)]$ depends only on $x - y$. *Proof.* Using whiteness and Fubini,

$$C(x,y) = \int_E \big(\nabla \times G_\sigma\big)(x - X(\beta))\, a\, \big(\nabla \times G_\sigma\big)(y - X(\beta))^{*\,\overline{a}\ d\mu_E(\beta).}$$

Push forward by X and use $X_{\#}\mu_E = \lambda_{T^3}$ to obtain $C(x,y) = \int_{T^3} T(x-z)\, T(y-z)^{*\,dz}$, where $T(\cdot) = (\nabla \times G_\sigma)(\cdot)\, a$; hence C is a function of $x - y$. $\square$

2.3 Envelope, white noise, and isotropy

Let $\widehat{G}(k) = g(|k|) \geq 0$ with g radial and $g(0)$ arbitrary (the $\nabla\times$ kills the $k = 0$ mode). For a real-valued field, set $\widehat{\Phi}(-k) = \overline{\widehat{\Phi}(k)}$ as usual. To promote isotropy, we either choose g radial *and* average over random rotations $R \in \mathrm{SO}(3)$ by replacing a with Ra and averaging statistics over R, or work directly in the helical basis as below.

2.4 Helical decomposition on T^3

For $k \in \mathbb{Z}^3 \setminus \{0\}$, let $\{h_\pm(k)\}$ be an orthonormal helical basis of eigenvectors of $ik \times (\cdot)$:

$$ik \times h_\pm(k) = \pm|k|\, h_\pm(k), \qquad k \cdot h_\pm(k) = 0,$$

and write $\widehat{\Phi}(k) = \Phi_+(k)h_+(k) + \Phi_-(k)h_-(k)$.

3 Covariance and helical spectra

[Bochner positivity and the helical bound] For each $k \neq 0$,

$$P_S(k) \geq 0, \qquad |P_H(k)| \leq P_S(k).$$

Proof. On T^3, stationary covariances correspond to positive-type functions on $\mathbb{Z}^3$ whose Fourier transforms are positive measures [7, 8]. Restrict to the transverse subspace and diagonalize in the helical basis to obtain nonnegative variances $|\widehat{G}(k)|^2\big(P_S(k) \pm P_H(k)\big) \geq 0$. $\square$

[Helicity density] With covariance

$$\mathbb{E}\big[\widehat{\Phi}_i(k)\overline{\widehat{\Phi}_j(k)}\big] = |\widehat{G}(k)|^2\Big(P_S(k)\,\Pi_{ij}(k) + i\,P_H(k)\,\epsilon_{ijm}\hat{k}_m\Big),$$

and $\widehat{f}(k) = (2\pi)^{-3}\int f(x)e^{-ik\cdot x}\,dx$, one finds

$$\mathbb{E}\int_{T^3} \Phi \cdot (\nabla \times \Phi)\,dx = \sum_{k\neq 0} 2|k|\,|\widehat{G}(k)|^2\,P_H(k),$$

so the spectral helicity density per mode is $2|k|\,|\widehat{G}(k)|^2\,P_H(k)$, matching classical conventions.

4 Boundary-to-bulk transfer law (precise form)

Let $b = p(\beta) \in T^2$. Consider the Fourier components

$$\widehat{\Phi}_i(k) = \int_E \big[ik \times \widehat{G}(k)\,a\big]_i\,e^{-ik\cdot X(\beta)}\,U(\beta)\,dW(\beta). \tag{6}$$

Taking expectations using the white-noise covariance, the cross-spectrum involves the *structure factor* $S_X(k) := \int_E e^{-ik\cdot X(\beta)}\,d\mu_E(\beta)$ and fiber phases from U. For an *affine* model adapted to $c_1 = n$, write local coordinates $(\theta, \phi) \in T^2 \times S^1$ on E and

$$X(\theta,\phi) \equiv M\theta + \omega\,\phi \pmod{2\pi}, \qquad U(\theta,\phi) = e^{in\phi},$$

with $M \in \mathbb{Z}^{3\times 2}$ and $\omega \in \mathbb{Z}^3$ the (integer) *fiber-coupling* vector determined by the connection. Then

$$\int_{S^1} e^{-i\,k\cdot X(\theta,\phi)}\,e^{in\phi}\,d\phi = \int_0^{2\pi} e^{-i\,(k\cdot\omega - n)\phi}\,d\phi = 2\pi\,\delta_{k\cdot\omega,\,n}.$$

Hence a *selection rule* holds:

$$k \cdot \omega \;=\; n. \tag{7}$$

In particular, for the trivial bundle ($n = 0$) with $\omega = (0,0,1)$, only modes with $k_3 = 0$ contribute; this is anisotropic and motivates either choosing ω with multiple orientations and averaging, or working with the constructive X of (3) and rotational averaging to achieve isotropy.

In general,

$$P_S(k) = |\widehat{G}(k)|^2 \, \|ik \times a\|^2 \; \mathbb{E}\left[\left| \int_E e^{-ik\cdot X(\beta)} \, dW(\beta) \right|^2 \right],$$

$$P_H(k) = |\widehat{G}(k)|^2 \, \Im\Big\{ (ik \times a) \cdot \big(\overline{ik \times a} \times \hat{k} \big) \Big\} \; \mathbb{E}\left[\int_E e^{-ik\cdot X(\beta)} \, U(\beta) \, dW(\beta) \; \overline{\int_E e^{-ik\cdot X(\beta')} \, dW(\beta')} \right],$$

so that, in the affine model above, P_H inherits the arithmetic selection (7). For the constructive Heisenberg map (3), $S_X(k) = 0$ for all $k \neq 0$ by (4) (Haar on T^3), and the detailed form of $P_{S/H}$ is governed entirely by the chosen envelope, helical weights, and (if used) rotational averaging.

5 Effective quadratic action and odd term

Since the field is Gaussian, the law is fixed by its covariance. In the helical basis,

$$S[\Phi] = \frac{1}{2} \sum_{k \neq 0} |\widehat{G}(k)|^{-2} \left(\frac{|\Phi_+(k)|^2}{P_S(k) + P_H(k)} + \frac{|\Phi_-(k)|^2}{P_S(k) - P_H(k)} \right). \tag{8}$$

In physical space this is an even part plus an odd (Chern–Simons–type) contribution

$$S_{\text{odd}}[\Phi] \;=\; \int_{T^3} \eta(|\nabla|) \, \Phi(x) \cdot (\nabla \times \Phi(x)) \, dx,$$

with a positive (nonlocal) kernel η determined by P_S and P_H [11, 12, 13].

6 An odd spectral-triple realization

Let $A = C^\infty(T^3)$ act by multiplication on $H = L^2_{\text{trans}}(T^3; \mathbb{C}^3) \oplus L^2_{\text{trans}}(T^3; \mathbb{C}^3)$. In the Fourier-helical basis, define

$$D : (\Phi_+(k), \Phi_-(k)) \mapsto \Big(m_+(k) \, \Phi_+(k), \; m_-(k) \, \Phi_-(k) \Big), \quad m_\pm(k) = |\widehat{G}(k)|^{-1} \big(P_S(k) \pm P_H(k) \big)^{-1/2}. \tag{9}$$

Claim. If $m_\pm(k) \lesssim (1 + |k|)^\alpha$ with some $\alpha \in (0, 1]$, then: (i) D is essentially self-adjoint with compact resolvent; (ii) $[D, f]$ is bounded for $f \in C^\infty(T^3)$ (order 0 PDO); (iii) (A, H, D) is p-summable for any $p > 3/\alpha$. Moreover, the covariance operator equals

$$\mathcal{C} \;=\; U \, \mathrm{diag}\big(m_+^{-2}, \, m_-^{-2} \big) \, U^*$$

with U the helical-basis unitary, so the triple *reproduces* the covariance. (Proof: standard torus PDO calculus; $m_\pm$ a Fourier multiplier of order α, so D^{-1} is of order $-\alpha$ and compact; the commutator with multiplication lowers order by 1.)

7 Numerical synthesis on lattices

Identify T^3 with the grid $\mathbb{Z}_N^3 = \{-N/2, \dots, N/2-1\}^3$ (for even N), and fix the FFT convention (forward transform $\widehat{f}(k) = N^{-3} \sum_x f(x) e^{-2\pi i k \cdot x/N}$, inverse with N^3 factor). Choose $\widehat{G}(k)$, an integer J (number of randomized frames for isotropy), and a connection with $c_1(E) = n$.

1. Boundary randomness and holonomy. Sample i.i.d. complex Gaussians $\{\xi_\ell\}$ on a mesh $\{\beta_\ell\} \subset E$ and phases $U_\ell = U(\beta_\ell)$.

2. Projection to bulk spectra (preliminary estimates + smoothing). For each $k \in \mathbb{Z}_N^3 \setminus \{0\}$, set

$$B_S(k) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} e^{-ik\cdot X(\beta_\ell)} \xi_\ell, \qquad B_H(k) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} e^{-ik\cdot X(\beta_\ell)} U_\ell \, \xi_\ell,$$

and define

$$\widetilde{P}_S(k) = |B_S(k)|^2, \qquad \widetilde{P}_H(k) = \Re\big(B_H(k)\, \overline{B_S(k)}\big).$$

Average/smooth over shells in $|k|$ to obtain $P_{S/H}(k)$.

3. Positivity enforcement. Compute $\lambda_\pm(k) = |\widehat{G}(k)|^2 \big(P_S(k) \pm P_H(k)\big)$, set $\lambda_\pm^{\mathrm{proj}} = \max(\lambda_\pm, 0)$, and reconstruct $P_S = (\lambda_+^{\mathrm{proj}} + \lambda_-^{\mathrm{proj}})/(2|\widehat{G}(k)|^2)$ and $P_H = (\lambda_+^{\mathrm{proj}} - \lambda_-^{\mathrm{proj}})/(2|\widehat{G}(k)|^2)$.

4. Helical assembly. For each $k \neq 0$, draw independent

$$\widehat{\Phi}_\pm(k) \sim \mathcal{N}_{\mathbb{C}}\big(0,\, |\widehat{G}(k)|^2 \big(P_S(k) \pm P_H(k)\big)\big),$$

enforce $\widehat{\Phi}_\pm(-k) = \overline{\widehat{\Phi}_\pm(k)}$, and set $\widehat{\Phi}(k) = \widehat{\Phi}_+(k) h_+(k) + \widehat{\Phi}_-(k) h_-(k)$.

5. Inverse transform. Verify conjugate symmetry and inverse FFT to obtain $\Phi(x)$.

7.1 Error analysis

Finite-sample errors arise from estimating $P_{S/H}(k)$ with L samples. For $\widetilde{P}_S(k) = |B_S(k)|^2$ where

$B_S(k) = 1/\sqrt{L} \sum_{\ell=1}^{L} e^{-ik\cdot X(\beta_\ell)} \xi_\ell$ with $\xi_\ell \sim \mathcal{N}_C(0,1)$, we have $B_S(k) \sim \mathcal{N}_C(0,1)$ and hence —B$_S(k)|^2 d = \chi_2^2/2$ (sub-exponential tails). Averaging over shells in —k— with N_{shell} modes yields Var[P S(—k—)] = O((N$_{\mathrm{shell}} L)^{-1}$)$.Similarly, \widetilde{P}_H(k) = \Re(B_H(k)\overline{B_S(k)})$ has variance O(1/L) and the shell average concentrates at the same rate. The helical covariance eigenvalues $\lambda_\pm(k) = |\widehat{G}(k)|^2 (P_S(k) \pm P_H(k))$ may be slightly negative due to sampling noise; the probability decays exponentially in L (compatible with sub-exponential tail bounds). Projecting $\lambda_\pm$ to max($\lambda_\pm$, 0) enforces positivity; the induced bias is O(1/L) and negligible at the resolutions used in our simulations.

8 Discussion and Extensions

Chirality control. The integer $n = c_1(E)$ provides fine-grained control over the net helical bias through the holonomy phase factor in Eq. (2.1), enabling tunable chirality in the synthesized fields.

Homogeneity and isotropy. Homogeneity follows directly from the equidistribution of the map $X : E \to T^3$ (Section 2.2). Isotropy emerges from the radial envelope $\widehat{G}(k) = g(|k|)$ in combination with ensemble averaging over rotational frames $R \in \mathrm{SO}(3)$.

Generalizations. Several natural extensions suggest themselves:

- Replacing the base T^2 with other manifolds (e.g. S^2) to explore different holographic geometries;
- Employing non-Abelian fiber groups beyond U(1), which could enrich the range of possible phase lifts;
- Allowing time-dependent connections and maps X_t, leading to dynamic holographic models of stochastic fields.

Outlook. Although the present framework is entirely classical and stochastic, the role of chirality and the appearance of odd (Chern–Simons–type) terms may suggest analogies with structures in quantum field theory. Making such connections precise would require a suitable quantization procedure and is left for future investigation.

9 Normalization cross-check

Under the Fourier conventions used, the energy and helicity normalizations are as follows. The total energy is

$\mathrm{E} = {}_{k0}|\widehat{G}(k)|((PS(k) + PH(k)) + (PS(k) - PH(k))) = 2_{k0}|\widehat{G}(k)|PS(k),$

since the helical modes are orthogonal. For the helicity, as derived in Remark 3.2.

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