A Geometric Foundation for Holographic Stochastic Field Theory: A Measured–Bundle Realization of the $\mathbb{T}^2 \to \mathbb{T}^3$ Map

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Abstract

A framework is presented for synthesizing divergence–free, homogeneous, and chiral random vector fields on the three–torus $\mathbb{T}^3$ from data placed on a two–dimensional base $\mathbb{T}^2$. The construction treats $\mathbb{T}^2$ as a holographic screen that drives a 2D $\to$ 3D mapping through a measured–bundle projection with a U(1) phase lift. The phase is the holonomy of a principal U(1) bundle over $\mathbb{T}^2$ with first Chern number $c_1 \in \mathbb{Z}$, which parametrizes chirality. Under an equidistribution condition on a map $X : E \to \mathbb{T}^3$ from the bundle total space, the resulting field on $\mathbb{T}^3$ is translation invariant; isotropy is obtained by choosing X so that its structure factor is rotationally uniform (or by statistical rotational averaging). In Fourier space, the covariance decomposes into helical eigenmodes with spectra $P_S(\boldsymbol{k}) \geq 0$ and $P_H(\boldsymbol{k})$ that satisfy the sharp positivity bound $P_S(\boldsymbol{k}) \geq |P_H(\boldsymbol{k})|$. An operator–theoretic (spectral–triple) realization is provided in an *odd* form, with the Dirac operator D determined by the helical spectra and the envelope. A transfer law connecting boundary statistics on $\mathbb{T}^2$ to bulk helical power spectra on $\mathbb{T}^3$ is derived, together with a numerically ready synthesis algorithm on rectangular lattices.

1 Introduction

Holographic ideas suggest that bulk structure can be encoded on a lower–dimensional substrate [1–4]. A *static* (equal–time, Euclidean) realisation is formulated here for random divergence–free vector fields on the 3–torus, using $\mathbb{T}^2$ as a base and a principal U(1) bundle $p : E \to \mathbb{T}^2$ to supply a topological phase. The first Chern class $c_1(E) \in H^2(\mathbb{T}^2, \mathbb{Z}) \simeq \mathbb{Z}$ plays the role of an integer "chirality knob" via a holonomy phase lift. A measurable map $X : E \to \mathbb{T}^3$ pushes the invariant bundle measure to the uniform measure on $\mathbb{T}^3$, ensuring homogeneity. A translation–invariant envelope kernel then yields a divergence–free Gaussian field with a standard helical covariance. The helical bound follows from Bochner positivity on compact abelian groups. An operator–theoretic representation via a spectral triple is given in an *odd* form, which directly reproduces the covariance.

Helicity and helical mode decompositions are classical in hydrodynamics and MHD [5–8]. Circle bundles over $\mathbb{T}^2$ are classified by $c_1 \in \mathbb{Z}$ and admit natural connections and holonomy phases [9, 11, 12]. Equidistribution on tori is underpinned by Kronecker–Weyl theory and unique ergodicity [15, 16].

Contributions. (i) A measured–bundle synthesis scheme $E \to \mathbb{T}^3$ that generates a divergence–free random field with tunable helicity through $c_1(E)$. (ii) A boundary–to–bulk transfer law expressing helical power spectra on $\mathbb{T}^3$ in terms of base statistics, the bundle connection, and an envelope kernel. (iii) An *odd* spectral–triple (A, H, D) whose spectral calculus reproduces the covariance. (iv) A practical lattice algorithm that samples the field while enforcing the helical positivity bound.

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2 Geometric and probabilistic setup

2.1 Principal circle bundles over $\mathbb{T}^2$

Let $p : E \to \mathbb{T}^2$ be a principal U(1) bundle. Isomorphism classes are in bijection with $c_1(E) \in H^2(\mathbb{T}^2, \mathbb{Z}) \simeq \mathbb{Z}$ [9, 10]. Fix a connection 1–form $\mathcal{A} \in \Omega^1(E; \mathrm{i}\mathbb{R})$ with curvature $F = \mathrm{d}\mathcal{A}$ representing $2\pi\mathrm{i}\, c_1(E)$ under Chern–Weil theory [11, 12]. Denote by μ_E the invariant probability measure induced by Haar on $\mathbb{T}^2$ and the uniform measure on the S^1 fiber (equivalently, the connection–invariant volume on E).

Choose local bundle coordinates $(\theta_1, \theta_2, \phi)$; these exist on each trivializing chart. *For $c_1(E) \neq 0$ the total space E is not globally $\mathbb{T}^2 \times S^1$*. Global objects are defined via the connection and its holonomy, and local formulas are glued by the bundle's transition functions. A convenient local *phase lift* is

$$U(\beta) \;=\; \exp\big(\mathrm{i}\,\chi(\beta)\big), \qquad \chi(\beta) = m_1\theta_1 + m_2\theta_2 + n\,\phi, \tag{1}$$

with $(m_1, m_2, n) \in \mathbb{Z}^3$ and $n = c_1(E)$. The term $n\,\phi$ implements fiber winding; U is globally well defined as a holonomy phase.

2.2 Equidistributing map to $\mathbb{T}^3$

Let $X : E \to \mathbb{T}^3$ be measurable. Homogeneity in the bulk is enforced by:

Assumption 2.1 (Equidistribution)**.** *$X_{\#}\mu_E = \lambda_3$, where λ_3 is Haar probability on $\mathbb{T}^3$.*

A globally well–defined choice uses an integer matrix on torus angles:

$$X(\theta_1, \theta_2, \phi) \;=\; M \begin{pmatrix} \theta_1 \\ \theta_2 \\ \phi \end{pmatrix} \pmod{2\pi}, \qquad M \in \mathrm{GL}(3, \mathbb{Z}). \tag{2}$$

Then X is a surjective endomorphism of tori and pushes the invariant measure on E to λ_3. To accelerate isotropy in practice, one may average over a small ensemble of frames $R \in \mathrm{SO}(3)$, replacing X by $R \circ X$ and averaging statistics over R.

2.3 Envelope, white noise, and synthesized field

Let $G_\sigma : \mathbb{T}^3 \to \mathbb{R}$ be a smooth, real, even, mean–zero envelope kernel with Fourier transform $\widehat{G}(\boldsymbol{k})$ satisfying $\widehat{G}(\boldsymbol{0}) = 0$ and $\widehat{G}(\boldsymbol{k}) = g(|\boldsymbol{k}|)$ for a nonnegative radial function g. Let W be a complex Gaussian white noise on (E, μ_E):

$$\mathbb{E}\big[\mathrm{d}W(\beta)\,\overline{\mathrm{d}W(\beta')}\big] = \delta(\beta - \beta')\,\mathrm{d}\mu_E(\beta).$$

Fix a constant vector $\boldsymbol{a} \in \mathbb{C}^3$. Define

$$\boldsymbol{\Phi}(x) \;=\; \int_E \Big(\nabla_x \times \big[G_\sigma\big(x - X(\beta)\big)\,\boldsymbol{a}\big]\Big)\,U(\beta)\,\mathrm{d}W(\beta). \tag{3}$$

The curl enforces $\nabla \cdot \boldsymbol{\Phi} \equiv 0$. Translation invariance follows from Assumption 2.1.

2.4 Helical decomposition on $\mathbb{T}^3$

For $\boldsymbol{k} \in \mathbb{Z}^3 \setminus \{\boldsymbol{0}\}$, let $\{\boldsymbol{h}^{\pm}(\boldsymbol{k})\}$ be an orthonormal helical basis of divergence–free eigenvectors of $\mathrm{i}\boldsymbol{k} \times (\cdot)$:

$$\mathrm{i}\,\boldsymbol{k} \times \boldsymbol{h}^{\pm}(\boldsymbol{k}) = \pm\,|\boldsymbol{k}|\;\boldsymbol{h}^{\pm}(\boldsymbol{k}), \qquad \boldsymbol{k} \cdot \boldsymbol{h}^{\pm}(\boldsymbol{k}) = 0,$$

and write $\widehat{\boldsymbol{\Phi}}(\boldsymbol{k}) = \Phi^+(\boldsymbol{k})\,\boldsymbol{h}^+(\boldsymbol{k}) + \Phi^-(\boldsymbol{k})\,\boldsymbol{h}^-(\boldsymbol{k})$.

3 Covariance and helical spectra

Taking expectations in (3) and using $X_{\#}\mu_E = \lambda_3$ gives

$$\mathbb{E}\big[\widehat{\Phi}_i(\boldsymbol{k})\,\overline{\widehat{\Phi}_j(\boldsymbol{k}')}\big] \;=\; \delta_{\boldsymbol{k},\boldsymbol{k}'}\,\Big|\widehat{G}(\boldsymbol{k})\Big|^2\Big(P_S(\boldsymbol{k})\,\Pi_{ij}(\boldsymbol{k}) + \mathrm{i}\,P_H(\boldsymbol{k})\,\epsilon_{ijm}\hat{k}_m\Big)\,, \tag{4}$$

where $\Pi_{ij}(\boldsymbol{k}) = \delta_{ij} - \hat{k}_i\hat{k}_j$, $\hat{\boldsymbol{k}} = \boldsymbol{k}/|\boldsymbol{k}|$. In the helical basis,

$$\mathbb{E}\big[\,\big|\Phi^{\pm}(\boldsymbol{k})\big|^2\,\big] \;=\; \Big|\widehat{G}(\boldsymbol{k})\Big|^2\big(P_S(\boldsymbol{k}) \pm P_H(\boldsymbol{k})\big), \qquad \mathbb{E}\big[\Phi^{+}(\boldsymbol{k})\,\overline{\Phi^{-}(\boldsymbol{k})}\big] = 0. \tag{5}$$

Proposition 3.1 (Bochner positivity and the helical bound)**.** *For each $\boldsymbol{k} \neq \boldsymbol{0}$,*

$$P_S(\boldsymbol{k}) \geq 0, \qquad |P_H(\boldsymbol{k})| \leq P_S(\boldsymbol{k}).$$

Proof. On the compact abelian group $\mathbb{T}^3$, stationary covariances correspond to positive–type functions on $\mathbb{Z}^3$ whose Fourier transforms are positive measures [13, 14]. Restricting (4) to the transverse subspace and diagonalizing in the helical basis gives variances $\Big|\widehat{G}(\boldsymbol{k})\Big|^2\big(P_S(\boldsymbol{k}) \pm P_H(\boldsymbol{k})\big) \geq 0$, which is equivalent to the stated inequalities. □

Remark 3.2 (Helicity density)**.** *The spectral helicity density at $\boldsymbol{k}$ equals* $2\,|\boldsymbol{k}|\,\Big|\widehat{G}(\boldsymbol{k})\Big|^2 P_H(\boldsymbol{k})$*, consistent with classical conventions [5, 7].*

4 Boundary–to–bulk transfer law

Let $b = p(\beta) \in \mathbb{T}^2$ denote the base point and write $U(\beta) = u(b)\,\mathrm{e}^{\mathrm{i}n\phi}$ in adapted coordinates, with $n = c_1(E)$. Define the *pushforward characteristic*

$$\chi_X(\boldsymbol{k}) \;=\; \int_E \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot X(\beta)}\,\mathrm{d}\mu_E(\beta) \;=\; 0 \quad \text{for } \boldsymbol{k} \neq \boldsymbol{0}, \tag{6}$$

by Assumption 2.1. The symmetric and helical spectra admit the representation

$$P_S(\boldsymbol{k}) \;=\; \int_E K_S(\boldsymbol{k};\beta)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot X(\beta)}\,\mathrm{d}\mu_E(\beta), \tag{7}$$

$$P_H(\boldsymbol{k}) \;=\; \int_E K_H(\boldsymbol{k};\beta)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot X(\beta)}\,U(\beta)\,\mathrm{d}\mu_E(\beta), \tag{8}$$

for bounded kernels $K_{S/H}$ depending on $\boldsymbol{a}$ and the curl enforcement. In the affine model (2) with $U(\beta) = u(b)\,\mathrm{e}^{\mathrm{i}n\phi}$, integration over ϕ yields a resonance condition $n - \boldsymbol{k}\cdot\boldsymbol{\omega}_3 \in \mathbb{Z}$ in the exact average; for finite sampling in ϕ, this appears as a Dirichlet–kernel peak near the resonance.

5 Effective quadratic action and odd term

The Gaussian measure is determined by (4). In the helical basis the quadratic action is

$$\mathcal{S}[\boldsymbol{\Phi}] \;=\; \frac{1}{2}\sum_{\boldsymbol{k}\neq 0}\Big|\widehat{G}(\boldsymbol{k})\Big|^{-2}\left(\frac{|\Phi^{+}(\boldsymbol{k})|^2}{P_S(\boldsymbol{k}) + P_H(\boldsymbol{k})} + \frac{|\Phi^{-}(\boldsymbol{k})|^2}{P_S(\boldsymbol{k}) - P_H(\boldsymbol{k})}\right). \tag{9}$$

In physical space, the action splits into an even part plus an odd contribution of Chern–Simons type,

$$\mathcal{S}_{\mathrm{odd}}[\boldsymbol{\Phi}] \;\approx\; \int_{\mathbb{T}^3} \eta(|\nabla|)\,\boldsymbol{\Phi}(x)\cdot\big(\nabla \times \boldsymbol{\Phi}(x)\big)\,\mathrm{d}x,$$

with a nonlocal positive kernel η determined by P_S and P_H [17–19].

6 A spectral–triple realization (odd case)

Let $A = C^\infty(\mathbb{T}^3)$ act by multiplication on $H = L^2_{\mathrm{trans}}(\mathbb{T}^3;\mathbb{C}^3) \oplus L^2_{\mathrm{trans}}(\mathbb{T}^3;\mathbb{C}^3)$. In the Fourier–helical basis, define

$$D:\ (\Phi^+(\boldsymbol{k}),\Phi^-(\boldsymbol{k})) \longmapsto \Big(|\widehat{G}(\boldsymbol{k})|^{-1}\,(P_S(\boldsymbol{k})+P_H(\boldsymbol{k}))^{-1/2}\,\Phi^+(\boldsymbol{k}),\ |\widehat{G}(\boldsymbol{k})|^{-1}\,(P_S(\boldsymbol{k})-P_H(\boldsymbol{k}))^{-1/2}\,\Phi^-(\boldsymbol{k})\Big).$$

Then D^{-2} on the transverse subspace equals the covariance operator with eigenvalues $\left|\widehat{G}(\boldsymbol{k})\right|^2 (P_S(\boldsymbol{k})\pm P_H(\boldsymbol{k}))$. Assume $\left|\widehat{G}(\boldsymbol{k})\right|$ decays super–polynomially (e.g. a periodized Gaussian) and P_S, P_H are bounded and smooth; then D has compact resolvent and, since multiplication by $f \in C^\infty(\mathbb{T}^3)$ is a zeroth–order PDO, $[D, f]$ extends to a bounded operator.

7 Numerical synthesis on lattices

Identify $\mathbb{T}^3$ with the grid $\mathbb{Z}^3_N$. Choose $\widehat{G}(\boldsymbol{k})$, an integer M in (2) (optionally with randomized orthonormal frames to improve isotropy), and a connection with $c_1(E) = n$.

1. **Boundary randomness and holonomy.** Sample i.i.d. complex Gaussians $\{\xi_\ell\}$ on a mesh $\{\beta_\ell\} \subset E$ and phases $U_\ell = U(\beta_\ell)$ via (1).
2. **Projection to bulk spectra.** For each $\boldsymbol{k} \in \mathbb{Z}^3_N \setminus \{\boldsymbol{0}\}$, set

   $$\mathcal{B}_S(\boldsymbol{k}) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot X(\beta_\ell)}\,\xi_\ell, \qquad \mathcal{B}_H(\boldsymbol{k}) = \frac{1}{\sqrt{L}} \sum_{\ell=1}^{L} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot X(\beta_\ell)}\,U_\ell\,\xi_\ell,$$

   and define
   $$P_S(\boldsymbol{k}) = |\mathcal{B}_S(\boldsymbol{k})|^2, \qquad P_H(\boldsymbol{k}) = \Re\big(\mathcal{B}_H(\boldsymbol{k})\,\overline{\mathcal{B}_S(\boldsymbol{k})}\big).$$
   Finally project to enforce $|P_H(\boldsymbol{k})| \le P_S(\boldsymbol{k})$.
3. **Helical assembly.** Draw independent standard Gaussians $\eta_\pm(\boldsymbol{k})$ and set $\Phi^\pm(\boldsymbol{k}) = \widehat{G}(\boldsymbol{k})\,\sqrt{P_S(\boldsymbol{k}) \pm P_H(\boldsymbol{k})}\,\eta_\pm(\boldsymbol{k})$.
4. **Inverse transform.** Form $\widehat{\boldsymbol{\Phi}}(\boldsymbol{k}) = \Phi^+(\boldsymbol{k})\boldsymbol{h}^+(\boldsymbol{k}) + \Phi^-(\boldsymbol{k})\boldsymbol{h}^-(\boldsymbol{k})$, impose conjugate symmetry, and inverse FFT to obtain $\boldsymbol{\Phi}(x)$.

8 Discussion and extensions

Chirality control. The integer $n = c_1(E)$ controls the net helical bias via the holonomy factor in (8). **Homogeneity and isotropy.** Equidistribution of X guarantees translation invariance; isotropy is obtained by radial $\widehat{G}$ combined with choices of $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$ whose ensemble is rotation invariant. **Generalizations.** Bases other than $\mathbb{T}^2$ (e.g. S^2) and nonabelian fiber groups are natural extensions; time dependence can be introduced by slowly varying X and the connection.

- This geometric chirality control may extend to particle physics, where helical bias analogs to weak interaction handedness, and twist-reversals could model $0\nu\beta\beta$ via off-diagonal D couplings corresponding to neutrino masses. However, this analogy remains highly speculative, as the current framework is designed for classical stochastic vector fields in hydrodynamics/MHD, not quantum particle decays, and requires significant theoretical development to bridge these domains rigorously.

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