EN.601.454/654 Augmented Reality

Assignment 2

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1. (Written) Matrices & Transformations

1.1 **Show that matrix multiplication is not commutative, i.e. find two matrices $A, B \in \mathbb{R}^{2\times 2}$ such that $AB \neq BA$. (2 points)**

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$

$BA = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$

$\because \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix}$

$\boxed{\therefore AB \neq BA}$

1.2 **However, in a special case, it is commutative. Consider $A, B \in \mathbb{R}^{n\times n}$ with $A^T = A$ and $B^T = B$ (symmetric matrices), show that if AB is also symmetric, then $AB = BA$. (2 points)**

$\because$ $A, B \in \mathbb{R}^{n\times n}$ and they both are symmetric matrices

$\therefore$ $A^T = A$, $B^T = B$ ①

$\because$ AB is also symmetric

$\therefore AB = (AB)^T$ ②

$\because (AB)^T = B^T A^T$

$\therefore AB = B^T A^T$

let equation ① put in

$\boxed{AB = B^T A^T = BA}$

1.3 The matrix multiplication can be expressed in a summation form: e.g. for $X = UV$, the element x_{ij} can be expressed as $x_{ij} = \sum_{k=1}^{m} u_{ik}v_{kj}$, where $X \in \mathbb{R}^{n\times n}, U \in \mathbb{R}^{n\times m}, V \in \mathbb{R}^{m\times n}$. By using the summation form, for matrices $A, B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{m\times k}$, show that $(A + B)C = AC + BC$. (2 points)

$\because A, B \in R^{n\times m}, \ C \in R^{m\times k}$

let $Y = A+B$, where $y_{ij} = a_{ij} + b_{ij}$

$$
\begin{aligned}
(A+B)C &= YC \\
&= \textstyle\sum_{p=1}^{m} y_{ip} c_{pj} = \sum_{p=1}^{m} (a_{ip}+b_{ip}) c_{pj} \\
&= \textstyle\sum_{p=1}^{m} (a_{ip}c_{pj} + b_{ip}c_{pj}) \\
&= \textstyle\sum_{p=1}^{m} a_{ip}c_{pj} + \sum_{p=1}^{m} b_{ip}c_{pj} \\
&= \boxed{AC + BC}
\end{aligned}
$$

1.4 Compute the rank and determinant of the following matrices if it exists and decide if the matrix is invertible or not. If so, compute the inverse. (6 points)

(a) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ (1 point)

(b) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ (1 points)

(c) $\begin{bmatrix} 3 & 1 & 3 \\ 9 & 4 & 9 \\ 15 & 7 & 15 \end{bmatrix}$ (2 points)

(d) $\begin{bmatrix} 132 & 165 & 198 \\ 348 & 435 & 522 \\ 564 & 705 & 846 \end{bmatrix}$ (2 points)

(a)

$\boxed{\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = 1\times 4 - 2\times 3 = -2}$

$\because \det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) \neq 0$

$\therefore$ it's invertible.

Apply row echelon form

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{R_2 - 3\times R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$\boxed{\operatorname{rank}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \# \text{ of non-zero rows} = 2}$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{1\times 4 - 3\times 2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$= \boxed{\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}}$

continue

(b)

let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$\boxed{\text{The determinant of } A \text{ doesn't exist since Matrix } A \text{ is not a Square matrix}}$

Apply row echelon form

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \xrightarrow[R_2 \to R_2 - 3 \times R_1]{R_3 \to R_3 - 5 \times R_1} \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$\boxed{\operatorname{rank}(A) = 2}$

$\boxed{\text{Since } A \text{ is not a square matrix, its inverse doesn't exist}}$

(c)

let $A = \begin{bmatrix} 3 & 1 & 3 \\ 9 & 4 & 9 \\ 15 & 7 & 15 \end{bmatrix}$

$\boxed{\det(A) = 3 \cdot (4 \times 15 - 9 \times 7) - 1 \cdot (9 \times 15 - 9 \times 15) + 3 \cdot (9 \times 7 - 4 \times 15) = 0}$

Apply row echelon form

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 9 & 4 & 9 \\ 15 & 7 & 15 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 5R_1]{R_2 \to R_2 - 3R_1} \begin{bmatrix} 3 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_2} \begin{bmatrix} 3 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{3}R_1}$$

$$\begin{bmatrix} 1 & \frac{1}{3} & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\boxed{\operatorname{rank}(A) = 2}$

$\boxed{\text{Since } \det(A) = 0, \text{ it doesn't have inverse}}$

(d)

let $A = \begin{bmatrix} 132 & 165 & 198 \\ 348 & 435 & 522 \\ 564 & 705 & 846 \end{bmatrix}$

$\boxed{\det(A) = 132 \cdot (435 \times 846 - 522 \times 705) - 165 \cdot (348 \times 846 - 522 \times 564) + 198 \cdot (348 \times 705 - 435 \times 564) = 0}$

Apply row echelon form

$A = \begin{bmatrix} 132 & 165 & 198 \\ 348 & 435 & 522 \\ 564 & 705 & 846 \end{bmatrix} \xrightarrow[\substack{R_2 \to \frac{1}{29} R_2 \\ R_3 \to \frac{1}{47} R_3}]{R_1 \to \frac{1}{11} R_1} \begin{bmatrix} 12 & 15 & 18 \\ 12 & 15 & 18 \\ 12 & 15 & 18 \end{bmatrix} \xrightarrow[R_2 \to R_2 - R_1]{R_3 \to R_3 - R_1} \begin{bmatrix} 12 & 15 & 18 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to \frac{1}{12} R_1}$

$\begin{bmatrix} 1 & \frac{4}{3} & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\boxed{\text{rank}(A) = 1}$

$\boxed{\text{Since } \det(A) = 0, \text{ it doesn't have inverse}}$

1.5 Express the following 3D rotations in matrix form (6 points)

(a) Rotate around x-axis by 30 degrees, $\mathbf{R}_x$. (2 point)

(b) Rotate around y-axis by 45 degrees, $\mathbf{R}_y$. (2 point)

(c) Rotate around z-axis by 60 degrees, $\mathbf{R}_z$. (2 point)

(a) $R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$, $\theta = 30^\circ \Rightarrow R_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

(b) $R_y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$, $\theta = 45^\circ \Rightarrow R_y = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$

(c) $R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\theta = 60^\circ \Rightarrow R_z = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1.6 Use the above matrices to show that $\mathbf{R}_x\mathbf{R}_y\mathbf{R}_z \neq \mathbf{R}_z\mathbf{R}_y\mathbf{R}_x$. (2 points)

$$R_xR_yR_z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{4} & \frac{\sqrt{3}}{2} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{6}}{4} & \frac{1}{2} & \frac{\sqrt{6}}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{8}+\frac{3}{4} & -\frac{\sqrt{6}}{8}+\frac{\sqrt{3}}{4} & -\frac{\sqrt{2}}{4} \\ -\frac{\sqrt{6}}{8}+\frac{\sqrt{3}}{4} & \frac{3\sqrt{2}}{8}+\frac{1}{4} & \frac{\sqrt{6}}{4} \end{bmatrix} = \boxed{\begin{bmatrix} 0.3536 & -0.6124 & 0.7071 \\ 0.9268 & 0.1268 & -0.3536 \\ 0.1268 & 0.7803 & 0.6124 \end{bmatrix}}$$

$$R_zR_yR_x = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} \\ \frac{\sqrt{6}}{4} & \frac{1}{2} & \frac{\sqrt{6}}{4} \\ -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 0.3536 & -0.5732 & 0.7392 \\ 0.6124 & 0.7392 & 0.2803 \\ -0.7071 & 0.3536 & 0.6124 \end{bmatrix}}$$

$$\boxed{\therefore\ R_xR_yR_z \neq R_zR_yR_x}$$

2. Homogeneous Coordinates

(a) For some $\alpha \in \mathbb{R}$ and $t \in \mathbb{R}^n$, express the mapping given by

$$x \mapsto \alpha x + t, \quad \forall x \in \mathbb{R}^n \tag{1}$$

in homogeneous coordinates (*i.e.* as a matrix in $\mathbb{R}^{(n+1)\times(n+1)}$). (2 points)

From the question, there's no rotation but scaling and translation.

The mapping is below:

$$\boxed{T = \begin{bmatrix} \alpha I_{n\times n} & t \\ 0^T & 1 \end{bmatrix}, \text{ where } I_{n\times n} \text{ is } n\times n \text{ identity matrix}}$$

(b) Given a point

$$x = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix} \in \mathbb{R}^3$$

and a transformation

$$T = \begin{bmatrix} 20 & 0 & 5 & 0 \\ 0 & 20 & 5 & 0 \\ 0 & 0 & -10 & -100 \\ 0 & 0 & -1 & 0 \end{bmatrix} \in \mathbb{R}^{4\times 4},$$

express x in homogeneous coordinates, apply T, and project the result back into $\mathbb{R}^3$. (3 points)

$$x \xrightarrow{\text{homogeneous}} \begin{bmatrix} 10 \\ 20 \\ 30 \\ 1 \end{bmatrix} \in R^4$$

$$T \cdot x = \begin{bmatrix} 20 & 0 & 5 & 0 \\ 0 & 20 & 5 & 0 \\ 0 & 0 & -10 & -100 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ 20 \\ 30 \\ 1 \end{bmatrix} = \begin{bmatrix} 350 \\ 550 \\ -400 \\ -30 \end{bmatrix}$$

$\because -30 \neq 0$

$$\therefore \boxed{\text{final result} = \begin{bmatrix} -\frac{35}{3} \\ -\frac{55}{3} \\ \frac{40}{3} \end{bmatrix} \in R^3}$$

(c) Let R be an $n-$dimensional rotation matrix. Compute the inverse of

$$\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}, \quad t \in \mathbb{R}^n.$$

(4 points)

let $A \in R^{(n+1)\times(n\times1)}$ which is the inverse of $\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$

$$A \cdot \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} = I_{(n+1)\times(n+1)} = \begin{bmatrix} I_{n\times n} & 0_{n\times 1} \\ 0^T & 1 \end{bmatrix}$$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a \in R^{n\times n}$, $b \in R^{n\times 1}$, $c \in R^{1\times n}$, $d \in R$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} I_{n\times n} & 0_{n\times 1} \\ 0^T & 1 \end{bmatrix}$$

$$\begin{cases} a \cdot R = I_{n\times n} \\ a \cdot t + b = 0_{n\times 1} \\ c \cdot R = 0^T \\ c \cdot t + d = 1 \end{cases} \Rightarrow \begin{cases} a = I_{n\times n} \cdot R^{-1} = R^{-1} \\ b = -a \cdot t = -I_{n\times n} R^{-1} \cdot t = -R^{-1} t \\ c = 0^T \cdot R^{-1} \\ d = 1 - c \cdot t \end{cases}$$

$\because$ R is rotation matrix (orthogonal matrix) and non-zero

$\therefore$ $R^T = R^{-1}$

$$\therefore \begin{cases} a = R^{-1} = R^T \\ b = -R^T t \\ c = 0_{1\times n} \\ d = 1 \end{cases}$$

$\therefore$ The inverse of $\begin{bmatrix} R & t \\ 0^T & 1 \end{bmatrix}$ is $\boxed{\begin{bmatrix} R^T & -R^T t \\ 0^T & 1 \end{bmatrix}}$

(d) Consider a 3D point $\vec{P} = \begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix}$ which is observed by two cameras with projection matrices:

$$\mathbf{P}_1 = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{P}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

where the image plane is assumed to be located at $z = 1$. Compute the corresponding projected points $\vec{p}_1$ and $\vec{p}_2$ in the image planes of the first and second camera, respectively. [Remember the projected points are in $\mathbb{R}^2$.] (6 points)

According to the equation

$$x = P \cdot X$$

$$\therefore \vec{p}_1 = P_1 \cdot \vec{P} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \longrightarrow \boxed{\vec{p}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}}$$

$$\vec{p}_2 = P_2 \cdot \vec{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \longrightarrow \boxed{\vec{p}_2 = \begin{bmatrix} -2 \\ -\frac{1}{2} \end{bmatrix}}$$

3. Parameter Estimation

(a) What is the minimum number of point-to-point correspondence required for homography estimation? Why? (2 points)

(a)

$\boxed{\text{4 paired points}}$ is the minimum number for homography estimation, because the transformation matrix has 8 DoF and 2 independent equations. As a result, 4 points × 2 independent equations can get 8 values.

(b)

(b) Let

$$\mathbf{H} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

be the homography between the two planar objects in Figure 1. Assume $p_1 = (0,0)^T$, $p_2 = (0,1)^T$, $p_3 = (1,1)^T$ and $p_4 = (1,0)^T$; and $p_1{}' = (1,1)^T$, $p_2{}' = (1,0)^T$, $p_3{}' = (2,0)^T$ and $p_4{}' = (2,1)^T$. Find $\mathbf{H}$ by fixing $h_{33} = 1$ and using the below equations. (4 points)

$$p_x{}' = \frac{h_{11}p_x + h_{12}p_y + h_{13}}{h_{31}p_x + h_{32}p_y + h_{33}} \quad (2)$$

$$p_y{}' = \frac{h_{21}p_x + h_{22}p_y + h_{23}}{h_{31}p_x + h_{32}p_y + h_{33}} \quad (3)$$

According to the problem statement, 4 corresponse are used to obtain H. and we will 8 equations; $h_{33} = 1$

Put P_1, P_1'; P_2, P_2'; P_3, P_3'; P_4, P_4' into the equation

$$P_1' = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{h_{11}\cdot 0 + h_{12}\cdot 0 + h_{13}}{h_{31}\cdot 0 + h_{32}\cdot 0 + 1} \\ \frac{h_{21}\cdot 0 + h_{22}\cdot 0 + h_{23}}{h_{31}\cdot 0 + h_{32}\cdot 0 + 1} \end{bmatrix} = \begin{bmatrix} h_{13} \\ h_{23} \end{bmatrix} \Rightarrow h_{13} = h_{23} = 1$$

$$P_2' = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{h_{11}\cdot 0 + h_{12}\cdot 1 + h_{13}}{h_{31}\cdot 0 + h_{32}\cdot 1 + 1} \\ \frac{h_{21}\cdot 0 + h_{22}\cdot 1 + h_{23}}{h_{31}\cdot 0 + h_{32}\cdot 1 + 1} \end{bmatrix} = \begin{bmatrix} \frac{h_{12} + h_{13}}{h_{32} + 1} \\ \frac{h_{22} + h_{23}}{h_{32} + 1} \end{bmatrix} = \begin{bmatrix} \frac{h_{12} + 1}{h_{32} + 1} \\ \frac{h_{22} + 1}{h_{32} + 1} \end{bmatrix}$$

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$$P_3' = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \dfrac{h_{11}\cdot 1 + h_{12}\cdot 1 + h_{13}}{h_{31}\cdot 1 + h_{32}\cdot 1 + 1} \\ \dfrac{h_{21}\cdot 1 + h_{22}\cdot 1 + h_{23}}{h_{31}\cdot 1 + h_{32}\cdot 1 + 1} \end{bmatrix} = \begin{bmatrix} \dfrac{h_{11} + h_{12} + 1}{h_{31} + h_{32} + 1} \\ \dfrac{h_{21} + h_{22} + 1}{h_{31} + h_{32} + 1} \end{bmatrix}$$

$$P_4' = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{h_{11}\cdot 1 + h_{12}\cdot 0 + h_{13}}{h_{31}\cdot 1 + h_{32}\cdot 0 + 1} \\ \dfrac{h_{21}\cdot 1 + h_{22}\cdot 0 + h_{23}}{h_{31}\cdot 1 + h_{32}\cdot 0 + 1} \end{bmatrix} = \begin{bmatrix} \dfrac{h_{11}+1}{h_{31}+1} \\ \dfrac{h_{21}+1}{h_{31}+1} \end{bmatrix}$$

Conclusion:

$$\begin{cases} \dfrac{h_{12}+1}{h_{32}+1} = 1 \\ \dfrac{h_{22}+1}{h_{32}+1} = 0 \\ \dfrac{h_{11}+h_{12}+1}{h_{31}+h_{32}+1} = 2 \\ \dfrac{h_{21}+h_{22}+1}{h_{31}+h_{32}+1} = 0 \\ \dfrac{h_{11}+1}{h_{31}+1} = 2 \\ \dfrac{h_{21}+1}{h_{31}+1} = 1 \end{cases} \xRightarrow[h_{32}+1 \neq 0]{h_{31}+h_{32}+1 \neq 0} \begin{cases} h_{12} = h_{32} \\ h_{22} = -1 \\ h_{11}+h_{12}+1 = 2h_{31}+2h_{32}+2 \\ h_{21}+h_{22}+1 = 0 \\ h_{11}+1 = 2h_{31}+2 \\ h_{21} = h_{31} \end{cases} \Rightarrow \begin{cases} h_{21} = 0 \\ h_{31} = 0 \\ h_{11} = 1 \\ h_{12} = 0 \\ h_{32} = 0 \\ h_{23} = h_{13} = 1 \\ h_{22} = -1 \end{cases}$$

$$\therefore H = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) ***(UG Optional)*** Why do equations (2) and (3) hold? (2 points)

$\vec{P'} = H\vec{P}$ ← map $\begin{bmatrix} P_x \\ P_y \end{bmatrix} \in R^2 \longrightarrow \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix} \in R^3$

$$= \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} P_x \\ P_y \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} h_{11}P_x + h_{12}P_y + h_{13} \\ h_{21}P_x + h_{22}P_y + h_{23} \\ h_{31}P_x + h_{32}P_y + h_{33} \end{bmatrix}$$

Then we should map homogeneous ⟶ Euclidean space

$$P' = \begin{bmatrix} \frac{h_{11}P_x + h_{12}P_y + h_{13}}{h_{31}P_x + h_{32}P_y + h_{33}} \\ \frac{h_{21}P_x + h_{22}P_y + h_{23}}{h_{31}P_x + h_{32}P_y + h_{33}} \\ 1 \end{bmatrix} \longrightarrow \boxed{P' = \begin{bmatrix} P'_x \\ P'_y \end{bmatrix} = \begin{bmatrix} \frac{h_{11}P_x + h_{12}P_y + h_{13}}{h_{31}P_x + h_{32}P_y + h_{33}} \\ \frac{h_{21}P_x + h_{22}P_y + h_{23}}{h_{31}P_x + h_{32}P_y + h_{33}} \end{bmatrix}}$$

(d) ***(UG Optional)*** Why can we fix $h_{33} = 1$? Is it always good to fix this for the estimation? Why or why not? What is the physical meaning when $h_{33} = 0$? (3 points)

As far as I am thinking, since we can scale the transformation matrix H and it won't change the interpretation of the homogeneous transformation. Even though h_{33} is not 1, we can multiply a scalar $\frac{1}{h_{33}}$ to make the h_{33} to be 1 and this won't change the transformation. In my opinion, it's not always good to fix h_{33}, because when h_{33} is 0, it has a transformation mapping the origin to the line at infinity which may cause failure if we fix the $h_{33} = 1$. When $h_{33} = 0$, this means it's the normal direction to the principal plane and the coordinate origin is mapped to a point at infinity.

(e) Four points can give us an unique solution, but it's unlikely the estimation will be robust due to noise and outliers. In general, all corner correspondences on the checkerboard are used for homography estimation, resulting in an overconstrained system. Suggest a suitable method for robust estimation in this paradigm, and explain your reasoning. (1 point)

We could apply a $\boxed{\text{normalization step}}$ on the data points. For the basic DLT algorithm, the result is dependent on the origin and scale of the coordinate system in the image, the normalization step could remove the effect of original image coordinates and converge solution to correct results.

(f) Will homography estimation work if we have four point-to-point correspondences between two views of a non-planar 3D shape? Why or why not? (2 points)

According to the equation of homograph

$$x_i' = HX_i \qquad X_i', X_i \in R^{4\times1}, \quad H \in R^{4\times4}, \qquad X_i' = \begin{bmatrix} x_i' \\ y_i' \\ z_i' \\ w_i' \end{bmatrix}$$

$$[x_i'] H X_i = 0$$

$$\underbrace{\begin{bmatrix} 0^T & -x_i' X_i^T & w_i' X_i^T & -z_i' X_i^T \\ x_i' X_i^T & 0^T & z_i' X_i^T & y_i' X_i^T \\ -w_i' X_i^T & -z_i' X_i^T & 0^T & -x_i' X_i^T \\ z_i' X_i^T & -y_i' X_i^T & x_i' X_i^T & 0^T \end{bmatrix}}_{4\times16} \underbrace{\begin{bmatrix} h_{1k}^T \\ h_{2k}^T \\ h_{3k}^T \\ h_{4k}^T \end{bmatrix}}_{16\times1} = 0$$

h has 15 DoF, 3 independent equations/points

# of correspondence = 15 / 3 = 5

As a result, 4 point-to-point correspondence doesn't work well on 3D homograph, but 5 works well.

(g) Aside from camera calibration and image rectification, suggest another application for homography estimation.(1 point)

$\boxed{\text{Image registration}}$ is an application for homography estimation. We can use homography estimation to get a good transformation and warp one image to be like another image.