Eulerian graphs and Hamiltonian graphs

Eulerian Circuit

circuit containing all edges of the graph
↓
closed trail.

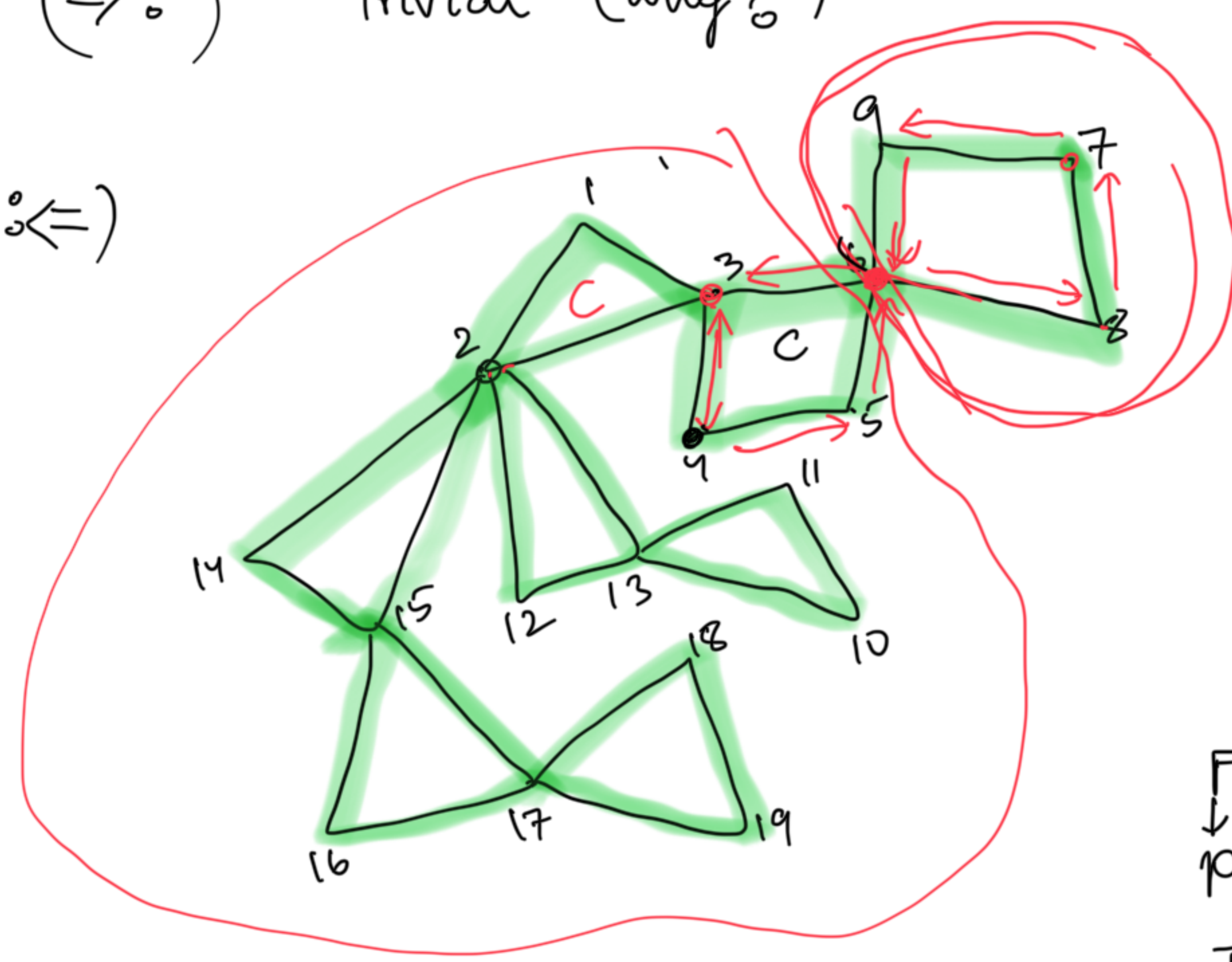
Graph possessing an Eulerian circuit is called an Eulerian graph.

Theorem:-

$\boxed{\text{Graph having Eulerian circuit / Eulerian graph} \iff d(v) \text{ is even for every } v}$

($\Rightarrow$:) Trivial (why?)

(:$\Leftarrow$)



Fleury's Algorithm

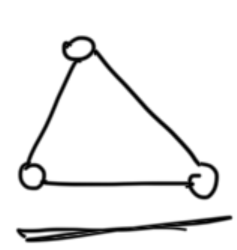
F
$p \Rightarrow q$ T

T F F

The proof is by induction on m (number of edges)
wlg. assume the graph to be connected.

$\cancel{m=0}$, $\cancel{m=1}$, $\cancel{m=2}$, $m \geq 3$

$m = 3$

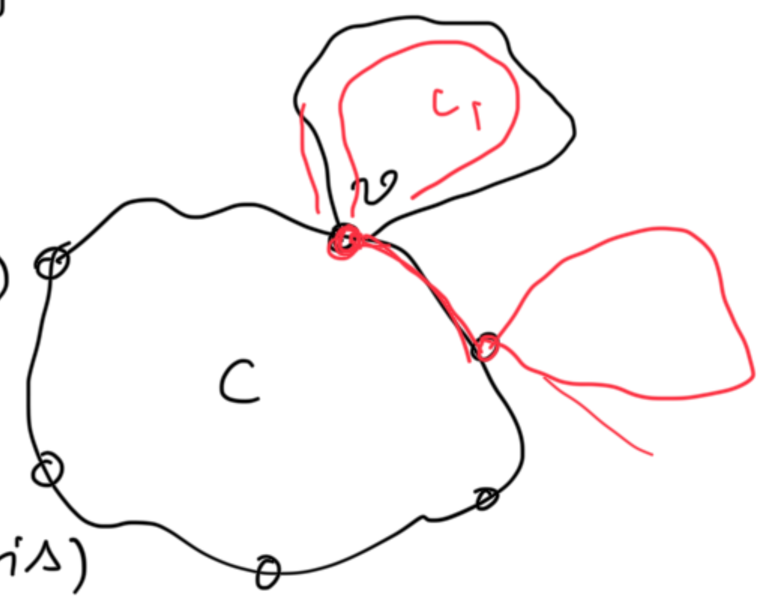


$m \leq k \rightarrow$ hypothesis true
G : even

$m > k \rightarrow G'$: even

Since $d(v) \geq 2$, $\exists$ a cycle C in G'

Let $H = G - E(C)$



$d_H(v) = 0$ or $2k$ $\ \ \& \ m(H) < m(G')$

$\Rightarrow$ Each component of H is Eulerian (by induction hypothesis)

Now to obtain an Eulerian circuit of G', we traverse C, and

(1) when a component of H is entered for the first time, we traverse through the Eulerian circuit of that component.

- When we complete the traversal of C, we get an Eulerian circuit of G'
  $\Rightarrow$ G' is Eulerian.

Hamiltonian cycle and Hamiltonian graph.

$\downarrow$ cycle containing all vertices of the graph

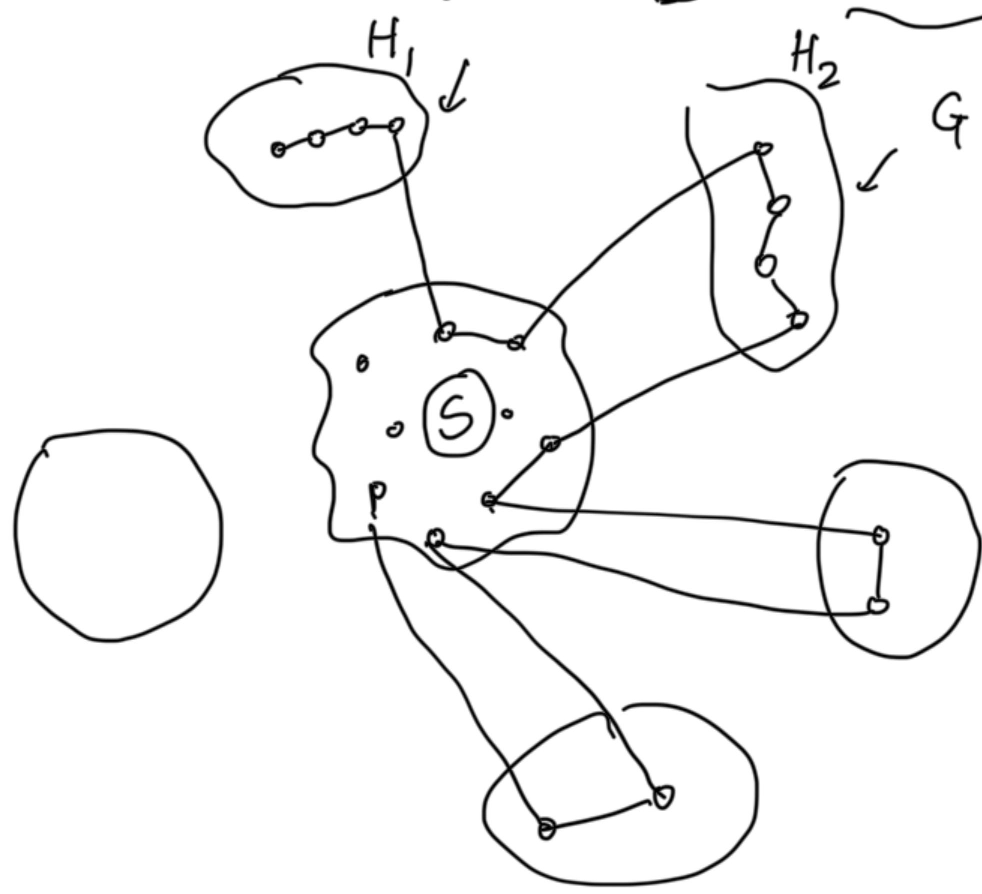
A graph having a Hamiltonian cycle.

Hamiltonian path

$\downarrow$ a path containing all the vertices of the graph.

G is Hamiltonian $\Rightarrow$ G is 2-connected

Suppose G has a Ham. Cycle $\underline{C}$ $\Phi \neq S \subseteq V(G)$



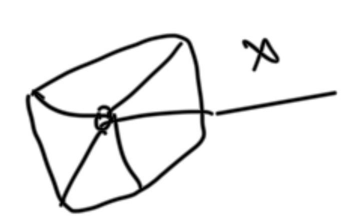
$c(H) \to$ # of components of H

Lemma:- Let G be a Hamiltonian graph. Then for every $\Phi \neq S \subseteq V(G)$,

$$c(G-S) \leq |S| \quad \checkmark \times$$

Proof:· Easy proof.

$(: \Leftarrow)$ $\times$



$S = \{v\}$

$c(G-S) \leq |S|$

$\Downarrow$

$\delta(G) \geq 2$ / 2-connected.

Some Sufficient conditions for Hamiltonicity

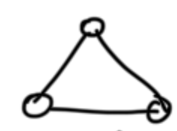
Lemma:-

Let G be a graph having n vertices and m edges. If $m > \binom{n-1}{2} + 1$, then G is Hamiltonian.

Proof: wlg, $n \geq 3$

$$m > \binom{2}{2} + 1 \Rightarrow m > 2$$

$$\Rightarrow m \geq 3$$



only graph.

Hamiltonian graph.

$n = k \Rightarrow$ If $m > \binom{k-1}{2} + 1$, Then G is Hamiltonian

$\boxed{n = k+1}$

Consider a graph G having $k+1$ vertices and $m > \binom{k}{2} + 1$

$$\underset{G}{\binom{k}{2} + 1} = \underset{K_n}{\binom{k+1}{2}} - \underset{G^c}{(k-1)}$$

$$|E(G^c)| < (k-1)$$

$$\sum_{v \in V(G)} d_{G^c}(v) < 2(k-1) < 2k < 2(k+1)$$

$\exists$ a v st $d_{G^c}(v) \leq 1$ (By Pigeonhole principle)

$$\Downarrow$$

$$d_G(v) \geq k-1$$

$k-1$ $\quad k$

$\frac{k(k-1)}{2} - (k-1)$

$\Rightarrow (k-1)\left(\frac{k}{2} - 1\right)$

$\underline{\text{Case-1}}$ $\quad d_G(v) = k-1$

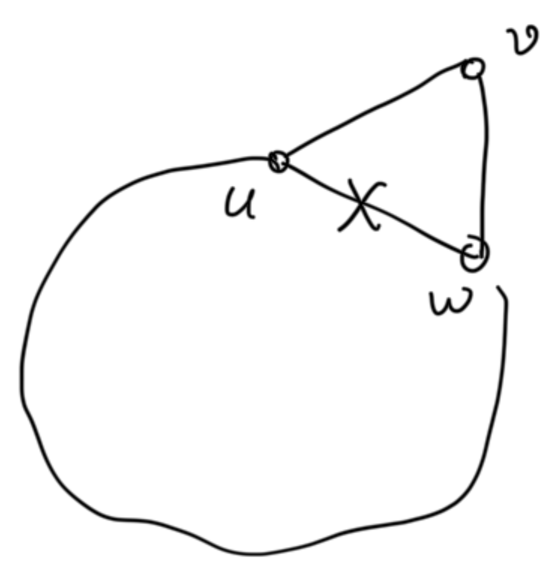
$$\underline{|E(G-v)|} = E(G) - (k-1) > \binom{k}{2} + 1 - k + 1$$
$$> \underbrace{\binom{k-1}{2} + 1}$$

By induction $G-v$ has a Ham. Cycle C

Since $d_G(v) \geq k-1$ and $V(G) = C \cup \{v\}$, there exist $u, w \in C$ st $uw \in E(C)$ and v is adjacent to u & w.

$$C^* = C - \{uw\} \cup \{uv, vw\}$$

C^* → Ham. Cycle of G.



$\underline{\text{Case-2}}$ $\quad d_G(v) = k$

$$|E(G-v)| > \binom{k-1}{2} \quad \checkmark \text{ check}$$

$\underline{\text{Case 2.1}}$ $\quad G-v$ is complete

↓

has Ham. Cycle C

$$C^* = C - \{u, w\} \cup \{uv, vw\}$$

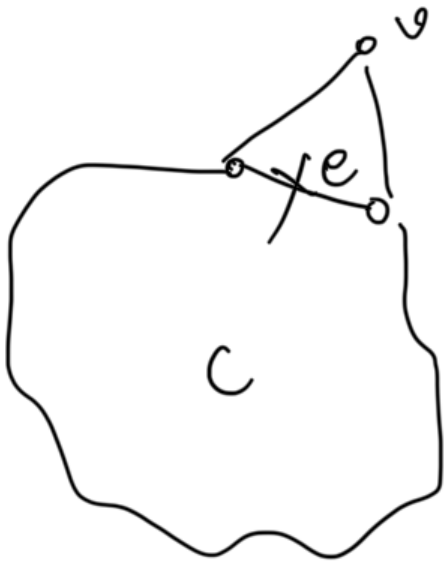
$\underline{\text{Case 2.2}}$ $\quad G-v$ is not complete.

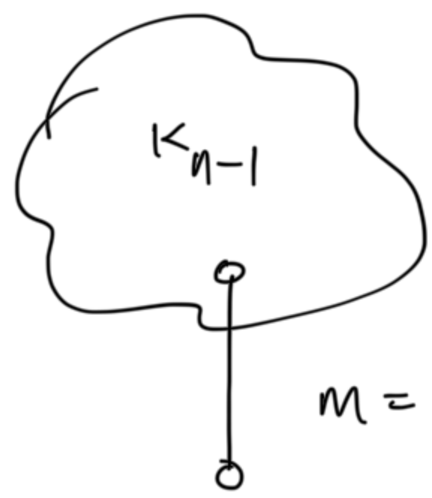
$G-v+e$ satisfies induction hypothesis

has Ham. cycle C

$e \in C$ $\qquad$ $e \notin C$

$\downarrow$ as pe $\quad C^* = C - \{u_0 w\} \cup \{uv, vw\}$





$m = \binom{n-1}{2} + 1$

Theorem:- Suppose G is a simple graph $n \geq 3$ vertices,
If $\delta(G) \geq \frac{n}{2}$, then G has a Ham. Cycle.

$\downarrow$ Dirac's Theorem.

Proof:-

Consider a maximal path $P = x_1 - \cdots - x_k$

$$S = \{x_i \mid \underline{1 \leq i \leq k-1},\ x_i x_k \in E(G)\} = N(x_k)$$

$$T = \{x_i \mid \underline{1 \leq i \leq k-1},\ x_{i+1} x_1 \in E(G)\}$$

$$\underset{d(x_k)}{\overset{\shortparallel}{|S|}} \geq \frac{n}{2} \qquad |T| = d(x_1) \geq \frac{n}{2}$$

$$|S \cup T| \leq k-1 < n$$

$$\begin{aligned} \underline{|S \cap T|} &= |S| + |T| - |S \cup T| \\ &\geq \frac{n}{2} + \frac{n}{2} - |S \cup T| \\ &> n - (n) \\ &> 0 \end{aligned}$$

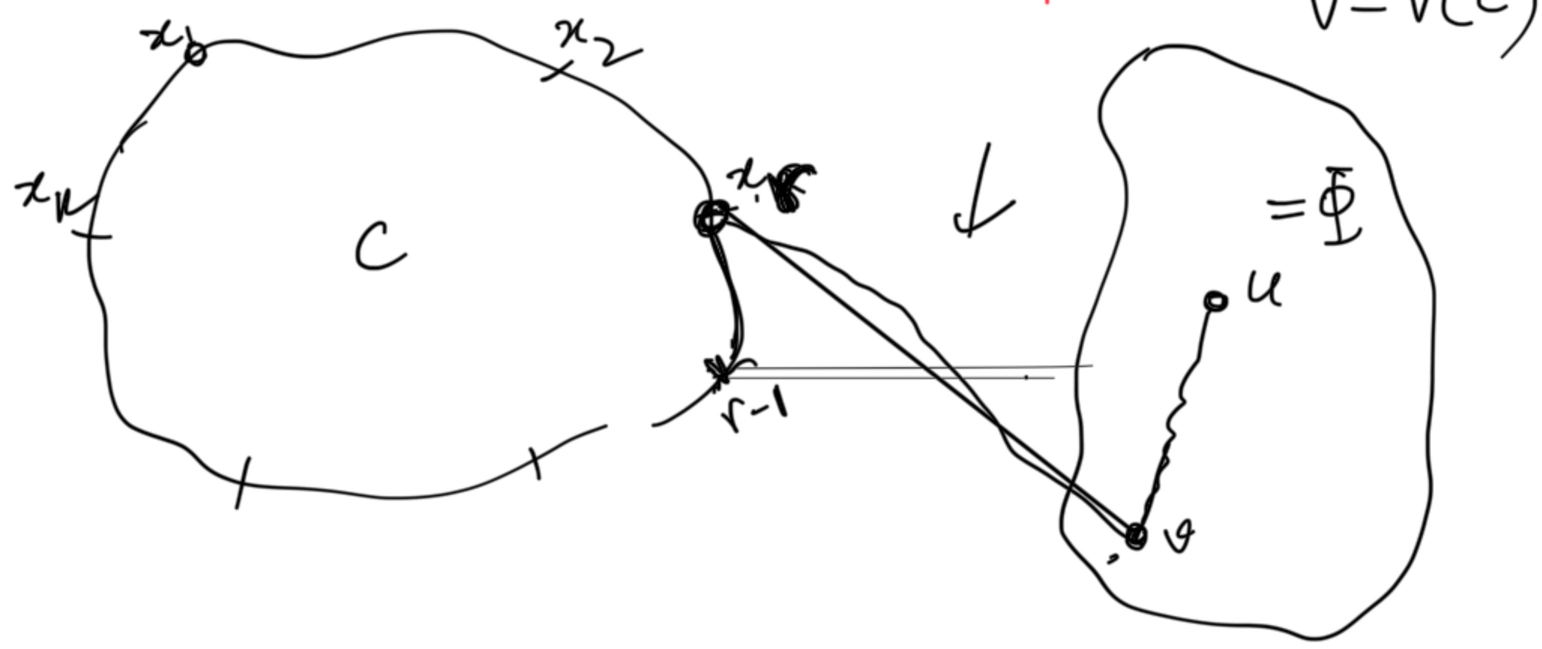
$$\Downarrow \quad \boxed{S \cap T \neq \Phi}$$

$\Rightarrow \exists \; i \in \{1, \dots, k-1\}$ st $x_i \in S \cap T$

$\Rightarrow x_i x_k, \; x_{i+1} x_1 \in E(G)$ by choice of S and T.

$C = x_1 \, x_{i+1} \, x_{i+2} \cdots x_k \, x_i \, x_{i-1} \cdots x_1$ is a cycle.

<u>claim</u>: C is a Ham cycle of G.



Let $u \in V - V(C)$, Since G is connected, $\exists$ a path b/w x_1 and u. Let v be the vertex from $V - V(C)$ that is adjacent to x_j for some $x_j \in C$. Such a vertex v exists.

Now the path from v to x_j ~~such that~~ along the cycle C and the edge $v x_j$ is a path of length $k+1$

$\Rightarrow \Leftarrow$ the maximality of P.

$\Rightarrow V - V(C) = \Phi$

$\Rightarrow C$ is a Ham. Cycle.

Ore's Theorem

Let G be a simple graph, $n \geq 3$ vertices. If u and v are distinct nonadjacent vertices such that $d(u) + d(v) \geq n$, then G has a Ham. cycle if and only if $G \vee \{uv\}$ has a Ham. cycle.

$\downarrow$

Proof :-