# SOLUTION MID SEMESTER
## GRAPH THEORY (MCD502)
## WINTER SEMESTER, 2022-23

**Q1**

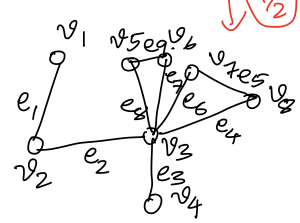
WALK:- It is an alternating sequence of vertices and edges in a graph starting and ending with vertices ½

TRAIL:- It is a walk in which no edge is repeated. ½

PATH:- It is a walk in which no vertex is repeated. ½

CIRCUIT:- It is a closed trail i.e. a trail whose starting and ending vertices are same. ½

CYCLE:- It is a closed path ½




$v_2 e_1 v_1 e_1 v_2 e_2 v_3 e_8 v_5 e_9 v_6 v_3$ is a walk, not a path or trail. ½

$v_1 e_1 v_2 e_2 v_3 e_8 v_5 e_9 v_6 v_3$ is a trail. (not a path) ½

$v_1 e_1 v_2 e_2 v_3 \longrightarrow$ a path. ½

$v_3 e_8 v_5 e_9 v_6 e_7 v_3 e_6 v_7 e_5 v_8 e_4 v_3 \longrightarrow$ Circuit (not a cycle) ½

$v_3 e_8 v_5 e_9 v_6 e_7 v_3 \longrightarrow$ cycle. ½

**Q2** (a)

$\cancel{5}$, 5, 4, 3, 2, 2, 2, 1 is graphic

$\Leftrightarrow$ subtracting and sorting.

$\cancel{4}$, 3, 2, 2, 1, 1, 1 is graphic

$\Leftrightarrow$

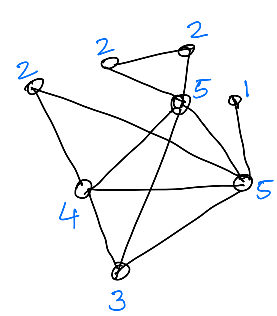
$\cancel{2}$, 1, 1, 1, 1, 0 is graphic

$\Leftrightarrow$

✓ 1, 1, 0, 0, 0 is graphic (True)

Hence the given sequence is graphic.

2




1

(b) $\cancel{5}, 5, 5, 3, 2, 2, 1, 1$ is graphic

$\Leftrightarrow$

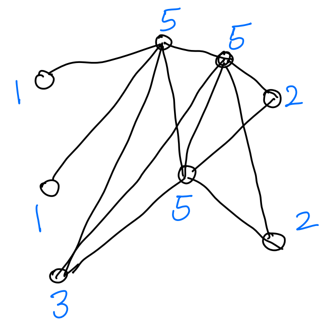
$\cancel{4}, 4, 2, 1, 1, 1, 1$ is graphic 2

$\Leftrightarrow$

$\cancel{3}, 1, 1, 1, 0, 0,$ is graphic

$\Leftrightarrow$

$0, 0, 0, 0, 0$ is graphic
(True)




1

Hence the given sequence is graphic

Q3

Given that $G$ is an acyclic graph with $n$ vertices

($\Rightarrow$:) Suppose that $G$ has exactly $n-1$ edges.

Assume that $G$ is not connected and $H_1, \dots, H_k$, $k \geq 2$ are the components of $G$.

Since $G$ is acyclic, each $H_i$, $i \in \{1, 2, \dots k\}$ is acyclic and hence each $H_i$ is a tree

$\Rightarrow$ $|E(H_i)| = |V(H_i)| - 1$ for each $i$ 1

$$|E(G)| = \sum_{i=1}^{k} |E(H_i)| = \sum_{i=1}^{k} |V(H_i)| - 1 = n - k$$

$\Rightarrow$ $n - 1 = n - k \Rightarrow \boxed{k = 1}$ a contradiction

$\Rightarrow$ $G$ is connected

$\Rightarrow$ For every $u, v \in V(G)$, $\exists$ a $u,v$-path.

Suppose there are two $u,v$-paths $P_1$ & $P_2$ between $u$ & $v$. Then $P_1 \cup P_2$ contains a cycle which 1

Contradicts that $G$ is acyclic

$\Rightarrow$ For every $u, v \in V(G)$, there is a unique $u,v$-path in $G$.

($\Leftarrow$:) Assume that in $G$, for every $u, v \in V(G)$, there is a unique path

$\Rightarrow$ $G$ is connected

$\Rightarrow$ $|E(G)| \geq n-1$ 2

If $|E(G)| \geq n$, then $G$ contains a cycle which is a contradiction

$\Rightarrow$ $G$ has exactly $n-1$ edges

Q4

Let $u, v \in V(G)$ be arbitrary. and $uv \notin E(G)$ (1)

Now $|N(u) \cup N(v)| = |N(u)| + |N(v)| - |N(u) \cap N(v)|$

Notice that $|N(u) \cup N(v)| \leq n-2$ (since $uv \notin E(G)$) (1)

Since $\delta(G) \geq \frac{n-1}{2}$, we have

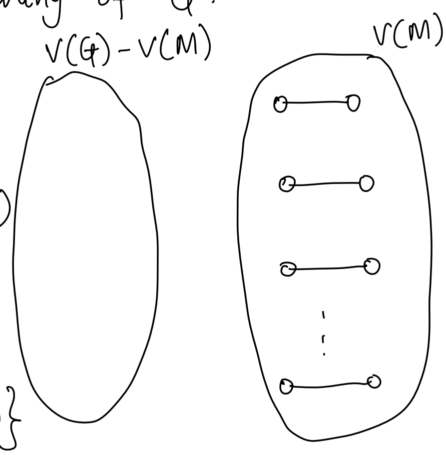
$$|N(u) \cap N(v)| = |N(u)| + |N(v)| - |N(u) \cup N(v)|$$
$$\geq \frac{n-1}{2} + \frac{n-1}{2} - (n-2)$$
$$\geq 1$$
(1)

Let $u, v \in V(G)$ be arbitrary. Then either $uv \in E(G)$ or $|N(u) \cap N(v)| \geq 1$. This implies we can get always an $u,v$-path in $G$ (1)

$\Rightarrow$ $G$ is connected.

Q5:-

Let $M$ be a $\max^m$ matching of $G$.




Note that for $u, v \in V(G) - V(M)$, $uv \notin E(G)$.

Since $G$ has no isolated vertices, for every $x \in V(G) - V(M)$ $\exists$ a $y \in V(M)$ st $xy \in E(G)$.

Let's denote such an edge by $e_x$ for $x$.

Now $L = M \cup \{e_x \mid x \in V(G) - V(M)\}$ forms an edge cover of $G$

$$\Rightarrow \quad \beta'(G) \leq |L| \leq |M| + |V(G) - V(M)|$$
$$= \alpha'(G) + |V(G)| - 2\alpha'(G)$$
$$= |V(G)| - \alpha'(G). \quad \text{———} \ (1)$$
(2)

Now consider that $L$ is a minimum edge cover of $G$. Then we can observe that for $e = xy$, if $e \in L$ then $x$ and $y$ cannot be simultaneously endpoints of other edges of $L$; otherwise $L - \{e\}$ is still an edge cover which is a contradiction.

$\Rightarrow$ If we consider any component of $L$, then it is a star.

Let $R =$ No. of components of $L$

$n_i =$ No. of vertices of the $i^{th}$ component.

$$n_1 + n_2 + \cdots + n_R = |V(G)|$$

$$|L| = n_1 - 1 + n_2 - 1 + \cdots + n_R - 1 = |V(G)| - R$$

Construct a matching $M$ by taking one edge from each component of $L$.

$$\Rightarrow \alpha'(G) \geq |M| = R = |V(G)| - |L|$$
$$= |V(G)| - \beta'(G) \quad \text{——— (2)}$$

(2)

From equations (1) & (2), we have

$$\alpha'(G) + \beta'(G) = |V(G)|$$

Q6

$$L(K_5) = \begin{bmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{bmatrix}$$

(2)

Any co-factor of $L(K_5)$ = No. of spanning trees of $K_5$

$$(1,1)\text{-cofactor} = \begin{vmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 15 & -5 & -5 \\ 0 & 5 & -5 & 0 \\ 0 & 0 & 5 & -5 \\ -1 & -1 & -1 & -4 \end{vmatrix}$$

$$= +1 \begin{vmatrix} 15 & -5 & -5 \\ 5 & -5 & 0 \\ 0 & 5 & -5 \end{vmatrix}$$

$$= \quad 15\,(25) - 5\,(25 + 25)$$
$$= \quad 5 \times 25 = 125$$
2

**Q7** ($\Rightarrow$:) Suppose that $G$ is bipartite. Then any cycle of $G$ will start and end at the same partite set. This makes the length of cycle even

$\Rightarrow$ $G$ does not contain any odd cycle.
2

($\Leftarrow$:) Assume that $G$ does not contain any odd cycle.

Let $u \in V(G)$ be arbitrary.

$X = \{ v \in V(G) \mid$ length of shortest $u,v$-path is odd$\}$.

$Y = \{ v \in V(G) \mid$ length of the shortest $u,v$-path is even$\}$.
1

By convention, we take $u \in Y$.

wlg, let $x, y \in X$ and $P_1$ = shortest $u,x$-path

$P_2$ = shortest $u,y$-path.
2

If $xy \in E(G)$, then $P_1 \cup P_2 \cup \{xy\}$ is a closed odd walk which will contain an odd cycle.

This is a contradiction. $\Rightarrow$ $xy \notin E(G)$

Similarly for $x', y' \in Y$, it can be proved that $x'y' \notin E(G)$

Therefore, $G$ is a bipartite graph.