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20W-Lecture-...

Lecture 7: Monday January 27

2.5 Stochastic Convergences

Limiting process which involves sequence of random variables; tools for studying the asymptotic behavior of a statistic $T_n = T(X_1, \cdots, X_n)$ when the sample size n gets large: the behavior of the sequence $T_1, T_2, \cdots, T_n, \cdots$. $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

(1) Convergence in probability

Definition: Let X, X_1, X_2, $\cdots$ be a sequence of random variables. If for any $\varepsilon > 0$,

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0\,,$$

or equivalently,

$$\lim_{n\to\infty} P(|X_n - X| < \varepsilon) = 1\,,$$

we say "X_n converges to X in probability", denoted by $X_n \xrightarrow{p} X$.

The limit X could be a constant: $X_n \xrightarrow{p} c$ if

$$\lim_{n\to\infty} P(|X_n - c| \geq \varepsilon) = 0$$

for any $\varepsilon > 0$.

The weak law of large numbers (WLLN):

Let $X_1, X_2, \cdots$ be iid random variables with common mean $\mu = E(X_i)$ and common variance $\sigma^2 = Var(X_i) < +\infty$. Then

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{p} \mu$$

Need to show: for $\forall \varepsilon > 0$,

$P(|\bar{X}_n - \mu| > \varepsilon) \longrightarrow 0$ as $n \to +\infty$

$P(|X - \mu| > \varepsilon) \leq \frac{Var(X)}{\varepsilon^2}$ $\mu = E(X)$

$E(\bar{X}_n) = \mu$

$Var(\bar{X}_n) = \frac{\sigma^2}{n}$

$\leq \frac{Var(\bar{X}_n)}{\varepsilon^2} = \frac{\frac{\sigma^2}{n}}{\varepsilon^2}$

(2) Convergence in distribution

Example: $X_n \sim t(n)$, then $F_n(x) \to F_Z(x)$, where $Z \sim N(0,1)$.

$f_n(x) = C_n \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}} \to \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

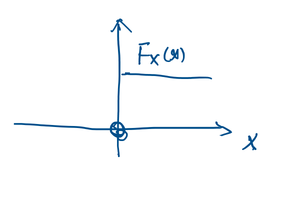
$F_n(x) = \int_{-\infty}^{x} f_n(t)\,dt \to \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dt.$

Example. $Z_1, Z_2, \cdots$, are iid $N(0,1)$. $\bar{Z}_n \sim N(0, 1/n)$. Let the cdf of $\bar{Z}_n$ be $F_n(x)$. Find $\lim_{n\to\infty} F_n(x)$.

$X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$

$\bar{X}_n \sim N(\mu, \frac{\sigma^2}{n})$

$\bar{Z}_n \sim N(0, \frac{1}{n})$



Q: $F_n(x) \to F_X(x)$ where $x = 0$.

$F_X(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$

$F_n(x) = P(\bar{Z}_n \leq x) = P\left(\frac{\bar{Z}_n - 0}{1/\sqrt{n}} \leq \frac{x - 0}{1/\sqrt{n}}\right)$ $\sqrt{n}x$

$= \Phi(\sqrt{n}\,x)$

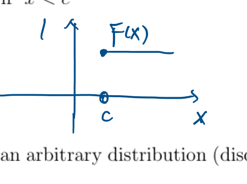
$= \begin{cases} 1 & \text{if } x > 0 \\ \frac{1}{2} & \text{if } x = 0 \\ 0 & \text{if } x < 0 \end{cases}$ as $n \to +\infty$

(The limit is not a cdf!).

Definition: Suppose that X_n has cdf $F_n(x)$ and X has cdf $F(x)$. If $\lim_{n\to\infty} F_n(x) = F(x)$ for all x where $F(x)$ is continuous, we say "X_n converges to X in distribution", denoted by $X_n \xrightarrow{d} X$.

If $X = c$, then

$$F(x) = P(X \leq x) = \begin{cases} 1 & \text{if } x \geq c \\ 0 & \text{if } x < c \end{cases}$$



The Central Limit Theorem (CLT):

Let $X_1, X_2, \cdots$ be iid random variables from X with an arbitrary distribution (discrete or continuous). Let $\mu = E(X)$ and $\sigma^2 = Var(X) < +\infty$. Then the sample mean $\bar{X}_n = n^{-1}\sum_{i=1}^{n} X_i$ satisfies

$$W_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

if $X_1, \ldots, X_n$ iid $N(\mu, \sigma^2)$, then $\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$\mu = E(X_i)$; $\sigma^2 = Var(X_i)$

$E(\bar{X}_n) = \mu$; $Var(\bar{X}_n) = \frac{\sigma^2}{n}$ $W_n = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}$

$E(W_n) = 0$, $Var(W_n) = 1$

Need to show:

$F_{W_n}(x) \longrightarrow \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$

as $n \to +\infty$

We show:

$M_{W_n}(t) \to M_Z(t) = e^{\frac{t^2}{2}}$

$W_n = \sqrt{n}\,\frac{\bar{X}_n - \mu}{\sigma}$ $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

$= \sqrt{n} \cdot \frac{1}{n}\sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$

$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i$

$Z_i = \frac{X_i - \mu}{\sigma}$, $i = 1, 2, \ldots$

(i) $Z_1, Z_2, Z_3 \cdots$ iid

(ii) $E(Z_i) = 0$; $Var(Z_i) = 1$

(iii) $E(Z_i^2) = 1$

(iv) let $M(t)$ be the MGF of Z_i

$M(0) = 1$. $M'(0) = E(Z_i) = 0$. $M''(0) = E(Z^2) = 1$

$M(t) = E(e^{tX})$

$W_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i$

MGF of Z_i: $M(t)$

MGF of $Z_1 + Z_2 + \cdots + Z_n$: $[M(t)]^n$.

MGF of W_n:

$M_{W_n}(t) = \left[M\left(\frac{t}{\sqrt{n}}\right)\right]^n$

$M(t) = M(0) + M'(0)t + \frac{M''(0)t^2}{2} + \cdots$

$M_{W_n}(t) = [M(t)]^n$.

$= \left[1 + \frac{1}{2}\frac{t^2}{n} + \cancel{\cdots}\right]^n \to e^{\frac{t^2}{2}}$ as $n \to +\infty$

(3) Useful results on stochastic convergences

(a) "$X_n \xrightarrow{p} X$" implies "$X_n \xrightarrow{d} X$".

if "$P(|X_n - X| > \varepsilon) \to 0$ for $\forall \varepsilon > 0$".

then "$F_n(x) \to F(x)$ for any x where $F(x)$ is continuous".

$F_n(x) = P(X_n \leq x)$ ...... $P(X \leq x) = F(x)$

$= P(X_n \leq x, |X_n - X| \leq \varepsilon) + P(X_n \leq x, |X_n - X| > \varepsilon)$

$= A + B$.

$B = P(X_n \leq x, |X_n - X| > \varepsilon)$.

$\leq P(|X_n - X| > \varepsilon)$

$\to 0$, as $n \to \infty$.

$A = P(X_n \leq x, |X_n - X| < \varepsilon) = P(X_n \leq x, X_n - \varepsilon \leq X \leq X_n + \varepsilon) \leq P(X \leq x + \varepsilon)$.

$\lim_{n\to\infty} F_n(x) \leq F(x + \varepsilon)$

$F(x - \varepsilon) \leq \underline{\lim}_{n\to\infty} F_n(x) \leq \overline{\lim}_{n\to\infty} F_n(x) \leq F(x + \varepsilon)$

let $\varepsilon \to 0$:

$F(x - 0) \leq \underline{\lim} F_n(x) \leq \overline{\lim} F_n(x) \leq F(x + 0)$

$\lim_{n\to\infty} F_n(x) = F(x)$

where $F(x)$ is cts at x.

(b) "$X_n \xrightarrow{d} X$" does not imply "$X_n \xrightarrow{p} X$".

Example: $X_1, \ldots, X_n$ iid $N(0,1)$. $F_n(x) = \Phi(x)$

So $X_n \xrightarrow{d} N(0,1)$

For $\varepsilon > 0$: $P(|X_n - X| \geq \varepsilon) = ?$

X and X_n are indep. $X \sim N(0,1)$; $X_n \sim N(0,1)$

$X_n - X \sim N(0, 2)$.

$P(|X_n - X| \geq \varepsilon) \not\to 0$.

(c) "$X_n \xrightarrow{d} c$" and "$X_n \xrightarrow{p} c$" are equivalent.

Need to show that "$X_n \xrightarrow{d} c$" implies "$X_n \xrightarrow{p} c$".

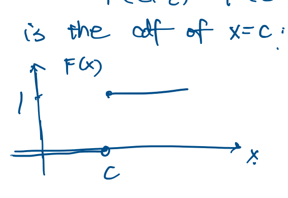
$\forall \varepsilon > 0$, $P(|X_n - c| \leq \varepsilon) \longrightarrow 1$.

$= P(c - \varepsilon \leq X_n \leq c + \varepsilon)$.

$= F_n(c + \varepsilon) - F_n(c - \varepsilon)$.

$\to F(c + \varepsilon) - F(c - \varepsilon)$ as $n \to +\infty$.

$F(x)$ is the cdf of $X = c$:



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(d) **CLT** implies **WLLN**.

$\sqrt{n}\,\frac{\bar{X}_n - \mu}{\sigma} \xrightarrow{d} N(0,1)$.

WLLN: $\bar{X}_n \xrightarrow{P} \mu$.

$\forall \varepsilon > 0$. $P(|\bar{X}_n - \mu| \leq \varepsilon) \to 1$.

$P(|\bar{X}_n - \mu| \leq \varepsilon) = P\left(\frac{|\bar{X}_n - \mu|}{\sigma/\sqrt{n}} \leq \frac{\varepsilon}{\sigma/\sqrt{n}}\right)$

$= P\left(-\frac{\varepsilon}{\sigma/\sqrt{n}} \leq W_n \leq \frac{\varepsilon}{\sigma/\sqrt{n}}\right)$

$= F_{W_n}\left(\frac{\varepsilon}{\sigma/\sqrt{n}}\right) - F_{W_n}\left(-\frac{\varepsilon}{\sigma/\sqrt{n}}\right) \to \Phi(+\infty) - \Phi(-\infty) = 1$

(e) Slutsky's Theorem

Let X_n, $n = 1, 2, \cdots$ and Y_n, $n = 1, 2, \cdots$ be two sequence of random variables with $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$. Then

(i) $X_n + Y_n \xrightarrow{d} X + c$

(ii) $X_n Y_n \xrightarrow{d} cX$

(iii) $X_n / Y_n \xrightarrow{d} X/c$ $(c \neq 0)$

(i) $P(X_n \leq x) \to P(X \leq x)$

$F_n(x) \to F(x)$

$Y_n \xrightarrow{P} c$.

$P(X_n + Y_n \leq t) \to P(X + c \leq t)$

$LHS = P(X_n + Y_n \leq t, |Y_n - c| \leq \varepsilon) + P(X_n + Y_n \leq t, |X_n - c| > \varepsilon)$ ... $\leq P(X + c \leq t + \varepsilon)$

Example. Let $(X_1, \cdots, X_n)$ be an iid sample from X with $\mu = E(X)$ and $Var(X) = \sigma^2$. Let $\bar{X}_n$ and S_n^2 be the sample mean and the sample variance. We have

(i) $\sqrt{n}(\bar{X}_n - \mu)/\sigma \xrightarrow{d} N(0,1)$; (CLT)

(ii) $S_n^2 \xrightarrow{p} \sigma^2$;

(iii) $S_n \xrightarrow{p} \sigma$ $(S_n = \sqrt{S_n^2})$.

By the Slutsky's Theorem, (i) and (iii) lead to $\sqrt{n}(\bar{X}_n - \mu)/S_n \xrightarrow{d} N(0,1)$

$\sqrt{n}\,\frac{\bar{X}_n - \mu}{S_n} = \underbrace{\sqrt{n}\,\frac{\bar{X}_n - \mu}{\sigma}}_{\xrightarrow{d} N(0,1)} \cdot \underbrace{\frac{\sigma}{S_n}}_{\xrightarrow{P} 1}$

(f) The Delta-Method

Suppose that a constant sequence a_n and a random sequence X_n satisfy (as $n \to +\infty$)

i) $a_n \to +\infty$, $\sqrt{n}(\bar{X}_n - \mu)$

ii) $a_n(X_n - c) \xrightarrow{d} N(0, \sigma^2)$ where c is a constant.

Suppose that $g(x)$ is differentiable at $x = c$. Then

$$a_n[g(X_n) - g(c)] \xrightarrow{d} N(0, [g'(c)\sigma]^2)$$

$\sqrt{n}\,[g(\bar{X}_n) - g(\mu)] \xrightarrow{d} N(0, [g'(\mu)\sigma]^2)$

Example. Suppose that $(X_1, \cdots, X_n)$ is an iid sample from X with $\mu = E(X)$ and $\sigma^2 = Var(X)$. Let $\bar{X}$ be the sample mean. Find the asymptotic distribution of $\sqrt{n}(\bar{X}_n^2 - \mu^2)$.

$g(\bar{X}_n) = \bar{X}_n^2$

$g'(\mu) = 2\mu$

$\sqrt{n}(\bar{X}_n^2 - \mu^2) \xrightarrow{d} N(0, (2\mu\sigma)^2)$

$(0, 4\mu^2\sigma^2)$

$a_n \to +\infty$

$a_n(X_n - c) \xrightarrow{d} N(0, \sigma^2)$

$a_n(X_n - c) = O_p(1)$ something goes to zero as you move alone.

$X_n - c = O_p\left(\frac{1}{a_n}\right) = o_p(1)$

The Taylor series expansion:

$g(x) = g(x_0) + g'(x_0)(x - x_0) + \frac{1}{2}g''(x_0)\cancel{(x - x_0)^2} + \cancel{\cdots}$

$g(X_n) = g(c) + g'(c)(X_n - c) + o_p(X_n - c)$

$a_n[g(X_n) - g(c)] = g'(c)\underbrace{[a_n(X_n - c)]}_{\xrightarrow{d} N(0,\sigma^2)} + \underbrace{o_p[a_n(X_n - c)]}_{\xrightarrow{P} 0}$