Coin Toss Experiment — LLN & CLT

Purpose

Hi! I'm Muhammed İkbal. This project demonstrates, in one go,
the **Law of Large Numbers (LLN)** and the **Central Limit Theorem (CLT)**
using a simple coin-toss experiment.

You press Enter to advance through increasing sample sizes:
N = 10^2, 10^3, 10^4, 10^5.

For each N, you get two visual panels:

- **(A) LLN** — Running mean of a single length-N sequence of Bernoulli(0.5) trials. You will see the running average stabilizing around *p = 0.5* as n grows.
- **(B) CLT** — Sampling distribution of p-hat across M independent experiments. Over the histogram we overlay the Normal approximation $\mathcal{N}(p, p(1-p)/n)$.

The whole point is to let you see both laws in action.

How to Run

1. Install dependencies: pip install numpy matplotlib
2. Run the script: python coin-ex.ipynb
3. Press Enter to cycle through N, or type q to quit.

Mathematical Commentary — by Muhammed İkbal Yılmaz

Setup

Let $X_i \sim \text{Bernoulli}(p)$, $i = 1, 2, \ldots, n$, independent.
We observe heads (1) or tails (0).
The sample mean is:

$$\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Law of Large Numbers (LLN)

As n increases along a single realization:

$$\hat{p}_n \;\to\; p \quad \text{(almost surely)}.$$

In practice, you will see the running mean curve stabilize around $p = 0.5$.
This addresses **convergence of one path**.

Central Limit Theorem (CLT)

Across many independent experiments (each of size n):

$$\sqrt{n}\,(\hat{p}_n - p) \quad \Rightarrow \quad \mathcal{N}(0,\ p(1-p))$$

Equivalently:

$$\hat{p}_n \ \approx \ \mathcal{N}\left(p,\ \frac{p(1-p)}{n}\right).$$

Hence, the histogram of $\hat{p}$ looks Gaussian with variance shrinking as $1/n$. The overlaid bell curve is $\mathcal{N}(p,\ p(1-p)/n)$. This addresses **distribution across many paths**.

Why Both?

- **LLN** explains stabilization of the average in a single long run.
- **CLT** explains the spread of averages across many independent runs at fixed (n).

Seeing them together avoids the common confusion between convergence of a sequence (LLN) and the distributional shape of sample averages (CLT).

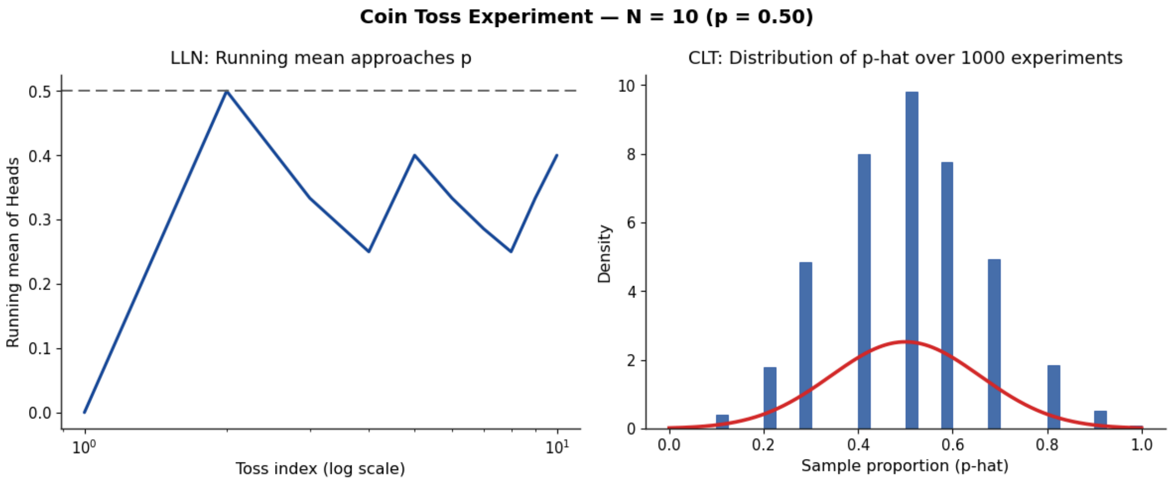
In [4]:

```
import sys
import math
import numpy as np
import matplotlib.pyplot as plt

# ============================================================
# Coin Toss Experiment — LLN & CLT  (Author: Muhammed İkbal Yılmaz)
# Added feature: after each run, print counts of heads/tails
# ============================================================

# ----------------------- Configuration -----------------------
NS = [10, 100, 1_000, 10_000]    # sample sizes per Enter
M_REPS = 1_000                   # number of experiments for the CLT histo
PROB_HEAD = 0.5                  # fair coin probability
RNG_SEED = 42                    # set None for fresh randomness
BINS = 40                        # histogram bins for p-hat

# Visual palette
DEEP_BLUE = "#0B3D91"
DEEP_BLUE_EDGE = "#0B3D91"
NORMAL_RED = "#D62828"

plt.rcParams.update({
    "figure.figsize": (12, 5),
    "axes.titlesize": 13,
    "axes.labelsize": 11.5,
```

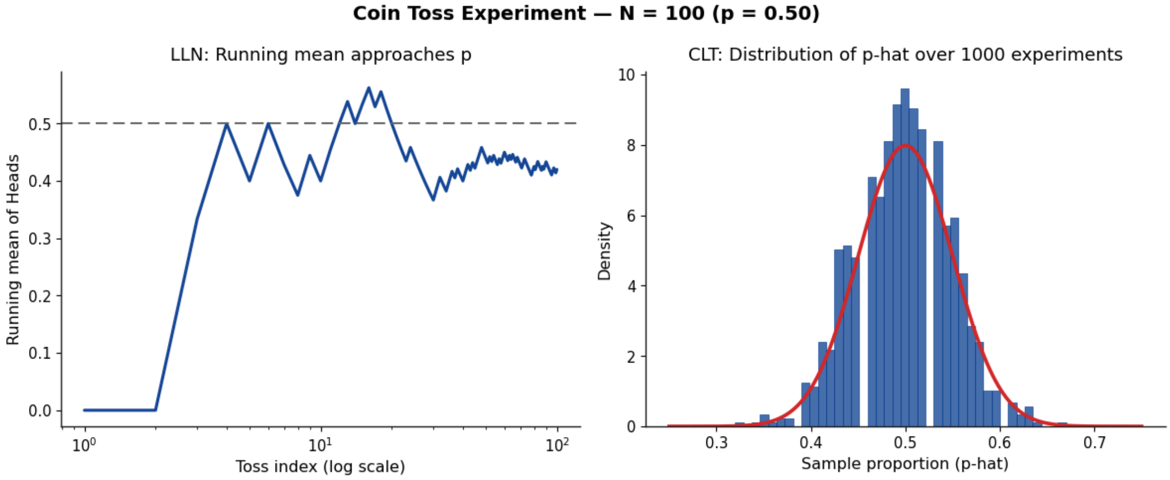
```
    "xtick.labelsize": 10.5,
    "ytick.labelsize": 10.5,
    "figure.autolayout": False,
})

rng = np.random.default_rng(RNG_SEED)


# ------------------------ Styling Helpers ----------------------
def style_axes_minimal(ax, keep=("left", "bottom")):
    for side in ("top", "right", "left", "bottom"):
        ax.spines[side].set_visible(False)
    for side in keep:
        ax.spines[side].set_visible(True)
        ax.spines[side].set_linewidth(1.0)
        ax.spines[side].set_color("#222222")
    ax.yaxis.set_ticks_position("left" if "left" in keep else "none")
    ax.xaxis.set_ticks_position("bottom" if "bottom" in keep else "none")
    ax.grid(False)


# ------------------------ Core Logic ---------------------------
def single_run_sequence(n: int, p: float = PROB_HEAD, rng=None) -> np.nda
    """Generate one sequence of n coin tosses (1=heads, 0=tails)."""
    if rng is None:
        rng = np.random.default_rng()
    return (rng.random(n) < p).astype(int)

def running_mean_from_sequence(seq: np.ndarray) -> np.ndarray:
    cumsum = np.cumsum(seq)
    k = np.arange(1, len(seq) + 1, dtype=float)
    return cumsum / k

def sampling_distribution_p_hat(n: int, m: int = M_REPS, p: float = PROB_
    if rng is None:
        rng = np.random.default_rng()
    samples = rng.random((m, n)) < p
    return samples.mean(axis=1).astype(float)

def normal_pdf(x: np.ndarray, mu: float, sigma: float) -> np.ndarray:
    coef = 1.0 / (math.sqrt(2.0 * math.pi) * sigma)
    z = (x - mu) / sigma
    return coef * np.exp(-0.5 * z * z)

def plot_lln_and_clt(seq: np.ndarray, m_reps: int = M_REPS, p: float = PR
    n = len(seq)
    running_mean = running_mean_from_sequence(seq)
    p_hats = sampling_distribution_p_hat(n, m_reps, p, rng=rng)

    fig, axes = plt.subplots(1, 2, figsize=(12, 5))
    fig.suptitle(f"Coin Toss Experiment — N = {n} (p = {p:.2f})", y=0.98,

    # Panel A: LLN
    ax = axes[0]
    ax.plot(np.arange(1, n + 1), running_mean, linewidth=2.2, color=DEEP_
    ax.axhline(p, linestyle="--", linewidth=1.4, color="#666666", dashes=
    ax.set_xscale("log")
    ax.set_xlabel("Toss index (log scale)")
    ax.set_ylabel("Running mean of Heads")
    ax.set_title("LLN: Running mean approaches p", pad=8)
```

```
    style_axes_minimal(ax, keep=("left", "bottom"))

    # Panel B: CLT
    ax2 = axes[1]
    ax2.hist(
        p_hats,
        bins=BINS,
        density=True,
        alpha=0.75,
        color=DEEP_BLUE,
        edgecolor=DEEP_BLUE_EDGE,
        linewidth=0.6,
    )
    ax2.set_xlabel("Sample proportion (p-hat)")
    ax2.set_ylabel("Density")
    ax2.set_title(f"CLT: Distribution of p-hat over {m_reps} experiments"

    mu = p
    sigma = math.sqrt(p * (1.0 - p) / n)
    x_min = max(0.0, mu - 5 * sigma)
    x_max = min(1.0, mu + 5 * sigma)
    x = np.linspace(x_min, x_max, 500)
    ax2.plot(x, normal_pdf(x, mu, sigma), linewidth=2.6, color=NORMAL_RED

    style_axes_minimal(ax2, keep=("left", "bottom"))
    fig.tight_layout(pad=1.2)
    plt.show()


# ----------------------- CLI / Interaction -------------------
def main():
    print("\nCoin Toss Experiment — LLN & CLT (Author: Muhammed İkbal Yıl
    print("Press <Enter> to run the next N, or type 'q' to quit.\n")
    for n in NS:
        msg = input(f"Ready for N = {n} tosses (M = {M_REPS} experiments
        if msg.strip().lower() == "q":
            print("Exiting. Bye!")
            return
        seq = single_run_sequence(n, p=PROB_HEAD, rng=rng)
        heads = int(seq.sum())
        tails = n - heads
        print(f"\nResults for N = {n}: Heads = {heads}, Tails = {tails}\n
        plot_lln_and_clt(seq, m_reps=M_REPS, p=PROB_HEAD, rng=rng)
    print("All experiments completed. Exiting.")


if __name__ == "__main__":
    try:
        main()
    except KeyboardInterrupt:
        print("\nInterrupted. Bye!")
        sys.exit(0)
```

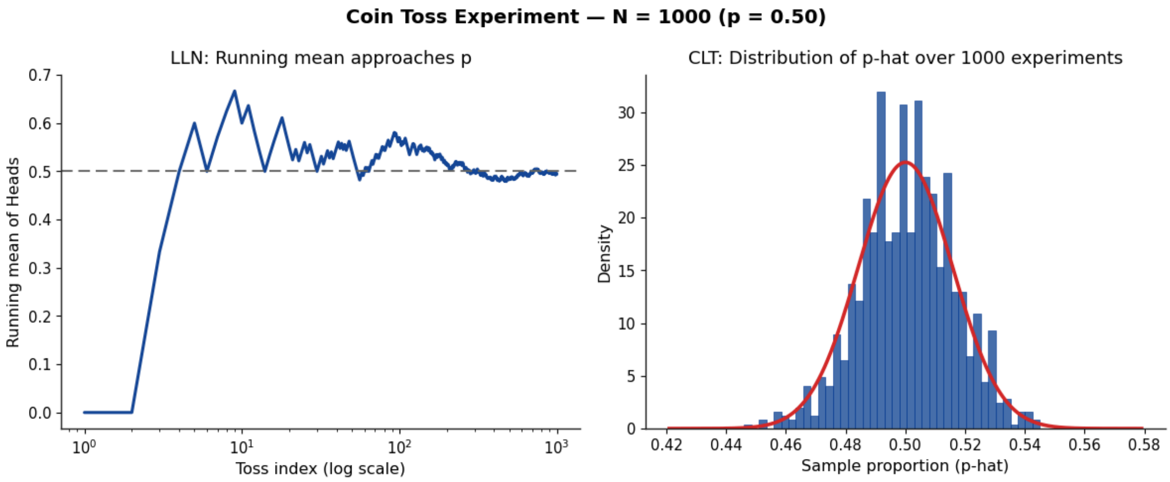
```
Coin Toss Experiment — LLN & CLT (Author: Muhammed İkbal Yılmaz)
Press <Enter> to run the next N, or type 'q' to quit.

Results for N = 10: Heads = 4, Tails = 6
```



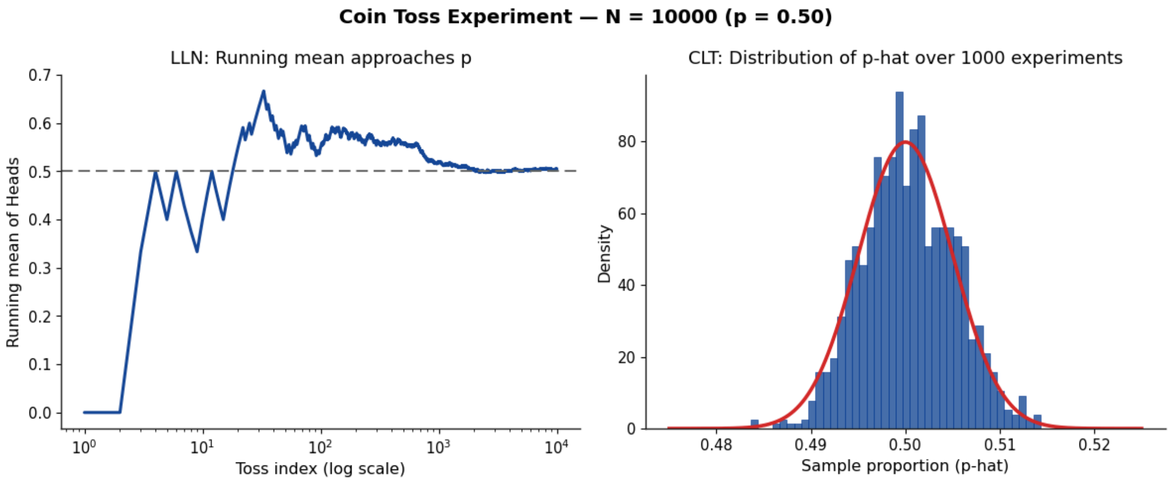

```
Results for N = 100: Heads = 42, Tails = 58
```




```
Results for N = 1000: Heads = 494, Tails = 506
```




```
Results for N = 10000: Heads = 5048, Tails = 4952
```




```
All experiments completed. Exiting.
```

It is observed that as the number of coin tosses increases, the empirical probability of getting heads or tails converges towards the theoretical value of one of the two possible outcomes, namely 0.5. Formally, by the Law of Large Numbers, $\hat{p}_n = \frac{1}{n}\sum_{i=1}^{n} X_i \to 0.5$ as $n \to \infty$. At the same time, the histogram plots of the sample proportions, although relatively sharp, exhibit a stable distribution that aligns with the Central Limit Theorem, which states that $\hat{p}_n \sim \mathcal{N}(0.5,\ 0.25/n)$. Especially after the $10{,}000^{\text{th}}$ toss, the observed probability essentially settles at 0.5, confirming that empirical results converge to theoretical expectations as the sample size grows.