Tutorial 1 Problems

for discussion on Week 45

Sets, countability, events and σ-algebras (σ-fields)

1. Describe the sample space Ω for the following experiments:

   (a) Toss of 3 indistinguishable coins;

   (b) 3 tosses of one coin;

   (c) Drawing 2 balls from an urn containing 2 black and 2 white balls;

   (d) A distance from a landed dart to the 'bullseye' (the centre of the dartboard);

   (e) Picking a random ark from a circle (give at least two ways of defining Ω!)

2. Prove the second de Morgan's law: $(A \cap B)^{\mathbf{c}} = A^{\mathbf{c}} \cup B^{\mathbf{c}}$.

3. Prove that a Cartesian product of two countable sets is countable.

4. Prove that a Cartesian product of countably many finite sets with at least two elements in each is uncountable.

5. Describe the σ-field of subsets of $\Omega = [0, 1]$ generated by the system of singletons, i.e. the sets $\{x\}$, $x \in [0, 1]$.

6. Let $[\Omega_1, \mathcal{F}_1]$, $[\Omega_2, \mathcal{F}_2]$ be two measurable spaces. Call a *rectangle* the sets from $\Omega_1 \times \Omega_2$ of the type $A \times B = \{(x, y) :\ x \in A,\ y \in B\}$ for some $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$. Show that the system of rectangles is a semi-ring, but generally *not* a σ-field. (Just construct a simple counterexample). When *is* it a σ-field?

7. Prove that the intersection of two (actually, of any system of) σ-fields of a same sample space Ω is a σ-field.

8. Prove that the Borel σ-field of subsets of $\mathbb{R}$ is countably generated, i.e. there is a countable system of subsets of $\mathbb{R}$ which also generates it. **NB.** Prove that any open interval can be represented as a countable union of intervals with rational endpoints.

9. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. Prove that

(a) for any growing system of sets $F_i \in \mathcal{F}$, i.e. $F_{i+1} \subseteq F_i$ for all i,

$$\mathbf{P}(\cap_i F_i) = \lim_{i\to\infty} \mathbf{P}(F_i);$$

(b) for any decaying system of sets O_i, i.e. $O_i \subseteq O_{i+1}$ for all i,

$$\mathbf{P}(\cup_i O_i) = \lim_{i\to\infty} \mathbf{P}(O_i).$$

10. Prove Problem 1.2.4 from GS [1], (i.e. Problem 4 for Section 1.2 – see page 4).

Probability and its main properties

1. Make sure you can easily do Classwork>Study 1.1: Elementary probability on the VLE: `vle.math.chalmers.se`

2. Our experiment is to choose a random ark on the unit circle in the plane centred at the origin which *starts* at $(1, 0)$ and goes along the upper unit semi-circle in the contra-clockwise direction. One way to describe such an ark is by its length $l \in [0, \pi] = \Omega_1$. The other one is by identifying the end point by its first coordinate: x (so that the end point has coordinates $(x, \sqrt{1 - x^2})$). Then $x \in [-1, 1] = \Omega_2$. A *randomly selected ark* corresponds in both models to the uniform measure on the sample space, i.e. the Lebesgue measure scaled to have total mass of Ω to be 1. Let B be the event that the chosen ark has length at most $\pi/3$. Indicate graphically event B as a subset of Ω_1 and of Ω_2 and compute the probability of B in both models. Are these the same? Can you explain this?

3. A right tetrahedron (a prisme with 4 faces) is tossed. Three its sides are coloured red, green and blue, respectively, and the forth sides contains all of these colours. Let R (resp. G, B) denote the event that a face containing red (resp. blue, green) colour touches the ground. Show that these events are pairwise independent, but not mutually independent.

[1] In what follows, GS stands for the course book: Geoffrey Grimmett and David Stirzaker, *Probability and Random Processes*, Oxford University Press, 3rd edition, 2001. Problem x.y.z means Problem z for Section x.y in this book

4. Show that if two events A and B cannot happen at the same time and their probabilities are non-zero, then they are dependent (i.e. not independent).

5. Problem 1.7.4 from GS.
NB. Notice that often a correct mathematical formulation of the problem is half its solution! Imagine an experiment when someone asks you every morning if you prefer a cup of coffee (x) to tea (y), and presumably your answer may depend on whether you slept well tonight (C). Denote by X an event that at a randomly selected day you prefer coffee to tea. Is $\mathbf{P}(X) = 1$ under the given conditions?

Symmetric experiment and combinatorics

1. A cube with all its faces coloured is cut into 1000 equal size small cubes. Find the probability that a small ball chosen at random has exactly 2 coloured faces.

2. Problem 1.8.1 from GS