MATHEMATICAL LOGIC FOR COMPUTER SCIENCE (A.Y. 20/21)

HANDOUT N. 14

Abstract. The Compactness Theorem. Applications of Compactness.

1. The Compactness Theorem

Consider a theory T consisting of infinitely many sentences $S_0, S_1, S_2, \ldots$ in some language $\mathcal{L}$. If there is a structure $\mathfrak{A}$ such that $\mathfrak{A} \models T$ then, obviously, for any finite $T_0 \subseteq T$, we also have $\mathfrak{A} \models T_0$.

On the other hand, suppose that for every finite subset T_0 of T, there is a model, say $\mathfrak{A}_{T_0}$ such that $\mathfrak{A}_{T_0} \models T$. Then is it the case that there is a model $\mathfrak{A}$ such that $\mathfrak{A} \models T$?

To appreciate the non-triviality of the question consider an enumeration $S_0, S_1, S_2, S_3, \ldots$ of all finite subsets of a countable theory T. The assumption is the following: for every n there is a model, say $\mathfrak{A}_n$ of S_n, i.e., the following map is defined:

$$S_n \mapsto \mathfrak{A}_n.$$

Note however that the association of a model $\mathfrak{A}_n$ to the finite subtheory S_n of T can be highly non-uniform: two models $\mathfrak{A}_n$ and $\mathfrak{A}_m$ for $n \neq m$ can be completely different from one another, including being defined on different domains, giving different interpretations of the symbols of the language, etc. Then it is not at all evident that we can be sure that a "global" model $\mathfrak{A}$ for T exists. If we take the problem as an algebraic/structural one, we would need a method to "glue" together the infinitely many "local" models $\mathfrak{A}_0, \mathfrak{A}_1, \mathfrak{A}_2, \ldots$ of the finite subtheories $S_0, S_1, S_2, \ldots$ into a single "global" model $\mathfrak{A}$ for the theory T. It is indeed possible to do so, and the crucial concept is that of an ultrafilter.

However, the following proposition shows that if we translate the semantic question into a syntactic one, using the equivalence between "T is consistent" and "T has a model", guaranteed by the Completeness Theorem, the question has an easy answer. The answer is essentially dependent on the finitary concept of formal proof.

Theorem 1.1 (Compactness Theorem)**.** *A theory T has a model iff every finite subset of T has a model.*

Proof. The non trivial direction is from right to left. Suppose that every finite subset T^* of T has a model. Then T^* is consistent. Suppose that T has no model. By Completeness this is true iff T is inconsistent. Let π be a proof of a contradiction from premises in T. Since proofs are finite, there exists a finite subset T_π of T such that π is a proof of a contradiction from premises in T_π. But we have seen that T_π is consistent, since it has a model. Contradiction. □

The Compactness Theorem is a powerful tool for proving non-axiomatizability of properties over the class of all structures, for building models of theories with particular properties, and for proving theorems in mathematics.

The following equivalent formulation of the Compactness Theorem is sometimes useful.

Proposition 1.2. *The Compactness Theorem is equivalent to the following double implication:*

$$T \models E \text{ if and only if for some finite } T_0 \subseteq T \text{ we have } T_0 \models E.$$

Proof. Exercise. □

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2. (Non) Expressibility/Axiomatizability

Given a property P of (arbitrary) structures, it is natural to ask if P can be expressed/defined by first-order sentences, i.e. if there is a set of sentences T that satisfies

$$\mathfrak{A} \models T \iff \mathfrak{A} \text{ has the property } P.$$

If such a set T exists, we say that T *axiomatizes* P and P is a *axiomatizable property.*

One can ask if there is a *finite* set of sentences that axiomatizes P. This is equivalent to asking if there is a *single* sentence E such that

$$\mathfrak{A} \models E \iff \mathfrak{A} \text{ has the property } P.$$

In this case we say that P is *finitely axiomatizable.*

The Compactness Theorem is a valid tool to prove that some property P is *not* finitely axiomatizable over the class of arbitrary structures. Note that the Compactness Theorem is not a good tool for studying expressibility of queries over classes of finite structures because the model that the Compactness Theorem guarantees to exist is usually an infinite model.

Remark 2.1. Each class $\mathcal{K}$ of structures obviously determines a property of structures, the property $P_{\mathcal{K}}$ of being a member of $\mathcal{K}$. There exist algebraic general conditions that ensure $P_{\mathcal{K}}$ is finitely axiomatizable (Keisler's Theorem).

2.1. Example 1: Finiteness.

We begin with a very simple example: The property of *having finite domain* is not axiomatizable.

We argue by way of contradiction. We suppose that the property of being of finite domain is axiomatizable and that T is a set of sentences that axiomatizes it. By definition this means that, for any structure $\mathfrak{A}$, the following holds:

$$\mathfrak{A} \models T \Leftrightarrow \mathfrak{A} \text{ has finite domain.}$$

We then consider the theory T augmented by sentences expressing the fact that there exists at least n distinct objects. Such a sentence is easily expressible in first-order logic with equality:

$$A_n := \exists x_1 \dots \exists x_n (\bigwedge_{1 \le i < j \le n} x_i \neq x_j).$$

We then consider the theory

$$T \cup \{A_n \,:\, n \in \mathbf{N}\}.$$

Consider now a finite subset X of this theory. X consists of a finite number of sentences from T, say $S_1, \dots, S_p$ and a finite number of sentences from $\{A_n \,:\, n \in \mathbf{N}\}$, say $A_{n_1}, \dots, A_{n_q}$. We claim that X has a model. First, the sentences $S_1, \dots, S_p$ are true in any finite structure, by hypothesis on T. Second, if m is the maximum of $n_1, \dots, n_q$, then any structure with more that m elements satisfies $A_{n_1}, \dots, A_{n_q}$. Thus, any finite structure with more than m elements satisfies X.

We have thus proved that any finite subset of $T \cup \{A_n \,:\, n \in \mathbf{N}\}$ has a model. By the Compactness Theorem, it then follows that the whole theory has a model. Let this model be $\mathfrak{A}$. On the one hand, since $\mathfrak{A} \models T$, it must be the case that the domain of $\mathfrak{A}$ is finite. On the other hand, since $\mathfrak{A} \models A_n$ for all $n \in \mathbf{N}$, it must be the case that the domain of $\mathfrak{A}$ is infinite. Therefore $\mathfrak{A}$ cannot exist. Thus, our hypothesis that T exists is contradictory.

The argument generalizes as in the following theorem.

Theorem 2.2. *Let T be a set of sentences. If T has arbitrarily large finite models, then T has an infinite model.*

Proof. The argument is identical to the one above. □

The above proof is prototypical and can be applied in a variety of situations. The general form of the proof is the following. Let T axiomatize a property P. We identify a (usually infinite) set of sentences T' such that: the set T' is globally incompatible with property P, while each finite portion of T' is compatible with P. These are the basic ingredients of a typical non-expressibility/non-axiomatizability proof by Compactness.

Note that Compactness does not work well if we care about non-expressibility over finite structures: in fact, the Compactness Theorem gives no guarantee on whether the model satisfying the theory is infinite or finite.

We have seen that the property of *having infinite domain* is axiomatizable (by the theory T above). Is the property also finitely axiomatizable? Obviously not, because otherwise the negation of the axioms would be a finite axiomatization for the property of having finite domain! (N.B. this reasoning applies because we are talking about a finite axiomatization, which is equivalent to an axiomatization by a single axiom).

Proposition 2.3. *The property of having infinite domain (i.e., the property of being an infinite set) is axiomatizable but is not finitely axiomatizable.*

The argument used just above is general and can be summarized as in the following proposition.

Lemma 2.4. *If a property P and its complement $\neg P$ are both axiomatizable, then P is finitely axiomatizable.*

Proof. Exercise! □

Remark 2.5. An argument similar to the above shows that the property of *being a well-ordering* is not finitely axiomatizable. A binary relation $<$ on a set X is a well-ordering if $<$ is anti-symmetrical and satisfies trichotomy (i.e., is a strict total order) and every nonempty subset of X has a minimum element. Equivalently, if there are no infinite descending chains of elements of X.

2.2. **Example 2: Connectivity of graphs.** A (simple) graph $G = (V, E)$ is connected if and only if for every $v, w \in V$ with $v \neq w$ there exists $n \in \mathbf{N}$ and vertices $x_1, \ldots, x_n \in V$ (all distinct from each other and from v, w) such that $E(v, x_1), \ldots, E(x_n, w)$ (if $n = 0$ the sentence is just $E(v, w)$).

Proposition 2.6. *Connectivity is not axiomatizable.*

Proof. Suppose by way of contradiction that it is, and let T be an axiomatization. The following formula (with two free variables v and w) intuitively says that there is no path of length $\leq n+1$ between v and w.

$$P_n(v, w) = \neg(\exists x_1 \ldots \exists x_n)(E(v, x_1) \wedge \bigwedge_{i=1}^{n-1} E(x_i, x_1) \wedge E(x_n, w)).$$

We consider the expansion of the language of graphs with two new constants, a and b. We consider the theory following theory in the new language:

$$T \cup \{P_n(a, b)\}_{n \in \mathbf{N}^+}.$$

By Compactness we prove that this theory is consistent. Pick an arbitrary finite (non empty) subset X of this theory. X consists of some sentences from T, say $S_1, \ldots, S_p$ and some sentences from $\{P_n(a, b) : n \in \mathbf{N}\}$, say $P_{n_1}(a, b), \ldots, P_{n_q}(a, b)$. To satisfy $S_1, \ldots, S_p$ it is sufficient to pick any connected graph. To satisfy $P_{n_1}(a, b), \ldots, P_{n_q}(a, b)$, it is sufficient to pick a graph in which at least two vertices are at distance larger than $m = \max(n_1, \ldots, n_q)$. Thus, to satisfy X it is sufficient to pick a connected graph with at least two vertices at distance larger than m.

By Compactness, the whole theory $T \cup \{P_n(a, b)\}_{n \in \mathbf{N}^+}$ is then satisfiable. Let $\mathfrak{G}$ be a model of this theory. On the one hand, $\mathfrak{G} \models T$ so it has to be a connected graph. On the other hand, $\mathfrak{G} \models P_n(a, b)$ for every $n \in \mathbf{N}$, so it must contain two vertices, $a^{\mathfrak{G}}$ and $b^{\mathfrak{G}}$ at distance larger than any natural number, i.e., two vertices with no path between them. This contradiction shows that T cannot exist. □

3. Proofs by Compactness

The Compactness Theorem can be used as a tool to prove theorems of classical mathematics in different areas. The general scheme is the following.

Let A be a mathematical structure (e.g. a set, a graph, a group, a vector space, etc.) for which there is an adequate notion of substructure (e.g. subset, subgraph, subgroup, subspace etc.). Let P be a significant property of structures of the type of A. In many cases of interest, the Compactness Theorem can be used to prove implications of the following form.

If every finite substructure of A has the property P then A has the property P.

The proof scheme is general. Given a structure A of the type in question, we define a theory T_A such that if T_A has a model is then A has the property P. To prove that T_A has a model it is enough – by Compactness – to prove that each finite subset $T_0 \subseteq T$ of T_A has a model. But to prove this latter fact, in many cases, if T_A is defined in an appropriate way, the premise of the implication that we are proving is sufficient: every finite substructure of A has the property P.

k-colorability of graphs. A graph is k-colorable if there is a partition of the vertices of G in k classes such that there are no edges between vertices of the same class. In other words: it is possible to color the vertices of G with k colors in such a way that if two vertices are adjacent, then they have different colors.

FIGURE 1. Nicholas De Bruijn (1918-2012)

Theorem 3.1 (Erdős-De Brujin). *Let G be an infinite graph. G is k-colorable if and only if every finite subgraph of G is k-colorable.*

Using the Compactness Theorem we prove the theorem. Given that we have limited ourselves to countable languages, we prove the theorem for countable graphs (but the argument does generalize easily to arbitrary graphs).

Fix an arbitrary countable graph $G = (V, E)$. We associate to G a first-order theory with identity T_G as follows. The language of the theory has a binary relation symbol for which they assume the axioms of identity. For the sake of simplicity we denote this symbol with $=$. The language has a binary relation symbol E (edges), k unary predicates $C_1, \dots, C_k$ (the color classes), and a constant c_v for each vertex of the graph G. The theory T_G is given by the following sentences.

(1) $\forall x \neg E(x, x)$
(2) $\forall x \forall y (E(x, y) \to E(y, x))$
(3) $\forall x (C_1(x) \vee \cdots \vee C_k(x))$
(4) $\forall x \neg (C_i(x) \wedge C_j(x))$ for $1 \leq i < j \leq k$
(5) $\forall x \forall y (C_i(x) \wedge C_i(y) \to \neg E(x, y))$ for $1 \leq i \leq k$
(6) $c_v \neq c_w$, for each pair of distinct vertices v, w in G
(7) $E(c_v, c_w)$ for each pair of vertices of G connected by an edge in G.

A (normal) model of this theory is a graph (because it satisfies axioms 1 and 2) and is k-colorable (because it satisfies the axioms 3,4,5) that contains G as a subgraph (because it satisfies the axioms 6 and 7). Therefore if the theory has a model, then G is k-colorable. We prove that there is a model. For this – by the Completeness Theorem – it is enough to prove that T_G is consistent. To prove that T_G is consistent – by the Compactness Theorem – it is enough to prove that each finite subset of T_G is consistent. In any finite subset T_0 of T_G only a finite number of constants c_v, $v \in V$, are mentioned. Let $c_{v_1}, \dots, c_{v_t}$ be these constants. The subgraph of G induced by these vertices is a finite subgraph of G and by hypothesis it is k-colorable. This subgraph is a model of T_0.

Ramsey's Theorem. Ramsey's Theorem is a fundamental result in Combinatorics, with many applications in other areas of Mathematics and Theoretical Computer Science. It was originally introduced by Frank P. Ramsey as a tool for proving a decidability result for a fragment of first-order logic. It was later re-discovered by Hungarian mathematicians, giving rise to an autonomous area of Combinatorics, called Ramsey Theory.

FIGURE 2. Frank Plumpton Ramsey (1903-1930)

Theorem 3.2 (Infinite Ramsey's Theorem)**.** *For all colorings of the unordered pairs of natural numbers in two colors $f : [\mathbf{N}]^2 \to 2$ there exists an infinite subset H of $\mathbf{N}$ such that f restricted to the pairs of elements of H (i.e., to $[H]^2$) is constant.*

A set H as above is called *monochromatic* or *homogeneous* for the coloring. In terms of graphs the theorem states that any coloring in two colors of the edges of the complete graph on the natural numbers admits an infinite monochromatic clique. Alternatively the theorem can be rephrased any countably infinite graph contains an infinite clique or an infinite independent set.

Proof. Let x_0 be 0. Partition $\mathbf{N} \setminus \{0\}$ in two parts

$$X = \{x > x_0 \, : \, f(x_0, x) = 0\}$$

$$Y = \{x > x_0 \, : \, f(x_0, x) = 1\}$$

Since $X \cup Y$ is infinite, at least one of X and Y is infinite. Pick one such set and call it A_1. Let x_1 be the minimum of A_1. Partition $A_1 \setminus \{x_1\}$ in two parts as follows:

$$X = \{x \in A_1 \, : \, x > x_1 \wedge f(x_1, x) = 0\}$$

$$Y = \{x \in A_1 \, : \, x > x_1 \wedge f(x_1, x) = 1\}$$

Again, one of the two must be infinite. Pick it and call it A_2. Let x_2 be its minimum.

Proceeding in this fashion we obtain an infinite sequence of nested sets

$$\mathbf{N} \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$$

and corresponding minima

$$\sigma = x_0 < x_1 < x_2 < x_3 < \dots.$$

The sequence σ satisfies that the coloring f on pairs of its elements only depend on the minimum of the pair: if $i < j < h$ then $f(x_i, x_j) = f(x_i, x_h)$. In other words the sequence induces a coloring g of $\mathbf{N}$ in 2 colors, setting $g(i) = c_i$ where $c_i \in \{0, 1\}$ is the unique color assigned by f to any pair of the form $\{x_i, x_j\}$ with $i < j$. By the Infinite Pigeonhole Principle infinitely many x_is in σ have the same color. It is then easy to verify that the original coloring f is constant on all pairs of those elements. □

Note that the above proof works for any countable infinite set X instead of $\mathbf{N}$ with the same proof (start with $x_0 = \min(X)$).

Ramsey's Theorem also has a finite version. The first non-trivial case is the following: if I color in 2 colors (say RED and BLUE) the edges of the complete graph on 6 vertices then necessarily there exists a monochromatic triangle. The popular version of this theorem is the following: at a party of 6 people there are necessarily 3 people who mutually know each other or 3 people that mutually do not know each other. The general version of the Finite Ramsey's Theorem says that we can shoot for any $m \geq 3$ provided that we take a large enough complete graph.

We show how it can be proved by Compactness from the Infinite version.

Theorem 3.3 (Finite Ramsey's Theorem)**.** *For any $m \in \mathbf{N}$ there exists an n so large that the following holds: For all colorings of the unordered pairs of natural numbers from $\{1, \dots, n\}$ in two colors $f : [\{1, \dots, n\}]^2 \to 2$ there exists a subset H of $\{1, \dots, n\}$ of cardinality m such that f restricted to the pairs of elements of H (i.e., to $[H]^2$) is constant.*

Proof. Reason by way of contradiction. Suppose that the Theorem is false. Then there is an m such that no n satisfies the statement of the theorem with respect to m. That is, For each n there exists a coloring f_n of pairs of numbers in $\{1, \dots, n\}$ such that no subset of $\{1, \dots, n\}$ of cardinality m is monochromatic for f_n. I.e., for any subset S of $\{1, \dots, n\}$, if S has cardinality m then it contains some a, b, c, d such that $f_n(a, b) \neq f_n(c, d)$. We thus have a sequence of counterexamples to the validity of the theorem, indexed by $n \in \mathbf{N}$. We use Compactness to show that these can be glued together in a single coloring of all pairs of natural numbers that gives a counterexample to the validity of the Infinite Ramsey's Theorem. This gives a contradiction.

Consider the language $\mathcal{L}$ consisting of binary relation symbols R and B. The intended meaning of $R(x, y)$ is that the pair $\{x, y\}$ has color 1 (RED), the intended meaning of $B(x, y)$ is that the pair $\{x, y\}$ has color 0 (BLUE). Note that in any structure interpreting $\mathcal{L}$ the interpretation of the symbol R and B can be thought of as a coloring in two colors of the unordered pairs of elements of the structure.

Fix m as above. We need to express the following facts:

(1) R and B are a coloring of unodered pairs,
(2) There is no monochromatic set of cardinality m.

Both points are easy to express in $\mathcal{L}$ (Exercise). We consider the theory T consisting of the following sentences:

$$\forall x \forall y((x = y) \to (\neg R(x, y) \land \neg B(x, y))$$
$$\forall x \forall y(\neg(x = y) \to (R(x, y) \leftrightarrow R(y, x))$$
$$\forall x \forall y(\neg(x = y) \to (B(x, y) \leftrightarrow B(y, x))$$
$$\forall x \forall y(\neg(x = y) \to (R(x, y) \lor B(x, y))$$
$$\forall x \forall y((\neg(x = y) \to (R(x, y) \leftrightarrow \neg B(x, y))$$
$$\forall x_1 \dots \forall x_m (\bigwedge_{i,j \in [1,n]} \neg(x_i = x_j) \to (\bigvee_{i,j \in [1,m]} R(x_i, x_j) \land \bigvee_{i,j \in [1,m]} B(x_i, x_j)))$$

For each n we include a sentence F_n saying that there are at least n distinct elements.

By choice of m we have that the theory T is finitely satisfiable. In fact, it has arbitrarily large finite models, given by the counterexamples to the finite Ramsey Theorem.

Let T_0 be a finite subset of T. Let n_0 be maximal such that F_{n_0} appears in T_0. Consider the coloring $f_{n_0} : [\{1, \dots, n_0\}]^2 \to 2$ giving a counterexample to the Finite Ramsey's Theorem for value m. We obtain a structure satisfying T_0 setting the domain to $\{1, \dots, n_0\}$, and interpreting R and B as the coloring f_{n_0} such that (i, j) are in the interpretation of R iff $f_{n_0}(i, j) = 1$ and (i, j) are in the interpretation of B if and only if $f_{n_0}(i, j) = 0$.

By Compactness the theory T has a model. Let $\mathfrak{A} \models T$. $\mathfrak{A}$ is necessarily infinite (since it satisfies all the F_ns). The interpretations $R^{\mathfrak{A}}$ and $B^{\mathfrak{A}}$ define a 2-coloring of the unordered pairs of A. We can assume wlog that A is countable, by restricting to a countable subset.

We can then apply the Infinite Ramsey's Theorem to this coloring and obtain an infinite subset $H \subseteq A$ such that all pairs from it have the same color under the coloring induced by $R^{\mathfrak{A}}$, $B^{\mathfrak{A}}$. But H contradicts the fact that there is no monochromatic set of size m. □

4. Extra: Non-standard Models of Arithmetic

The Compactness Theorem also gives us a tool to answer the following natural question: is it possible to axiomatize the structure of the natural numbers up to isomorphism? That is, can we write a first-order theory T such that all models of T are isomorphic copies of the "natural" structure which we call "the natural numbers"? The answer is a big NO.

We consider the language of arithmetic $\mathcal{L} = \{0.1, , \times, <\}$. For any natural $n \in \mathbf{N}$, denote by $\overline{n}$ the closed term of $\mathcal{L}$ of the form $(1 + (1 + (\dots(1 + 1)\dots)$ with n occurrences of 1. We call the terms of this type *numerals*. The structure $\mathbb{N} = \{\mathbf{N}, 0, 1, +, \times, <\}$ is an adequate structure for $\mathcal{L}$ and $\overline{n}^{\mathbb{N}}$ is n. This structure is called the *standard model*.

Imagine we want to write a theory T such that all its models are isomorphic to the standard model. (This would, by the way, imply that the theory T is complete.) To this aim, we choose the strongest possible theory, i.e. we take all sentences that are true in the standard model. We show that even with this choice it is impossible to force all models of the theory to be isomorphic to the standard natural numbers.

Let $Th(\mathbb{N})$ be the set of sentences in $\mathcal{L}$ true in the structure $\mathbb{N}$. Let c be a new constant. We consider the theory

$$T \cup \{\overline{n} < c \ : \ n \in \mathbf{N}\}.$$

By Compactness, the theory T is consistent. In every structure $\mathfrak{A}$ that satisfies T, the element that interprets the constant c is larger (in the sense of $<^{\mathfrak{A}}$) than all elements that interpret the terms $\overline{n}$. In addition, all the sentences true in the standard model $\mathbb{N}$ are also true in $\mathfrak{A}$. A model of this type is called *non-standard*.

Given a non-standard model $\mathfrak{A}$, call the *standard numbers* of $\mathfrak{A}$ the interpretations of the numerals, i.e.

$$\{a \in A : \text{ There exists } n \in \mathbf{N}(a = \overline{n}^{\mathfrak{A}})\}.$$

We will see that in general a non-standard model of $Th(\mathbf{N})$ begins with a set isomorphic to $\mathbf{N}$, i.e. with a copy of $\mathbf{N}$, but continues in a quite different way!

Given that $\mathfrak{A} \models Th(\mathbf{N})$, we have that

$$\mathfrak{A} \models \forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z),$$

and

$$\mathfrak{A} \models \forall x \forall y (x < y \vee x = y \vee y < x).$$

therefore $<^{\mathfrak{A}}$ is a linear order on the domain A.

We also have

$$\mathfrak{A} \models \forall x (0 < x \vee 0 = x),$$

and therefore $0^{\mathfrak{A}}$ is the minimum element of A with respect to the order $<^{\mathfrak{A}}$.

Given that for every $n \in \mathbf{N}$, we have the following:

$$\mathfrak{A} \models \neg \exists x (\overline{n} < x \wedge x < \overline{n+1}),$$

then there are no elements of A between two standard numbers.

Given that we have

$$\mathfrak{A} \models \forall x (x \neq 0 \rightarrow \exists y (y + 1 = x)),$$

then every element of A has an *immediate predecessor*.

Given that for every $n \in \mathbf{N}$, we have

$$\mathfrak{A} \models \overline{n} + 1 = \overline{n+1},$$

then the successor of a standard number is a standard number.

Therefore the model $\mathfrak{A}$ begins with a copy of $\mathbf{N}$ (the map $n \mapsto \overline{n}^{\mathfrak{A}}$ is an order isomorphism).

Moreover, if $a \in A$ is a non-standard number, then also the predecessor of A is a non-standard number. Thus below each non-standard number $a \in A$ there is an infinite decreasing chain with respect to the order $<^{\mathfrak{A}}$:

$$a > a_1 > a_2 > \dots$$

Therefore the order $<^{\mathfrak{A}}$ is not a well-ordering of A.

It can be observed however that any subset of A definable by a formula has a minimum. Let $X = \{\in \mathfrak{A} \models F(a)\}$. Given that in $Th(\mathbb{N})$ we have the Least Element Principle, i.e. the formula

$$\exists x F(x) \rightarrow \exists x (F(x) \wedge \forall y (F(y) \rightarrow x \leq y)),$$

it follows that X has a minimum in A with respect to $<^{\mathfrak{A}}$.

From this observation it follows that the set of standard numbers is not definable! If $F(x)$ defines the standard numbers in $\mathfrak{A}$ then $\neg F(x)$ defines the non-standard numbers. Then there should be a minimum non-standard number. But we have seen above that such a number does not exist.