MATHEMATICAL LOGIC FOR COMPUTER SCIENCE (A.Y. 20/21)

HANDOUT N. 13

Abstract. We introduce the first-order Predicate Calculus (with equality) and prove its basic properties. In particular we prove the Deduction Theorem. We state the Completeness Theorem and prove Lindenbaum's Lemma.

1. Formal Proofs: Predicate Calculus

We define the concept of formal logic proof (first-order). The concept is an abstraction from the informal notion of proof in Mathematics and exact sciences in general.

A proof is essentially as a *finite* sequence of propositions such that each proposition has a precise justification for being in the proof: it is either a truth we assume (an axiom), or is obtained from one or more of the previous formulas by application of an inference rule. We will further distinguish between general or logical axioms, which represent truths that we give for granted in every reasoning independently of the specific domain (we will call these the *logical axioms*), and domain-specific assumptions about the domain or structure about which we are reasoning – e.g., vector spaces, groups, natural numbers, graphs, etc. (these will be called *domain-specific* or *non-logical* axioms.

Furthermore, the system that we define has the extra properties that the axioms are algorithmically recognizable (i.e. they are decidable) and that the correct application of an inference rule is algorithmically testable (i.e. decidable). This is meant to formalize the fact that we expect that a proof can be rigorously checked for correctness (typically by a human being).

Axioms are so designed as to capture all and only the logically valid formulas. We'll show, with the Completeness Theorem, that it is exactly so: a formula is provable if and only if it is valid. In this way we would have replaced a semantic and infinitary concept (that of validity in all models) by a syntactic and finitary concept (that of provability). An important corollary is that the set of valid formulas is algorithmically enumerable.[1]

1.1. **Hilbert Calculus.** The particular type formalism which we now introduce is due to David Hilbert and is based on axioms and deduction rules. We will see other formalisms for the Predicate Calculus and we will use in different contexts the most suitable one. They are all equivalent with respect to the set of provable formulas. We focus on the essential properties that we shall need in what follows. We will not write many formal proofs in the calculus. The calculus is a mathematical tool that we use to prove theorems *about* logic more than a tool for proving theorems. We will limit ourselves to some basic properties. The axioms of the Predicate Calculus are the following (where F and G are arbitrary formulas). The first group are propositional axioms, the second group are predicative axioms. A third group, the axioms of identity is added when one chooses to deal with the relation of equality $=$ as a special logical symbol , as we did in the definition of the syntax and semantics of First-Order Logic.

(Schemes of) Propositional Axioms.

(1) $F \to (G \to F)$

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[1] Historically this was a major step in trying to solve in the positive the Decision Problem (is there an algorithm for answering whether a formula is a logical validity) since there was some hope that there could be a deterministic procedure to decide whether a formal proof of a given formula exists.

(2) $(F \to (G \to H) \to ((F \to G) \to (F \to H)$
(3) $(\neg F \to \neg G) \to ((\neg F \to G) \to F)$
(4) $(F \wedge G) \to F$
(5) $(F \wedge G) \to G$
(6) $(H \to F) \to ((H \to G) \to (H \to (F \wedge G)))$
(7) $F \to (F \vee G)$
(8) $G \to (F \vee G)$
(9) $(F \to H) \to ((G \to H) \to ((F \vee G) \to H)$

(Schemes of) Predicative Axioms.
(10) $\forall x F \to F[x/t]$ with t a term free for x in F.
(11) $\forall x(F \to G) \to (F \to (\forall x)G)$ with F with no free occurrences of x.

The above expressions are schemes for axioms. It is meant that each formula obtained by the above schemes by replacing coherently the letters F, G, H by first-order formulas is *an axiom* of the Predicate Calculus.

The axioms above do not have clauses for $\exists$. This is only a choice of convenience, since we can define $\exists$ as $\neg\forall\neg$. Alternatively, one can introduce as axiom the following double implication

$$\exists x F \leftrightarrow \neg\forall x \neg F.$$

It is an instructive exercise to justify the constraints in the last two axioms (10) and (11). In general we wish that our axioms and rules preserve logical validity (see below). To understand the raionale behing the extra conditions on the axioms (10) and (11) it is useful to see what happens if these conditions are dropped. For axioms (10), we should find an $F(x)$ and a t not free fro x in F such that the instance of axioms (10) applied to this choice of F and t would give rise to a formula that is not logically valid. A similar analysis can be done for axioms (11).

If we choose (as we did) to have $=$ as special symbol in the language, we add to the calculus the following axioms. In this case the calculus is called Predicate Calculus with equality.

Axioms of Equality
(10) For each function symbol f, the following statement is an axiom.

$$\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n (x_1 = y_1 \wedge \cdots \wedge x_n = y_n \to (f(x_1, \dots, x_n) = f(y_1, \dots, y_n).$$

(11) For each relation symbol R, the following statement is an axiom.

$$\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_n (x_1 = y_1 \wedge \cdots \wedge x_n = y_n \to (R(x_1, \dots, x_n) \leftrightarrow R(y_1, \dots, y_n).$$

We define two Inference Rules (or Deduction Rules).

- Modus Ponens: From F and $(F \to G)$ we deduce G.
- Generalization: From F we deduce $\forall x F$ (x a variable).

The formal system (i.e., a decidable set of axioms and inference rules) just defined is called the **First-order Predicate Calculus (with equality)**. For accuracy, to each fixed first-order language $\mathcal{L}$ we can associate *a* Predicate Calculus by considering formulas in $\mathcal{L}$ only.

We denote the set of logical axioms described above by **LA** (mnemonic for logical axioms).

The Predicate Calculus intends to capture the concept of logical validity. The provable theorems are theorems of pure logic. The concept of proof extends easily to mathematical proofs based on non-logical premises. This is very useful to capture common mathematical proofs (which usually start from axioms concerning a given theory or mathematical object, e.g. groups, vector spaces, natural numbers, etc.).

To formalize this idea we use sets of sentences, which formalize the domain-specific axioms or hypotheses on which we base our reasoning. We therefore give the definition of the notion "F is provable from premises in Γ".

Definition 1.1 (Proof/Derivation from hypotheses)**.** Let Γ be a set of formulas.

A *proof* the Predicate Calculus from premises/hypotheses in Γ is a finite sequence of formulas $(F_1, \ldots, F_n)$, where for every $i \in [1, n]$ we have

- F_i is an axiom, or
- F_i is in Γ, or
- There exists $j < i$ such that $F_i = \forall x F_j$ (Generalization), or
- There exist $j, k < i$ such that F_k is $F_j \to F_i$ (Modus Ponens).

A formula F is *provable* from Γ if *there exists* a proof $(F_1, \ldots, F_n)$ in the Predicate Calculus from premises in Γ such that F_n is F. We denote this fact by $\Gamma \vdash F$.

If Γ is the empty set we are dealing with proofs based on the logical axioms only. In this case we write $\vdash F$ and say that F is provable in the Predicate Calculus.

A formula provable in the Predicate Calculus is also called a *logical theorem*, or a *theorem of the Predicate Calculus.*

1.2. **Fundamental properties of the Predicate Calculus.** Let P be a propositional formula containing the propositional variables $p_1, \ldots, p_n$. Let $F_1, \ldots, F_n$ be first-order formulas. Let F be the formula obtained by substituting in P each p_i by the formula F_i. If P is a tautology then F is called a first-order instance of a propositional tautology.

Remark 1.2. Every instance of a propositional tautology is a theorem of the Predicate Calculus.

This observation is based on the fact that the propositional axioms allow to prove only and all instances of propositional tautologies. A propositional formula (built from propositional variables using the Boolean connectives) is a tautology (i.e., is true for all assignments of Boolean values to its propositional variable) if and only if it is derivable from the propositional axioms. This latter result is the Completeness Theorem for Propositional Logic.

Remark 1.3. Every theorem of the Predicate Calculus is logically valid.

This observation is based on the fact that the axioms are logically valid and that the inference rules preserve validity: e.g., if $\models F$ then $\models \forall x F$ (by definition of $\models$). This shows that the rule of Generalization preserves logical validity.

For the concept of proof from hypotheses we have the analogous remark.

Remark 1.4. If $\Gamma \vdash F$ then $\Gamma \models F$.

Remark 1.5. The Predicate Calculus is consistent, i.e. for no formula F it is the case that $\vdash F$ and $\vdash \neg F$.

This obviously follows from the previous observation. It should be noted that this proof of the consistency of the Predicate Calculus is far from being constructive, but is based on the infinitary concept of validity in all models. Several purely syntactic inductive proofs of consistency for the Predicate Calculus exist.

Remark 1.6. The set of theorems of the Predicate Calculus is computably enumerable.

We can enumerate all and only all the theorems by the following mechanical procedure. We fix an enumeration of the axioms of the Predicate Calculus, $(A_1, A_2 \ldots)$. We get an enumeration of theorems as follows. Put A_1 in the list. Add all the formulas obtained by applying Modus Ponens or a single application of Generalization with v_1 as the quantified variable. Add A_2 to the list. Add all the formulas obtained by applying to the formulas of the new list Modus Ponens or an application of Generalization with v_1 or v_2 as the quantified variable. And so on...

1.3. **Some noteworthy derived rules.** **Instantiation Rule** Let t be free for x in $F(x)$. Then

$$(\forall x)F(x) \vdash F(t).$$

The rule follows directly by Modus Ponens by an axiom.

A particular case is when t is x, and then we have $(\forall x)F(x) \vdash F$.

Existential Rule Let t be free for x in F. Let $F[x/t]$ be obtained from F by replacing all free occurrences of x with t (t may also not appear in $F[x/t]$). Then

$$F[x/t] \vdash (\exists x)F.$$

One can prove $\vdash F[x/t] \to (\exists x)F$. Use the axiom $(\forall x)\neg F \to \neg F[x/t]$, the tautology $(A \to \neg B) \to (B \to \neg A)$ and Modus Ponens, to get $\vdash F[x/t] \to \neg(\forall x)\neg F$.

A particular case is $F(t) \vdash (\exists x)F(x)$ with t free for x in $F(x)$. If t is x, we get $F(x) \vdash (\exists x)F(x)$.

1.4. **Deduction Theorem.**

Theorem 1.7 (Predicative Deduction Theorem)**.** *Let E be a sentence, G a formula and Γ a collection of formulas. If*

$$\Gamma, E \vdash G$$

then

$$\Gamma \vdash E \to G.$$

Proof. Let $(D_1, \ldots, D_n)$ be a proof of G from Γ, E. Then $D_n = G$. By induction we prove that

$$\Gamma \vdash E \to D_i.$$

Suppose D_i is either an axiom or an element of Γ. Then we use the axiom $D_i \to (E \to D_i)$.

Suppose D_i equal to E. Then we use the fact that $\vdash E \to E$ is provable.

Let $j, k < i$ be such that $D_k = D_j \to D_i$. By inductive hypothesis we have $\Gamma \vdash E \to (D_j \to D_i)$ and $\Gamma \vdash E \to D_j$. We use the axiom

$$E \to (D_j \to D_i) \to ((E \to D_j) \to (E \to D_i)$$

and Modus Ponens for obtaining the desired result.

Let $j < i$ such that D_i is $\forall x D_j$. By inductive hypothesis we $\Gamma \vdash E \to D_j$. Since E is a sentence, x is not a free variable of E. We use the axiom $(\forall x)(E \to D_j) \to (E \to \forall x D_j)$ and the fact that, by Generalization from $\Gamma \vdash E \to D_j$ we get $\Gamma \vdash \forall x(E \to D_j)$. □

In the predicative case the Deduction Theorem *not true in its general form.* It is an interesting exercise to find a counterexample.

2. Completeness Theorem

We prove Gödel's Completeness Theorem.

In its first formulation the Completeness Theorem is the equivalence of the following two points:

(A) Sentence E is provable from theory T (i.e., $T \vdash E$).
(B) Sentence E is valid in all models of T (i.e., $T \models E$).

In particular, if theory T is empty, we have the equivalence between

E is a theorem of the predicate calculus $\Leftrightarrow$ E is true in all structures.

We observe that

- $\vdash E$ abbreviates an existential quantification on a countable number of finite objects: *there exists a formal derivation $(D_1, \ldots, D_n)$ with conclusion E.*
- $\models E$ abbreviates a universal quantification on an uncountable collection of infinite objects: *for each structure $\mathfrak{A}$ (adequate for the language of E), E is true in $\mathfrak{A}$.*

A theory T is called *(syntactically) consistent* if there is no sentence F such that $T \vdash F$ and $T \vdash \neg F$, i.e., T proves no contradiction.

We will prove the Completeness Theorem in the following equivalent formulation: we will show that points (C) and (D) are equivalent.

(C) Theory T does not prove contradictions (i.e., T is consistent).
(D) Theory T has a model (i.e., T is satisfiable).

The two double implications ("A if and only if B" and "C if and only if D") are two formulations of the Completeness Theorem. The two formulations are equivalent. In particular, $A \Rightarrow B$ is equivalent to $D \Rightarrow C$ and $B \Rightarrow A$ is equivalent to $C \Rightarrow D$.

It is an instructive exercise to prove the equivalence of the two formulations.

Since the notion of consistency is crucial it is important to isolate some of its basic properties.

Remark 2.1. By definition, a theory T is *consistent* if for no sentence F we have that both $T \vdash F$ and $T \vdash \neg F$ hold. This is tantamount to saying that there is a sentence E such that $T \nvdash E$. In other words, a theory is consistent if and only if it does not prove all sentences (if and only if there exists a sentence that the theory does not prove). It is enough to observe that $A \to (\neg A \to E)$ is a tautology (called *ex falso quodlibet*).

Remark 2.2. $T \cup \{E\}$ is consistent if and only if $T \nvdash \neg E$. Suppose $T \cup \{E\}$ is consistent and suppose by way of contradiction (abbreviated bwoc from now on) that $T \vdash \neg E$. Then $T \cup \{E\} \vdash E \wedge \neg E$ and is inconsistent. Contradiction. Suppose now that $T \nvdash \neg E$ but that, bwoc, $T \cup \{E\}$ is inconsistent. Then $T \cup \{E\} \vdash \neg E$. Therefore $T \vdash E \to \neg E$. Therefore $T \vdash \neg E$ (by propositional logic). This contradicts the hypothesis. Contradiction.

The hard-to-prove implication is: $C \to D$. To prove this we need to assume that there is a formal deduction of a contradiction on the premises in T from the existence of a model that satisfies T.

2.1. **The Complete Extension of a Theory.** Now we prove that in the space of the extensions of a theory there is always a maximal object. We say that a theory T' *extends* a theory T if $T \subseteq T'$.

We say that a theory T is *(syntactically) complete* if for every sentence E in the language of T, either $T \vdash T$ or $T \vdash \neg E$.

Lemma 2.3 (Lindenbaum Lemma)**.** *Any consistent theory has a consistent and complete extension.*

Proof. Let T be a consistent theory in the language $\mathcal{L}$ (countable). Let $\{E_1, E_2 \dots, \}$ be an enumeration of all sentences in $\mathcal{L}$. Define $S_0 := T$. Given S_n, $n \geq 0$, we define S_{n+1} as follows.

$$S_{n+1} := \begin{cases} S_n \cup \{E_{n+1}\} & \text{if } S_n \cup \{E_{n+1}\} \text{ is consistent,} \\ S_n & \text{otherwise.} \end{cases}$$

The definition is sound because only one between $S_n \cup \{E_{n+1}\}$ and $S_n \cup \{\neg E_{n+1}\}$ is consistent. The condition that $S_n \cup \{E_{n+1}\}$ is consistent is equivalent to assuming $S_n \nvdash \neg E_{n+1}$.

Let $S_\infty = \bigcup_{n \in \mathbf{N}} S_n$. S_∞ is a consistent and complete set of sentences. (Exercise). $\square$