MLCS - Homework 2

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1 EF-games and (non)-expressibility

Exercise 1.6. Consider the following two graphs:

1. $\mathfrak{G}_1$ is a line of length $4n$;
2. $\mathfrak{G}_2$ consists of a line of length $2n$ and a cycle of length $2n$ (the two components are disjoint).

Analyze the EF games on these structures for $n = 1, 2$ and write a sentence of minimal quantifier rank distinguishing the two structures for $n = 1, 2$.

BONUS: Formulate a generalization of your observations and prove that the Acyclicity query is not expressible in the language of graphs over finite graphs, using EF-games.

Solution. Let's discuss every point separately.

Part 1: $n = 1$.
In this case, $\mathfrak{G}_1$ is a line of length 4 (i.e., a graph of containing 5 nodes connected by 4 edges):

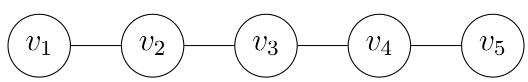

Figure 1: Graph $\mathfrak{G}_1$, consisting of a line of length 4.

While $\mathfrak{G}_2$ is the disjoint structure composed by a line of length 2 and a cycle of length 2:

Figure 2: Graph $\mathfrak{G}_2$, consisting of a line of length 2 and cycle of length 2.

Let's now simulate a match of the EF-games, analyzing each round individually from both Spoiler's and Discriminator's point of view. For each round, we will consider all possible scenarios, in order to completely solve the game. We will adopt the usual EF rules:

- Spoiler always starts first.
- If a player chooses his node from one of the two structures, the other player must necessarily choose his node from the other structure.

- Spoiler always tries to win the game in as few moves as possible, while Discriminator always tries to prolong the game as long as possible.

Round 1:

During the first round, each player freely chooses a node from one of the structures. There is no special case to mention.

Round 2:

During the second round, Spoiler can either select a node connected to the first one, or select a node disconnected from the first one. Either way, Discriminator can respond with a similar move, since both structures contain both connected and disconnected nodes.

Round 3:

During the third round, there are three possible scenarios. Namely:

1. Spoiler and Discriminator choose two connected nodes in the first two rounds, in the third round they choose a third node connected to either of the previous two.

2. Spoiler and Discriminator choose two connected nodes in the first two rounds, in the third round they choose a third node not connected to either of the previous two.

3. Spoiler and Discriminator choose two disconnected nodes in the first two rounds, in the third round they choose a third node not connected to either of the previous two.

It's easy to see that all the mentioned scenarios are possible, since both structures contain three connected nodes (e.g., v_1, v_2, v_3 in $\mathfrak{G}_1$, and u_1, u_2, u_3 in $\mathfrak{G}_2$), two connected nodes and a third one disconnected from the first two (e.g., v_1, v_2, v_4 in $\mathfrak{G}_1$, and u_1, u_2, e_1 in $\mathfrak{G}_2$), and three disconnected nodes (e.g., v_1, v_3, v_5 in $\mathfrak{G}_1$, and u_1, u_3, w_1 in $\mathfrak{G}_2$).

Moreover, without loss of generality, we can consider these three as the only possible scenarios. In fact, even if there are still cases that don't follow within those three (e.g., Spoiler and Discriminator choose two disconnected nodes, in the third round they choose a third node connected only to one of the first two), the outcome will still be beneath the cases we described.

Round 4:

Spoiler chooses three connected nodes in structure $\mathfrak{G}_1$ in the first three rounds, in the fourth round he chooses a fourth node connected to either of the previous three. Therefore, Spoiler wins, since there are no 4 connected nodes in structure $\mathfrak{G}_2$.

Conclusion:

We saw that Discriminator can win all 3-rounds games on these structures, while he has no winning condition on some of the 3-rounds games. Therefore, $\mathfrak{G}_1$ and $\mathfrak{G}_2$ cannot be distinguished by a sentence of a quantifier-rank 3 or less (i.e., $\mathfrak{G}_1 \equiv_3 \mathfrak{G}_2$), and there is a sentence of quantifier-rank 4 with which we can distinguish the two structure (i.e., $\mathfrak{G}_1 \not\equiv_4 \mathfrak{G}_2$).

Specifically, the sentence we empirically used during the game is the one expressing the existence of 4 connected nodes. Formally:

$$\exists v_1 \exists v_2 \exists v_3 \exists v_4 (E(v_1, v_2) \wedge E(v_2, v_3) \wedge E(v_3, v_4)),$$

where $E(a, b)$ expresses the existence of an edge between nodes a and b.

Part 2: $n = 2$.

For $n = 2$, $\mathfrak{G}_1$ is a line of length 8:

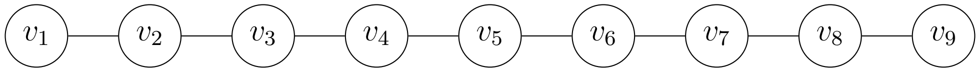

Figure 3: Graph $\mathfrak{G}_1$, consisting of a line of length 8.

While $\mathfrak{G}_2$ is the disjoint structure composed by a line of length 4 and a cycle of length 4:

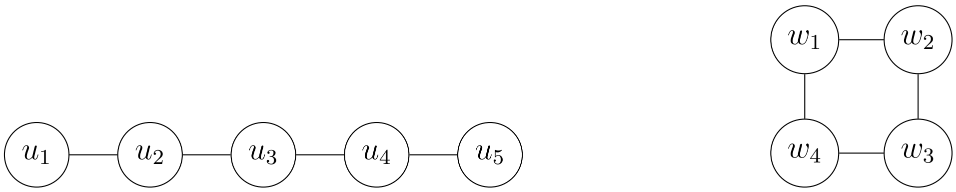

Figure 4: Graph $\mathfrak{G}_2$, consisting of a line of length 4 and cycle of length 4.

As we did in the $n = 1$ case, we now simulate a match of the EF-games, analyzing each round individually from both Spoiler's and Discriminator's point of view.

Round 1, **Round 2** and **Round 3**:

Equal to the rounds of the $n = 1$ case.

Round 4:

Spoiler chooses three nodes in the cycle of $\mathfrak{G}_2$ (e.g., w_1, w_2, w_3) during the first three rounds. In the fourth round he chooses the fourth remaining node in the cycle, creating a cycle of length 4. Therefore, Spoiler wins, since there is no cycle of length 4 in structure $\mathfrak{G}_1$.

Conclusion:

Similarly to the $n = 1$ case, we saw that Discriminator can win all 3-rounds games on these structures, while he has no winning condition on some of the 3-rounds games. Therefore, $\mathfrak{G}_1$ and $\mathfrak{G}_2$ cannot be distinguished by a sentence of a quantifier-rank 3 or less (i.e., $\mathfrak{G}_1 \equiv_3 \mathfrak{G}_2$), and there is a sentence of quantifier-rank 4 with which we can distinguish the two structure (i.e., $\mathfrak{G}_1 \not\equiv_4 \mathfrak{G}_2$).

Specifically, the sentence we empirically used during the game is the one expressing the existence of 4-cycle. Formally:

$$\exists v_1 \exists v_2 \exists v_3 \exists v_4 (E(v_1, v_2) \wedge E(v_2, v_3) \wedge E(v_3, v_4) \wedge E(v_4, v_1)),$$

where $E(a, b)$ expresses the existence of an edge between nodes a and b.

$\square$