MLCS - Homework 1

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1 Group 1

Exercise 2. Show that the following sentence is unsatisfiable, where S is any formula with two free variables: $\exists x \forall y (S(x, y) \leftrightarrow \neg S(y, y))$.

Solution. Given any choice of x, it is sufficient to pick $y = x$, in which case we would get

$$S(x, x) \leftrightarrow \neg S(x, x).$$

Since this is obviously a contradiction, it means that it exists y such that the formula S is false, meaning that it is unsatisfiable. □

Exercise 4. Is the following formula logically valid for any formula F and any term t?

$$\forall x F(x) \rightarrow F(t).$$

If not, give an example of a formula F, a structure $\mathfrak{A}$ and an assignment α witnessing this fact.

Solution. If we analyze the premise of the implication, we notice that we only have two cases:

1. $F(x)$ is true for every x (i.e., the premise is true).

2. It exists x such that $F(x)$ is false (i.e., the premise is false).

In case 1, we have that for every choice of x the formula F is true, meaning that it is also true for the given term t. Therefore, the implication is true, because we are considering an implication between a two true statements.

In case 2, the premise of the implication is false, therefore we can conclude that the whole implication is true. □

Exercise 9. In the language $\mathcal{L} = \{<\}$ of **DLO**, write a sentence that distinguishes $(\mathbf{N}, <)$ from $(\mathbf{Q}, <)$ i.e., that is true in one structure but not in the other.

Solution. The following sentence is true in $(\mathbf{N}, <)$, but not in $(\mathbf{Q}, <)$:

$$\forall x \exists y (\neg(x = y) \rightarrow y < x).$$

This represents the fact that $\mathbb{Q}$ has no left endpoint, while $\mathbb{N}$ has 0 (or 1, depending on whether we consider the set to have or not the element 0), which is lower than any other element. □

2 Group 2

Exercise 3. Is the structure $\mathcal{Q} = (\mathbb{Q}, +, \times, 0, 1)$ a substructure $\mathcal{R} = (\mathbb{R}, +, \times, 0, 1)$? Is it an elementary substructure?

Solution. My assumption is that $\mathcal{Q}$ is a substructure of $\mathcal{R}$, but that it is not an elementary substructure. Let's prove both points.

To prove that $\mathfrak{B}$ is a substructure of $\mathfrak{A}$, we need to check if

1. $B \subseteq A$.
2. For every constant symbol c, $c^{\mathfrak{A}} = c^{\mathfrak{B}}$.
3. Every relation $R^{\mathfrak{B}}$ (resp. function $f^{\mathfrak{B}}$) is the restriction of $R^{\mathfrak{A}}$ (resp. $f^{\mathfrak{A}}$) to B.

For the first point, there is nothing to prove. The second point is trivially true, since we only have two constants in both structures (i.e., 0 and 1) and they correspond. The third point (?)

To prove that the substructure $\mathfrak{B}$ is not elementary, we can consider the formula

$$\forall x \exists y ((x > 0) \rightarrow (y \times y = x)).$$

This is clearly true for $\mathcal{R}$, but it is not for $\mathcal{Q}$. In fact, we can consider $x = 2$ and it doesn't exist any y such that $y \times y = 2$. $\square$

Exercise 4. Prove that the following structures are not isomorphic:

(1) $(\mathbb{N} +, \times, 0, 1, <)$ and $(\mathbb{Q}, +, \times, 0, 1, <)$.

(2) $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$.

(3) $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$.

(Hint: in some cases you can use the fact that if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic then they satisfy the same sentences).

Solution. Let's consider case by case.

To prove that $(\mathbb{N} +, \times, 0, 1, <)$ and $(\mathbb{Q}, +, \times, 0, 1, <)$ are not isomorphic, we can consider the sentence that expresses the existence of additive inverse:

$$\forall x \exists y (x + y = 0)$$

is satisfied in $(\mathbb{Q}, +, \times, 0, 1, <)$, but not in $(\mathbb{N} +, \times, 0, 1, <)$. We know that two structures are isomorphic if and only if they satisfy the same sentences, therefore we can conclude that these two are not.

To prove that $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$ are not isomorphic, we can proceed in a similar way, using the sentence expressing the existence of the left-end point:

$$\ldots$$

$\square$