MATHEMATICAL LOGIC FOR COMPUTER SCIENCE (A.Y. 20/21)

HANDOUT N. 6

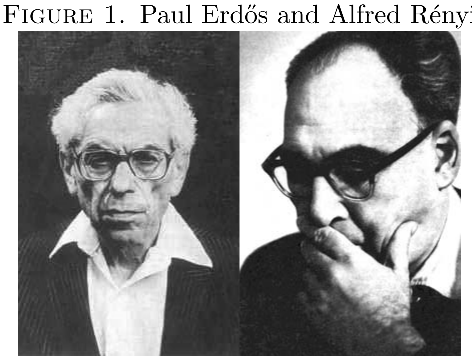
Abstract. Countable random graphs. The Random Graph Property. Extension Axioms and the Random Graph Theory. All countable models of the Random Graph Theory are isomorphic.

1. The Random Graph

A graph is a pair (V, E) where V is a set (the vertices of the graph) and E is a binary irreflexive and symmetric relation on $V \times V$. A graph $G = (V, E)$ is countable if V is countable.

Random graphs were first studied by Erdős and Rényi. Given a countable set V of vertices we can impose a graph structure on V (i.e., an edge relation E) at random by picking edges independently with probability $1/2$ from the set of unordered pars of vertices. A graph obtained in this manner is a random graph. Erdős and Rényi established the following surprising result.

Figure 1. Paul Erdős and Alfred Rényi



Theorem 1.1 (Erdős and Rényi, 1963)**.** *There exists a graph G such that any countable random graph is isomorphic to G with probability* 1*, i.e., almost surely!*

An important ingredient of the proof is the following property, which we call the Random Graph Property, also known as "Alice's Restaurant Property":

> A graph $G = (V, E)$ has the **Random Graph Property** if and only if for any pair of disjoint finite sets $A, B \subseteq V$, there is a vertex $v \in V$ not in A nor B such that v is connected to all vertices in A and disconnected from all vertices in B.

Erdős and Rényi established their theorem in two steps:

(1) With probability 1 a countable random graph satisfies the Random Graph Property.
(2) Any two countable graphs satisfying the Random Graph Property are isomorphic.

The first point is obtained by a probabilistic argument (for a proof see, e.g., Diestel, *Graph Theory*, Lemma 11.3.2). We will establish the second point by a back-and-forth proof. To this end we first define an appropriate theory to express the Random Graph Property.

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2. Extension Axioms and the Random Graph Theory

A predicate language adequate for speaking about graphs is the language $\mathcal{L}_G = \{E\}$, where E is a binary relation symbol. The following sentences force any structure $\mathfrak{A} = (A, R^{\mathfrak{A}})$ adequate for the language of graph to actually be a graph:

$$\forall x \neg E(x,x),$$
$$\forall x \forall y (E(x,y) \leftrightarrow E(y,x)).$$

We call these the Graph Axioms.

We now wish to define a theory (set of sentences) that forces any of its models to satisfy the Random Graph Property. Since direct quantification over all finite sets is not available in the formalism of first-order logic, we use a different strategy: we deal separately with any choice of possible cardinalities of the sets A and B.

For each $m, n \in \mathbf{N}$ we define a sentence $S_{n,m}$ as follows.

If $m, n > 0$, then $S_{n,m}$ is the following sentence:

$$\forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_m (\{x_1, \dots, x_n\} \cap \{y_1, \dots, y_m\} = \varnothing \to$$
$$\exists z (z \notin \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_m\} \wedge \bigwedge_{1 \le i \le n} E(z, x_i) \wedge \bigwedge_{1 \le j \le m} \neg E(z, y_j)).$$

Note that $\{x_1, \dots, x_n\} \cap \{y_1, \dots, y_m\} = \varnothing$ and $z \notin \{x_1, \dots, x_n\} \cup \{y_1, \dots, y_m\}$ are not formulas in our language. But it is easy to write a formula that expresses these properties using pairwise inequalites (Exercise).

If $n = 0$ and $m > 0$, then $S_{n,m}$ is the following sentence:

$$\forall y_1 \dots \forall y_m \exists z (z \notin \{y_1, \dots, y_m\} \bigwedge_{1 \le j \le m} \neg E(z, y_j)).$$

If $n > 0$ and $m = 0$, then $S_{n,n}$ is the following sentence:

$$\forall x_1 \dots \forall x_n \exists z (z \notin \{x_1, \dots, x_n\} \bigwedge_{1 \le i \le n} E(z, x_i)).$$

If $n = 0 = m$ then $S_{n,m}$ is $\exists z(z = z)$.

The $S_{n,m}$s are called **Extension Axioms**.

NB In class we also added clauses asserting that the xs are all distinct and similarly for the ys, so as to force that we are referring to at least n many (elements corresponding to the) xs and to at least m many elements corresponding to the ys. It is no harm *not* to use these clauses; the intended case of a subset of cardinality n and a subset of cardinality m are will anyways be a particular case of the formulas given above.

It is now easy to see that any countable structure $\mathfrak{A} = (A, E^{\mathfrak{A}})$ adequate for the language of graphs satisfies the Random Graph Property if it satisfies the Graph Axioms and all sentences $S_{n,m}$ (the Extension Axioms).

Let T be the theory consisting of the Graph Axioms and all Extension Axioms. We call T the **Random Graph Theory**.

3. Countable models of the Random Graph Theory

Theorem 3.1. *All countable models of the Random Graph Axioms are isomorphic.*

Proof. Let $\mathfrak{G} = (V_0, E^{\mathfrak{G}})$ and $\mathfrak{H} = (V_1, E^{\mathfrak{H}})$ be two countable models of the Random Graph Axioms. Let $(a_n)_{n \in \mathbf{N}}$ be an enumeration of V_0 with no repetitions and let $(b_n)_{n \in \mathbf{N}}$ be an enumeration of V_1 with no repetitions.

Base Case: Set $p_0 = a_0$ and $q_0 = b_0$.

Inductive Case: Suppose we have defined $\{p_0, \dots, p_n\}$ in V_0 and $\{q_0, \dots, q_n\}$ in V_1. We define p_{n+1} and q_{n+1} as follows.

If n is even: First observe that some vertices of V_0 are unused, i.e., are not among $\{p_0, \dots, p_n\}$, since V_0 is infinite countable. Thus for all but finitely many $i \in \mathbf{N}$ we have $a_i \in V_0 \setminus \{p_0, \dots, p_n\}$. Let p_{n+1} be the one with the smallest index (i.e., the first element of $V_0 \setminus \{p_0, \ldots, p_n\}$ that appears in the fixed injective listing $(a_n)_{n\in\mathbf{N}}$ of V_0.

Now we split $\{p_0, \ldots, p_n\}$ in two disjoint parts according to connectedness with p_{n+1} in the structure $\mathfrak{G}$. More precisely, we define

$$X = \{m \,:\, m \leq n \,\&\, (p_m, p_{n+1}) \in E^{\mathfrak{G}}\}$$
$$Y = \{m \,:\, m \leq n \,\&\, (p_m, p_{n+1}) \notin E^{\mathfrak{G}}\}$$

It is easy to observe that X and Y are finite and disjoint. Consider the sets induced by X, Y in $\mathfrak{H}$:

$$A = \{q_m \,:\, m \in X\}$$
$$B = \{q_m \,:\, m \in Y\}$$

It is easy to observe that A and B are two disjoint sets in $\mathfrak{H}$ (Exercise: spell out the details needed here).

Since $\mathfrak{H}$ models the Random Graph Axioms, the extension axiom $S_{|A|,|B|}$ is such that its premises are satisfied by the elements of A and the elements of B. Therefore $\mathfrak{H}$ also satisfies the consequence of the extension axiom, and therefore we can pick a q_{n+1} in V_1 that is connected (in the sense of $E^{\mathfrak{H}}$) to all elements of A and disconnected (in the sense of $E^{\mathfrak{H}}$) to all elements of B.

If n is odd: We do as in the previous case but exhchanging the roles of $\mathfrak{G}$ and $\mathfrak{H}$.

It is easy to verify that the map $p_i \mapsto q_i$ is an isomorphism between $\mathfrak{G}$ and $\mathfrak{H}$. $\square$

Corollary 3.2. *The set of sentences satisfied in all countable models of the Random Graph Theory is a complete theory.*