MLCS - Homework 3

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1 Computable, c.e. and not computable Problems

Exercise 1.2. Show by informal arguments the following points.

(1) If $L_1 \cap L_2$ is not computable and L_2 is computable then L_1 is not computable.

(2) If $L_1 \cup L_2$ is not computable and L_2 is computable then L_1 is not computable.

(3) If $L_1 \setminus L_2$ is not computable and L_2 is computable then L_1 is not computable.

(4) If $L_2 \setminus L_1$ is not computable and L_2 is computable then L_1 is not computable.

(5) If L_1 and L_2 are computably enumerable then $L_1 \cup L_2$ and $L_1 \cap L_2$ are computably enumerable.

Solution. The first four points can be proved proceeding by contradiction, slightly tweeking the rationale at each case.

Part 1:
Let's assume by contradiction that L_1 is computable. By hypothesis, we also know that L_2 is computable, therefore there exist two Turing Acceptors M_1 and M_2 such that, for $i = 1, 2$:

$$\text{if } x \in L_i \text{ then } M_i \text{ accepts } x; \text{ if } x \notin L_i \text{ then } M_i \text{ rejects } x.$$

Let's consider now a new Turing Acceptor M, defined as follows:

$$M(x) = \begin{cases} M_1(x) & \text{if } x \in L_1 \cap L_2 \\ M_1(x) & \text{if } x \in L_2 \setminus L_1 \\ M_2(x) & \text{if } x \in L_1 \setminus L_2 \\ M_2(x) & \text{if } x \notin L_1 \cup L_2 \end{cases}$$

where $M_1(x)$ and $M_2(x)$ are respectively the results of the computation of the Turing machines M_1 and M_2 on the element x.

The existence of such machine is, by definition, equivalent to the fact that $L_1 \cap L_2$ is decidable, since the elements in $L_1 \cap L_2$ are going to be accepted by the machine M and all the elements outside $L_1 \cap L_2$ are going to be rejected. This is absurd by hypothesis. Therefore, we can conclude that L_1 is not computable.

Notice also that the for the first and last cases in the definition of M, we could have chosen either M_1 and M_2, getting the same results.

Part 2:

Similarly to the previous point, we can proceed by contradiction, assuming that L_1 is computable, and considering a new Turing Acceptor M:

$$M(x) = \begin{cases} M_1(x) & \text{if } x \in L_1 \\ M_2(x) & \text{if } x \in L_2 \\ M_2(x) & \text{if } x \notin L_1 \cup L_2 \end{cases}$$

The existence of this machine would be equivalent to the fact that $L_1 \cup L_2$ is computable, which again would be absurd. Therefore, we can conclude that L_1 is not computable.

$\square$

2 *NP* problems