HOMEWORK N. 4, MATHEMATICAL LOGIC FOR COMPUTER SCIENCE 2020/2021

DEADLINE: MAY 31 2021.

Choose three exercises, at least one from each group!

1. Compactness

Exercise 1.1. Let T be a theory that axiomatizes a property P of structures. Prove that if P is also finitely axiomatizable then P is finitely axiomatizable by a subset of T.

Exercise 1.2. Let T_1 and T_2 be two theories in a language $\mathcal{L}$. Suppose that for each structure $\mathfrak{A}$ adequate for $\mathcal{L}$, $\mathfrak{A} \models T_1$ if and only if $\mathfrak{A} \not\models T_2$. Then T_1 and T_2 are finitely axiomatizable.

(Hint: Reason by way of contradiction and use the Compactness Theorem to obtain a model of an unsatisfiable theory).

Exercise 1.3. Show that the property of being a non-bipartite graph is not finitely axiomatizable.

(Hint: use a particular property of bipartite graphs concerning cycles and a standard Compactness argument).

Exercise 1.4. Let $<$ be a strict total order on a set X (that is, $<$ is anti-symmetric, irreflexive and total). We call such a relation *nice* if it admits no infinite decreasing sequences. Prove by a Compactness argument that the notion of being a nice relation is not axiomatizable.

Exercise 1.5. Prove the following: If a property P and its complement are axiomatizable then P is finitely axiomatizable.

Exercise 1.6. Let $p_0, p_1, p_2, \ldots$ be the list of all prime numbers in increasing order. Show that for any subset $S \subseteq \mathbf{N}$ there is a model of arithmetic (i.e. a model of all sentences true in the standard model) that contains an element c such that c is divisible by p_i for all and only the p_is such that $i \in S$.

(Hint: use an extra constant and Compactness)

Exercise 1.7. Let T be a theory that has some finite models and some infinite models. Let E be a sentence such that is $\mathfrak{A} \models T$ and $\mathfrak{A}$ is infinite then $\mathfrak{A} \models E$. Show that there is a bound $b \in \mathbf{N}$ such that if $\mathfrak{A} \models T$ and $\mathfrak{A}$ is of cardinality $\geq b$ then $\mathfrak{A} \models E$.

(Hint: Compactness and some of its corollaries).

Exercise 1.8. Assuming the Infinite Ramsey's Theorem give a proof by Compactness of the following principle:

For all m, there exists an n such that for all colorings $f : [1, \ldots, n]^2 \to \{0, 1\}$ there exists a set $H \subseteq [1, \ldots, n]$ such that $|H| \geq m$ and $|H| > \min(H)$ and f is constant on $[H]^2$.

Exercise 1.9. Consider the class of undirected graphs with no self-loop.

A graph is **acyclic** if, for each $n \geq 3$ *it does not* contain distinct vertices $x_1, \ldots, x_n$ such that x_i is adjacent to x_{i+1} for each $1 \leq i < n$ and x_n is adjacent to x_1. Prove that the property of being an acyclic graph is not finitely axiomatizable in the first-order language of graphs.

(Hint: Use Compactness).

Exercise 1.10. A subset S of vertices of an undirected graph is called a **star** if there exists an element $v \in S$ such that for each other $w \in S$, w is adjacent to v and to no other vertex.

The property P of "not containing a star of even cardinality" is axiomatizable in the language of graphs by the following theory: $\{A_n : n \in \mathbf{N}\}$ where A_n expresses "There is no star of cardinality $2n$" (i.e., "There are no 2n distinct elements forming a star").

Is $\neg P$ axiomatizalbe?
Is P finitely axiomatizable?

Exercise 1.11. A strict total order on a set X (i.e., an anti-symmetric, reflexive and total) binary relation is called *a well-ordering* if it admits no infinite strictly decreasing sequences. Prove that the property of being a well-ordering is not axiomatizable (in the language of orders).

(Hint: Expand by constant(s) and use Compactness).

Exercise 1.12. A subset S of vertices of an undirected graph is called a *clique* if each vertex in S is adjacent to any other vertex in S. The property P of "not containing cliques of even cardinality" is axiomatizable in the language of graphs by the theory $\{A_n : n \in \mathbf{N}\}$ where A_n expresses "There is no clique of cardinality $2n$".

Is $\neg P$ axiomatizable?
Is P finitely axiomatizable?

Exercise 1.13. In the language $\mathcal{L}$ containing the symbol $R(x, y)$ (and identity $=$ as a logical symbol) write a sentence E such that the following set

$$S = \{n \in \mathbf{N}^+ : \text{ esiste } \mathfrak{A} \text{ t.c. } \mathfrak{A} \models E \text{ e } |A| = n\}$$

is the set of even positive integers ($\mathbf{N}^+$ denotes the set of positive integers).

Does E finitely axiomatize the property of having even cardinality domain?

(Hint: axiomatize R as a special equivalence relation.)

Exercise 1.14. Is there a theory T with infinite models, at least one finite model but such that T does not have finite models of arbitrarily high cardinality?

Exercise 1.15. Let $\mathfrak{A}$ be a non-standard model of arithmetic and let $F(x)$ be a formula with one free variable. If it is the case that for infinitely many $n \in \mathbf{N}$ we have $\mathfrak{A} \models F[\binom{x}{\overline{n}^{\mathfrak{A}}})$ then there is a non-standard element $a \in A$ such that $\mathfrak{A} \models F[\binom{x}{a})]$. In other words: if a formula is satisfied by infinitely many standard numbers then it is satisfied also by a non-standard number.

(Hint: reason on the properties of sets that are definable in a non-standard model of arithmetic).

2. Group 2

Recall that a theory T is ω-consistent if there is not formula $A(x)$ such that for all $n \in \mathbf{N}$, $T \vdash A(\overline{n})$ and $T \vdash \exists x \neg A(x)$.

Exercise 2.1. Prove that ω-consistency implies consistency.

Exercise 2.2. Consider Gödel's unprovable sentence G for some theory $T \supseteq \mathbf{MA}$, satisfying:

$$T \vdash G \leftrightarrow \forall x \neg Proof_T(x, \overline{code(G)}).$$

Prove that if T is consistent then $T + \{\neg G\}$ is consistent but not ω-consistent.

Exercise 2.3. Apply the fix-point theorem to obtain sentences in the language of arithmetic that express the following:

(1) I am decidable in **MA** (i.e. either provable or disprovable).
(2) I am undecidable in **MA**.
(3) I am not refutable in **MA** (i.e. I am consistent with **MA**).
(4) I am provable in **MA**.

For each of the above say as much as possible of the following questions: Is the sentence provable, refutable or undecidable? Is the sentence true in the standard model?

Exercise 2.4. A theory T is called 1-consistent if the following holds: For every formula $R(x)$ of type Δ_0 (i.e., with bounded quantifiers only), if for all $n \in \mathbf{N}$ we have $T \vdash R(\overline{n})$, then $T \nvdash \exists x \neg R(x)$. Let T be a theory in the language of arithmetic such that for every sentence E of type Σ_1, if $\mathbf{N} \models E$ then $T \vdash E$. Prove that, for every sentence A, $T \cup \{A\}$ is 1-consistent if and only if for every sentence E of type Π_1 true in $\mathbf{N}$, $T \cup \{A, E\}$ is consistent. (A sentence of type Π_1 is of the form $\forall x_1 \ldots \forall x_k H$ where H contains only bounded quantifiers. Note that the negation of a Σ_1 formula is Π_1, and viceversa).

Exercise 2.5. Recall the Rosser sentence (see handout notes).

$$E := (\forall y)(F(\overline{m}, y) \to (\exists z)(z \leq y \wedge H(\overline{m}, z))),$$

where m is the code of the formula $(\forall y)(F(x, y) \to (\exists z)(z \leq y \wedge H(x, z)))$ F represents the relations R and H represents the relation S introduced in the handouts. Show that if $T \supseteq \mathsf{MA}$ is consistent then $T \nvdash E$ and $T \nvdash \neg E$. One can prove that if T is consistent then $\neg E$ is not provable.

Exercise 2.6. Deduce from Tarski's Theorem (on the non-representability of the theorems of a theory within a theory - proved in class) the following version of Godel's Theorem: If T is an ω-consistent decidable set of sentence extending MA then T is incomplete. 7

(Hint: use the fact that the relation $(a, b) \in R$ iff "a is the code of a proof in T of sentence coded by b" is computable).

Exercise 2.7. Argue that if F is a Σ_1-sentence then if $\mathcal{N} \models F$ then $\mathsf{MA} \vdash F$. (You can consider F with just one existential quantifier). Argue that this implies that for some class of sentences of arithmetic, if $T \nvdash \neg E$ then E is true.

Exercise 2.8. Indicate whether the following points are true or false assuming that MA is consistent, and explain why (E is a sentence in the language of MA).

(1) If $\mathsf{MA} \vdash E$ then $\mathsf{MA} \not\models \neg E$.
(2) If there exists a structure $\mathfrak{A}$ such that $\mathfrak{A} \models \mathsf{MA}$ and $\mathfrak{A} \models E$ then $\mathbf{N} \not\models \neg E$.
(3) If $\mathbf{N} \models E$ then $\mathsf{MA} \cup \{\neg E\}$ has not models.
(4) If $\mathbf{N} \not\models E$ then $\mathsf{MA} \nvdash E$.
(5) If $\mathbf{N} \not\models E$ then $\mathsf{MA} \nvdash \neg E$.
(6) If $\mathbf{N} \models A \to B$ then, if $\mathsf{MA} \vdash A$, then $\mathsf{MA} \vdash B$.