MLCS - Homework 1

Michele Conti
1599133

1 Group 1

Exercise 2. Show that the following sentence is unsatisfiable, where S is any formula with two free variables: $\exists x \forall y (S(x,y) \leftrightarrow \neg S(y,y))$.

Solution. Given any choice of x, it is sufficient to pick $y = x$, in which case we would get

$$S(x,x) \leftrightarrow \neg S(x,x).$$

Since this is obviously a contradiction, it means that it exists y such that the formula S is false, meaning that it is unsatisfiable. □

Exercise 4. Is the following formula logically valid for any formula F and any term t?

$$\forall x F(x) \rightarrow F(t).$$

If not, give an example of a formula F, a structure $\mathfrak{A}$ and an assignment α witnessing this fact.

Solution. If we analyze the premise of the implication, we notice that we only have two cases:

1. $F(x)$ is true for every x (i.e., the premise is true).
2. It exists x such that $F(x)$ is false (i.e., the premise is false).

In case 1, we have that for every choice of x the formula F is true, meaning that it is also true for the given term t. Therefore, the implication is true, because we are considering an implication between a two true statements.

In case 2, the premise of the implication is false, therefore we can conclude that the whole implication is true. □

Exercise 9. In the language $\mathcal{L} = \{<\}$ of **DLO**, write a sentence that distinguishes $(\mathbf{N}, <)$ from $(\mathbf{Q}, <)$ i.e., that is true in one structure but not in the other.

Solution. The following sentence is true in $(\mathbf{N}, <)$, but not in $(\mathbf{Q}, <)$:

$$\forall x \exists y (\neg(x = y) \rightarrow y < x).$$

This represents the fact that $\mathbb{Q}$ has no left endpoint, while $\mathbb{N}$ has 0 (or 1, depending on whether we consider the set to have or not the element 0), which is lower than any other element. □

2 Group 2

Exercise 3. Is the structure $\mathcal{Q} = (\mathbb{Q}, +, \times, 0, 1)$ a substructure $\mathcal{R} = (\mathbb{R}, +, \times, 0, 1)$? Is it an elementary substructure?

Solution. My assumption is that $\mathcal{Q}$ is a substructure of $\mathcal{R}$, but that it is not an elementary substructure. Let's prove both points.

Part 1: $\mathfrak{B}$ is a substructure of $\mathfrak{A}$.

Recalling the definition of substructure, we need to prove that:

1. $B \subseteq A$.
2. For every constant symbol c, $c^{\mathfrak{A}} = c^{\mathfrak{B}}$.
3. Every relation $R^{\mathfrak{B}}$ (resp. function $f^{\mathfrak{B}}$) is the restriction of $R^{\mathfrak{A}}$ (resp. $f^{\mathfrak{A}}$) to B.

For the first point, there is nothing to prove.

The second point is trivially true, since we only have two constants in both structures (i.e., 0 and 1) and they correspond.

For the last point, since both structures only have the sum and multiplication operations in the language, we need to check if the sum and multiplication operations of the first structure $\mathcal{Q}$ (to which we'll refer to respectively with $+^{\mathcal{Q}}$ and $\times^{\mathcal{Q}}$) are the restrictions of the sum and multiplication operations of the second structure $\mathcal{R}$ (to which we'll refer to respectively with $+^{\mathcal{R}}$ and $\times^{\mathcal{R}}$).

Part 2: Substructure $\mathfrak{B}$ of $\mathfrak{A}$ is not elementary.

To prove that the substructure $\mathfrak{B}$ is not elementary, we can consider the formula

$$\forall x \exists y((x > 0) \rightarrow (y \times y = x)).$$

This is clearly true for $\mathcal{R}$, but it is not for $\mathcal{Q}$. In fact, we can consider $x = 2$ and it doesn't exist any y such that $y \times y = 2$. $\square$

Exercise 4. Prove that the following structures are not isomorphic:

(1) $(\mathbb{N} +, \times, 0, 1, <)$ and $(\mathbb{Q}, +, \times, 0, 1, <)$.

(2) $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$.

(3) $(\mathbb{Q}, <)$ and $(\mathbb{R}, <)$.

(Hint: in some cases you can use the fact that if $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic then they satisfy the same sentences).

Solution. Let's consider case by case.

To prove that $(\mathbb{N} +, \times, 0, 1, <)$ and $(\mathbb{Q}, +, \times, 0, 1, <)$ are not isomorphic, we can consider the sentence that expresses the existence of additive inverse:

$$\forall x \exists y(x + y = 0)$$

is satisfied in $(\mathbb{Q}, +, \times, 0, 1, <)$, but not in $(\mathbb{N} +, \times, 0, 1, <)$. We know that two structures are isomorphic if and only if they satisfy the same sentences, therefore we can conclude that these two are not.

To prove that $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$ are not isomorphic, we can proceed in a similar way, using the sentence expressing the existence of the left-end point:

$$\ldots$$

$\square$