MATHEMATICAL LOGIC FOR COMPUTER SCIENCE
(A.Y. 20/21)

HANDOUT N. 2

ABSTRACT. Semantics of first-order logic. Satisfaction, truth, validity.

1. STRUCTURES, SATISFACTION, TRUTH, VALIDITY

We fix a language $\mathcal{L} = \{R_i, f_j, c_k \ : \ I \in i, j \in J, k \in K\}$, where I, J, K are subsets of $\mathbf{N}$. A *structure* $\mathfrak{A}$ *for* the language $\mathcal{L}$ (or *adequate for* $\mathcal{L}$, or an $\mathcal{L}$-structure) consists of

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- a nonempty set A, called the *domain* of the structure.
- for each R_i of arity k, a relation of size k in A, which we denote by $R_i^{\mathfrak{A}}$.
- for each f_j of arity k, a function of k arguments from A^k to A, which we denote by $f_j^{\mathfrak{A}}$.
- for each $k \in K$, an element of A, which we denote by $c_k^{\mathfrak{A}}$.

Recall that a relation of dimension k on a set A is a set of ordered sequences of size k of elements of A, i.e. a subset of the cartesian product A^k (the set of all ordered k-tuples of elements of A).

Example 1.1. If $\mathcal{L}$ is the language of graphs, i.e. $\{E\}$, where E is a binary relation symbol, any structure of type $\mathfrak{A} = (A, E^{\mathfrak{A}})$, with $E^{\mathfrak{A}} \subseteq A \times A$ is adequate for $\mathcal{L}$. Any structure of this kind can be viewed as a graph. If $E^{\mathfrak{A}}$ is symmetric, we can identify $\mathfrak{A}$ with an undirected graph. Standard mathematical practice gives us an intuitive understanding of what it means that a formula such as $F = \forall x \forall y (\neg(x = y) \to E(x, y))$ is true or false in a given graph G, for example $G = (V, E) = (\{1, 2, 3, 4\}, \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 4), (4, 2)\})$, which we read as the undirected graph with four vertices and four undirected edges. It is intuitively clear that F is false in this graph, since it asserts that the graph is complete.

We also have an intuitive understanding of how to evaluate whethere a formula like

$$\exists y \exists z (\neg(y = z) \wedge E(x, y) \wedge E(x, z))$$

is true or false in a graph. Let G' be (V', E') where $V' = V \cup \{5\}$ and $E' = E \cup \{(4, 5), (5, 4)\}$. In this case we notice that the truth in G' of the formula above depends on what value we assign to the only free variable x appearing in the formula. If the variable x is interpreted as one of the vertices $1, 2, 3, 4$ then the formula is true because we can find two witnesses for the existential quantifiers; while if x is interpreted as 5 then it is impossible to find values for y and z that make the rest of the formula true (vertex 5 has only one neighbour in G').

The definitions we give below aim at making this intuitive way of evalutating the truth of formulas precise and completely rigorous, in order to turn it into an object of mathematical study.

We want to define the relation of *truth* or *validity* of a formula F in a structure $\mathfrak{A}$, which we denote with $\mathfrak{A} \models F$. For this purpose we define a ternary relation of *satisfiability* (or *truth*) of a formula *in a structure relatively to an assignment* of elements to variables. This is necessary to deal with formulas containing free variables, whose meaning depends on an interpretation/assignment of the variables.

An *assignment* α in $\mathfrak{A}$ is a map that associates to each variable an element of A, i.e.,

$$\alpha : \{v_n \ : \ n \in \mathbf{N}\} \longrightarrow A.$$

Remark 1.2. An assignment extends uniquely to the terms by setting $\alpha(c)$ equal to $c^{\mathfrak{A}}$ for each constant symbol c and, inductively, defining $\alpha(f(t_1, \ldots, t_k))$ equal to $f^{\mathfrak{A}}(\alpha(t_1), \ldots, \alpha(t_k))$ for each k-ary function symbol f. We use α to indicate the extension of assignment α to terms.

If α is an assignment, v a variable and a an element of domain A, we denote by $\alpha\binom{x}{a}$ the assignment that differs from α only because it assigns the element a to the variable x.

The following fundamental definition is due to Alfred Tarski (1933).

Definition 1.3 (Satisfaction/Truth relative to an assignment). We define the relation "the structure $\mathfrak{A}$ satisfies the formula F with respect to the assignment α" (or "F is true in $\mathfrak{A}$ relative to α"), which we denote by $\mathfrak{A} \models F[\alpha]$ (or $\mathfrak{A} \models^\alpha F$), as follows, by induction on the structure of the formula.

> $\mathfrak{A} \models (t = s)[\alpha]$ if and only if $\alpha(t)$ is equal to $\alpha(s)$.
> $\mathfrak{A} \models R(t_1, \ldots, t_k)[\alpha]$ if and only if $(\alpha(t_1), \ldots, \alpha(t_k)) \in R^{\mathfrak{A}}$.
> $\mathfrak{A} \models (\neg G)[\alpha]$ if and only if it is not the case that $\mathfrak{A} \models G[\alpha]$ (we write $\mathfrak{A} \not\models G[\alpha]$ in this case).
> $\mathfrak{A} \models (G \wedge H)[\alpha]$ if and only if $\mathfrak{A} \models G[\alpha]$ and $\mathfrak{A} \models H[\alpha]$.
> $\mathfrak{A} \models (G \vee H)[\alpha]$ if and only if $\mathfrak{A} \models G[\alpha]$ or $\mathfrak{A} \models H[\alpha]$.
> $\mathfrak{A} \models (G \rightarrow H)[\alpha]$ if and only if: If $\mathfrak{A} \models G[\alpha]$ then $\mathfrak{A} \models H[\alpha]$
> $\mathfrak{A} \models (G \leftrightarrow H)[\alpha]$ if and only if: $\mathfrak{A} \models G[\alpha]$ if and only if $\mathfrak{A} \models H[\alpha]$.
> $\mathfrak{A} \models \exists v G[\alpha]$ if and only if there exists $a \in A$ such that $\mathfrak{A} \models G[\alpha\binom{v}{a}]$.
> $\mathfrak{A} \models \forall v G[\alpha]$ if and only if for every $a \in A$ it is the case that $\mathfrak{A} \models G[\alpha\binom{v}{a}]$.

Note that in the case of a formula $(G \rightarrow H)$ the implication if ... then in the definition of satisfaction it is understood as so-called the material implication, i.e. as either $\mathfrak{A} \not\models G[\alpha]$ or $\mathfrak{A} \models H[\alpha]$. The symbol $\not\models$ stands for: $\models$ does not hold.

Example 1.4. Let $\mathcal{L} = \{c, f, R\}$ where f is a function symbol with one argument and R a binary relation symbol. Let F be the formula $\forall y(R(x, y) \rightarrow \exists z(R(x, z) \wedge R(z, y)))$ and G the sentence $\forall x \forall y(x = c \vee R(f(x), x))$. Let $\mathfrak{A}$ be the structure with domain $\mathbb{Z}$, which interprets c to 0, f as $x \mapsto x - 1$ and R as the standard $<$ relation on integers. $\mathfrak{A} \models F[\alpha]$ if and only if for every integer a if $\alpha(x) < a$ then there exists an integer b such that $\alpha(x) < b$ and $b < a$. $\mathfrak{A} \models G[\alpha]$ if and only if for every integer a for every integer b, if a is not 0 then $a - 1$ is less than a.

Remark 1.5. The fact that $\mathfrak{A} \models F[\alpha]$ holds or not depends only on the values of α on the free variables that appear in F. In other words, if the free variables F are contained in $\{x_1, \ldots, x_n\}$ and α and β are two assignments that coincide on the values assigned to the variables $x_1, \ldots, x_n$, then $\mathfrak{A} \models F[\alpha]$ if and only if $\mathfrak{A} \models F[\beta]$. Therefore we can write $\mathfrak{A} \models F[\alpha(x_1), \ldots, \alpha(x_n)]$ stating explicitly the elements assigned to the variables that really matter. We also note that if F is sentence, then $\mathfrak{A} \models F[\alpha]$ applies to all assignments or to anyone!!!!

Definition 1.6 (Satisfaction/Truth of a Formula in a Structure under an Assignment). If $\mathfrak{A} \models F[\alpha]$ for some assignment α, we say that α *satisfies* the formula F in $\mathfrak{A}$ (or that F is *true* in $\mathfrak{A}$ relative to α). In this case F is said to be *satisfiable in* $\mathfrak{A}$. A formula is *satisfiable* if it is satisfiable in some structure.

Definition 1.7 (Satisfaction/Truth of a Formula in a Structure). A formula F is *true* in a structure $\mathfrak{A}$ if for all assignments α in $\mathfrak{A}$, $\mathfrak{A} \models F[\alpha]$. In this case we say that $\mathfrak{A}$ is a model of F and we write $\mathfrak{A} \models F$.

Note that a sentence is either true or not true (false) in any given structure interpreting its language. On the other hand a formula can be satisfied with respect to some assignments and not satisfied with respect to others. So, for sentences S we have, for any structure $\mathfrak{A}$, that either $\mathfrak{A} \models S$ or $\mathfrak{A} \models \neg S$; while for formulas F we can have that neither $\mathfrak{A} \models F$ nor $\mathfrak{A} \models \neg F$.

We now define the notion of validity. This notion is intended to capture the idea of a logical truth. The intuition is that a formula is true on logical grounds (rather than domain-dependent ones) if it is true in all possible interpretations (all possible worlds).

Definition 1.8 (Satisfaction/Truth/Validity of a Formula (= truth in all structures))**.** A formula F is *valid* (or *true*, or *logically valid*) if it is satisfied in all structures with respect to all assignments (i.e. if F is true in all structures). We also say in this case that it is a *logical truth* and we write $\models F$.

Remark 1.9. Observe that F is valid if and only if $\neg F$ is not satisfiable.

Remark 1.10. If F is a formula then for some $x_1, \ldots, x_n$, F is $F(x_1, \ldots, x_n)$. Then $\forall x_1 \ldots \forall x_n F(x_1, \ldots, x_n)$ is a sentence. We call it the *universal closure* of F. Suppose that $\mathfrak{A} \models \forall x_1 \ldots \forall x_n F(x_1, \ldots, x_n)$. Then for all assignment α, for all choices of $a_1, \ldots, a_n \in A$, $\mathfrak{A} \models F(x_1, \ldots, x_n)[\alpha\binom{x_1, \ldots, x_n}{a_1, \ldots, a_n}]$, i.e. for any assignment α, $\mathfrak{A} \models F(x_1, \ldots, x_n)[\alpha]$. Summing up: if the universal closure of a formula is valid then the formula is satisfied in all structures by all assignments; i.e., the formula is valid. Vice versa, suppose $F(x_1, \ldots, x_n)$ is satisfied in all structures by all assignments. Then it is easy to check that the sentence $\forall x_1 \ldots \forall x_n F(x_1, \ldots, x_n)$ is valid.

Definition 1.11 (Satisfaction/Truth/Validity of a set of Formulas)**.** Let Γ be a set of formulas (either finite or infinite). We write $\mathfrak{A} \models \Gamma[\alpha]$ if for each formula F in Γ, $\mathfrak{A} \models F[\alpha]$. If $\mathfrak{A} \models \Gamma[\alpha]$ holds for all α, then we write $\mathfrak{A} \models \Gamma$.