Problem 1:

(a) $f_i'(x) + \sum_{k=0}^{2} a_k f_{i+k}(x) = O(h^{\alpha})$

$\Rightarrow$

	f_i	f_i'	f_i''	f_i'''
f_i'	0	1	0	0
$a_0 f_i$	a_0	0	0	0
$a_1 f_{i+1}$	a_1	$a_1 h$	$\frac{1}{2}a_1 h^2$	$\frac{1}{6}a_1 h^3$
$a_2 f_{i+2}$	a_2	$2a_2 h$	$2a_2 h^2$	$\frac{4}{3}a_2 h^3$

$$\Rightarrow f_i' + \sum_{k=0}^{2} a_k f_{i+k} = (a_0 + a_1 + a_2) f_i + (1 + a_1 h + 2a_2 h) f_i' + \left(\tfrac{1}{2}a_1 h^2 + 2a_2 h^2\right) f_i'' + \left(\tfrac{1}{6}a_1 h^3 + \tfrac{4}{3}a_2 h^3\right) f_i''' + \cdots$$

$$\Rightarrow \begin{cases} a_0 + a_1 + a_2 = 0 \\ 1 + a_1 h + 2a_2 h = 0 \\ \frac{1}{2}a_1 h^2 + 2a_2 h^2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = -\frac{2}{h} \\ a_2 = \frac{1}{2h} \\ a_3 = \frac{3}{2h} \end{cases}$$

$$\boxed{\Rightarrow f_i'(x) = \frac{-3f_i + 4f_{i+1} - f_{i+2}}{2h} + O(h^2) \quad \Rightarrow \text{leading error: } \frac{1}{3}h^2 f_i'''}$$

(b) $f(x) = e^{ikx}$

Consider a domain with length L and N intervals (uniform) $\Rightarrow h = \frac{L}{N}$

① consider FDA in (a):

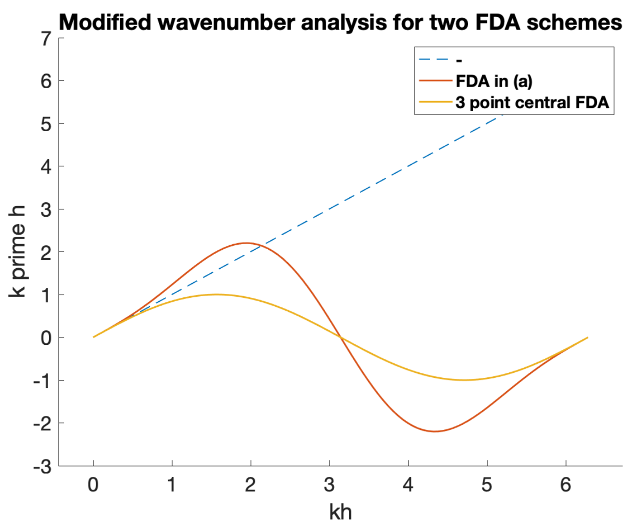
$$f_j'(x) = \frac{-3f_j + 4f_{j+1} - f_{j+2}}{2h} = \frac{1}{2h}\left(-3e^{ikx_j} + 4e^{ik(x_j+h)} - e^{ik(x_j+2h)}\right)$$

$$= \frac{e^{ikx_j}}{2h}\left(-3 + 4e^{ikh} - e^{2ikh}\right)$$

$$= i\cdot\left[\frac{1}{2ih}\left(-3 + 4e^{ikh} - e^{2ikh}\right)\right]\cdot e^{ikx_j} \quad \Rightarrow \boxed{k_1' = \frac{1}{2ih}\left(-3 + 4e^{ikh} - e^{2ikh}\right)}$$

② consider 3-point central FDA:

$$f_j'(x) = \frac{f_{j+1} - f_{j-1}}{2h} = i\cdot\left[\frac{1}{2ih}\left(e^{ikh} - e^{-ikh}\right)\right]\cdot e^{ikx_j} \quad \Rightarrow \boxed{k_2' = \frac{1}{2ih}\left(e^{ikh} - e^{-ikh}\right)}$$



(c) consider the 3-point central formula for $f''(x)$

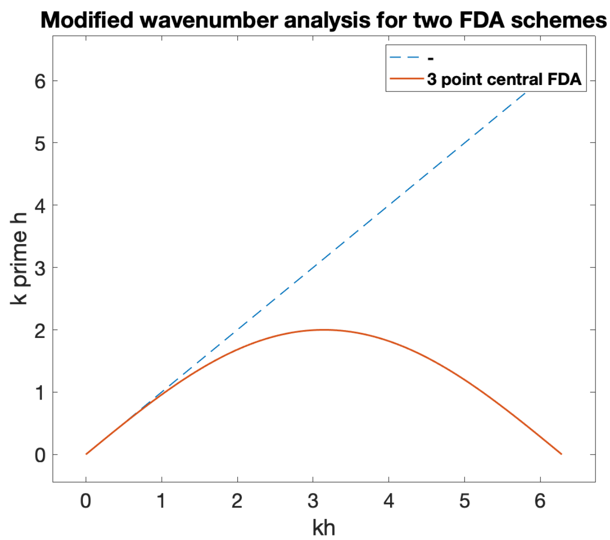
exact derivative: $f''(x) = i^2 k^2 e^{ikx} = -k^2 e^{ikx}$

$$f_j''(x) = \frac{f_{j+1} - 2f_j + f_{j-1}}{h^2} = \frac{1}{h^2}\left(e^{ik(x_j+h)} - 2e^{ikx_j} + e^{ik(x_j-h)}\right)$$

$$= \frac{1}{h^2} e^{ikx_j} \cdot \left(e^{ikh} - 2 + e^{-ikh}\right)$$

$$= -\left[\frac{-1}{h^2}\left(e^{ikh} - 2 + e^{-ikh}\right)\right] e^{ikx_j} \Rightarrow \boxed{k'^2 = \frac{-1}{h^2}\left(e^{ikh} - 2 + e^{-ikh}\right)}$$

comment: with the wavenumber increasing, the accuracy decrease.



Probem 2:

(a) $y_{n+1}=y_n+h[\theta f_{n+1}+(1-\theta)f_n]$

$\Rightarrow y_{n+1}=y_n+\lambda h[\theta y_{n+1}+(1-\theta)y_n]$

$\Rightarrow (1-\lambda h\theta)y_{n+1}=[1+(1-\theta)\lambda h]y_n$

$\Rightarrow y_{n+1}=\dfrac{1+(1-\theta)\lambda h}{1-\theta\lambda h}y_n$

$\Rightarrow \sigma=\dfrac{1+(1-\theta)h(\lambda_R+i\lambda_I)}{1-\theta h(\lambda_R+i\lambda_I)}=\dfrac{Ae^{i\alpha}}{Be^{i\beta}}$

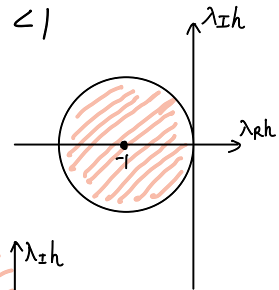
we want $|\sigma|^2=\dfrac{|A|^2}{|B|^2}=\dfrac{[1+(1-\theta)\lambda_R h]^2+[(1-\theta)\lambda_I h]^2}{(1-\theta h\lambda_R)^2+(\lambda_I\theta h)^2}<1$

① $\theta=0$

$|\sigma|^2=(1+\lambda_R h)^2+(\lambda_I h)^2$

$\Rightarrow$ conditionally stable

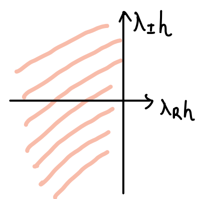
Region of stability



② $\theta=\frac{1}{2}$

$|\sigma|^2=\dfrac{(1+\frac{1}{2}\lambda_R h)^2+(\frac{1}{2}\lambda_I h)^2}{(1-\frac{1}{2}\lambda_R h)^2+(\frac{1}{2}\lambda_I h)^2}<1$ always

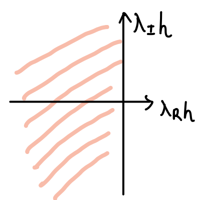
$\Rightarrow$ unconditionally stable



③ $\theta=1$

$|\sigma|^2=\dfrac{1}{(1-\lambda_R h)^2+(\lambda_I h)^2}<1$ for all $\lambda_R<0$

$\Rightarrow$ unconditionally stable



(b) $|\sigma|^2=\dfrac{[1+(1-\theta)\lambda_R h]^2+[(1-\theta)\lambda_I h]^2}{(1-\theta h\lambda_R)^2+(\lambda_I\theta h)^2}$ With $\lambda_R=-\frac{1}{\tau}$, $\lambda_I=\omega$

$\Rightarrow |\sigma|^2=\dfrac{[1-\frac{h}{\tau}(1-\theta)]^2+[(1-\theta)\omega h]^2}{(1+\frac{h}{\tau}\theta)^2+(\omega\theta h)^2}=\dfrac{[1+\frac{h}{\tau}(\theta-1)]^2+[(\theta-1)\omega h]^2}{(1+\frac{h}{\tau}\theta)^2+(\omega\theta h)^2}$

If $\theta>1$, then $(1+\frac{h}{\tau}\theta)^2>[1+\frac{h}{\tau}(\theta-1)]^2$, $(\omega\theta h)^2>[(\theta-1)\omega h]^2$

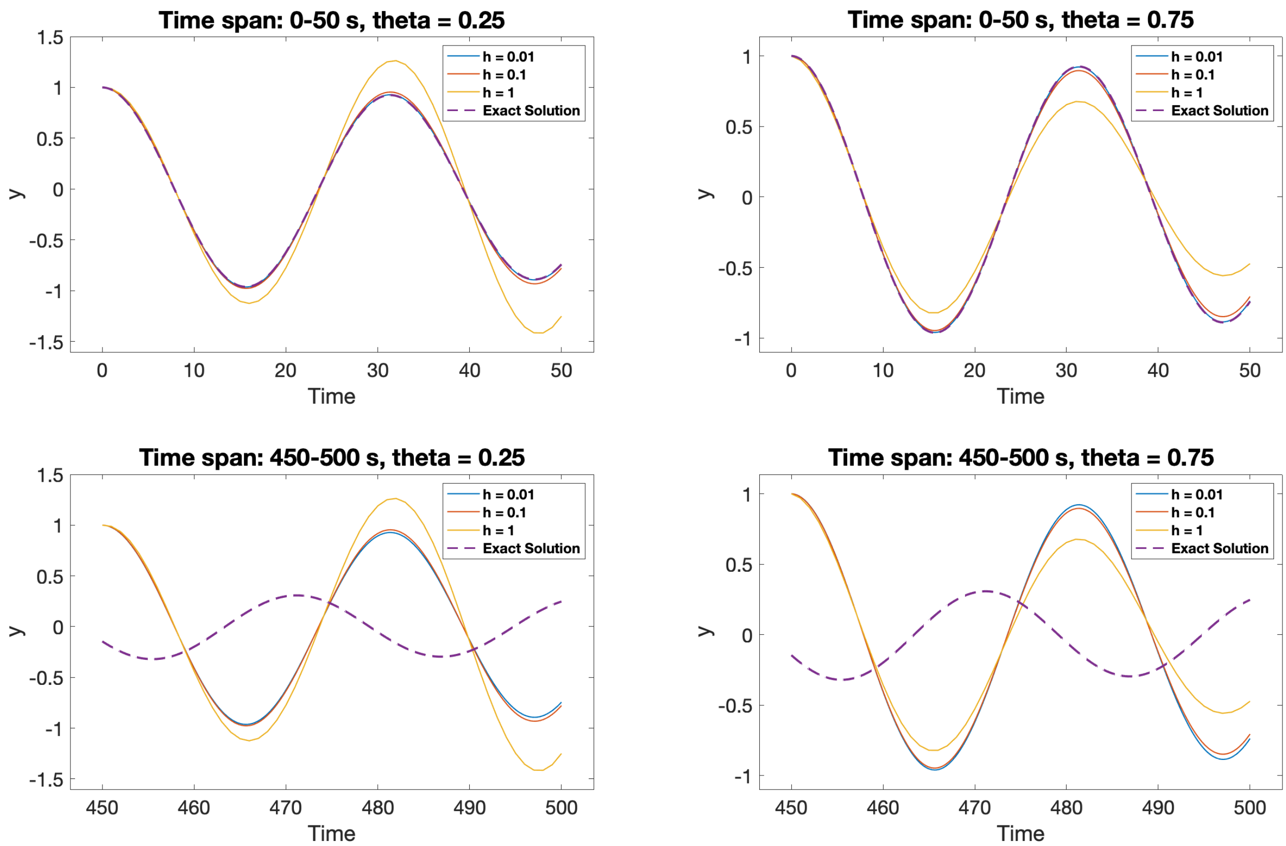
then $|\sigma|^2<1$ always

$\Rightarrow$ $\boxed{\text{Eq(1) is unconditionally stable}}$

(c) exact solution:

$y'=(-\frac{1}{\tau}+i\omega)y$, $y(0)=1$ $\Rightarrow y=e^{(i\omega-\frac{1}{\tau})t}$

$\Rightarrow$ see plots in the next page



(d) $y' = i\omega y \quad , \quad y(0)=1$

$$\sigma = \frac{\left[1-\frac{h}{\tau}(1-\theta)\right] + \left[(1-\theta)\omega h\right] i}{\left(1+\frac{h}{\tau}\theta\right) + (\omega\theta h) i} = \frac{1 + \left(\frac{1}{4}\omega h\right) i}{1 + \left(\frac{3}{4}\omega h\right) i} = A e^{i\alpha}$$

$$|\sigma|^2 = \frac{\left[1-\frac{h}{\tau}(1-\theta)\right]^2 + \left[(1-\theta)\omega h\right]^2}{\left(1+\frac{h}{\tau}\theta\right)^2 + (\omega\theta h)^2} = \frac{1 + \frac{1}{16}\omega^2 h^2}{1 + \frac{9}{16}\omega^2 h^2}$$

Amplitude Error $= |\sigma| - 1$

Phase Error $= \alpha - \omega h$

with $\omega h \ll 1$, we have $\sigma \approx 1$, $\alpha \approx 0$

$$\Rightarrow \begin{cases} \text{Amplitude Error} \approx 0 \\ \text{Phase Error} \approx 0 \end{cases}$$

3. (20 points) *A third-order Runge-Kutta scheme (RK3) is used to integrate the model linear problem:*

$$y' = \lambda y \quad ; \quad y(0) = 1\,. \tag{3}$$

RK3 scheme is given below :

$$\begin{aligned} f_1 &= f(y_n) \\ f_2 &= f(y_n + (8/15)hf_1) \\ f_3 &= f(y_n + (1/4)hf_1 + (5/12)hf_2) \\ y_{n+1} &= y_n + h(f_1/4 + 3f_3/4) \end{aligned}$$

a) *Obtain the stability restriction on the time step h for $\lambda \in \mathbb{C}$. Plot the solution as a stability diagram. What is the restriction on h when $\lambda \in \mathbb{R}$?*

b) *Implement the RK3 scheme to solve Equation (1). Verify the order of truncation error in your RK3 solution by comparing it to the exact solution.*

Problem 3:

(a) $f_1 = f(y_n) = \lambda y_n$

$f_2 = f(y_n + \frac{8}{15} h \cdot \lambda y_n)$

$= \lambda (1 + \frac{8}{15} h\lambda) y_n$

$f_3 = f(y_n + \frac{1}{4} h f_1 + \frac{5}{12} h f_2)$

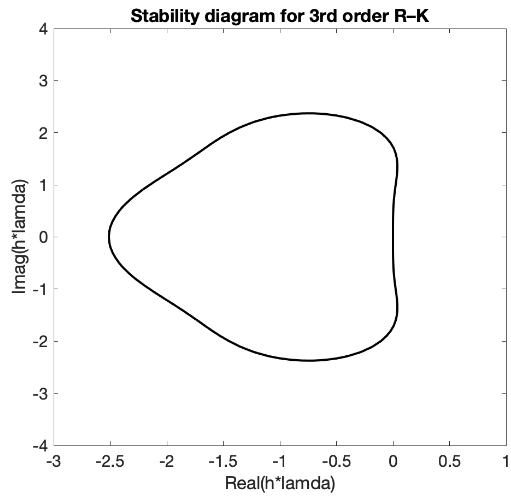
$= \lambda \left(1 + \frac{2}{3} h\lambda + \frac{2}{9} h^2 \lambda^2\right) y_n$

$\Rightarrow y_{n+1} = y_n + h \left(\frac{1}{4} f_1 + \frac{3}{4} f_3\right)$

$= \left[1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3\right] y_n$

$\Rightarrow \sigma = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3$

For h: $\left|1 + h + \frac{1}{2} h^2 + \frac{1}{6} h^3\right| \le 1$



(b) T.E.: $O(h^4)$