Count of Matches for a Highly Ambiguous Regular Expression

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Abstract

We evaluate the number of matches of the regular expression `(a?)^n (a*)^n` for the string `a^n`. We show the total match count is the dot product of two vectors taken from Pascal's Triangle. A formula is given for the total count and values are calculated for *n=1..10*.

Problem Statement

Let the exponential meta-operator '`^`' mean repetition. So '`a^4`' for a string means repeating the '`a`' character 4 times `'aaaa'`, and '`(a?)^4`' for a regex means '`(a?a?a?a?)`'. We will consider a regex of the form '`(a?)^n (a*)^n`' matching a string of '`a^n`', which is a highly ambiguous exaggeration of the example given in [Cox].

Definitions

Consider the match counts for each operator in the regular expression:

- Optional quantifier *zero or one* '`?`' matches 0 or 1 characters.
  The counts for the first half of the expression '`?^n`' are a sequence of *n* binary digits
- Star quantifier *zero or more* '`*`' matches *0..n* characters.
  The counts for the second half '`?^n`' are a sequence of *n* numbers in the range *0..n*.

Count the total number of ways to get a specific partial sum of matches *k=0..n* for each half of the expression. Assemble these counts into two vectors of *n+1* values over index *k=0..n*:

- $S_{?n}[k]$ ways for '`?^n`' to match *k* '`n`' characters.
- $S_{\star n}[k]$ ways for '`*^n`' to match *k* '`n`' characters

For a successful match, the two counts for each half of the expression must add up to *n:* if the second half matches *k*, the first half must have matched *n-k.*
So the total count is the pairwise multiplication of the S vectors:

Total count : $$S_n = \sum_{k=0..n} S_{?n}[n-k] \times S_{\star n}[k]$$

? Quantifiers

The count value $S_{?n}[k]$ is:

- The number of ways to get *n* *zero-or-one* matches accepting a total of *k* characters.
- The count of *n*-digit binary numbers that have *k* bits set (1s).
- The number of ways of choosing *k* from *n*, which is the binomial coefficient nCk:

$$\boldsymbol{S_{?n}[k] = nCk}$$

The table of binomial coefficients is just Pascal's Triangle with the recurrence relation:

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nCk = (n-1)C(k-1) + (n-1)Ck
```

The vector of counts $S_{?n}[k]$ is nCk for $k=0..n$, which is just the n^{th} diagonal in Pascal's Triangle.

The diagonal vector is symmetric because: `nCk = nC(n-k)` and so $S_{?n}[k] = S_{?n}[n-k]$ which means we can invert the $S_{?n}$ vector index and justify the dot product formulation:

Total count: $$\boldsymbol{S_n = \sum_{k=0..n} S_{?n}[k] \times S_{\star n}[k] = S_{?n} \bullet S_{\star n}}$$

* Quantifiers

The count value $S_{*n}[k]$ is:

- The number of ways to get n *zero-or-more* matches accepting a total of k characters.
- *Sum of digits* problem: the ways n numbers in the range *0..n* can have sum of *k.*

Construct a recurrence relation for the sum of digits problem. Each set of *n-1* numbers with a sum in the range *0..k* is uniquely made up to sum *k* by adding the n^{th} number *n-k:*

$$S_{*n}[k] \quad = \quad \sum\nolimits_{k=0..n} \; S_{*n-1}[k] \quad = \quad S_{*n}[k-1] \; + \; S_{*n-1}[k]$$

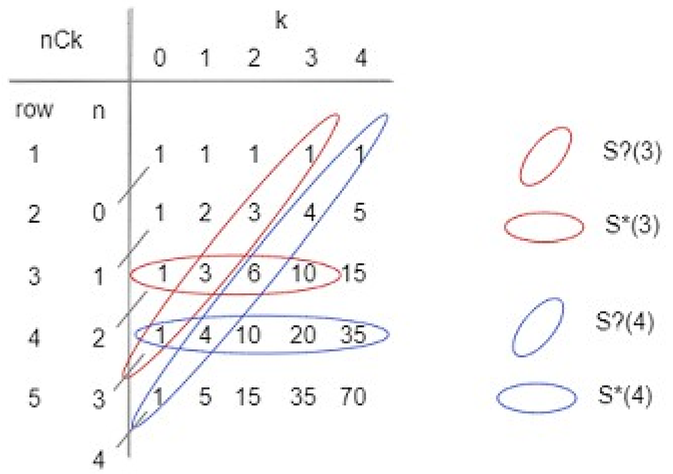
Each entry is the sum of values in the row above, up to and including the same column (*k*), and hence also the sum of the two terms to the left (*k-1*) and above (*k*).
The recurrence is grounded at 1 on the left and at the top:

$$\forall_{k=0..n} \quad S_{*1}[k] \; = \; 1, \qquad \forall_{n>0} \quad S_{*n}[0] \; = \; 1$$

This is just Pascal's Triangle accessed by 1-based row number, rather than the 0-based n^{th} diagonal.
A entry in the table has (*row,column*) coordinates *(n+k-1,k)*, so the count is a binomial coefficient:

$$\boldsymbol{S_{*n}[k] \; = \; (n+k-1)Ck}$$

Vectors in Pascal's Triangle



Final Formula

Total count: $\quad \boldsymbol{S_n \; = \; S_{?n} \bullet S_{*n} \; = \; \sum_{k=0..n} \; nCk \times (n+k-1)Ck}$

Specific Examples

S(1) = [1,1] • [1,1] = 1+1 = 2
S(2) = [1,2,1] • [1,2,3] = 1+4+3 = 8
S(3) = [1,3,3,1] • [1,3,6,10] = 1+9+18+10 = 38
S(4) = [1,4,6,4,1] • [1,4,10,20,35] = 1+16+60+80+35 = 192
S(5) = [1,5,10,10,5,1] • [1,5,15,35,70,126] = 1+25+150+350+350+126 = 1,002
S(6) = [1,6,15,20,15,6,1] • [1,6,21,56,126,252,462] = 1+36+315+1120+1890+1512+462 = 5,336

n	1	2	3	4	5	6	7	8	9	10
Sn	2	8	38	192	1,002	5,336	28,814	157,184	864,146	4,780,008

References

[Cox] "*Regular Expression Matching Can Be Simple And Fast*", Russ Cox, January 2007 [web].