Count of Matches for a
Highly Ambiguous Regular Expression

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Abstract

We evaluate the number of matches of the regular expression `(a?){m}(a*){m}`, meaning *m* repetitions of each part, against the string '`a...`', containing *m* copies of the character `'a'`.

The problem is a highly ambiguous exaggeration of the example given in [Cox].

We show the total count of matches is the dot product of two vectors taken from Pascal's Triangle.

A formula is given for the total count and values are calculated for *m=1..10*.

Definitions

Consider the match counts for each operator in the regular expression:

- Optional quantifier *zero or one* '`?`' matches 0 or 1 characters.
  The counts for the first half of the expression '`?{m}`' form a sequence of *m* binary digits
- Kleene Star quantifier *zero or more* '`*`' matches *0..m* characters.
  The counts for the second half '`*{m}`' form a sequence of *m* numbers in the range *0..m*.

Count the total number of ways to get a specific partial sum of matches *k=0..m* for each half of the expression. Assemble these counts into two vectors of m*+1* values over index *k=0..m*:

- $S_{?m}[k]$ ways for '`?{m}`' to match *k* characters.
- $S_{\star m}[k]$ ways for '`*{m}`' to match *k* characters

For a successful match, the two counts for each half of the expression must add up to *m.*
So if the second half matches *k*, the first half must have matched *m-k.*
Then the total count is the pairwise multiplication of the S vectors:

$$\text{Total count :} \quad S_m = \sum_{k=0..m} S_{?m}[m-k] \times S_{\star m}[k]$$

? Quantifiers

The count value $S_{?m}[k]$ is:

- The number of ways to get *m* zero-or-one matches accepting a total of *k* characters.
- The count of *m*-digit binary numbers that have *k* bits set (1s).
- The number of ways of choosing *k* from *m*, which is the binomial coefficient *nCk,* with *n=m:*

$$S_{?m}[k] = mCk$$

The table of binomial coefficients is just Pascal's Triangle with the recurrence relation:

$$nCk = (n-1)C(k-1) + (n-1)Ck$$

The vector of counts $S_{?m}[k]$ is mCk for $k=0..m$, which is just a diagonal vector in Pascal's Triangle.

The diagonal vector is symmetric because: $nCk = nC(n-k)$ therefore $S_{?m}[k] = S_{?m}[m-k]$
which means we can invert the $S_{?m}$ vector index and justify the dot product formulation:

$$\text{Total count:} \quad \boldsymbol{S_n = \sum_{k=0..m} S_{?m}[k] \times S_{\star m}[k] = S_{?m} \bullet S_{\star m}}$$

*** Quantifiers**

The count value $S_{*m}[k]$ is:

- The number of ways to get m zero-or-more matches accepting a total of *k* characters.
- *Sum of digits* problem: the ways m numbers each in the range *0..m* can have sum *k.*

Construct a recurrence relation for the sum of digits problem. Each set of *m-1* numbers with a sum in the range *0..k* is uniquely made up to sum *k* by adding the m^{th} number *m-k:*

$$S_{*m}[k] \quad = \quad \sum_{k=0..m} \; S_{*m-1}[k] \quad = \quad S_{*m}[k-1] \; + \; S_{*m-1}[k]$$

Each entry is the sum of values in the row above, up to and including the same column *k*, and hence also the sum of the two terms to the left (*m, k-1*) and above (*m-1, k*).
The recurrence is grounded at 1 on the left and at the top:

$$\forall_{k=0..m} \quad S_{*1}[k] = 1, \qquad \forall_{m>0} \quad S_{*m}[0] = 1$$

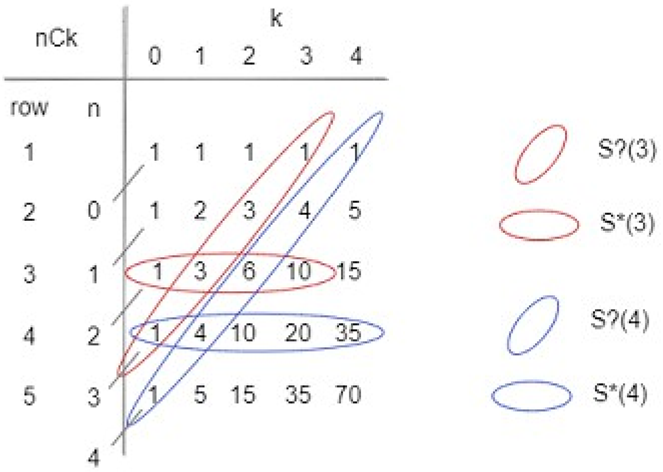
This is just Pascal's Triangle accessed by the 1-based row number *m*, rather than *n*, the 0-based diagonal. The indices are linked by `n=m+k-1`, so the count is the binomial coefficient:

$$\boldsymbol{S_{*m}[k] \; = \; (m+k-1)Ck}$$

Final Formula

Total count: $\quad \boldsymbol{S_m \;=\; S_{?m} \bullet S_{*m} \;=\; \sum_{k=0..m} mCk \times (m+k-1)Ck}$

Vectors in Pascal's Triangle



S(1) = [1,1] • [1,1] = 1+1 = 2
S(2) = [1,2,1] • [1,2,3] = 1+4+3 = 8
S(3) = [1,3,3,1] • [1,3,6,10] = 1+9+18+10 = 38
S(4) = [1,4,6,4,1] • [1,4,10,20,35] = 1+16+60+80+35 = 192
S(5) = [1,5,10,10,5,1] • [1,5,15,35,70,126] = 1+25+150+350+350+126 = 1,002
S(6) = [1,6,15,20,15,6,1] • [1,6,21,56,126,252,462] = 1+36+315+1120+1890+1512+462 = 5,336

n	1	2	3	4	5	6	7	8	9	10
Sn	2	8	38	192	1,002	5,336	28,814	157,184	864,146	4,780,008

References

[Cox] "*Regular Expression Matching Can Be Simple And Fast*", Russ Cox, January 2007 [web].