# Count of Matches for a Highly Ambiguous Regular Expression

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**Abstract**

We evaluate the number of matches of the regular expression `(a?)^n (a*)^n` for the string `a^n`. We show the total match count is the dot product of two vectors taken from Pascal's Triangle. A formula is given for the total count and values are calculated for *n=1..10*.

**Problem Statement**

Let the exponential meta-operator '^' mean repetition. So '`a^4`' for a string means repeating the '`a`' character 4 times `'aaaa'`, and '`(a?)^4`' for a regex means '`(a?a?a?a?)`'. We will consider a regex of the form '`(a?)^n (a*)^n`' matching a string of '`a^n`', which is a highly ambiguous exaggeration of the example given in [Cox].

**Definitions**

Consider the match counts for each operator in the regular expression:

- Optional quantifier *zero or one* '`?`' matches 0 or 1 characters.
  The counts for the first half of the expression '`?^n`' are a sequence of *n* binary digits
- Star quantifier *zero or more* '`*`' matches *0..n* characters.
  The counts for the second half '`?^n`' are a sequence of *n* numbers in the range *0..n*.

Count the total number of ways to get a specific partial sum of matches *k=0..n* for each half of the expression. Assemble these counts into two vectors of *n+1* values over index *k=0..n*:

- $S_{?n}[k]$ ways for '`?^n`' to match *k* characters.
- $S_{*n}[k]$ ways for '`*^n`' to match *k* characters

For a successful match, the two counts for each half of the expression must add up to *n:* if the second half matches *k*, the first half must have matched *n-k.*
*S*o the total count is the pairwise multiplication of the S vectors:

Total count : $S_n = \sum_{k=0..n} S_{?n}[n-k] \times S_{*n}[k]$

**? Quantifiers**

The count value $S_{?n}[k]$ is:

- The number of ways to get *n zero-or-one* matches accepting a total of *k* characters.
- The count of *n*-digit binary numbers that have *k* bits set (1s).
- The number of ways of choosing *k* from *n*, which is the binomial coefficient $nCk$:

$$\boldsymbol{S_{?n}[k] = nCk}$$

The table of binomial coefficients is just Pascal's Triangle with the recurrence relation:

```
nCk = (n-1)C(k-1) + (n-1)Ck
```

The vector of counts $S_{?n}[k]$ is $nCk$ for $k=0..n$, which is just the $n^{th}$ diagonal in Pascal's Triangle.

The diagonal vector is symmetric because: `nCk = nC(n-k)` and so $S_{?n}[k] = S_{?n}[n-k]$ which means we can invert the $S_{?n}$ vector index and justify the dot product formulation:

Total count: $\boldsymbol{S_n = \sum_{k=0..n} S_{?n}[k] \times S_{*n}[k] = S_{?n} \bullet S_{*n}}$

*** Quantifiers**

The count value $S*_n[k]$ is:

- The number of ways to get *n zero-or-more* matches accepting a total of *k* characters.
- *Sum of digits* problem: the ways *n* numbers in the range *0..n* can have sum of *k.*

Construct a recurrence relation for the sum of digits problem. Each set of *n-1* numbers with a sum in the range *0..k* is uniquely made up to sum *k* by adding the $n^{th}$ number *n-k:*

$$S*_n[k] \;=\; \sum_{k=0..n} S*_{n-1}[k] \;=\; S*_n[k-1] + S*_{n-1}[k]$$

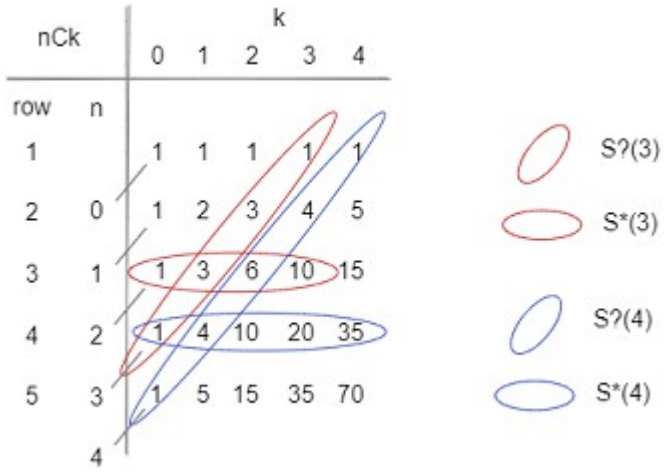
Each entry is the sum of values in the row above, up to and including the same column (*k*), and hence also the sum of the two terms to the left (*k-1*) and above (*k*).
The recurrence is grounded at 1 on the left and at the top:

$$\forall_{k=0..n} \quad S*_1[k] = 1, \qquad \forall_{n>0} \quad S*_n[0] = 1$$

This is just Pascal's Triangle accessed by 1-based row number, rather than the 0-based $n^{th}$ diagonal. A entry in the table has (*row,column*) coordinates *(n+k-1,k)*, so the count is a binomial coefficient:

$$S*_n[k] \;=\; (n+k-1)Ck$$

**Vectors in Pascal's Triangle**




**Final Formula**

Total count: $S_n \;=\; S_{?n} \bullet S*_n \;=\; \sum_{k=0..n} nCk \times (n+k-1)Ck$

**Specific Examples**

| | | | | | |
|---|---|---|---|---|---|
| S(1) = | [1,1] • [1,1] | = | 1+1 | = | 2 |
| S(2) = | [1,2,1] • [1,2,3] | = | 1+4+3 | = | 8 |
| S(3) = | [1,3,3,1] • [1,3,6,10] | = | 1+9+18+10 | = | 38 |
| S(4) = | [1,4,6,4,1] • [1,4,10,20,35] | = | 1+16+60+80+35 | = | 192 |
| S(5) = | [1,5,10,10,5,1] • [1,5,15,35,70,126] | = | 1+25+150+350+350+126 | = | 1,002 |
| S(6) = | [1,6,15,20,15,6,1] • [1,6,21,56,126,252,462] | = | 1+36+315+1120+1890+1512+462 | = | 5,336 |

| *n* | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| *Sn* | 2 | 8 | 38 | 192 | 1,002 | 5,336 | 28,814 | 157,184 | 864,146 | 4,780,008 |

**References**


[Cox] "*Regular Expression Matching Can Be Simple And Fast*", Russ Cox, January 2007 [web].