

Assume two matrices which we plan to multiply A which is $n \times m$ and B which is $m \times p$. Both are block matrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad (1)$$

Where A_{11} is a $k_1 \times k_2$ matrix and B_{11} is a $\ell_1 \times \ell_2$ matrix. Note that $0 \leq k_2, \ell_1 \leq m$ but the ordering of k_2 and ℓ_1 is unknown. Multiplication with $k_2 = \ell_1$, is commonly used:

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix} \quad (2)$$

(“conformable partitioning” is the term used in *Elementary Matrix Theory* - Howard Eves). I haven’t found anything written on non-conformable partitioned multiplication but admittedly haven’t looked very hard.

For the general case ($k_2 \neq \ell_1$), we still get back a 2×2 block matrix, which we will denote C .

$$AB = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \quad (3)$$

C_{11} is a $k_1 \times \ell_2$ sub-matrix; the sizes of the other partitions can be derived from this. The domains of these 12 submatrices can be partitioned using:

$$\begin{aligned} N_1 &= \llbracket 0, k_1 \rrbracket & N_2 &= \llbracket k_1, n \rrbracket \\ P_1 &= \llbracket 0, \ell_2 \rrbracket & P_2 &= \llbracket \ell_2, p \rrbracket \\ M_1 &= \llbracket 0, k_2 \rrbracket & M_2 &= \llbracket k_2, \ell_1 \rrbracket & M_3 &= \llbracket \ell_1, m \rrbracket \end{aligned}$$

So we can rewrite A and B as:

$$A = A_{11}^{N_1 \times M_1} \oplus A_{12}^{N_1 \times (M_2 \oplus M_3)} \oplus A_{21}^{N_2 \times M_1} \oplus A_{22}^{N_2 \times (M_2 \oplus M_3)} \quad (4)$$

$$B = B_{11}^{(M_1 \oplus M_2) \times P_1} \oplus B_{12}^{(M_1 \oplus M_2) \times P_2} \oplus B_{21}^{M_3 \times P_1} \oplus B_{22}^{M_3 \times P_2} \quad (5)$$

Here \oplus is *not* the direct sum of matrices, but hybrid function pointwise sum. \times is the Cartesian product of intervals.

For $i, j \in \{1, 2\}$ we can write blocks in C as:

$$C_{i,j} = A_{i,1}^{N_i \times M_1} B_{1,j}^{M_1 \times P_j} + A_{i,2}^{N_i \times M_2} B_{1,j}^{M_2 \times P_j} + A_{i,2}^{N_i \times M_3} B_{2,j}^{M_3 \times P_j} \quad (6)$$

If $k_2 = \ell_1$ then $M_2 = \emptyset$. We can think of multiplying a $n \times 0$ matrix by a $0 \times p$ matrix as giving a $n \times p$ matrix which is the sum over an empty set, 0 everywhere.

If $k_2 < \ell_1$ then this is like treating A and B instead as 2×3 and 3×2 block matrices. M_1 , M_2 and M_3 are all positive intervals and form conformable blocks.

If $k_2 > \ell_1$ then M_2 is a negative interval which doesn't have a good interpretation. To simplify $C_{i,j}$, we can use the partition $\{M_1 \oplus M_2, \ominus M_2, M_2 \oplus M_2\}$ which contains only positive intervals.

First we rewrite the hybrid functions in the first and third terms:

$$C_{i,j} = \left(A_{i,1}^{N_i \times M_1 \oplus M_2} \oplus A_{i,1}^{N_i \times \ominus M_2} \right) \left(B_{1,j}^{M_1 \oplus M_2 \times P_j} \oplus B_{1,j}^{\ominus M_2 \times P_j} \right) + A_{i,2}^{N_i \times M_2} B_{1,j}^{M_2 \times P_j} \\ + \left(A_{i,2}^{N_i \times \ominus M_2} \oplus A_{i,2}^{N_i \times M_3 \oplus M_2} \right) \left(B_{2,j}^{\ominus M_2 \times P_j} \oplus B_{2,j}^{M_3 \oplus M_2 \times P_j} \right) \quad (7)$$

Which are now the product of conformable 1×2 and 2×1 block matrices. This gives us a 1×1 block matrix :

$$C_{i,j} = A_{i,1}^{N_i \times M_1 \oplus M_2} B_{1,j}^{M_1 \oplus M_2 \times P_j} + A_{i,1}^{N_i \times \ominus M_2} B_{1,j}^{\ominus M_2 \times P_j} + A_{i,2}^{N_i \times M_2} B_{1,j}^{M_2 \times P_j} \\ + A_{i,2}^{N_i \times \ominus M_2} B_{2,j}^{\ominus M_2 \times P_j} + A_{i,2}^{N_i \times M_3 \oplus M_2} B_{2,j}^{M_3 \oplus M_2 \times P_j} \quad (8)$$

Then we merge the middle three terms:

$$C_{i,j} = \left(A_{i,1}^{N_i \times \ominus M_2} \oplus A_{i,2}^{N_i \times M_2} \oplus A_{i,2}^{N_i \times \ominus M_2} \right) \left(B_{1,j}^{\ominus M_2 \times P_j} \oplus B_{1,j}^{M_2 \times P_j} \oplus B_{2,j}^{\ominus M_2 \times P_j} \right) \\ + A_{i,1}^{N_i \times M_1 \oplus M_2} B_{1,j}^{M_1 \oplus M_2 \times P_j} + A_{i,2}^{N_i \times M_3 \oplus M_2} B_{2,j}^{M_3 \oplus M_2 \times P_j} \quad (9)$$

$A_{i,2}$ occurs twice with opposite sign on the same region and so cancels itself. As does $B_{1,j}$:

$$C_{i,j} = \left(A_{i,1}^{N_i \times \ominus M_2} \oplus A_{i,2}^{N_i \times \emptyset} \right) \left(B_{2,j}^{\ominus M_2 \times P_j} \oplus B_{1,j}^{\emptyset \times P_j} \right) \\ + A_{i,1}^{N_i \times M_1 \oplus M_2} B_{1,j}^{M_1 \oplus M_2 \times P_j} + A_{i,2}^{N_i \times M_3 \oplus M_2} B_{2,j}^{M_3 \oplus M_2 \times P_j} \quad (10)$$

And functions over empty domains can be removed:

$$= A_{i,1}^{N_i \times M_1 \oplus M_2} B_{1,j}^{M_1 \oplus M_2 \times P_j} + A_{i,1}^{N_i \times \ominus M_2} B_{2,j}^{\ominus M_2 \times P_j} + A_{i,2}^{N_i \times M_3 \oplus M_2} B_{2,j}^{M_3 \oplus M_2 \times P_j} \quad (11)$$

Formally this relies on the following identities:

$$M^B \oplus M^C = M^{B \oplus C} \quad (12)$$

(we can split up a matrix into blocks)

$$M_1^A \oplus M_2^\emptyset = M_1^A = M_2^\emptyset \oplus M_1^A \quad (13)$$

(empty blocks can be ignored)

$$M_1^{A \times B} M_2^{B \times D} + M_3^{A \times C} M_4^{C \times D} = (M_1^{A \times B} \oplus M_3^{A \times C}) (M_2^{B \times D} \oplus M_4^{C \times D}) \quad (14)$$