DOMAIN DECOMPOSITION, INTEGRATION, AND INCLUSION-EXCLUSION
(Thesis format: Monograph)

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Graduate Program in Computer Science

A thesis submitted in partial fulfillment
of the requirements for the degree of
Masters of Science

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Domain Decomposition, Integration, and Inclusion-Exclusion

is accepted in partial fulfillment of the
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Masters of Science

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Abstract

Mathematic notation has been dominated by sets and, when repeated elements are required, sequences generally make an appearance. Historical inertia has caused these structures to be used in many situations where they are ill-suited often evidenced by phrases like ”without loss of generality”, ”up to ordering of terms”, ”up to a sign”. However, by tackling these problems instead with more apt data structures, we can eliminate some of these stipulations and more formally reduce symmetric cases in reasoning. In particular, this thesis will deal with *hybrid sets* (that is, signed multisets), as well *hybrid functions* (that is, functions with hybrid sets for their domain) with applications in piecewise functions, integration on manifolds, (...). More than just an aesthetic change, by allowing negative multiplicity (even if it would not make physical sense), we may symblically manipulate structures in ways that might otherwise be cumbersome or inefficient.

Keywords: Hybrid set, Signed multiset, Integration on chains, Inclusion-Exclusion, (...)

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Chapter 1

Introduction

1.1 Motivation

Some data structures in mathematics are more well-loved than others and none more than Cantor's set. The very foundations of mathematics lie in set theory: numbers are defined in terms of sets as are ordered tuples which in turn lead to relations, functions, sequences and from there branches into countless other structures. But this trunk typically omits a satisfactory treatment of several *generalized sets*. For example, it is difficult to begin speaking about *hybrid sets* without immediately punctuating, "that is, multisets with negative multiplicity". It is a statement of progress that multisets have even entered into (relatively) common mathematical parlance.

Still, sequences need no introduction and are generally relied on when a structure allowing repeated elements is needed. The addition of an ordering is often not even needed but "surely it can't hurt?" Consider the *Fundamental Theorem of Arithmetic*:

> "Every positive integer, except 1, is a product of primes." ... "The standard form of *n* is unique; apart from rearrangement of factors, *n* can be expressed as a product of primes in one way only." (Hardy and Wright 1979, p.2-3)

By recognizing the possibility of rearranging factors, the authors implicitly define type the "product of primes" as a sequence. But for iterated commutative operators (e.g. $\sum, \prod \bigcap, \bigcup$), the order of terms is irrelevant. So then, why order terms to begin with? Reisig [14] uses multisets to define relation nets where "... several individuals of some sort do not have to be distinguished" and furthermore "One should not be forced to distinguish individuals if one doesn't wish to. This would lead to overspecification". The same applies here. Secondly, iterated operators over an empty set is simply the respective identity ($\prod_{x \in \emptyset} x = 1$) , and so 1 *is* a product of primes. Despite the empty sequence being just as well-defined as the empty set; it tends to be treated as an aberrant case. Some definitions even disregard the singleton sequence to say, "is prime or the product of primes".

Stripped of these qualifications we are left with simply:

> "Every positive integer is the product of a unique multiset of primes."

Although the definition is equivalent, by using appropriate data structures, the result is more elegant and less case-based reasoning for later uses of the definition. In this spirit that we will graft hybrid sets into areas of mathematics where conventional structures don't fit as tightly as we'd like.

1.2 Objectives

This thesis will include and extend the work of [8] on hybrid sets and their applications. In particular, integration is a natural application of signed domains that had not been explored from this perspective. Take the identity:

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x \tag{1.1}$$

The domain of integration on the left-side is considered to be the interval $[a, b]$ or in set builder notation, $\{x \in \mathbb{R} \mid a \leq x \leq b\}$. It would follow then that the right hand should have domain $[b, a]$. Under traditional set definitions, this is not be well defined. Using hybrid sets allows us to give meaning to an inverted interval. When generalized to higher dimensions, this goal becomes an attempt to unify the *Lebesgue integral* and *integration of forms*. Such a model would allow for integation of differential forms over subsets of manifolds.

I don't currently have a set of objectives for the Petri net chapter. Another couple lines will go here once I have a better idea what the chapter will look like.

1.3 Related Work

It is difficult to date the origin of multisets. The term was coined by N.G. de Bruijn in corresponces with Donald Knuth, [12] but thought of as a "collection of objects that may or may not be distinguished" is as old as tally marks. In regards to the generalization to *signed* multisets, Hailperin [11] suggests that Boole's 1854 *Laws of Thought* [7] is a treatise of signed multisets. Whether this was Boole's intent is debated [cite]. Sets with negative membership explicitly began to appear in [18] and were formalized under the name Hybrid sets in Blizard's extensive work with generalized sets [5, 6] Although hybrid set and signed multiset are the most common nomenclature, other names appearing in literature include multiset (specifying positive when for unsigned multisets) [14] and integral multiset [19].

Existing explicit applications of hybrid sets are currently limited. Loeb *et al.* [10, 13] use hybrid sets to generalize several combinatoric identities to negative values. Bailey *et al.* [1] and Banâtre *et al.* [2] have also had success with hybrid sets in chemical programming. Representing a solution is represented as a collection of atoms and molecules, negative multiplicities are treated as "antimatter" . For a deeper overview and systemization of generalized sets, see [16, 17]. Finally, Bartoletti [3, 4] and Schmidt [15] also investigate petri and relation nets (to be examined in chapter 5), albeit not from the perspective of hybrid sets.

1.4 Thesis Outline

In chapter 2, the foundations for hybrid sets and functions with hybrid set domains will be laid. Some immediate applications to piecewise functions will be presented. In chapter 3, the algebra of domains will be more deeply explored with generalized partitions and inclusion-exclusion and present an method for computing the support of a hybrid set. In chapter 4, hybrid functions will be applied towards integration. Starting from foundations we will use hybrid functions to unify integration of forms with integration on subsets of manifolds and prove Stokes' theorem on the new model. Finally in chapter 5, hybrid sets will be applied to Petri and relation nets with the scope of ...

Chapter 2

Generalized Partitions

2.1 Piecewise Functions

To Do: Blending "piecewise are everywhere"

The perennial example of a piecewise function is abs : $\mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ given in the form:

$$\text{abs}(x) = \begin{cases} -x & : x < 0 \\ x & : x \geq 0 \end{cases} \tag{2.1}$$

To evaluate abs for an argument x, one must first determine which sub-function to use. If $x < 0$ then the first case is evaluated and abs will return the result of $x \mapsto -x$. Otherwise, if $x \geq 0$ the second case is evaluated and the result of $x \mapsto x$ is returned. Rather than as a condition, we could just as easily think of "$x < 0$" and "$x \geq 0$" as partitions of the real line. Evaluation then occurs by checking whether $x \in \mathbb{R}_+ \cup \{0\}$ or $x \in \mathbb{R}_-$. In general, a piecewise function f will take the form:

$$f(x) = \begin{cases} f_1(x) & : x \in P_1 \\ f_2(x) & : x \in P_2 \\ \vdots & \qquad \vdots \\ f_n(x) & : x \in P_n \end{cases} \tag{2.2}$$

where the set $\{P_i\}$ forms a partition of the domain of f and for each f_i is defined over all of the corresponding P_i. To formalize this we require the ability to restrict a function's domain and join disjoint pieces together.

Definition Given a function $f : X \to Y$ for any subset of the domain, $Z \subset X$, the *restriction of f to Z* is the function $f|_Z : Z \to Y$, such that $f|_Z(x) = f(x)$ for all $x \in Z$.

Definition Define $\bar{\Phi}$, the *join* of two functions, f and g by:

$$f \,\bar{\Phi}\, g = \begin{cases} f(x) & \text{if } g(x) = \perp \\ g(x) & \text{if } f(x) = \perp \\ \perp & otherwise \end{cases} \tag{2.3}$$

Is there a way to define without using piecewise functions?

which would allow us to re-write our previous definition of (2.2) as:

$$f = f|_{P_1} \,\bar{\Phi}\, f|_{P_2} \,\bar{\Phi}\, \ldots \,\bar{\Phi}\, f|_{P_n} \tag{2.4}$$

To Do: problem with this approach + join

But we must be careful as this definition is not associative.

Let $x \in A \cap B \cap C$, then $((f|_A \,\bar{\Phi}\, g|_B) \,\bar{\Phi}\, h|_C)(x) = h(x)$ but $(f|_A \,\bar{\Phi}\, (g|_B \,\bar{\Phi}\, h|_C))(x) = f(x)$

Other conventions exist, for example *Maple*'s `piecewise(cond_1, f_1, cond_2, f_2, ..., cond_n, f_n, f_otherwise)` effectively uses a short-circuted $\bar{\Phi}$; it takes the first sub-function, `f_i`, such that the corresponding condition, `cond_i`, evaluates to *true*.

This approach simply trades associativity for commutivity.

In this section we will construct a formal system to manipulate partial functions more elegantly.

2.2 Hybrid Sets

To Do: join

We shall consider *hybrid sets*: an extension of multisets which has multiplicities ranging over $\mathbb{N}_0$, a hybrid set has multiplicities over all of $\mathbb{Z}$.

However first we must establish *partial sets* (unrelated to a *poset* or "partially ordered set").

Similar to a partial function being only partially defined over its domain, for a partial set, there may exist some items for which membership is undefined.

For an underlying set U we will consider a hybrid set as a function $U \to \mathbb{Z}$ as a way to track the multiplicities of any particular element.

Definition Let U be a universe, then any function $U \to \mathbb{Z}$ is called a *hybrid set*.

On its own, this definition does us very little good; much of the usefulness of sets is derived from their rich notation.

Definition Let H be a hybrid set. Then we say that $H(x)$ is the *multiplicity* of the element x. We write, $x \in^n H$ if $H(x) = n$. Furthermore we will use $x \in H$ to denote $H(x) \neq 0$ (or equivalently, $x \in^n H$ for $n \neq 0$). Conversely, $x \notin H$ denotes $x \in^0 H$ or $H(x) = 0$. The symbol $\emptyset$ will be used to denote the empty hybrid set for which all elements have multiplicity 0. Finally the support of a hybrid set, is the (non-hybrid) set $\operatorname{supp} H$ where $x \in \operatorname{supp} H$ if and only if $x \in H$

We will use the notation:

$$H = \{\!\!\{\, x_1^{m_1}, x_2^{m_2}, \ldots \,\}\!\!\}$$

to describe the hybrid set H where the element x_i has multiplicity m_i. We allow for repetitions in $\{x_i\}$ but interpret the overall multiplicity of an element x_i by the sum of multiplicities among copies. Using Iverson brackets:

$$H(x) = \sum_{x_i \in^{m_i} H} [x = x_i]\, m_i \tag{2.5}$$

For example, $H = \{\!\!\{ a^1, a^1, b^{-2}, a^3, b^1 \}\!\!\} = \{\!\!\{ a^5, b^{-1} \}\!\!\}$. A writing in which $x_i \neq x_j$ for all $i \neq j$ is refered to as a *normalized form* of a hybrid set. For normalized hybrid sets it follows that $H(x_i) = m_i$.

Traditional sets use the operations $\cup$ union, $\cap$ intersection, and $\setminus$ complementation. In the same way a hybrid set is a function $H : U \to \mathbb{Z}$, a set could be considered as function $S : U \to \{0, 1\}$. Then set operations correspond to pointwise `OR`, `AND`, and `NOT`. That is, for two sets A and B, then $(A \cup B)(x) = A(x)$ `OR` $B(x)$. One could easily extend union and intersection to hybrid sets using pointwise min and max [cite], but it would make more sense to have operations corresponding to primitive operations in $\mathbb{Z}$ instead. Thus we will define $\oplus$, $\ominus$, and $\otimes$ by pointwise $+$, $-$, and $\cdot$.

Definition For any two hybrid sets A and B over a common universe U, we define the operations $\oplus, \ominus, \otimes : \mathbb{Z}^U \times \mathbb{Z}^U \to \mathbb{Z}^U$ such that for all $x \in U$:

$$(A \oplus B)(x) = A(x) + B(x) \tag{2.6}$$

$$(A \ominus B)(x) = A(x) - B(x) \tag{2.7}$$

$$(A \oplus B)(x) = A(x) \cdot B(x) \tag{2.8}$$

We also define, $\ominus A$ as $\emptyset \ominus A$ and for $c \in \mathbb{Z}$:

$$(cA)(x) = c \cdot A(x) \tag{2.9}$$

Definition We say A and B are *disjoint* if and only if $A \otimes B = \emptyset$

Taken alone, hybrid sets can be used to model various objects.

2.2.1 Example: *Rational Arithmetic*

Any positive rational number can be represented as a hybrid set over the set of primes and vice versa (i.e. $(\mathbb{Z}^{\mathbb{P}}, \oplus) \simeq (\mathbb{Q}_{+}, \cdot)$). For any rational number a/b, both a and b being integers will have a prime decomposition: $a = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots$ and $b = q_1^{n_1} \cdot q_2^{n_2} \cdot \ldots$. Then there is an isomorphism:

$$f(a/b) = \{\!\!\{ p_1^{m_1}, p_2^{m_2}, \ldots \}\!\!\} \ominus \{\!\!\{ q_1^{n_1}, q_2^{n_2}, \ldots \}\!\!\} \tag{2.10}$$

Example Concretely, we have:

$$20/9 \cdot 15/8 = \{\!\!\{ 5^1, 2^2, 3^{-2} \}\!\!\} \oplus \{\!\!\{ 5^1, 3^1, 2^{-3} \}\!\!\} = \{\!\!\{ 5^2, 2^{-1}, 3^{-1} \}\!\!\} = 25/6$$

Typically one would need to specify equivalence classes on $\mathbb{Q}$ to consolidate the identity $ca/cb = a/b$. With hybrid set representation, this identity comes for free: any common factor between numerator and denominator will cancel result in cancelling multiplicities. For example, $2/4 = \{\!\!\{ 2^1, 2^{-2} \}\!\!\}$ which is the un-normalized form of $\{\!\!\{ 2^{-1} \}\!\!\} = 1/2$.

Is there a (nice!) way to extend this for 0 and negative $\mathbb{Q}$ that preserves uniqueness up to normalization?

2.2.2 Example: *Rational Polynomials*

Hybrid sets can also be used to represent the roots and asymptotes of a rational polynomial.

Blending

Example Concretely:

$$\frac{(x-2)}{(x-1)^2(x+1)} = \{\!\!\{ 2^1, 1^{-2}, -1^{-1} \}\!\!\} \tag{2.11}$$

Definition Generalized partition

To Do

Definition Reducibility, $\mathcal{R}(H) = \mathrm{supp}(H)$

any set partition is a generalized set partition

a generalized set partition of a reducible partition is a set position iff each generalized partition is reducible

2.3 Hybrid Functions

Next we consider functions which have hybrid sets as their domain which we will call hybrid functions

Definition A hybrid set over $S \times T$ is called a *hybrid (binary) relation*.

Example Algebra of orderings:

For some ordered set S if we define the hyrbid relation $[>] = \left\{\!\left| \, (x,y)^1, (y,x)^{-1} : x > y \, \right|\!\right\}$. Immediately we have:

$$[<] = \ominus[>] \tag{2.12}$$

If $\mathrm{supp}[>] = S \times S$ then $[>]$ is a *total ordering*

$[\leq] = [<] \oplus [=]$

$[\leq] = [=] \ominus [>] = [=] \ominus ([\geq] \ominus [=]) = 2[=] \ominus [\geq]$

Definition Let H be a hybrid relation. For all x, y, z if $(x, y) \in H$ and $(x, z) \in H$ implies $y = z$

reword then H is said to be a *hybrid function.*

Although this tells us what *is* and *is not* a hybrid function, it is not the most useful definition to work with. Generally, we already have a function in mind which we would like to use over a hybrid domain.

Theorem 2.3.1 *Let H be a hybrid set over U, $f : B \to S$ be a function where $B \subseteq U$ and S a set. Then*

$$f^H := \bigoplus_{x \in B} H(x) \left\{\!\left| \, (x, f(x))^1 \, \right|\!\right\} \tag{2.13}$$

theorem? reword *is a hybrid function.*

This definition of a hybrid function should be less thought of as a true function and more as the *graph of a* function. To return the functional behavior we extend the definition of $\mathcal{R}$ from the previous section.

Definition If H is a reducible hybrid set, then f^H is a reducible. Additionally, if f^H is reducible, we override $\mathcal{R}$ by:

$$\mathcal{R}(f^H)(x) = f|_{\mathrm{supp}(H)}(x) \tag{2.14}$$

Note that $\mathcal{R}$ can only be applied if at all points $H(x)$ is 0 or 1; an irreducible hybrid function cannot be reduced! Unlike the join for regular functions, $\bar{\obar}$ (2.3), the join of hybrid function is simply defined and does not rely on piecewise functions buried in definitions.

Definition The *join*, $f^F \obar g^G$ of two hybrid functions f^F and g^G is the hybrid relation given by:

$$f^F \obar g^G := f^F \oplus g^G \tag{2.15}$$

The join of two hybrid relations is identically defined.

It is important to note that the join operator is closed under hybrid relations but not under hybrid functions. For any two hybrid functions the result will be a hybrid relation but not necessarily another hybrid function. So this definition is still nearly as "dangerous" as $\bar{\obar}$, non-hybrid function join. We must still be wary of overlapping regions but there are some cases where we can be guaranteed to get a hybrid function.

All of the everything

Let A and B be hybrid sets over U and let $f : U \to S$ a function.

$f^A \obar f^B = f^{A \oplus B}$ is always a hybrid function. Every element in $\mathrm{supp}(f^A \obar f^B)$ is using the same map f, so there cannot be disagreement among points.

Inductively, this holds for any number of pieces.

For any generalized partition P, given by $P = P_1 \oplus P_2 \oplus ... \oplus P_n$, we have

$$f^P = f^{P_1} \obar f^{P_2} \obar ... \obar f^{P_n} \tag{2.16}$$

For $g : U \to S$ another function then $f^A \oplus g^B = (f \bar{\oplus} g)^{A \oplus B}$ if and only if A and B are disjoint (that is, $A \otimes B = \emptyset$).

But the join of two non-disjoint functions may still be a hybrid function even if their respective functions do not agree at all points; as long as they agree on all points in the "intersection" the functions can be safely joined.

Definition We say that two hybrid functions f^A *and* g^B *are compatible* if and only if $f(x) = g(x)$ for all $x \in \operatorname{supp}(A \otimes B)$.

As with our definition of disjointness, the pointwise product $\otimes$, of hybrid sets acts as an analog for intersection $\cap$, of sets.

Note that any two hybrid functions with disjoint domains will be compatible.

Theorem 2.3.2 *Let f^A and g^B be two hybrid functions. Then $f^A \oplus g^B$ is a hybrid function if and only if f^A and g^B are compatible.*

Notion of compatibility is not associative.

Consider

$$(f^H \oplus g^H) \oplus g^{\ominus H} = f^H \oplus (g^H \oplus g^{\ominus H}) = f^H \oplus g^{\emptyset} = f^H \tag{2.17}$$

Although f^H and g^H may be mutually incompatible, their join is compatible with $g^{\ominus H}$.

Similarly, we can also see above that *reducibility* does not lift through $\oplus$.

Is *lift* the right word? - I don't know enough category theory

2.3.1 Example: *Piecewise functions on generalized partitions*

(2 pages)

$$
\begin{aligned}
(f * g)(x) &= \lBrace\, f_1(x)^{A_1}, f_2(x)^{A_2} \,\rBrace * \lBrace\, g_1(x)^{B_1}, g_2(x)^{B_2} \,\rBrace \\
&= \lBrace\, (f_1(x) * g_1(x))^{A_1} \,\rBrace \obar^* \lBrace\, (f_2(x) * g_1(x))^{B_1 \ominus A_2} \,\rBrace \obar^* \lBrace\, (f_2(x) * g_2(x))^{B_2} \,\rBrace
\end{aligned}
$$

Formula for $f * g$ where $f = f_1^{P_1} \obar f_2^{P_2} \obar f_n^{P_n}$ and $g = g_1^{Q_1} \obar \ldots \obar g_m^{Q_m}$

$$
\begin{aligned}
f * g = &\lBrace\, (f_1 * g_m)^{P_1} \,\rBrace \obar^* \ldots \obar^* \lBrace\, (f_{n-1} * g_m)^{P_{n-1}} \,\rBrace \\
&\obar^* \lBrace\, (f_n * g_1)^{Q_1} \,\rBrace \obar^* \ldots \obar^* \lBrace\, (f_n * g_{m-1})^{Q_{m-1}} \,\rBrace \\
&\obar^* \lBrace\, (f_n * g_n)^{U \ominus (P_1 \oplus \ldots \oplus Q_{n-1} \oplus Q_1 \oplus \ldots \oplus Q_{m-1})} \,\rBrace
\end{aligned}
$$

Can't actually do this example until $\obar^*$ is introduced, which doesn't work with basic hybrid functions

2.4 Pseudo-functions and Hybrid Forms

Definition Refinement

Example Refinement of intervals

Definition Pseudo-function

Properties of pseudo-functions

Definition Hybrid Form

Definition $(f^F \oplus^* g^G)(x) = (F(x) + G(x)) \left\{\!\!\left| \, (x, (f * g)(x))^1 \, \right|\!\!\right\}$

Returning to the example of the sign function from section 2.1, we could think of the function as the join of 3 different hybrid functions:

$$\mathrm{sign} = -1^{(-\infty,0)} \oplus 0^{\{0\}} \oplus 1^{(0,\infty)} \tag{2.18}$$

but we could also consider it as the "joined sum" of $-1^{(-\infty,0]}$ and $1^{[0,\infty)}$.

(3 pages)

2.4.1 Example: *Piecewise functions revisited*

Repeat piece-wise function example with unsafe points (1 page)

$$(-x^2 + 2)^{[-1,1]} \oplus \left(\frac{1}{x^2}\right)^{\mathbb{R}\ominus[1,1]} \tag{2.19}$$

Chapter 3

Symbolic Linear Algebra

Common practice to use "…" in vectors and matrices.

3.1 Oriented Intervals

Definition Given a totally ordered set $(X, \leq)$ *(and with an implied strict ordering $<$)*, for any $a, b \in X$, an **interval between a and b** is the set of elements in X between a and b, up to inclusion of a and b themselves. Formally:

$$[a, b] = \{x \in X \mid a \leq x \leq b\} \tag{3.1}$$

$$[a, b) = \{x \in X \mid a \leq x < b\} \tag{3.2}$$

$$(a, b] = \{x \in X \mid a < x \leq b\} \tag{3.3}$$

$$(a, b) = \{x \in X \mid a < x < b\} \tag{3.4}$$

It should be noted that for $b < a$, $[b, a]$ is the empty set. Also, the interval $[a, a]$ contains a single point while (a, a), $(a, a]$, and $[a, a)$ are all empty. As intervals are simply sets, they can naturally be interpreted as hybrid sets. If $a \leq b \leq c$, for intervals $[a, b)$ and $[b, c)$ using the hybrid set operator $\oplus$, one has $[a, b) \oplus [b, c) = [a, c)$ In this case, $\oplus$ behaves like concatenation but this is not always true. When $a \leq c \leq b$ then $[a, b) \oplus [b, c) = [a, b)$. When working with intervals, a case-based approach to consider relative ordering of endpoints easily becomes quite cumbersome. Thus we turn to oriented intervals.

Definition We define **oriented intervals** with $a, b \in X$, a totally ordered set, using hybrid set point-wise subtraction as follows:

$$[\![a, b)\!) = [a, b) \ominus [b, a) \tag{3.5}$$

$$(\!(a, b]\!] = (a, b] \ominus (b, a] \tag{3.6}$$

$$[\![a, b]\!] = [a, b] \ominus (b, a) \tag{3.7}$$

$$(\!(a, b)\!) = (a, b) \ominus [b, a] \tag{3.8}$$

For any oriented interval, at most one interval term will be non-empty. However when using symbolic terms,

Several results follow immediately from this definition.

Theorem 3.1.1 *For all $a, b, c \in \mathbb{R}$,*

$$[\![a, b)\!) = \ominus[\![b, a)\!) \tag{3.9}$$

$$(\!(a, b]\!] = \ominus(\!(b, a]\!] \tag{3.10}$$

$$[\![a, b]\!] = \ominus(\!(a, b)\!) \tag{3.11}$$

Like their unoriented analogues, the oriented intervals $[\![a, a)\!)$ and $(\!(a, a]\!]$ are both empty and $[\![a, a]\!]$ contains a single point (with multiplicity 1). However, unlike traditional intervals $(\!(a, a)\!)$ is *not* empty but is equivalent to but rather, $(\!(a, a)\!) = \ominus[\![a, a]\!] = \{\!\!\{ a^{-1} \}\!\!\}$.

more

Theorem 3.1.2 *For all $a, b, c \in \mathbb{R}$ (regardless of relative ordering),*

$$[\![a, b)\!) \oplus [\![b, c)\!) = [\![a, c)\!) \tag{3.12}$$

Proof $[\![a,b)\!) \oplus [\![b,c)\!)$

$= ([a,b) \ominus [b,a)) \oplus ([b,c) \ominus [c,b))$

$= ([a,b) \oplus [b,c)) \ominus ([c,b) \oplus [b,a))$

If $a \geq c$ then $[c,a) = \emptyset$ and so $[\![a,c)\!) = [a,c)$.

Case 1: $a \leq b \leq c$ then $[c,b) = [b,a) = \emptyset$ and $[a,b) \oplus [b,c) = [a,c)$

Case 2: $b \leq a \leq c$ then $[b,c) \ominus [b,a) = [b,a) \oplus [a,c) \ominus [b,a) = [a,c)$

Case 3: $a \leq c \leq b$ then $[a,b) \ominus [c,b) = ([a,c) \oplus [c,b)) \ominus [c,b) = [a,c)$

Similar arguments will show that when $c \geq a$, that $[\![a,b)\!) \oplus [\![b,c)\!) = \ominus[a,c)$.

This sort of reasoning is routine but a constant annoyance when dealing with intervals and is exactly the reason we want to be working with oriented intervals. Many similar formulations such as $[\![a,b]\!] \oplus (\!(b,c)\!) = [\![a,c)\!)$ are also valid for any ordering of a, b, c. We will not enumerate all possible cases here.

Note about partitions

3.2 Symbolic Vectors

Vectors as hybrid functions

We will use the following n-dimensional vectors as a running example in this section:

$$U^T = [u_1, u_2, \ldots, u_k, u'_1, u'_2, \ldots, u_{n-k}] \tag{3.13}$$

$$V^T = [v_1, v_2, \ldots, v_\ell, v'_1, v'_2, \ldots, v_{n-\ell}] \tag{3.14}$$

Using intervals, these vectors can be represented by hybrid functions over their indices.

For example

$$U^T = (i \mapsto u_i)^{[\![1,k]\!]} \oplus (i \mapsto u_{i-k})^{(\!(k,n]\!]} \tag{3.15}$$

$$V^T = (i \mapsto v_i)^{[\![1,\ell]\!]} \oplus (i \mapsto v_{i-\ell})^{(\!(\ell,n]\!]} \tag{3.16}$$

Although for clarity and succinctness we will use (u_i) instead of $(i \mapsto u_i)$.

However we are more interested in performing arithmetic with these vectors.

3.2.1 Vector Addition

Consider pointwise vector addition $U^T + V^T$:

$$U^T + V^T = \left((u_i)^{[\![1,k]\!]} \oplus (u'_{i-k})^{(\!(k,n]\!]}\right) \oplus^{+} \left((v_i)^{[\![1,\ell]\!]} \oplus (v'_{i-\ell})^{(\!(\ell,n]\!]}\right) \tag{3.17}$$

$$= \left((u_i)^{[\![1,k]\!]} \oplus (u'_{i-k})^{(\!(k,\ell]\!]} \oplus (u'_{i-k})^{(\!(\ell,n]\!]}\right) \oplus^{+} \left((v_i)^{[\![1,k]\!]} \oplus (v_i)^{(\!(k,\ell]\!]} \oplus (v'_{i-\ell})^{(\!(\ell,n]\!]}\right) \tag{3.18}$$

$$= \left((u_i + v_i)^{[\![1,k]\!]} \oplus^{+} (u'_{i-k} + v_i)^{(\!(k,\ell]\!]} \oplus^{+} (u'_{i-k} + v'_{i-\ell})^{(\!(\ell,n]\!]}\right) \tag{3.19}$$

This formulation is not unique.

The choice to partition $[\![1,n]\!]$ into $[\![1,k]\!] \oplus (\!(k,\ell]\!] \oplus (\!(\ell,n]\!]$ was arbitrary.

We can just as easily partition $[\![1,n]\!]$ into $[\![1,\ell]\!] \oplus (\!(\ell,k]\!] \oplus (\!(k,n]\!]$ to get the equivalent expression:

$$U^T + V^T = \Big((u_i + v_i)^{[\![1,\ell]\!]} \oplus^+ (u_i + v'_{i-\ell})^{(\!(\ell,k]\!]} \oplus^+ (u'_{i-k} + v'_{i-\ell})^{(\!(k,n]\!]} \Big) \tag{3.20}$$

We must be careful while evaluating these expressions to not forget that $(u'_{i-k} + v_i)$ is actually shorthand for the function:

$$i \mapsto u'_{i-k} + v_i$$

For example, consider the concrete example where $n = 5$, $k = 4$ and $\ell = 1$ so that $U^T = [u_1, u_2, u_3, u_4, u'_1]$ and $V^T = [v_1, v'_1, v'_2, v'_3, v'_4]$.

We will also only assume that the functions u_i, u'_i, v_i and v'_i are defined only on the intervals in which they appear (e.g. u_5 is undefined, as is v'_1).

Then the expression in (3.19) becomes:

$$(u_i + v_i)^{[\![1,4]\!]} \oplus^+ (u'_{i-4} + v_i)^{(\!(4,1]\!]} \oplus^+ (u'_{i-4} + v'_{i-1})^{(\!(1,5]\!]} \tag{3.21}$$

None of the individual subterms cannot be evaluated directly.

In the first term, v_i is not totally defined over the interval $[\![1, 4]\!]$.

In the third term, on the interval $(\!(1, 5]\!]$, u'_{i-4} would even evaluated on negative indices.

However, these unevaluable terms also appear in the middle term however the interval $(\!(4, 1]\!]$ is a negatively oriented interval and the offending points cancel!

3.2.2 Inner Product

The inner product or dot product of two vectors is given by:

$$A \cdot B = \sum_i A_i B_i \tag{3.22}$$

Returning to the running example of U^T and V^T, as defined in (3.13) and (3.14) respectively, we will consider $U^T \cdot V^T$.

Definition Let $X = \{\!\{ x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n} \}\!\}$ be a hybrid set with elements x_i in a $\mathbb{Z}$-module. Given a hybrid function over X, f^X, we define the **sum over f^X**, denoted with $\sum$, as

$$\sum\left(f^X\right) := \sum_{i=1}^{n} (m_i \cdot f(x_i)) \tag{3.23}$$

The **product over f^X**, denoted with $\prod$ is defined similarly.

Then the dot product of U^T and V^T becomes very familiar:

$$U^T \cdot V^T = \sum\left((u_i v_i)^{[\![1,k]\!]} \oplus^{\times} (u'_{i-k} v_i)^{(\!(k,\ell]\!]} \oplus^{\times} (u'_{i-k} v'_{i-\ell})^{(\!(\ell,n]\!]}\right) \tag{3.24}$$

The inner expression is identical to $U^T + V^T$ except for a replacement of $+$ with $\times$.

3.2.3 Outer Product

$U \cdot V^T$ instead of $U^T \cdot V$. But should be part of matrix multiplication?

3.3 Abstract Matrices

It is common practice in mathematics to represent matrices symbolically with sub-matrices such as:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{3.25}$$

If A is an $n \times n$ matrix then A_1, A_2, A_3, A_4 are not entries but $(k \times \ell)$, $(n - k \times \ell)$, $(k \times n - \ell)$ and $(n - k \times n - \ell)$ matrices respectively. Ellipses are also routinely used for interpolating over regions of a matrix, as in:

$$M = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ & \ddots & \vdots \\ 0 & & x_{nn} \end{bmatrix} \tag{3.26}$$

Chapter 4

Integration I

In this chapter we will present an introduce integration over oriented intervals and generalise to higher dimensions. For the time being, we will focus on the intuition behind this and only worry about axis-aligned n-cubes. Following this, in the next chapter, we will delve more deeply from a measure theoretic perspective and integration over more general shapes.

4.1 Single variable integration

Given a function f with real variable x and an interval $[a, b)$ on the (extended) real line, a traditional **definite integral** would be of the form:

$$\int_a^b f(x)\,dx \quad \text{or} \quad \int_{[a,b)} f(x)\,dx$$

Which we interpret as the signed area bounded by f between $x = a$ and $x = b$. However, defining this definite integral using (unoriented) intervals like this is a bit of a misnomer. In the case where $a \geq b$ one would typically use the identity:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{4.1}$$

to evaluate the integral. But, as we saw in the previous chapter, when $a \geq b$, the interval $[a, b)$ is the empty set! We can't translate equation (4.1) to an identity along the lines of:

$$\int_{[a,b)} f(x)\ dx = -\int_{[b,a)} f(x)\ dx \tag{4.2}$$

since at least one of $[a, b)$ or $[b, a)$ will be always be empty. Although this notation generally appears in context of Lebesgue integrals, Riemann integrals (which generally use $\int_a^b$ ... instead) simply hide this mis-use of intervals. For example, it is typically glossed over that when doing the formal Riemann sum for an integral $\int_a^b f(x)\ dx$, one would use the tagged partition given as a series x_i such that $a = x_1 < x_2 < ... < x_n = b$.

Find one that's not wikipedia! - Unintentionally we are claiming that $\int_1^0 f(x)\ \ dx\ \ = \int_{[1,0]} = \int_{\emptyset} f(x)\ dx = 0$

If oriented intervals are used instead of traditional intervals, then the identity from equation (4.1) can instead just be a result of *bi-linearity*.

$$\int_{[\![a,b)\!)} f(x)\ dx = -\int_{\ominus[\![a,b)\!)} f(x)\ dx = -\int_{[\![b,a)\!)} f(x)\ dx \tag{4.3}$$

Definition Let $[\![a, b]\!]$ be an interval on $\mathbb{R}$ then the boundary function ∂ is the linear map such that:

$$\partial(\ [\![a, b]\!]\ ) = \left\{\!\!\left|\ a^1, b^{-1}\ \right|\!\!\right\} \tag{4.4}$$

By linearity we also have:

$\partial(\!(a, b)\!) = \partial(\ \ominus[\![b, a]\!]\ ) = \ominus\partial(\ [\![b, a]\!]\ ) = \ominus \left\{\!\!\left|\ b^1, a^{-1}\ \right|\!\!\right\} = \left\{\!\!\left|\ a^1, b^{-1}\ \right|\!\!\right\}$.

Using this we also have:

$$\partial[\![a, b)\!) = \partial([\![a, c]\!] \oplus (\!(c, b)\!)) = \partial[\![a, c]\!] \oplus \partial(\!(c, b)\!) = \left\{\!\!\left|\ a^1, c^{-1}\ \right|\!\!\right\} \oplus \left\{\!\!\left|\ c^1, b^{-1}\ \right|\!\!\right\} = \left\{\!\!\left|\ a^1, b^{-1}\ \right|\!\!\right\}$$

And by a similar proof for $\partial(\!(a, c]\!]$, we conclude that:

$$\partial[\![a, b]\!] = \partial[\![a, b)\!) = \partial(\!(a, b]\!] = \partial(\!(a, b)\!) \tag{4.5}$$

Isolated points do not affect the boundary of an oriented interval. This should have been obvious from the definition to begin with. The interval $[\![a, a]\!]$ is a hybrid set which contains only the element a with multiplicity one. From equation XX,

$$\partial[\![a, a]\!] = \{\!\!\{ a^1, a^{-1} \}\!\!\} = \emptyset \tag{4.6}$$

So whether we use $\int_a^b$ to denote the integral over the intervals $[\![a, b]\!]$, $[\![a, b)\!)$ or $(\!(a, b)\!)$ the boundary is unchanged and so the integral will evaluate identically.

The hybrid sets $[\![a, b]\!] \oplus [\![b, c]\!]$ and $(\!(a, b)\!) \oplus (\!(b, c)\!)$ have identical multiplicities almost everywhere. At b, they will differ by 2. Although simple arguments resolve this issue, by using left-closed, right-open oriented intervals we can bypass these arguments altogether when showing

$$\int_{[\![a,b)\!)} f(x)\,dx + \int_{[\![b,c)\!)} f(x)\,dx = \int_{[\![a,c)\!)} f(x)\,dx \tag{4.7}$$

4.2 Higher dimension intervals

For now, we will concern ourselves only with oriented, n-dimensional, axis-aligned rectangles in $\mathbb{R}^n$. In one dimension, the previously discussed oriented intervals cover most of the "obvious shapes" one would be interested in. Moving to two dimensions, there are many more "obvious shapes" to consider but we will temporarily ignore triangles, circles and even rectangles that are tilted. We could also use triangles instead of rectangles as our primitive of choice. This would generalize to n-simplexes in higher dimensions. Since n-simplexes and n-cubes end up b[9] But, first we must introduce some notation to describe these and higher dimension rectangles.

At the moment the only rectangles we have defined are the one-dimensional "oriented interval". Hence we will also refer to this as a 1-cube. We construct higher dimensional n rectangles using the Cartesian product.

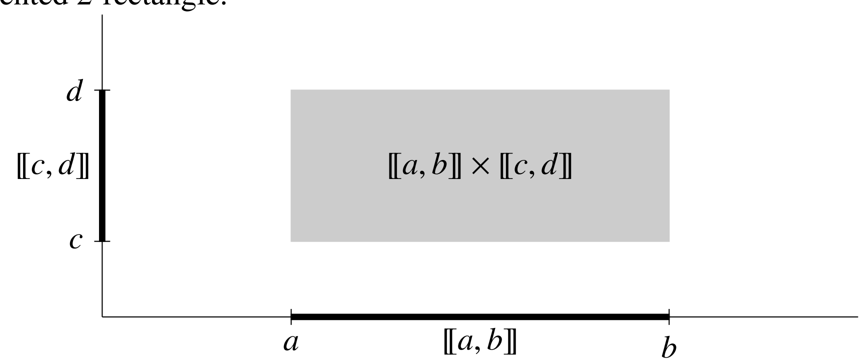
Definition Let $X = \{\!\!\{ x_1^{m_1}, ..., x_k^{m_k} \}\!\!\}$ and $Y = \{\!\!\{ y_1^{n_1}, ..., y_\ell^{n_\ell} \}\!\!\}$ be hybrid sets. We define the **Carte-**

sian product of hybrid sets X and Y, denoted with $\times$ operator as:

$$X \times Y = \{\!\!\{ (x,y)^{m \cdot n} \ : \ x^m \in X, y^n \in Y \}\!\!\} \tag{4.8}$$

If $[\![a,b]\!]$ and $[\![c,d]\!]$ are both positively oriented 1-rectangles then their Cartesian product is shown in Figure 4.1 is clearly a two dimensional rectangle or *2-rectangle*. Taking the Cartesian product of a 2-rectangle and 1-rectangle gives a 3-rectangle in $\mathbb{R}^3$. We should note here that we do not distinguish between $((x,y),z)$ and $(x,(y,z))$ but rather we treat both as different names for the ordered triple (x,y,z). We similarly associate parentheses in higher dimensions as well.

Figure 4.1: The Cartesian product of two positively oriented 1-rectangles $[\![a,b]\!]$ and $[\![c,d]\!]$ is a positively oriented 2-rectangle.



Theorem 4.2.1 *The Cartesian product of a k-rectangle in $\mathbb{R}^m$ (where, $k \leq m$) and ℓ-rectangle in $\mathbb{R}^n$ (again, $\ell \leq n$) is a $(k+\ell)$-rectangle in $\mathbb{R}^{m+n}$.*

For completeness we will also define a 0-rectangle as a hybrid set containing a single point with multiplicity 1 or -1. Firstly this allows us to embed k-rectangles in $\mathbb{R}^n$. For example $[\![a,b]\!] \times [\![c,d]\!] \times \left\{\!\!\left\{ e^1 \right\}\!\!\right\}$ is the product of two 1-rectangles and a 0-rectangle (all over $\mathbb{R}$) and so it is a 2-rectangle over $\mathbb{R}^3$. Specifically, it is the 2-rectangle $[\![a,b]\!] \times [\![c,d]\!]$ on the plane $z = 3$. This also illustrates the principle that given a k-rectangle in $\mathbb{R}^n$ where $n > k$ we can always find a k dimensional subspace which also contains the rectangle. We will re-use the interval notation from earlier although one should be careful to “type-check” while interpreting. When a and b are real numbers then we continue to use the definition $[\![a,b]\!] = [a,b) \ominus [b,a)$. However, when

$\boldsymbol{a}$ and $\boldsymbol{b}$ are n-tuples (for example, coordinates in $\mathbb{R}^n$ then this is *not* the oriented line interval $[\boldsymbol{a}, \boldsymbol{b}) \ominus [\boldsymbol{b}, \boldsymbol{a})$.

Definition Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ be ordered n-tuples then we use the notation:

$$[\![\boldsymbol{a}, \boldsymbol{b}]\!] = [\![a_1, b_1]\!] \times [\![a_2, b_2]\!] \times \ldots \times [\![a_n, b_n]\!] \tag{4.9}$$

The dimension of $[\![\boldsymbol{a}, \boldsymbol{b}]\!]$ is equal to the number of indices where a_i and b_i are distinct. For any i where $a_i = b_i$, the corresponding term: $[\![a_i, b_i]\!]$ will be a hybrid set containing a single point, that is, a 0-rectangle. The orientation of $[\![\boldsymbol{a}, \boldsymbol{b}]\!]$ is based on the number of negatively oriented intervals $[\![a_i, b_i]\!]$. Should there be an odd number of indices i such that $a_i > b_i$ then $[\![\boldsymbol{a}, \boldsymbol{b}]\!]$ will also be negatively oriented. Otherwise, it will be positively oriented.

4.3 Riemann Integral on n-cubes

Now that we have oriented n-dimensional cubes, we would like to define the integral over one. For now, we will content ourselves with the Riemann integral and Euclidean volume. More complex domains and other metrics will be handled in later chapters with push-backs and the Lebesgue integral. The volume of an oriented n-cube in $\mathbb{R}^n$ we define to be the product of its side lengths. Formally,

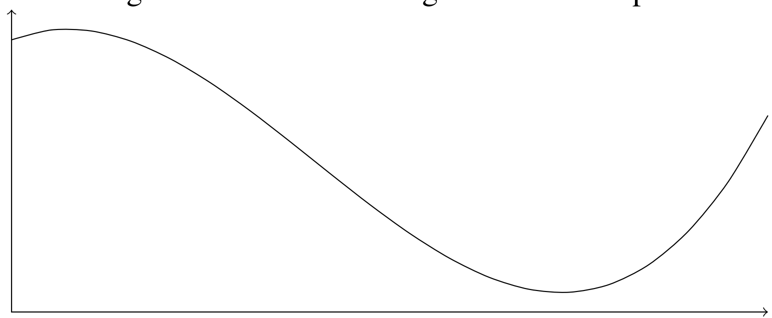
Definition Let $[\![\boldsymbol{a}, \boldsymbol{b}]\!]$ be a k-cube in $\mathbb{R}^n$ again with $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$. We denote the **volume of $[\![\boldsymbol{a}, \boldsymbol{b}]\!]$** with vol and define it as:

$$\mathrm{vol}(\ [\![\boldsymbol{a}, \boldsymbol{b}]\!]\ ) = (b_1 - a_1) \cdot (b_2 - a_2) \cdot \ldots \cdot (b_n - a_n) \tag{4.10}$$

For any $k < n$, a k-cube will have volume zero. In at least one dimension, the cube will be degenerate (i.e. $a_i = b_i$) and so will contribute zero to the product. Additionally, one can also observe that $\mathrm{vol}(\ominus[\![\boldsymbol{a}, \boldsymbol{b}]\!]) = -\mathrm{vol}([\![\boldsymbol{a}, \boldsymbol{b}]\!]$.

image instead of tikz?

Figure 4.2: Riemann Integral visual example.



Given an n-cube $\llbracket \boldsymbol{a}, \boldsymbol{b} \rrbracket$ we must cut each $\llbracket a_i, b_i \rrbracket$ into partitions. Previously we used generalized partitions and did not mind if pieces overlapped or exceeded the original range. However, for building our Riemann sums, we are only interested in partitions in the traditional, non-intersecting sense.

Definition Given an oriented interval $\llbracket a, b \rrbracket$ of the real line, we say that a partition of $\llbracket a, b \rrbracket$, $\{P_i\}_{i=1}^n$ is an **interval partition of $\llbracket \boldsymbol{a}, \boldsymbol{b} \rrbracket$** if its pieces are:

1. *Oriented intervals*: P_i is an oriented interval of the real line for all i.
2. *Disjoint*: $P_i \otimes P_j = \emptyset$ for all i, j

We denote the set of all such partition as $\mathcal{P}\llbracket a, b \rrbracket$.

This greatly restricts the types of partitions we have access to. Every interval partition will be — up to substitution of "$\rrbracket$, $\llparenthesis$" in place of "$\rrparenthesis$, $\llbracket$" — of the form:

$$\left\{ \llbracket a, x_1 \rrparenthesis,\ \llbracket x_1, x_2 \rrparenthesis,\ \llbracket x_2, x_3 \rrparenthesis,\ \ldots,\ \llbracket x_{n-1}, b \rrbracket \right\} \tag{4.11}$$

where x_i is a monotone sequence (that is, either non-increasing or non-decreasing). This is not to say that $P_i = \llbracket x_{i-1}, x_i \rrparenthesis$ as the pieces of P may not be given in this order. Regardless of the ordering, we select partitions $P^j \in \mathcal{P}\llbracket a^j, b^j \rrparenthesis$ for each dimension of $\llbracket a, b \rrparenthesis$. To build our mesh, we construct smaller n-cubes $I_{i_1,\ldots,i_n}$ using the Cartesian product of pieces:

$$I_{i_1,\ldots,i_n} = i_1 \times \ldots \times i_n \tag{4.12}$$

where each i_j is taken from P^j. We are now ready to construct Riemann sums.

Definition Given $P = \{P_j\}_{j=1}^n$ where $P_j \in \mathcal{P}[\![a_j, b_j]\!]$, and $f : [\![\boldsymbol{a}, \boldsymbol{b}]\!] \to \mathcal{R}$ then we define a Riemann sum f_P to be:

$$f_P = \sum_{i_1 \in P_1} \ldots \sum_{i_n \in P_n} f(x_{i_1,\ldots,i_n}) \mathrm{vol}(I_{i_1,\ldots,i_n}) \tag{4.13}$$

where $x_{i_1,\ldots,i_n}$ is any point chosen from $I_{i_1,\ldots,i_n}$.

Note that we specify *a* Riemann sum, not *the* Riemann sum. There are several ways to choose $x_{i_1,\ldots,i_n} \in I_{i_1,\ldots,i_n}$ and different samplings can lead to different Riemann sums for the same partition and same function. In $\mathbb{R}^1$, several common ways to sample include the left and right Riemann sums (which sample at the left and right boundaries), the trapezoidal Riemann sum (which averages the left and right) the upper Riemann sum (which samples at $\max(f(x_{i_1,\ldots,i_n}))$) and the lower Riemann sum (which samples at $\min(f(x_{i_1,\ldots,i_n}))$).

Recall our discussion of refinements from Chapter 2. Given 2 partitions P and P' of the same set then we say $P \leq P'$ if P' is a refinement of P. In this way we can induce a partial ordering on $\mathcal{P}[\![a, b]\!]$. There is a unique smallest element in this partial ordering which is $[\![a, b]\!]$ itself; every partition by definition will refine $[\![a, b]\!]$. Additionally, given $P, P' \in \mathcal{P}[\![a, b]\!]$ then propose that we can always find Q that simultaneously refines both. If $P = P'$ then trivially $Q = P$ is a common refinement. Otherwise, we take:

$$Q = [\![a, b]\!] \otimes \bigoplus_{p \in P} \bigoplus_{p' \in P} p \otimes p' \tag{4.14}$$

The heavy lifting here is done by the fact that every point in $[\![a, b]\!]$ is covered by exactly one $p \in P$ and exactly one $p' \in P$. The product of any two partition pieces will then be the smallest common interval with multiplicity 1. Multiplying the entire thing by $[\![a, b]\!]$ is done to correct the sign. As we go up in our ordering, our mesh becomes increasingly fine. Taking the Riemann sum of the suprema of this poset gives us the Riemann integral.

Definition The Riemann integral of a function $f : \mathbb{R}^n \to \mathbb{R}$ over a k-cube $[\![\boldsymbol{a}, \boldsymbol{b}]\!]$ where P is $\sup\{\mathcal{P}[\![a, b]\!]\}$

$$\int_{[\![a,b]\!]} f(x)\,dx = f_{\sup\{\mathcal{P}[\![a,b]\!]\}} \tag{4.15}$$

We say that a function is **Riemann integrable** if the upper and lower Riemann sums converge.

4.4 Boundary Operator

In one dimension, the boundary of an interval was quite straight-forward. For a positively oriented interval, the boundary was composed of two points; the right end-point was positive and the left end-point was negative. From the perspective of k-rectangles, the ∂ operator has mapped an oriented 1-rectangle to a set of oriented 0-rectangles. We will now generalize the boundary to map an oriented n-rectangle to an $(n-1)$-rectangle.

Definition Let $[\![\boldsymbol{a}, \boldsymbol{b}]\!]$ be a a k-rectangle in $\mathbb{R}^n$. Additionally, let $i_1, i_2, \ldots, i_k$ be the unique non-decreasing sequence of indices such that $a_{i_j} \neq b_{i_j}$. The **boundary of $[\![\boldsymbol{a}, \boldsymbol{b}]\!]$** , denoted the operator ∂ is given by

$$\begin{aligned}\partial\left([\![\boldsymbol{a}, \boldsymbol{b}]\!]\right) = \bigoplus_{j=1}^{k} (-1)^j \Big(& [\![\boldsymbol{a}^{[\![1,i_j)\!)}, \boldsymbol{b}^{[\![1,i_j)\!)}]\!] \times \left\{\!\!\left\{ a_{i_j} \right\}\!\!\right\} \times [\![\boldsymbol{a}^{(\!(i_j,n]\!]}, \boldsymbol{b}^{(\!(i_j,n]\!]}]\!] \\ & \ominus [\![\boldsymbol{a}^{[\![1,i_j)\!)}, \boldsymbol{b}^{[\![1,i_j)\!)}]\!] \times \left\{\!\!\left\{ b_{i_j} \right\}\!\!\right\} \times [\![\boldsymbol{a}^{(\!(i_j,n]\!]}, \boldsymbol{b}^{(\!(i_j,n]\!]}]\!] \Big)\end{aligned} \tag{4.16}$$

The above equation will require a bit of unpacking to digest featuring oriented intervals in two different contexts. The first appears in the superscripts of $\boldsymbol{a}$ and $\boldsymbol{b}$. The intervals $[\![1, i_j)\!)$ and $(\!(i_j, n]\!]$ are and is an interval over vector indices just as in Chapter 3. Thus, the term $\boldsymbol{a}^{[\![1,i_j)\!)}$ refers to the vector $(a_1, a_2, \ldots, a_{i-1})$ while the term $\boldsymbol{b}^{(\!(i,n]\!]}$ refers to $(b_{i+1}, b_{i+2}, \ldots, b_n)$. This provides a compact notation to partition the original range of indices into 3 pieces: $[\![1, i)\!)$, $[\![i, i]\!]$, and

$(\!(i,n]\!]$. The first and last portions are isolated but left untouched, but the central $[\![i,i]\!]$ term is then replaced with the 0-rectangles a_i or b_i. Formally, we are actually using the hybrid sets $\{\!\!\{ (a_i)^1 \}\!\!\}$ and $\{\!\!\{ (b_i)^1 \}\!\!\}$ but we omit the explicit multiplicity of one (and will continue to do so through out the section) to lighten notation.

Each Cartesian product forms a $(k-1)$-rectangular face of the k-rectangle which we shall show. Let $[\![\boldsymbol{a},\boldsymbol{b}]\!]$ be a k rectangle in $\mathbb{R}^n$. Following from definitions we have:

$$[\![\boldsymbol{a},\boldsymbol{b}]\!] = [\![\boldsymbol{a}^{[\![1,i)\!)},\boldsymbol{b}^{[\![1,i)\!)}]\!] \times [\![\boldsymbol{a}^{[\![i,i]\!]},\boldsymbol{b}^{[\![i,i]\!]}]\!] \times [\![\boldsymbol{a}^{(\!(i,n]\!]},\boldsymbol{b}^{(\!(i,n]\!]}]\!] \tag{4.17}$$

If a_i and b_i are distinct then $[\![\boldsymbol{a}^{[\![i,i]\!]},\boldsymbol{b}^{[\![i,i]\!]}]\!]$ is a 1-rectangle. Since both a_i and b_i are 0-rectangles and expressions agree everywhere else, then the following are both $(k-1)$-rectangles:

$$[\![\boldsymbol{a}^{[\![1,i)\!)},\boldsymbol{b}^{[\![1,i)\!)}]\!] \times \{\!\!\{ a_i \}\!\!\} \times [\![\boldsymbol{a}^{(\!(i,n]\!]},\boldsymbol{b}^{(\!(i,n]\!]}]\!] \tag{4.18}$$

$$[\![\boldsymbol{a}^{[\![1,i)\!)},\boldsymbol{b}^{[\![1,i)\!)}]\!] \times \{\!\!\{ b_i \}\!\!\} \times [\![\boldsymbol{a}^{(\!(i,n]\!]},\boldsymbol{b}^{(\!(i,n]\!]}]\!] \tag{4.19}$$

However if $a_i = b_i$ then $[\![\boldsymbol{a}^{[\![i,i]\!]},\boldsymbol{b}^{[\![i,i]\!]}]\!]$ is a 0-rectangle and so the expressions in (4.10) and (4.11) are both k-rectangles! Since $a_i = b_i$, the two expressions are identical, so their difference is zero and the terms disappear.

4.4.1 Example: *Boundary of a 1-rectangle*

Let $\boldsymbol{a} = (a_1)$ and $\boldsymbol{b} = (b_1)$ be trivial 1-tuples. It follows that:

$$\begin{aligned}
\partial(\ [\![\boldsymbol{a},\boldsymbol{b}]\!]\ ) = (-1)^i(&[\![\boldsymbol{a}^{[\![1,1)\!)},\boldsymbol{b}^{[\![1,1)\!)}]\!] \times \{\!\!\{ a_1 \}\!\!\} \times [\![\boldsymbol{a}^{(\!(1,1]\!]},\boldsymbol{b}^{(\!(1,1]\!]}]\!] \\
&\ominus [\![\boldsymbol{a}^{[\![1,1)\!)},\boldsymbol{b}^{[\![1,1)\!)}]\!] \times \{\!\!\{ b_1 \}\!\!\} \times [\![\boldsymbol{a}^{(\!(1,1]\!]},\boldsymbol{b}^{(\!(1,1]\!]}]\!]) \\
= \ &\ominus [\![\boldsymbol{a}^{\emptyset},\boldsymbol{b}^{\emptyset}]\!] \times \{\!\!\{ a_1 \}\!\!\} \times [\![\boldsymbol{a}^{\emptyset},\boldsymbol{b}^{\emptyset}]\!] \\
&\oplus [\![\boldsymbol{a}^{\emptyset},\boldsymbol{b}^{\emptyset}]\!] \times \{\!\!\{ b_1 \}\!\!\} \times [\![\boldsymbol{a}^{\emptyset},\boldsymbol{b}^{\emptyset}]\!] \\
= \ &\{\!\!\{ b_1 \}\!\!\} \ominus \{\!\!\{ a_1 \}\!\!\}
\end{aligned}$$

Again, we omit the braces from the hybrid sets $\Big\{\!\!\Big| (a_1)^1 \Big|\!\!\Big\}$ and $\Big\{\!\!\Big| (b_1)^1 \Big|\!\!\Big\}$. Recalling from the previous section in equation (4.4), we can see that the results agree:

$$\partial(\, [\![a_1, b_1]\!]\,) = \{\!|\, b_1 \,|\!\} \ominus \{\!|\, a_1 \,|\!\} = \Big\{\!\!\Big| (b_1)^1 \Big|\!\!\Big\} \ominus \Big\{\!\!\Big| (a_1)^1 \Big|\!\!\Big\} = \Big\{\!\!\Big| (a_1)^{-1}, (b_1)^1 \Big|\!\!\Big\} \tag{4.20}$$

4.4.2 Example: *Boundary of a 3-rectangle*

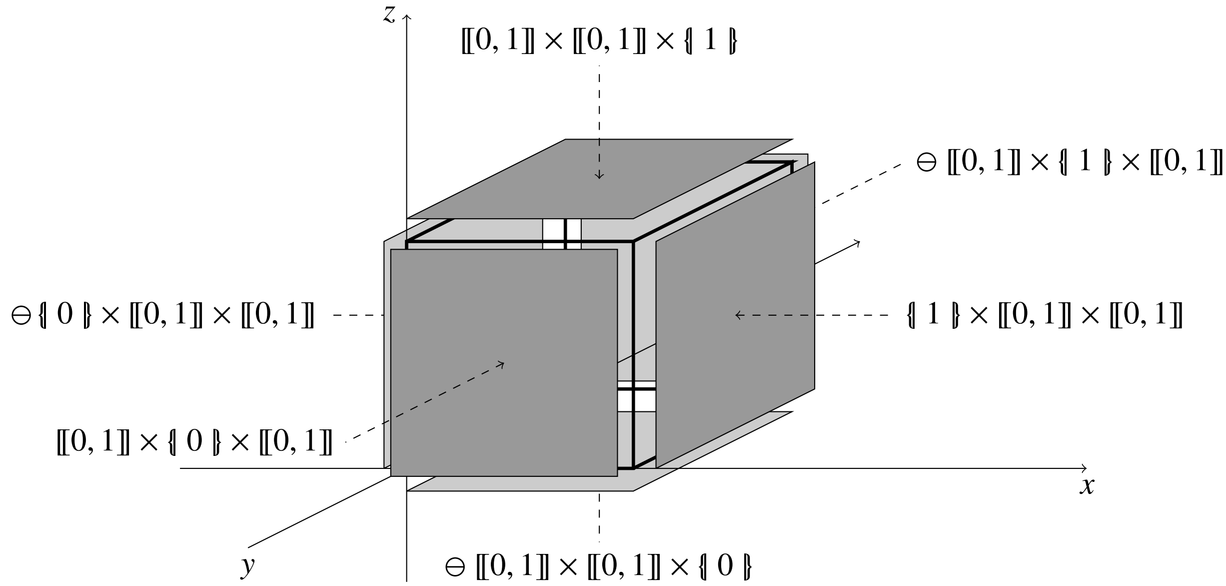
Let $\boldsymbol{a} = (0, 0, 0)$ and $\boldsymbol{b} = (1, 1, 1)$. Omitting the intermediate step, we find the boundary of $[\![\boldsymbol{a}, \boldsymbol{b}]\!]$ to be:

$$\begin{aligned}\partial(\, [\![\boldsymbol{a}, \boldsymbol{b}]\!]\,) = \; &\ominus \; (\{\!|\, 0 \,|\!\} \times [\![0,1]\!] \times [\![0,1]\!]) \; \oplus \; (\{\!|\, 1 \,|\!\} \times [\![0,1]\!] \times [\![0,1]\!]) \\ &\oplus \; ([\![0,1]\!] \times \{\!|\, 0 \,|\!\} \times [\![0,1]\!]) \; \ominus \; ([\![0,1]\!] \times \{\!|\, 1 \,|\!\} \times [\![0,1]\!]) \\ &\ominus \; ([\![0,1]\!] \times [\![0,1]\!] \times \{\!|\, 0 \,|\!\}) \; \oplus \; ([\![0,1]\!] \times [\![0,1]\!] \times \{\!|\, 1 \,|\!\})\end{aligned}$$

This may not be the most enlightening expression on its own. In Figure 4.3 below, the 3-rectangle given by $[\![\boldsymbol{a}, \boldsymbol{b}]\!]$ can be seen as a cube in three dimensions. Physically, the 3-rectangle is a solid cube and includes all interior points. The boundary meanwhile are just the rectangular outer faces, which conveniently, there are also six to match the six terms of $\partial[\![\boldsymbol{a}, \boldsymbol{b}]\!]$.

There are several ways to interpret and visualize the $\oplus$ and $\ominus$ sign associated with each face. Most naturally in $\mathbb{R}^3$ for 2-rectangles is to give each a front and back side with the sign determining which to use. Alternatively, a 2-rectangle has a boundary formed by 1-rectangles which when drawn as arrows, will all meet head-to-tail. This induces a clockwise or counter-clockwise cycle around the edge of the rectangle and so $\circlearrowright$ and $\circlearrowleft$ are also commonly used. This can be seen in Figure 4.4. One may even notice that the normals produced by both are the same and choose to use that. These are all conceptual tools, which are convenient to use particularly in $\mathbb{R}^2$ and $\mathbb{R}^3$. There may not be such a nice physical interpretation in other spaces.

Figure 4.3: The unit cube in $\mathbb{R}^3$ with positive orientation can be represented as the 3-rectangle: $[\![(0,0,0),(1,1,1)]\!]$ is shown as a wire-frame. The six faces that make up its boundary are shaded and labeled with their corresponding terms.



4.5 Chains

In fact, we have already seen k-chains without mentioning them explicitly. The boundary of a k-cube was the sum $\oplus$, of $2k$ $(k-1)$ cubes. Chains do not have to be boundaries however, any linear combination of k-cubes will do.

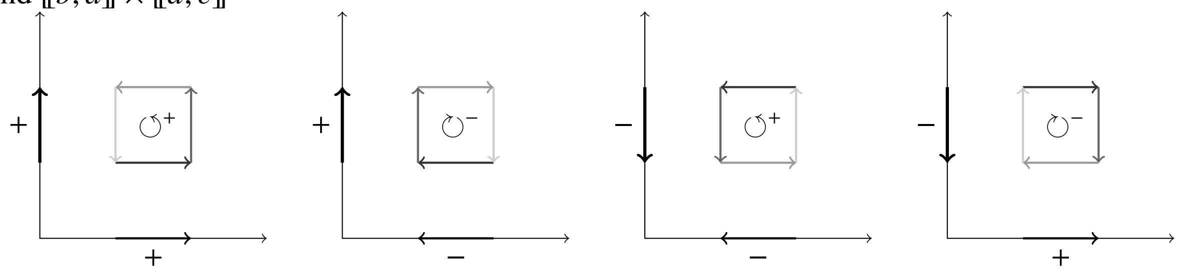
Definition We denote the Abelian group of of all k-cubes in X as $C_k(X)$ (omitting X when obvious by context). An element $c \in C_k(\mathrm{X})$ is called a ***k*-chain on** X and is of the form:

$$c = \bigoplus_{c_i \in X} \lambda_i c_i \tag{4.21}$$

with integer coefficients λ_i and k-cubes in c_i. If coefficients λ_i are ± 1 and c is *locally finite* (i.e. each c_i intersects with only finitely many c_j that have non-zero coefficients) then we say that c is a **domain of integration**.

Since k-chains are just linear combinations of k-cubes, we naturally extend many of our definitions linearly as well. The integral $\int_c f$ of a k-chain $c = \bigoplus_i \lambda_i c_i$ is defined as $\lambda_i \int_{c_i} f +$

Figure 4.4: One way of visualizing the orientation of 2-rectangles using clockwise and counter-clockwise cycles of arrows for 1-rectangles. The boundary of $[\![a,b]\!]\times[\![c,d]\!]$ becomes the cycle: $(a,c) \to (b,c) \to (b,d) \to (a,d) \to (a,c)$. Showing the relationship between $[\![a,b]\!] \times [\![c,d]\!]$ and $[\![b,a]\!] \times [\![d,c]\!]$



$\lambda_2 \int_{c_2} f + \ldots$. Doing the same for the boundary operator ∂ we have:

$$\partial_k : C_k \to C_{k-1}$$
$$\partial_k(c) = \bigoplus_{i=1}^{k} \lambda_i \partial_k(c_i)$$

Elegantly, the boundary operator now maps k-chains to $(k-1)$-chains!

$$\ldots \xleftarrow{\partial_{k-1}} C_{k-1} \xleftarrow{\partial_k} C_k \xleftarrow{\partial_{k+1}} C_{k+1} \xleftarrow{\partial_{k+2}} \ldots \tag{4.22}$$

The most natural next question becomes *"What does $\partial_{k-1}(\partial_k(c))$ look like?"*

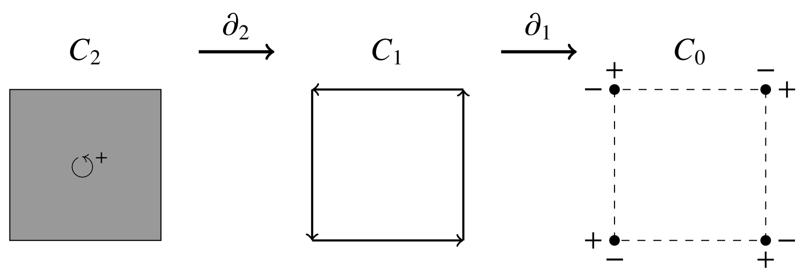
4.5.1 Example: *Boundary of a boundary (of a 2-cube)*

Let $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$. We wish to compute $\partial_1(\partial_2(\ [\![\boldsymbol{a}, \boldsymbol{b}]\!]\ ))$

$$
\begin{aligned}
\partial_1(\partial_2(\llbracket a_1, b_1\rrbracket \times \llbracket a_2, b_2\rrbracket)) = {} & \ominus \partial_1(\{\!\{\, 0 \,\}\!\} \times \llbracket 0,1\rrbracket) \,\oplus\, \partial_1(\{\!\{\, 1 \,\}\!\} \times \llbracket 0,1\rrbracket) \\
& \oplus \partial_1(\llbracket 0,1\rrbracket \times \{\!\{\, 0 \,\}\!\}) \,\ominus\, \partial_1(\llbracket 0,1\rrbracket \times \{\!\{\, 1 \,\}\!\}) \qquad (4.23)\\
= {} & \ominus(\ominus\{\!\{\, (0,0) \,\}\!\} \oplus \{\!\{\, (0,1) \,\}\!\}) \,\oplus\, (\ominus\{\!\{\, (1,0) \,\}\!\} \oplus \{\!\{\, (1,1) \,\}\!\}) \\
& \oplus(\ominus\{\!\{\, (0,0) \,\}\!\} \oplus \{\!\{\, (1,0) \,\}\!\}) \,\ominus\, (\ominus\{\!\{\, (0,1) \,\}\!\} \oplus \{\!\{\, (1,1) \,\}\!\}) \qquad (4.24)\\
= {} & \emptyset \qquad (4.25)
\end{aligned}
$$

When moving from (4.21) to (4.22), in addition to applying ∂_1 we also simplify, $\{\!\{\, x \,\}\!\} \times \{\!\{\, y \,\}\!\} = \{\!\{\, (x, y) \,\}\!\}$. The identity "$\partial\partial = 0$" is not unique to 2-cubes but holds for higher dimensions as well.

Figure 4.5: The boundary of 2-cube gives a cycle of oriented edges. Taking the boundary of again, at each corner, the negative boundary of one edge will be canceled by the positive boundary of the preceding edge.



Let $[\![\boldsymbol{a},\boldsymbol{b}]\!]$ be a k-rectangle in $\mathbb{R}^n$. Then we have:

$$\begin{aligned}\partial_k\partial_{k-1}\left([\![\boldsymbol{a},\boldsymbol{b}]\!]\right) = \bigoplus_{j=1}^{k}(-1)^j \ \Big(&\partial_{n-1}\left([\![\boldsymbol{a}^{[\![1,n]\!]},\ \ \boldsymbol{b}^{[\![1,i_j)\!)} \oplus \boldsymbol{a}^{[\![i_j]\!]} \oplus \boldsymbol{b}^{(\!(i_j,n]\!]}]\!]\right) \\ &\ominus\ \partial_{n-1}\left([\![\boldsymbol{a}^{[\![1,i_j)\!)} \oplus \boldsymbol{b}^{[\![i_j]\!]} \oplus \boldsymbol{a}^{(\!(i_j,n]\!]},\ \ \boldsymbol{b}^{[\![1,n]\!]}]\!]\right)\Big)\end{aligned} \tag{4.26}$$

$$\begin{aligned}= \bigoplus_{j=1}^{k}\bigoplus_{\ell=1}^{k-1}(-1)^{j+\ell}\ &[\![(\boldsymbol{a}^{[\![1,n]\!]}),\ \ (\boldsymbol{b}^{[\![1,i_j)\!)\,\oplus\,(\!(i_j,i_{j,\ell})\!)\,\oplus\,(\!(i_{j,\ell},n]\!]} \oplus \boldsymbol{a}^{[\![i_j]\!]\,\oplus\,[\![i_{j,\ell}]\!]})]\!] \\ &\ominus [\![(\boldsymbol{a}^{[\![1,i_{j,\ell})\!)\,\oplus\,(\!(i_{j,\ell},n]\!]} \oplus \boldsymbol{b}^{[\![i_{j,\ell}]\!]}),\ \ (\boldsymbol{b}^{[\![1,i_j)\!)\,\oplus\,(\!(i_j,n]\!]} \oplus \boldsymbol{a}^{[\![i_j]\!]})]\!] \\ &\ominus [\![(\boldsymbol{a}^{[\![1,i_j)\!)\,\oplus\,(\!(i_j,n]\!]} \oplus \boldsymbol{b}^{[\![i_j]\!]}),\ \ (\boldsymbol{b}^{[\![1,i_{j,\ell})\!)\,\oplus\,(\!(i_{j,\ell},n]\!]} \oplus \boldsymbol{a}^{[\![i_{j,\ell}]\!]})]\!] \\ &\oplus [\![(\boldsymbol{a}^{[\![1,i_j)\!)\,\oplus\,(\!(i_j,i_{j,\ell})\!)\,\oplus\,(\!(i_{j,\ell},n]\!]} \oplus \boldsymbol{b}^{[\![i_j]\!]\,\oplus\,[\![i_{j,\ell}]\!]}),\ \ (\boldsymbol{b}^{[\![1,n]\!]})]\!]\end{aligned} \tag{4.27}$$

Note that we have i_j and i'_ℓ; after applying the first boundary operator, one dimension of the k-cube is degenerate. Hence for each sequence: $\{i_j\}_{j=1}^{k}$ we construct $\{i_{j,\ell}\}_{\ell=1}^{k-1}$ given by:

$$i_{j,1},\ldots,i_{j,k-1} = i_1,\ldots,\widehat{i_j},\ldots,i_k \tag{4.28}$$

The double sum iterates over all pairs but $\oplus$ commutes so the $(k-2)$-cube with degenerate dimensions $[\![i_j]\!] \oplus [\![i_{j,\ell}]\!]$ will be iterated over twice. The sequences depend on one another so it is not as simple as simply swapping ℓ and j:

$$[\![i_j]\!] \oplus [\![i_{j,\ell}]\!] = \begin{cases}[\![i_\ell]\!] \oplus [\![i_{\ell,j-1}]\!] & j > \ell \\ [\![i_{\ell+1}]\!] \oplus [\![i_{\ell+1,j}]\!] & j \leq \ell\end{cases} \tag{4.29}$$

So each term representing a $(k-2)$-cube will occur twice in the sum. Once with the iteration (j,ℓ) and once with $(\ell, j-1)$ or $(\ell+1, j)$. In either case, $(-1)^{j+\ell}$ is inverted meaning the two cubes will cancel. Leaving us with the boundary of a boundary being empty. By linearity this extends to all chains as well as the sum of empty sets is of course still empty.

Definition A **chain complex** is a sequence of Abelian groups $\ldots, A_2, A_1, A_0, A_{-1}, A_{-2}, \ldots$ which are connected by homomorphisms $d_n : A_n \to A_{n-1}$ such that $d_n \circ d_{n+1} = 0$ for all n. Typically written out as:

$$\dots \xleftarrow{d_{k-1}} A_{k-1} \xleftarrow{d_k} A_k \xleftarrow{d_{k+1}} A_{k+1} \xleftarrow{d_{k+2}} \dots \tag{4.30}$$

A **cochain complex** is a sequence of Abelian groups $\dots, A^{-2}, A^{-1}, A^0, A^1, A^2, \dots$ which are connected by homomorphisms $d^n : A^n \to A^{n+1}$ such that $d^n \circ d^{n-1} = 0$ for all n. Typically written out as:

$$\dots \xrightarrow{d^{k-1}} A^{k-1} \xrightarrow{d^k} A^k \xrightarrow{d^{k+1}} A^{k+1} \xrightarrow{d^{k+2}} \dots \tag{4.31}$$

$(C_\bullet, \partial_\bullet)$ is just one instance of a chain complex known as the *"cubic homology"*. In the next chapter we will look at the more general *"cubic singular homology"*. As well as the related cochain complex: the *"De Rham cohomology"* and how the two relate.

Chapter 5

Integration II

5.1 Differential Forms

Definition A k-form β on the open set $\Omega \subset \mathbb{R}^n$ has the form:

open set or Lebesgue measurable sets??

$$\beta = \sum_j b_j(x)\, \mathrm{d}x_{j_1} \wedge \ldots \wedge \mathrm{d}x_{j_k} \tag{5.1}$$

where $j = (j_1, \ldots, j_k)$ is a k dimensional multi-index. We say that $\beta \in \Lambda^k(\Omega)$

We have not yet defined the $\wedge$ operator.

Anti-commutative: $\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$. Which implies for any permutation σ of $\{1, \ldots, k\}$:

$$dx_1 \wedge \ldots \wedge dx_k = \operatorname{sgn}(\sigma)\, dx_{\sigma(1)} \wedge \ldots \wedge dx_{\sigma(k)} \tag{5.2}$$

Anti-commutativity additionally implies that for all x_i, $dx_i \wedge dx_i = 0$.

Let $\alpha = \sum_i a_i(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_\ell} \in \Lambda^\ell(\Omega)$ and $\beta = \sum_j b_j(x)\, \mathrm{d}x_{j_1} \wedge \ldots \wedge \mathrm{d}x_{j_k} \in \Lambda^k(\Omega)$ then define:

$$\alpha \wedge \beta := \sum_{i,j} a_i(x) b_j(x)\; dx_{i_1} \wedge ... \wedge dx_{i_\ell} \wedge dx_{j_1} \wedge ... \wedge dx_{j_k} \tag{5.3}$$

Thus we can think of $\wedge$ as mapping a k-form and an ℓ-form to a $(k+\ell)$-form, $\wedge : \Lambda^\ell(\Omega) \times \Lambda^k(\Omega) \to \Lambda^{k+\ell}(\Omega)$. By anti-commutativity we have:

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha \tag{5.4}$$

Definition Let α be a k-form on $\Omega \subset \mathbb{R}^n$ of the form $\alpha = A(x)\, \mathrm{d}x_1 \wedge ... \wedge \mathrm{d}x_n$. If $A \in \mathcal{L}^1(\Omega, \mathrm{d}x)$ then we define:

$$\int_\Omega \alpha = \int_\Omega A(x)\, \mathrm{d}x \tag{5.5}$$

Where the left-hand side is the integral of a k-form and the right-hand side is a Lebesgue integral. For any $\beta \in \Lambda^k(\Omega)$ we extend this definition linearly as the sum of integrals.

Define dx from $x_1, ..., x_n$. Need to lift sign change from permutations

5.2 Pull-backs

Benefit of differential forms is how cleanly they handle changes in coordinates.

Definition $F : X \to \Omega$ Define the pullback $F^*\beta$

$$F^*\beta = \sum_j b_j(F(x))(F^*\mathrm{d}x_{j_1}) \wedge ... \wedge (F^*\mathrm{d}x_{j_k}) \tag{5.6}$$

and

$$F^*\mathrm{d}x_j = \sum_\ell \frac{\partial F^j}{\partial x_\ell}\, \mathrm{d}x_\ell \tag{5.7}$$

Which can be reduced by:

$$F^*\beta = \sum_j b_j(F(x))(F^*\mathrm{d}x_{j_1}) \wedge ... \wedge (F^*\mathrm{d}x_{j_k}) \tag{5.8}$$

$$= \sum_j b_j(F(x))\left(\sum_\ell \frac{\partial F^{j_1}}{\partial x_\ell}\,\mathrm{d}x_\ell\right) \wedge ... \wedge \left(\sum_\ell \frac{\partial F^{j_k}}{\partial x_\ell}\,\mathrm{d}x_\ell\right) \tag{5.9}$$

$$= ... \tag{5.10}$$

$$= \sum_j b_j(F(x))\,\det(J_F)\;\mathrm{d}x_{j_1} \wedge ... \wedge \mathrm{d}x_{j_k} \tag{5.11}$$

Which is significant given the change of variable formula for integration:

$$\int_{\phi(U)} f(v)\,dv = \int_U f(\phi(u))\,|\det\phi'(u)|\,du \tag{5.12}$$

Theorem 5.2.1 *Let $F : X \to \Omega$ be an (orientation-preserving diffeomorphism) and α an integrable n-form on Ω then*

$$\int_\Omega \alpha = \int_X F^*\alpha \tag{5.13}$$

More algebra of differential forms

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta) \tag{5.14}$$

Definition Exterior derivative

...

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^j \alpha \wedge (d\beta) \tag{5.15}$$

...

$$F^*(d\beta) = dF^*\beta \tag{5.16}$$

5.3 Integration over Manifolds

Example Integrate an atlas with overlapping charts (using inclusion-exclusion)

5.4 Stokes' Theorem

5.5 A non-trivial example

??????????

Example Tricky Integration Using Stokes' theorem and Inclusion/Exclusion to evaluate a tricky theorem like:

$$f(z) = \frac{z^2}{(z^2 + 2z + 2)} \tag{5.17}$$

(2-3 pages)

Chapter 6

Integration III

"Chapter goals"

Section 4.1 will cover a conventional treatment of the Lebesgue integral, for those already familiar with the construction, this may be skipped.

6.1 Sigma Algebras

Before we can talk about the Lebesgue integral we must first set the stage, so to speak.

Definition Let X be a non-empty set. A **σ-algebra on the set** X, Σ, is a family of subsets of X such that:

1. Σ is non-empty
2. *Closed under complement.* If $E \in \Sigma$, then $X \setminus E \in \Sigma$.
3. *Closed under countable union.* If $E_1, E_2, ... \in \Sigma$ then $(E_1 \cup E_2 \cup ...) \in \Sigma$.

The pair (X, Σ) is called a **measurable space** and elements of Σ are called the **measurable sets** (of X).

It can easily be shown through the use of De Morgan's laws that a σ-algebra is also closed under countable intersection as well.

Example $\{\emptyset, X\}$ is a σ-algebra on X. In fact, X and $\emptyset$ are members of *every* σ-algebra on X.

Example 2^X is a σ-algebra on X.

However, we would also like to be able to construct more interesting σ-algebras.

Definition Given an arbitrary family of subsets $F \subseteq 2^X$, there is a unique smallest σ-algebra containing F which is called the **σ-algebra generated by** F and we will denote as $\sigma(F)$.

Can be constructed by taking the intersection of all σ-algebras containing F.

Of particular interest to us is the σ-algebra generated by a topology, $\mathcal{T}(X)$.

Borel σ-algebra: $\mathcal{B}(X) = \sigma(\mathcal{T}(X))$

6.2 The Lebesgue Integral

Definition Let (X, μ) be a measure space and $S \in \mathcal{L}(X, \mu)$ a measurable set. If $\mathbb{I}_S$ is indicator function $\mathbb{I}_S : X \to \{0, 1\}$ given by $\mathbb{I}_S(x) = 1$ if $x \in S$ and $\mathbb{I}_S(x) = 0$ otherwise. Then we define:

$$\int_X \mathbb{I}_S \, d\mu := \mu(S) \tag{6.1}$$

We define a simple function as a function which maps to a finite set. countable?

Theorem 6.2.1 *If φ is a simple function, then we it can be represented as a finite sum of indicator functions:*

$$\varphi := \sum_{k=0}^{n} a_k \mathbb{I}_{E_k} \tag{6.2}$$

Where $\{a_k\}_{k=0}^n$ is the set of values in the range of φ and $E_k = \varphi^{-1}(\{a_k\})$, the set of all points which map to a_k.

This allows us to define the integral of a simple function.

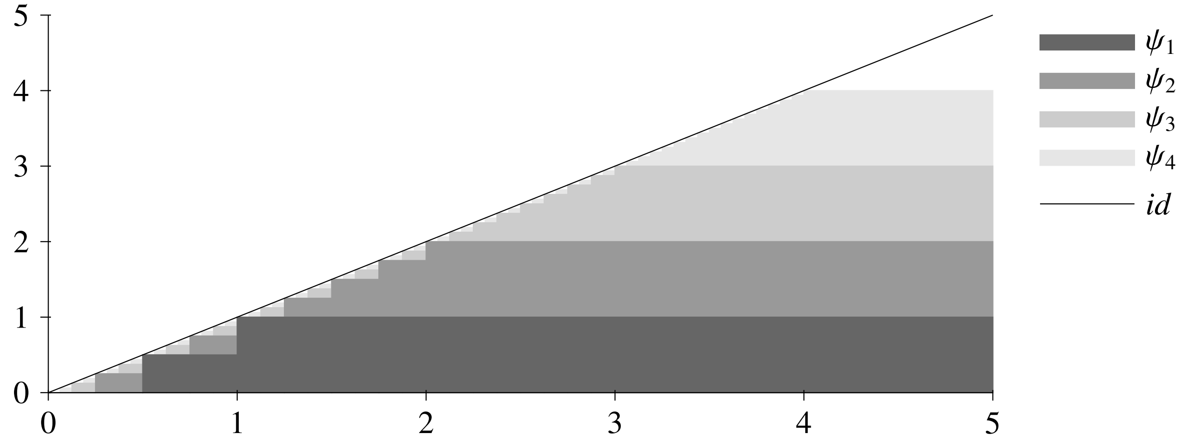


Figure 6.1: asdfasdf

Definition Let $\varphi = \sum_{k=0}^{n} a_k \mathbb{I}_{E_k}$ be a simple function. We define the integral:

$$\int_X \varphi \, d\mu = \int_X \sum_{k=0}^{n} a_k \mathbb{I}_{E_k} d\mu = \sum_{k=0}^{n} a_k \int_X \mathbb{I}_{E_k} d\mu \tag{6.3}$$

Next we define integral of a non negative functions in $\bar{\mathbb{R}}$

Definition Let (X, μ) be a measure space and $f : X \to [0, \infty]$.

$$\int_X f \, d\mu := \sup \left\{ \int_X \varphi : \varphi \text{ simple, and } 0 \leq \varphi \leq f \right\} \tag{6.4}$$

We use the simple function ψ_n defined as:

$$\psi_n = \sum_{k=0}^{n2^n-1} \left[\left(\frac{k}{2^n} \right)^{\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)} \right] + n^{[n,\infty]} \tag{6.5}$$

Notationally, this definition is rather heavy but is easily understood geometrically as seen in Figure 4.1

figure description

Which for any non-negative f allows us to define $\varphi_n = \psi_n \circ f$ a simple function.

Obviously we have φ_n simple since ψ_n is simple.

Since $\psi_n(x) \leq x$ for all x then we also have $0 \leq \varphi_n \leq f$.

Most importantly we have $0 \leq f(x) - \varphi_n(x) \leq 2^{-n}$ and so φ_n, uniformly approaches f as n approaches infinity.

Theorem 6.2.2 *Let f be a function mapping to $\mathbb{R}$. Then $f = f^+ - f^-$ where f^+ and f^- are non-negative functions given by:*

$$f^+(x) := \max(0, f(x)) \tag{6.6}$$

$$f^-(x) := \max(0, -f(x)) \tag{6.7}$$

Now we have everything we need to define integrals for arbitrary functions mapping to $\mathbb{R}$:

Definition asdfasd

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu \tag{6.8}$$

This was a standard treatment of Lebesgue integration.

Mathematically it is very simple to extend this to integrals of hybrid sets over $L(X, \mu)$ by linearity. That is, given $\Gamma \in \mathbb{Z}^{\mathcal{L}(X,\mu)}$:

$$\int_\Gamma f d\mu = \sum_{\sigma \in \Gamma} \Gamma(\sigma) \int_\sigma f d\mu \tag{6.9}$$

Chapter 7

Inclusion/Exclusion for Parallel

Chapter 8

Conclusions

The primary objective of this thesis was to extend [8] by investigating further applications of hybrid sets and functions. Although this paper was focused on results for integration and Petri net graphs, this is not to take away from the "smaller" results shown along the way. As a first example, we showed that, (notational choice: $\mathbb{Z}^{\mathbb{P}}$, $h(\mathbb{P})$, $\mathcal{H}(\mathbb{P})$?), the hybrid sets over prime numbers is equivalent to $\mathbb{Q}_{+}$.

Hybrid sets come into their own within the context of hybrid functions as we saw with arithmetic on piecewise functions and symbolic matrices. Combined with tricks from linear algebra, the usage of hybrid functions allowed for large decreases in both cases. Hybrid pseudo-functions leave a function associated with an element unevalutated and allow for algebra to be performed on the domains before requesting any functions be evaluated.

Hybrid functions were shown to be a good model for domains of integration. An atlas can succintly be defined in terms of a set of hybrid relation over a universe of Euclidean rectangles mapping to a common manifold. Principle of Inclusion-Exclusion was used instead of the typical *partitions of unity* to reduce the atlas to it's support. Unlike some of the previous examples, any leftover negative terms produced by PIE are completely well-founded and have natural geometric interpretations. Moreover, ∂, the boundary operator on a k-chain explicitly constructs them. The beauty (and usefulness) of ∂ was then shown with a proof of the generalized Stokes' theorem. Which in turn we used in conjunction with generalized partitions to transform an otherwise difficult to compute integral.

Finally, we showed a novel formulation of Petri net graphs. Instead of considering transitions as a special type of node, we represented transitions along with corresponding arc weights as a single hybrid set. Conditions for liveness and coverability were also discussed. Relaxing the condition of non-negative markings gives way to *lending Petri nets* [3] [4] for which hybrid sets were even better suited. Unfortunately, I just discovered the Bartoletti papers and have not had a chance to read more than the abstract.

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