In other chapters, I've been using the notation $[\![...)\!)$ for **oriented intervals** defined as:

$$[\![a,b)\!) = [a,b) \ominus [b,a) \tag{1}$$

Only one of the non-oriented (traditional) intervals $[a,b) = \{x | a \le x < b\}$ or $[b,a) = \{x | b \le x < a\}$ will be non-empty. The multiplicity of $[\![a,b)\!)$ will be +1 or -1 between a and b depending. This gives us some nice properties like:

$$[\![a,b)\!) = \ominus[\![b,a)\!) \tag{2}$$
$$[\![a,c)\!) = [\![a,b)\!) \oplus [\![b,c)\!) \tag{3}$$

The convolution of two functions is defined by:

$$(F * G)(t) = \int_{-\infty}^{\infty} F(\tau)G(t-\tau)d\tau \tag{4}$$

If F and G are the sum of disjoint sub-functions $f_1, f_2, \ldots$ and $g_1, g_2, \ldots$ then:

$$(F * G)(t) = \sum_i \sum_j \int_{-\infty}^{\infty} f_i(\tau)g_j(t-\tau)d\tau \tag{5}$$

So we only need to worry about convolving "one-piece" functions then we take the sum of all pairs. So for two one piece functions:

$$F(t) = \begin{cases} f(t) & t \in [a_f, b_f) \\ 0 & \text{otherwise} \end{cases} \qquad G(t) = \begin{cases} g(t) & t \in [a_g, b_g) \\ 0 & \text{otherwise} \end{cases}$$

the typical approach would be to commute F and G so that $b_f - a_f < b_g - a_g$ so that:

$$(F * G)(t) = \begin{cases} \int_{a_f}^{x-a_g} f(\tau)\ g(t-\tau)\ d\tau & (a_f + a_g) \le t < (b_f + a_g) \\ \int_{a_f}^{b_f} f(\tau)\ g(t-\tau)\ d\tau & (b_f + a_g) \le t < (a_f + b_g) \\ \int_{x-b_g}^{b_f} f(\tau)\ g(t-\tau)\ d\tau & (a_f + b_g) \le t < (b_f + b_g) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

However with hybrid functions we can (regardless of relative length) write:

$$\begin{aligned} (F\ *\ G)(t) = \mathcal{R}_+ \Bigg(& \left(\int_{a_f}^{t-a_g} f(\tau)\ g(t-\tau)\ d\tau \right)^{[\![a_f+a_g,\ b_f+a_g)\!)} \\ & \oplus \left(\int_{a_f}^{b_f} f(\tau)\ g(t-\tau)\ d\tau \right)^{[\![b_f+a_g,\ a_f+b_g)\!)} \\ & \oplus \left(\int_{t-b_g}^{b_f} f(\tau)\ g(t-\tau)\ d\tau \right)^{[\![a_f+b_g,\ b_f+b_g)\!)} \Bigg)(t) \end{aligned} \tag{7}$$

Let F, G be defined as:

$$F = f_1^{(-\infty,-1)} \oplus f_2^{[-1,1)} \oplus f_3^{[1,\infty)} \tag{8}$$
$$G = 0^{(-\infty,-1)} \oplus g_1^{[-1,0)} \oplus g_2^{[0,1)} \oplus 0^{[1,\infty)} \tag{9}$$

then

$$\begin{aligned}(f_2 * g_1)(t) = \mathcal{R}_+ \Bigg(& \left(\int_{[\![-1,t+1))} f_2(\tau)\; g_1(t-\tau)\; d\tau\right)^{[\![-2,0))} \\ & \oplus \left(\int_{[\![-1,1))} f_2(\tau)\; g_1(t-\tau)\; d\tau\right)^{[\![0,-1))} \\ & \oplus \left(\int_{[\![t,1))} f_2(\tau)\; g_1(t-\tau)\; d\tau\right)^{[\![-1,1))} \Bigg)(t)\end{aligned} \tag{10}$$

Suppose we wanted to calculate the exact value at $-.5$. The point is in both $[\![-2,0))$ and $[\![-1,1))$ once and in $[\![0,-1))$ with multiplicity negative one. After $+$-reducing we have:

$$\begin{aligned}(f_2 * g_1)(-.5) &= \int_{[\![-1,.5))} f_2(\tau)\; g_1(-.5-\tau)\; d\tau - \int_{[\![-1,1))} f_2(\tau)\; g_1(-.5-\tau)\; d\tau \\ &\quad + \int_{[\![-.5,1))} f_2(\tau)\; g_1(-.5-\tau)\; d\tau \qquad (11) \\ &= \int_{[\![-1,.5))\ominus[\![-1,1))\oplus[\![-.5,1))} f_2(\tau)\; g_1(-.5-\tau)\; d\tau \qquad (12) \\ &= \int_{[\![-.5,.5))} f_2(\tau)\; g_1(-.5-\tau)\; d\tau \qquad (13)\end{aligned}$$

Note that the terms in (11) are not directly evaluable; g_1 may not be well-defined outside of $[-1,0)$. However, since the integrands are all identical, we can collect all the terms and the offending domains cancel.

But there is a potential problem that I've been fudging over when computing $(f_3 * g_1)$ or any interval with infinite bounds. (7) gives the bounds $[\![0,\infty-1))$, $[\![\infty-1,1))$ and $[\![1,\infty+0))$. By treating $\infty - 1 = \infty$, the numbers again cancel so that we have $(f_3 * g_1)(2) = \int_{[\![2,3))} f_3(\tau) g_1(t-\tau)\; d\tau$. I'm not sure if this is *actually* a problem or not. This paper was making it out to be a problem, but that might only be due to the worrying about $b_f - a_f < b_g - a_g$ (which we can ignore). I'd like to avoid breaking it into 12 cases; it doesn't seem like the right way.

Algorithms for symbolic linear convolution:

http://ptolemy.eecs.berkeley.edu/publications/papers/94/symbolic_convolution/asilomar94.pdf