DOMAIN DECOMPOSITION, INTEGRATION, AND
INCLUSION-EXCLUSION
(Thesis format: Monograph)

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Graduate Program in Computer Science

A thesis submitted in partial fulfillment
of the requirements for the degree of
Masters of Science

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Domain Decomposition, Integration, and Inclusion-Exclusion

is accepted in partial fulfillment of the
requirements for the degree of
Masters of Science

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Abstract

Mathematic notation has been dominated by sets and, when repeated elements are required, sequences generally make an appearance. Historical inertia has caused these structures to be used in many situations where they are ill-suited often evidenced by phrases like "without loss of generality", "up to ordering of terms", "up to a sign". However, by tackling these problems instead with more apt data structures, we can eliminate some of these stipulations and more formally reduce symmetric cases in reasoning. In particular, this thesis will deal with *hybrid sets* (that is, signed multisets), as well *hybrid functions* (that is, functions with hybrid sets for their domain) with applications in piecewise functions, integration on manifolds, (...). More than just an aesthetic change, by allowing negative multiplicity (even if it would not make physical sense), we may symblically manipulate structures in ways that might otherwise be cumbersome or inefficient.

Keywords: Hybrid set, Signed multiset, Integration on chains, Inclusion-Exclusion, (...)

Contents

List of Figures

Chapter 1

Introduction

1.1 Motivation

Some data structures in mathematics are more well-loved than others and none more than Cantor's set. The very foundations of mathematics lie in set theory: numbers are defined in terms of sets as are ordered tuples which in turn lead to relations, functions, sequences and from there branches into countless other structures. But this trunk typically omits a satisfactory treatment of several *generalized sets*. For example, it is difficult to begin speaking about *hybrid sets* without immediately punctuating, "that is, multisets with negative multiplicity". It is a statement of progress that multisets have even entered into (relatively) common mathematical parlance.

Still, sequences need no introduction and are generally relied on when a structure allowing repeated elements is needed. The addition of an ordering is often not even needed but "surely it can't hurt?" Consider the *Fundamental Theorem of Arithmetic*:

> "Every positive integer, except 1, is a product of primes." ... "The standard form of
> n is unique; apart from rearrangement of factors, n can be expressed as a product
> of primes in one way only."   (Hardy and Wright 1979, p.2-3)

By recognizing the possibility of rearranging factors, the authors implicitly define type the "product of primes" as a sequence. But for iterated commutative operators (e.g. $\sum, \prod \cap, \cup$), the order of terms is irrelevant. So then, why order terms to begin with? Reisig [15] uses

1

multisets to define relation nets where "... several individuals of some sort do not have to be distinguished" and furthermore "One should not be forced to distinguish individuals if one doesn't wish to. This would lead to overspecification". The same applies here. Secondly, iterated operators over an empty set is simply the respective identity $(\prod_{x \in \emptyset} x = 1)$, and so 1 *is* a product of primes. Despite the empty sequence being just as well-defined as the empty set; it tends to be treated as an aberrant case. Some definitions even disregard the singleton sequence to say, "is prime or the product of primes".

Stripped of these qualifications we are left with simply:

"Every positive integer is the product of a unique multiset of primes."

Although the definition is equivalent, by using appropriate data structures, the result is more elegant and less case-based reasoning for later uses of the definition. In this spirit that we will graft hybrid sets into areas of mathematics where conventional structures don't fit as tightly as we'd like.

1.2   Objectives

This thesis will include and extend the work of [8] on hybrid sets and their applications. In particular, integration is a natural application of signed domains that had not been explored from this perspective. Take the identity:

$$\int_a^b f(x)\, \mathrm{d}x = -\int_b^a f(x)\, \mathrm{d}x \tag{1.1}$$

The domain of integration on the left-side is considered to be the interval $[a, b]$ or in set builder notation, $\{x \in \mathbb{R} \mid a \leq x \leq b\}$. It would follow then that the right hand should have domain $[b, a]$. Under traditional set definitions, this is not be well defined. Using hybrid sets allows us to give meaning to an inverted interval. When generalized to higher dimensions, this goal becomes an attempt to unify the *Lebesgue integral* and *integration of forms*. Such a model

would allow for integation of differential forms over subsets of manifolds.

I don't currently have a set of objectives for the Petri net chapter. Another couple lines will go here once I have a better idea what the chapter will look like.

1.3 Related Work

It is difficult to date the origin of multisets. The term was coined by N.G. de Bruijn in corresponces with Donald Knuth, [13] but thought of as a "collection of objects that may or may not be distinguished" is as old as tally marks. In regards to the generalization to *signed* multisets, Hailperin [12] suggests that Boole's 1854 *Laws of Thought* [7] is a treatise of signed multisets. Whether this was Boole's intent is debated [cite]. Sets with negative membership explicitly began to appear in [21] and were formalized under the name Hybrid sets in Blizard's extensive work with generalized sets [5, 6] Although hybrid set and signed multiset are the most common nomenclature, other names appearing in literature include multiset (specifying positive when for unsigned multisets) [15] and integral multiset [22].

Existing explicit applications of hybrid sets are currently limited. Loeb *et al.* [10, 14] use hybrid sets to generalize several combinatoric identities to negative values. Bailey *et al.* [1] and Banâtre *et al.* [2] have also had success with hybrid sets in chemical programming. Representing a solution is represented as a collection of atoms and molecules, negative multiplicities are treated as "antimatter" . For a deeper overview and systemization of generalized sets, see [17, 18]. Finally, Bartoletti [3, 4] and Schmidt [16] also investigate petri and relation nets (to be examined in chapter 5), albeit not from the perspective of hybrid sets.

1.4 Thesis Outline

In chapter 2, the foundations for hybrid sets and functions with hybrid set domains will be laid. Some immediate applications to piecewise functions will be presented. In chapter 3, the algebra of domains will be more deeply explored with generalized partitions and inclusion-exclusion

and present an method for computing the support of a hybrid set. In chapter 4, hybrid functions will be applied towards integration. Starting from foundations we will use hybrid functions to unify integration of forms with integration on subsets of manifolds and prove Stokes' theorem on the new model. Finally in chapter 5, hybrid sets will be applied to Petri and relation nets with the scope of ...

Chapter 2

Hybrid Set Theory

The motivation behind hybrid sets and functions can be traced to wanting a better approach to piecewise functions. Piecewise functions are enormously useful constructions in many areas (...) The perennial example of a piecewise function is abs : $\mathbb{R} \to \mathbb{R}_+ \cup \{0\}$ given in the form:

$$\text{abs}(x) = \begin{cases} -x & : x < 0 \\ x & : x \geq 0 \end{cases} \tag{2.1}$$

To evaluate abs for an argument x, one must first determine which sub-function to use. If $x < 0$ then the first case is evaluated and abs will return the result of $x \mapsto -x$. Otherwise, if $x \geq 0$ the second case is evaluated and the result of $x \mapsto x$ is returned. Rather than as a condition, we could just as easily think of "$x < 0$" and "$x \geq 0$" as partitions of the real line. Evaluation then occurs by checking whether $x \in \mathbb{R}_+ \cup \{0\}$ or $x \in \mathbb{R}_-$. In general, a piecewise function f will take the form:

$$f(x) = \begin{cases} f_1(x) & : x \in P_1 \\ f_2(x) & : x \in P_2 \\ \vdots & \quad \vdots \\ f_n(x) & : x \in P_n \end{cases} \tag{2.2}$$

5

where the set $\{P_i\}$ forms a partition of the domain of f and for each f_i is defined over all of the corresponding P_i. To formalize this we require the ability to restrict a function's domain and join disjoint pieces together.

Definition Given a function $f : X \to Y$ for any subset of the domain, $Z \subset X$, the *restriction of f to Z* is the function $f|_Z : Z \to Y$, such that $f|_Z(x) = f(x)$ for all $x \in Z$.

Definition Define $①$, the *join* of two functions, f and g by:

$$f ① g = \begin{cases} f(x) & \text{if } g(x) = \bot \\ g(x) & \text{if } f(x) = \bot \\ \bot & \text{otherwise} \end{cases} \tag{2.3}$$

This allows us to re-write our previous definition of (2.2) as:

$$f = f|_{P_1} ① f|_{P_2} ① \dots ① f|_{P_n} \tag{2.4}$$

But we must be careful as this definition is not associative. Given functions f, g, h each restricted to the sets A, B, C. Then for any element in the intersection $x \in A \cap B \cap C$:

$$((f|_A ① g|_B) ① h|_C)(x) = h(x) \tag{2.5}$$

But by rearranging the parentheses we find,

$$(f|_A ① (g|_B ① h|_C))(x) = f(x) \tag{2.6}$$

Other conventions exist, for example *Maple*'s `piecewise(cond_1, f_1, cond_2, f_2, ..., cond_n, f_n, f_otherwise)` effectively uses a short-circuted $①$. Each of the conditionals, `cond_i` are evaluated in order and when one evaluates to true, the corresponding `f_i` is evelated. Finally if all of `cond_i` evaluate to false then `f_otherwise` is used. This approach simply trades associativity for commutivity.

More importantly, when dealing with multiple piecewise functions, expressions cannot be easily simplified without an explosion in terms. For example, given f, g two piecewise functions $f = \left(f_1|_{P_1} \oplus f_2|_{P_2} \oplus f_3|_{P_3} \right)$ and $g = \left(g_1|_{Q_1} \oplus g_2|_{Q_2} \right)$ their sum $(f + g)$ is:

$$
\begin{aligned}
(f + g) = \; & (f_1 + g_1)|_{P_1 \cap Q_1} \;\; \oplus \;\; (f_1 + g_2)|_{P_1 \cap Q_2} \;\; \oplus \;\; (f_1 + g_3)|_{P_1 \cap Q_3} \\
& (f_2 + g_1)|_{P_2 \cap Q_1} \;\; \oplus \;\; (f_2 + g_2)|_{P_2 \cap Q_2} \;\; \oplus \;\; (f_2 + g_3)|_{P_2 \cap Q_3}
\end{aligned}
\tag{2.7}
$$

That is, we first create a common refinement by the pairwise intersection of each P_i with each Q_j. Over each intersection we then take the sum of the respective sub-functions. In general, assuming no degenerate intersections, the sum of an n piece function and m piece function will give a piecewise function with $n \times m$ pieces. To flatten an expression containing multiple piecewise functions we can very quickly find ourselves with an unreasonable number of pieces. If we abandon boolean sets, we can construct common refinements much more efficiently as will be shown.

2.1 Hybrid Sets

We shall consider *hybrid sets*. One can define traditional sets by an indicator function which maps each member element to 1 and each non member to 0. A *multiset* or *bag* extends this by allowing multiple copies of the same element. The indicator function of a multiset would therefore range over $\mathbb{N}_0$, the set of non-negative integers. For example the types of coins in a currency might be represented as a set: {\$0.01, \$0.05, \$0.10, \$0.25, \$1.00}. A multiset might represent the physical coins in my pocket: $2 \times \$0.05, 5 \times \$0.10, 1 \times \$1.00$. Hybrid sets take this one step further allowing for an element to occur *negatively many* times.

Definition Let U be a universe, then any function $U \to \mathbb{Z}$ is called a **hybrid set**.

By universe, we just mean a (traditional) set that *contains everything we may be interested in*. The size of a universe is context dependant but we will rarely be interested in everything

in the universe or its exact size. It is simply a compact way for us to talk about "everything else". This definition of hybrid sets does little good on its own; much of the usefulness of sets is derived from their rich notation. So with that said,

Definition Let H be a hybrid set. Then we say that $H(x)$ is the **multiplicity** of the element x. We write, $x \in^n H$ if $H(x) = n$. Furthermore we will use $x \in H$ to denote $H(x) \neq 0$ (or equivalently, $x \in^n H$ for $n \neq 0$). Conversely, $x \notin H$ denotes $x \in^0 H$ or $H(x) = 0$. The symbol $\emptyset$ will be used to denote the empty hybrid set for which all elements have multiplicity 0. Finally the **support of a hybrid set** is the (non-hybrid) set supp H, where $x \in$ supp H if and only if $x \in H$

We will use the notation:
$$H = \left\{\!\!\left| \; x_1^{m_1}, x_2^{m_2}, \ldots \; \right|\!\!\right\}$$

to describe the hybrid set H where the element x_i has multiplicity m_i. We allow for repetitions in $\{x_i\}$ but interpret the overall multiplicity of an element x_i as the sum of multiplicities among copies. This is, (using Iverson brackets):

$$H(x) = \sum_{x_i \in^{m_i} H} [x = x_i] \, m_i \tag{2.8}$$

For example, $H = \left\{\!\!\left| \; a^1, a^1, b^{-2}, a^3, b^1 \; \right|\!\!\right\} = \left\{\!\!\left| \; a^5, b^{-1} \; \right|\!\!\right\}$. A writing in which $x_i \neq x_j$ for all $i \neq j$ is refered to as a *normalized form* of a hybrid set. For normalized hybrid sets it follows that $H(x_i) = m_i$.

Traditional sets use the operations $\cup$ union, $\cap$ intersection, and $\setminus$ complementation. In the same way a hybrid set is a function $H : U \to \mathbb{Z}$, a set could be considered as function $S : U \to \{0, 1\}$. Then set operations correspond to point-wise OR, AND, and NOT. That is, for two sets A and B, then $(A \cup B)(x) = A(x)$ OR $B(x)$. One could easily extend union and intersection to hybrid sets using point-wise min and max [cite], but it would make more sense to have operations corresponding to primitive operations in $\mathbb{Z}$ instead. Thus we will define $\oplus$,

$\ominus$, and $\otimes$ by point-wise $+$, $-$, and $\times$ respectively.

Definition For any two hybrid sets A and B over a common universe U, we define the operations $\oplus, \ominus, \otimes : \mathbb{Z}^U \times \mathbb{Z}^U \to \mathbb{Z}^U$ such that for all $x \in U$:

$$(A \oplus B)(x) = A(x) + B(x) \tag{2.9}$$

$$(A \ominus B)(x) = A(x) - B(x) \tag{2.10}$$

$$(A \oplus B)(x) = A(x) \cdot B(x) \tag{2.11}$$

We also define, $\ominus A$ as $\emptyset \ominus A$ and for $c \in \mathbb{Z}$:

$$(cA)(x) = c \cdot A(x) \tag{2.12}$$

Definition We say ***A* and *B* are disjoint** if and only if $A \otimes B = \emptyset$

For boolean sets A and B, disjointness would be defined by $A \cap B = \emptyset$. If we consider these boolean sets as simply hybrid sets with multiplicity 0 or 1 then the operations $\cap$ and $\otimes$ identically correspond to elementwise `AND`. From these definitions, we can use hybrid sets to model various objects that would traditionally be described otherwise.

2.1.1   Example: *Rational Arithmetic*

Any positive rational number can be represented as a hybrid set over the set of primes and vice versa (i.e. we have the group isomorphism $(\mathbb{Z}^{\mathbb{P}}, \oplus) \simeq (\mathbb{Q}_+, \cdot)$). For any rational number a/b, both a and b being integers will have a prime decomposition: $a = p_1^{m_1} \cdot p_2^{m_2} \cdot \ldots$ and $b = q_1^{n_1} \cdot q_2^{n_2} \cdot \ldots$. Then there is an isomorphism:

$$f(a/b) = \left\{\!\!\left\{ p_1^{m_1}, p_2^{m_2}, \ldots \right\}\!\!\right\} \ominus \left\{\!\!\left\{ q_1^{n_1}, q_2^{n_2}, \ldots \right\}\!\!\right\} \tag{2.13}$$

Example Concretely, we have:

$$20/9 \cdot 15/8 = \left\{\!\!\left\{\, 5^1, 2^2, 3^{-2} \,\right\}\!\!\right\} \oplus \left\{\!\!\left\{\, 5^1, 3^1, 2^{-3} \,\right\}\!\!\right\} = \left\{\!\!\left\{\, 5^2, 2^{-1}, 3^{-1} \,\right\}\!\!\right\} = 25/6$$

Typically one would need to specify equivalence classes on $\mathbb{Q}$ to consolidate the identity $ca/cb = a/b$, but with hybrid set representation, this identity comes for free. Any common factor between numerator and denominator will cancel result in cancelling multiplicities. Equivalence of rational numbers is identical to equivalence on hybrid sets and a normalized hybrid set For example, $2/4 = \left\{\!\!\left\{\, 2^1, 2^{-2} \,\right\}\!\!\right\}$ which is the un-normalized form of $\left\{\!\!\left\{\, 2^{-1} \,\right\}\!\!\right\} = 1/2$.

One should note that this does not however extend (nicely) for zero and negative $\mathbb{Q}$. Allowing for our hybrid sets to contain -1 would mean we can represent negative rational numbers at the expense of having unique a unique representation. The representations of $1/2 = \left\{\!\!\left\{\, 2^{-1} \,\right\}\!\!\right\}$ and $-1/-2 = \left\{\!\!\left\{\, -1^2, 2^{-1} \,\right\}\!\!\right\}$ no longer agree! Allowing our hybrid sets to contain 0 leads to rational numbers which are not well defined. We would like our rational numbers to be members of $\mathbb{Z} \times \mathbb{Z} - \{0\}$ but our construction does not let us discriminate.

2.1.2   Example: *Rational Polynomials*

Similarly, we can also use hybrid sets to represent monic rational polynomials by encoding the roots and asymptotes. For example:

$$\frac{(x-2)}{(x-1)^2(x+1)} = \left\{\!\!\left\{\, 2^1, 1^{-2}, -1^{-1} \,\right\}\!\!\right\} \tag{2.14}$$

Although neither model is a revolutionary method of looking at rational numbers and polynomials, hopefully they demonstrate that hybrid set are not an arcane construction. Negative multiplicities of elements can arise very naturally in many contexts.

In traditional set theory, a partition of X is a collection of subsets of X such that the subsets are mutually disjoint and their union is equivalent to X. When dealing with hybrid sets we no longer have union but use point-wise sum instead.

Definition A **generalized partition P of a hybrid set $H(x)$** is a sequence of hybrid sets $P = \{P_i\}_{i=1}^{n}$ such that

$$P_1 \oplus P_2 \oplus \ldots \oplus P_n = H \tag{2.15}$$

We say that **P is a strict partition of H** if P_i and P_j are disjoint for all $i \neq j$.

Traditional boolean sets can be trivially converted to hybrid sets. For a set X, this is done simply by taking $H(x) = 1$ if $x \in X$ and $H(x) = 0$ if $x \notin X$. In this way any traditional set partition is a strict, generalized set partition. By principle of inclusion-exclusion,

$$P_i \cup P_j = P_i \oplus P_j \ominus (P_i \cap P_j)$$

Since P_i and P_j are disjoint then we have that $\bigcup_i P_i = \bigoplus_i P_i$ and thus a strict partition. What about the converse, how do generalized partitions relate to partitions? First we must consider that not all hybrid sets can be cast back down to traditional sets.

Definition Given a hybrid set H over universe U, if for all $x \in U$ $H(x) = 1$ or $H(x) = 0$ then we say that **$H(x)$ is reducible**. If H is reducible then we denote the **reduction of H** by $\mathcal{R}(H)$ as the (non-hybrid) set over U with the same membership.

If a hybrid set H is reducible, we still cannot be guaranteed that a generalized partition P of H will be a strict. Consider the reducible hybrid set $[0, 1]$ (that is, $H(x) = 1$ if $0 \leq x \leq 1$ and $H(x) = 0$ otherwise). Then $P = \{[0, 2], \ominus(1, 2]\}$ is a generalized partition of H. A generalized partition of a reducible set is strict if and only if each generalized partition is reducible. Generalized partitions and reducibility will be very useful to us over the next section.

2.2 Hybrid Relations

A function is typically defined as a mapping from elements of one set to another set. We will consider functions which have hybrid sets as their domain (but still map to traditional sets) which we will call *hybrid functions*. But first we must establish *hybrid relations*.

Definition For two sets S and T, a hybrid set over their Cartesian product $S \times T$ is called a **hybrid (binary) relation between S and T**. We denote the set of all such hybrid relations by $\mathbb{Z}^{S \times T}$.

2.2.1 Example: *Algebra of Orderings*?!?!

Consider a set S with some ordering $>$. Then we can define the hybrid relations $[>]$ and $[=]$ as a hybrid relations between S and itself (i.e. elements of $\mathbb{Z}^{S \times S}$). Which we define as,

$$[>] = \left\{\!\!\left\{ (x, y)^1, (y, x)^{-1} : x > y \right\}\!\!\right\} \tag{2.16}$$

$$[=] = \left\{\!\!\left\{ (x, x)^1, : x \in S \right\}\!\!\right\} \tag{2.17}$$

These hybrid relations can then be algebraically manipulated to create other relations on S. For example, we can construct their respective dual relations $[<]$ and $[\neq]$ as well as the non-strict $[\geq]$:

$$[<] = \ominus[>] \tag{2.18}$$
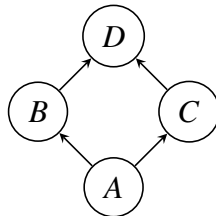
$$[\neq] = (S \times S) \ominus [=] \tag{2.19}$$

$$[\geq] = [=] \oplus [>] \tag{2.20}$$

We can even use supp to define the notion of a total ordering. We say that $\geq$ is a total ordering of S if:

$$\text{supp}[\geq] = S \times S \tag{2.21}$$

Figure 2.1: poset caption

2.3   Hybrid Functions

Definition A **hybrid function from S to T** is a hybrid relation H between S and T such that $(x, y) \in H$ and $(x, z) \in H$ implies $y = z$. We denote the set of all such hybrid functions by $\mathbb{Z}^{S \to T}$.

Although this tells us what *is* and *is not* a hybrid function, it is not the most useful definition to work with. We would like to think of a hybrid function not as a mapping from a hybrid set to a boolean set but as a function between two sets with a multiplicity attached to each mapping. From this perspective, we would generally have a function and hybrid set already in mind. Given a hybrid set H over U and a function $f : B \to S$ be a function where $B \subseteq U$ and S a set. Then we denote by f^H the hybrid function from B to S defined by:

$$f^H := \bigoplus_{x \in B} H(x) \left\{\!\left| (x, f(x))^1 \right|\!\right\} \tag{2.22}$$

There is little literature or established use for hybrid functions; their primary use to us will be as something that we can turn back into traditional functions. One may notice that we have defined hybrid functions by their graph, taking the reduction (if it exists) of a hybrid function one naturally gets the graph of a function.

Definition If H is a reducible hybrid set, then f^H **is a reducible hybrid function**. Additionally, if f^H is reducible, we extend $\mathcal{R}$ by:

$$\mathcal{R}(f^H)(x) = f|_{\mathcal{R}(H)}(x) \tag{2.23}$$

Since $\mathcal{R}(H)$ only exists if $H(x)$ is everywhere 0 or 1; $\mathcal{R}(f^H)$ only makes sense when f^H is reducible. Assuming that we end up with a reducible function, hybrid functions make an excellent primitives to construct piecewise functions. Earlier in (2.3) we defined the *join of functions* to allow us to construct piecewise functions from *restricted functions*. The join of two hybrid functions is quite trivially defined.

Definition The **join**, $f^F \oslash g^G$ of two hybrid functions f^F and g^G is the hybrid relation given by their point-wise sum.

$$f^F \oslash g^G := f^F \oplus g^G \tag{2.24}$$

However, we will immediately dispense with using $\oslash$ altogether and simply use $\oplus$ in order to prevent confusion between hybrid functional join (e.g. $f^F \oplus g^G$) and traditional functional join (e.g. $f|_F \oslash g|_G$). It is important to note that the join operator is closed under hybrid relations but not under hybrid functions. For any two hybrid *functions* the result will be a hybrid *relation* but not necessarily another hybrid function. As with traditional functional join, we must still be wary of overlapping regions but non-disjoint hybrid domains are not nearly as "dangerous". For intersecting regions we do not have to choose between commutivity and associativity, we can have both. In general, all that we can say the result is a hybrid relation but there are many cases where we can guarantee hybrid function status is preserved.

Theorem 2.3.1 *Let A and B be hybrid sets over U and let* $f : U \to S$ *be a function. Then* $f^A \oplus f^B$ *is a hybrid function. Moreover,*

$$f^A \oplus f^B = f^{A \oplus B} \tag{2.25}$$

It should not be a surprise that the same function over different restrictions combine to form a function. Let $(x, f(x)) \in f^A$ and $(y, f(y)) \in B$. Clearly, if $x = y$ then $f(x) = f(y)$. Since f is common between both hybrid functions, there cannot be disagreement among points. Additionally,

$$
\begin{aligned}
f^A \oplus f^B &= \bigoplus_{x \in U} A(x) \left\{\!\!\left| (x, f(x))^1 \right|\!\!\right\} \oplus \bigoplus_{x \in U} B(x) \left\{\!\!\left| (x, f(x))^1 \right|\!\!\right\} \\
&= \bigoplus_{x \in U} (A(x) + B(x)) \left\{\!\!\left| (x, f(x))^1 \right|\!\!\right\} \\
&= \bigoplus_{x \in U} (A \oplus B)(x) \left\{\!\!\left| (x, f(x))^1 \right|\!\!\right\}
\end{aligned}
\tag{2.26}
$$

Inductively, this holds for any number of hybrid sets with a common function. For instance, given any generalized partition $P = P_1 \oplus P_2 \oplus \ldots \oplus P_n$, we have the following equation reminiscent of (2.4):

$$f^P = f^{P_1} \oplus f^{P_2} \oplus \ldots \oplus f^{P_n} \tag{2.27}$$

Joining a function with itself is not the most interesting construction. Piecewise functions are useful for their ability to tie together two *different* functions. If two functions have separate, non-overlapping regions, then our definition is again trivial.

Theorem 2.3.2 *Given two function $f : U \to S$ and $g : U \to S$. The following identity holds if and only if A and B are disjoint:*

$$f^A \oplus g^B = (f \oplus g)^{A \oplus B} \tag{2.28}$$

Notice here the use of $\oplus$ on the right hand side. Here $\oplus$ is traditional (non-hybrid) function join. Since A and B are disjoint, the problems that arose earlier are not an issue. In fact, it doesn't even matter which convention we use. Disjointness is still too strong of a condition for two hybrid functions to be compatible. The join of two non-disjoint functions may still be a hybrid function even if their respective functions do not agree at *all* points. So long as long as they agree on all points in overlapping regions intersection the functions can be safely joined.

Definition For hybrid functions f^A and g^B, $f^A \oplus g^B$ is a hybrid function if and only if for all $x \in \mathrm{supp}(A \otimes B)$, we have $f(x) = g(x)$. We say that f^A **and** g^B **are compatible**.

As with our definition of disjointness, we use the pointwise product $\otimes$ of hybrid sets as an analog for intersection $\cap$ of sets. This definition also generalizes the two cases we have already seen in theorems 2.3.1 and 2.3.2. The hybrid functions f^A and f^B are clearly compatible because f agrees with itself everywhere. For A and B disjoint, f^A and g^B are also compatible since $A \otimes B$ is empty; there are no mutual points for f and g to disagree over.

Compatibility becomes less clear when we begin to consider multiple hybrid functions. For one, compatibility is not associative. Consider the following sequence:

$$(f^H \oplus g^H) \oplus g^{\ominus H} = f^H \oplus (g^H \oplus g^{\ominus H}) = f^H \oplus g^{\emptyset} = f^H \tag{2.29}$$

g^H and $g^{\ominus H}$ are clearly compatible by 2.3.1. The resulting $g^{\emptyset}$ is compatible with *any* hybrid function. Although f^H and g^H could be mutually incompatible; we can only certain that $f^H \oplus g^H$ is a hybrid relation. Regardless of their compatibility, when this is joined with $g^{\ominus H}$, the g terms cancel and we are left with *is* a hybrid function.

2.4 Hybrid Function Fold

Compatibility and reducibility are one way of collapsing a hybrid function to a traditional function. Another approach is to fold or aggregate a hybrid function using some operator. To aid in this we will first introduce some notation for iterated operators.

Definition Let $\star : S \times S \to S$ be an operator on S. Then, for $n > 0$, $n \in \mathbb{Z}$ we use $\star^n : S \times S \to S$ to denote iterated $\star$. So,

$$x \star^n y = (((x \overbrace{\star y) \star y) \star \ldots \star y}^{n \text{ times}})\tag{2.30}$$

If $\star$ has an identity $e_\star$, then we extend $x \star^0 y = x$. If $\star$ is invertible, then we use $\star^{-1}$ to denote its inverse and $\star^{-n}$ for iterated $\star^{-1}$. Finally, we allow $\star^n$ to be a unary operator, which we define by $\star^n x = e_\star \star^n x$.

Assuming $\star^m$ and $\star^n$ are both defined (e.g. if $m, n \leq 0$ then $\star$ must be invertible), we have $(x \star^m y) \star^n y = x \star^{m+n} y$.

For non-associative magmas, it may be of interest to instead define $\star^T$ for some tree T. For example $\star^n$ above is analogous to Haskell's `foldl`. There are applications where `foldr`:

$(x \star (y \star (y \star \ldots \star y)))$ or a balanced expression tree like `foldt` might be desired. Most applications however will be interested in group operators and so we will not explore this.

Definition We say that a hybrid relation $f^A = f_1^{A_1} \oplus f_2^{A_2} \oplus \ldots$ over $T \times S$ is $\star$**-reducible** if $(S, \star)$ is an abelian group or if $(S, \star)$ is an abelian semi-group and f^A is everywhere non-negative. If f^A is $\star$-reducible we define its $\star$**-reduction**, $\mathcal{R}_\star(f^A) : T \to S$ as a (non-hybrid) function given by:

$$\mathcal{R}_\star(f^A)(x) = \left(\star^{A_1(x)} f_1(x) \star^{A_2(x)} f_2(x) \star \ldots \right)\Big|_{\mathrm{supp}A} \tag{2.31}$$

If f^A is reducible then it is trivially $\star$-reducible and we have:

$$\mathcal{R}(f^A) = \mathcal{R}_\star(f^A) \tag{2.32}$$

If a hybrid function is already "flattened", then reducing it will do nothing. So clearly $\mathcal{R}_\star$ is a projection since it is idempotent (i.e. $\mathcal{R}_\star(\mathcal{R}_\star(f^A)) = \mathcal{R}_\star(f^A)$). Moreover,

$$\mathcal{R}_\star(\mathcal{R}_\star(f^F) \oplus \mathcal{R}_\star(g^G)) = \mathcal{R}_\star(f^F \oplus g^G) \tag{2.33}$$

$$\boxed{\mathcal{R}_\times(\mathcal{R}_+ \oplus \mathcal{R}_+)}$$

2.4.1 Example: *Sign function*

One would typically write the sign function out as a piecewise function over 3 regions of the extended real line: $(-\infty, 0)$, $\{0\}$, and $(0, \infty)$. Or alternatively using a +-reduction using 2 pieces: $(-\infty, 0]$ and $[0, \infty)$.

$$\mathrm{sign} = -1^{(-\infty,0)} \oplus 0^{\{0\}} \oplus 1^{(0,\infty)} = \mathcal{R}_+ \left(-1^{(-\infty,0]} \oplus 1^{[0,\infty)} \right) \tag{2.34}$$

Evaluation of the sign at the points 1 or 0 is performed as follows:

$$\mathrm{sign}(1) = \mathcal{R}_+ \left(-1^{(-\infty,0]} \oplus 1^{[0,\infty)} \right)(1) = \mathcal{R}_+ \left(-1^0 \oplus 1^1 \right) = +^0(-1) +^1 (1) = 1$$

$$\text{sign}(0) = \mathcal{R}_+\left(-1^{(-\infty,0]} \oplus 1^{[0,\infty)}\right)(0) = \mathcal{R}_+\left(-1^1 \oplus 1^1\right) = +^1(-1) +^1 (1) = 0$$

Going from 3 regions to 2 may seem a small step. However, consider two piecewise functions with n and m regions respectively. Taking the sum of these two functions would lead to a new piecewise function with $n \cdot m$ regions. We will show that $\star$-reductions can allow us to reduce this to only a linear increase!

2.4.2   Example: *Piecewise functions on generalized partitions*

Let $A_1 = [0, a)$, $A_2 = [0, 1] \setminus A_1$, $B_1 = [0, b)$ and $B_2 = [0, 1] \setminus B_1$. As well as piecewise functions f and g given as:

$$f(x) = f_1^{A_1} \oplus f_2^{A_2} = \begin{cases} f_1(x) & x \in A_1 \\ \\ f_2(x) & x \in A_2 \end{cases} \quad \text{and} \quad g(x) = g_1^{B_1} \oplus g_2^{B_2} = \begin{cases} g_1(x) & x \in B_1 \\ \\ g_2(x) & x \in B_2 \end{cases}$$

If one were interested in computing the piecewise function $(f+g)$, the naïve method would be to take the pairwise sum of each sub-function. Then restrict each sum to the intersection of respective partitions as below:

$$(f + g)(x) = (f_1 + g_1)^{A_1 \cap B_1} \oplus (f_1 + g_2)^{A_1 \cap B_2} \oplus (f_2 + g_1)^{A_2 \cap B_1} \oplus (f_2 + g_2)^{A_2 \cap B_2} \tag{2.35}$$

We will take another approach. First, we can partition $[0, 1]$ into the generalized partition A_1, $B_1 \ominus A_1$, B_2. The source of this particular partition will remain mysterious for now but observe that we can still construct the partitions: $B_1 = (B_1 \ominus A_1) \oplus A_1$ and $A_2 = (B_1 \ominus A_1) \oplus B_2$. And so we can represent f and g from above with a common partition by using:

$$f = f_1^{A_1} \oplus f_2^{(B_1 \ominus A_1) \oplus B_2} = f_1^{A_1} \oplus f_2^{B_1 \ominus A_1} \oplus f_2^{B_2} \tag{2.36}$$

$$g = g_1^{A_1 \oplus (B_1 \ominus A_1)} \oplus g_2^{B_2} = g_1^{A_1} \oplus g_1^{B_1 \ominus A_1} \oplus g_2^{B_2} \tag{2.37}$$

Since we have a common partition we can avoid computing pairwise intersections alto-gether and simply add each sub-function to the corresponding sub-function which shares a partition. Since $\{A_1, B_1, (B_1 \ominus A_1)\}$ is a generalized partition, we will need to flatten the ex-pression back down to get a traditional function. We use $\mathcal{R}_+$ for this so that negative regions properly cancel.

$$(f + g)(x) = \mathcal{R}_+ \left((f_1 + g_1)^{A_1} \oplus (f_2 + g_1)^{B_1 \ominus A_1} \oplus (f_2 + g_2)^{B_2} \right) \tag{2.38}$$

This equation holds regardless of the relative ordering of a and b. Suppose $x \in A_1 \cap B_1$ Then we have $A_1(x) = 1$ and $(B_1 \ominus A_1)(x) = B_2(x) = 0$. And so $(f + g)(x) = (f_1 + g_1)(x)$. Similarly, if $x \in B_1 \cap A_2$ or $x \in A_2 \cap B_2$ then only $(B_1 \ominus A_1)(x)$ or $B_2(x)$ will respectively be non-zero. However, if $x \in B_2 \cap A_1$ then we have $A_1(x) = 1$, $(B_1 \ominus A_1)(x) = -1$ and $B_2(x) = 1$. Simplifying this expression yields:

$$(f + g) = +^1(f_1 + g_1) +^{-1} (f_2 + g_1) +^1 (f_2 + g_2) = (f_1 + g_2) \tag{2.39}$$

In the above example only three regions could simultaneously exist. If $a < b$ then $A_1 \cap B_2 = \emptyset$ but if $b < a$ then $A_2 \cap B_1 = \emptyset$. Although it might seem that the three terms are a result of three regions, in the general case where A_1 and B_1 could be arbitrary subsets, we still only have three terms. We will even extend this to any generalized partition. First we will formalize some ideas we have already seen.

Definition A **refinement** of a generalized partition $P = \{P_i\}_{i \in I}$ is another generalized partition $Q = \{Q_j\}_{j \in J}$ such that, for every P_i in P there is a subset of Q: $\{Q_j\}_{j \in J_i}$, $J_i \subseteq J$ such that for some integers $\{a_{i,j}\}_{j \in J_i}$

$$P_i = \bigoplus_{j \in J_i} a_{i,j} Q_j \tag{2.40}$$

Given a *set* of generalized partitions a **common refinement** is a generalized partition which is a refinement of every partition in the set. A refinement Q of P is **strict** if $\text{supp}(Q) = \text{supp}(P)$.

In our previous example we used $\{A_1, B_1 \ominus A_1, B_2\}$ which was a common refinement of both $\{A_1, A_2\}$ and $\{B_1, B_2\}$. A_1 and B_2 have trivial representations while $A_2 = (B_1 \ominus A_1) \oplus B_2$ and $B_1 = A_1 \oplus (B_1 \ominus A_1)$. Another common refinement would be the trivial $\{A_1, B_1, A_2, B_2\}$ This is less preferable due to not only containing 4 regions instead of 3 but also the pointwise sum is $[0, 1]^2$ instead of the reducible $[0, 1]$.

We can now formally phrase our problem. Let $A = \{A_i\}_{i \in [n]}$ and $B = \{B_j\}_{j \in [m]}$ be two generalized partitions of a hybrid set U (where $[n] = \{1, \dots, n\}$ for $n \in \mathbb{N}$). We wish to find a generalized partition C of U which is a *common*, *strict* refinement of A and B and has minimal cardinality. These conditions can be summarized into the following linear system of $n + m + 1$ simultaneous equations:

$$U = \bigoplus_i C_i \tag{2.41}$$

$$\forall i \in [n] : A_i = \bigoplus_j a_{i,j} C_j \tag{2.42}$$

$$\forall j \in [m] : B_j = \bigoplus_i b_{i,j} C_i \tag{2.43}$$

for some integers $a_{i,j}$ and $b_{i,j}$. Since we know that A and B are each partitions of U, only $n + m - 1$ can be independent. If there are additional dependencies betwen A and B then this can be even lower. Expressed as a linear system we have:

$$M \cdot \begin{pmatrix} C_1 \\ \vdots \\ C_{n+m-1} \end{pmatrix} = \begin{pmatrix} U \\ A_1 \\ \vdots \\ A_{n-1} \\ B_1 \\ \vdots \\ B_{m-1} \end{pmatrix} \quad \text{where} \quad M = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_{1,1} & a_{1,2} & \cdots & a_{1,n+m-1} \\ \vdots & & & \vdots \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n+m-1} \\ b_{1,1} & b_{1,2} & \cdots & b_{1,n+m-1} \\ \vdots & & & \vdots \\ b_{m-1,1} & b_{m-1,2} & \cdots & b_{m-1,n+m-1} \end{pmatrix} \tag{2.44}$$

By definition, M is an integer matrix. But we are actually more interested in its inverse M^{-1} as this will give us values for C_i relative to $(U, A_1, \dots, A_{m-1}, B_1, \dots, B_{m-1})$. To stay in the realm

of hybrid sets, we would also like to enforce that M^{-1} is also an integer matrix. Assuming this, then the determinant of M must be ± 1. Further restricting ourselves to upper triangular matrices we can choose M to be all 1's along the diagonal as well as the top row. Which has the following inverse:

$$
\begin{pmatrix}
1 & \cdots & \cdots & \cdots & 1 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}^{-1}
=
\begin{pmatrix}
1 & -1 & \cdots & \cdots & -1 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix}
\tag{2.45}
$$

Thus we find that for the partitions $A = \{A_i\}_{i \in [n]}$ and $B = \{B_j\}_{j \in [m]}$ we can *always* use the strict, common and minimal refinement:

$$
\left\{ (U \ominus A_1 \ominus \ldots \ominus A_{n-1} \ominus B_1 \ominus \ldots \ominus B_{n-1}), \quad A_1, A_2, \ldots, A_{n-1}, \quad B_1, B_2, \ldots, B_{m-1} \right\}
\tag{2.46}
$$

Finally, to generalize our example from earlier, let $f = f_1^{P_1} \oplus f_2^{P_2} \oplus f_n^{P_n}$ and $g = g_1^{Q_1} \oplus \ldots \oplus g_m^{Q_m}$ be two piecewise functions over a common domain U. We can compute $f \star g$ by:

$$
\begin{aligned}
f \star g = \ \mathcal{R}_\star \Big(& (f_1 \star g_m)^{P_1} \oplus \ldots \oplus (f_{n-1} \star g_m)^{P_{n-1}} \\
& \oplus (f_n \star g_1)^{Q_1} \oplus \ldots \oplus (f_n \star g_{m-1})^{Q_{m-1}} \\
& \oplus (f_n \star g_n)^{U \ominus (P_1 \oplus \ldots \oplus Q_{n-1} \oplus Q_1 \oplus \ldots \oplus Q_{m-1})} \Big)
\end{aligned}
\tag{2.47}
$$

This is not a unique formulation. Any upper triangular matrix containing only $1, 0$ and -1 and all 1's in the first row would also be acceptable. Each would lead to slightly different forms of (2.47). This particular choice is merely one of the simplest options. Certain applications could lead to other choices of M for reasons of memory efficiency, but it is not clear this is needed.

2.5   Pseudo-functions

One last detail remains to be settled, the sub-functions f_i and g_j may not be defined outside of P_i and Q_j respectively. Consider the equation in (2.47), evaluated at $x \in P_1 \cap Q_1$. Then we have the following terms with non-zero multiplicity.

$$(f \star g)(x) = \mathcal{R}_\star \left((f_1(x) \star g_m(x))^1 \oplus (f_n(x) \star g_1(x))^1 \oplus (f_n(x) \star g_m(x))^{-1} \right) \qquad (2.48)$$

We would like for the $g_m(x)$ in the first term to cancel with the $g_m(x)$ in the third. If q_m (and f_n) are defined and finite at all points in U, then there is no problem. However if $g_m(x) = \infty$, then we need to make cumbersome arguments for: $\infty + y - \infty = y$. To resolve this, we use a lambda-lifting trick to have a hybrid relation over the domain and *the function itself* rather than the domain and the image implied by the function.

Definition Using the same notation as in our definition from (2.22), we define a pseudo-function $\tilde{f}^A$ as:

$$\tilde{f}^A = \bigoplus_{x \in B} A(x) \left\{\!\left| (x, f)^1 \right|\!\right\} \qquad (2.49)$$

One should notice the similarity between (2.49) and (2.22). The difference is that we have replaced $(x, f(x))$ with the unevaluated (x, f). This formally makes $\tilde{f}^A$ a hybrid relation over $U \times (U \to S)$ as opposed to a hybrid function over $U \times S$. To evaluate $\tilde{f}^A$ we map back to f^A and evaluate that. This mapping between $(x, f(x))$ and (x, f) is very natural and we will perform it unceremoniously, often using f^A and $\tilde{f}^A$ interchangeably. Properties of hybrid functions such as compatibility and reducibility will be lifted to hybrid pseudo-functions by this as well.

2.5.1   Example: *Piecewise functions revisited*

We will repeat the example from 2.4.2 but concretely using the following function:

$$f(x) = \begin{cases} (2 - x^2) & -1 \le x \le 1 \\[2mm] 1/x^2 & \text{otherwise} \end{cases} \tag{2.50}$$

Graphically, this is a mundane-looking bell-shaped curve shown in black in Figure 2.2. Although f is defined for all of $\mathbb{R}$, this is not the case for its sub-functions. The plots in red and blue show the behaviour of $(2 - x^2)$ and $1/x^2$ outside of their defined ranges in f.

Figure 2.2: A piecewise rational function is shown in black. The plots in red and blue are continuations of $(2 - x^2)$ and $1/x^2$ respectively.
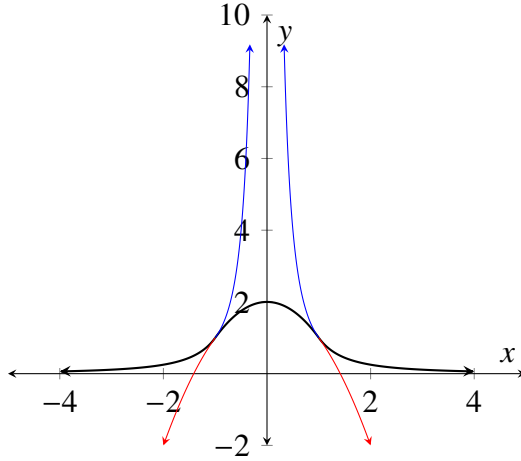
Represented as a hybrid pseudo-functions we have $\tilde{f}$ as:

$$\tilde{f} = \tilde{f_1}^{A_1} \oplus \tilde{f_2}^{A_2} = (x \mapsto 2 - x^2)^{[-1,1]} \oplus (x \mapsto 1/x^2)^{(-\infty,-1)\oplus(1,\infty)} \tag{2.51}$$

which we are interested in multiplying by the unevaluated absolute value function:

$$\tilde{g} = \tilde{\text{abs}} = (x \mapsto x)^{[0,\infty)} \oplus (x \mapsto -x)^{(-\infty,0)} \tag{2.52}$$

First we must find a minimal common refinement. Using the formula in (2.46), we decide

to use the partition:

$$P_1 = [-1, 1] \tag{2.53}$$

$$P_2 = [0, \infty) \tag{2.54}$$

$$P_3 = \mathbb{R} \ominus [-1, 1] \ominus [0, \infty) \tag{2.55}$$

where P_3 could also be simplified into disjoint regions by $P_3 = (-\infty, -1) \ominus [0, 1]$. Thus we have the expression

$$\begin{aligned}
(\tilde{f} \times \tilde{g}) = \mathcal{R}_\times \Big(&\big((x \mapsto 2 - x^2) \times (x \mapsto -x)\big)^{P_1} \\
&\oplus \big((x \mapsto 1/x^2) \times (x \mapsto -x)\big)^{P_2} \\
&\oplus \big((x \mapsto 1/x^2) \times (x \mapsto -x)\big)^{P_3} \Big)
\end{aligned} \tag{2.56}$$

If we use pseudo-functions then we assign multiplicities to the still unevaluated sub-functions. So, for example to evaluate $(f \times g)$ at 0, we would evaluate each of P_1, P_2 and P_3 at 0 and find:

$$\begin{aligned}
(\tilde{f} \times \tilde{g})(0) = \mathcal{R}_\times \Big(&\big((x \mapsto 2 - x^2) \times (x \mapsto -x)\big)^{1} \\
&\oplus \big((x \mapsto 1/x^2) \times (x \mapsto -x)\big)^{1} \\
&\oplus \big((x \mapsto 1/x^2) \times (x \mapsto -x)\big)^{-1} \Big)(0)
\end{aligned} \tag{2.57}$$

Once we have this, we can then evaluate the $\times$-reduction on the still unevaluated functions. Clearly $x \mapsto -x$ occurs with cancelling signs as does $x \mapsto 1/x^2$. This leaves us with the product of two unevaluated functions:

$$(\tilde{f} \times \tilde{g})(0) = \big((x \mapsto 2 - x^2) \times (x \mapsto x)\big)(0) \tag{2.58}$$

After these cancellations occur, then it is finally safe to evaluate at a point and we find

$(\tilde{f} \times \tilde{g})(0) = 0$ as expected. To contrast this, if one were to attempt to evaluate a the (non-pseudo) hybrid function version of $\tilde{f} \times \tilde{g}$, instead of (2.57) one would have the expression:

$$(f \times g)(0) = \mathcal{R}_{\times} \left((2 \times 0)^1 \oplus (\text{``}\infty\text{''} \cdot 0)^1 \oplus (\text{``}\infty\text{''} \cdot 0)^{-1} \right) \tag{2.59}$$

Simplifying the $\times$-reduction leads to both ∞/∞ and $0/0$. Although they *should* obviously cancel, the behaviour is not well defined.

Chapter 3

Symbolic Linear Algebra

In mathematics literature, is common practice to use ellipses: "..." in matrices. For example:

$$
\begin{bmatrix}
A_{1,1} & A_{1,2} & \ldots & A_{1,n} \\
0 & A_{2,2} & \ldots & A_{2,n} \\
\vdots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & A_{n,n}
\end{bmatrix}
\tag{3.1}
$$

and symbolic blocks as in

$$
\left[
\begin{array}{c|c}
A & B \\
\hline
C & D
\end{array}
\right]
\tag{3.2}
$$

where A, B, C, D are sub matrices.

Computer algebra has unsatisfactory structures to represent these

more

3.1 Oriented Intervals

Definition Given a totally ordered set $(X, \leq)$ *(and with an implied strict ordering <)*, for any $a, b \in X$, an **interval between *a* and *b*** is the set of elements in X between a and b, up to inclusion of a and b themselves. Formally:

$$[a, b] = \{x \in X \mid a \leq x \leq b\} \tag{3.3}$$

$$[a, b) = \{x \in X \mid a \leq x < b\} \tag{3.4}$$

$$(a, b] = \{x \in X \mid a < x \leq b\} \tag{3.5}$$

$$(a, b) = \{x \in X \mid a < x < b\} \tag{3.6}$$

It should be noted that for $b < a$, $[b, a]$ is the empty set. Also, the interval $[a, a]$ contains a single point while (a, a), $(a, a]$, and $[a, a)$ are all empty. As intervals are simply sets, they can naturally be interpreted as hybrid sets. If $a \leq b \leq c$, for intervals $[a, b)$ and $[b, c)$ using the hybrid set operator $\oplus$, one has $[a, b) \oplus [b, c) = [a, c)$ In this case, $\oplus$ behaves like concatenation but this is not always true. When $a \leq c \leq b$ then $[a, b) \oplus [b, c) = [a, b)$. When working with intervals, a case-based approach to consider relative ordering of endpoints easily becomes quite cumbersome. Thus we turn to oriented intervals.

1 more line

Definition We define **oriented intervals** with $a, b \in X$, where X is a totally ordered set, using hybrid set point-wise subtraction as follows:

$$[\![a, b)\!) = [a, b) \ominus [b, a) \tag{3.7}$$

$$(\!(a, b]\!] = (a, b] \ominus (b, a] \tag{3.8}$$

$$[\![a, b]\!] = [a, b] \ominus (b, a) \tag{3.9}$$

$$(\!(a, b)\!) = (a, b) \ominus [b, a] \tag{3.10}$$

For any choice of distinct a and b, exactly one term will be empty. Unlike traditional inter-
views where $[a, b)$ would be empty if $b < a$, the oriented interval $[\![a, b)\!)$ simply has elements
with negative multiplicity. Several results follow immediately from this definition.

Theorem 3.1.1 *For all $a, b, c \in \mathbb{R}$,*

$$[\![a, b)\!) = \ominus[\![b, a)\!) \tag{3.11}$$

$$(\!(a, b]\!] = \ominus(\!(b, a]\!] \tag{3.12}$$

$$[\![a, b]\!] = \ominus(\!(a, b)\!) \tag{3.13}$$

$$(\!(a, b)\!) = \ominus[\![a, b]\!] \tag{3.14}$$

We should make a note here how oriented intervals behave when $a = b$. Like their unori-
ented analogues, the oriented intervals $[\![a, a)\!)$ and $(\!(a, a]\!]$ are both empty while $[\![a, a]\!]$ contains
the point at a (with multiplicity 1). However, unlike traditional intervals $(\!(a, a)\!)$ is *not* empty
but rather, $(\!(a, a)\!) = \ominus[\![a, a]\!] = \left\{\!\left| \, a^{-1} \, \right|\!\right\}$.

> more

Theorem 3.1.2 *For all $a, b, c \in \mathbb{R}$ (regardless of relative ordering),*

$$[\![a, b)\!) \oplus [\![b, c)\!) = [\![a, c)\!) \tag{3.15}$$

Proof $[\![a, b)\!) \oplus [\![b, c)\!)$

   $= ([a, b) \ominus [b, a)) \oplus ([b, c) \ominus [c, b))$

   $= ([a, b) \oplus [b, c)) \ominus ([c, b) \oplus [b, a))$

   If $a \geq c$ then $[c, a) = \emptyset$ and so $[\![a, c)\!) = [a, c)$.

Case 1: $a \leq b \leq c$ then $[c, b) = [b, a) = \emptyset$ and $[a, b) \oplus [b, c) = [a, c)$

Case 2: $b \leq a \leq c$ then $[b, c) \ominus [b, a) = [b, a) \oplus [a, c) \ominus [b, a) = [a, c)$

Case 3: $a \leq c \leq b$ then $[a, b) \ominus [c, b) = ([a, c) \oplus [c, b)) \ominus [c, b) = [a, c)$

Similar arguments will show that when $c \geq a$, that $[\![a, b)\!) \oplus [\![b, c)\!) = \ominus[a, c)$.

This sort of reasoning is routine but a constant annoyance when dealing with intervals and is exactly the reason we want to be working with oriented intervals. Many similar formulations such as $[\![a, b]\!] \oplus (\!(b, c)\!) = [\![a, c)\!)$ are also valid for any ordering of a, b, c. We will not enumerate all possible cases here.

Note about partitions

3.2   Symbolic Vectors

Vectors as hybrid functions

We will use the following n-dimensional vectors as a running example in this section:

$$U^T = [u_1, u_2, \ldots, u_k, u'_1, u'_2, \ldots, u_{n-k}] \tag{3.16}$$

$$V^T = [v_1, v_2, \ldots, v_\ell, v'_1, v'_2, \ldots, v_{n-\ell}] \tag{3.17}$$

Using intervals, these vectors can be represented by hybrid functions over their indices.

For example

$$U^T = (i \mapsto u_i)^{[\![1,k]\!]} \oplus (i \mapsto u_{i-k})^{(\!(k,n]\!]} \tag{3.18}$$

$$V^T = (i \mapsto v_i)^{[\![1,\ell]\!]} \oplus (i \mapsto v_{i-\ell})^{(\!(\ell,n]\!]} \tag{3.19}$$

Although for clarity and succinctness we will use (u_i) instead of $(i \mapsto u_i)$.

However we are more interested in performing arithmetic with these vectors.

3.2.1   Vector Addition

Consider pointwise vector addition $U^T + V^T$:

$$U^T + V^T = \left((u_i)^{[\![1,k]\!]} \oplus (u'_{i-k})^{(\!(k,n]\!]}\right) \oplus^+ \left((v_i)^{[\![1,\ell]\!]} \oplus (v'_{i-\ell})^{(\!(\ell,n]\!]}\right) \tag{3.20}$$

$$= \left((u_i)^{[\![1,k]\!]} \oplus (u'_{i-k})^{(\!(k,\ell]\!]} \oplus (u'_{i-k})^{(\!(\ell,n]\!]}\right) \oplus^+ \left((v_i)^{[\![1,k]\!]} \oplus (v_i)^{(\!(k,\ell]\!]} \oplus (v'_{i-\ell})^{(\!(\ell,n]\!]}\right) \tag{3.21}$$

$$= \left((u_i + v_i)^{[\![1,k]\!]} \oplus^+ (u'_{i-k} + v_i)^{(\!(k,\ell]\!]} \oplus^+ (u'_{i-k} + v'_{i-\ell})^{(\!(\ell,n]\!]}\right) \tag{3.22}$$

This formulation is not unique.

The choice to partition $[\![1,n]\!]$ into $[\![1,k]\!] \oplus (\!(k,\ell]\!] \oplus (\!(\ell,n]\!]$ was arbitrary.

We can just as easily partition $[\![1,n]\!]$ into $[\![1,\ell]\!] \oplus (\!(\ell,k]\!] \oplus (\!(k,n]\!]$ to get the equivalent

expression:

$$U^T + V^T = \left((u_i + v_i)^{[\![1,\ell]\!]} \oplus^+ (u_i + v'_{i-\ell})^{(\!(\ell,k]\!]} \oplus^+ (u'_{i-k} + v'_{i-\ell})^{(\!(k,n]\!]}\right) \tag{3.23}$$

We must be careful while evaluating these expressions to not forget that $(u'_{i-k} + v_i)$ is actually shorthand for the function:

$$i \mapsto u'_{i-k} + v_i$$

For example, consider the concrete example where $n = 5$, $k = 4$ and $\ell = 1$ so that $U^T = [u_1, u_2, u_3, u_4, u'_1]$ and $V^T = [v_1, v'_1, v'_2, v'_3, v'_4]$.

We will also only assume that the functions u_i, u'_i, v_i and v'_i are defined only on the intervals in which they appear (e.g. u_5 is undefined, as is v'_1).

Then the expression in (3.19) becomes:

$$(u_i + v_i)^{[\![1,4]\!]} \oplus^+ (u'_{i-4} + v_i)^{(\!(4,1]\!]} \oplus^+ (u'_{i-4} + v'_{i-1})^{(\!(1,5]\!]} \tag{3.24}$$

None of the individual subterms cannot be evaluated directly.

In the first term, v_i is not totally defined over the interval $[\![1, 4]\!]$.

In the third term, on the interval $(\!(1, 5]\!]$, u'_{i-4} would even evaluated on negative indices.

However, these unevaluable terms also appear in the middle term however the interval $(\!(4, 1]\!]$ is a negatively oriented interval and the offending points cancel!

3.2.2   Inner Product

The inner product or dot product of two vectors is given by:

$$A \cdot B = A^T B = \sum_i A_i B_i \tag{3.25}$$

Returning to the running example of U^T and V^T, as defined in (3.13) and (3.14) respectively, we will consider $U^T \cdot V^T$.

Definition Let $X = \left\{\!\!\left| x_1^{m_1}, x_2^{m_2}, \ldots, x_n^{m_n} \right|\!\!\right\}$ be a hybrid set with elements x_i in a $\mathbb{Z}$-module. Given a hybrid function over X, f^X, we define the **sum over f^X**, denoted with $\sum$, as

$$\sum \left(f^X\right) := \sum_{i=1}^{n} (m_i \cdot f(x_i)) \tag{3.26}$$

The **product over f^X**, denoted with $\prod$ is defined similarly.

Then the dot product of U^T and V^T becomes very familiar:

$$U^T \cdot V^T = \sum \left((u_i v_i)^{[\![1,k]\!]} \; \mathbb{O}^{\times} \; (u'_{i-k} v_i)^{(\!(k,\ell]\!]} \; \mathbb{O}^{\times} \; (u'_{i-k} v'_{i-\ell})^{(\!(\ell,n]\!]} \right) \tag{3.27}$$

The inner expression is identical to $U^T + V^T$ except for a replacement of + with $\times$.

3.2.3   Outer Product

While the inner product took two n-vectors and returned a single number, the outer product would take those two n-vectors and return an $n \times n$ matrix. Alternatively, one could think of U^T and V^T as $1 \times n$ matrices. Then the inner product is the 1×1 matrix given by $U^T \cdot V$ while the outer product is given by $U \cdot V^T$. Formally we define:

Definition Let $A^T = [a_1, a_2, \ldots, a_n]$ and $B^T = [b_1, b_2, \ldots, b_m]$, then the **outer product** (or *tensor product*) $\boxtimes$ is given by:

$$A \boxtimes B = A\,B^T = \begin{bmatrix} a_1 b_1 & \ldots & a_1 b_m \\ \vdots & \ddots & \vdots \\ a_n b_1 & \ldots & a_n b_m \end{bmatrix} \tag{3.28}$$

Hybrid sets won't allow us to do anything differently.

But it will provide a nice lead in to our notation for matrix multiplication.

3.3   Abstract Matrices

It is common practice in mathematics to represent matrices symbolically with sub-matrices such as:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{3.29}$$

If A is an $n \times n$ matrix then A_1, A_2, A_3, A_4 are not entries but $(k \times \ell)$, $(n - k \times \ell)$, $(k \times n - \ell)$ and $(n - k \times n - \ell)$ matrices respectively. Ellipses are also routinely used for interpolating over regions of a matrix, as in:

$$M = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ & \ddots & \vdots \\ 0 & & x_{nn} \end{bmatrix} \tag{3.30}$$

3.3.1   Matrix Addition

3.3.2   Matrix Multiplication

Chapter 4

Integration I

Given a function f with real variable x and an interval $[a, b)$ on the (extended) real line, a traditional **definite integral** would be of the form:

$$\int_a^b f(x)\, dx \quad \text{or} \quad \int_{[a,b)} f(x)\, dx$$

Which we interpret as the signed area bounded by f between $x = a$ and $x = b$. However, defining this definite integral using (unoriented) intervals like this is a bit of a misnomer. In the case where $a \geq b$ it is typical to use the identity:

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx \tag{4.1}$$

to evaluate the integral. But, as we saw in the previous chapter, when $a \geq b$, the interval $[a, b)$ is the empty set, hence we *cannot* make the similar translation to

$$\int_{[a,b)} f(x)\, dx = -\int_{[b,a)} f(x)\, dx \tag{4.2}$$

Using sets as domains of integration like this generally appears in context of Lebesgue integrals, but Riemann integrals (which generally use $\int_a^b f(x)\, dx$ instead) often just hide their

mis-use of intervals. For example, it is typically glossed over that when doing the formal Riemann sum for an integral $\int_a^b f(x)\ dx$, one would use the tagged partition given as a series x_i such that $a = x_1 < x_2 < ... < x_n = b$. (citations needed)

Another advantage to using oriented sets is a more natural language for manipulating domains of integration than sets. There is no common understanding of the sum of two sets, but since we have the point-wise sum $\oplus$ for hybrid sets, we simply say that the integral operator is *bi-linear*. By this we mean, it is linear with respect to both operands: the function it is integrating over:

$$\int_X f(x) + g(x)\ dx = \int_X f(x)\ dx + \int_X g(x)\ dx \tag{4.3}$$

and the domain of integration:

$$\int_{X \oplus Y} f(x)\ dx = \int_X f(x)\ dx + \int_Y f(x)\ dx \tag{4.4}$$

By definition, this immediately gives way to the very useful identities:

$$\int_{[\![a,c)\!)} f(x)\ dx = \int_{[\![a,b)\!)} f(x)\ dx + \int_{[\![b,c)\!)} f(x)\ dx \tag{4.5}$$

$$\int_{[\![a,b)\!)} f(x)\ dx = \int_{\ominus[\![b,a)\!)} f(x)\ dx = - \int_{[\![b,a)\!)} f(x)\ dx \tag{4.6}$$

for all a, b and c.

In one dimension, many of these changes may seem trivial advances but i n higher dimensions, the oriented and measure-theoretic approaches diverge. [19] Using hybrid sets as domains of integration allow us to use the best features of both. In this chapter we will present an introduce integration over oriented intervals and generalize to oriented n-cubes in higher dimensions. We will also explore the boundary operator and the cubic homology formed by n-cubes. This will provide a base for the following chapter to investigate the cubic singular homology, integration of forms and Stokes' theorem. In Chapter 6, we will introduce the oriented Lebesgue integral.

4.1   Higher dimension intervals

For now, we will concern ourselves only with oriented, n-dimensional, axis-aligned rectangles in $\mathbb{R}^n$. In one dimension, the previously discussed oriented intervals cover most of the "obvious shapes" one would be interested in. Moving to two dimensions, there are many more "obvious shapes" to consider but we will temporarily ignore triangles, circles and even rectangles that are tilted. We could also use triangles instead of rectangles as our primitive of choice. This would generalize to n-simplexes in higher dimensions. Since n-simplexes and n-cubes end up b[9] But, first we must introduce some notation to describe these and higher dimension rectangles.

At the moment the only rectangles we have defined are the one-dimensional "oriented interval". Hence we will also refer to this as a 1-cube. We construct higher dimensional n rectangles using the Cartesian product.

Definition Let $X = \left\{\!\left\{\, x_1^{m_1}, ..., x_k^{m_k} \,\right\}\!\right\}$ and $Y = \left\{\!\left\{\, y_1^{n_1}, ..., y_\ell^{n_\ell} \,\right\}\!\right\}$ be hybrid sets. We define the **Cartesian product of hybrid sets X and Y**, denoted with $\times$ operator as:

$$X \times Y = \left\{\!\left\{\, (x, y)^{m \cdot n} \ : \ x^m \in X, y^n \in Y \,\right\}\!\right\} \tag{4.7}$$

If $[\![a, b]\!]$ and $[\![c, d]\!]$ are both positively oriented 1-rectangles then their Cartesian product is shown in Figure 4.1 is clearly a two dimensional rectangle or *2-rectangle*. Taking the Cartesian product of a 2-rectangle and 1-rectangle gives a 3-rectangle in $\mathbb{R}^3$. We should note here that we do not distinguish between $((x, y), z)$ and $(x, (y, z))$ but rather we treat both as different names for the ordered triple (x, y, z). We similarly associate parentheses in higher dimensions as well.

Theorem 4.1.1 *The Cartesian product of a k-rectangle in $\mathbb{R}^m$ (where, $k \leq m$) and ℓ-rectangle in $\mathbb{R}^n$ (again, $\ell \leq n$) is a $(k + \ell)$-rectangle in $\mathbb{R}^{m+n}$.*

For completeness we will also define a 0-rectangle as a hybrid set containing a single point with multiplicity 1 or -1. Firstly this allows us to embed k-rectangles in $\mathbb{R}^n$. For example $[\![a, b]\!] \times [\![c, d]\!] \times \left\{\!\left\{\, e^1 \,\right\}\!\right\}$ is the product of two 1-rectangles and a 0-rectangle (all over $\mathbb{R}$) and so

Figure 4.1: The Cartesian product of two positively oriented 1-rectangles $[\![a,b]\!]$ and $[\![c,d]\!]$ is a positively oriented 2-rectangle.
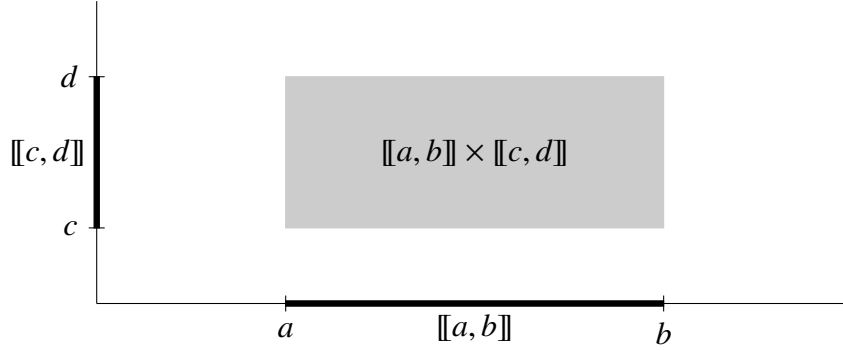
it is a 2-rectangle over $\mathbb{R}^3$. Specifically, it is the 2-rectangle $[\![a,b]\!] \times [\![c,d]\!]$ on the plane $z = 3$. This also illustrates the principle that given a k-rectangle in $\mathbb{R}^n$ where $n > k$ we can always find a k dimensional subspace which also contains the rectangle. We will re-use the interval notation from earlier although one should be careful to "type-check" while interpreting. When a and b are real numbers then we continue to use the definition $[\![a,b]\!] = [a,b) \ominus [b,a)$. However, when $\boldsymbol{a}$ and $\boldsymbol{b}$ are n-tuples (for example, coordinates in $\mathbb{R}^n$ then this is *not* the oriented line interval $[\boldsymbol{a},\boldsymbol{b}) \ominus [\boldsymbol{b},\boldsymbol{a})$.

Definition Let $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ and $\boldsymbol{b} = (b_1, b_2, \ldots, b_n)$ be ordered n-tuples then we use the notation:

$$[\![\boldsymbol{a},\boldsymbol{b}]\!] = [\![a_1, b_1]\!] \times [\![a_2, b_2]\!] \times \ldots \times [\![a_n, b_n]\!] \tag{4.8}$$

The dimension of $[\![\boldsymbol{a},\boldsymbol{b}]\!]$ is equal to the number of indices where a_i and b_i are distinct. For any i where $a_i = b_i$, the corresponding term: $[\![a_i, b_i]\!]$ will be a hybrid set containing a single point, that is, a 0-rectangle. The orientation of $[\![\boldsymbol{a},\boldsymbol{b}]\!]$ is based on the number of negatively oriented intervals $[\![a_i, b_i]\!]$. Should there be an odd number of indices i such that $a_i > b_i$ then $[\![\boldsymbol{a},\boldsymbol{b}]\!]$ will also be negatively oriented. Otherwise, it will be positively oriented.

4.2   Riemann Integral on n-cubes

Now that we have oriented n-dimensional cubes, we would like to define the integral over one. For now, we will content ourselves with the Riemann integral and Euclidean volume. More complex domains and other metrics will be handled in later chapters with push-backs and the Lebesgue integral. The volume of an oriented n-cube in $\mathbb{R}^n$ we define to be the product of its side lengths. Formally,

Definition Let $[\![a, b]\!]$ be a k-cube in $\mathbb{R}^n$ again with $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$. We denote the **volume of $[\![a, b]\!]$** with vol and define it as:

$$\text{vol}(\ [\![a, b]\!]\ ) = (b_1 - a_1) \cdot (b_2 - a_2) \cdot \ldots \cdot (b_n - a_n) \tag{4.9}$$

For any $k < n$, a k-cube will have volume zero. In at least one dimension, the cube will be degenerate (i.e. $a_i = b_i$) and so will contribute zero to the product. Additionally, one can also observe that $\text{vol}(\ominus[\![a, b]\!]) = -\text{vol}([\![a, b]\!])$.

Figure 4.2: Upper and lower Riemann sums shown for the same sequence shown with light and dark rectangles respectively. A function over an oriented interval is Riemann integrable if the two sums converge.
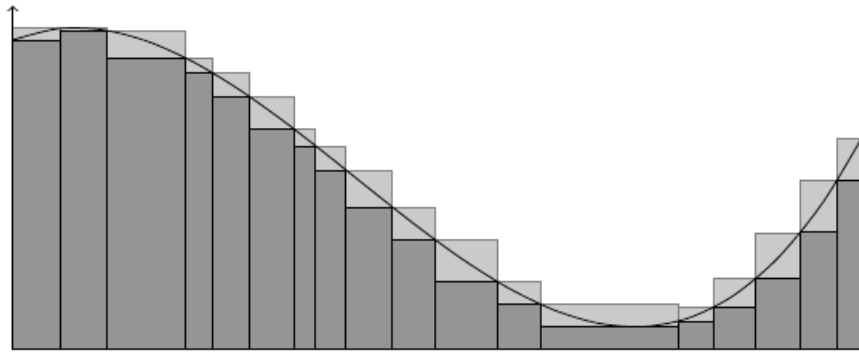
Given an n-cube $[\![a, b]\!]$ we must cut each $[\![a_i, b_i]\!]$ into partitions. Previously we used generalized partitions and did not mind if pieces overlapped or exceeded the original range. However, for building our Riemann sums, we are only interested in partitions in the traditional, non-intersecting sense.

Definition Given an oriented interval $[\![a, b]\!]$ of the real line, we say that a partition of $[\![a, b]\!]$, $\{P_i\}_{i=1}^n$ is an **interval partition of $[\![a, b]\!]$** if its pieces are:

1. *Oriented intervals*: P_i is an oriented interval of the real line for all i.

2. *Disjoint*: $P_i \otimes P_j = \emptyset$ for all i, j

We denote the set of all such partition as $\mathcal{P}[\![a, b]\!]$.

This greatly restricts the types of partitions we have access to. Every interval partition will be — up to substitution of "$]\!]$, $(\![$" in place of "$)\!)$, $[\![$" — of the form:

$$\left\{ [\![a, x_1)\!), \ [\![x_1, x_2)\!), \ [\![x_2, x_3)\!), \ \ldots, \ [\![x_{n-1}, b]\!] \right\} \tag{4.10}$$

where x_i is a monotone sequence (that is, either non-increasing or non-decreasing). This is not to say that $P_i = [\![x_{i-1}, x_i)\!)$ as the pieces of P may not be given in this order. Regardless of the ordering, we select partitions $P^j \in \mathcal{P}[\![a^j, b^j)\!)$ for each dimension of $[\![a, b)\!)$. To build our mesh, we construct smaller n-cubes $I_{i_1,\ldots,i_n}$ using the Cartesian product of pieces:

$$I_{i_1,\ldots,i_n} = i_1 \times \ldots \times i_n \tag{4.11}$$

where each i_j is taken from P^j. We are now ready to construct Riemann sums.

Definition Given $P = \{P_j\}_{j=1}^n$ where $P_j \in \mathcal{P}[\![a_j, b_j]\!]$, and $f : [\![a, b]\!] \to \mathcal{R}$ then we define a Riemann sum f_P to be:

$$f_P = \sum_{i_1 \in P_1} \ldots \sum_{i_n \in P_n} f(x_{i_1,\ldots,i_n}) \mathrm{vol}(I_{i_1,\ldots,i_n}) \tag{4.12}$$

where $x_{i_1,\ldots,i_n}$ is any point chosen from $I_{i_1,\ldots,i_n}$.

Note that we specify *a* Riemann sum, not *the* Riemann sum. There are several ways to choose $x_{i_1,\ldots,i_n} \in I_{i_1,\ldots,i_n}$ and different samplings can lead to different Riemann sums for the same partition and same function. In $\mathbb{R}^1$, several common ways to sample include the left and right

Riemann sums (which sample at the left and right boundaries), the trapezoidal Riemann sum (which averages the left and right) the upper Riemann sum (which samples at $\max(f(x_{i_1,\dots,i_n}))$) and the lower Riemann sum (which samples at $\min(f(x_{i_1,\dots,i_n}))$).

Recall our discussion of refinements from Chapter 2. Given 2 partitions P and P' of the same set then we say $P \preceq P'$ if P' is a refinement of P. In this way we can induce a partial ordering on $\mathcal{P}[\![a, b]\!]$. There is a unique smallest element in this partial ordering which is $[\![a, b]\!]$ itself; every partition by definition will refine $[\![a, b]\!]$. Additionally, given $P, P' \in \mathcal{P}[\![a, b]\!]$ then propose that we can always find Q that simultaneously refines both. If $P = P'$ then trivially $Q = P$ is a common refinement. Otherwise, we take:

$$Q = [\![a, b]\!] \otimes \left(\bigoplus_{p \in P} \bigoplus_{p' \in P} p \otimes p' \right) \tag{4.13}$$

The heavy lifting here is done by the fact that every point in $[\![a, b]\!]$ is covered by exactly one $p \in P$ and exactly one $p' \in P$. The product of any two partition pieces will then be the smallest common interval with multiplicity 1. Multiplying the entire thing by $[\![a, b]\!]$ is done to correct the sign. As we go up in our ordering, our mesh becomes increasingly fine. Taking the Riemann sum of the suprema of this poset gives us the Riemann integral.

Definition The Riemann integral of a function $f : \mathbb{R}^n \to \mathbb{R}$ over a k-cube $[\![\boldsymbol{a}, \boldsymbol{b}]\!]$ where $P = \sup \{ \mathcal{P}[\![a, b]\!] \}$

$$\int_{[\![a,b]\!]} f(x) \, dx = f_P \tag{4.14}$$

We say that a function is **Riemann integrable** if the upper and lower Riemann sums converge.

4.3 Boundary Operator

In one dimension, the boundary of an interval was quite straight-forward. For a positively oriented interval, the boundary was composed of two points; the right end-point was positive and the left end-point was negative. From the perspective of k-rectangles, the ∂ operator has mapped an oriented 1-rectangle to a set of oriented 0-rectangles. We will now generalize the boundary to map an oriented n-rectangle to an $(n-1)$-rectangle.

Definition Let $[\![a, b]\!]$ be a a k-rectangle in $\mathbb{R}^n$. Additionally, let $i_1, i_2, \ldots, i_k$ be the unique non-decreasing sequence of indices such that $a_{i_j} \neq b_{i_j}$. The **boundary of $[\![a, b]\!]$** , denoted the operator ∂ is given by

$$\partial\left([\![a, b]\!]\right) = \bigoplus_{j=1}^{k} (-1)^j \left([\![(a^{[\![1,n]\!]}), \quad (b^{[\![1,i_j)\!\!)} \oplus a^{\{\!|\, i_j\, |\!\}} \oplus b^{(\!(i_j,n]\!]})]\!] \right.$$

$$\left. \ominus [\![(a^{[\![1,i_j)\!\!)} \oplus b^{\{\!|\, i_j\, |\!\}} \oplus a^{(\!(i_j,n]\!]}), \quad (b^{[\![1,i_j)\!\!)})]\!] \right) \qquad (4.15)$$

The above equation will require a bit of unpacking to digest featuring oriented intervals in two different contexts. The first appears in the superscripts of a and b. The intervals $[\![1, i_j)\!\!)$ and $(\!(i_j, n]\!]$ are and is an interval over vector indices just as in Chapter 3. Thus, the term $a^{[\![1,i_j)\!\!)}$ refers to the vector $(a_1, a_2, \ldots, a_{i_j-1})$ while the term $b^{(\!(i_j,n]\!]}$ refers to $(b_{i_j+1}, b_{i_j+2}, \ldots, b_n)$. This provides a compact notation to partition the original range of indices into 3 pieces: $[\![1, i_j)\!\!)$, $\{\!|\, i_j\, |\!\}$, and $(\!(i_j, n]\!]$. Formally, we are actually using the hybrid sets $\{\!|\, (i_j)^1\, |\!\}$ but we omit multiplicity of one (and will continue to do so through out the section) to lighten notation.

Next we use the pointwise sum $\oplus$ we reconstruct n-dimensional vectors from our pieces. We then construct a $(k-1)$-rectangle using these vectors as in (4.8). Hence we will have terms of the forms:

$$[\![a_1, b_1]\!] \times \ldots \times [\![a_{i_{j-1}}, b_{i_{j-1}}]\!] \times [\![a_{i_j}, a_{i_j}]\!] \times [\![a_{i_{j-1}}, b_{i_{j-1}}]\!] \times \ldots \times [\![a_n, b_n]\!] \qquad (4.16)$$

$$[\![a_1, b_1]\!] \times \ldots \times [\![a_{i_{j-1}}, b_{i_{j-1}}]\!] \times [\![b_{i_j}, b_{i_j}]\!] \times [\![a_{i_{j-1}}, b_{i_{j-1}}]\!] \times \ldots \times [\![a_n, b_n]\!] \qquad (4.17)$$

In each Cartesian product, the terms at i_j: $[\![a_{i_j}, a_{i_j}]\!]$ and $[\![b_{i_j}, b_{i_j}]\!]$ are both 0-cubes. Since we defined the sequence i_j by $a_{i_j} \neq b_{i_j}$, these 0-rectangles are replacing 1-cubes in $[\![a, b]\!]$. Hence we are indeed left with a $(k-1)$-cube.

4.3.1   Example: *Boundary of a 1-rectangle*

Let $a = (a_1)$ and $b = (b_1)$ be trivial 1-tuples. Then $[\![a, b]\!] = [\![a_1, b_1]\!]$ It follows that:

$$
\begin{aligned}
\partial(\,[\![a, b]\!]\,) &= (-1)^i([\![a^{[\![1,1)\!)}, b^{[\![1,1)\!)}]\!] \times \{\!\!\{\, a_1 \,\}\!\!\} \times [\![a^{((1,1]\!]}, b^{((1,1]\!]}]\!] \\
&\qquad \ominus [\![a^{[\![1,1)\!)}, b^{[\![1,1)\!)}]\!] \times \{\!\!\{\, b_1 \,\}\!\!\} \times [\![a^{((1,1]\!]}, b^{((1,1]\!]}]\!]) \\
&= \ominus [\![a^{\emptyset}, b^{\emptyset}]\!] \times \{\!\!\{\, a_1 \,\}\!\!\} \times [\![a^{\emptyset}, b^{\emptyset}]\!] \\
&\qquad \oplus [\![a^{\emptyset}, b^{\emptyset}]\!] \times \{\!\!\{\, b_1 \,\}\!\!\} \times [\![a^{\emptyset}, b^{\emptyset}]\!] \\
&= \{\!\!\{\, b_1 \,\}\!\!\} \ominus \{\!\!\{\, a_1 \,\}\!\!\} \\
&= \left\{\!\!\left\{\, a^{-1}, b^1 \,\right\}\!\!\right\}
\end{aligned}
$$

One may notice a relationship between this result and the fundamental theorem of calculus:

$$
\int_a^b F'(x)\, dx = F(b) - F(a) \tag{4.18}
$$

Which one could easily rewrite as $\int_{[\![a,b]\!]} F'(x)\, dx = \sum(\partial([\![a, b]\!]))$. Indeed, this is why we have defined the boundary function as such, but more general statements await. We defined the boundary for not just intervals on $\mathbb{R}$ but k-cubes in $\mathbb{R}^n$.

4.3.2   Example: *Boundary of a 3-rectangle*

Let $a = (0, 0, 0)$ and $b = (1, 1, 1)$. Omitting the intermediate step, we find the boundary of $[\![a, b]\!]$ to be:

$$\partial(\,[\![a, b]\!]\,) = \ominus\; (\{\,0\,\} \times [\![0, 1]\!] \times [\![0, 1]\!]) \;\oplus\; (\{\,1\,\} \times [\![0, 1]\!] \times [\![0, 1]\!])$$

$$\oplus\; ([\![0, 1]\!] \times \{\,0\,\} \times [\![0, 1]\!]) \;\ominus\; ([\![0, 1]\!] \times \{\,1\,\} \times [\![0, 1]\!])$$

$$\ominus\; ([\![0, 1]\!] \times [\![0, 1]\!] \times \{\,0\,\}) \;\oplus\; ([\![0, 1]\!] \times [\![0, 1]\!] \times \{\,1\,\})$$

This may not be the most enlightening expression on its own. In Figure 4.3 below, the 3-rectangle given by $[\![a, b]\!]$ can be seen as a cube in three dimensions. Physically, the 3-rectangle is a solid cube and includes all interior points. The boundary meanwhile are just the rectangular outer faces, which conveniently, there are also six to match the six terms of $\partial[\![a, b]\!]$.

Figure 4.3: The unit cube in $\mathbb{R}^3$ with positive orientation can be represented as the 3-rectangle: $[\![(0, 0, 0), (1, 1, 1)]\!]$ is shown as a wire-frame. The six faces that make up its boundary are shaded and labeled with their corresponding terms.
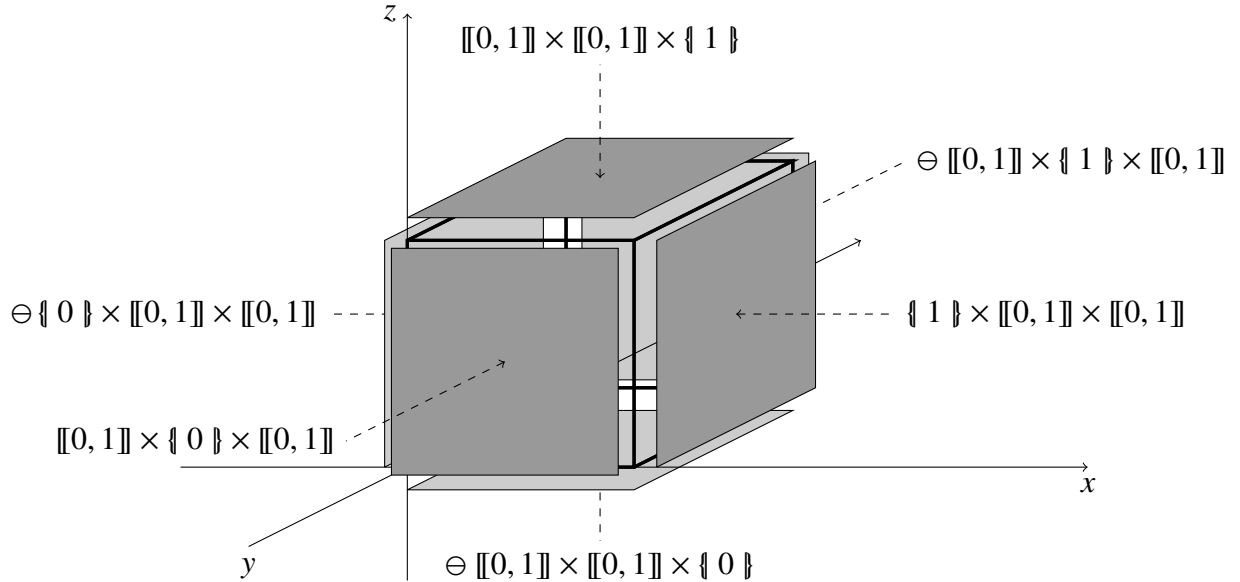
There are several ways to interpret and visualize the $\oplus$ and $\ominus$ sign associated with each face. Most naturally in $\mathbb{R}^3$ for 2-rectangles is to give each a front and back side with the sign

determining which to use. Alternatively, a 2-rectangle has a boundary formed by 1-rectangles which when drawn as arrows, will all meet head-to-tail. This induces a clockwise or counter-clockwise cycle around the edge of the rectangle and so ↻ and ↺ are also commonly used. This can be seen in Figure 4.4. One may even notice that the normals produced by both are the same and choose to use that. These are all conceptual tools, which are convenient to use particularly in $\mathbb{R}^2$ and $\mathbb{R}^3$. There may not be such a nice physical interpretation in other spaces.

Figure 4.4: One way of visualizing the orientation of 2-rectangles using clockwise and counter-clockwise cycles of arrows for 1-rectangles. The boundary of $[\![a, b]\!] \times [\![c, d]\!]$ becomes the cycle: $(a, c) \rightarrow (b, c) \rightarrow (b, d) \rightarrow (a, d) \rightarrow (a, c)$. Showing the relationship between $[\![a, b]\!] \times [\![c, d]\!]$ and $[\![b, a]\!] \times [\![d, c]\!]$
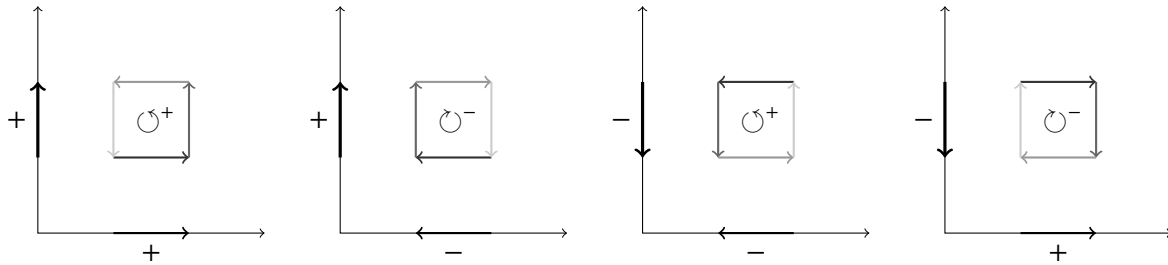
4.4   Chains

In fact, we have already seen k-chains without mentioning them explicitly. The boundary of a k-cube was the sum $\oplus$, of $2k$ $(k - 1)$ cubes. Chains do not have to be boundaries however, any linear combination of k-cubes will do.

Definition We denote the Abelian group of of all k-cubes in X as $C_k(X)$ (omitting X when obvious by context). An element $c \in C_k(X)$ is called a **k-chain on X** and is of the form:

$$c = \bigoplus_{c_i \in X} \lambda_i c_i \tag{4.19}$$

with integer coefficients λ_i and k-cubes in c_i. If coefficients λ_i are ± 1 and c is *locally finite* (i.e. each c_i intersects with only finitely many c_j that have non-zero coefficients) then we say that c is a **domain of integration**.

Since k-chains are just linear combinations of k-cubes, we naturally extend many of our definitions linearly as well. The integral $\int_c f$ of a k-chain $c = \bigoplus_i \lambda_i c_i$ is defined as $\lambda_i \int_{c_i} f + \lambda_2 \int_{c_2} f + \dots$. Doing the same for the boundary operator ∂ we have:

$$\partial_k : C_k \to C_{k-1}$$

$$\partial_k(c) = \bigoplus_{i=1}^{k} \lambda_i \partial_k(c_i)$$

Elegantly, the boundary operator now maps k-chains to $(k-1)$-chains!

$$\dots \xleftarrow{\partial_{k-1}} C_{k-1} \xleftarrow{\partial_k} C_k \xleftarrow{\partial_{k+1}} C_{k+1} \xleftarrow{\partial_{k+2}} \dots \tag{4.20}$$

The most natural next question becomes *"What does $\partial_{k-1}(\partial_k(c))$ look like?"*

4.4.1   Example: *Boundary of a boundary (of a 2-cube)*

Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$. We wish to compute $\partial_1(\partial_2(\ [\![a, b]\!]\ ))$

$$\partial_1(\partial_2([\![a_1, b_1]\!] \times [\![a_2, b_2]\!])) = \ominus \partial_1(\{\!\!\{ 0 \}\!\!\} \times [\![0, 1]\!]) \oplus \partial_1(\{\!\!\{ 1 \}\!\!\} \times [\![0, 1]\!])$$

$$\oplus \partial_1([\![0, 1]\!] \times \{\!\!\{ 0 \}\!\!\}) \ominus \partial_1([\![0, 1]\!] \times \{\!\!\{ 1 \}\!\!\}) \tag{4.21}$$

$$= \ominus (\ominus \{\!\!\{ (0,0) \}\!\!\} \oplus \{\!\!\{ (0,1) \}\!\!\}) \oplus (\ominus \{\!\!\{ (1,0) \}\!\!\} \oplus \{\!\!\{ (1,1) \}\!\!\})$$

$$\oplus (\ominus \{\!\!\{ (0,0) \}\!\!\} \oplus \{\!\!\{ (1,0) \}\!\!\}) \ominus (\ominus \{\!\!\{ (0,1) \}\!\!\} \oplus \{\!\!\{ (1,1) \}\!\!\}) \tag{4.22}$$

$$= \emptyset \tag{4.23}$$

When moving from (4.21) to (4.22), in addition to applying ∂_1 we also simplify, $\{\!\!\{ x \}\!\!\} \times \{\!\!\{ y \}\!\!\} = \{\!\!\{ (x, y) \}\!\!\}$. The identity "$\partial\partial = 0$" is not unique to 2-cubes but holds for higher dimensions as well.

Figure 4.5: The boundary of 2-cube gives a cycle of oriented edges. Taking the boundary of again, at each corner, the negative boundary of one edge will be canceled by the positive boundary of the preceding edge.
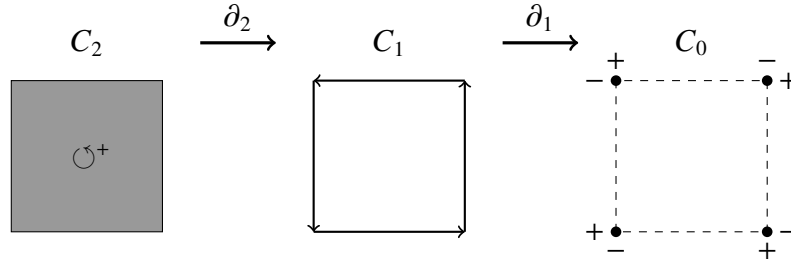
Let $[\![a, b]\!]$ be a k-rectangle in $\mathbb{R}^n$. Then we have:

$$\partial_k \partial_{k-1} \left([\![a, b]\!]\right) = \bigoplus_{j=1}^{k} (-1)^j \left(\partial_{n-1} \left([\![a^{[\![1,n]\!]}, \quad b^{[\![1,i_j)\!)} \oplus a^{[\![i_j]\!]} \oplus b^{(\!(i_j,n]\!]}]\!] \right) \right.$$
$$\left. \ominus \partial_{n-1} \left([\![a^{[\![1,i_j)\!)} \oplus b^{[\![i_j]\!]} \oplus a^{(\!(i_j,n]\!]}, \quad b^{[\![1,n]\!]}]\!] \right) \right) \tag{4.24}$$

$$= \bigoplus_{j=1}^{k} \bigoplus_{\ell=1}^{k-1} (-1)^{j+\ell} \; [\![(a^{[\![1,n]\!]}), \quad (b^{[\![1,i_j)\!) \oplus (\!(i_j,i_{j,\ell})\!) \oplus (\!(i_{j,\ell},n]\!]} \oplus a^{[\![i_j]\!]} \oplus [\![i_{j,\ell}]\!]})]\!]$$
$$\ominus [\![(a^{[\![1,i_{j,\ell})\!) \oplus (\!(i_{j,\ell},n]\!]} \oplus b^{[\![i_{j,\ell}]\!]}), \quad (b^{[\![1,i_j)\!) \oplus (\!(i_j,n]\!]} \oplus a^{[\![i_j]\!]})]\!]$$
$$\ominus [\![(a^{[\![1,i_j)\!) \oplus (\!(i_j,n]\!]} \oplus b^{[\![i_j]\!]}), \quad (b^{[\![1,i_{j,\ell})\!) \oplus (\!(i_{j,\ell},n]\!]} \oplus a^{[\![i_{j,\ell}]\!]})]\!]$$
$$\oplus [\![(a^{[\![1,i_j)\!) \oplus (\!(i_j,i_{j,\ell})\!) \oplus (\!(i_{j,\ell},n]\!]} \oplus b^{[\![i_j]\!] \oplus [\![i_{j,\ell}]\!]}), \quad (b^{[\![1,n]\!]})]\!] \tag{4.25}$$

Note that we have i_j and i'_ℓ; after applying the first boundary operator, one dimension of the k-cube is degenerate. Hence for each sequence: $\{i_j\}_{j=1}^{k}$ we construct $\{i_{j,\ell}\}_{\ell=1}^{k-1}$ given by:

$$i_{j,1}, \ldots, i_{j,k-1} = i_1, \ldots, \widehat{i_j}, \ldots, i_k \tag{4.26}$$

The double sum iterates over all pairs but $\oplus$ commutes so the $(k-2)$-cube with degenerate dimensions $[\![i_j]\!] \oplus [\![i_{j,\ell}]\!]$ will be iterated over twice. The sequences depend on one another so it

is not as simple as simply swapping ℓ and j:

$$[\![i_j]\!] \oplus [\![i_{j,\ell}]\!] = \begin{cases} [\![i_\ell]\!] \oplus [\![i_{\ell,j-1}]\!] & j > \ell \\[2mm] [\![i_{\ell+1}]\!] \oplus [\![i_{\ell+1,j}]\!] & j \le \ell \end{cases} \tag{4.27}$$

So each term representing a $(k-2)$-cube will occur twice in the sum. Once with the iteration (j, ℓ) and once with $(\ell, j-1)$ or $(\ell + 1, j)$. In either case, $(-1)^{j+\ell}$ is inverted meaning the two cubes will cancel. Leaving us with the boundary of a boundary being empty. By linearity this extends to all chains as well as the sum of empty sets is of course still empty.

Definition A **chain complex** is a sequence of Abelian groups $\ldots, A_2, A_1, A_0, A_{-1}, A_{-2}, \ldots$ which are connected by homomorphisms $d_n : A_n \to A_{n-1}$ such that $d_n \circ d_{n+1} = 0$ for all n. Typically written out as:

$$\ldots \xleftarrow{d_{k-1}} A_{k-1} \xleftarrow{d_k} A_k \xleftarrow{d_{k+1}} A_{k+1} \xleftarrow{d_{k+2}} \ldots \tag{4.28}$$

A **cochain complex** is a sequence of Abelian groups $\ldots, A^{-2}, A^{-1}, A^0, A^1, A^2, \ldots$ which are connected by homomorphisms $d^n : A^n \to A^{n+1}$ such that $d^n \circ d^{n-1} = 0$ for all n. Typically written out as:

$$\ldots \xrightarrow{d^{k-1}} A^{k-1} \xrightarrow{d^k} A^k \xrightarrow{d^{k+1}} A^{k+1} \xrightarrow{d^{k+2}} \ldots \tag{4.29}$$

$(C_\bullet, \partial_\bullet)$ is just one instance of a chain complex known as the *"cubic homology"*. In the next chapter we will look at the more general *"cubic singular homology"*. As well as the related cochain complex: the *"De Rham cohomology"* and how the two relate.

Chapter 5

Integration II

Earlier we established integration of functions and treated the "dx" nearly as punctuation. Another more flexible and approach is to use *differential forms*.

5.1 Differential Forms

Definition A **(differential) 0-form** β on $\mathbb{R}^n$ is a function $\beta : \mathbb{R}^n \to \mathbb{R}$. A **(differential) 1-form** ω on $\mathbb{R}^n$ is an expression of the form:

$$\omega = f_1(\mathbf{x}) \, dx_1 + f_2(\mathbf{x}) \, dx_2 + \ldots + f_n(\mathbf{x}) \, dx_n \tag{5.1}$$

There is very little to say about 0-forms, they are just functions on $\mathbb{R}^n$. 1-forms look very much like something we're used to integrating with. Of course if all but one of f_i are zero then we have a *basic 1-form*, $\omega = f_i \, dx_i$ which is nothing new to integrate over. Using the linearity of integration, general 1-forms can easily be seperated into a sum of integrals on basic 1-forms. Typically, 1-forms are encountered inexplicitly in a multivariate calcululus class with Green's theorem:

$$\int_D \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \right) dx \, dy = \int_{\partial D} f_1(x, y) \, dx + f_2(x, y) \, dy \qquad \text{(Green's Theorem)}$$

> Formally, $T_x(\mathbb{R}^n) \to \mathbb{R}$.

We can add differential forms but rather than multiplication we use the $\wedge$ product.

To create higher order p-forms we use the wedge operator $\wedge$.

First of all, the wedge product is primarily defined by being *anti-commutative* or *skew-symmetric*. That is, $dx \wedge dy = -dy \wedge dx$. Several results immediately follow from this. When used on two identical dx, we have $dx \wedge dx = -dx \wedge dx$ and so the result must be zero. Additionally, for any permutation σ of $[p]$:

$$dx_1 \wedge \ldots \wedge dx_p = \text{sgn}(\sigma)\, dx_{\sigma(1)} \wedge \ldots \wedge dx_{\sigma(p)} \tag{5.2}$$

Definition Given a k-rectangle $\Omega \in \mathbb{R}^n$ with coordinates $\mathrm{x} = (x_1, x_2, \ldots, x_n)$ A **differential** p**-form** β over Ω has the form:

$$\beta = \sum_{j_1 \in [n]} \ldots \sum_{j_p \in [n]} b_{(j_1, \ldots, j_p)}(\mathrm{x})\, \mathrm{d}x_{j_1} \wedge \ldots \wedge \mathrm{d}x_{j_p} \tag{5.3}$$

Typically, we will take $j = (j_1, \ldots, j_p)$ and express β as, $\sum_j b_j(x)\, dx_{j_1} \wedge \ldots \wedge dx_{j_p}$. We denote the space of all p-forms on Ω by $\Lambda^p(\Omega)$.

dx^i and dx^j are trivial differential 1-forms by taking $f_i = 1$ and $f_j = 1$ respectively. When we take the wedge product of these, we end up with the 2-form $dx^i \wedge dx^j$. It should then seem natural to think of $\wedge$ as an operator which maps a k-form and an ℓ-form to a $(k + \ell)$-form. For general p and q-forms we define the wedge product.

Definition Let $\alpha = \sum_i a_i(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p} \in \Lambda^p(\Omega)$ and $\beta = \sum_j b_j(x)\, dx_{j_1} \wedge \ldots \wedge dx_{j_q} \in \Lambda^q(\Omega)$. We extend the wedge product $\wedge : \Lambda^p(\Omega) \times \Lambda^q(\Omega) \to \Lambda^{p+q}(\Omega)$ by:

$$\alpha \wedge \beta = \sum_{i,j} a_i(x) b_j(x)\, dx_{i_1} \wedge \ldots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_q} \tag{5.4}$$

Although we take all possible $\binom{n}{q} \cdot \binom{n}{p}$ pairs of $a_i(x)dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ and $b_i(x)dx_{i_1} \wedge \ldots \wedge dx_{i_p}$, most of the possible terms will end up being zero. If *any* of the terms in dx_i appears in dx_j, then the wedge product will be zero and no term will be contributed. As such, if $q + p > n$, there will be a duplicate in every term and so the entire sum will be zero. When all is said and done, at most we will have $\binom{n}{p+q}$ terms. Rather than the skew-symmetry we had when commuting $dx \wedge dy$, in higher dimensions the sign depends on $p \cdot q$ of the p-form and q-form we are commuting. Specifically,

$$\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha \tag{5.5}$$

This can be easily seen by commuting each of $dx_{j_1}, \ldots dx_{j_q}$ terms each past $dx_{i_1} \wedge \ldots \wedge dx_{i_p}$. So we are commuting q terms each past p terms, reversing the sign each time for a net $(-1)^{pq}$.

The wedge product is only one part of our algebra of differential forms. We have several other nice identities for its behaviour with addition and multiplication. For the following, we consider f to be a function on $\mathbb{R}^n$. Additionally we consider the differential forms ω_1 and ω_2 to be k-forms, α to be a p-form and β to be an q-form. Then we have the following:

$$(\omega_1 + \omega_2) \wedge \alpha \;=\; \omega_1 \wedge \alpha + \omega_2 \wedge \alpha \tag{5.6}$$

$$(\omega_1 \wedge \alpha) \wedge \beta \;=\; \omega_1 \wedge (\alpha \wedge \beta) \tag{5.7}$$

$$(f \cdot \omega_1) \wedge \alpha \;=\; f \cdot (\omega_1 \wedge \alpha) \;=\; \omega_1 \wedge (f \cdot \alpha) \tag{5.8}$$

These should all be quite obvious from definitions. We should also note the identities which are *not* present. We have defined the sum of ω_1 and ω_2: two differential forms which are the same dimension but not the sum of α and β: differential forms with different dimension. It is clear how one would add two differential forms of the same dimension as both were defined as sums to begin with. We also do not define the multiplication of $\cdot$ two differential forms but we

multiplying a form by a function is simply:

$$(f \cdot \alpha)(x) = \sum_i f(x) \cdot a_i(x) \, dx_{i_1} \wedge \ldots \wedge dx_{i_p} \tag{5.9}$$

Integrating over a k-form is quite simple. First we shall consider integrating a k-form over a k-rectangle in $\mathbb{R}^k$. Such a k-form is also known as a *top-dimensional form*. As we saw previously, any form of higher degree must be zero. If ω is such a top form then we can always write

$$\omega = f \, dx_1 \wedge \ldots \wedge dx_k \tag{5.10}$$

for some function f. Other presentations of ω exist, but we can always achieve such a presentation by commuting over $\wedge$ to the canonical ordering $x_1, \ldots, x_n$. Once we have the k-form in this presentation, we just remove the wedges and evaluate the integral using the integrand $f \, dx_1 \, dx_2 \ldots dx_k$.

Definition Let α be a k-form on $\Omega \subset \mathbb{R}^n$ of the form $\alpha = A(x) \, dx_1 \wedge \ldots \wedge dx_n$. If $A \in \mathcal{L}^1(\Omega, dx)$ then we define:

$$\int_\Omega \alpha = \int_\Omega A(x) \, dx \tag{5.11}$$

Where the left-hand side is the integral of a k-form and the right-hand side is a Lebesgue integral. For any $\beta \in \Lambda^k(\Omega)$ we extend this definition linearly as the sum of integrals.

5.2 Singular Cubes

Up until now we have been dealing with the very small set of axis aligned cubes.

Definition Manifold

Definition Given a map $\Phi : X \to Y$, $x \in X$ pushes forward to $\Phi(x) \in Y$

Forms pull back from Y to X

Benefit of differential forms is how cleanly they handle changes in coordinates.

This is generally done through the use of pull-backs.

A pullback $\varphi^* f$ can be thought of as *functional precomposition*: $(\varphi^* f)(x) = f(\varphi(x))$

Gets its name from pulling F back through ω

Definition $F : X \to \Omega$ Define the pullback $F^*\beta$

$$F^*\beta = \sum_j b_j(F(x))(F^* dx_{j_1}) \wedge \ldots \wedge (F^* dx_{j_k}) \tag{5.12}$$

and

$$F^* dx_j = \sum_\ell \frac{\partial F^j}{\partial x_\ell} dx_\ell \tag{5.13}$$

Which can be reduced by:

$$F^*\beta = \sum_j b_j(F(x))(F^* dx_{j_1}) \wedge \ldots \wedge (F^* dx_{j_k}) \tag{5.14}$$

$$= \sum_j b_j(F(x))\left(\sum_\ell \frac{\partial F^{j_1}}{\partial x_\ell} dx_\ell\right) \wedge \ldots \wedge \left(\sum_\ell \frac{\partial F^{j_k}}{\partial x_\ell} dx_\ell\right) \tag{5.15}$$

$$= \ldots \tag{5.16}$$

$$= \sum_j b_j(F(x)) \det(J_F) \, dx_{j_1} \wedge \ldots \wedge dx_{j_k} \tag{5.17}$$

Which is significant given the change of variable formula for integration:

$$\int_{\phi(U)} f(v) \, dv = \int_U f(\phi(u)) \, |\det\phi'(u)| \, du \tag{5.18}$$

Theorem 5.2.1 *Let $F : X \to \Omega$ be an (orientation-preserving diffeomorphism) and α an integrable n-form on Ω then*

$$\int_\Omega \alpha = \int_X F^*\alpha \tag{5.19}$$

More algebra of differential forms

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta) \tag{5.20}$$

Definition Exterior derivative

...

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^j \alpha \wedge (d\beta) \tag{5.21}$$

...

$$F^*(d\beta) = dF^*\beta \tag{5.22}$$

5.3   Integration on Singular Cubes

Need introduction to manifolds somewhere.

We have already seen standard n-cubes in the previous chapter. Now that we have pull-backs we can define integration on the more general *singular cube* But first, what is a singular cube.

Definition Singular cube: differentiable map c from standard k-cube to k-dimensional manifold.

It is a common abuse of notation to use c to also to refer to the image $c([\![0, 1]\!]^k) \subseteq M$. The choice of using specifically the standard k-cube is arbitrary. A differentiable map f from $[\![a, b]\!]$ can always be composed with $g : t \mapsto ta + (1 - t)b$ to get singular cube $c = f \circ g$. Gives us innumerably more shapes to work with. Hemisphere for example is the singular 2-cube given by $(r, \varphi) \to r \cos(\pi\varphi)x + r \sin(\pi\varphi)y$

Definition Let c be a singular k-cube and ω a k-form on the image of c. Then we define the integral using $c^*(\omega)$ the pull-back of ω along c:

$$\int_c \omega = \int_{c(\llbracket 0,1 \rrbracket^k)} \omega = \int_{\llbracket 0,1 \rrbracket^k} c^*\omega \tag{5.23}$$

5.4   Integration over Manifolds

A manifold is nothing more than a collection of local charts diffeomorphic to $\mathbb{R}^n$

So we use pullbacks to turn a set of rectangles into a manifold while simultaneously applying the manifold's pullback.

Integration on oriented manifolds is just a careful application of Theorem 5.2.1

Example Integrate an atlas with overlapping charts (using inclusion-exclusion)

partitions of unity

5.5   Stokes' Theorem

$$\int_{\partial M} \omega = \int_M d\omega \tag{5.24}$$

Generalization of FTC, Greens, ...

Stokes' on n-cubes

Simplicial cubes.

Manifolds

5.5.1   Example: *Contour Integration*

??????????

Using Stokes' theorem and Inclusion/Exclusion to evaluate a tricky theorem like:

$$f(z) = \frac{z^2}{(z^2 + 2z + 2)} \tag{5.25}$$

(2-3 pages)

Chapter 6

Integration III

"Chapter goals"

Section 4.1 will cover a conventional treatment of the Lebesgue integral, for those already familiar with the construction, this may be skipped.

6.1 Sigma Algebras

Before we can talk about the Lebesgue integral we must first set the stage, so to speak.

Definition Let X be a non-empty set. A σ-**algebra on the set** X, Σ, is a family of subsets of X such that:

1. Σ is non-empty

2. *Closed under complement.* If $E \in \Sigma$, then $X \setminus E \in \Sigma$.

3. *Closed under countable union.* If $E_1, E_2, ... \in \Sigma$ then $(E_1 \cup E_2 \cup ...) \in \Sigma$.

The pair (X, Σ) is called a **measurable space** and elements of Σ are called the **measurable sets** (of X).

It can easily be shown through the use of De Morgan's laws that a σ-algebra is also closed under countable intersection as well.

Example $\{\emptyset, X\}$ is a σ-algebra on X. In fact, X and $\emptyset$ are members of *every* σ-algebra on X.

Example 2^X is a σ-algebra on X.

However, we would also like to be able to construct more interesting σ-algebras.

Definition Given an arbitrary family of subsets $F \subseteq 2^X$, there is a unique smallest σ-algebra containing F which is called the σ**-algebra generated by** F and we will denote as $\sigma(F)$.

Can be constructed by taking the intersection of all σ-algebras containing F.

Of particular interest to us is the σ-algebra generated by a topology, $\mathcal{T}(X)$.

Borel σ-algebra: $\mathcal{B}(X) = \sigma(\mathcal{T}(X))$

6.2 The Lebesgue Integral

Definition Let (X, μ) be a measure space and $S \in \mathcal{L}(X, \mu)$ a measurable set. If $\mathbb{I}_S$ is indicator function $\mathbb{I}_S : X \to \{0, 1\}$ given by $\mathbb{I}_S(x) = 1$ if $x \in S$ and $\mathbb{I}_S(x) = 0$ otherwise. Then we define:

$$\int_X \mathbb{I}_S \, d\mu := \mu(S) \tag{6.1}$$

We define a simple function as a function which maps to a finite set.

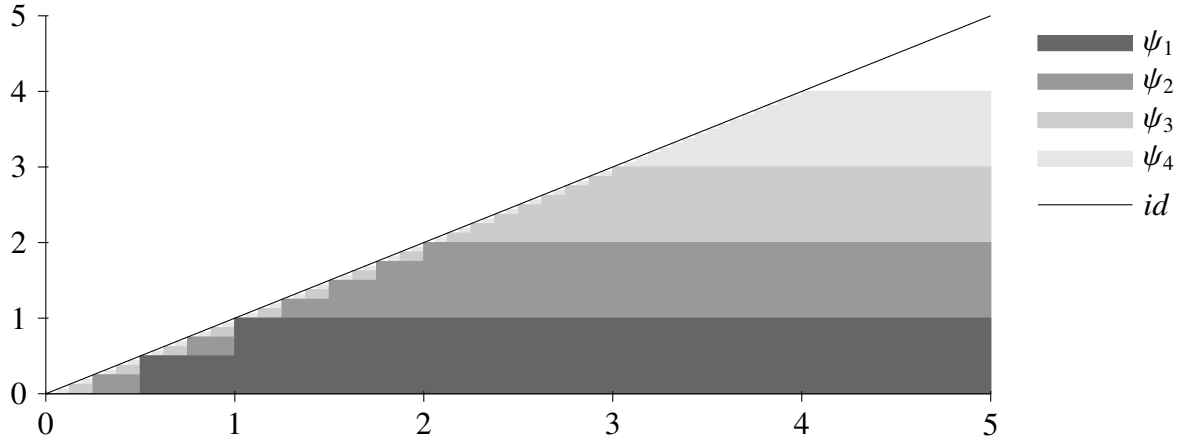**Theorem 6.2.1** *If φ is a simple function, then we it can be represented as a finite sum of indicator functions:*

$$\varphi := \sum_{k=0}^{n} a_k \mathbb{I}_{E_k} \tag{6.2}$$

Where $\{a_k\}_{k=0}^{n}$ is the set of values in the range of φ and $E_k = \varphi^{-1}(\{a_k\})$, the set of all points which map to a_k.

This allows us to define the integral of a simple function.

Figure 6.1: asdfasdf

Definition Let $\varphi = \sum_{k=0}^{n} a_k \mathbb{I}_{E_k}$ be a simple function. We define the integral:

$$\int_X \varphi \, d\mu = \int_X \sum_{k=0}^{n} a_k \mathbb{I}_{E_k} d\mu = \sum_{k=0}^{n} a_k \int_X \mathbb{I}_{E_k} d\mu \tag{6.3}$$

Next we define integral of a non negative functions in $\bar{\mathbb{R}}$

Definition Let (X, μ) be a measure space and $f : X \to [0, \infty]$.

$$\int_X f \, d\mu := \sup \left\{ \int_X \varphi : \varphi \text{ simple, and } 0 \leq \varphi \leq f \right\} \tag{6.4}$$

We use the simple function ψ_n defined as:

$$\psi_n = \sum_{k=0}^{n2^n - 1} \left[\left(\frac{k}{2^n} \right)^{\left[\frac{k}{2^n}, \frac{k+1}{2^n} \right)} \right] + n^{[n,\infty]} \tag{6.5}$$

Notationally, this definition is rather heavy but is easily understood geometrically as seen in Figure 4.1

figure description

Which for any non-negative f allows us to define $\varphi_n = \psi_n \circ f$ a simple function.

Obviously we have φ_n simple since ψ_n is simple.

Since $\psi_n(x) \leq x$ for all x then we also have $0 \leq \varphi_n \leq f$.

Most importantly we have $0 \le f(x) - \varphi_n(x) \le 2^{-n}$ and so φ_n, uniformly approaches f as n approaches infinity.

Theorem 6.2.2 *Let f be a function mapping to $\mathbb{R}$. Then $f = f^+ - f^-$ where f^+ and f^- are non-negative functions given by:*

$$f^+(x) := \max(0, f(x)) \tag{6.6}$$

$$f^-(x) := \max(0, -f(x)) \tag{6.7}$$

Now we have everything we need to define integrals for arbritrary functions mapping to $\mathbb{R}$:

Definition asdfasd

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu \tag{6.8}$$

This was a standard treatment of Lebesgue integration.

Mathematically it is very simple to extend this to integrals of hybrid sets over $L(X, \mu)$ by linearity. That is, given $\Gamma \in \mathbb{Z}^{\mathcal{L}(X,\mu)}$:

$$\int_\Gamma f \, d\mu = \sum_{\sigma \in \Gamma} \Gamma(\sigma) \int_\sigma f \, d\mu \tag{6.9}$$

Chapter 7

Convolution

Convolution is an operation which takes two functions and produces a third. It takes one of the two input functions, and modifies one by mirroring and translating. The resulting function is then the overlap between one of these functions as a function of the translation. Visually, this can be seen below in Figure 7.1.

+1 line

Figure 7.1: The convolution of the box signal $f(t) = g(t) = \left(0^{((-\infty,-0.5))} \oplus 1^{[\![-0.5,0.5]\!]} \oplus 0^{((0.5,\infty))}\right)$ with itself. *(from wikipedia; need to create new versions)*
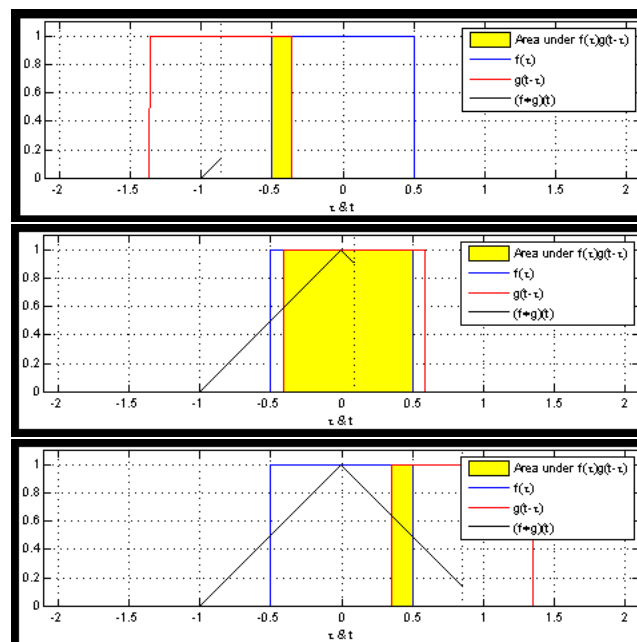
Formally this, equates to the following definition for convolution over continuous domains:

Definition The **convolution** $*$, of two functions F and G is defined as:

$$(F * G)(t) = \int_{-\infty}^{\infty} F(\tau)\, G(t - \tau)\, d\tau \tag{7.1}$$

and in the case of discrete linear convolution, summation would replace integration. In this equation, t represents the translation of G as well as the input for $(F * G)$ while τ is internal to the integral and varies over the real line.

Figure 7.2: 512x512px "Lena"(a) with a 1px (b) and 5px (c) Gaussian blur applied. Gaussian blurring is accomplished by convolving an image with a Gaussian kernel and is commonly used in image processing to reduce noise prior to edge detection.

(a) Original Image   (b) 1px Gaussian blur   (c) 5px Gaussian blur

Convolution has applications in many areas of mathematics and engineering. One very common use in image processing is in blurring. *Gaussian blurring* is the result of a 2-dimensional convolution of an image with the Gaussian distribution function:

$$G(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) \tag{7.2}$$

Blurring an image in this way reduces noise and greatly increases the efficacy of subsequent edge detection. In statistics, a (simple) *moving average* can be represented as a convolution by a rectangular pulse while more generally, weighted moving averages can be made by convolving with other functions.

7.1   Convolution of Piecewise Functions

CAS such as Maple and Mathematica are quite adept at solving integrals. Convolution of elementary functions generally poses no problem. When convolving two piecewise continuous functions, many possible intervals arise and the conditionals that arise can quickly overwhelm them unaided.

cite, rewrite?

We are interested in *Symbolic Linear Convolution* (of piecewise continuous functions). First we consider the typical approach for convolution of "one piece" functions [11, 20]. Given two functions, F and G defined as:

$$F(x) = f^{[a_f, b_f)}(x) = \begin{cases} f(x) & a_f \leq x < b_f \\ 0 & \text{otherwise} \end{cases} \tag{7.3}$$

$$G(x) = g^{[a_g, b_g)}(x) = \begin{cases} g(x) & a_g \leq x < b_g \\ 0 & \text{otherwise} \end{cases} \tag{7.4}$$

we would like to compute the convolution $(F * G)$. In addition, we generally assume that $b_f - a_f \leq b_g - a_g$ (i.e. F is non-zero over a shorter interval). If this is not the case, convolution is commutative so we simply rearrange: $F * G = G * F$. To see this, simply apply the substitute $\tau' = t - \tau$ in equation (7.1):

$$(F * G)(t) = \int_{-\infty}^{\infty} F(\tau)G(t - \tau) \, d\tau = \int_{\infty}^{-\infty} F(t - \tau')G(\tau') \, (-1)d\tau' = (G * F)(t) \tag{7.5}$$

Thus we can assume that our static function is also the function with the shorter interval. Since F and G are zero outside of their respective intervals, we do not need to integrate over the entire real line. F is our static function, so $[a_f, b_f)$ would be sufficient. This is still more than we necessarily need though,

$$
\begin{aligned}
(F * G)(t) &= \int_{-\infty}^{\infty} F(\tau)\, G(t - \tau)\, d\tau \\[2mm]
&= \int_{a_f}^{b_f} f(\tau)\, G(t - \tau)\, d\tau \\[2mm]
&= \begin{cases}
\int_{a_f}^{x - a_g} f(\tau)\, g(t - \tau)\, d\tau & (a_f + a_g) \le t < (b_f + a_g) \\[2mm]
\int_{a_f}^{b_f} f(\tau)\, g(t - \tau)\, d\tau & (b_f + a_g) \le t < (a_f + b_g) \\[2mm]
\int_{x - b_g}^{b_f} f(\tau)\, g(t - \tau)\, d\tau & (a_f + b_g) \le t < (b_f + b_g) \\[2mm]
0 & \text{otherwise}
\end{cases}
\end{aligned}
\tag{7.6}
$$

This is the way it's typically done.

7.2 Hybrid Function Convolution

With hybrid functions we do not have to worry about the relative length of f and g intervals.

Instead for hybrid functions $F = f^{[a_f, b_f)}$ and $G = g^{[a_f, b_f)}$:

$$
\begin{aligned}
(F * G)(t) = \mathcal{R}_+ \Bigg(&\left(\int_{a_f}^{x - a_g} f(\tau)\, g(t - \tau)\, d\tau \right)^{[\![a_f + a_g,\, b_f + a_g)\!)} \\
&\oplus \left(\int_{a_f}^{b_f} f(\tau)\, g(t - \tau)\, d\tau \right)^{[\![b_f + a_g,\, a_f + b_g)\!)} \\
&\oplus \left(\int_{x - b_g}^{b_f} f(\tau)\, g(t - \tau)\, d\tau \right)^{[\![a_f + b_g,\, b_f + b_g)\!)} \Bigg)(t)
\end{aligned}
\tag{7.7}
$$

When $b_f - a_f \le b_g - a_g$ then the intervals will be disjoint and the expression is identical to 2.61

Otherwise, the interval $[\![b_f + a_g,\ a_f + b_g)\!)$ will have a negative orientation.

For $(a_f + b_g) \le t < (b_f + a_g)$ then t will be in all three regions and we have

7.3 Infinite Intervals

7.4 Example

Chapter 8

Conclusions

The primary objective of this thesis was to extend [8] by investigating further applications of hybrid sets and functions. Although this paper was focused on results for integration and Petri net graphs, this is not to take away from the "smaller" results shown along the way. As a first example, we showed that, (notational choice: $\mathbb{Z}^{\mathbb{P}}$, $h(\mathbb{P})$, $\mathcal{H}(\mathbb{P})$?), the hybrid sets over prime numbers is equivalent to $\mathbb{Q}_+$.

Hybrid sets come into their own within the context of hybrid functions as we saw with arithmetic on piecewise functions and symbolic matrices. Combined with tricks from linear algebra, the usage of hybrid functions allowed for large decreases in both cases. Hybrid pseudo-functions leave a function associated with an element unevalutated and allow for algebra to be performed on the domains before requesting any functions be evaluated.

Hybrid functions were shown to be a good model for domains of integration. An atlas can succintly be defined in terms of a set of hybrid relation over a universe of Euclidean rectangles mapping to a common manifold. Principle of Inclusion-Exclusion was used instead of the typical *partitions of unity* to reduce the atlas to it's support. Unlike some of the previous examples, any leftover negative terms produced by PIE are completely well-founded and have natural geometric interpretations. Moreover, ∂, the boundary operator on a k-chain explicitly constructs them. The beauty (and usefulness) of ∂ was then shown with a proof of the gener-

alized Stokes' theorem. Which in turn we used in conjunction with generalized partitions to transform an otherwise difficult to compute integral.

Finally, we showed a novel formulation of Petri net graphs. Instead of considering transitions as a special type of node, we represented transitions along with corresponding arc weights as a single hybrid set. Conditions for liveness and coverability were also discussed. Relaxing the condition of non-negative markings gives way to *lending Petri nets* [3] [4] for which hybrid sets were even better suited. Unfortunately, I just discovered the Bartoletti papers and have not had a chance to read more than the abstract.

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