I'm investigating a number of uses for **hybrid sets**: sets with integer ($\pm$) multiplicity rather than boolean and hybrid functions: functions with hybrid set domains. Union and intersection make sense for boolean sets when the correspond to boolean `AND` and `OR`. For hybrid sets we use $\oplus$, $\otimes$ and $\ominus$ for pointwise sum, multiplication and subtraction.

So we can use $[\![...)\!)$ for **oriented intervals** defined as:

$$[\![a, b)\!) = [a, b) \ominus [b, a) \tag{1}$$

Only one of the non-oriented (traditional) intervals $[a, b) = \{x | a \leq x < b\}$ or $[b, a) = \{x | b \leq x < a\}$ will be non-empty. The multiplicity of $[\![a, b)\!)$ will be $+1$ or -1 between a and b depending. This gives us some nice properties like:

$$[\![a, b)\!) = \ominus[\![b, a)\!) \tag{2}$$

$$[\![a, c)\!) = [\![a, b)\!) \oplus [\![b, c)\!) \tag{3}$$

A hybrid function is generally written as f^A where A is a hybrid set. Formally it is a hybrid set over ordered pairs that look like $(x, f(x))$ (or sometimes leaving f unevaluated for technical reasons(x, f)). If A is "reducible" (+1 or 0 everywhere) then it behaves like restricting a function to the set A. A non-reducible set can be reduced through $\mathcal{R}_+$ by adding when multiplicity is positive and subtracting when multiplicity is negative (more generally some binary operation and it's inverse).

The convolution of two functions is defined by:

$$(F * G)(t) = \int_{-\infty}^{\infty} F(\tau) G(t - \tau) d\tau \tag{4}$$

If F and G are the sum of disjoint sub-functions $f_1, f_2, \ldots$ and $g_1, g_2, \ldots$ then:

$$(F * G)(t) = \sum_i \sum_j \int_{-\infty}^{\infty} f_i(\tau) g_j(t - \tau) d\tau \tag{5}$$

So we only need to worry about convolving "one-piece" functions then we take the sum of all pairs. So for two one piece functions:

$$F(t) = \begin{cases} f(t) & t \in [a_f, b_f) \\ 0 & \text{otherwise} \end{cases} \qquad G(t) = \begin{cases} g(t) & t \in [a_g, b_g) \\ 0 & \text{otherwise} \end{cases}$$

the typical approach would be to commute F and G so that $b_f - a_f < b_g - a_g$ so that:

$$(F * G)(t) = \begin{cases} \int_{a_f}^{x - a_g} f(\tau)\, g(t - \tau)\, d\tau & (a_f + a_g) \leq t < (b_f + a_g) \\ \int_{a_f}^{b_f} f(\tau)\, g(t - \tau)\, d\tau & (b_f + a_g) \leq t < (a_f + b_g) \\ \int_{x - b_g}^{b_f} f(\tau)\, g(t - \tau)\, d\tau & (a_f + b_g) \leq t < (b_f + b_g) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

However with hybrid functions we can (regardless of relative length) write:

$$
\begin{aligned}
(F\ *\ G)(t) = \mathcal{R}_+ \Bigg(& \left(\int_{a_f}^{t-a_g} f(\tau)\ g(t-\tau)\ d\tau \right)^{[\![a_f+a_g,\ b_f+a_g))} \\
& \oplus \left(\int_{a_f}^{b_f} f(\tau)\ g(t-\tau)\ d\tau \right)^{[\![b_f+a_g,\ a_f+b_g))} \\
& \oplus \left(\int_{t-b_g}^{b_f} f(\tau)\ g(t-\tau)\ d\tau \right)^{[\![a_f+b_g,\ b_f+b_g))} \Bigg)(t)
\end{aligned} \tag{7}
$$

Let F, G be defined as:

$$
F = f_1^{(-\infty,-1)} \oplus f_2^{[-1,1)} \oplus f_3^{[1,\infty)} \tag{8}
$$

$$
G = 0^{(-\infty,-1)} \oplus g_1^{[-1,0)} \oplus g_2^{[0,1)} \oplus 0^{[1,\infty)} \tag{9}
$$

then

$$
\begin{aligned}
(f_2 * g_1)(t) = \mathcal{R}_+ \Bigg(& \left(\int_{[\![-1,t+1))} f_2(\tau)\ g_1(t-\tau)\ d\tau \right)^{[\![-2,0))} \\
& \oplus \left(\int_{[\![-1,1))} f_2(\tau)\ g_1(t-\tau)\ d\tau \right)^{[\![0,-1))} \\
& \oplus \left(\int_{[\![t,1))} f_2(\tau)\ g_1(t-\tau)\ d\tau \right)^{[\![-1,1))} \Bigg)(t)
\end{aligned} \tag{10}
$$

Suppose we wanted to calculate the exact value at $-.5$. The point is in both $[\![-2,0))$ and $[\![-1,1))$ once and in $[\![0,-1))$ with multiplicity negative one. After $+$-reducing we have:

$$
\begin{aligned}
(f_2 * g_1)(-.5) = & \int_{[\![-1,.5))} f_2(\tau)\ g_1(-.5-\tau)\ d\tau - \int_{[\![-1,1))} f_2(\tau)\ g_1(-.5-\tau)\ d\tau \\
& + \int_{[\![-.5,1))} f_2(\tau)\ g_1(-.5-\tau)\ d\tau && (11) \\
= & \int_{[\![-1,.5))\ominus[\![-1,1))\oplus[\![-.5,1))} f_2(\tau)\ g_1(-.5-\tau)\ d\tau && (12) \\
= & \int_{[\![-.5,.5))} f_2(\tau)\ g_1(-.5-\tau)\ d\tau && (13)
\end{aligned}
$$

Note that the terms in (11) are not directly evaluable; g_1 may not be well-defined outside of $[-1,0)$. However, since the integrands are all identical, we can collect all the terms and the offending domains cancel.

So relative length doesn't matter between f and g. Now, for what I've been working on this weekend: indeterminate arithmetic (e.g. $+\infty - \infty$) is not a problem either. Suppose $a_f = -\infty$ and $b_g = \infty$ are both infinite while b_f and a_g are both finite. Which gives the three intervals:

$$[\![-\infty, b_f + a_g)\!), [\![b_f + a_g, \perp)\!), [\![\perp, \infty)\!) \tag{14}$$

Even though $a_f + b_g$ is unevaluable, the two $\perp$ points are derived identically and so (3) gives us:

$$[\![b_f + a_g, \perp)\!) \oplus [\![\perp, \infty)\!) = [\![b_f + a_g, \infty)\!) \tag{15}$$

If we assume that t is finite then the domains on the integrals in the 2nd and 3rd terms of (7) are identical:

$$[\![a_f, b_f)\!) = [\![-\infty, b_f)\!) = [\![t - b_g, b_f)\!) \tag{16}$$

These two functions are identical which is important for the identity for hybrid functions:

$$f^A \oplus f^B = f^{A \oplus B} \tag{17}$$

Finally:

$$\begin{aligned}(f^{[\![-\infty, b_f)\!)} * g^{[\![a_g, \infty)\!)})(t) = \mathcal{R}_+ \Bigg(&\left(\int_{[\![-\infty, t-a_g)\!)} f(\tau)\, g(t-\tau)\, d\tau \right)^{[\![-\infty,\, b_f + a_g)\!)} \\ &\oplus \left(\int_{[\![-\infty, b_f)\!)} f(\tau)\, g(t-\tau)\, d\tau \right)^{[\![b_f + a_g,\, \infty)\!)} \Bigg)(t)\end{aligned} \tag{18}$$

Each of the four end-points can either be finite or infinite ($+\infty$ if right end-point, or $-\infty$ if left) so there are $2^4 = 16$ cases.

So far, all cases seem to work, either producing indeterminate end-points which can be removed as above or making empty regions like: $[\![\infty, \infty)\!)$ that can just be ignored.