GENERALIZED INCLUSION-EXCLUSION

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Motivation

Let f and g be two piecewise functions each of two pieces:

$$f(x) = \begin{cases} f_1 & x \in [0, a] \\ f_2 & x \in (a, 1] \end{cases}$$

$$g(x) = \begin{cases} g_1 & x \in [0, b] \\ g_2 & x \in (b, 1] \end{cases}$$

Their sum $(f+g)(x)$, is a piecewise function of up to four pieces:

$$(f+g)(x) = \begin{cases} f_1 + g_1 & x \in [0,a] \cap [0,b] \\ f_1 + g_2 & x \in [0,a] \cap (b,1] \\ f_2 + g_1 & x \in (a,1] \cap [0,b] \\ f_2 + g_2 & x \in (a,1] \cap (b,1] \end{cases}$$

(some pieces may be degenerate)

More generally, when f and g have n and m pieces respectively, $(f+g)$ is a piecewise function of $n \cdot m$ pieces.

Piecewise functions are very useful theoretical constructions but pairwise intersection is not a feasible method for constructing common refinements. [1]

OUTLINE

1. Hybrid Sets and Functions
2. Symbolic Block Matrix Algebra
3. Integration over Hybrid Domains
4. Convolution of Piecewise Interval Functions

HYBRID SETS

Definition

Let U be a universe, then any function $U \to \mathbb{Z}$ is called a hybrid set. We denote the collection of all hybrid sets over a universe by $\mathbb{Z}^U$.

Definition

For a hybrid set H, the value $H(x) \in \mathbb{Z}$ is said to be the multiplicity of the element x.

- $x \in^n H \iff H(x) = n$ [3]
- $x \notin H \iff H(x) = 0 \iff x \in^0 H$
- $x \in H \iff H(x) \neq 0 \iff x \in^n H$ for $n \neq 0$

We will use the notation

$$H = \{\!\!|\ x_1{}^{m_1}, x_2^{m_2}, \ldots\ |\!\!\}$$

to denote the hybrid set given by $H(x) = \sum_{x_i = x} m_i$. This representation is *not* unique:

$$\{\!\!|\ a^1, a^1, b^{-2}, a^3, b^1\ |\!\!\} = \{\!\!|\ a^5, b^{-1}\ |\!\!\}$$

We say that H is in normalized form if all x_i are unique.

We *could* define union: $\cup$, intersection: $\cap$, and complement: $\setminus$ ...

NOT UNCOMMON TO DO SO FOR MULTI-SETS, BUT...

For multi-sets A and B, (i.e. $A, B : U \to \mathbb{N}$):

$$(A \cup B)(x) = \max(A(x), B(x))$$

$$(A \cap B)(x) = \min(A(x), B(x))$$

[3]

...THEY MAKE MORE SENSE WITH BOOLEAN PRIMITIVES.

For Boolean sets A and B, (i.e. $A, B : U \to \{0, 1\}$):

$$(A \cup B)(x) = A(x) \vee B(x)$$

$$(A \cap B)(x) = A(x) \wedge B(x)$$

$$(A \setminus B)(x) = A(x) \wedge \neg B(x)$$

Definition

For hybrid sets A and B, (i.e. $A, B : U \to \mathbb{Z}$):

$$
\begin{aligned}
(A \oplus B)(x) &= A(x) + B(x) \\
(A \ominus B)(x) &= A(x) - B(x) \\
(A \otimes B)(x) &= A(x) \cdot B(x)
\end{aligned}
$$

$$(cA)(x) = c \cdot A(x)$$

THE EMPTY (HYBRID) SET

$\emptyset \in \mathbb{Z}^U$ is the zero function (i.e. maps all of U to 0).

$\ominus A = \emptyset \ominus A$

If $A \otimes B = \emptyset$ then A and B are said to be disjoint.

A family of non-empty sets P, is a partition of a set X if and only if:

- $\bigcup_{A \in P} A = X$
- $A, B \in P$ then $A = B$ or $A \cap B = \emptyset$

Sets in a partition sum together to form a larger set with disjointness required to not "double count" any points. Why not subtract those intersections instead?

Definition

A generalized partition of X is a family of non-empty hybrid sets P such that

$$\bigoplus_{A \in P} A = X$$

(If the elements of P are disjoint, then P is a strict partition of X)

HYBRID FUNCTIONS

Definition

For sets S and T, a hybrid set over the Cartesian product $S \times T$ is called a hybrid relation between S and T.

If $(x, y) \in H$ and $(x, z) \in H$ implies that $y = z$ then H is a hybrid function from S to T. Given a function f and a hybrid set H, let f^H denote the following hybrid function:

$$f^H = \bigoplus_{x \in S} H(x) \{\!|\ (x, f(x))^1\ |\!\}$$

$$\widetilde{f}^H = \bigoplus_{x \in S} H(x) \{\!|\ (x, f)^1\ |\!\}$$

Definition

For a hybrid set H, the support of H is the (non-hybrid) set $\operatorname{supp} H = \{x \mid x \in H\}$.

If $H = 1_{\operatorname{supp}(H)}$ then we say H is a reducible hybrid set.

If A is reducible then f^A is a reducible hybrid function. In this case we define:

$$\mathcal{R}(f^A)(x) = f|_{\operatorname{supp}(A)}(x)$$

Definition

For a group $(G, *, e_*)$ with $x, y, z \in G$, define iterated $*$ $*^n$, as:

$$x *^n y = (((x \overbrace{*y) * y) * \ldots * y)}^{n \text{ times}}$$
$$*^n x = e_* *^n x \qquad (e_* \text{ is the identity})$$
$$x *^{-n} y = x *^n z \qquad (z * y = e_*)$$

Definition

For *any* hybrid relation $f_1^{A_1} \oplus f_2^{A_2} \oplus \ldots$, the $*$-reduction $\mathcal{R}_*$, is defined:

$$\mathcal{R}_*(f_1^{A_1} \oplus f_2^{A_2} \oplus \ldots)(x) = \left(*^{A_1(x)} f_1(x) *^{A_2(x)} f_2(x) * \ldots \right)\Big|_{\operatorname{supp} \bigoplus A_i}$$

Example: Piecewise Functions using Generalized Partitions

Let A_1, A_2, B_1, and B_2 be the intervals:

$$A_1 = [0, a) \qquad A_2 = [0, 1] \setminus A_1$$
$$B_1 = [0, b) \qquad B_2 = [0, 1] \setminus B_1$$

and f and g piecewise functions given by:

$$f(x) = f_1^{A_1} \oplus f_2^{A_2} = \begin{cases} f_1(x) & x \in A_1 \\ f_2(x) & x \in A_2 \end{cases}$$

$$g(x) = g_1^{B_1} \oplus g_2^{B_2} = \begin{cases} g_1(x) & x \in B_1 \\ g_2(x) & x \in B_2 \end{cases}$$

$(f+g)(x)$

Using Strict Partitions:

$$(f+g)(x) = (f_1+g_1)|_{A_1\cap B_1} \oplus (f_1+g_2)|_{A_1\cap B_2} \oplus (f_2+g_1)|_{A_2\cap B_1} \oplus (f_2+g_2)|_{A_2\cap B_2}$$

Using Generalized Partitions:

$$(f+g) = \mathcal{R}_+\left((f_1+g_1)^{A_1} \oplus (f_2+g_1)^{B_1 \ominus A_1} \oplus (f_2+g_2)^{B_2}\right)$$

Suppose $x \in A_1 \cap B_2$:

$$\begin{aligned}
(f+g)(x) &= \mathcal{R}_+ \left((f_1+g_1)^{A_1} \oplus (f_2+g_1)^{B_1 \ominus A_1} \oplus (f_2+g_2)^{B_2}\right)(x) \\
&= \mathcal{R}_+ \left((f_1+g_1)^{A_1(x)} \oplus (f_2+g_1)^{(B_1 \ominus A_1)(x)} \oplus (f_2+g_2)^{B_2(x)}\right)(x) \\
&= \mathcal{R}_+ \left((f_1+g_1)^{1} \oplus (f_2+g_1)^{-1} \oplus (f_2+g_2)^{1}\right)(x) \\
&= \left(+^1(f_1+g_1) +^{-1} (f_2+g_1) +^1 (f_2+g_2)\right)(x) \\
&= \left((f_1+g_1) - (f_2+g_1) + (f_2+g_2)\right)(x) \\
&= (f_1+g_2)(x)
\end{aligned}$$

Strict Partitions:

$$f * g = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} (f_i * g_j)\big|_{A_i \cap B_j} \qquad O(nm)$$

Generalized Partitions:

$$f * g = \mathcal{R}_*\left(\bigoplus_{i=1}^{n-1} (f_i * g_m)^{A_i} \oplus \bigoplus_{i=1}^{m-1} (f_n * g_i)^{B_i} \right. \qquad O(n + m - 1)$$
$$\left. \oplus\ (f_n * g_m)^{U \ominus (A_1 \oplus \ldots \oplus A_{n-1} \oplus B_1 \oplus \ldots \oplus B_{m-1})} \right)$$

SYMBOLIC BLOCK MATRICES

$$M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|ccc} A_{1,1} & \dots & A_{1,m} & B_{1,1} & \dots & B_{1,q} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{n,1} & \dots & A_{n,m} & B_{n,1} & \dots & B_{n,q} \\ \hline C_{1,1} & \dots & C_{1,m} & D_{1,1} & \dots & D_{1,q} \\ \vdots & & \vdots & \vdots & & \vdots \\ C_{r,1} & \dots & C_{r,m} & D_{r,1} & \dots & D_{r,q} \end{array}\right]$$

Definition

Given a totally ordered set $(X, \leq)$ and $a, b \in X$ the intervals between a and b are the following sets:

$$\begin{aligned}
[a, b]_X &= \{x \in X \mid a \leq x \leq b\} \\
[a, b)_X &= \{x \in X \mid a \leq x < b\} \\
(a, b]_X &= \{x \in X \mid a < x \leq b\} \\
(a, b)_X &= \{x \in X \mid a < x < b\}
\end{aligned}$$

Omitting subscripts when the choice of X is either obvious or unimportant.

INTERVALS DON'T ALWAYS BEHAVE NICELY

For $a < b$:

$$[b, a] = \emptyset$$

For any a, b, c:

$$[a, b) \oplus [b, c) = \begin{cases} [a, c) & a \leq b \leq c \\ [a, b) & a \leq c \leq b \\ [b, c) & b \leq a \leq c \\ \emptyset & \text{otherwise} \end{cases}$$

Definition

With $(X, \leq)$, a and b as before, the oriented intervals between a and b are:

$$\llbracket a, b \rrparenthesis_X = [a, b)_X \ominus [b, a)_X$$
$$\llparenthesis a, b \rrbracket_X = (a, b]_X \ominus (b, a]_X$$
$$\llbracket a, b \rrbracket_X = [a, b]_X \ominus (b, a)_X$$
$$\llparenthesis a, b \rrparenthesis_X = (a, b)_X \ominus [b, a]_X$$

$$\llbracket a, b \rrparenthesis = \ominus \llbracket b, a \rrparenthesis$$
$$\llparenthesis a, b \rrbracket = \ominus \llparenthesis b, a \rrbracket$$
$$\llbracket a, b \rrbracket = \ominus \llparenthesis a, b \rrparenthesis$$
$$\llparenthesis a, b \rrparenthesis = \ominus \llbracket a, b \rrbracket$$

$$\llbracket a, b \rrparenthesis \oplus \llbracket b, c \rrparenthesis = \llbracket a, c \rrparenthesis$$

Using oriented intervals, the vectors U and V can be written as hybrid functions:

$$\begin{aligned} U &= [u_1, u_2, \ldots, u_k, u'_1, u'_2, \ldots, u_{n-k}] \\ &= (i \mapsto u_i)^{[\![1,k]\!]} \oplus (i \mapsto u'_{i-k})^{(\!(k,n]\!]} \\ &= (u_i)^{[\![1,k]\!]} \oplus (u'_{i-k})^{(\!(k,n]\!]} \end{aligned}$$

$$\begin{aligned} V &= [v_1, v_2, \ldots, v_\ell, v'_1, v'_2, \ldots, v_{n-\ell}] \\ &= (i \mapsto v_i)^{[\![1,\ell]\!]} \oplus (i \mapsto v'_{i-\ell})^{(\!(\ell,n]\!]} \\ &= (v_i)^{[\![1,\ell]\!]} \oplus (v'_{i-\ell})^{(\!(\ell,n]\!]} \end{aligned}$$

$$(U + V) = \mathcal{R}_+ \left((u_i + v_i)^{[\![1,\ell]\!]} \oplus (u_i + v'_{i-\ell})^{(\!(\ell,k]\!]} \oplus (u'_{i-k} + v'_{i-\ell})^{(\!(k,n]\!]} \right)$$

Definition

The Cartesian product of hybrid sets $X \in \mathbb{Z}^S$ and $Y \in \mathbb{Z}^T$, is a hybrid set over $S \times T$ and denoted with $\times$ operator:

$$X \times Y = \{\!\!|\ (x, y)^{m \cdot n} \mid x \in^m X, y \in^n Y\ |\!\!\}$$

$$\begin{aligned} X \times Y \times Z &= \left\{\!\!\left|\ ((x, y), z)^{(m \cdot n) \cdot p} | x \in^m X, y \in^n Y, z \in^p Z\ \right|\!\!\right\} \\ &= \left\{\!\!\left|\ (x, (y, z))^{m \cdot (n \cdot p)} | x \in^m X, y \in^n Y, z \in^p Z\ \right|\!\!\right\} \end{aligned}$$

$$[\![\boldsymbol{a}, \boldsymbol{b}]\!] = [\![(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n)]\!] = [\![a_1, b_1]\!] \times [\![a_2, b_2]\!] \times \ldots \times [\![a_n, b_n]\!]$$

MATRIX ADDITION IS THE SAME, JUST WITH MORE TERMS

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \qquad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

$$\mathcal{U} = [\![0, n)\!)_{\mathbb{N}_0} \times [\![0, m)\!)_{\mathbb{N}_0}$$

$$\mathcal{A}_{11} = [\![0, q)\!) \times [\![0, r)\!) \qquad \mathcal{B}_{11} = [\![0, s)\!) \times [\![0, t)\!)$$

$$\mathcal{A}_{12} = [\![0, q)\!) \times [\![r, m)\!) \qquad \mathcal{B}_{12} = [\![0, s)\!) \times [\![t, m)\!)$$

$$\mathcal{A}_{21} = [\![q, n)\!) \times [\![0, r)\!) \qquad \mathcal{B}_{21} = [\![s, n)\!) \times [\![0, t)\!)$$

$$\mathcal{A}_{22} = [\![q, n)\!) \times [\![r, m)\!) \qquad \mathcal{B}_{22} = [\![s, n)\!) \times [\![t, m)\!)$$

$$A = A_{11}^{\mathcal{A}_{11}} \oplus A_{12}^{\mathcal{A}_{12}} \oplus A_{21}^{\mathcal{A}_{21}} \oplus A_{22}^{\mathcal{A}_{22}} \qquad B = B_{11}^{\mathcal{B}_{11}} \oplus B_{12}^{\mathcal{B}_{12}} \oplus B_{21}^{\mathcal{B}_{21}} \oplus B_{22}^{\mathcal{B}_{22}}$$

MATRIX ADDITION IS THE SAME, JUST WITH MORE TERMS

Let $\mathcal{P} = \mathcal{U} \ominus (\mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{B}_{11} \oplus \mathcal{B}_{12} \oplus \mathcal{B}_{21})$

$$\begin{aligned}(A+B) = \mathcal{R}_{+} \big(& (A_{11}+B_{22})^{\mathcal{A}_{11}} \oplus (A_{12}+B_{22})^{\mathcal{A}_{12}} \oplus (A_{21}+B_{22})^{\mathcal{A}_{21}} \\ & \oplus (A_{22}+B_{11})^{\mathcal{B}_{11}} \oplus (A_{22}+B_{12})^{\mathcal{B}_{12}} \oplus (A_{22}+B_{21})^{\mathcal{B}_{21}} \\ & \oplus \, (A_{22}+B_{22})^{\mathcal{P}} \big)\end{aligned}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \in \mathbb{R}^{n \times m} \qquad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \in \mathbb{R}^{m \times p}$$

$$A_{11} \in \mathbb{R}^{q \times r} \qquad A_{12} \in \mathbb{R}^{q \times (m-r)}$$

$$A_{21} \in \mathbb{R}^{(n-q) \times r} \qquad A_{22} \in \mathbb{R}^{(n-q) \times (m-r)}$$

$$B_{11} \in \mathbb{R}^{s \times t} \qquad B_{12} \in \mathbb{R}^{s \times (p-t)}$$

$$B_{21} \in \mathbb{R}^{(m-s) \times t} \qquad B_{22} \in \mathbb{R}^{(m-s) \times (p-t)}$$

The product will *always* be a 2×2 block matrix:

$$AB = \left[\begin{array}{c|c} ? & ? \\ \hline ? & ? \end{array}\right]$$

But the contents of those blocks depend on r and s. If $r = s$:

$$AB = \left[\begin{array}{c|c} (A_{11}B_{11} + A_{12}B_{21}) & (A_{11}B_{12} + A_{12}B_{22}) \\ \hline (A_{21}B_{11} + A_{22}B_{21}) & (A_{21}B_{12} + A_{22}B_{22}) \end{array}\right]$$

If $r > s$:

$$A = \left[\begin{array}{cc|c} A_{11}^{(1)} & A_{11}^{(2)} & A_{12} \\ \hline A_{21}^{(1)} & A_{21}^{(2)} & A_{22} \end{array}\right] \qquad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21}^{(1)} & B_{22}^{(1)} \\ B_{21}^{(2)} & B_{22}^{(2)} \end{array}\right]$$

$$AB = \left[\begin{array}{c|c} \left(A_{11}^{(1)}B_{11} + A_{11}^{(2)}B_{21}^{(1)} + A_{12}B_{21}^{(2)}\right) & \left(A_{11}^{(1)}B_{12} + A_{11}^{(2)}B_{22}^{(1)} + A_{12}B_{22}^{(2)}\right) \\ \hline \left(A_{21}^{(1)}B_{11} + A_{21}^{(2)}B_{21}^{(1)} + A_{22}B_{21}^{(2)}\right) & \left(A_{21}^{(1)}B_{12} + A_{21}^{(2)}B_{22}^{(1)} + A_{22}B_{22}^{(2)}\right) \end{array}\right]$$

If $r < s$:

$$A = \left[\begin{array}{c|cc} A_{11} & A_{12}^{(1)} & A_{12}^{(2)} \\ \hline A_{21} & A_{21}^{(1)} & A_{22}^{(2)} \end{array}\right] \qquad B = \left[\begin{array}{c|c} B_{11}^{(1)} & B_{12}^{(1)} \\ B_{11}^{(2)} & B_{12}^{(2)} \\ \hline B_{21} & B_{22} \end{array}\right]$$

$$AB = \left[\begin{array}{c|c} \left(A_{11}B_{11}^{(1)} + A_{12}^{(1)}B_{11}^{(2)} + A_{12}^{(2)}B_{21}\right) & \left(A_{11}B_{12}^{(1)} + A_{12}^{(1)}B_{12}^{(2)} + A_{12}^{(2)}B_{22}\right) \\ \hline \left(A_{21}B_{11}^{(1)} + A_{22}^{(1)}B_{11}^{(2)} + A_{22}^{(2)}B_{21}\right) & \left(A_{21}B_{12}^{(1)} + A_{22}^{(1)}B_{12}^{(2)} + A_{22}^{(2)}B_{22}\right) \end{array}\right]$$

SETTING UP REGIONS OF a, b, c

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \cdot \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array}\right]$$

$$N_1 = [\![0, q)\!)_{\mathbb{N}_0} \qquad N_2 = [\![q, n)\!)_{\mathbb{N}_0}$$

$$M_1 = [\![0, r)\!)_{\mathbb{N}_0} \qquad M_2 = [\![r, s)\!)_{\mathbb{N}_0} \qquad M_3 = [\![s, m)\!)_{\mathbb{N}_0}$$

$$P_1 = [\![0, t)\!)_{\mathbb{N}_0} \qquad P_2 = [\![t, p)\!)_{\mathbb{N}_0}$$

$$A = A_{11}^{N_1 \times M_1} \oplus A_{12}^{N_1 \times (M_2 \oplus M_3)} \oplus A_{21}^{N_2 \times M_1} \oplus A_{22}^{N_2 \times (M_2 \oplus M_3)}$$

$$B = B_{11}^{(M_1 \oplus M_2) \times P_1} \oplus B_{12}^{(M_1 \oplus M_2) \times P_2} \oplus B_{21}^{M_3 \times P_1} \oplus B_{12}^{M_3 \times P_2}$$

$$C = C_{11}^{N_1 \times P_1} \oplus C_{12}^{N_1 \times P_2} \oplus C_{21}^{N_2 \times P_1} \oplus C_{22}^{N_2 \times P_2}$$

$$
\begin{aligned}
C_{i,j}^{N_i \times P_j}(x, y) = \sum_{M} \mathcal{R}_{\times} \Bigg(\; & A_{i,1}^{N_1 \times M_1}\Big|_{X=x} \oplus B_{1,j}^{M_1 \times P_1}\Big|_{Y=y} \\
\oplus\; & A_{i,2}^{N_1 \times M_2}\Big|_{X=x} \oplus B_{1,j}^{M_2 \times P_1}\Big|_{Y=y} \\
\oplus\; & A_{i,2}^{N_1 \times M_3}\Big|_{X=x} \oplus B_{2,j}^{M_3 \times P_1}\Big|_{Y=y} \; \Bigg)
\end{aligned}
$$

$$M = \begin{bmatrix} M[0,0] & \dots & M[0,n] \\ \vdots & & \vdots \\ M[m,0] & \dots & M[m,n] \end{bmatrix}$$

$$M|_{X=i} = \begin{bmatrix} M[i,0] & \dots & M[i,n] \end{bmatrix} \qquad M|_{Y=j} = \begin{bmatrix} M[0,j] \\ \vdots \\ M[m,j] \end{bmatrix}$$

This transforms $M : (X \times Y) \to Z$ into:

$$M|_{X=x} : Y \to (X \to Z) \qquad M|_{Y=y} : X \to (Y \to Z)$$

$\mathcal{R}_\times$ will group elements by m:

$$m \mapsto (x \mapsto A[x][m]) \qquad\qquad m \mapsto (y \mapsto B[m][y])$$

And then flatten with iterated $\times$:

$$(x \mapsto A[x][m]) \times (y \mapsto B[m][y]) = (x, y) \mapsto A[x][m] \cdot B[m][y]$$

Resulting in terms of the form:

$$m \mapsto \Big((x, y) \mapsto A[x][m] \cdot B[m][y]\Big)$$

Finally sum for $m \in M$:

$$(x, y) \mapsto \sum_{m \in M} A[x][m] \cdot B[m][y]$$

EQUATION FOR C

$$
\begin{aligned}
C_{i,j}^{N_i \times P_j}(x,y) = \sum_{M} \mathcal{R}_{\times} \Bigg(\; & A_{i,1}^{N_1 \times M_1}\Big|_{X=x} \oplus B_{1,j}^{M_1 \times P_1}\Big|_{Y=y} \\
\oplus \; & A_{i,2}^{N_1 \times M_2}\Big|_{X=x} \oplus B_{1,j}^{M_2 \times P_1}\Big|_{Y=y} \\
\oplus \; & A_{i,2}^{N_1 \times M_3}\Big|_{X=x} \oplus B_{2,j}^{M_3 \times P_1}\Big|_{Y=y} \Bigg)
\end{aligned}
$$

Example: Matrix Multiplication

EXAMPLE: MATRIX MULTIPLICATION

$$Q = \left[\begin{array}{cc|c} a_1 & a_2 & b_1 \\ a_3 & a_4 & b_2 \\ \hline c_1 & c_2 & d_1 \\ c_3 & c_4 & d_2 \end{array}\right] \qquad R = \left[\begin{array}{c|cccc} e_1 & f_1 & f_2 & f_3 & f_4 \\ \hline g_1 & h_1 & h_2 & h_3 & h_4 \\ g_2 & h_5 & h_6 & h_7 & h_8 \end{array}\right]$$

$$N_1 = [\![0, 1]\!] = \{\!|\ 0^1, 1^1\ |\!\} \quad N_2 = [\![2, 3]\!] = \{\!|\ 2^1, 3^1\ |\!\}$$

$$M_1 = [\![0, 1]\!] = \{\!|\ 0^1, 1^1\ |\!\} \quad M_2 = (\!(1, 1)\!) = \{\!|\ 1^{-1}\ |\!\} \quad M_3 = [\![1, 2]\!] = \{\!|\ 1^1, 2^1\ |\!\}$$

$$P_1 = [\![0, 0]\!] = \{\!|\ 0^1\ |\!\} \quad P_2 = [\![1, 4]\!] = \{\!|\ 1^1, 2^1, 3^1, 4^1\ |\!\}$$

$$Q = A^{N_1 \times M_1} \oplus B^{N_1 \times (M_2 \oplus M_3)} \oplus C^{N_2 \times M_1} \oplus D^{N_2 \times (M_2 \oplus M_3)}$$

$$R = E^{(M_1 \oplus M_2) \times P_1} \oplus F^{(M_1 \oplus M_2) \times P_2} \oplus G^{M_3 \times P_1} \oplus H^{M_3 \times P_2}$$

$$Q \cdot R = S = {S_1}^{N_1 \times P_1} \oplus {S_2}^{N_1 \times P_2} \oplus {S_3}^{N_2 \times P_1} \oplus {S_4}^{N_2 \times P_2}$$

EXAMPLE: EVALUATING S_1

$$\begin{aligned} {S_1}^{N_1 \times P_1}(i,j) = \sum_{M_1 \oplus M_2 \oplus M_3} \mathcal{R}_\times \Big(& A^{N_1 \times M_1}\big|_{X=i} \oplus E^{M_1 \times P_1}\big|_{Y=j} \oplus \\ & B^{N_1 \times M_2}\big|_{X=i} \oplus E^{M_2 \times P_1}\big|_{Y=j} \oplus \\ & B^{N_1 \times M_3}\big|_{X=i} \oplus G^{M_3 \times P_1}\big|_{Y=j} \Big) \end{aligned}$$

EXAMPLE: EVALUATING S_1

$$= \sum_{\{\!|\, 0^1, 1^1, 2^1 \,|\!\}} \mathcal{R}_\times \Bigg(\left\{\!\!\left| \left(0 \mapsto \begin{bmatrix} a_1 \\ a_3 \end{bmatrix}\right)^{+1}, \left(1 \mapsto \begin{bmatrix} a_2 \\ a_4 \end{bmatrix}\right)^{+1} \right|\!\!\right\} \qquad \left(A^{N_1 \times M_1}\big|_{X=i} \right)$$

$$\oplus \left\{\!\left| (0 \mapsto [e_1])^{+1}, (1 \mapsto [e_\perp])^{+1} \right|\!\right\} \qquad \left(E^{M_2 \times P_1}\big|_{Y=j} \right)$$

$$\oplus \left\{\!\!\left| \left(1 \mapsto \begin{bmatrix} b_\perp \\ b_\perp \end{bmatrix}\right)^{-1} \right|\!\!\right\} \qquad \left(B^{N_1 \times M_2}\big|_{X=i} \right)$$

$$\oplus \left\{\!\left| (1 \mapsto [e_\perp])^{-1} \right|\!\right\} \qquad \left(E^{M_2 \times P_1}\big|_{Y=j} \right)$$

$$\oplus \left\{\!\!\left| \left(1 \mapsto \begin{bmatrix} b_\perp \\ b_\perp \end{bmatrix}\right)^{+1}, \left(2 \mapsto \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\right)^{+1} \right|\!\!\right\} \qquad \left(B^{N_1 \times M_3}\big|_{X=i} \right)$$

$$\oplus \left\{\!\left| (1 \mapsto [g_1])^{+1}, (2 \mapsto [g_2])^{+1} \right|\!\right\} \Bigg) \qquad \left(G^{M_3 \times P_1}\big|_{Y=j} \right)$$

$$= \sum_{\{\!|\; 0^1, 1^1, 2^1 \;|\!\}} \{\!|\; \left(0 \mapsto \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} \times^{+1} [e_1]\right),$$

$$\left(1 \mapsto \begin{bmatrix} a_2 \\ a_4 \end{bmatrix} \times^{+1} [e_\perp] \times^{-1} \begin{bmatrix} b_\perp \\ b_\perp \end{bmatrix} \times^{-1} [e_\perp] \times^{+1} \begin{bmatrix} b_\perp \\ b_\perp \end{bmatrix} \times^{+1} [g_1]\right),$$

$$\left(2 \mapsto \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \times^{+1} [g_2]\right) \;|\!\}$$

$$= \sum_{\{\!|\; 0^1, 1^1, 2^1 \;|\!\}} \{\!|\; \left(0 \mapsto \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} [e_1]\right), \quad \left(1 \mapsto \begin{bmatrix} a_2 \\ a_4 \end{bmatrix} [g_1]\right), \quad \left(2 \mapsto \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} [g_2]\right) \;|\!\}$$

EXAMPLE: EVALUATING S_1

$$
\begin{aligned}
{S_1}^{N_1 \times P_1} &= \begin{bmatrix} a_1 \\ a_3 \end{bmatrix} [e_1] + \begin{bmatrix} a_2 \\ a_4 \end{bmatrix} [g_1] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} [g_2] \\
&= \begin{bmatrix} a_1 e_1 + a_2 g_1 + b_2 g_2 \\ a_3 e_1 + a_4 b_1 + b_2 g_2 \end{bmatrix}
\end{aligned}
$$

EXAMPLE: EVALUATING S_1

$$S = \left[\begin{array}{c|c} \begin{matrix} a_1 e_1 + a_2 g_1 + b_2 g_2 \\ a_3 e_1 + a_4 b_1 + b_2 g_2 \end{matrix} & S_2 \\ \hline S_3 & S_4 \end{array}\right]$$

INTEGRATION OVER HYBRID DOMAINS

$$\int_a^b f(x)\,\mathrm{d}x \neq \int_{[a,b)} f(x)\,\mathrm{d}x$$

$$\int_a^b f(x)\,\mathrm{d}x = \int_{[\![a,b))} f(x)\,\mathrm{d}x$$

For a measure space (X, Σ, μ) the integral of an indicator function 1_S of a measurable set $S \in \Sigma$ is just the measure of S:

$$\int 1_S \, \mathrm{d}\mu = \mu(S)$$

A simple function s is a function such that for a family of measurable sets $\{A_k\}_{k=0}^{n}$ and matching coefficients $\{a_k\}_{k=0}^{n}$.

$$s = \sum_{k=0}^{n} a_k 1_{A_k}$$

The integral of a simple function is:

$$\int s \, \mathrm{d}\mu = \sum_{k=0}^{n} a_k \cdot \mu(A_k)$$

μ extends easily from $\mu : \Sigma \to \mathbb{R}$ to the signed measure $\mu : \mathbb{Z}^{\Sigma} \to \mathbb{R}$:

$$\mu(H) = \sum_{x \in^m H} m \cdot \mu(x)$$

Which allows us to integrate simple functions over hybrid set domains:

$$\int_H s \, \mathrm{d}\mu = \sum_{k=0}^{n} a_k \cdot \mu(H \otimes A_k)$$

For non-negative function f:

$$\int_H f \, \mathrm{d}\mu = \sup \left\{ \int_H s \, \mathrm{d}\mu \;\middle|\; s \text{ simple, and } 0 \leq s \leq f \right\}$$

If f takes negative values then we can split it into two functions:

$$f^+(x) = \max(0, f(x))$$
$$f^-(x) = \max(0, -f(x))$$

...and integrate each as non-negative functions:

$$\int_H f \, \mathrm{d}\mu = \int_H f^+ \, \mathrm{d}\mu - \int_H f^- \, \mathrm{d}\mu$$

$$\int_{1}^{0} 1_{\mathbb{R}\setminus\mathbb{Q}} \,\mathrm{d}\mu = \int_{[\![1,0]\!]} 1_{\mathbb{R}\setminus\mathbb{Q}} \,\mathrm{d}\mu = \mu([\![1,0]\!] \otimes \mathbb{R} \setminus \mathbb{Q}) = -1$$

Oriented Intervals can also be used to cleanly express chains.

Definition

For a k-rectangle in $\mathbb{R}^n$ $[\![\boldsymbol{a},\boldsymbol{b}]\!]_{\mathbb{R}^n} = [\![a_1,b_1]\!]_{\mathbb{R}} \times \ldots [\![a_n,b_n]\!]_{\mathbb{R}}$, Define ∂, the boundary of $[\![\boldsymbol{a},\boldsymbol{b}]\!]$ as:

$$\partial\left([\![\boldsymbol{a},\boldsymbol{b}]\!]\right) = \bigoplus_{j=1}^{k}(-1)^j \left(\left[\!\left[\left(\boldsymbol{a}^{[\![1,n]\!]_{\mathbb{N}}}\right), \quad \left(\boldsymbol{b}^{[\![1,i_j)\!)_{\mathbb{N}}} \oplus \boldsymbol{a}^{\{\!|\ i_j\ |\!\}} \oplus \boldsymbol{b}^{(\!(i_j,n]\!]_{\mathbb{N}}}\right) \right]\!\right]_{\mathbb{R}^n} \right.$$
$$\left. \ominus \left[\!\left[\left(\boldsymbol{a}^{[\![1,i_j)\!)_{\mathbb{N}}} \oplus \boldsymbol{b}^{\{\!|\ i_j\ |\!\}} \oplus \boldsymbol{a}^{(\!(i_j,n]\!]_{\mathbb{N}}}\right), \quad \left(\boldsymbol{b}^{[\![1,n]\!]_{\mathbb{N}}}\right) \right]\!\right]_{\mathbb{R}^n} \right)$$

Where the sequence $i_1,\ldots,i_k$ is the unique non-decreasing sequence of indices such that $a_{i_j} \neq b_{i_j}$.

Definition

A *k-chain* C, is a linear combination of k-rectangles $\{c_i\}$, with integer coefficients $\{\lambda_i\}$:

$$C = \bigoplus_i \lambda_i c_i$$

$$\int_{\partial C} \omega = \int_C \mathrm{d}\omega \qquad \text{(Stokes' Theorem)}$$

SYMBOLIC CONVOLUTION

Definition

The convolution $*$, of two functions F and G is defined as:

$$(F * G)(t) = \int_{-\infty}^{\infty} F(\tau)\, G(t - \tau)\, \mathrm{d}\tau$$

Definition

An *n-piece interval function* is a function f of the form:

$$f = \sum_{i=1}^{n} f_i^{P_i}$$

where $\{P_i\}_{i=1}^{n}$ are disjoint intervals and

$$f_i^{P_i}(x) = \begin{cases} f_i(x) & x \in P_i \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}\left(\left(\sum_i f_i^{P_i}\right) * \left(\sum_j g_j^{Q_j}\right)\right)(t) &= \int_{-\infty}^{\infty} \left(\sum_i f_i^{P_i}\right)(\tau) \cdot \left(\sum_j g_j^{Q_j}\right)(t-\tau)\,\mathrm{d}\tau \\ &= \sum_i \sum_j \int_{-\infty}^{\infty} f_i^{P_i}(\tau) \cdot g_j^{Q_j}(t-\tau)\,\mathrm{d}\tau \\ &= \sum_i \sum_j \left(f_i^{P_i} * g_j^{P_j}\right)\end{aligned}$$

$$
\begin{aligned}
(F * G)(t) &= \int_{-\infty}^{\infty} F(\tau)G(t-\tau)\,\mathrm{d}\tau \\
&= \int_{\infty}^{-\infty} F(t-\tau')G(\tau')\,(-1)\,\mathrm{d}\tau' && (\tau' = t - \tau) \\
&= \int_{-\infty}^{\infty} G(\tau')F(t-\tau')\,\mathrm{d}\tau' \\
&= (G * F)(t)
\end{aligned}
$$

TRADITIONAL FORMULATION

Commute F and G so that $b_f - a_f \leq b_g - a_g$, then use either:

$$(F * G)(t) = \begin{cases} \int_{a_f}^{t-a_g} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau & (a_f + a_g) \leq t < (b_f + a_g) \\ \int_{a_f}^{b_f} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau & (b_f + a_g) \leq t < (a_f + b_g) \\ \int_{t-b_g}^{b_f} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau & (a_f + b_g) \leq t < (b_f + b_g) \\ 0 & \text{otherwise} \end{cases}$$

[4, 5]

$$(F * G)(t) = \begin{cases} \int_{\max(a_f,\, t-b_g)}^{\min(b_f, t-a_g)} f(\tau) \cdot g(t-\tau)\, \mathrm{d}\tau & (a_f + a_g) \leq t < (b_f + b_g) \\ 0 & \text{otherwise} \end{cases}$$

[6, 7]

$$(f^{[a_f,b_f)} * g^{[a_g,b_g)})(t) = \mathcal{R}_+ \Bigg(\left(\int_{[\![a_f,\, t-a_g)\!)} f(\tau)\, g(t-\tau)\,\mathrm{d}\tau \right)^{[\![a_f+a_g,\, b_f+a_g)\!)} \oplus \left(\int_{[\![a_f,\, b_f)\!)} f(\tau)\, g(t-\tau)\,\mathrm{d}\tau \right)^{[\![b_f+a_g,\, a_f+b_g)\!)} \oplus \left(\int_{[\![t-b_g,\, b_f)\!)} f(\tau)\, g(t-\tau)\,\mathrm{d}\tau \right)^{[\![a_f+b_g,\, b_f+b_g)\!)} \Bigg)(t)$$

Example: Convolution

Suppose $(a_f + b_g) \leq t < (b_f + a_g)$:

$$
\begin{aligned}
(f^{[a_f, b_f)} \ast g^{[a_g, b_g)})(t) = \mathcal{R}_+ \Bigg(& \left(\int_{[\![a_f,\, t-a_g))} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau \right)^{[\![a_f+a_g,\, b_f+a_g))} \\
& \oplus \left(\int_{[\![a_f,\, b_f))} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau \right)^{[\![b_f+a_g,\, a_f+b_g))} \\
& \oplus \left(\int_{[\![t-b_g,\, b_f))} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau \right)^{[\![a_f+b_g,\, b_f+b_g))} \Bigg)(t)
\end{aligned}
$$

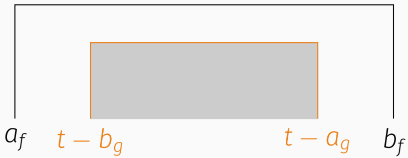
$$
\begin{aligned}
(f^{[a_f,b_f)} * g^{[a_g,b_g)})(t) = \mathcal{R}_+ \Bigg(& \left(\int_{[\![a_f,\, t-a_g)\!)} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau\right)^{+1} \\
& \oplus \left(\int_{[\![a_f,\, b_f)\!)} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau\right)^{-1} \\
& \oplus \left(\int_{[\![t-b_g,\, b_f)\!)} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau\right)^{+1} \Bigg)(t)
\end{aligned}
$$

EXAMPLE: CONVOLUTION

$$
\begin{aligned}
(f^{[a_f,b_f)} * g^{[a_g,b_g)})(t) &= \Bigg(\int_{[\![a_f,\, t-a_g)\!)} f(\tau)\, g(t-\tau)\,\mathrm{d}\tau \\
&\qquad - \int_{[\![a_f,\, b_f)\!)} f(\tau)\, g(t-\tau)\,\mathrm{d}\tau \\
&\qquad + \int_{[\![t-b_g,\, b_f)\!)} f(\tau)\, g(t-\tau)\,\mathrm{d}\tau \Bigg)(t) \\
&= \Bigg(\int_{[\![a_f,\, t-a_g)\!) \ominus [\![a_f,\, b_f)\!) \oplus [\![t-b_g,\, b_f)\!)} f(\tau)\, g(t-\tau)\,\mathrm{d}\tau \Bigg)(t) \\
&= \Bigg(\int_{[\![t-b_g,\, t-a_g)\!)} f(\tau)\, g(t-\tau)\,\mathrm{d}\tau \Bigg)(t)
\end{aligned}
$$

$a_f < b_f$ and $a_g < b_g$ by construction.

$(a_f + b_g) \leq t < (b_f + a_g) \implies a_f \leq (t - b_g)$ and $(t - a_g) < b_f$.



$$\left(f^{[a_f, b_f)} * g^{[a_g, b_g)}\right)(t) = \left(\int_{[\![t - b_g,\, t - a_g)\!)} f(\tau)\, g(t - \tau)\, \mathrm{d}\tau \right)(t)$$

IF WE ALLOW $a_f = -\infty$ AND $b_g = \infty$
THEN $(a_f + b_g)$ **IS UNDEFINED**.

NORMALLY, THIS WOULD BE A PROBLEM. [5]

EXAMPLE: INFINITE ENDPOINTS

Suppose $a_f = -\infty$ and $b_g = \infty$.

$$(f^{[a_f, b_f)} \ast g^{[a_g, b_g)})(t) = \mathcal{R}_+ \Bigg(\left(\int_{[\![a_f,\, t - a_g)\!)} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau \right)^{[\![a_f + a_g,\, b_f + a_g)\!)} \oplus \left(\int_{[\![a_f,\, b_f)\!)} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau \right)^{[\![b_f + a_g,\, a_f + b_g)\!)} \oplus \left(\int_{[\![t - b_g,\, b_f)\!)} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau \right)^{[\![a_f + b_g,\, b_f + b_g)\!)} \Bigg)(t)$$

...and assume t, a_g, b_g are all finite.

$$(f^{[-\infty,b_f)} \ast g^{[a_g,\infty)})(t) = \mathcal{R}_+ \left(\left(\int_{[\![-\infty,\, t-a_g)\!)} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau \right)^{[\![-\infty,\, b_f+a_g)\!)} \oplus \left(\int_{[\![-\infty,\, b_f)\!)} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau \right)^{[\![b_f+a_g,\, a_f+b_g)\!)} \oplus \left(\int_{[\![-\infty,\, b_f)\!)} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau \right)^{[\![a_f+b_g,\, \infty)\!)} \right)(t)$$

The first and second terms collapse by $f^A \oplus f^B = f^{A \oplus B}$.

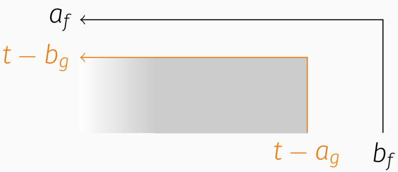
Now we have two oriented intervals with endpoints that cancel:

$$[\![b_f + a_g,\ a_f + b_g)\!) \oplus [\![a_f + b_g,\ \infty)\!) = [\![b_f + a_g,\ \infty)\!)$$

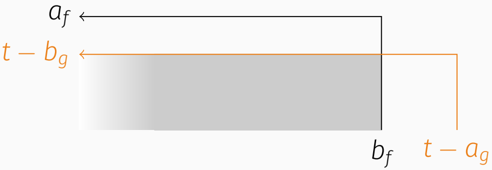
$$(f^{[-\infty, b_f)} * g^{[a_g, \infty)})(t) = \mathcal{R}_+ \left(\left(\int_{[\![-\infty,\ t-a_g)\!)} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau \right)^{[\![-\infty,\ b_f+a_g)\!)} \oplus \left(\int_{[\![-\infty,\ b_f)\!)} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau \right)^{[\![b_f+a_g,\ \infty)\!)} \right)(t)$$

EXAMPLE: INFINITE ENDPOINTS

$t \in [\![-\infty, b_f + a_g)\!)$: $\int_{-\infty}^{t-a_g} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau$



$t \in [\![b_f + a_g, \infty)\!)$: $\int_{-\infty}^{b_f} f(\tau)\, g(t-\tau)\, \mathrm{d}\tau$



SUMMARY

Defined and introduced notation for hybrid sets and functions.

Introduced $\mathcal{R}_*$ in place of $\oplus^*$.

Showed how this can be applied towards piecewise functions.

Introduced oriented intervals.

Used hybrid functions to better compute block matrix addition.

Used hybrid functions for block matrix multiplication.

Hybrid sets over a measure space allow for signed Lebesgue integrals.

Oriented intervals can be used to represent chains.

Convolution of symbolic piecewise interval functions can be performed by decomposing into “one-piece” functions.

Using hybrid functions, represent convolution with minimal cases without needing to commute operands.

This method handles symbolic and infinite endpoints.

Questions?

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