

DOMAIN DECOMPOSITION, INTEGRATION, AND  
INCLUSION-EXCLUSION  
(Thesis format: Monograph)

by

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Graduate Program in Computer Science

A thesis submitted in partial fulfillment  
of the requirements for the degree of  
Masters of Science

The School of Graduate and Postdoctoral Studies  
The University of Western Ontario  
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**Domain Decomposition, Integration, and Inclusion-Exclusion**

is accepted in partial fulfillment of the  
requirements for the degree of  
Masters of Science

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# Abstract

Mathematic notation has been dominated by sets and, when repeated elements are required, sequences generally make an appearance. Historical inertia has caused these structures to be used in many situations where they are ill-suited often evidenced by phrases like "without loss of generality", "up to ordering of terms", "up to a sign". However, by tackling these problems instead with more apt data structures, we can eliminate some of these stipulations and more formally reduce symmetric cases in reasoning. In particular, this thesis will deal with *hybrid sets* (that is, signed multisets), as well *hybrid functions* (that is, functions with hybrid sets for their domain) with applications in piecewise functions, integration on manifolds, (...). More than just an aesthetic change, by allowing negative multiplicity (even if it would not make physical sense), we may symbolically manipulate structures in ways that might otherwise be cumbersome or inefficient.

**Keywords:** Hybrid set, Signed multiset, Integration on chains, Inclusion-Exclusion, (...)

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# Chapter 1

## Introduction

### 1.1 Motivation

Some data structures in mathematics are more well-loved than others and none more than Cantor's set. The very foundations of mathematics lie in set theory: numbers are defined in terms of sets as are ordered tuples which in turn lead to relations, functions, sequences and from there branches into countless other structures. But this trunk typically omits a satisfactory treatment of several *generalized sets*. For example, it is difficult to begin speaking about *hybrid sets* without immediately punctuating, "that is, multisets with negative multiplicity". It is a statement of progress that multisets have even entered into (relatively) common mathematical parlance.

Still, sequences need no introduction and are generally relied on when a structure allowing repeated elements is needed. The addition of an ordering is often not even needed but "surely it can't hurt?" Consider the *Fundamental Theorem of Arithmetic*:

"Every positive integer, except 1, is a product of primes." ... "The standard form of  $n$  is unique; apart from rearrangement of factors,  $n$  can be expressed as a product of primes in one way only." (Hardy and Wright 1979, p.2-3)

By recognizing the possibility of rearranging factors, the authors implicitly define type the "product of primes" as a sequence. But for iterated commutative operators (e.g.  $\sum, \prod, \cap, \cup$ ), the order of terms is irrelevant. So then, why order terms to begin with? Reisig [13] uses

multisets to define relation nets where “... several individuals of some sort do not have to be distinguished” and furthermore “One should not be forced to distinguish individuals if one doesn’t wish to. This would lead to overspecification”. The same applies here. Secondly, iterated operators over an empty set is simply the respective identity ( $\prod_{x \in \emptyset} x = 1$ ), and so 1 is a product of primes. Despite the empty sequence being just as well-defined as the empty set; it tends to be treated as an aberrant case. Some definitions even disregard the singleton sequence to say, “is prime or the product of primes”.

Stripped of these qualifications we are left with simply:

“Every positive integer is the product of a unique multiset of primes.”

Although the definition is equivalent, by using appropriate data structures, the result is more elegant and less case-based reasoning for later uses of the definition. In this spirit that we will graft hybrid sets into areas of mathematics where conventional structures don’t fit as tightly as we’d like.

## 1.2 Objectives

This thesis will include and extend the work of [8] on hybrid sets and their applications. In particular, integration is a natural application of signed domains that had not been explored from this perspective. Take the identity:

$$\int_a^b f(x) \, dx = - \int_b^a f(x) \, dx \quad (1.1)$$

The domain of integration on the left-side is considered to be the interval  $[a, b]$  or in set builder notation,  $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ . It would follow then that the right hand should have domain  $[b, a]$ . Under traditional set definitions, this is not well defined. Using hybrid sets allows us to give meaning to an inverted interval. When generalized to higher dimensions, this goal becomes an attempt to unify the *Lebesgue integral* and *integration of forms*. Such a model



would allow for integration of differential forms over subsets of manifolds.

I don't currently have a set of objectives for the Petri net chapter. Another couple lines will go here once I have a better idea what the chapter will look like.

## 1.3 Related Work

It is difficult to date the origin of multisets. The term was coined by N.G. de Bruijn in correspondence with Donald Knuth, [11] but thought of as a “collection of objects that may or may not be distinguished” is as old as tally marks. In regards to the generalization to *signed* multisets, Hailperin [10] suggests that Boole's 1854 *Laws of Thought* [7] is a treatise of signed multisets. Whether this was Boole's intent is debated [cite]. Sets with negative membership explicitly began to appear in [17] and were formalized under the name Hybrid sets in Blizard's extensive work with generalized sets [5, 6] Although hybrid set and signed multiset are the most common nomenclature, other names appearing in literature include multiset (specifying positive when for unsigned multisets) [13] and integral multiset [18].

Existing explicit applications of hybrid sets are currently limited. Loeb *et al.* [9, 12] use hybrid sets to generalize several combinatoric identities to negative values. Bailey *et al.* [1] and Banâtre *et al.* [2] have also had success with hybrid sets in chemical programming. Representing a solution is represented as a collection of atoms and molecules, negative multiplicities are treated as “antimatter”. For a deeper overview and systemization of generalized sets, see [15, 16]. Finally, Bartoletti [3, 4] and Schmidt [14] also investigate petri and relation nets (to be examined in chapter 5), albeit not from the perspective of hybrid sets.

## 1.4 Thesis Outline

In chapter 2, the foundations for hybrid sets and functions with hybrid set domains will be laid. Some immediate applications to piecewise functions will be presented. In chapter 3, the algebra of domains will be more deeply explored with generalized partitions and inclusion-exclusion

and present an method for computing the support of a hybrid set. In chapter 4, hybrid functions will be applied towards integration. Starting from foundations we will use hybrid functions to unify integration of forms with integration on subsets of manifolds and prove Stokes' theorem on the new model. Finally in chapter 5, hybrid sets will be applied to Petri and relation nets with the scope of ...

# Chapter 2

## Generalized Partitions

### 2.1 Piecewise Functions

To Do: Blending “piecewise are everywhere”

The perennial example of a piecewise function is  $\text{abs} : \mathbb{R} \rightarrow \mathbb{R}_+ \cup \{0\}$  given in the form:

$$\text{abs}(x) = \begin{cases} -x & : x < 0 \\ x & : x \geq 0 \end{cases} \quad (2.1)$$

To evaluate  $\text{abs}$  for an argument  $x$ , one must first determine which sub-function to use. If  $x < 0$  then the first case is evaluated and  $\text{abs}$  will return the result of  $x \mapsto -x$ . Otherwise, if  $x \geq 0$  the second case is evaluated and the result of  $x \mapsto x$  is returned. Rather than as a condition, we could just as easily think of “ $x < 0$ ” and “ $x \geq 0$ ” as partitions of the real line. Evaluation then occurs by checking whether  $x \in \mathbb{R}_+ \cup \{0\}$  or  $x \in \mathbb{R}_-$ . In general, a piecewise function  $f$  will take the form:

$$f(x) = \begin{cases} f_1(x) & : x \in P_1 \\ f_2(x) & : x \in P_2 \\ \vdots & \vdots \\ f_n(x) & : x \in P_n \end{cases} \quad (2.2)$$

where the set  $\{P_i\}$  forms a partition of the domain of  $f$  and for each  $f_i$  is defined over all of the corresponding  $P_i$ . To formalize this we require the ability to restrict a function's domain and join disjoint pieces together.

**Definition** Given a function  $f : X \rightarrow Y$  for any subset of the domain,  $Z \subset X$ , the *restriction of  $f$  to  $Z$*  is the function  $f|_Z : Z \rightarrow Y$ , such that  $f|_Z(x) = f(x)$  for all  $x \in Z$ .

**Definition** Define  $\bar{\oplus}$ , the *join* of two functions,  $f$  and  $g$  by:

$$f \bar{\oplus} g = \begin{cases} f(x) & \text{if } g(x) = \perp \\ g(x) & \text{if } f(x) = \perp \\ \perp & \text{otherwise} \end{cases} \quad (2.3)$$

Is there a way to define without using piecewise functions?

which would allow us to re-write our previous definition of (2.2) as:

$$f = f|_{P_1} \bar{\oplus} f|_{P_2} \bar{\oplus} \dots \bar{\oplus} f|_{P_n} \quad (2.4)$$

To Do: problem with this approach + join

But we must be careful as this definition is not associative.

Let  $x \in A \cap B \cap C$ , then  $((f|_A \bar{\oplus} g|_B) \bar{\oplus} h|_C)(x) = h(x)$  but  $(f|_A \bar{\oplus} (g|_B \bar{\oplus} h|_C))(x) = f(x)$

Other conventions exist, for example *Maple's* `piecewise(cond_1, f_1, cond_2, f_2, ..., cond_n, f_n, f_otherwise)` effectively uses a short-circuited  $\bar{\oplus}$ ; it takes the first sub-function,  $f_i$ , such that the corresponding condition,  $cond_i$ , evaluates to *true*.

This approach simply trades associativity for commutivity.

In this section we will construct a formal system to manipulate partial functions more elegantly.

## 2.2 Hybrid Sets

To Do: join

We shall consider *hybrid sets*: an extension of multisets which has multiplicities ranging over  $\mathbb{N}_0$ , a hybrid set has multiplicities over all of  $\mathbb{Z}$ .

However first we must establish *partial sets* (unrelated to a *poset* or “partially ordered set”).

Similar to a partial function being only partially defined over its domain, for a partial set, there may exist some items for which membership is undefined.

For an underlying set  $U$  we will consider a hybrid set as a function  $U \rightarrow \mathbb{Z}$  as a way to track the multiplicities of any particular element.

**Definition** Let  $U$  be a universe, then any function  $U \rightarrow \mathbb{Z}$  is called a *hybrid set*.

On its own, this definition does us very little good; much of the usefulness of sets is derived from their rich notation.

**Definition** Let  $H$  be a hybrid set. Then we say that  $H(x)$  is the *multiplicity* of the element  $x$ . We write,  $x \in^n H$  if  $H(x) = n$ . Furthermore we will use  $x \in H$  to denote  $H(x) \neq 0$  (or equivalently,  $x \in^n H$  for  $n \neq 0$ ). Conversely,  $x \notin H$  denotes  $x \in^0 H$  or  $H(x) = 0$ . The symbol  $\emptyset$  will be used to denote the empty hybrid set for which all elements have multiplicity 0. Finally the support of a hybrid set, is the (non-hybrid) set  $\text{supp } H$  where  $x \in \text{supp } H$  if and only if  $x \in H$

We will use the notation:

$$H = \{ \{ x_1^{m_1}, x_2^{m_2}, \dots \} \}$$

to describe the hybrid set  $H$  where the element  $x_i$  has multiplicity  $m_i$ . We allow for repetitions in  $\{x_i\}$  but interpret the overall multiplicity of an element  $x_i$  by the sum of multiplicities among copies. Using Iverson brackets:

$$H(x) = \sum_{x_i \in m_i H} [x = x_i] m_i \quad (2.5)$$

For example,  $H = \langle a^1, a^1, b^{-2}, a^3, b^1 \rangle = \langle a^5, b^{-1} \rangle$ . A writing in which  $x_i \neq x_j$  for all  $i \neq j$  is referred to as a *normalized form* of a hybrid set. For normalized hybrid sets it follows that  $H(x_i) = m_i$ .

Traditional sets use the operations  $\cup$  union,  $\cap$  intersection, and  $\setminus$  complementation. In the same way a hybrid set is a function  $H : U \rightarrow \mathbb{Z}$ , a set could be considered as function  $S : U \rightarrow \{0, 1\}$ . Then set operations correspond to pointwise OR, AND, and NOT. That is, for two sets  $A$  and  $B$ , then  $(A \cup B)(x) = A(x) \text{ OR } B(x)$ . One could easily extend union and intersection to hybrid sets using pointwise min and max [cite], but it would make more sense to have operations corresponding to primitive operations in  $\mathbb{Z}$  instead. Thus we will define  $\oplus$ ,  $\ominus$ , and  $\otimes$  by pointwise  $+$ ,  $-$ , and  $\cdot$ .

**Definition** For any two hybrid sets  $A$  and  $B$  over a common universe  $U$ , we define the operations  $\oplus, \ominus, \otimes : \mathbb{Z}^U \times \mathbb{Z}^U \rightarrow \mathbb{Z}^U$  such that for all  $x \in U$ :

$$(A \oplus B)(x) = A(x) + B(x) \quad (2.6)$$

$$(A \ominus B)(x) = A(x) - B(x) \quad (2.7)$$

$$(A \otimes B)(x) = A(x) \cdot B(x) \quad (2.8)$$

We also define,  $\ominus A$  as  $\emptyset \ominus A$  and for  $c \in \mathbb{Z}$ :

$$(cA)(x) = c \cdot A(x) \quad (2.9)$$

**Definition** We say  $A$  and  $B$  are *disjoint* if and only if  $A \otimes B = \emptyset$

Taken alone, hybrid sets can be used to model various objects.

### 2.2.1 Example: *Rational Arithmetic*

Any positive rational number can be represented as a hybrid set over the set of primes and vice versa (i.e.  $(\mathbb{Z}^{\mathbb{P}}, \oplus) \simeq (\mathbb{Q}_+, \cdot)$ ). For any rational number  $a/b$ , both  $a$  and  $b$  being integers will have a prime decomposition:  $a = p_1^{m_1} \cdot p_2^{m_2} \cdot \dots$  and  $b = q_1^{n_1} \cdot q_2^{n_2} \cdot \dots$ . Then there is an isomorphism:

$$f(a/b) = \left\{ p_1^{m_1}, p_2^{m_2}, \dots \right\} \ominus \left\{ q_1^{n_1}, q_2^{n_2}, \dots \right\} \quad (2.10)$$

**Example** Concretely, we have:

$$20/9 \cdot 15/8 = \left\{ 5^1, 2^2, 3^{-2} \right\} \oplus \left\{ 5^1, 3^1, 2^{-3} \right\} = \left\{ 5^2, 2^{-1}, 3^{-1} \right\} = 25/6$$

Typically one would need to specify equivalence classes on  $\mathbb{Q}$  to consolidate the identity  $ca/cb = a/b$ . With hybrid set representation, this identity comes for free: any common factor between numerator and denominator will cancel result in cancelling multiplicities. For example,  $2/4 = \left\{ 2^1, 2^{-2} \right\}$  which is the un-normalized form of  $\left\{ 2^{-1} \right\} = 1/2$ .

Is there a (nice!) way to extend this for 0 and negative  $\mathbb{Q}$  that preserves uniqueness up to normalization?

### 2.2.2 Example: *Rational Polynomials*

Hybrid sets can also be used to represent the roots and asymptotes of a rational polynomial.

Blending

**Example** Concretely:

$$\frac{(x-2)}{(x-1)^2(x+1)} = \left\{ 2^1, 1^{-2}, -1^{-1} \right\} \quad (2.11)$$

**Definition** Generalized partition

To Do

**Definition** Reducibility,  $\mathcal{R}(H) = \text{supp}(H)$

any set partition is a generalized set partition

a generalized set partition of a reducible partition is a set position iff each generalized partition is reducible

## 2.3 Hybrid Functions

Next we consider functions which have hybrid sets as their domain which we will call hybrid functions

**Definition** A hybrid set over  $S \times T$  is called a *hybrid (binary) relation*.

**Example** Algebra of orderings:

For some ordered set  $S$  if we define the hybrid relation  $[>] = \{ (x, y)^1, (y, x)^{-1} : x > y \}$ .

Immediately we have:

$$[<] = \Theta[>] \quad (2.12)$$

If  $\text{supp}[>] = S \times S$  then  $[>]$  is a *total ordering*

$$[\leq] = [<] \oplus [=]$$

$$[\leq] = [=] \Theta [>] = [=] \Theta ([\geq] \Theta [=]) = 2[=] \Theta [\geq]$$

**Definition** Let  $H$  be a hybrid relation. For all  $x, y, z$  if  $(x, y) \in H$  and  $(x, z) \in H$  implies  $y = z$

**reword** then  $H$  is said to be a *hybrid function*.

Although this tells us what *is* and *is not* a hybrid function, it is not the most useful definition to work with. Generally, we already have a function in mind which we would like to use over a hybrid domain.

**Theorem 2.3.1** Let  $H$  be a hybrid set over  $U$ ,  $f : B \rightarrow S$  be a function where  $B \subseteq U$  and  $S$  a set. Then

$$f^H := \bigoplus_{x \in B} H(x) \{ (x, f(x))^1 \} \quad (2.13)$$

**theorem?** is a *hybrid function*.

**reword**



This definition of a hybrid function should be less thought of as a true function and more as the *graph of a function*. To return the functional behavior we extend the definition of  $\mathcal{R}$  from the previous section.

**Definition** If  $H$  is a reducible hybrid set, then  $f^H$  is a reducible. Additionally, if  $f^H$  is reducible, we override  $\mathcal{R}$  by:

$$\mathcal{R}(f^H)(x) = f|_{\text{supp}(H)}(x) \quad (2.14)$$

Note that  $\mathcal{R}$  can only be applied if at all points  $H(x)$  is 0 or 1; an irreducible hybrid function cannot be reduced! Unlike the join for regular functions,  $\bar{\oplus}$  (2.3), the join of hybrid function is simply defined and does not rely on piecewise functions buried in definitions.

**Definition** The *join*,  $f^F \oplus g^G$  of two hybrid functions  $f^F$  and  $g^G$  is the hybrid relation given by:

$$f^F \oplus g^G := f^F \oplus g^G \quad (2.15)$$

The join of two hybrid relations is identically defined.

It is important to note that the join operator is closed under hybrid relations but not under hybrid functions. For any two hybrid functions the result will be a hybrid relation but not necessarily another hybrid function. So this definition is still nearly as “dangerous” as  $\bar{\oplus}$ , non-hybrid function join. We must still be wary of overlapping regions but there are some cases where we can be guaranteed to get a hybrid function.

All of the everything

Let  $A$  and  $B$  be hybrid sets over  $U$  and let  $f : U \rightarrow S$  a function.

$f^A \oplus f^B = f^{A \oplus B}$  is always a hybrid function. Every element in  $\text{supp}(f^A \oplus f^B)$  is using the same map  $f$ , so there cannot be disagreement among points.

Inductively, this holds for any number of pieces.

For any generalized partition  $P$ , given by  $P = P_1 \oplus P_2 \oplus \dots \oplus P_n$ , we have

$$f^P = f^{P_1} \oplus f^{P_2} \oplus \dots \oplus f^{P_n} \quad (2.16)$$

For  $g : U \rightarrow S$  another function then  $f^A \oplus g^B = (f \bar{\oplus} g)^{A \oplus B}$  if and only if  $A$  and  $B$  are disjoint (that is,  $A \otimes B = \emptyset$ ).

But the join of two non-disjoint functions may still be a hybrid function even if their respective functions do not agree at all points; as long as they agree on all points in the “intersection” the functions can be safely joined.

**Definition** We say that two hybrid functions  $f^A$  and  $g^B$  are *compatible* if and only if  $f(x) = g(x)$  for all  $x \in \text{supp}(A \otimes B)$ .

As with our definition of disjointness, the pointwise product  $\otimes$ , of hybrid sets acts as an analog for intersection  $\cap$ , of sets.

Note that any two hybrid functions with disjoint domains will be compatible.

**Theorem 2.3.2** *Let  $f^A$  and  $g^B$  be two hybrid functions. Then  $f^A \oplus g^B$  is a hybrid function if and only if  $f^A$  and  $g^B$  are compatible.*

Notion of compatibility is not associative.

Consider

$$(f^H \oplus g^H) \oplus g^{\ominus H} = f^H \oplus (g^H \oplus g^{\ominus H}) = f^H \oplus g^{\emptyset} = f^H \quad (2.17)$$

Although  $f^H$  and  $g^H$  may be mutually incompatible, their join is compatible with  $g^{\ominus H}$ .

Similarly, we can also see above that *reducibility* does not lift through  $\oplus$ .

Is *lift* the right word? - I don't know enough category theory

### 2.3.1 Example: *Piecewise functions on generalized partitions*

(2 pages)

$$\begin{aligned}
 (f * g)(x) &= \left\| f_1(x)^{A_1}, f_2(x)^{A_2} \right\| * \left\| g_1(x)^{B_1}, g_2(x)^{B_2} \right\| \\
 &= \left\| (f_1(x) * g_1(x))^{A_1} \right\| \oplus^* \left\| (f_2(x) * g_1(x))^{B_1 \oplus A_2} \right\| \oplus^* \left\| (f_2(x) * g_2(x))^{B_2} \right\|
 \end{aligned}$$

Formula for  $f * g$  where  $f = f_1^{P_1} \oplus f_2^{P_2} \oplus f_n^{P_n}$  and  $g = g_1^{Q_1} \oplus \dots \oplus g_m^{Q_m}$

$$\begin{aligned}
 f * g &= \left\| (f_1 * g_m)^{P_1} \right\| \oplus^* \dots \oplus^* \left\| (f_{n-1} * g_m)^{P_{n-1}} \right\| \\
 &\quad \oplus^* \left\| (f_n * g_1)^{Q_1} \right\| \oplus^* \dots \oplus^* \left\| (f_n * g_{m-1})^{Q_{m-1}} \right\| \\
 &\quad \oplus^* \left\| (f_n * g_n)^{U \ominus (P_1 \oplus \dots \oplus P_{n-1} \oplus Q_1 \oplus \dots \oplus Q_{m-1})} \right\|
 \end{aligned}$$

Can't actually do this example until  $\oplus^*$  is introduced, which doesn't work with basic hybrid functions

## 2.4 Pseudo-functions and Hybrid Forms

**Definition** Refinement

**Example** Refinement of intervals

**Definition** Pseudo-function

Properties of pseudo-functions

**Definition** Hybrid Form

**Definition**  $(f^F \oplus^* g^G)(x) = (F(x) + G(x)) \left\| (x, (f * g)(x))^1 \right\|$

Returning to the example of the sign function from section 2.1, we could think of the function as the join of 3 different hybrid functions:

$$\text{sign} = -1^{(-\infty, 0)} \oplus 0^{\{0\}} \oplus 1^{(0, \infty)} \quad (2.18)$$

but we could also consider it as the “joined sum” of  $-1^{(-\infty, 0]}$  and  $1^{[0, \infty)}$ .

(3 pages)

### 2.4.1 Example: *Piecewise functions revisited*

Repeat piece-wise function example with unsafe points (1 page)

$$(-x^2 + 2)^{[-1,1]} \oplus \left( \frac{1}{x^2} \right)^{\mathbb{R} \ominus [1,1]} \quad (2.19)$$

# Chapter 3

## Abstract Matrices

### 3.1 Oriented Intervals

**Definition** Given a totally ordered set  $(X, \leq)$  (and with an implied strict ordering  $<$ ), for any  $a, b \in X$ , an **interval between  $a$  and  $b$**  is the set of elements in  $X$  between  $a$  and  $b$ , up to inclusion of  $a$  and  $b$  themselves. Formally:

$$[a, b] = \{x \in X \mid a \leq x \leq b\} \quad (3.1)$$

$$[a, b) = \{x \in X \mid a \leq x < b\} \quad (3.2)$$

$$(a, b] = \{x \in X \mid a < x \leq b\} \quad (3.3)$$

$$(a, b) = \{x \in X \mid a < x < b\} \quad (3.4)$$

It should be noted that for  $b < a$ ,  $[b, a]$  is the empty set. Also, the interval  $[a, a]$  contains a single point while  $(a, a)$ ,  $(a, a]$ , and  $[a, a)$  are all empty. As intervals are simply sets, they can naturally be interpreted as hybrid sets. If  $a \leq b \leq c$ , for intervals  $[a, b)$  and  $[b, c)$  using the hybrid set operator  $\oplus$ , one has  $[a, b) \oplus [b, c) = [a, c)$ . In this case,  $\oplus$  behaves like concatenation but this is not always true. When  $a \leq c \leq b$  then  $[a, b) \oplus [b, c) = [a, b)$ . When working with intervals, a case-based approach to consider relative ordering of endpoints easily becomes quite

cumbersome. Thus we turn to oriented intervals.

**Definition** We define **oriented intervals** with  $a, b \in X$ , a totally ordered set, using hybrid set point-wise subtraction as follows:

$$\llbracket a, b \rrbracket = [a, b] \ominus [b, a] \quad (3.5)$$

$$\ll(a, b] = (a, b] \ominus (b, a] \quad (3.6)$$

$$\llbracket a, b \rrbracket = [a, b] \ominus (b, a) \quad (3.7)$$

$$\ll(a, b) = (a, b) \ominus [b, a] \quad (3.8)$$

For any oriented interval, at most one interval term will be non-empty. However when using symbolic terms,

Several results follow immediately from this definition.

**Theorem 3.1.1** *For all  $a, b, c \in \mathbb{R}$ ,*

$$\llbracket a, b \rrbracket = \ominus \llbracket b, a \rrbracket \quad (3.9)$$

$$\ll(a, b] = \ominus \ll(b, a] \quad (3.10)$$

$$\llbracket a, b \rrbracket = \ominus \ll(a, b) \quad (3.11)$$

Like their unoriented analogues, the oriented intervals  $\llbracket a, a \rrbracket$  and  $\ll(a, a]$  are both empty and  $\llbracket a, a \rrbracket$  contains a single point (with multiplicity 1). However, unlike traditional intervals  $\ll(a, a)$  is *not* empty but is equivalent to but rather,  $\ll(a, a) = \ominus \llbracket a, a \rrbracket = \left\{ a^{-1} \right\}$ .

**Theorem 3.1.2** *For all  $a, b, c \in \mathbb{R}$  (regardless of relative ordering),*

$$\llbracket a, b \rrbracket \oplus \llbracket b, c \rrbracket = \llbracket a, c \rrbracket \quad (3.12)$$

**Proof**  $\llbracket a, b \rrbracket \oplus \llbracket b, c \rrbracket$

$$= ([a, b] \ominus [b, a]) \oplus ([b, c] \ominus [c, b])$$

$$= ([a, b] \oplus [b, c]) \ominus ([c, b] \oplus [b, a])$$

If  $a \geq c$  then  $[c, a] = \emptyset$  and so  $\llbracket a, c \rrbracket = [a, c)$ .

**Case 1:**  $a \leq b \leq c$  then  $[c, b] = [b, a] = \emptyset$  and  $[a, b] \oplus [b, c] = [a, c)$

**Case 2:**  $b \leq a \leq c$  then  $[b, c] \ominus [b, a] = [b, a] \oplus [a, c] \ominus [b, a] = [a, c)$

**Case 3:**  $a \leq c \leq b$  then  $[a, b] \ominus [c, b] = ([a, c] \oplus [c, b]) \ominus [c, b] = [a, c)$

Similar arguments will show that when  $c \geq a$ , that  $\llbracket a, b \rrbracket \oplus \llbracket b, c \rrbracket = \ominus[a, c)$ .

This sort of reasoning is routine but a constant annoyance when dealing with intervals and is exactly the reason we want to be working with oriented intervals. Many similar formulations such as  $\llbracket a, b \rrbracket \oplus \llbracket b, c \rrbracket = \llbracket a, c \rrbracket$  are also valid for any ordering of  $a, b, c$ . We will not enumerate all possible cases here.



## 3.2 Symbolic Vector Algebra

We will use the following  $n$ -dimensional vectors as a running example in this section:

$$U^T = [u_1, u_2, \dots, u_k, u'_1, u'_2, \dots, u_{n-k}] \quad (3.13)$$

$$V^T = [v_1, v_2, \dots, v_\ell, v'_1, v'_2, \dots, v_{n-\ell}] \quad (3.14)$$

Using intervals, these vectors can be represented by hybrid functions over their indices.

For example

$$U^T = (i \mapsto u_i)^{\llbracket 1, k \rrbracket} \oplus (i \mapsto u_{i-k})^{\langle\langle k, n \rangle\rangle} \quad (3.15)$$

$$V^T = (i \mapsto v_i)^{\llbracket 1, \ell \rrbracket} \oplus (i \mapsto v_{i-\ell})^{\langle\langle \ell, n \rangle\rangle} \quad (3.16)$$

Although for clarity and succinctness we will use  $(u_i)$  instead of  $(i \mapsto u_i)$ .

However we are more interested in performing arithmetic with these vectors.

### 3.2.1 Vector Addition

Consider pointwise vector addition  $U^T + V^T$ :

$$U^T + V^T = \left( (u_i)^{\llbracket 1, k \rrbracket} \oplus (u'_{i-k})^{\langle\langle k, n \rangle\rangle} \right) \oplus^+ \left( (v_i)^{\llbracket 1, \ell \rrbracket} \oplus (v'_{i-\ell})^{\langle\langle \ell, n \rangle\rangle} \right) \quad (3.17)$$

$$= \left( (u_i)^{\llbracket 1, k \rrbracket} \oplus (u'_{i-k})^{\langle\langle k, \ell \rrbracket} \oplus (u'_{i-k})^{\langle\langle \ell, n \rangle\rangle} \right) \oplus^+ \left( (v_i)^{\llbracket 1, k \rrbracket} \oplus (v_i)^{\langle\langle k, \ell \rrbracket} \oplus (v'_{i-\ell})^{\langle\langle \ell, n \rangle\rangle} \right) \quad (3.18)$$

$$= \left( (u_i + v_i)^{\llbracket 1, k \rrbracket} \oplus^+ (u'_{i-k} + v_i)^{\langle\langle k, \ell \rrbracket} \oplus^+ (u'_{i-k} + v'_{i-\ell})^{\langle\langle \ell, n \rangle\rangle} \right) \quad (3.19)$$

This formulation is not unique.

The choice to partition  $\llbracket 1, n \rrbracket$  into  $\llbracket 1, k \rrbracket \oplus \langle\langle k, \ell \rrbracket \oplus \langle\langle \ell, n \rangle\rangle$  was arbitrary.

We can just as easily partition  $\llbracket 1, n \rrbracket$  into  $\llbracket 1, \ell \rrbracket \oplus \langle\langle \ell, k \rrbracket \oplus \langle\langle k, n \rangle\rangle$  to get the equivalent expression:

$$U^T + V^T = \left( (u_i + v_i)^{\llbracket 1, \ell \rrbracket} \oplus^+ (u_i + v'_{i-\ell})^{\langle \ell, k \rangle} \oplus^+ (u'_{i-k} + v'_{i-\ell})^{\langle k, n \rangle} \right) \quad (3.20)$$

We must be careful while evaluating these expressions to not forget that  $(u'_{i-k} + v_i)$  is actually shorthand for the function:

$$i \mapsto u'_{i-k} + v_i$$

For example, consider the concrete example where  $n = 5$ ,  $k = 4$  and  $\ell = 1$  so that  $U^T = [u_1, u_2, u_3, u_4, u'_1]$  and  $V^T = [v_1, v'_1, v'_2, v'_3, v'_4]$ .

We will also only assume that the functions  $u_i$ ,  $u'_i$ ,  $v_i$  and  $v'_i$  are defined only on the intervals in which they appear (e.g.  $u_5$  is undefined, as is  $v'_1$ ).

Then the expression in (3.19) becomes:

$$(u_i + v_i)^{\llbracket 1, 4 \rrbracket} \oplus^+ (u'_{i-4} + v_i)^{\langle 4, 1 \rangle} \oplus^+ (u'_{i-4} + v'_{i-1})^{\langle 1, 5 \rangle} \quad (3.21)$$

None of the individual subterms cannot be evaluated directly.

In the first term,  $v_i$  is not totally defined over the interval  $\llbracket 1, 4 \rrbracket$ .

In the third term, on the interval  $\langle 1, 5 \rangle$ ,  $u'_{i-4}$  would even be evaluated on negative indices.

However, these unevaluable terms also appear in the middle term however the interval  $\langle 4, 1 \rangle$  is a negatively oriented interval and the offending points cancel!

### 3.2.2 Inner Product

The inner product or dot product of two vectors is given by:

$$A \cdot B = \sum_i A_i B_i \quad (3.22)$$

Returning to the running example of  $U^T$  and  $V^T$ , as defined in (3.13) and (3.14) respectively, we will consider  $U^T \cdot V^T$ .

**Definition** Let  $X = \{x_1^{m_1}, x_2^{m_2}, \dots, x_n^{m_n}\}$  be a hybrid set with elements  $x_i$  in a  $\mathbb{Z}$ -module. Given a hybrid function over  $X$ ,  $f^X$ , we define the **sum over**  $f^X$ , denoted with  $\sum$ , as

$$\sum(f^X) := \sum_{i=1}^n (m_i \cdot f(x_i)) \quad (3.23)$$

The **product over**  $f^X$ , denoted with  $\prod$  is defined similarly.

Then the dot product of  $U^T$  and  $V^T$  becomes very familiar:

$$U^T \cdot V^T = \sum \left( (u_i v_i)^{\llbracket 1, k \rrbracket} \oplus^{\times} (u'_{i-k} v_i)^{\llbracket k, \ell \rrbracket} \oplus^{\times} (u'_{i-k} v'_{i-\ell})^{\llbracket \ell, n \rrbracket} \right) \quad (3.24)$$

The inner expression is identical to  $U^T + V^T$  except for a replacement of  $+$  with  $\times$ .

### 3.2.3 Outer Product

Does this even work?

It is common practice in mathematics to represent matrices symbolically with sub-matrices such as:

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \quad (3.25)$$

If  $A$  is an  $n \times n$  matrix then  $A_1, A_2, A_3, A_4$  are not entries but  $(k \times \ell)$ ,  $(n - k \times \ell)$ ,  $(k \times n - \ell)$  and  $(n - k \times n - \ell)$  matrices respectively. Ellipses are also routinely used for interpolating over regions of a matrix, as in:

$$M = \begin{bmatrix} x_{11} & \dots & x_{1n} \\ & \ddots & \vdots \\ 0 & & x_{nn} \end{bmatrix} \quad (3.26)$$

# Chapter 4

## Integration I

In this chapter we will present and introduce integration over oriented intervals and generalise to higher dimensions. For the time being, we will focus on the intuition behind this and only worry about axis-aligned  $n$ -cubes. Following this, in the next chapter, we will delve more deeply from a measure theoretic perspective and integration over more general shapes.

### 4.1 Single variable integration

Given a function  $f$  with real variable  $x$  and an interval  $[a, b)$  on the (extended) real line, a traditional **definite integral** would be of the form:

$$\int_a^b f(x) dx \quad \text{or} \quad \int_{[a,b)} f(x) dx$$

Which we interpret as the signed area bounded by  $f$  between  $x = a$  and  $x = b$ . However, defining this definite integral using (unoriented) intervals like this is a bit of a misnomer. In the case where  $a \geq b$  one would typically use the identity:

$$\int_a^b f(x) dx = - \int_b^a f(x) dx \tag{4.1}$$

to evaluate the integral. But, as we saw in the previous chapter, when  $a \geq b$ , the interval  $[a, b]$  is the empty set! We can't translate equation (4.1) to an identity along the lines of:

$$\int_{[a,b)} f(x) dx = - \int_{[b,a)} f(x) dx \quad (4.2)$$

since at least one of  $[a, b)$  or  $[b, a)$  will be always be empty. Although this notation generally appears in context of Lebesgue integrals, Riemann integrals (which generally use  $\int_a^b \dots$  instead) simply hide this mis-use of intervals. For example,

Find one that's not wikipedia!

- tagged partition given as a series  $x_i$  such that  $a = x_1 < x_2 < \dots < x_n = b$  which we say is the integral  $\int_a^b$

- Unintentionally we are claiming that  $\int_1^0 f(x) dx = \int_{[1,0]} = \int_{\emptyset} f(x) dx = 0$

We will put off formal definitions until the next chapter.

If oriented intervals are used instead of traditional intervals, then the identity from equation (4.1) can instead just be a result of *bi-linearity*.

$$\int_{\llbracket a,b \rrbracket} f(x) dx = - \int_{\ominus \llbracket a,b \rrbracket} f(x) dx = - \int_{\llbracket b,a \rrbracket} f(x) dx \quad (4.3)$$

**Definition** Let  $\llbracket a, b \rrbracket$  be an interval on  $\mathbb{R}$  then the boundary function  $\partial$  is the linear map such that:

$$\partial(\llbracket a, b \rrbracket) = \left\{ a^1, b^{-1} \right\} \quad (4.4)$$

By linearity we also have:

$$\partial(\llbracket a, b \rrbracket) = \partial(\ominus \llbracket b, a \rrbracket) = \ominus \partial(\llbracket b, a \rrbracket) = \ominus \left\{ b^1, a^{-1} \right\} = \left\{ a^1, b^{-1} \right\}.$$

Using this we also have:

$$\partial(\llbracket a, b \rrbracket) = \partial(\llbracket a, c \rrbracket \oplus \llbracket c, b \rrbracket) = \partial \llbracket a, c \rrbracket \oplus \partial \llbracket c, b \rrbracket = \left\{ a^1, c^{-1} \right\} \oplus \left\{ c^1, b^{-1} \right\} = \left\{ a^1, b^{-1} \right\}$$

And by a similar proof for  $\partial(\llbracket a, c \rrbracket)$ , we conclude that:

$$\partial \llbracket a, b \rrbracket = \partial \llbracket a, b \rrbracket = \partial(\llbracket a, b \rrbracket) = \partial(\llbracket a, b \rrbracket) \quad (4.5)$$

Informally, isolated points do not affect the boundary of an oriented interval. This should have been obvious from the definition to begin with. The interval  $\llbracket a, a \rrbracket$  is a hybrid set which contains only the element  $a$  with multiplicity one. From equation XX,

$$\partial \llbracket a, a \rrbracket = \{ a^1, a^{-1} \} = \emptyset \quad (4.6)$$

So whether we use  $\int_a^b$  to denote the integral over the intervals  $\llbracket a, b \rrbracket$ ,  $\llbracket a, b \rangle$  or  $\langle a, b \rrbracket$  the boundary is unchanged and so the integral will evaluate identically.

The hybrid sets  $\llbracket a, b \rrbracket \oplus \llbracket b, c \rrbracket$  and  $\langle a, b \rangle \oplus \langle b, c \rangle$  have identical multiplicities almost everywhere. At  $b$ , they will differ by 2. Although simple arguments resolve this issue, by using left-closed, right-open oriented intervals we can bypass these arguments altogether when showing

$$\int_{\llbracket a, b \rangle} f(x) dx + \int_{\llbracket b, c \rangle} f(x) dx = \int_{\llbracket a, c \rangle} f(x) dx \quad (4.7)$$

Somewhere in this section:

- “For now we will ignore the underlying integration mechanics and focus on the intuition of using oriented intervals. In chapter X we will return to this topic from a measure theoretic perspective to establish the Lebesgue integral.”

## 4.2 $n$ -cubes

For now, we will concern ourselves only with oriented,  $n$ -dimensional, axis-aligned rectangles in  $\mathbb{R}^n$ . In one dimension, the previously discussed oriented intervals cover most obvious shapes. Moving to two dimensions, there are many more obvious shapes to consider but we will temporarily ignore triangles, circles and even rectangles that are tilted. First we must introduce some notation to describe these and higher dimension rectangles.

At the moment the only rectangles we have defined are the one-dimensional “oriented interval”. Hence we will also refer to this as a 1-rectangle. We construct higher dimensional  $n$

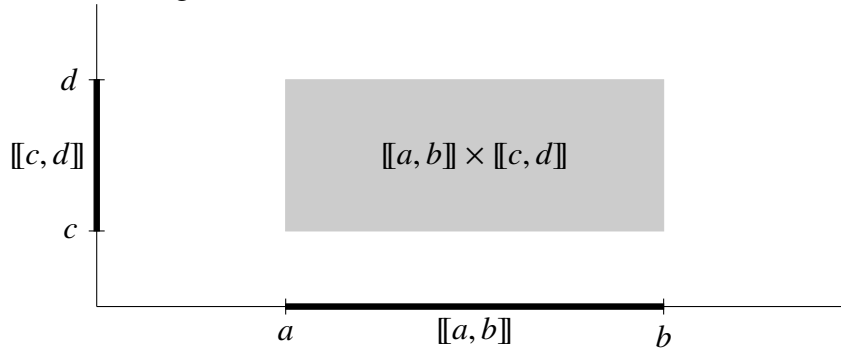
rectangles using the Cartesian product.

**Definition** Let  $X = \{x_1^{m_1}, \dots, x_k^{m_k}\}$  and  $Y = \{y_1^{n_1}, \dots, y_\ell^{n_\ell}\}$  be hybrid sets. We define the **Cartesian product of hybrid sets  $X$  and  $Y$** , denoted with  $\times$  operator as:

$$X \times Y = \{(x, y)^{m \cdot n} : x^m \in X, y^n \in Y\} \quad (4.8)$$

If  $\llbracket a, b \rrbracket$  and  $\llbracket c, d \rrbracket$  are both positively oriented 1-rectangles then their Cartesian product is shown in Figure 4.1 is clearly a two dimensional rectangle or *2-rectangle*. Taking the Cartesian product of a 2-rectangle and 1-rectangle gives a 3-rectangle in  $\mathbb{R}^3$ . We should note here that we do not distinguish between  $((x, y), z)$  and  $(x, (y, z))$  but rather we treat both as different names for the ordered triple  $(x, y, z)$ . We similarly associate parentheses in higher dimensions as well.

Figure 4.1: The Cartesian product of two positively oriented 1-rectangles  $\llbracket a, b \rrbracket$  and  $\llbracket c, d \rrbracket$  is a positively oriented 2-rectangle.



**Theorem 4.2.1** *The Cartesian product of a  $k$ -rectangle in  $\mathbb{R}^m$  (where,  $k \leq m$ ) and  $\ell$ -rectangle in  $\mathbb{R}^n$  (again,  $\ell \leq n$ ) is a  $(k + \ell)$ -rectangle in  $\mathbb{R}^{m+n}$ .*

For completeness we will also define a 0-rectangle as a hybrid set containing a single point with multiplicity 1 or  $-1$ . Firstly this allows us to embed  $k$ -rectangles in  $\mathbb{R}^n$ . For example  $\llbracket a, b \rrbracket \times \llbracket c, d \rrbracket \times \{e^1\}$  is the product of two 1-rectangles and a 0-rectangle (all over  $\mathbb{R}$ ) and so it is a 2-rectangle over  $\mathbb{R}^3$ . Specifically, it is the 2-rectangle  $\llbracket a, b \rrbracket \times \llbracket c, d \rrbracket$  on the plane  $z = 3$ . This also illustrates the principle that given a  $k$ -rectangle in  $\mathbb{R}^n$  where  $n > k$  we can always find



a  $k$  dimensional subspace which also contains the rectangle. We will re-use the interval notation from earlier although one should be careful to “type-check” while interpreting. When  $a$  and  $b$  are real numbers then we continue to use the definition  $\llbracket a, b \rrbracket = [a, b] \ominus [b, a]$ . However, when  $a$  and  $b$  are  $n$ -tuples (for example, coordinates in  $\mathbb{R}^n$  then this is *not* the oriented line interval  $[a, b] \ominus [b, a]$ ).

**Definition** Let  $a = (a_1, a_2, \dots, a_n)$  and  $b = (b_1, b_2, \dots, b_n)$  be ordered  $n$ -tuples then we use the notation:

$$\llbracket a, b \rrbracket = \llbracket a_1, b_1 \rrbracket \times \llbracket a_2, b_2 \rrbracket \times \dots \times \llbracket a_n, b_n \rrbracket \quad (4.9)$$

The dimension of  $\llbracket a, b \rrbracket$  is equal to the number of indices where  $a_i$  and  $b_i$  are distinct. For any  $i$  where  $a_i = b_i$ , the corresponding term:  $\llbracket a_i, b_i \rrbracket$  will be a hybrid set containing a single point, that is, a 0-rectangle. The orientation of  $\llbracket a, b \rrbracket$  is based on the number of negatively oriented intervals  $\llbracket a_i, b_i \rrbracket$ . Should there be an odd number of indices  $i$  such that  $a_i > b_i$  then  $\llbracket a, b \rrbracket$  will also be negatively oriented. Otherwise, it will be positively oriented.

## 4.3 Boundary Operator

In one dimension, the boundary of an interval was quite straight-forward. For a positively oriented interval, the boundary was composed of two points; the right end-point was positive and the left end-point was negative. From the perspective of  $k$ -rectangles, the  $\partial$  operator has mapped an oriented 1-rectangle to a set of oriented 0-rectangles. We will now generalize the boundary to map an oriented  $n$ -rectangle to an  $(n - 1)$ -rectangle.

**Definition** Let  $\llbracket a, b \rrbracket$  be a  $k$ -rectangle in  $\mathbb{R}^n$ . Additionally, let  $i_1, i_2, \dots, i_k$  be the unique non-decreasing sequence of indices such that  $a_{i_j} \neq b_{i_j}$ . The **boundary of  $\llbracket a, b \rrbracket$** , denoted the

operator  $\partial$  is given by

$$\begin{aligned} \partial(\llbracket \mathbf{a}, \mathbf{b} \rrbracket) = & \bigoplus_{j=1}^k (-1)^j \left( \llbracket \mathbf{a}^{\llbracket 1, i_j \rrbracket}, \mathbf{b}^{\llbracket 1, i_j \rrbracket} \rrbracket \times \llbracket a_{i_j} \rrbracket \times \llbracket \mathbf{a}^{\llbracket i_j, n \rrbracket}, \mathbf{b}^{\llbracket i_j, n \rrbracket} \rrbracket \right. \\ & \left. \ominus \llbracket \mathbf{a}^{\llbracket 1, i_j \rrbracket}, \mathbf{b}^{\llbracket 1, i_j \rrbracket} \rrbracket \times \llbracket b_{i_j} \rrbracket \times \llbracket \mathbf{a}^{\llbracket i_j, n \rrbracket}, \mathbf{b}^{\llbracket i_j, n \rrbracket} \rrbracket \right) \end{aligned} \quad (4.10)$$

The above equation will require a bit of unpacking to digest featuring oriented intervals in two different contexts. The first appears in the superscripts of  $\mathbf{a}$  and  $\mathbf{b}$ . The intervals  $\llbracket 1, i_j \rrbracket$  and  $\llbracket i_j, n \rrbracket$  are and is an interval over vector indices just as in Chapter 3. Thus, the term  $\mathbf{a}^{\llbracket 1, i_j \rrbracket}$  refers to the vector  $(a_1, a_2, \dots, a_{i-1})$  while the term  $\mathbf{b}^{\llbracket i, n \rrbracket}$  refers to  $(b_{i+1}, b_{i+2}, \dots, b_n)$ . This provides a compact notation to partition the original range of indices into 3 pieces:  $\llbracket 1, i \rrbracket$ ,  $\llbracket i, i \rrbracket$ , and  $\llbracket i, n \rrbracket$ . The first and last portions are isolated but left untouched, but the central  $\llbracket i, i \rrbracket$  term is then replaced with the 0-rectangles  $a_i$  or  $b_i$ . Formally, we are actually using the hybrid sets  $\llbracket (a_i)^1 \rrbracket$  and  $\llbracket (b_i)^1 \rrbracket$  but we omit the explicit multiplicity of one (and will continue to do so through out the section) to lighten notation.

Each Cartesian product forms a  $(k - 1)$ -rectangular face of the  $k$ -rectangle which we shall show. Let  $\llbracket \mathbf{a}, \mathbf{b} \rrbracket$  be a  $k$  rectangle in  $\mathbb{R}^n$ . Following from definitions we have:

$$\llbracket \mathbf{a}, \mathbf{b} \rrbracket = \llbracket \mathbf{a}^{\llbracket 1, i \rrbracket}, \mathbf{b}^{\llbracket 1, i \rrbracket} \rrbracket \times \llbracket \mathbf{a}^{\llbracket i, i \rrbracket}, \mathbf{b}^{\llbracket i, i \rrbracket} \rrbracket \times \llbracket \mathbf{a}^{\llbracket i, n \rrbracket}, \mathbf{b}^{\llbracket i, n \rrbracket} \rrbracket \quad (4.11)$$

If  $a_i$  and  $b_i$  are distinct then  $\llbracket \mathbf{a}^{\llbracket i, i \rrbracket}, \mathbf{b}^{\llbracket i, i \rrbracket} \rrbracket$  is a 1-rectangle. Since both  $a_i$  and  $b_i$  are 0-rectangles and expressions agree everywhere else, then the following are both  $(k - 1)$ -rectangles:

$$\llbracket \mathbf{a}^{\llbracket 1, i \rrbracket}, \mathbf{b}^{\llbracket 1, i \rrbracket} \rrbracket \times \llbracket a_i \rrbracket \times \llbracket \mathbf{a}^{\llbracket i, n \rrbracket}, \mathbf{b}^{\llbracket i, n \rrbracket} \rrbracket \quad (4.12)$$

$$\llbracket \mathbf{a}^{\llbracket 1, i \rrbracket}, \mathbf{b}^{\llbracket 1, i \rrbracket} \rrbracket \times \llbracket b_i \rrbracket \times \llbracket \mathbf{a}^{\llbracket i, n \rrbracket}, \mathbf{b}^{\llbracket i, n \rrbracket} \rrbracket \quad (4.13)$$

However if  $a_i = b_i$  then  $\llbracket \mathbf{a}^{\llbracket i, i \rrbracket}, \mathbf{b}^{\llbracket i, i \rrbracket} \rrbracket$  is a 0-rectangle and so the expressions in (4.10) and (4.11) are both  $k$ -rectangles! Since  $a_i = b_i$ , the two expressions are identical, so their difference is

zero and the terms disappear.

### 4.3.1 Example: *Boundary of a 1-rectangle*

Let  $\mathbf{a} = (a_1)$  and  $\mathbf{b} = (b_1)$  be trivial 1-tuples. It follows that:

$$\begin{aligned}
 \partial(\llbracket \mathbf{a}, \mathbf{b} \rrbracket) &= (-1)^i(\llbracket \mathbf{a}^{(1,1)}, \mathbf{b}^{(1,1)} \rrbracket \times \llbracket a_1 \rrbracket \times \llbracket \mathbf{a}^{(1,1)}, \mathbf{b}^{(1,1)} \rrbracket \\
 &\quad \ominus \llbracket \mathbf{a}^{(1,1)}, \mathbf{b}^{(1,1)} \rrbracket \times \llbracket b_1 \rrbracket \times \llbracket \mathbf{a}^{(1,1)}, \mathbf{b}^{(1,1)} \rrbracket) \\
 &= \ominus \llbracket \mathbf{a}^0, \mathbf{b}^0 \rrbracket \times \llbracket a_1 \rrbracket \times \llbracket \mathbf{a}^0, \mathbf{b}^0 \rrbracket \\
 &\quad \oplus \llbracket \mathbf{a}^0, \mathbf{b}^0 \rrbracket \times \llbracket b_1 \rrbracket \times \llbracket \mathbf{a}^0, \mathbf{b}^0 \rrbracket \\
 &= \llbracket b_1 \rrbracket \ominus \llbracket a_1 \rrbracket
 \end{aligned}$$

Again, we omit the braces from the hybrid sets  $\llbracket (a_1)^1 \rrbracket$  and  $\llbracket (b_1)^1 \rrbracket$ . Recalling from the previous section in equation (4.4), we can see that the results agree:

$$\partial(\llbracket a_1, b_1 \rrbracket) = \llbracket b_1 \rrbracket \ominus \llbracket a_1 \rrbracket = \llbracket (b_1)^1 \rrbracket \ominus \llbracket (a_1)^1 \rrbracket = \llbracket (a_1)^{-1}, (b_1)^1 \rrbracket \quad (4.14)$$

### 4.3.2 Example: *Boundary of a 3-rectangle*

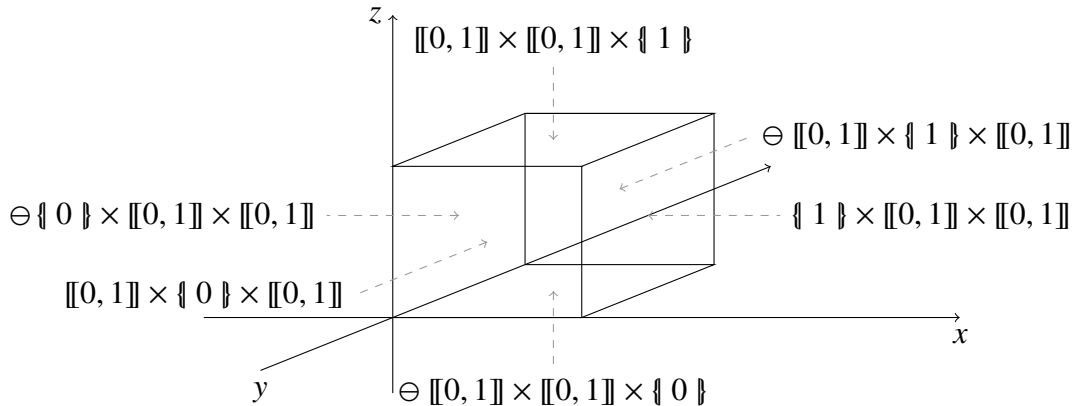
Let  $\mathbf{a} = (0, 0, 0)$  and  $\mathbf{b} = (1, 1, 1)$ . Omitting the intermediate step, we find the boundary of  $\llbracket \mathbf{a}, \mathbf{b} \rrbracket$  to be:

$$\begin{aligned}
 \partial(\llbracket \mathbf{a}, \mathbf{b} \rrbracket) &= \ominus (\llbracket 0 \rrbracket \times \llbracket 0, 1 \rrbracket \times \llbracket 0, 1 \rrbracket) \oplus (\llbracket 1 \rrbracket \times \llbracket 0, 1 \rrbracket \times \llbracket 0, 1 \rrbracket) \\
 &\quad \oplus (\llbracket 0, 1 \rrbracket \times \llbracket 0 \rrbracket \times \llbracket 0, 1 \rrbracket) \ominus (\llbracket 0, 1 \rrbracket \times \llbracket 1 \rrbracket \times \llbracket 0, 1 \rrbracket) \\
 &\quad \ominus (\llbracket 0, 1 \rrbracket \times \llbracket 0, 1 \rrbracket \times \llbracket 0 \rrbracket) \oplus (\llbracket 0, 1 \rrbracket \times \llbracket 0, 1 \rrbracket \times \llbracket 1 \rrbracket)
 \end{aligned}$$

This may not be the most enlightening expression on its own. In figure 4.2 below the 3-rectangle given by  $\llbracket \mathbf{a}, \mathbf{b} \rrbracket$  can be seen as a cube in three dimensions. Physically, the 3-rectangle is a solid cube and includes all interior points. The boundary meanwhile are just the outer

faces. The six terms in the sum are shown in Figure 4.2 with arrows to six corresponding faces. Clearly, each face is itself an axis aligned 2-rectangle!

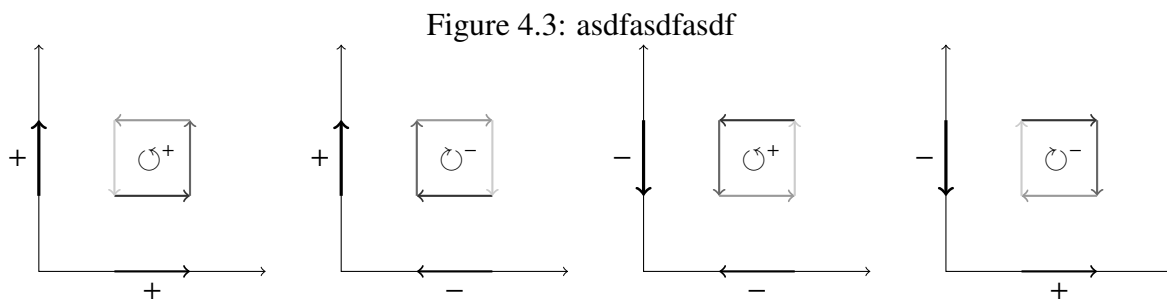
Figure 4.2: The unit cube given as the 3-rectangle:  $\llbracket (0, 0, 0), (1, 1, 1) \rrbracket$  and the 6 faces in its boundary.



float faces off of cube + fill

There are several ways to interpret and visualize the  $\oplus$  and  $\ominus$  sign associated with each face. Most naturally in  $\mathbb{R}^3$  for 2-rectangles is to give each a front and back side with the sign determining which to use. Alternatively, a 2-rectangle has a boundary formed by 1-rectangles which when drawn as arrows, will all meet head-to-tail. This induces a clockwise or counter-clockwise cycle around the edge of the rectangle and so  $\circlearrowright$  and  $\circlearrowleft$  are also commonly used. This can be seen in Figure 4.3.

These are all conceptual tools, subtraction may not have any physical interpretation (as we will see later with manifold points?).



## 4.4 Integrating $n$ -cubes

## 4.5 Chains

We have already seen chains without mentioning them explicitly.

To form the boundary, of a  $k$ -cube we took the sum of  $(k - 1)$ -cubes.

These sums are commonly known as chains.

**Definition**  $C_k$  is the Abelian group generated by  $k$ -cubes with  $\oplus$  hybrid set addition.

Since  $k$ -chains are linear sums of  $k$ -cubes.

It is natural to extend the boundary linearly to chains as well

And so  $\partial_k$  becomes a map from  $k$ -chains to  $(k - 1)$ -chains.

Sometimes expressed as

$$\dots \xleftarrow{\partial_{k-1}} C_{k-1} \xleftarrow{\partial_k} C_k \xleftarrow{\partial_{k+1}} C_{k+1} \xleftarrow{\partial_{k+2}} \dots \quad (4.15)$$

### 4.5.1 Example: *Boundary of a boundary*

Let  $\mathbf{a} = (a_1, a_2)$  and  $\mathbf{b} = (b_1, b_2)$ . We wish to compute  $\partial_1(\partial_2(\llbracket \mathbf{a}, \mathbf{b} \rrbracket))$

$$\begin{aligned} \partial_1(\partial_2(\llbracket a_1, b_1 \rrbracket \times \llbracket a_2, b_2 \rrbracket)) &= \ominus \partial_1(\llbracket 0 \rrbracket \times \llbracket 0, 1 \rrbracket) \oplus \partial_1(\llbracket 1 \rrbracket \times \llbracket 0, 1 \rrbracket) \\ &\quad \oplus \partial_1(\llbracket 0, 1 \rrbracket \times \llbracket 0 \rrbracket) \ominus \partial_1(\llbracket 0, 1 \rrbracket \times \llbracket 1 \rrbracket) \end{aligned} \quad (4.16)$$

$$\begin{aligned} &= \ominus (\ominus \llbracket (0, 0) \rrbracket \oplus \llbracket (0, 1) \rrbracket) \oplus (\ominus \llbracket (1, 0) \rrbracket \oplus \llbracket (1, 1) \rrbracket) \\ &\quad \oplus (\ominus \llbracket (0, 0) \rrbracket \oplus \llbracket (1, 0) \rrbracket) \ominus (\ominus \llbracket (0, 1) \rrbracket \oplus \llbracket (1, 1) \rrbracket) \end{aligned} \quad (4.17)$$

$$= \emptyset \quad (4.18)$$

When moving from (4.16) to (4.17), in addition to applying  $\partial_1$  we used the reduction,

$\|x\| \times \|y\| = \|(x, y)\|$ . This is not unique to 2-cubes but holds for higher dimensions as well.

Let  $\llbracket a, b \rrbracket$  be a  $k$ -rectangle in  $\mathbb{R}^n$ .

## 4.6 Homology??

# Chapter 5

## Integration II

“Chapter goals”

Section 4.1 will cover a conventional treatment of the Lebesgue integral, for those already familiar with the construction, this may be skipped.

### 5.1 Sigma Algebras

Before we can talk about the Lebesgue integral we must first set the stage, so to speak.

**Definition** Let  $X$  be a non-empty set. A  $\sigma$ -**algebra on the set  $X$** ,  $\Sigma$ , is a family of subsets of  $X$  such that:

1.  $\Sigma$  is non-empty
2. *Closed under complement.* If  $E \in \Sigma$ , then  $X \setminus E \in \Sigma$ .
3. *Closed under countable union.* If  $E_1, E_2, \dots \in \Sigma$  then  $(E_1 \cup E_2 \cup \dots) \in \Sigma$ .

The pair  $(X, \Sigma)$  is called a **measurable space** and elements of  $\Sigma$  are called the **measurable sets** (of  $X$ ).

It can easily be shown through the use of De Morgan’s laws that a  $\sigma$ -algebra is also closed under countable intersection as well.

**Example**  $\{\emptyset, X\}$  is a  $\sigma$ -algebra on  $X$ . In fact,  $X$  and  $\emptyset$  are members of *every*  $\sigma$ -algebra on  $X$ .

**Example**  $2^X$  is a  $\sigma$ -algebra on  $X$ .

However, we would also like to be able to construct more interesting  $\sigma$ -algebras.

**Definition** Given an arbitrary family of subsets  $F \subseteq 2^X$ , there is a unique smallest  $\sigma$ -algebra containing  $F$  which is called the  **$\sigma$ -algebra generated by  $F$**  and we will denote as  $\sigma(F)$ .

Can be constructed by taking the intersection of all  $\sigma$ -algebras containing  $F$ .

Of particular interest to us is the  $\sigma$ -algebra generated by a topology,  $\mathcal{T}(X)$ .

Borel  $\sigma$ -algebra:  $\mathcal{B}(X) = \sigma(\mathcal{T}(X))$

## 5.2 The Lebesgue Integral

**Definition** Let  $(X, \mu)$  be a measure space and  $S \in \mathcal{L}(X, \mu)$  a measurable set. If  $\mathbb{I}_S$  is indicator function  $\mathbb{I}_S : X \rightarrow \{0, 1\}$  given by  $\mathbb{I}_S(x) = 1$  if  $x \in S$  and  $\mathbb{I}_S(x) = 0$  otherwise. Then we define:

$$\int_X \mathbb{I}_S d\mu := \mu(S) \quad (5.1)$$

**countable?** We define a simple function as a function which maps to a finite set.

**Theorem 5.2.1** *If  $\varphi$  is a simple function, then we it can be represented as a finite sum of indicator functions:*

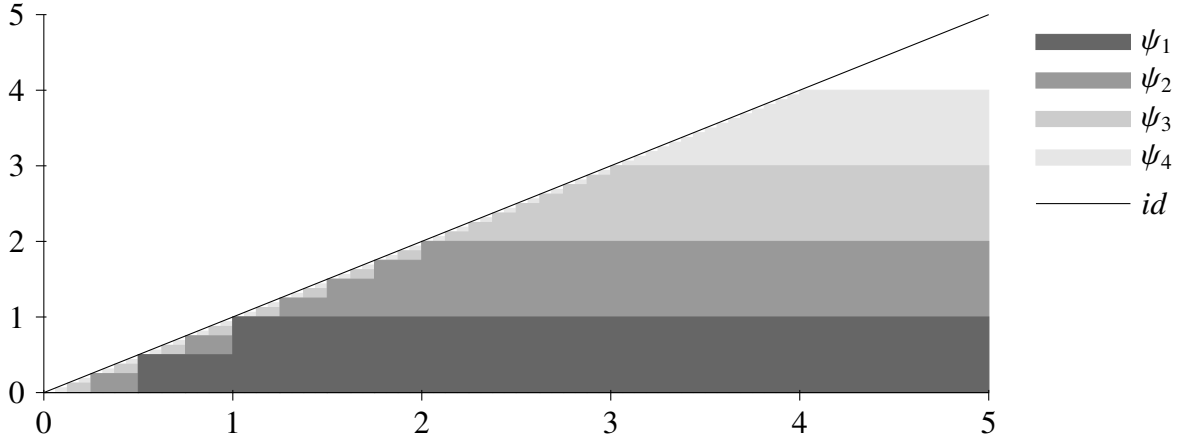
$$\varphi := \sum_{k=0}^n a_k \mathbb{I}_{E_k} \quad (5.2)$$

Where  $\{a_k\}_{k=0}^n$  is the set of values in the range of  $\varphi$  and  $E_k = \varphi^{-1}(\{a_k\})$ , the set of all points which map to  $a_k$ .

This allows us to define the integral of a simple function.



Figure 5.1: asdfasdf



**Definition** Let  $\varphi = \sum_{k=0}^n a_k \mathbb{I}_{E_k}$  be a simple function. We define the integral:

$$\int_X \varphi \, d\mu = \int_X \sum_{k=0}^n a_k \mathbb{I}_{E_k} \, d\mu = \sum_{k=0}^n a_k \int_X \mathbb{I}_{E_k} \, d\mu \quad (5.3)$$

Next we define integral of a non negative functions in  $\bar{\mathbb{R}}$

**Definition** Let  $(X, \mu)$  be a measure space and  $f : X \rightarrow [0, \infty]$ .

$$\int_X f \, d\mu := \sup \left\{ \int_X \varphi \, d\mu : \varphi \text{ simple, and } 0 \leq \varphi \leq f \right\} \quad (5.4)$$

We use the simple function  $\psi_n$  defined as:

$$\psi_n = \sum_{k=0}^{n2^n-1} \left[ \left( \frac{k}{2^n} \right)^{\left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right)} \right] + n^{[n, \infty]} \quad (5.5)$$

Notationally, this definition is rather heavy but is easily understood geometrically as seen in Figure 4.1

Which for any non-negative  $f$  allows us to define  $\varphi_n = \psi_n \circ f$  a simple function.

Obviously we have  $\varphi_n$  simple since  $\psi_n$  is simple.

Since  $\psi_n(x) \leq x$  for all  $x$  then we also have  $0 \leq \varphi_n \leq f$ .

figure  
descrip-  
tion

Most importantly we have  $0 \leq f(x) - \varphi_n(x) \leq 2^{-n}$  and so  $\varphi_n$ , uniformly approaches  $f$  as  $n$  approaches infinity.

**Theorem 5.2.2** *Let  $f$  be a function mapping to  $\mathbb{R}$ . Then  $f = f^+ - f^-$  where  $f^+$  and  $f^-$  are non-negative functions given by:*

$$f^+(x) := \max(0, f(x)) \quad (5.6)$$

$$f^-(x) := \max(0, -f(x)) \quad (5.7)$$

Now we have everything we need to define integrals for arbitrary functions mapping to  $\mathbb{R}$ :

**Definition** asdfasd

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu \quad (5.8)$$

This was a standard treatment of Lebesgue integration.

Mathematically it is very simple to extend this to integrals of hybrid sets over  $L(X, \mu)$  by linearity. That is, given  $\Gamma \in \mathbb{Z}^{\mathcal{L}(X, \mu)}$ :

$$\int_{\Gamma} f d\mu = \sum_{\sigma \in \Gamma} \Gamma(\sigma) \int_{\sigma} f d\mu \quad (5.9)$$

## 5.3 Differential Forms

**Definition** A  $k$ -form  $\beta$  on the open set  $\Omega \subset \mathbb{R}^n$  has the form:

open set or Lebesgue measurable sets??

$$\beta = \sum_j b_j(x) dx_{j_1} \wedge \dots \wedge dx_{j_k} \quad (5.10)$$

where  $j = (j_1, \dots, j_k)$  is a  $k$  dimensional multi-index. We say that  $\beta \in \Lambda^k(\Omega)$

We have not yet defined the  $\wedge$  operator.

Anti-commutative:  $dx \wedge dy = -dy \wedge dx$ . Which implies for any permutation  $\sigma$  of  $\{1, \dots, k\}$ :

$$dx_1 \wedge \dots \wedge dx_k = \text{sgn}(\sigma) dx_{\sigma(1)} \wedge \dots \wedge dx_{\sigma(k)} \quad (5.11)$$

Anti-commutativity additionally implies that for all  $x_i$ ,  $dx_i \wedge dx_i = 0$ .

Let  $\alpha = \sum_i a_i(x) dx_{i_1} \wedge \dots \wedge dx_{i_\ell} \in \Lambda^\ell(\Omega)$  and  $\beta = \sum_j b_j(x) dx_{j_1} \wedge \dots \wedge dx_{j_k} \in \Lambda^k(\Omega)$  then define:

$$\alpha \wedge \beta := \sum_{i,j} a_i(x) b_j(x) dx_{i_1} \wedge \dots \wedge dx_{i_\ell} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_k} \quad (5.12)$$

Thus we can think of  $\wedge$  as mapping a  $k$ -form and an  $\ell$ -form to a  $(k + \ell)$ -form,  $\wedge : \Lambda^\ell(\Omega) \times \Lambda^k(\Omega) \rightarrow \Lambda^{k+\ell}(\Omega)$ . By anti-commutativity we have:

$$\alpha \wedge \beta = (-1)^{k\ell} \beta \wedge \alpha \quad (5.13)$$

**Definition** Let  $\alpha$  be a  $k$ -form on  $\Omega \subset \mathbb{R}^n$  of the form  $\alpha = A(x) dx_1 \wedge \dots \wedge dx_n$ . If  $A \in \mathcal{L}^1(\Omega, dx)$  then we define:

$$\int_{\Omega} \alpha = \int_{\Omega} A(x) dx \quad (5.14)$$

Where the left-hand side is the integral of a  $k$ -form and the right-hand side is a Lebesgue integral. For any  $\beta \in \Lambda^k(\Omega)$  we extend this definition linearly as the sum of integrals.

Define  $dx$  from  $x_1, \dots, x_n$ . Need to lift sign change from permutations

## 5.4 Integration on Chains

Note on equivalence of  $k$ -simplices and  $k$ -rectangles.

**Definition**  $k$ -rectangle

**Definition** (singular)  $k$ -chain as a hybrid set (function) over the  $k$ -rectangles (with  $f$  a differentiable map)

**Definition** Integral on a chain

**Example** Integrate an atlas with overlapping charts (using inclusion-exclusion)

(6 pages)

## 5.5 Pull-backs

Benefit of differential forms is how cleanly they handle changes in coordinates.

**Definition**  $F : X \rightarrow \Omega$  Define the pullback  $F^*\beta$

$$F^*\beta = \sum_j b_j(F(x))(F^*dx_{j_1}) \wedge \dots \wedge (F^*dx_{j_k}) \quad (5.15)$$

and

$$F^*dx_j = \sum_\ell \frac{\partial F^j}{\partial x_\ell} dx_\ell \quad (5.16)$$

Which can be reduced by:

$$F^*\beta = \sum_j b_j(F(x))(F^*dx_{j_1}) \wedge \dots \wedge (F^*dx_{j_k}) \quad (5.17)$$

$$= \sum_j b_j(F(x)) \left( \sum_\ell \frac{\partial F^{j_1}}{\partial x_\ell} dx_\ell \right) \wedge \dots \wedge \left( \sum_\ell \frac{\partial F^{j_k}}{\partial x_\ell} dx_\ell \right) \quad (5.18)$$

$$= \dots \quad (5.19)$$

$$= \sum_j b_j(F(x)) \det(J_F) dx_{j_1} \wedge \dots \wedge dx_{j_k} \quad (5.20)$$

Which is significant given the change of variable formula for integration:

$$\int_{\phi(U)} f(v) dv = \int_U f(\phi(u)) |\det \phi'(u)| du \quad (5.21)$$

**Theorem 5.5.1** *Let  $F : X \rightarrow \Omega$  be an (orientation-preserving diffeomorphism) and  $\alpha$  an integrable  $n$ -form on  $\Omega$  then*

$$\int_\Omega \alpha = \int_X F^*\alpha \quad (5.22)$$

More algebra of differential forms

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta) \quad (5.23)$$

**Definition** Exterior derivative

...

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^j \alpha \wedge (d\beta) \quad (5.24)$$

...

$$F^*(d\beta) = dF^*\beta \quad (5.25)$$

## 5.6 Stokes' Theorem

**Definition**  $\partial$

**Example** Boundary of a rectangle

**Example** Boundary of a boundary

**Theorem 5.6.1**  $\partial^2 = 0$

**Proof** (6 pages)

## 5.7 A non-trivial example

**Example** Tricky Integration Using Stokes' theorem and Inclusion/Exclusion to evaluate a tricky theorem like:

$$f(z) = \frac{z^2}{(z^2 + 2z + 2)} \quad (5.26)$$

(2-3 pages)

## **Chapter 6**

### **Inclusion/Exclusion for Parallel**



# Chapter 7

## Conclusions

The primary objective of this thesis was to extend [8] by investigating further applications of hybrid sets and functions. Although this paper was focused on results for integration and Petri net graphs, this is not to take away from the “smaller” results shown along the way. As a first example, we showed that, (notational choice:  $\mathbb{Z}^{\mathbb{P}}$ ,  $h(\mathbb{P})$ ,  $\mathcal{H}(\mathbb{P})?$ ), the hybrid sets over prime numbers is equivalent to  $\mathbb{Q}_+$ .

Hybrid sets come into their own within the context of hybrid functions as we saw with arithmetic on piecewise functions and symbolic matrices. Combined with tricks from linear algebra, the usage of hybrid functions allowed for large decreases in both cases. Hybrid pseudo-functions leave a function associated with an element unevaluated and allow for algebra to be performed on the domains before requesting any functions be evaluated.

Hybrid functions were shown to be a good model for domains of integration. An atlas can succinctly be defined in terms of a set of hybrid relation over a universe of Euclidean rectangles mapping to a common manifold. Principle of Inclusion-Exclusion was used instead of the typical *partitions of unity* to reduce the atlas to its support. Unlike some of the previous examples, any leftover negative terms produced by PIE are completely well-founded and have natural geometric interpretations. Moreover,  $\partial$ , the boundary operator on a  $k$ -chain explicitly constructs them. The beauty (and usefulness) of  $\partial$  was then shown with a proof of the gener-

alized Stokes' theorem. Which in turn we used in conjunction with generalized partitions to transform an otherwise difficult to compute integral.

Finally, we showed a novel formulation of Petri net graphs. Instead of considering transitions as a special type of node, we represented transitions along with corresponding arc weights as a single hybrid set. Conditions for liveness and coverability were also discussed. Relaxing the condition of non-negative markings gives way to *lending Petri nets* [3] [4] for which hybrid sets were even better suited. Unfortunately, I just discovered the Bartoletti papers and have not had a chance to read more than the abstract.

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