Assume two matrices which we plan to multiply A which is $n \times m$ and B which is $m \times p$. Both are block matrices:

$$A = \left[\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right] \quad \text{and} \quad B = \left[\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array} \right] \tag{1}$$

Where A_{11} is a $k_1 \times k_2$ matrix and B_{11} is a $\ell_1 \times \ell_2$ matrix. Note that $0 \leq k_2, \ell_1 \leq m$ but the ordering of k_2 and ℓ_1 is unknown. Multiplication with $k_2 = \ell_1$, is commonly used:

$$AB = \left[\begin{array}{cc} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array} \right] \tag{2}$$

("conformable partitioning" is the term used in *Elementary Matrix Theory* - Howard Eves). I haven't found anything written on non-conformable partitioned multiplication but admittedly haven't looked very hard.

For the general case ($k_2 \neq \ell_1$), we still get back a 2×2 block matrix, which we will denote C.

$$AB = \left[\begin{array}{cc} C_{11} & C_{12} \\ C_{21} & C_{22} \end{array} \right] \tag{3}$$

C_{11} is a $k_1 \times \ell_2$ sub-matrix; the sizes of the other partitions can be derived from this. The domains of these 12 submatrices can be partitioned using:

$$N_1 = [\![0, k_1)\!) \quad N_2 = [\![k_1, n)\!)$$

$$P_1 = [\![0, \ell_2)\!) \quad P_2 = [\![\ell_2, p)\!)$$

$$M_1 = [\![0, k_2)\!) \quad M_2 = [\![k_2, \ell_1)\!) \quad M_3 = [\![\ell_1, m)\!)$$

So we can rewrite A and B as:

$$A = A_{11}^{N_1 \times M_1} \oplus A_{12}^{N_1 \times (M_2 \oplus M_3)} \oplus A_{21}^{N_2 \times M_1} \oplus A_{22}^{N_2 \times (M_2 \oplus M_3)} \tag{4}$$

$$B = B_{11}^{(M_1 \oplus M_2) \times P_1} \oplus B_{12}^{(M_1 \oplus M_2) \times P_2} \oplus B_{21}^{M_3 \times P_1} \oplus B_{12}^{M_3 \times P_2} \tag{5}$$

Here $\oplus$ is *not* the direct sum of matrices, but hybrid function pointwise sum. $\times$ is the Cartesian product of intervals.

For $i, j \in \{1, 2\}$ we can write blocks in C as:

$$C_{i,j} = A_{i,1}^{N_i \times M_1} B_{1,j}^{M_1 \times P_j} + A_{i,2}^{N_i \times M_2} B_{1,j}^{M_2 \times P_j} + A_{i,2}^{N_i \times M_3} B_{2,j}^{M_3 \times P_j} \tag{6}$$

If $k_2 = \ell_1$ then $M_2 = \emptyset$. We can think of multiplying a $n \times 0$ matrix by a $0 \times p$ matrix as giving a $n \times p$ matrix which is the sum over an empty set, 0 everywhere.

If $k_2 < \ell_1$ then this is like treating A and B instead as 2×3 and 3×2 block matrices. M_1, M_2 and M_3 are all positive intervals and form conformable blocks.

If $k_2 > \ell_1$ then M_2 is a negative interval which doesn't have a good interpretation. To simplify C_i, j, we can use the partition $\{M_1 \oplus M_2, \ominus M_2, M_2 \oplus M_2\}$ which contains only positive intervals.

First we rewrite the hybrid functions in the first and third terms:

$$\begin{aligned} C_{i,j} = &\left(A_{i,1}^{N_i \times M_1 \oplus M_2} \oplus A_{i,1}^{N_i \times \ominus M_2}\right)\left(B_{1,j}^{M_1 \oplus M_2 \times P_j} \oplus B_{1,j}^{\ominus M_2 \times P_j}\right) + A_{i,2}^{N_i \times M_2} B_{1,j}^{M_2 \times P_j} \\ &+ \left(A_{i,2}^{N_i \times \ominus M_2} \oplus A_{i,2}^{N_i \times M_3 \oplus M_2}\right)\left(B_{2,j}^{\ominus M_2 \times P_j} \oplus B_{2,j}^{M_3 \oplus M_2 \times P_j}\right) \end{aligned} \tag{7}$$

Which are now the product of conformable 1×2 and 2×1 block matrices. This gives us a 1×1 block matrix :

$$\begin{aligned} C_{i,j} = &A_{i,1}^{N_i \times M_1 \oplus M_2} B_{1,j}^{M_1 \oplus M_2 \times P_j} + A_{i,1}^{N_i \times \ominus M_2} B_{1,j}^{\ominus M_2 \times P_j} + A_{i,2}^{N_i \times M_2} B_{1,j}^{M_2 \times P_j} \\ &+ A_{i,2}^{N_i \times \ominus M_2} B_{2,j}^{\ominus M_2 \times P_j} + A_{i,2}^{N_i \times M_3 \oplus M_2} B_{2,j}^{M_3 \oplus M_2 \times P_j} \end{aligned} \tag{8}$$

Then we merge the middle three terms:

$$\begin{aligned} C_{i,j} = &\left(A_{i,1}^{N_i \times \ominus M_2} \oplus A_{i,2}^{N_i \times M_2} \oplus A_{i,2}^{N_i \times \ominus M_2}\right)\left(B_{1,j}^{\ominus M_2 \times P_j} \oplus B_{1,j}^{M_2 \times P_j} \oplus B_{2,j}^{\ominus M_2 \times P_j}\right) \\ &+ A_{i,1}^{N_i \times M_1 \oplus M_2} B_{1,j}^{M_1 \oplus M_2 \times P_j} + A_{i,2}^{N_i \times M_3 \oplus M_2} B_{2,j}^{M_3 \oplus M_2 \times P_j} \end{aligned} \tag{9}$$

$A_{i,2}$ occurs twice with opposite sign on the same region and so cancels itself. As does $B_{1,j}$:

$$\begin{aligned} C_{i,j} = &\left(A_{i,1}^{N_i \times \ominus M_2} \oplus A_{i,2}^{N_i \times \emptyset}\right)\left(B_{2,j}^{\ominus M_2 \times P_j} \oplus B_{1,j}^{\emptyset \times P_j}\right) \\ &+ A_{i,1}^{N_i \times M_1 \oplus M_2} B_{1,j}^{M_1 \oplus M_2 \times P_j} + A_{i,2}^{N_i \times M_3 \oplus M_2} B_{2,j}^{M_3 \oplus M_2 \times P_j} \end{aligned} \tag{10}$$

And functions over empty domains can be removed:

$$= A_{i,1}^{N_i \times M_1 \oplus M_2} B_{1,j}^{M_1 \oplus M_2 \times P_j} + A_{i,1}^{N_i \times \ominus M_2} B_{2,j}^{\ominus M_2 \times P_j} + A_{i,2}^{N_i \times M_3 \oplus M_2} B_{2,j}^{M_3 \oplus M_2 \times P_j} \tag{11}$$

Formally this relies on the following identities:

$$M^B \oplus M^C = M^{B \oplus C} \tag{12}$$

(we can split up a matrix into blocks)

$$M_1^A \oplus M_2^\emptyset = M_1^A = M_2^\emptyset \oplus M_1^A \tag{13}$$

(empty blocks can be ignored)

$$M_1^{A\times B} M_2^{B\times D} + M_3^{A\times C} M_4^{C\times D} = \left(M_1^{A\times B} \oplus M_3^{A\times C}\right)\left(M_2^{B\times D} \oplus M_4^{C\times D}\right) \quad (14)$$