Homework Set #3 – PHYS 6260
Prof. John Wise
Due Friday, January 31st, 11:59pm (Submit github URL to Canvas; all code on github)

- Your assignment should be uploaded as a **single Jupyter notebook** with all of the problems included.
- Please use the template notebook uploaded on Canvas as a starter.
- Comment your code through inline comments with # or markdown blocks, where the latter option is preferred.
- In the problem descriptions, "programs" are referring to single or multiple code blocks in a notebook.
- The materials that you are required to include are indicated at the end of each problem, next to the check symbol: ☑

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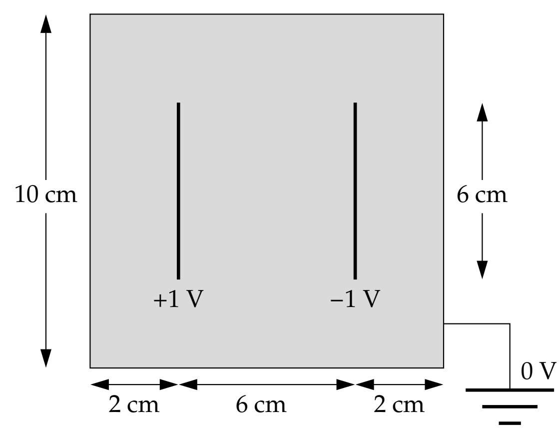
1. **Boundary value problem with a capacitor (25 points total):** Consider the following simple model of an electronic capacitor, consisting of two flat metal plots enclosed in a square metal box that is bounded to a voltage of 0 V, as shown in the figure to the right.



For simplicity, let's model the system in two dimensions. Using any of the methods we have studied, write a program to calculate the electrostatic potential in the box on a grid of 100×100 points, where the walls of the box are at voltage zeros and the two plates (which are of negligible thickness) are at voltages ± 1 V as shown. Have your program calculate the value of the potential at each grid point to a precision of 1 μV and then make both a density plot and surface plot of the result.

Hint: Notice that the capacitor plates are a fixed voltage, not fixed charge, so this problem differs from the in-class example with two charges. In effect, the capacitor plates are part of the boundary condition in this case: they behave the same way as the walls of the box, with potentials that are fixed at a certain value and cannot change.

☑ **For full credit**, include your program with comments and a density plot and a surface plot of the resulting electric potential.

2. **Thermal diffusion in the Earth's crust (25 points total):** A classic example of a diffusion problem with a time-varying boundary condition is the diffusion of heat into the crust of the Earth, as surface temperature varies with the seasons. Suppose the mean daily temperature at a particular point on the surface varies as:

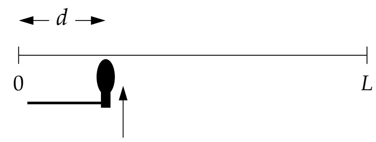
$$T_0(t) = A + B \sin \frac{2\pi t}{\tau}, \tag{1}$$

where $\tau = 365$ days, $A = 10°$ C, and $B = 12°$ C. At a depth of 20 m below the surface almost all annual temperature variation is ironed out and the temperature is, to a good approximation, a constant 11° C (which is higher than the mean surface temperature of 10° C—temperature increases with depth, due to heating from the hot core of the planet). The thermal diffusivity of the Earth's crust varies somewhat from place to place, but for our purposes we will treat it as a constant with the value $D = 0.1\ \text{m}^2\ \text{day}^{-1}$.

Write a program, or modify one of the in-class / lecture notes examples, to calculate the temperature profile of the crust as a function of depth up to 20 m and time up to 10 years. Start with temperature everywhere equal to 10° C, except a the surface and the deepest point, choose values for the number of grid points and the time-step h, then run your program for the first nine simulated years, to allow it to settle down into whatever pattern it reaches. Then for the tenth and final year, plot four temperature profiles taken at 3-month intervals on a single graph to illustrate how the temperature changes as a function of depth and time. In addition to these line plots, make a density plot showing temperature as a function of time (x-axis) and depth (y-axis).

☑ **For full credit**, include your program with comments, along with the requested two plots.

3. **FTCS solution of the wave equation (40 points total):** Consider a piano string of length L, initially at rest. At time $t = 0$, the string is struck by the piano hammer a distance d from the end of the string, as shown to the right. The string vibrates as a result of being struck, except at the ends, $x = 0$ and $x = L$, where it is held fixed.



(a) (30 points) Write a program that uses the FTCS method to solve the wave equation expressed as a set of simultaneous first-order equations (see Equation 6-9 in Lecture Notes #8) for the case $v = 100$ m s^{-1}, with the initial condition that $\phi(x) = 0$ everywhere but the velocity $\psi(x)$ is non-zero, with the profile

$$\psi(x) = C\,\frac{x(L-x)}{L^2}\exp\left[-\frac{(x-d)^2}{2\sigma^2}\right], \tag{2}$$

which is a Gaussian-like profile. Here $L = 1$ m, $d = 10$ cm, $C = 1$ m s^{-1}, and $\sigma = 0.3$ m. You will also need to choose a value for the timestep h – a reasonable choice is $h = 10^{-6}$ s. Make a plot of the vertical displacement and its time derivative as a function of distance at time $t = 0.02$ s.

(b) (10 points) Because the FTCS method is inherently unstable when applied to the wave equation, the noise should dominate the solution after some time. By trial and error (optional: use an automated method by computing the noise) by increasing the final time, find when the solution starts to become noisy and then overwhelmed by the high-frequency noise. Make three plots at three different times of the vertical displacement at this transition time, similar to the plot shown in 2nd PDE slide set. Denote the time of each solution in a legend.

☑ **For full credit**, include your program with comments along with the plots of vertical displacement and its time derivative at the requested times.

4. **Application question (10 points):** In a couple of paragraphs (about 250 words), discuss the aspect of energy conservation in numerical integration and how it is achieved numerically. In what types of systems would you desire energy conservation, and which methods would you use to achieve high accuracy in energy conservation? Alternatively, why can one neglect energy conservation in some numerical systems when it is a cornerstone of physics? You do not have to provide any code.