Computational Physics Term Project, Task 2: Progress Report

A Multi-Dimensional, Distributed, Non-Oscillatory Ideal Magnetohydrodynamics Framework

Neil Baker[1], **C. Michael Haynes**[2], **Corinne Hill**[3], **Eden Schapera**[1], **Kenneth Thompson**[3]

[1]School of Physics, Georgia Institute of Technology, Atlanta, GA, USA
[2]School of Earth and Atmospheric Sciences, Georgia Institute of Technology, Atlanta, GA, USA
[3]George W. Woodruff School of Mechanical Engineering: Nuclear and Radiological Engineering, Georgia Institute of Technology, Atlanta, GA, USA

1 Overview and Science Objectives

During the initial development of our numerical MHD solver, we uncovered several unforeseen technical challenges and computational restrictions, prompting us to modify the original scope of this project. Rather than developing a full 2-D MHD engine, we have focused our efforts on a 1-D solver, so we are restricted to cases with multiple simplifying symmetries. Within this framework, we investigate several such cases to probe and validate the physical accuracy of our numerical model.

In particular, we aim to address the following scientific questions:

Science Questions:

1. How does plasma behave in a quasi-1 Dimensional Slow Shock? (Falle et al., 1998)
2. How are slow shock plasma dynamics modified for a fast Shock Tube? (Brio & Wu, 1988)
3. How do rarefaction waves propagate in a MHD plasma? (e.g., Falle et al., 1998)
4. What is the mechanism for plasma confinement in the Tokamak reactor torus? (Chance et al., 1982; Nakazawa et al., 2000)

Following previous analytical models and theoretical analysis of MHD systems, we expect certain physical features to emerge in each test scenario. For slow shock tubes, we anticipate weaker compression and decreased tangential magnetic fields compared to stronger compression and tangential fields in fast shock tubes. Additionally, theoretical understanding suggests rarefaction waves exhibit smooth, continuous plasma expansion profiles. The efficacy of our model will be constrained by examining the output of our MHD engine and via direct model comparison (Balbás et al., 2004).

2 Technical Approach and Methodology

Our project relies on ideal MHD framework which assumes that the frequency of disturbances are small and that the displacement current is negligible. Our code takes in initialized cases of density, pressure, and velocity. Throughout the computation, the code tracks density, energy, magnetic field, pressure, and velocity.

2.1 Numerical Approach

We used the ideal MHD scheme of Balbás et al. (2004) as the backbone of numerical approach. Like these authors, we employed a non-oscillatory, staggered-mesh, predictor-corrector central differencing technique which means we did not need to process a Jacobian matrix at each timestep, instead opting for passing vectors instead. Following the derivations in Equations 18-19 from our proposal, we calculate Equation 20 from the proposal, now Equation 1, at each time step.

$$\bar{Z}_{j+\frac{1}{2}}^{n+1} = \frac{1}{\Delta x}\int_{I_{j+\frac{1}{2}}} Z(x,t_n)\,\mathrm{d}x - \frac{1}{\Delta x}\left[\int_{t^n}^{t^{n+1}} f(w(x_{j+1},t))\,\mathrm{d}t - \int_{t^n}^{t^{n+1}} f(w(x_j,t))\,\mathrm{d}t\right] \quad . \tag{1}$$

3 Preliminary Results

Our project can be found in this github repository: `https://github.com/mike-haynes2/mhd-solver.git`: the "stable" version and developmental versions are inside self-labeled subdirectories.

3.1 Configuration of our MHD system

Our first case is the Brio-Wu Fast Shock Tube (Brio & Wu, 1988) which has initial "negative" conditions of 1.0 for density (ro0_neg), 1.0 for pressure (pr0_neg), (0.0, 0.0, 0.0) for the velocity vector (u_vec_neg), ($\frac{2}{\sqrt{16\arctan(1)}}$, $\frac{4}{\sqrt{16\arctan(1)}}$, $\frac{2}{\sqrt{16\arctan(1)}}$) for the magnetic field vector (b_vec_neg). Similarly, initial conditions are also defined for the "positive" direction as 0.125 for density (ro0_pos), 0.1 for pressure (pr0_pos), (0.0, 0.0, 0.0) for the velocity vector (u_vec_pos), (0.75, -1.0, 0.0) for the magnetic field vector (b_vec_pos). There are other cases defined such as a slow shock tube and a refraction case, but we will focus on the Brio-Wu Shock Tube.

3.2 Current Status

We currently have seven python scripts to handle our 1D MHD computations. Equation 2.25 from Balbás et al. (2004) is referenced often and serves as the notation our code follows. It is the Jacobian-free form alternative shown as:

$$\left\{ f\left(w_j^n - \frac{\lambda\beta}{2} f_j' \right) \right\}' = \theta_j^{\beta-\frac{1}{2}} \Delta_0 \left(f\left(w_j^n - \frac{\lambda\beta}{2} \Delta_0 (f(w_j^n)) \right) \right), \quad \beta = \frac{1}{2}, 1$$

A brief overview of the python scripts:

`configuration.py`
Currently initialized for the Brio-Wu Shock Tube. The initial variables were found using Equation 4.4 in Balbás et al. (2004). Additional cases can be selected for the 1D MHD solver.

`spatial_integral.py`
Evaluates the first term of Equation 2.25 for each time step.

`temporal_integral.py`
Evaluates the second and third terms of Equation 2.25 for each time step.

`deltas.py`
A helper script to compute the integrals in `spatial_integral.py` and `temporal_integral.py`. A 3D function has also been written for future work. Based on these delta definitions in Balbás et al. (2004): $\Delta_{\pm} w_j = \pm(w_{j\pm1} - w_j)$ and $\Delta_0 = \frac{1}{2}(\Delta_+ + \Delta_-)$.

`minmod.py`
A flux limiter routine to compute the integrals in `spatial_integral.py` and `temporal_integral.py`. A 3D function has also been written for future work. Based on this equation in Balbás et al. (2004):

$$\left\| \sum_j p_j(x) \chi_j(x) \right\|_{TV} \leq \left\| \sum_j \bar{w}_j^n \chi_j(x) \right\|_{TV}$$

`reconstruct.py`
Computes the polynomial approximations for each quantity based on the second order reconstruction method shown as Equation 2.26 Balbás et al. (2004):

$$\int_n^{n+1} f(w(x_j, \tau))\, d\tau \approx \Delta t f(w_j^{n+\frac{1}{2}}) =: \Delta t f_j^{n+\frac{1}{2}}$$

It also contains four functions to reconstruct the quantities we are tracking after solving the general conservation equation.

`solver.py`
This is the main solver routine, handling time advancement for all quantities and tracking output.

3.3 Initial Results

Initial animations for $\mathbf{B}$, p, $\mathbf{u}$, and ρ can be found in the repository. We have implemented the Brio-Wu Shock Tube (Science Question 1), but not the other proposed science cases (see section 1). The proposed work will result in plots and animations for magnetic field, velocity, pressure, and density for all four case studies for our final presentation and report (i.e., Brio-Wu Fast Shock, Slow Shock, Refraction Waves, and the Tokamak Reactor geometry).

References

Balbás, J., Tadmor, E., & Wu, C.-C. (2004, November). Non-oscillatory central schemes for one- and two-dimensional MHD equations: I. *Journal of Computational Physics*, *201*(1), 261-285. doi: 10.1016/j.jcp.2004.05.020

Brio, M., & Wu, C. (1988). An upwind differencing scheme for the equations of ideal magnetohydrodynamics. *Journal of Computational Physics*, *75*(2), 400-422. Retrieved from `https://www.sciencedirect.com/science/article/pii/0021999188901209` doi: https://doi.org/10.1016/0021-9991(88)90120-9

Chance, M., Furth, H., Glasser, A., & Selberg, H. (1982, feb). Ideal- and resistive-mhd stability of one-dimensional tokamak equilibria. *Nuclear Fusion*, *22*(2), 187. Retrieved from `https://dx.doi.org/10.1088/0029-5515/22/2/001` doi: 10.1088/0029-5515/22/2/001

Falle, S. A. E. G., Komissarov, S. S., & Joarder, P. (1998, 06). A multidimensional upwind scheme for magnetohydrodynamics. *Monthly Notices of the Royal Astronomical Society*, *297*(1), 265-277. Retrieved from `https://doi.org/10.1046/j.1365-8711.1998.01506.x` doi: 10.1046/j.1365-8711.1998.01506.x

Nakazawa, S., Nakajima, N., Okamoto, M., & Ohyabu, N. (2000, April). One-dimensional simulation on stability of detached plasma in a tokamak divertor. *Plasma Physics and Controlled Fusion*, *42*(4), 401-413. doi: 10.1088/0741-3335/42/4/303