Homework Set #2 – PHYS 6260
Prof. John Wise
Due Friday, January 24th, 11:59pm (Submit github URL to Canvas; all code on github)

- Your assignment should be uploaded as a **single Jupyter notebook** with all of the problems included.
- Please use the template notebook uploaded on Canvas as a starter.
- Comment your code through inline comments with # or markdown blocks, where the latter option is preferred.
- In the problem descriptions, "programs" are referring to single or multiple code blocks in a notebook.
- The materials that you are required to include are indicated at the end of each problem, next to the check symbol: ☑

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1. **Adaptive integration (20 points):** Consider the integral

$$I = \int_0^1 \sin^2 \sqrt{100x}\, dx$$

Write a program that uses the adaptive trapezoidal rule method (Section 5.3 in the Lecture Notes #3) to calculate the value of this integral to an approximate accuracy of $\epsilon = 10^{-6}$. Start with one single integration slice ($N = 1$) and work up from there to 2, 4, 8, and so forth. Have your program print out the number of slices, its estimate of the integral, and its estimate of the error on the integral, for each value of the number of slices N, until the target accuracy is reached. *Hint:* The result should be around 0.45.

☑ **For full credit** in each part, include your program with comments and the requested information when refining your solution in each doubling of N.

2. **Heat capacity of a solid (20 points):** Debye's theory of solids gives the heat capacity of a solid at temperature T to be

$$C_{\mathrm{V}} = 9V\rho k_{\mathrm{B}} \left(\frac{T}{\theta_{\mathrm{D}}}\right)^3 \int_0^{\theta_{\mathrm{D}}/T} \frac{x^4 e^x}{(e^x - 1)^2}\, dx,$$

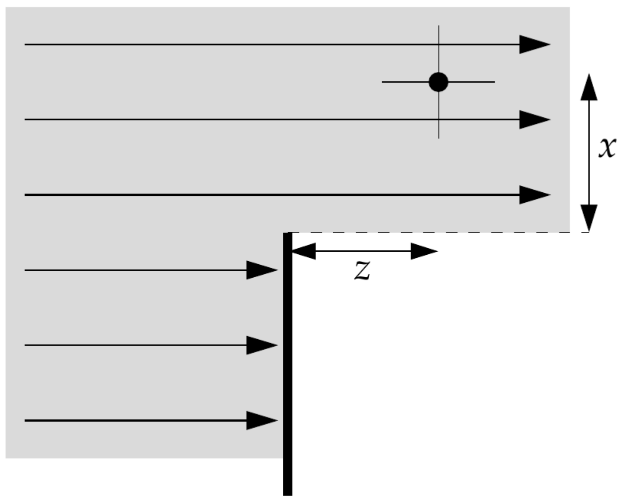
where V is the volume of the solid, ρ is the number density of atoms, k_{B} is Boltzmann's constant, and θ_{D} is the *Debye temperature*, a property of solids that depends on their density and sound speed.

(a) (15 points) Write a Python function `cv(T)` that calculates C_{V} for a given value of the temperature, for a sample consisting of 1000 cubic centimeters of solid aluminum, which has a number density of $\rho = 6.022 \times 10^{28}\ \mathrm{m}^{-3}$ and a Debye temperature $\theta_{\mathrm{D}} = 428$ K. Use Gaussian quadrature to evaluate the integral with $N = 50$ sample points. You may use the routines in the file `gaussxb.py` that we used in class and is provided in Canvas and in the git repository.

(b) (5 points) Use your function to make a graph of the heat capacity as a function of temperature from $T = 5$ K to $T = 500$ K.

☑ **For full credit**, include your program with comments and requested graph with the appropriate labels.

3. **Wave diffraction (20 points)** Suppose a plane wave of wavelength λ, such as light or a sound wave, is blocked by an object with a straight edge, represented by the solid line at the bottom of this figure:



The wave will be diffracted at the edge, and the resulting intensity at the position (x, z) marked by the dot is given by near-field diffraction theory to be

$$I = \frac{I_0}{8}\left\{[2C(u)+1]^2 + [2S(u)+1]^2\right\},$$

where I_0 is the intensity of the wave before diffraction and

$$u = x\sqrt{\frac{2}{\lambda z}}, \quad C(u) = \int_0^u \cos(\pi t^2/2)\,dt, \quad S(u) = \int_0^u \sin(\pi t^2/2)\,dt,$$

Write a program to calculate I/I_0 and make a plot of it as a function of x in the range $[-5\text{ m}, +5\text{ m}]$ for the case of a sound wave with wavelength $\lambda = 1$ m, measured $z = 3$ m past the straight edge. Calculate the integrals using Gaussian quadrature with $N = 50$ points. You should find significant variation in the intensity of the diffracted sound—enough that you could easily hear the effect if sound were diffracted, say, at the edge of a tall building.

☑ **For full credit**, include your program with comments and the requested graph with the appropriate labels.

4. **The Lotka-Volterra equations (30 points):** The Lotka-Volterra equations are a mathematical model of predator-prey interactions between biological species. let two variables x and y be proportional to the size of the populations of two species, traditionally called rabbits (the prey) and foxes (the predators). You could think of x and y being the population in thousands. Strictly the only allowed values of x and y would be multiples of 0.001 since you can only have whole numbers of animals! But 0.001 is a pretty close spacing of values, so it's a decent approximation to treat x and y as continuous real numbers so long as neither gets very close to zero.

In the Lotka-Volterra model, the rabbits reproduce at a rate proportional to their population, but are eaten by the foxes at a rate proportional to both their own population and the population of foxes:

$$\frac{dx}{dt} = \alpha x - \beta xy, \tag{1}$$

where α and β are constants. At the same time, the foxes reproduce at a rate proportional to the rate at which they eat rabbits—because they need food to grow and reproduce—but also die of old age at a rate proportional to their own population:

$$\frac{dy}{dt} = \gamma xy - \delta y, \tag{2}$$

where γ and δ are also constants.

(a) (25 points) Write a program to solve these equations using the fourth-order Runge-Kutta method for the case ($\alpha = 1, \beta = \gamma = 1/2, \delta = 2$), starting from the initial condition $x = y = 2$. Have the program make a graph showing both x and y as a function of time on the same axes from $t = 0 \to 30$.

Hint: Notice that the differential equations in this case do not depend explicitly on time t—in vector notation, the right-hand side of each equation is a function $f(\mathbf{r})$ with no t dependence. You may nonetheless find it convenient to define a Python function `f(r,t)` including the time variable, so that your program takes the same form as programs given earlier in this chapter. You don't have to do it that way, but it can avoid some confusion. Several of the following exercises have a similar lack of explicit time-dependence.

(b) (5 points) Describe in words what is going on in the system, in terms of prey and predators.

☑ **For full credit**, include your program with comments, along with your plot of x and y as a function of time for part (a), supplemented with your interpretation of the result in part (b).

5. **Application question (10 points):** In a couple of paragraphs (about 250 words), describe a time-dependent physical system that interests you, along with any quantities or ODE(s) that describe its state and evolution, respectively, also answering the following questions. You do not have to provide any code.

(a) Pick one quantity that is calculated through integration. Which integration method would you use and why?

(b) For the ODE(s), which method is best suited to evolve the system forward in time? Justify your choice by considering the necessary accuracy and computational speed.