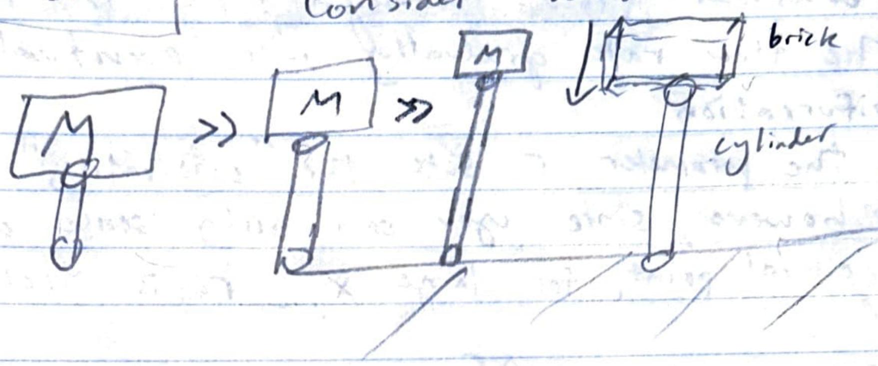
Lecture 4: 08/29

Strogatz Ch. 3: Bifurcations

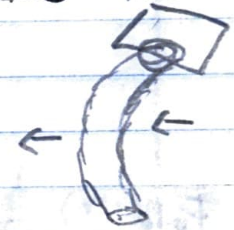
Leonard Euler

Consider a brick resting on a cylinder



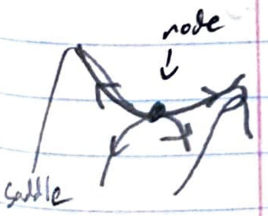
→ the cylinder will compress and deform axially, following a Hookean response (restoring force of compression resists brick mass)

→ Eventually, however, the mass overcomes the hookean response, and the rod will bend:

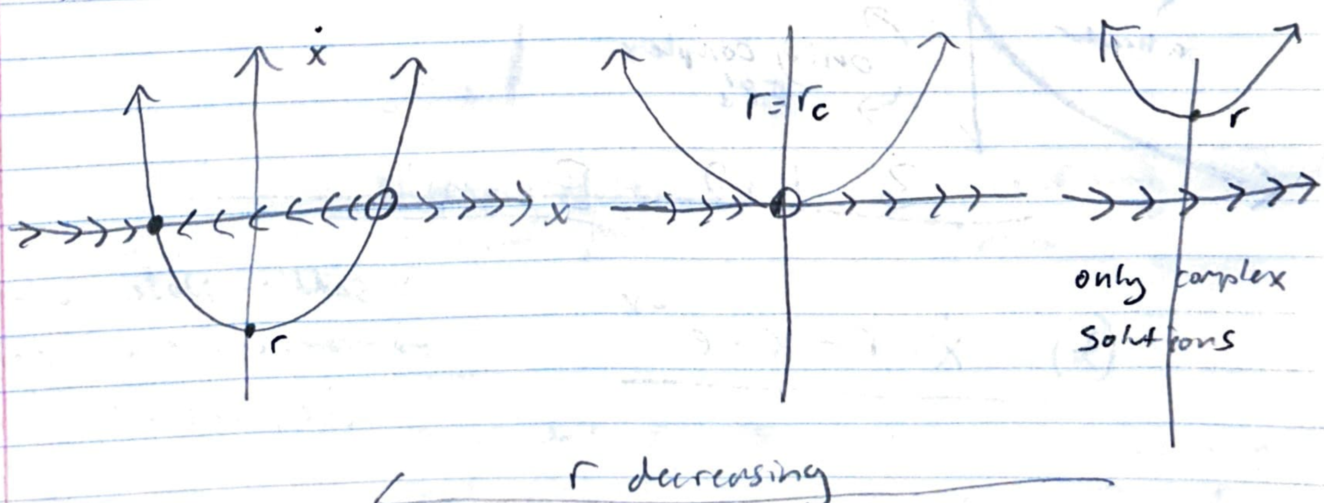


→ this bending happens suddenly, and represents a sharp break between two solutions: a bifurcation

3.1 Saddle-Node Bifurcation



$\dot{x} = r + x^2$ what happens when you vary 'r'

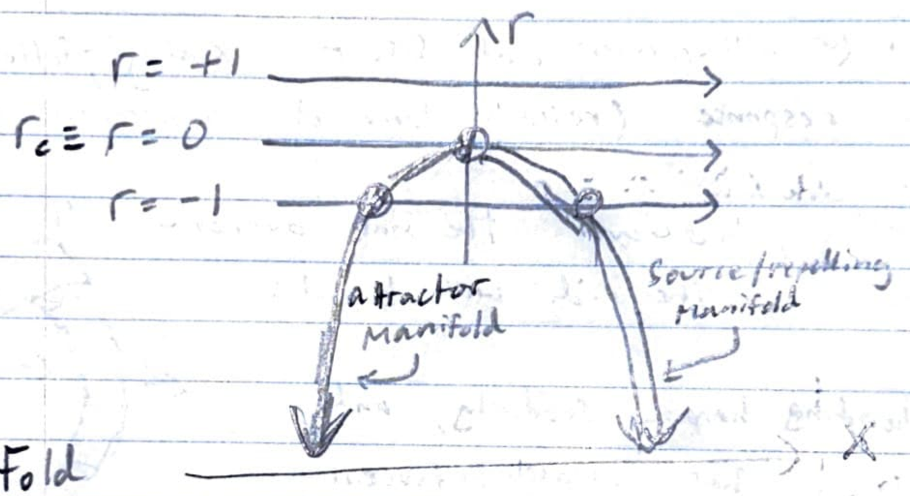


→ saddle point bifurcations are encountered often in higher / infinite dimensional problems, such as fluid flow

→ consider flow through pipe: in this case, increasing the flow rate gradually will eventually produce a bifurcation

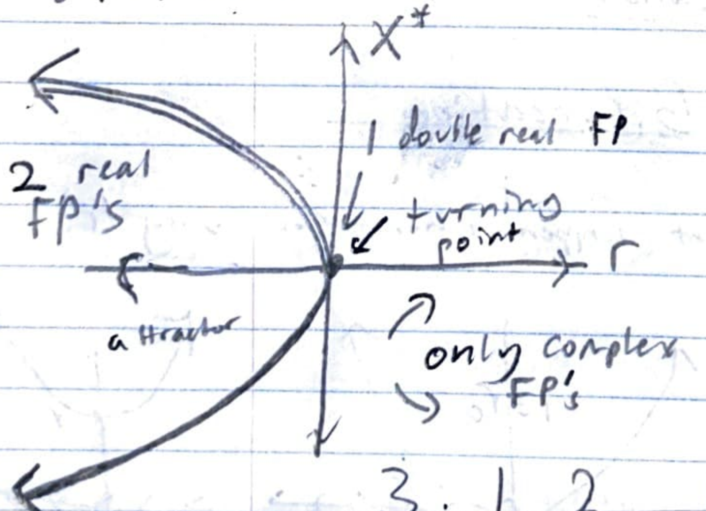
→ the parameter r sets the "criticality" of the system

→ however, since you can only sense criticality near the critical point, for large x, r is irrelevant / unimpactful



However, often we are interested in the variation of the parameter r (e.g., T)

also called Fold Bifurcations



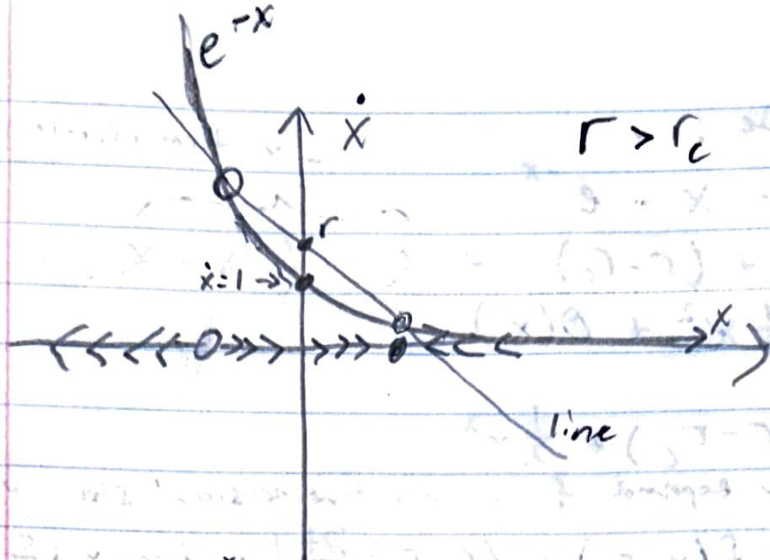
→ this represents one of many types of bifurcations, but across all forms they fundamentally share this property

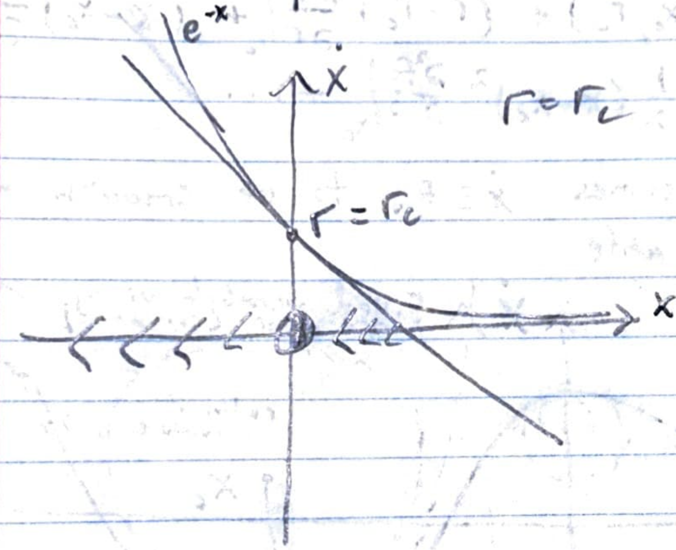
3.1.2 Example

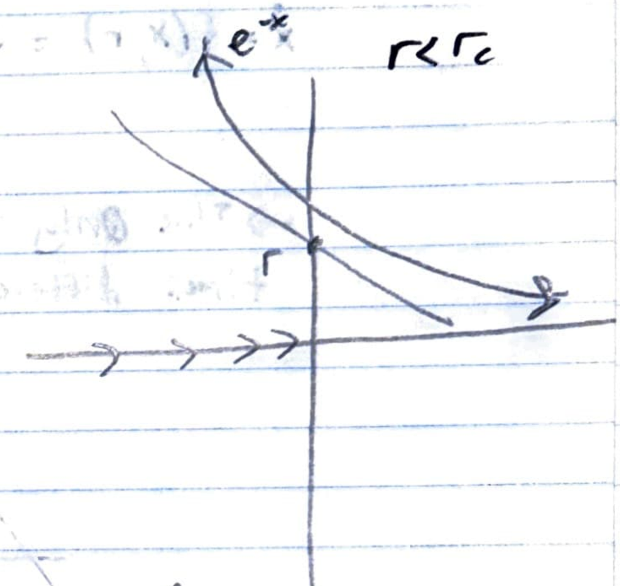
$$(*) \quad \dot{x} = r - x - e^{-x}$$

→ saddle-node, another example

→ transcendental equation (i.e., unsolvable in closed form)







$$\frac{d}{dx}\left(e^{-x}\right)\Big|_{x=x^*} \to = (-1)e^{-x^*} \qquad \frac{d}{dx}(r-x) = -1$$

$$\Rightarrow x^* = 0$$

$$\dot{x}(x^*) = 0 = r_c - x^* - e^{-x^*} = r_c - 0 - 1 \Rightarrow \underline{r_c = 1}$$

<u>tangency condition</u> allows you to calculate r_c

Normal Forms

→ the important dynamics occur in a small neighborhood around the critical point in state space

Example 3.1.2

→ consider a re-formulated eqn (*): translate the origin to be centered around the critical point r_c

We rewrite :

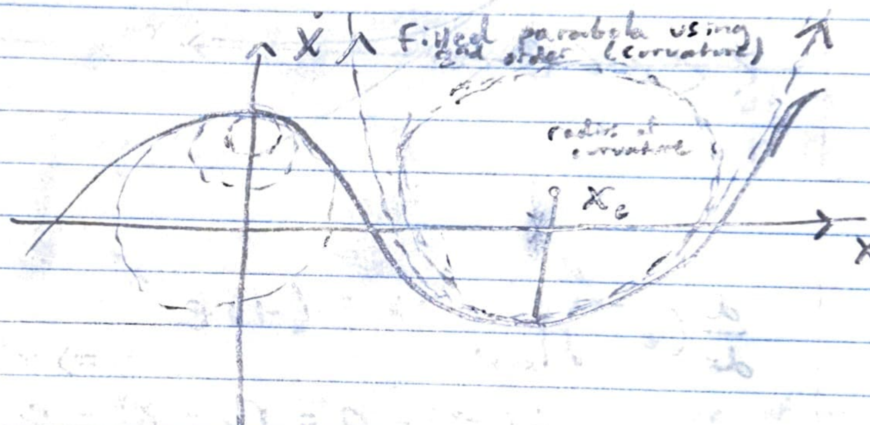
$$\dot{x} = r - x - e^{-x}$$
$$= r_c + (r - r_c) + \overbrace{x_c + (x - x_c)}^{-x \text{ term resealed}} + \overbrace{1 - x}^{\text{Taylor expand } e^{-x}} + \frac{1}{2}x^2 + O(x^3)$$

$$\dot{x} = (r - r_c) + \frac{1}{2}x^2$$

we can now expand f in multidimensional state space about (x_c, r_c)

$$\dot{x} = f(x, r) = f(x_c, r_c) + (r - r_c)\frac{\partial f}{\partial r}\Big|_{r_c} + (x - x_c)\frac{\partial f}{\partial x}\Big|_{x_c} + \frac{1}{2}(x - x_c)^2 \frac{\partial^2 f}{\partial x^2}\Big|_{x_c} + \ldots$$

→ This only assumes $\dot{x} \equiv f$ to be smooth and many-times differentiable



3.2 Transcritical Bifurcation

Consider $\dot{x} = x(r - x)$