Nonlinear Dynamics

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Lecture 1 : Strogatz Section 2.1

Flows on a Line

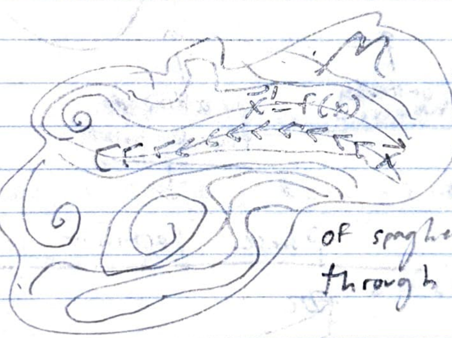
→ Follow book until 3rd dimension is introduced

Definition: Dynamical Systems ⟶ Deterministic
(i.e., not stochastic (noisy) or quantum mechanical)

Measurable space: (M, f) describe the system

1. State Space M, a set of numbers / coordinates $\vec{x} = (x_1, \dots, x_d)$ that represent properties of the system, defining a unique state. Can be thought of as a manifold
2. Governing Law: $x' = f(x)$

M: what am I describing?
f: how does it change



- Manifold representing state space with the constraint of $f(x)$
- If law is not time dependent, this space represents a "ball of spaghetti" with all possible points connected through orbits.

This is largely (effectively) a course in differential equations
⇒ time is continuous

→ $\dot{x} = f(x)$: represents a vector derivative, i.e., a velocity in state space. The tangents of each orbit are described by this eqn.

→ Consider a linear law f: (like Hooke's law)

- $f(x) = \sum^{d} a_i x_i = a_1 x_1 + \dots + a_d x_d$ → solve linear system w/ Laplace, Jordan Normal forms

→ or a nonlinear one:

- $f(x) = \sum^{d-1} b_{i,i+1} x_i x_{i+1} = b_{12} x_1 x_2 + b_{23} x_2 x_3 + \dots$
- $f(x) = a x^2$ → (Since 1910, has led to Fields Medals, Abel prizes, math today)

"Our mind is not quantum mind"

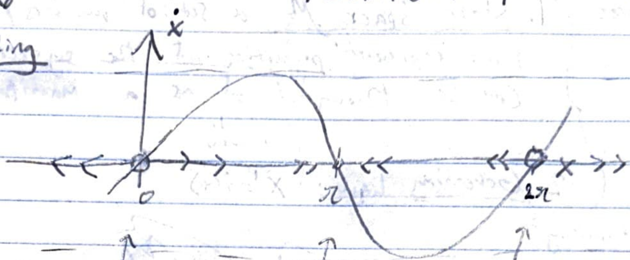
→ An example of flow on a line

f: $\dot{x} = \sin x$ $\qquad \mathcal{M} = \mathbb{R}^1$

$$\frac{dx}{dt} = \sin x \rightarrow dt = \frac{dx}{\sin x} = \csc x \, dx$$

$$\Rightarrow \int_0^t dt = \int_{x_0}^{x} dx \csc x \Rightarrow t = \ln \left| \frac{\csc x_0 + \cot x_0}{\csc x + \cot x} \right|$$ → but what does this tell us?

→ Seek qualitative understanding



unstable $\qquad$ attractor fixed point $\qquad$ unstable

↓

Phase portrait

Zeroes:

$Z = \{ k\pi : k \in \mathbb{N} \}$

set of measure zero: $\mu(Z) = 0$

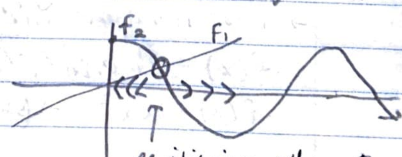
"everything falls into the hole"

→ Equilibria: stable points of the system, i.e., fixed points

↳ values of state space for which "nothing" happens

Stability: fixed points are stable, unstable, or "marginal" (saddle)

$$\dot{x} = \underbrace{x}_{\substack{\text{linear,} \\ \text{easy } (e^x) \\ f_1}} - \underbrace{\cos x}_{\substack{\text{subtract a} \\ \text{"spring"} \\ f_2}}$$



equilibrium when the two forces are equal

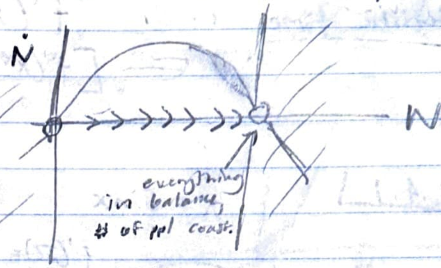
Lecture 2 08/21/25

2.3 Population (British Movement early 1900s)

Malthus: $\dot{N} = \underline{r} N$ $r > 0$: grows, $r < 0$: shrinks

Verlhulst: $r = r(N)$:

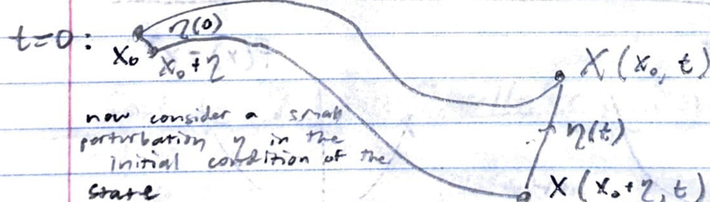
$\dot{N} = rN\left(1 - \frac{N}{K}\right)$



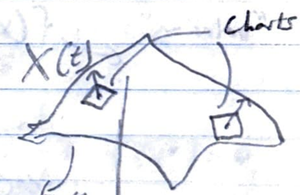
in reality, this model breaks down when considering the coupling b/t 2 populations

2.4 Stability, Linearization

· consider a state $X(x_0, t)$ @ time t & point x_0



now consider a small perturbation η in the initial condition of the state



This process is called linearization
→ sometimes it fails, when the 1st derivative is zero.

differentiable smooth Manifold. Hence, you can place a tangent plane (i.e., a chart) on any local neighborhood

Linearize via Taylor expansion   in higher dims, this becomes Jacobian

$$f(x_0 + \eta, t) = X(x_0, t) \quad f'(x_0, t)\,\eta(t)$$

$$\eta(t) = f'(x_0, t)\,\eta(0)$$

consider equilibrium X^*

$$\dot{X}(t) = f(x(t))$$

what happens close to η? $\dot{X}(x^*) = 0$

$\dot{X}(x^* + \eta)$

Q: is there ever a case where our linear / locally flat [illegible] assumption breaks down: i.e., when the

$$\dot{x}(t) = f(x(t))$$

$$= f(x^* + \eta(t))$$

$$\frac{d}{dt}(\dot{x} + \eta(t)) = \cancel{f(x^*)} + f'(x^*)\eta(t)$$

$\dot{x}$ is const $\rightarrow$ $\dot{\eta}(t) = f'(x^*)\eta(t)$

$\Rightarrow$ if:

$f'(x^*) < 0$ $\rightarrow$ stable: exponentially towards each other

$f'(x^*) = 0$ linear not sufficient

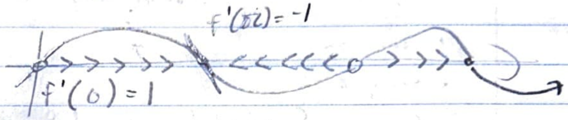
$f'(x^*) > 0$ points exponentially separate (unstable)

Characteristic time

$$\frac{1}{f'(x^*)} \leftarrow [f'(x^*)] = \frac{1}{T}$$

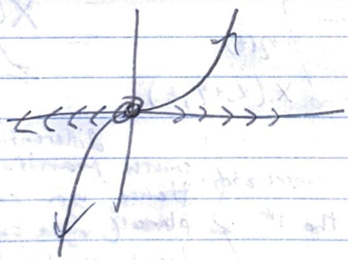
example 2.4.1 | $\dot{x} = \sin x$

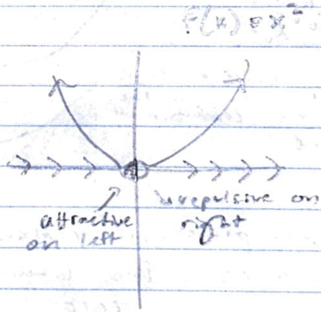
$$\dot{\eta} = f'(x^*)\eta$$



Non-linear Stability: example 2.4.3

"velocity" $\leftarrow$ $f(x) = x^3$



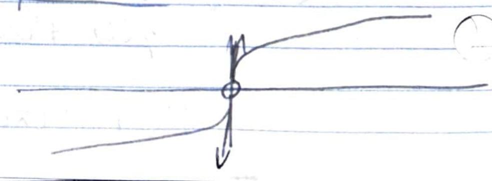


Section 2.5

example 2.5.1 | Non-uniqueness

$F(x) = x^{1/3}$ $\quad f'(0) = \infty$

$$\frac{dx}{dt} = x^{1/3} \rightarrow \int \frac{dx}{x^{1/3}} = \int dt$$



→ play w/ this eq.

→ predomy believes 3D smooth solutions to NS are not possible (Millenium prize)

The equation $\int \frac{dx}{x^{1/3}} = \int dt$ has many solutions:

$$\frac{3}{2} x^{2/3} = t + C$$

$$x = 0 \quad \forall t$$

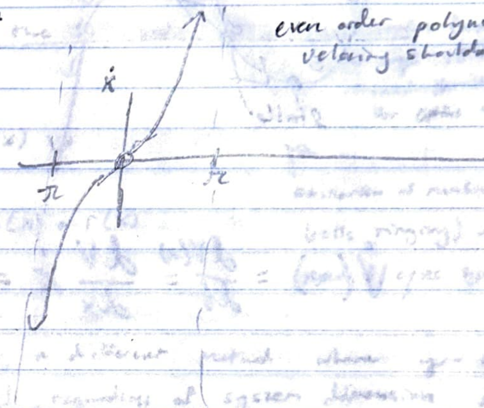
$$x(t) = \left(\frac{2}{3} t\right)^{3/2}$$

example 2.5.2 | Blow-Up (Millenium prize for NS)

$$\dot{x} = 1 + x^2$$

even order polynomial since velocity shouldn't be asymmetric

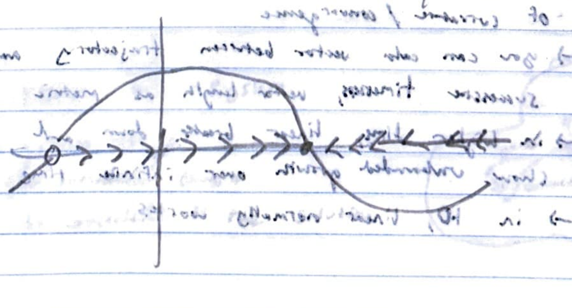
$$X(t) = \tan x$$



→ linear (sin x) near origin, but it blows up moving away from origin

Section 2.6 | No Oscillatory Systems

→ because an oscillation passes through $f(x^*) = 0$, must (i.e., is at rest @ ends of period), the oscillation cannot persist once the equilibrium point is reached.

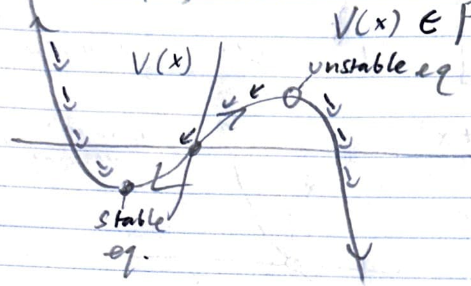


$f(x) = \dot{x}$

Section 2.7: Potential functions

$$f(x) = -\frac{dV(x)}{dx}$$ define $V(x)$ as a potential function.

For example, let $V(x) \in P^{(3)}(\mathbb{R})$



→ extremely useful in high dimensional problems

→ Nobel 2024: Hopfield
↳ computational neuroscience, describe memory in neurons as potential function

$$\dot{V}(x) = \frac{dV(x)}{dt} = \frac{dV}{dx}\dot{x} = -\left(\frac{dV}{dx}\right)^2$$

↳ potential function has to be positive, so $\dot{V} < 0$

Answer to Q: What happens when trajectories are large

- → Compare linear approximation to 2nd order term
- → when / before two are comparable, you determine "radius" of curvature / convergence
- → you can calc vector between trajectory and traj + ε at successive timesteps, vector length as metric
- → in higher dims, linear breaks down and hyperparam exponents show unbounded growth over infinite time
- → in 1D, linear normally works

Lecture 3: 08/26/25

Connection to Numerics

- ODEs

(x is time)

$$\frac{d^2y}{dx^2} = r(x) \quad \rightarrow \quad \frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(y(x)) \text{ or } r(y(x), x)$$

acceleration, forcing; acc., friction, forcing, driven or autonomous

→ dynamical systems can be recast as ODEs

→ consider systemizing the 2nd order ODE: (hamiltonian formulation)

at age 16 learned Hamiltonian dynamics for optics only, then extended

let

$$\frac{dy}{dx} = z(x)$$

$$\frac{dz}{dx} = -q(x)z(x) + r(x)$$

oscillation of membranes (i.e., church bells ringing) use 4th derivative, need 4 eqns system + 4 initial data

in late 18th century, a different method where you specify a single function (energy), regardless of system dimension, and from this equation (with variational techniques) an intelligent guess is a solution to the EOM: Variational principle to solve ODE systems

Newtonian → Lagrangian

Quantum mech → QFT

linear regressions → machine learning

This technique does not integrate the system, but writes a governing law.

Consider a system

with guesses o

+ slight variation o→ •

can be corrected to the solution

next thing is a fact of life...

we consider everything to be a continuum:

Discretization

Lattice methods

simple approach: 3D grid mesh

or a

hypercube lattice

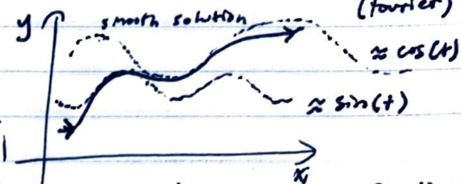
Versus

FFT: take smart subset of series truncation, using symmetries natural to sinusoids, that allows you to have a great representation rapidly

↳ very powerful; invented ≈ 1980

Spectral methods

(Fourier)



weighted sums of trig functions

$$y(x) = \sum_{-\infty}^{\infty} c_n b_n(x)$$ ← can integrate

spectral basis

$$\approx \sum_{-N/2}^{N/2} c_n b_n(x)$$

example bases

Fourier,
Chebyshev,
Hermite, etc...

"selecting the right set of coordinates/observables is a dark art, but it is critical" (Spectral representation ⟺ Latent representation)

Euler's Method

$x(0)$ 

$$t \to t_0 + k\Delta t_0$$

$$X(t) \approx x(k\Delta t) = x_k$$

$$\frac{dy}{dt} = \frac{y_{n+1} - y_n}{h} = f(x_n, y_n)$$

f is velocity field,

$$y_{n+1} = \underbrace{y_n + h f(x_n, y_n)}_{\text{Euler's estimate}} + \underbrace{\mathcal{O}(h^2)}_{\text{error}}$$

→ higher order discretizations, such as midpoint & RK4, exist