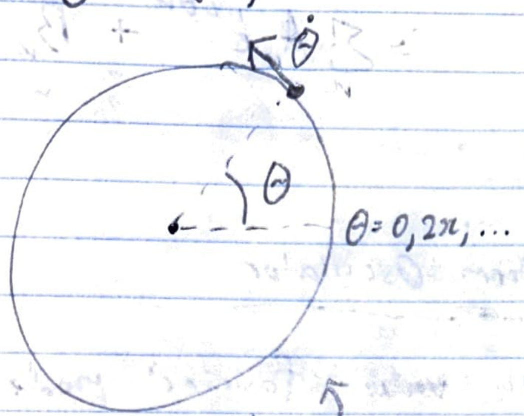
Lecture 09-04

Strogatz: Section 4

Flows on a Circle

→ 1-D problem with periodic boundary conditions

$\dot{\theta} = f(\theta)$

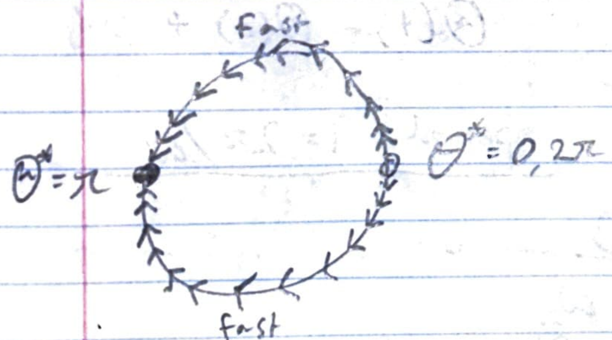


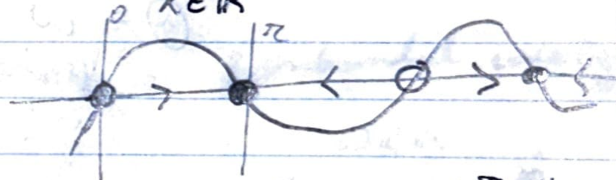
→ on an infinite line, there are no oscillations for systems $\dot{x} = f(x)$ since the system cannot "turn around"

→ on a circle, however, oscillations are permitted

compact line versus infinite line: ⟵⟶

$\dot{\theta} = \sin\theta$ ⟷ $\dot{x} = \sin x$ on infinite line, $x \in \mathbb{R}$

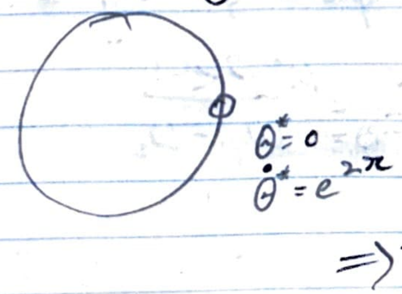




infinite periodicity on a 1-D line represents a problem in condensed matter physics, i.e., in a crystal with Å-scale periodicities

example 4.1.1: Continuity

$\dot{\theta} = \theta$



$\dot{x} = x \rightarrow x(t) = x_0 e^t$

⟹ This implies that $\dot{\theta} = f(\theta)$ requires something of f, a condition:

$\Rightarrow$ Thus, for a 1-D circle flow to have continuity, we must have that:

$$\dot{\Theta} = f(\Theta) \longrightarrow f(\theta_0) = f(\Theta_0 + 2\pi k)$$

$$= \sum_k a_k \sin(\theta k) + b_k \cos(\theta k)$$

$$= \sum_k A_k e^{i\theta k} + B_k e^{-i\theta k}$$

4.1 Oscillators

Uniform Oscillator

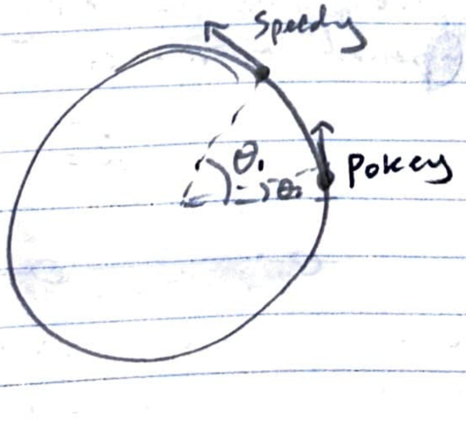
$\rightarrow$ The simplest, zeroth order fourier mode, is a constant ω. This constant is periodic over all periods.

$$\dot{\Theta} = \underset{\text{(frequency)}}{\omega} \rightarrow \Theta(t) = \Theta(0) + \omega t$$

period $T = 2\pi / \omega$

4.2 Beat Frequency

$$\dot{\Theta}_1 = \omega_1 \; ; \; \dot{\Theta}_2 = \omega_2$$



Phase difference

$$\phi(t) = \Theta_1(t) - \Theta_2(t)$$

$$\dot{\phi}(t) = \dot{\phi} = \omega_1 - \omega_2$$

$$T_{lap} = \frac{2\pi}{\omega_1 - \omega_2}$$

For $\omega_1 = 1$: lap every 2π if $\omega_2 = 0$

Cool aside on measuring brownian motion of ribosomes: see Predrag's undergrad senior thesis

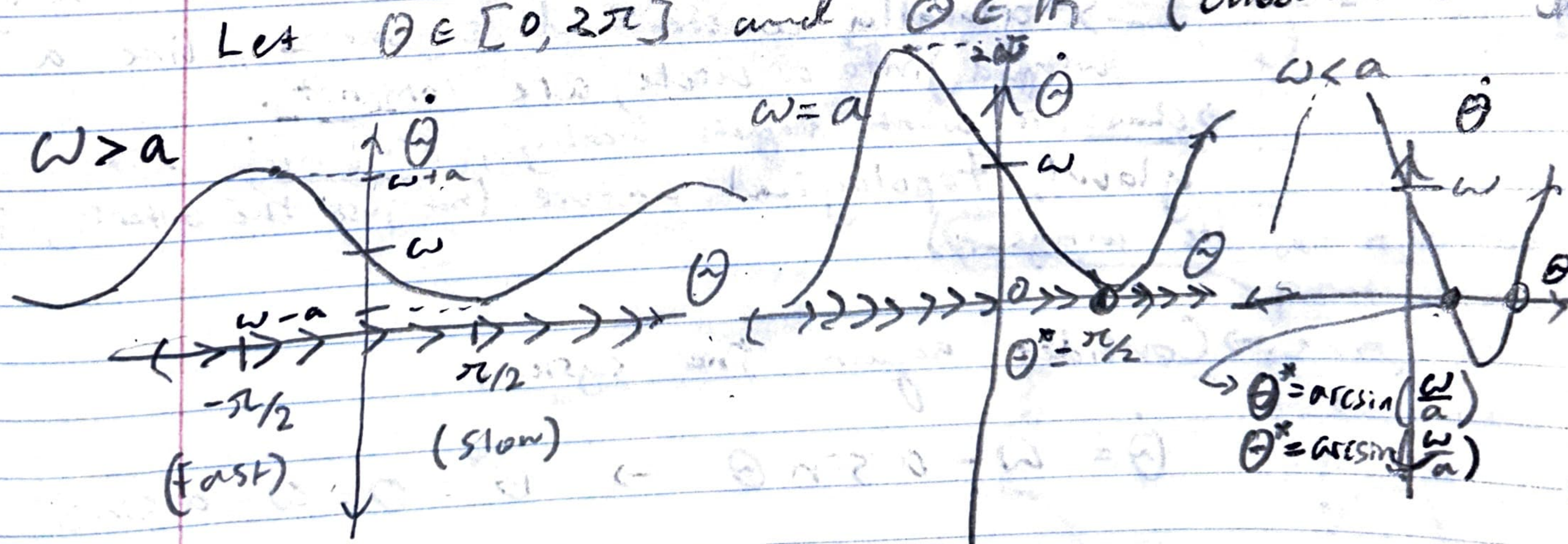
→ In nature, there are all kinds of clocks biologically
→ such clocks are driven by biochemical reactions, but perform a non-uniform oscillation

Beats, cont:

$$\dot{\theta} = \underbrace{\omega - a\sin\theta}$$

→ non-uniform nonlinear oscillator
→ over long times, the $a\sin\theta$ term averages to 0, but ω remains fixed
→ hence, ω is the mean frequency
→ a: amplitude of non-uniformity
→ when $\omega >> a$, small jitter
→ when $\omega << a$, oscillation dominates

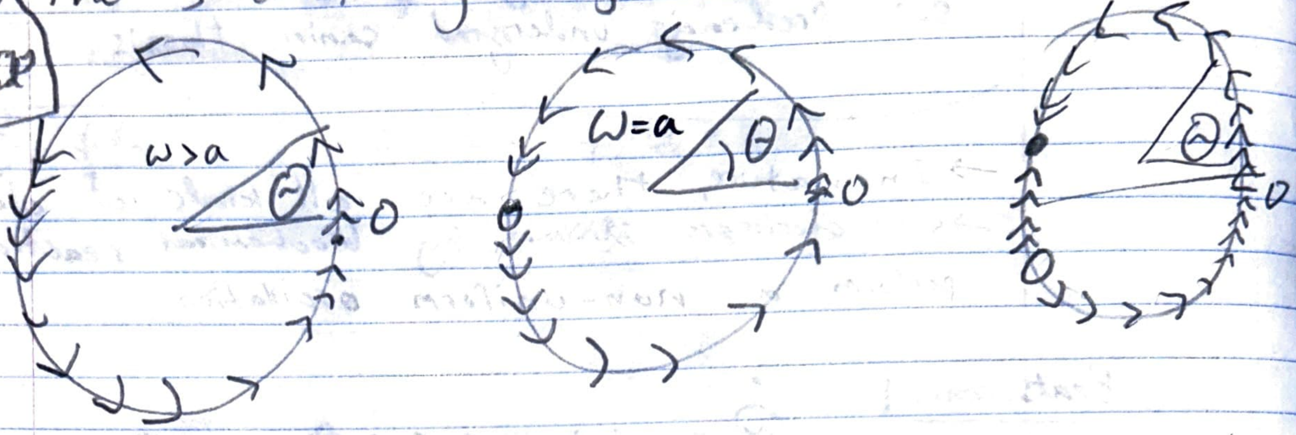
Let $\theta \in [0, 2\pi]$ and $\dot{\theta} \in \mathbb{R}^+$ (unbounded velocity)



→ in QFT, electromagnetism dictates that the nonlinearity is proportional to the fine structure constant (α, i.e. small) relative to mean photon field: weak coupling
→ however, 2 strong lasers overlapping yield strong coupling

HW Q: a harmonic oscillator is, by construction, a constrained 2-D system, is integrable if it is harmonic w/ a specific frequency
→ these systems need velocity coordinate too, since the energy is constrained

The 3 corresponding trajectories:



Stability Analysis

→ quantifying the degree of "attraction or repulsion" trajectories are from fixed points
→ a local concept, but the global topological structure is critical: the topological shapes, like infinite lines, drive the conditions of the systems
  ↳ globally constrained systems, like a line wrapped into a circle, are compact.
→ chaos: is what happens locally, combined with the global, topological nature. (not just the butterfly flapping its wings...)

→ Consider again the system

$$\dot{\Theta} = \omega - a\sin\Theta \quad \rightarrow \quad \dot{\Theta}^* = 0 = \omega - a\sin\Theta^*$$

$$\sin\Theta^* = \frac{\omega}{a}, \quad \cos\Theta^* = \pm\sqrt{1 - \frac{\omega^2}{a^2}}$$

→ these are the equilibria of the system.
→ Now, calculate/evaluate $df/d\Theta$

C

"Topology Changes Everything"

Stability: $\frac{df}{d\theta} = f'(\theta) = -a\cos\theta$

→ as long as amplitude dominates frequency ω: $a > \omega$

+ branch: $\cos\theta^* > 0$: stable

$$f'(\theta^*) = -a\sqrt{1-\frac{\omega^2}{a^2}}$$

− branch: $f'(\theta^*) = a\sqrt{1-\frac{\omega^2}{a^2}}$ unstable

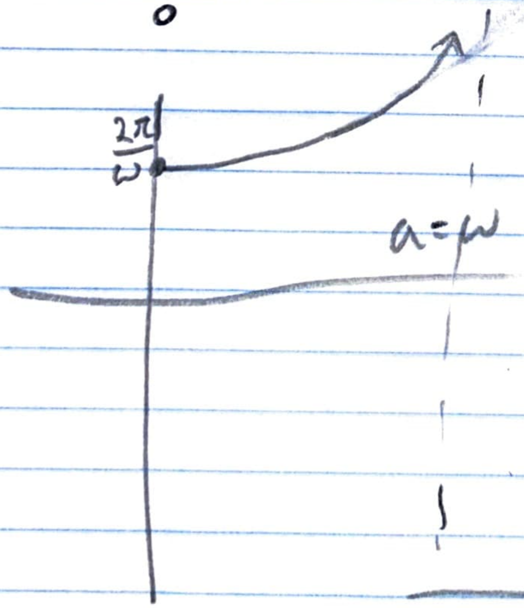
→ when $\omega = a$, $\cos\theta^* = \pm\sqrt{1-\frac{\omega^2}{a^2}} = \pm 0$ is a double root

$$T = \int_0^T dt = \int_0^{2\pi} \frac{dt}{d\theta}\, d\theta = \oint_0^{2\pi} \frac{d\theta}{f(\theta)}$$

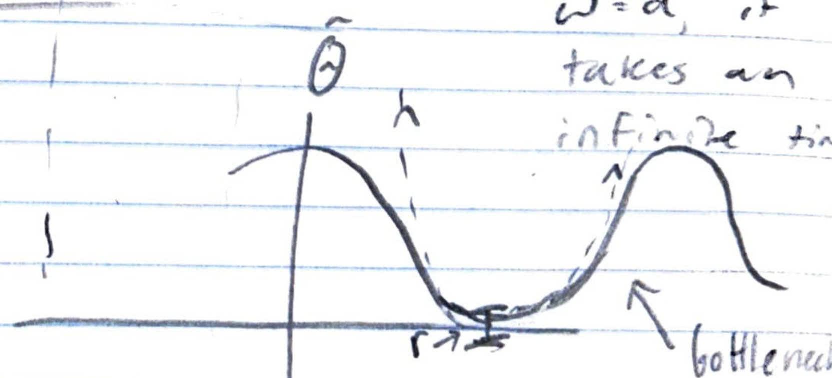
convert to frequency, introduce dynamical information

this only works if inverse of $\dot{\theta}$ is well-defined

$$= \oint_0^{2\pi} \frac{d\theta}{\omega - a\sin\theta} = \frac{2\pi}{(\sqrt{\omega+a})(\sqrt{\omega-a})} = \frac{2\pi}{\sqrt{\omega^2-a^2}}$$



at a critical point when $\omega = a$, it takes an infinite time



Mean-field Universality

↳ **Universality**: regardless of the system, the law describes all of them

$$\dot{\theta} = r + \theta^2$$