NEWTON AND THE APPLE

THE WHY, WHO AND WHAT OF CALCULUS

In the summer of 1666, Isaac Newton was sitting under an apple tree in his garden. He watched an apple fall to the ground and promptly invented the theory of gravity. That is, supposedly, how the story goes. In all likelihood, this old legend is largely false. But, however oversimplified this version of the events may be, it makes as good a starting point as any for an introduction to calculus. Because, the apple moves and speeds up as it falls. And this raises an important question - what exactly do we mean by "the speed of the apple" at any given moment during its flight? How can we calculate such a thing? And how do we model its motion?

The Historical Motivation for the Calculus. The story of Newton and the apple reminds us that mathematics does not develop in its own little vacuum. More often that not, a branch of mathematics has been developed to attack a class of problems arising in another field (often in the physical and social sciences). Elementary algebra was created to find answers to simple physical problems which required solving equations with one or two unknowns. Euclidean geometry came about because man needed to find perimeters, areas and volumes of common figures and find conditions under which, for example, two triangles were congruent or similar in shape. Trigonometry (which was introduced by astronomers) enabled man to determine the sizes and distances of the various heavenly bodies. In high school algebra and trigonometry, we also learn the fundamentals of another branch of mathematics called analytic geometry. We learn to graph linear equations such as $x+2y=5$, how to represent a circle of radius R with an equation of the form $x^2+y^2=R^2$, and how to determine which curves match the equations $y=\sin x$ and $y=\cos x$. The purpose of relating equations to curves is to enable us to use equations in the analysis and study of curves which model the paths of planets, projectiles and so on.

The seventeenth century was a pivotal time in the history of the world. The Enlightenment was sweeping through Europe, and scientists and philosophers were at the very center of the revolution. Astronomy took the center stage in providing the motivation for interesting mathematics problems. The heliocentric theory created by Copernicus and Kepler introduced the idea of the earth rotating on its own axis and revolving around the sun. The earlier theory of planetary motion, dating back to Ptolemy, which presupposed the earth absolutely fixed in space and in the center of the universe, was discarded. With Copernicus' new ideas, the theories and explanations for the motions of the planets which were developed by the Greeks were now invalid. Furthermore, Kepler showed that the path of each planet around the sun is an ellipse; however, there was no theoretical explanation for why the planets moved on such paths. The concept of universal gravitation as an explanation for the movement of the planets and moons was slowly starting to take steam. There were many important questions to be answered, and what is certain is that for seventeenth century scientists, the motion of the celestial bodies was the dominant problem of interest. Here are four examples of problems these scientists were trying to solve:

(1) Think back to the famous formula you learned in middle school: $d = v \times t$. Your teacher would have told you that this formula only applies with the velocity is constant. To put it another way, the formula only works if the graph of distance versus time is a straight line (in which case, the velocity is just the slope of the line). But of course, the motion of the planets (and other objects) involve changing velocity. And the tools that mathematicians had up to this point could not handle this more complicated case.

(2) A second major problem of the seventeenth century was the determination of tangents to various curves. This is a very practical problem. The invention of the telescope and microscope in the seventeenth century stimulated great interest in the action of lenses. To determine the course of a light ray after it strikes the surface of a lens, we must know the angle that the light ray makes with the lens - that is, the angle between the light ray and the tangent to the lens.

(3) A third type of problem besetting the seventeenth-century scientists can be called "maximization problems." As a planet moves about the sun, its distance from the sun varies. A basic question is what are the maximum and minimum distances of the planet from the sun?

(4) Finally, seventeenth century scientists were concerned with finding the lengths of curves and the areas and volumes of different shapes bounded by curves. Elementary mathematics can help in a few cases, but when curved surfaces are involved, simple geometry is almost helpless. For example, the shape of the earth is an oblate spheroid (a sphere somewhat flattened on the top and bottom). The calculation of the volume of this figure cannot be performed with elementary geometry. But it can be done with calculus.

The efforts to find the solutions to these four types of problems led mathematicians to the methods which we now call "calculus." Of course, similar problems continue to be important in our time, and the deep ideas of calculus are invaluable tools in helping us understand the world around us. The calculus has proved to be one of the richest mines that mathematicians have ever struck.

The Creators of the Calculus. Like almost all branches of mathematics, the calculus is the product of many men. The ideas of Fermat, Descartes, Pascal, Cavalieri, Huygens, Gregory, Wallis and many others played a fundamental role in the creation of calculus. However, there are two names which are most often mentioned in discussions on the origins of calculus - Isaac Newton and Godfried Leibniz. Interestingly, there has been considerable debate about who arrived at the ideas first. In fact, there was a major rift between Britain and Continental Europe with regards to who could lay claim to inventing calculus. In the Royal Society of London's *Philosophical Transactions* for 1708, there is a forgotten paper by the Oxford mathematician John Keill. In it, he refers to the calculus

> which Mr. Newton, beyond all doubt, first discovered...though the same Arithmetic was published later by Mr. Leibniz in the Acta Eruditorum with changes in the name and method of notation.

When Leibniz eventually saw this, in 1711, he took it as an accusation of plagiarism, and immediately put in an official complaint to the Royal Society, demanding an apology from Keill. A committee was set up to investigate the matter, but did not uphold Leibniz's complaint. This was hardly surprising because at the time Newton was President of the Royal Society, and he not only filled the committee with his own supporters, but wrote much of the final report himself!

We now know that Newton had many of the main results of calculus in 1665-6, long before Leibniz turned his attention to mathematics. But Newton allowed these results to be seen by a small, select number of contemporary mathematicians. Somewhat later, in 1674-6, Leibniz made many of his discoveries in calculus, while working in Paris. Towards the end of this period, Leibniz visited London on a diplomatic mission, and this lies at the dispute. He never met Newton; but he was shown some of Newton's early work in manuscript form. So while Leibniz was certainly the first to publish (in 1684), his detractors speculated on what he might have gleaned from the London visit. Perhaps the whole calculus dispute could have been avoided if Newton had published his work earlier. So why didn't he? Some scholars cite the dire state of the book trade following the Great Fire of London in 1666. But most see the explanation in Newton's extremely introverted and secretive character. He was known to be fearful, cautious and suspicious of others.

In many ways, the whole dispute between Newton and Leibniz was slightly absurd. After all, calculus did not just come out of nowhere, and the two of them were certainly standing on the shoulders of others. Yet it was these two who took a whole host of different ideas and created the calculus as a coherent subject. Thus, many historians today say that they invented calculus independently, though in different ways.

The Nature of the Calculus. So what exactly do we mean by the calculus? The word calculus comes from the Latin word for pebble, which became associated with mathematics because the early Greek mathematicians of about 600BC did arithmetic with the aid of pebbles. Today, a calculus can mean a procedure or a set of procedures to perform calculations. However, most often the word means the theory and procedures we are going to study which center around two very important concepts - the derivative and the integral. Usually we say *the calculus* to denote the theory and application of the derivative and the integral. The former gives us a way to understand *change* mathematically. The latter gives us a way to understand *accumulation* mathematically. And the calculus will help us to understand a deep connection between these two concepts.

The proper study of calculus involves at least three things. The first is the theory. The second is the technique. And the third is application. The calculus was created in response to scientific needs, and we should study many of the applications to gain appreciation of what can be accomplished with the subject. The calculus uses everything you have learned in your mathematical journeys so far - algebra, geometry, trigonometry and coordinate geometry. And it will also introduce new concepts, such as the derivative and the integral. But what is fundamental to both of these concepts is the concept of the limit. In the *Do Now* problem, we can see an example of how we can use limits to get us from an approximation of area to the exact area itself. This idea is as old as the Greeks (Archimedes first applied this to computing areas), but it lies at the very heart of everything we will do in calculus. In fact, one can really call calculus "the art of approximation." That is, using an appropriate approximation and taking the limit to get us an exact answer. And this "art of approximation" will show us how we can quantify two of the most basic concepts we deal with in everyday life - change and accumulation. During this year, we will learn how limits can help us to do deep and wonderful things as well as engage with some of the most beautiful and consequential ideas in human history.