Descartes' Rule of Signs

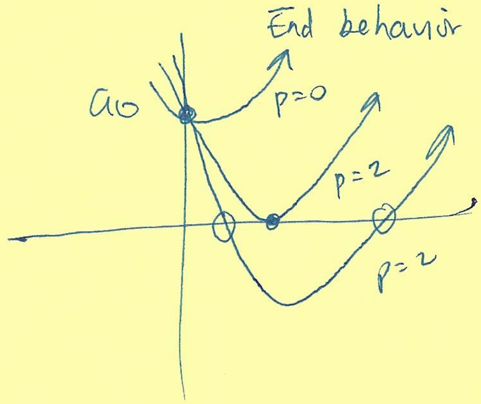
Thm: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = 0$ be a polynomial with $a_i \in \mathbb{R}$. Let δ be the number of sign changes in the sequence $a_n, a_{n-1}, \dots, a_0$ and let p be the number of positive real roots (counted with multiplicity). Then $\delta - p$ is a non-negative even number.

Proof: We have to show $\delta - p$ is even and $\delta - p \geq 0$. We can show the former "by illustration" and the latter by induction.

Consider $f(x) = x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0 = 0$.

$\boxed{\text{Case 1}}$ Let $a_0 > 0$. a_0 is the y-intercept of $f(x)$. Then $f(x)$ crosses the y-axis above the x-axis. We can then count p for different cases:

End behavior of $f(x)$: $x \to \infty \Rightarrow y \to \infty$



$\Leftarrow$ In every case, p is even counting multiplicity because of end-behavior.

Now for sign changes, we have the sequence $\underset{+}{1}, a_{n-1}, a_{n-2}, \dots, a_1, \underset{+}{a_0}$

Suppose all the a_i's are positive. Then $\delta = 0$

Now suppose there were only one negative coefficient somewhere:

$+ - + \cdots + \Rightarrow \delta = 2$

$+ + + \cdots - + + \Rightarrow \delta = 2$

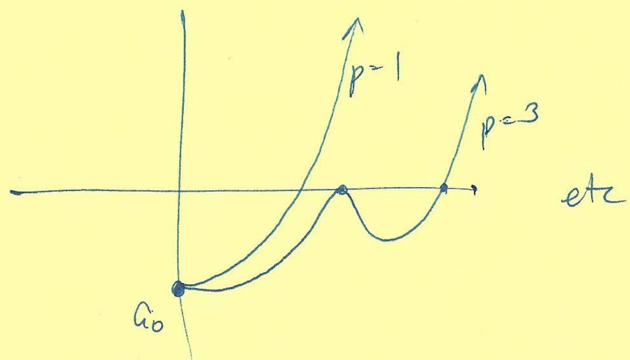
If there are two negatives:

$+ - - \cdots + + \Rightarrow \delta = 2$

$+ - + - \cdots + + \Rightarrow \delta = 4$

In general δ will always be even $\Rightarrow$ $\delta - p$ is even.

$\boxed{\text{Case 2}}$ Let $a_0 < 0$. Then:



$\Rightarrow$ p is always odd.

Similarly for sign changes, we have the sequence $1, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$

$+, \cdots \cdots -$

Suppose there are no negative coefficients $a_{n-1}, \ldots, a_1$. Then $\delta = 1$

If we introduce another negative: $\delta = 3$, etc... δ will always be odd $\Rightarrow$ $\delta - p$ is even

To show $\delta - p \geq 0$, we use induction.

① Let $n=1$: Then $f(x) = x + a_0 = 0$. Root @ $x = -a_0$.

$\Rightarrow p = 0$ and $\delta = 0 \Rightarrow \delta - p \geq 0$ ✓

② Assume the statement holds for $n = k-1$. Then $f(x) = x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_0$

Let p' be the number of positive roots and δ' be the number of sign changes in the sequence $1, a_{k-2}, a_{k-3}, \ldots, a_0$. Then we have $\delta' - p' \geq 0$.

③ Let $n = k$ and consider $f(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$.

Then $f'(x) = k x^{k-1} + (k-1)a_{k-1} x^{k-2} + \cdots + a_1$. The sequence of sign changes in $f'(x)$ will be equivalent to the sign changes in $f(x)$ up to the last coefficient a_0 (because we are multiplying by the positive powers arising from the power rule). Now in $f(x)$, we have one extra coefficient, so we will be adding at most one extra sign change. $\Rightarrow \delta \geq \delta'$.

Now let p be the number of positive roots of $f(x)$. Then $f'(x)$ must have $p-1$ positive roots. To see this, apply Rolle's Theorem. For any two roots of $f(x)$, say α_1 and α_2, we have $f(\alpha_1) = f(\alpha_2) = 0$. Since $f'(x)$ and $f(x)$ are continuous & differentiable, $\exists\ \beta$ such that $f'(\beta) = 0 \Rightarrow \beta$ is a root of $f'(x)$.

Thus, $f(x)$ will have one less root than $f(x)$. So we have

$$p' = p - 1.$$

Finally: we have $\delta \geq \delta'$ and $p' = p - 1$, so:

$$\delta \geq \delta' \geq p' = p - 1$$

inductive hypothesis

$$\Rightarrow \quad \delta \geq p - 1$$

$$\delta - p \geq -1.$$

However, we showed that $\delta - p$ is even so we must have

$$\boxed{\delta - p \geq 0}.$$

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