KNOWING WHAT TO TEACH

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> "It is clear that the chief end of mathematical study must be to make the students think."' - JW Young

Since the fall of 2012, I have had the opportunity to teach a variety of courses ranging from Differential Equations to Mathematics for Elementary School Teachers to a diverse group of students across multiple institutions in the Greater New York area. There are few experiences which have brought me more joy than the experience of showing a student how mathematics is not merely the rote manipulation of symbols, but a deep expression of the rigor, beauty and harmony found in the world itself. Over the years, my teaching style has evolved as I have gained more experience, and I have been able to grow in my ability to communicate mathematics effectively through the feedback and evaluations I have received from advisors and colleagues. But perhaps the greatest source of insight with regards to helping others learn and understand mathematics has come from conversations with my students (both past and present) on the nature of mathematics and its place in the world. My reflections on these conversations and subjects have led me to a basic philosophical goal as a mathematics instructor - to effectively support my students' engagement in deep mathematical ideas and to encourage peer-to-peer collaboration among them so as to make them better and deeper thinkers.

What then is my role as the instructor and what concrete steps do I take to achieve this end? Inside the classroom, I focus on incorporating three things into my teaching: 1. Creating interaction which emphasizes student thinking; 2. Teaching mathematics within a real-world context with a problem-centered approach; and 3. Teaching mathematics in historical context. Let me consider each of these in turn.

First, with regards to instructor-student interaction, I have seen a development in my teaching over the years where I have moved away from the traditional means of initiation and response between the instructor and the students and toward an emphasis on the use and development of student thinking. The former is often found in the traditional mathematics classroom: the instructor attempts to garner participation with prompts such as, "Does anyone have a question?" "How can we simplify this?" "What is the inverse of this matrix?" These questions lead to an interaction between the instructor and the student where the instructor initiates with the question, the student responds, and then the instructor determines whether the response is correct or incorrect. While this method can sometimes be useful, it fails to promote the deep engagement with the subject matter and collaboration between students which I believe makes mathematics education worthwhile. I now attempt (wherever I can, and to the best of my ability) to form my lecturing around helping students to share and deepen their thinking, as well as engage with the thinking of those around them. In teaching Introductory Linear Algebra, Differential Equations, and Calculus, I have learned several steps which I try to implement to achieve

this. The first important step is to create a classroom environment in which students feel comfortable in expressing their thinking. This type of classroom atmosphere stems largely from expressing my own enthusiasm and joy for the subject. Showing enthusiasm and approachability goes a long way toward transforming a student's experience from dry and difficult to engaging and thought-provoking. However, I have found that this environment can also be created by prompting students in the right way. I have discovered that questions such as, "Can you expand more on that idea?", "I know you haven't finished the problem yet, but what was your initial thinking?" and "We should not shy away from making mistakes - can you talk us through an initial approach you took that did not work?" to be particularly helpful. These types of prompts show the students that there is an expectation for their ideas to be shared, even if they are incorrect. Often times their thinking is fuzzy and lacks rigor; however, my goal in engaging them in discussion is not to evaluate them immediately, but to encourage them to make their contribution. Following this, a second step is to help students engage with the ideas and thinking of their peers. I use questions such as "Do we agree with his idea or thinking?", "Could someone re-express her argument in your own words?", and "Can you repeat that so the whole class can hear what you said?" to move students in this direction. I have seen the benefits of these kinds of interactions often in the classroom. Sometimes, one student arrives at a different conclusion and this leads to debate about who is correct and who can justify their thinking more precisely. Other times, a misunderstanding surfaces and a discussion ensues on how to be more precise and careful in the way we are communicating. All of these outcomes are productive in teaching students how to think mathematically. The third and final step to follow the previous two is to assist students in deepening their thinking and to build on and extend their ideas more formally. This is where I see the role of the instructor as being key. Once discussion has occurred, it becomes mathematically productive when it is followed on by solid reasoning. Therefore, I see my role as one in which I press students for reasoning and justification for their ideas. Questions such as "How can we know that this is correct?", "Does this work all the time?", "What is the connection to this other concept we were discussing?" and "Do we have to worry about this counterexample?" have been helpful in getting students to really develop their ideas and think through how to justify them. This also allows me to introduce the importance of rigor and proof, and show students that fundamentally, mathematics is more than just symbolic computation and is about argumentation and deep thinking. And it is here that formal and conventional definitions can be introduced in such a way that the students are (in some sense) discovering them for themselves and actually doing mathematics on their own. I have found that creating classroom interactions with students on these three levels has helped me tremendously to achieve the goal of engaging my students with deep mathematical ideas in collaboration with each other.

This leads me straight to my second point: in order to effectively create interactions of the kind mentioned above, I try to build my lectures around a particular problem or puzzle. Rather than simply present a technique to students and reinforce it with plenty of examples, I begin lectures (wherever possible) by presenting them with a motivating problem - or series of problems - which we need to solve together as a class. This allows room for discussion and interaction over what the right approach and technique could be to achieve that end. The goal, then, of the

lecture is to discover an appropriate technique or Theorem for the particular problem and justify its use through proof. In my experience, students find problems which come from a real-world context to be particularly engaging, and I attempt to use examples from a wide variety of fields including engineering, science, architecture, economics, business and others. A great benefit of using a problem-centered approach is that it often leads to very interesting follow-up questions from students. It is here that I can often point to other sub-disciplines of mathematics or science to pique the interest of the more curious students. Many have often followed up to learn more about this, and I have enjoyed fruitful discussion on a wide variety of topics with them.

Finally, in addition to teaching mathematics within a real-world context, I also attempt to provide as much historical context to my students as I can. Connecting mathematical ideas to to their place in history is not simply good pedagogy, but it also gives me (as the instructor) an opportunity to discuss broader ideas and themes about the people who developed mathematics. Many mathematicians were accomplished in other fields (philosophy, history, art, etc), and I have found that students enjoy understanding and discussing the connections mathematics has to other disciplines and the unique contribution it brings to thinking about and understanding the world.

A particular example of my use of these three teaching principles can be found in my presentation of Calculus and its foundations. I use the problem of finding the orbit of planet around the sun as the motivating problem the class has to solve. Deriving the equations from first principles allows for discussion on the role of mathematics in physics and modeling, and the importance of making good and accurate assumptions. Solving for a particular solution of the problem leads to the introduction of the derivative and integral, where students can engage with deep ideas about the rate of change of functions and the accumulation of quantities over time intervals. The discussion about some of the intricacies in calculus computations is always interesting for my students, and allows me to engage them in discussion about interesting results beyond the scope of the course (such as Lebesgue integration). Finally, discussing the life of Isaac Newton and the historical context for his examination of the problem always piques the interest of my students. Through this example and many others, I have found that teaching around these three principles has helped me to engage my students in mathematical thinking and collaboration with the ultimate goal of helping them to be strong thinkers and communicators. In my conversations with students over the years, these are the constant elements of my classes that they have pointed out to me which have been the most enjoyable and fruitful for them.

Of course, having as many principles and techniques as one would desire is ultimately useless if one does not understand his or her hearers. The most basic analysis begins with simply knowing who is in the room and how best to communicate to the group. Therefore, knowing my students is vital to the work I hope to accomplish in the classroom. Inside the classroom, as I have mentioned above, I devote much of my energy to creating an open environment for learning and developing community. It is often in this environment that I can have side conversations with students or even engage the class as a whole and learn about their strengths weaknesses, desires and ambitions. I have found that students respond well to a more informal and open classroom and, with some time, are quick to open up and

share with me and with their peers. In addition to this, I use surveys and forms throughout the duration of my class to gather data from students. Often these forms include questions where students can tell me more about their interests outside of school. This information has been extremely helpful for me in engaging students in conversations that are not purely academic in nature. As the rapport with my students grows stronger, I am able to bring that into the classroom and tailor my teaching to meet their specific needs. Outside of the classroom, I use office hours and conversations outside of the classroom to get to know my students on a deeper level. I consider my office hours to be an extension of my classroom, with whiteboards on the wall and various materials in the room so that students can use that space to work together and collaborate together. Even though most students come to office hours with questions which are specific to class content, I have had the opportunity to talk to many students during office hours on a variety of topics and offer them advice from my own life experience. These moments have been vital to helping me understand and relate to my students and have helped me craft my teaching to make it relevant and interesting for them. It would be an understatement for me to say that office hours are the best moments in my work as a teacher. I have found as much joy in building relationships with my students as I have in helping them grow in their knowledge of mathematics.