REPORT MEMORANDUM

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TO: **Maureen Ellison-Connolly**
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RE: **AT Course Feedback -- Nov 19, 2021 visit to Scarsdale High School Math Department**

Background

The Scarsdale High School Math Department offers 9-12th graders a curriculum that features a natural progression through standard areas of high school mathematics; students are placed into four levels ranging from *skills* to *advanced* on the basis of core competency upon enrollment as well as talent and interest for the discipline.

I was asked to review the course descriptions and syllabi for the two advanced topics courses MATHEMATICS 455 (AB Calculus) and MATHEMATICS 455 (BC Calculus). I was also invited to sit in on two lectures (Nista & Wagner) for the BC course and one lecture (Apostle) for the AB course. On the basis of this information and discussions with Department faculty, I have been asked to address eight (8) questions that are provided below together with my responses.

Questions and Responses

Q1: In what ways are high school students well prepared and in what ways ill prepared for college?

A1: Students coming from strong high schools such as the Scarsdale Program are exposed well to standard topics (e.g., in prerequisite mathematics, such topics as algebra, geometry, trigonometry, pre-calculus, and the elementary notions of calculus) and thereby prepared to work with these topics on the basis of skills required at onset of the college freshman year. Both the content stipulated in the descriptions / syllabi that were provided and the classroom observations suggest that Scarsdale students enjoy an excellent setting in which to acquire such skills (see below for further specificity).

Recent trends in higher education suggest students face challenges especially in adapting to a college environment with significantly less day-to-day / week-to-week tracking of their progress / performance and fewer checkpoints during the course of an academic term. There are of course students with weaker technical backgrounds that may face specific curricular challenges in a math/science-intensive major track, but this type of problem is likely to be of secondary (and probably minimal...) concern for Scarsdale graduates.

A question / perhaps food-for-thought for the High School Faculty: In what ways are the last two years at the school fostering the independence and good habits necessary to succeed as a young adult in college with minimal (at times, essentially nonexistent) oversight from professors? (i.e., What life skill exercises does the curriculum provide implicitly or explicitly, so as to give students an opportunity to develop in this way.)

Q2: At the College level, what are the key content and skills components necessary for success? What levels of achievement define or identify college level mastery of content/skills goals?

A2: Success in a college Calculus sequence taught at a high level is mostly a consequence of sufficiently developed (and reasonably sophisticated) logical reasoning skills / abilities. While it is true that prior exposure to basic concepts and computational principles in the discipline will facilitate a subsequent pass through this material, particularly in critical areas where the topics involve deep ideas and require additional mathematical maturity from the audience, the subject matter emerges from deductions based on a concise framework of well-motivated definitions and basic notions. As a result, proper placement of computational algorithms and techniques in the student's mind stems naturally from a fostered and developed comfort with the logical organization of the subject. Knowing well and being reasonably comfortable with where formulas for symbolic manipulation come from (how they can be derived from first principles, why they hold, when they fail) leads to a greater long-term retention of the latter algorithms; from a slightly more abstract/philosophical perspective, an approach of deducing and justifying the techniques of Calculus tends to produce graduates that can appreciate and explain to others what they have studied and how it applies to the world beyond the classroom.

In my courses, I usually give exams once monthly; throughout the progression of the academic term, I expect students to provide correct statements to major theorems - for instance, in Calculus I, I ask students to state the Intermediate Value Theorem and I typically repeat this prompt on each of the four course exams (including the Final Exam). I find that an emphasis on the accumulation of major theoretical results in the subject reinforces the importance of knowing where techniques come from for understanding how to apply them. (For example, is it reasonable to assign a particular root-finding problem to a computer? Should we implement Newton's method in order to solve $1/x=0$?)

From my own students, I expect progress in their ability to compose elegant solutions to challenging applications problems that require principles and techniques from Calculus; this notwithstanding, I also expect progress in their ability to explain how the theory works (and when / where it fails).

Q3: How might these content and skills expectations be reflected in instruction?

A3: I find it unlikely that a course would be able to place a proper emphasis on logical development and organization of the material unless it spends a significant (probably major) component of class time on lectures that discuss theory. The role of the practicum / recitation, where students engage actively with working out calculations or comparing their work on homework problems to that of their peers should not be understated, but it is natural to expect that active learning pedagogy will not on its own dissect a discipline as old and sophisticated as mathematics (or any of its sub-areas), which has developed over many decades or, in certain cases, over centuries. It is equally unreasonable to expect that 15-18 year old students would experimentally arrive at the nuances that the discipline involves without not only targeted guidance but also an explicit presentation component that affords students the opportunity to take in how the instructor, a trained professional mathematician, thinks about and organizes the subject matter.

This being said, speaking generally (not specifically to the lectures observed at Scarsdale High School), more can be done by us in the direction of engaging students during a lecture presentation so that they are more actively and critically involved in thinking about it. Questions such as "What is the converse of the sentence just provided?" or "What does this theorem say in the case where[...]?" and prompts such as "Provide an example of a scenario where the theorem's conclusions are not at hand, and explain in terms of where the theorem's hypotheses are violated," can guide students in real time and keep them engaged despite the instructor's lead. I've also had some success in formulating weaker / less relevant sentences (lesser theorems, not stated to the point) and on the basis of this, asking students to think critically about the sentence and to conjecture on possible ways of strengthening it. One can also ask for counterexamples to false (universally quantified) sentences, and this is another excellent way to keep students on their toes and engaged in the presentation.

During the visit, I sat in on presentations of the definition of functional average over an interval as well as the Mean Value Theorem for Integrals and proof of the latter. In both instances, continuity of the function was emphasized as a hypothesis. For example, a prompt for students to identify a function that is Riemann-integrable over an interval but fails to attain its average there as a value provides a loop-closing opportunity along the lines described above.

Q4: What types of assessments provide useful measures of the goals identified above?

A4: Over time I have found that pre-assessing students regarding their level of sophistication with logical reasoning gives me a sense about their baseline. In some cohorts, I have felt sufficiently comfortable / at ease with designing graded assessments that explicitly resolve this skill set further; in other instances, this is too much to expect early on in the term, especially across the board with all students, but it is a reasonable goal over the course of an academic year.

This being said, I would expect 17/18-year-old high school seniors to still struggle substantially with mathematical rigor and would incorporate it lightly without overemphasis. For instance, I think it is reasonable to assess student performance in such a way that a high B is still attainable without mastery (or even necessarily a distinctly measurable development of theoretical understanding), but that an A would be reserved for the latter. I believe that individual support for each student, such that encourages attention to logical reasoning and demonstrates the benefits of leaping in this direction can go a great way in instilling confidence and maximizing the acquisition of such skills, even in younger students. This must go hand-in-hand with the challenges to work on applied problems that require ingenuity and do not depend directly on mastery of the abstract material, so as to give students exposure to the elegance of applications as well as a sense of accomplishment even when a more theoretical understanding is not yet at hand.

Problems that challenge students to synthesize simple algebraic relationships prior to applications of the techniques of calculus and then carry out a calculation and analysis will typically satisfy well the applications component. While the composition of a logical argument is a heavy lift for a high school student, it is not unreasonable to ask that they identify a logical fallacy in an incorrectly formulated argument; or that they identify an unnecessary (or unnecessarily strong) hypothesis in a sentence and improve the statement; this kind of assessment tool does well in extending logical reasoning abilities and ascertaining student progress in this regard.

Q5: What are the recent trends in research, the new directions in the field, the new ways of thinking about the discipline? Are we in line with those trends?

A5: Data assimilation, data analysis, and data-driven modeling are among the newest trends across all areas of applied mathematics. I can speak to this phenomenology specifically within the context of applied dynamical systems, where my own training and research interests reside. It is evermore the case that models aimed at reproducing salient features of physical and biological systems are increasingly complex, to the point where machine computation is an unavoidable component to scientific inquiry. For instance, models of mechanisms for the processing of sensory input by the visual cortex must be constrained with extensive parameter fitting in order to render them in regimes that are physically relevant, and this often involves the incorporation of data input from lab measurements by way of adequate mathematical techniques involving linear algebra, probability / statistics, and analysis.

As discussed with colleagues during the visit's lunch break, the Scarsdale Program's plans to offer linear algebra training to students beyond the AT Calculus courses is an excellent forward-thinking step. It is increasingly likely that careers in applied science will require technical training in linear algebra and its applications, to the point where literacy in this subject area may become as critically necessary as other parts of the high school curriculum. The challenges involved in this effort obviously have to do with the advanced nature of the subject matter, and the reality that most high school students would not be ready to face it. This being said, it seems appropriate to introduce students to the basics of systems of linear equations and matrix theory, with an advanced track / cohort that may delve into subsequent topics such as linear independence, basis, dimension, and fundamental matrix subspaces.

Q6: What support measures do you have to help students at the college levels? In Scarsdale High School we have the Math Center and offer a tutorial model. Are there any recommendations you could suggest to better support students given your experience?

A6: The Cooper Union's School of Engineering has recently set up an academic resource center that provides freshmen and sophomores with tutoring support for the required chemistry, mathematics, and physics courses in the curriculum foundation. The Center is run through the Dean's Office and managed directly by the Associate Dean for Academic Affairs. Students in their junior and senior years whose records feature strong performance in math courses are invited to staff the Center.

From what I learned during my visit, Scarsdale High School has a better model in place - the Math tutoring is provided by a teaching professional that is solely focused on running this resource and is otherwise free from formal teaching assignments. I believe that this system has a greater potential to ensure adequate support for students that need active discussion and feedback outside of class time, particularly because of the greater level of expertise and experience in the support staff. I am considering suggesting this model to our administration.

Depending on how much time students spend together at school, or generally their proximity to each other off-campus, incorporating group collaboration structure for homework can improve learning and retention. When I have adopted such a strategy for my college-level courses, I have found that students seek my help together with their group-mates; the opportunity for them to discuss issues prior to engaging with me focuses their attention and prepares them to respond more actively to me. Additionally, office-hour discussions / tutoring are more efficient with this structure in place.

Q7: What do you see as the role of technology at the high school level compared to the college level? How is homework assigned and assessed? How do you assess/check for understanding?

A7: Whether at the high school or college level, technology can be a valuable aide in visualizing geometric representations of the material (graphs of functions, more generally curves in the plane, etc.); it can also bring to life computations that require a large number of repeated steps whose results should be shown to students without involving them in excessive arithmetic and busy work (e.g., an instantiation of Newton's Method in order to solve a specific non-linear equation of interest).

This being said, advanced topics courses would be best served to declare at onset (among others) the goals of *prediction* and *validation* of the computing environment. If students will have their hands on technology anyway, the goal of the mathematics course should be to address the "why" of what they see on the machine's display. Whether or not a course is declared entirely technology-free at the start (as are my Calculus courses at The Cooper Union), mathematics is ultimately about the acquisition of rigorous understanding.

Students are served best when their math courses avoid the trap of faux-proof, in which students compute limits or definite integrals or claim to ascertain general principles entirely through numerical experimentation. To fix this in classes that can't avoid student access to computing technology, math faculty should craft problems where blind numerical experimentation leads to an (obviously) incorrect result; a process of investigating what went wrong provides an effective way to motivate the acquisition of greater theoretical understanding. For instance: *The numerical implementation of the centered difference formula for the absolute value at x=0 gives a derivative of 0 there. It must be that the formula sometimes fails to compute a derivative faithfully. What are appropriate hypotheses that would ensure the formula is valid? Prove this.*

I typically assign, collect, and grade weekly homework sets. I tend to weigh homework at around 10-15% of the course grade, which means a typical week of homework is worth 1% or so of the final grade. (For reference, a final exam is typically weighed at 25-30%.)

My standard for grading any of the student work is based on the reasonable degree of confidence that I (the reader) am left with regarding the student's ability to explain what they are doing. In problems that mostly require algorithm execution and calculations, I expect the work to be well organized so that the reader can find all of the necessary steps. In problems that require the statement of a theorem, I look especially for correctly quantified sentences; during the lecture time, errors can be submitted (anonymously) to a group/class discussion, where peer instruction and active feedback form me can be a successful technique for resolving misconceptions.

Q8: Comments/impressions of the Scarsdale advanced topics program. (goals/objectives, curriculum, assessments, samples of student work)

A8: My review of the AT AB and BC Calculus syllabi indicates that the course goals / objectives and content are appropriate for (1) preparing students for the standardized AP exams for which results yield college credit and (2) generally broadening the mathematical sophistication of seniors as they complete their final year of high school and seek to continue their studies at the university level. The lectures adequately support students through the review of homework from prior lessons and then provide presentations of new ideas / topics; guided inquiry during the latter component seems well-placed, insomuch as a predominant proportion of the students present are engaged and able to contribute to the discussion as it unfolds. From what I observed, these advanced students may be in a position to resolve further and more-demanding challenges of the framework, which is a good outcome for the program. For instance, a discussion in the BC lectures featured a ground-up motivation for the definition of functional average over an interval for a Riemann-integrable function. Students arrived at the correct formula very efficiently after an experimental excursion into uniform sampling and a connection to discrete averages over finite sets. This made me wonder how students would resolve a challenge in which they were explicitly asked to comment on the necessity for uniformity of the sampling, in relation to departures that can occur when the latter is not at hand. This leads to natural questions such as "In what sense are averaging and integration analogous?" In turn, delving into such observations can lead to more generally useful perspectives on integration that transcend the single-variable framework - valuable intuition for subsequent studies in which "area under the curve" is no longer an adequate way to capture the meaning of the definite integral.

The courses that I teach indicate that college freshmen often treat calculus mechanistically, without regard to explanation, reasoning, and argumentation. A perfect example of this appears in the context of applied optimization problems, where many beginners will treat the zero derivative condition as a clear indication of optimality. For obvious reasons, this principle can fail. Without explicit knowledge of whether or not this is addressed in the training provided by the Scarsdale Math AT Program, one specific suggestion is to include parametric dependence in optimization problems and ask students to characterize parameter intervals for which optimality is attained at domain endpoints (as opposed to in the domain interior). Aside from stretching the students' abilities in abstract reasoning, problems like this provide special emphasis on the necessity to test candidates for optimality prior to drawing ultimate conclusions.

The suggestion offered above can be generalized considerably, so as to cover various other topics / Big Ideas covered in the courses. I would be very glad to discuss such ideas if this would be helpful to the colleagues at Scarsdale High School.