EXTRA CREDIT PROJECT - DEL FERRO AND CARDAN'S CUBICS

Instructions. Follow the steps below to derive del Ferro's formula for the solution of the depressed cubic $x^3 + px = q$, and then a formula for calculating the solution to a general cubic. Write your answers on a separate sheet of paper and be sure to show all your steps. Good luck!

Problem 1. The cubic solved by del Ferro has the general form $x^3 + px = q$, where p and q are non-negative.

(1) Assume that the solution can be written as the sum of two terms, namely, $x = u + v$. Substitute this into the depressed cubic, expand the terms and then factor to show the following equation:

$$u^3 + v^3 + (3uv + p)(u + v) = q$$

(2) Set $u^3 + v^3 = q$. Then explain why $3uv + p = 0$.

(3) Solve $3uv + p = 0$ for v in terms of p and u.

(4) Substitute your answer from (3) into the equation $u^3 + v^3 = q$ and show that

$$u^6 - qu^3 - \frac{p^3}{27} = 0$$

(5) Substitute $w = u^3$ in your answer to (4) to obtain a quadratic equation in w. Solve the quadratic equation, and substitute back to obtain:

$$u^3 = \frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$$

(6) Take the cube root of both sides and keep only the positive cube root. You should get:

$$u = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

(7) Since $u^3 + v^3 = q$, show that $v^3 = q - u^3$. Then substitute your answer in (6) for $u$and solve for v to get:

$$v = \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

(8) Finally, since $x = u + v$, show that the solution to the depressed cubic is

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

Problem 2. Use del Ferro's formula which you derived in Problem 1 to show that:

(1) $x^3 + 6x = 20$ has a solution at $x = 2$

(2) $x^3 = 6x + 20$ has a solution at $x = 3.4377073$

Problem 3. Cardan built on del Ferro's work to find a formula to find the roots of a general cubic. Follow the steps below to see how he did it!

(1) Consider $x^3 + a_1x^2 + a_2x + a_3 = 0$. Substitute $x = y - \frac{1}{3}a_1$ in to the formula. Expand and factor the terms using the powers y. You should get:

$$y^3 + \left(a_2 - \frac{1}{3}a_1^2\right) y = -\frac{2}{27}a_1^3 + \frac{1}{3}a_2a_1 - a_3$$

(2) Show that the answer in (1) is a depressed cubic, and identify the values of p and q.

Problem 4. Consider the cubic $x^3 - 15x^2 + 81x - 175 = 0$. Use Cardan's formula (and the steps you took in Problem 3) to show that the cubic has the solution $x = 7$.