Uniqueness of tangent flows in mean curvature flow

Michael Law (ANU)

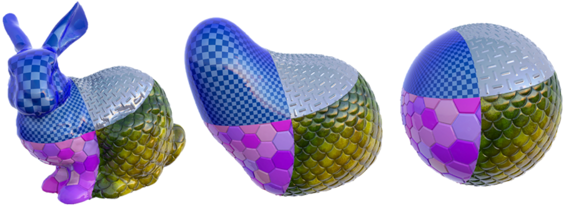
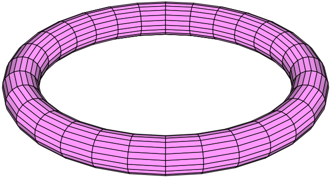
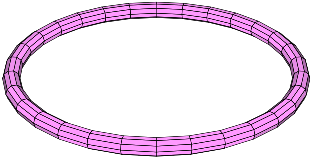
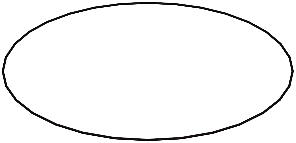


Image: https://www.cs.cmu.edu/~kmcrane/Projects/ConformalWillmoreFlow/.
This isn't actually mean curvature flow, but it looks cool.







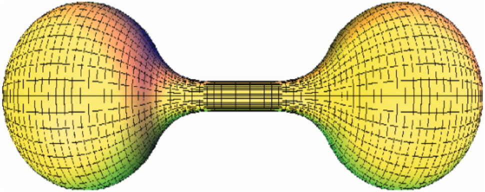
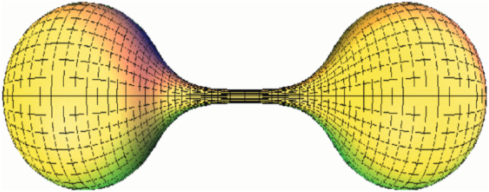


Image: [Colding–Minicozzi–Petersen, 2015]



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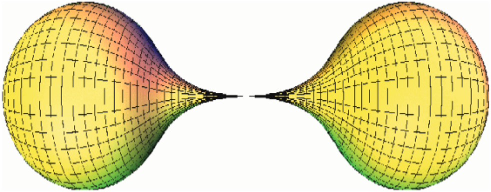

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A one-parameter family of hypersurfaces Σ_t evolves by **mean curvature flow** (MCF) if

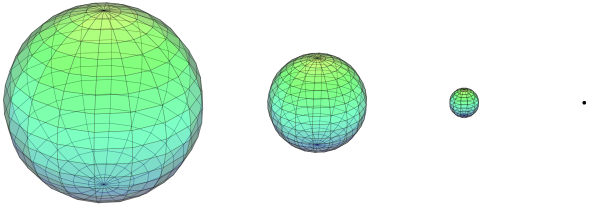
$$\left\langle \frac{\partial}{\partial t} x, \mathbf{n} \right\rangle = -H, \quad \forall x \in \Sigma_t,$$

where $H =$ mean curvature and $\mathbf{n} =$ outward unit normal to Σ_t.

Mean curvature flow

Example: the shrinking spheres

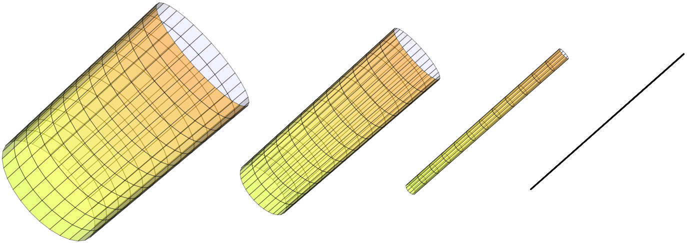
$$S^n_{\sqrt{1-2nt}}, \quad t \in \left[0, \frac{1}{2n}\right).$$



Mean curvature flow

Example: the shrinking cylinders

$$S^k_{\sqrt{1-2kt}} \times \mathbb{R}^{n-k}, \quad t \in \left[0, \frac{1}{2k}\right),$$



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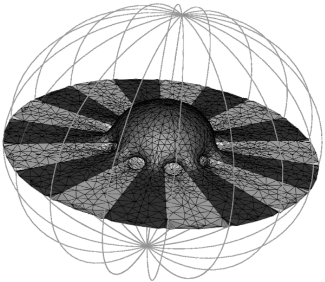

Image: [Ilmanen, 1995]

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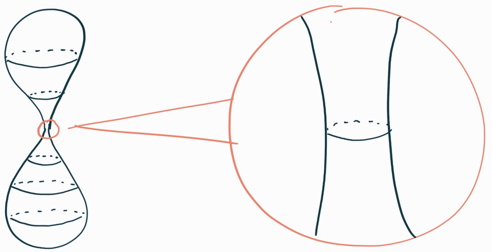
Zoom in during the flow to see what the singularity looks like.



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Convergence

Theorem. [Huisken, White, Ilmanen 1990–95]
For every sequence of times $s_i \to \infty$, there is a subsequence $s_{\varphi(i)}$ such that the RMCF $\widetilde{\Sigma}_{s_{\varphi(i)}}$ converges (in an appropriate sense) to a limiting hypersurface $\widetilde{\Sigma}_\infty$.

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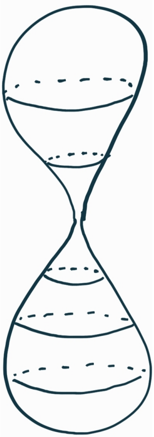
+ Tangent flows always exist.
+ Singularities are characterised as self-shrinkers.
- The tangent flow may depend on the sequence of times $s_i \to \infty$!

Uniqueness of tangent flows



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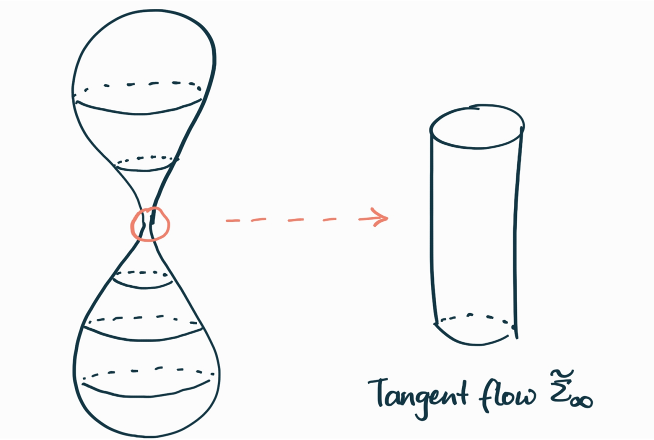

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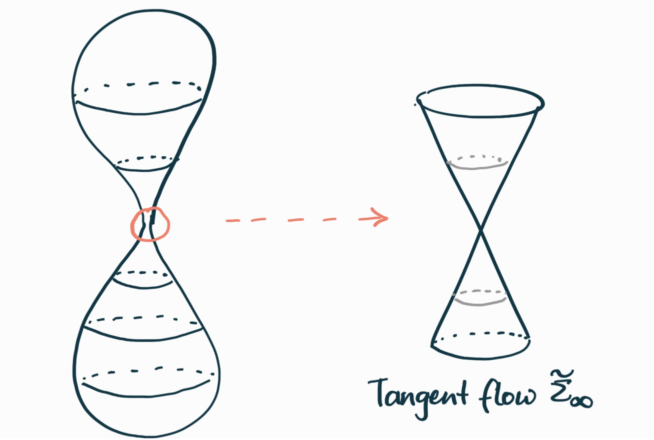

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Uniqueness of tangent flows holds for:

- Compact tangent flows; [Schulze 2014]
- Cylindrical tangent flows; [Colding–Minicozzi 2015]
- Asymptotically conical tangent flows. [Chodosh–Schulze 2021]

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These account for all singularities in MCF of a mean convex ($H \geq 0$) hypersurface. [Huisken 1990, Colding–Minicozzi 2012]

Łojasiewicz inequality in $\mathbb{R}^n$

Theorem. [Łojasiewicz 1959–65]
Let $f : \mathbb{R}^n \to \mathbb{R}$ be analytic, γ negative gradient flow line i.e.

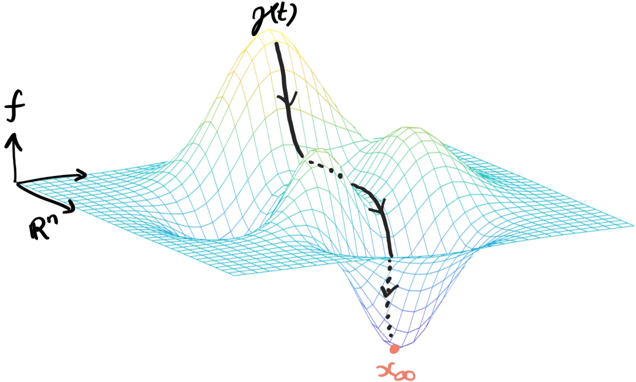
$$\gamma'(t) = -\nabla_{\gamma(t)} f.$$

If $\gamma(t_i) \to x_\infty$ along some $t_i \to \infty$, then

$$\lim_{t \to \infty} \gamma(t) = x_\infty.$$

Łojasiewicz inequality in $\mathbb{R}^n$

(f analytic, $\gamma(t_i) \to x_\infty$.)



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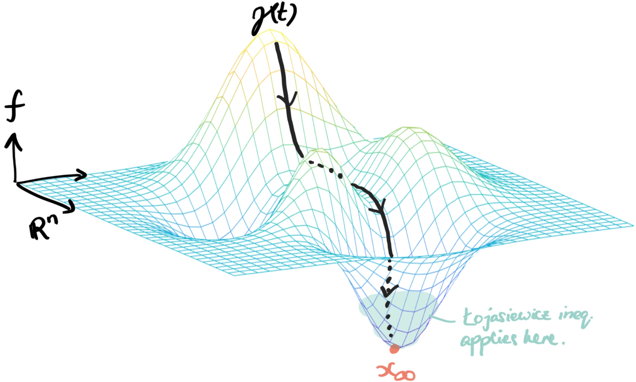

Theorem. (Gradient Łojasiewicz inequality) [Łojasiewicz 1959–65]
There exists $\beta \in (\frac{1}{2}, 1)$, $C > 0$ and a neighbourhood of x_∞ where

$$|f(x) - f(x_\infty)|^\beta \leq C|\nabla_x f|.$$

Łojasiewicz inequality in $\mathbb{R}^n$

Aside: we can't relax the requirement that f is analytic!

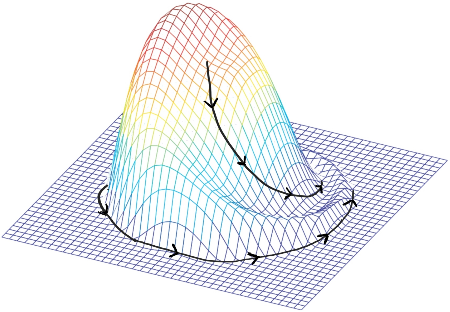

Image: [Absil–Mahony–Andrews, 2005]

A smooth function and negative gradient flow line which never converges.

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The (L^2) gradient of $\mathcal{F}$ at Σ is

$$\nabla_\Sigma \mathcal{F} = H - \frac{\langle x, \mathbf{n} \rangle}{2}.$$

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   $\Rightarrow$ tangent flows are accumulation points of the flow lines!

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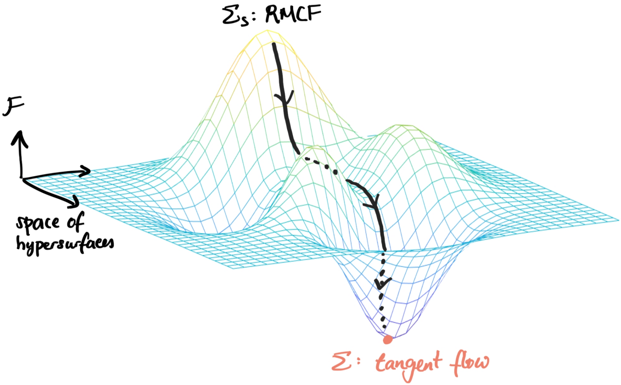

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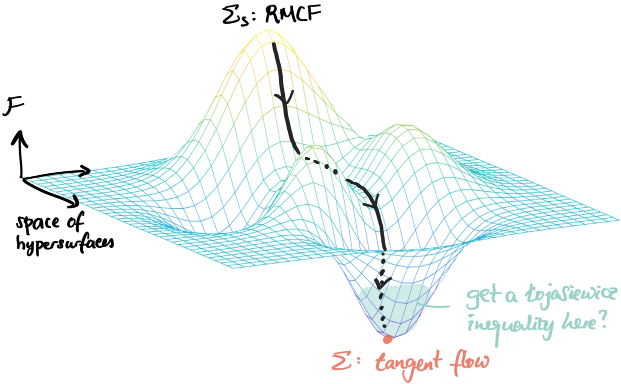

Aim: generalise the Łojasiewicz inequality to the space of hypersurfaces near Σ.

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Theorem. (Łojasiewicz–Simon inequality) [Simon, 1983]
If $\Sigma \subset \mathbb{R}^{n+1}$ is a compact tangent flow, and $\|u\|_{C^{2,\alpha}}$ is small, then

$$|\mathcal{F}(\Sigma_u) - \mathcal{F}(\Sigma)|^{\beta} \leq C\|\nabla_{\Sigma_u}\mathcal{F}\|_{L^2},$$

where $\beta \in (\frac{1}{2}, 1)$ and $C > 0$.

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where $\beta \in (\frac{1}{2}, 1)$ and $C > 0$.

Compare with the Łojasiewicz inequality in $\mathbb{R}^n$:

$$|f(x) - f(x_\infty)|^\beta \leq C|\nabla_x f|.$$

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- **Lemma:** Łojasiewicz–Simon gives an L^2 bound:

$$\sup_{s\in[\tau_1,\tau_2]} \|u(s) - u(\tau_1)\|_{L^2} \leq \underbrace{C(\mathcal{F}(\Sigma_{u(\tau_1)}) - \mathcal{F}(\Sigma))^{\theta}}_{\to 0}.$$

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- Iterate the **Lemma**s.

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we have $\Sigma_s \to \Sigma$ in L^2 as $s \to \infty$.

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(RMCF Σ_s, compact tangent flow Σ. Claim Σ is unique.)

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- If $s_i \to \infty$, then $\Sigma_{s_i} \to \Sigma$. $\square$

Back to Łojasiewicz–Simon

Łojasiewicz–Simon: Σ compact tangent flow, $\|u\|_{C^{2,\alpha}}$ small,

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(...not really.)

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- Adding these,

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$\square$

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If Σ is close to a cylinder Γ, then

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Theorem (real version). [Colding–Minicozzi, 2015]
Given n, there exist ε_0 and ℓ_0 with the following property. For all $\lambda_0 > 0$, $\varepsilon \leq \varepsilon_0$, $\ell \geq \ell_0$ and $K > 0$, there exists R_0 so that if $\Sigma \subset \mathbb{R}^{n+1}$ is a hypersurface with $\lambda(\Sigma) \leq \lambda_0$, $R \in [R_0, \mathbf{r}_{\varepsilon,\ell,K}(\Sigma)]$, and $\beta \in [0,1)$, then

$$|\mathcal{F}(\Sigma) - \mathcal{F}(\mathcal{C}_k)| \leq CR^\rho \left\{ \|\nabla_\Sigma \mathcal{F}\|_{L^2(B_R)}^{d_{\ell,n}\frac{3+\beta}{2+2\beta}} + e^{-\frac{d_{\ell,n}(3+\beta)R^2}{8(1+\beta)}} + e^{-\frac{(3+\beta)(R-1)^2}{16}} \right\},$$

where $C = C(n, \lambda_0, \varepsilon, \ell, K)$, $\rho = \rho(n)$ and $d_{\ell,n} \in (0,1) \nearrow 1$ as $\ell \to \infty$.