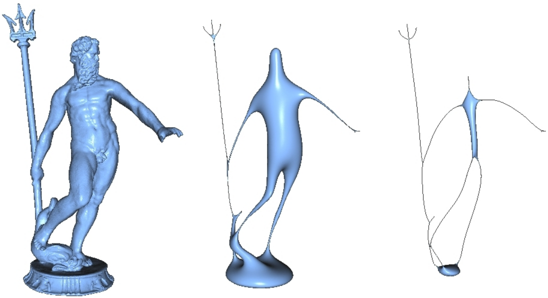
Singularities in the Mean Curvature Flow

Michael Law



Let M be a smooth n-dimensional manifold. A **mean curvature flow** (MCF) is a family of embeddings $\varphi_t : M \to \mathbb{R}^{n+1}$ satisfying

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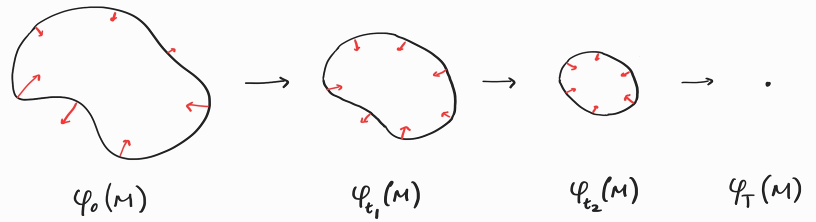


Convex regions move inward and concave regions move outward.

Given any initial condition $\varphi : M \to \mathbb{R}^{n+1}$, there exists a unique solution to MCF

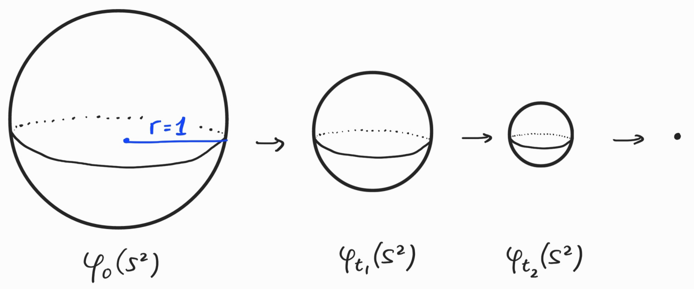
$$\frac{\partial}{\partial t}\varphi_t = -H\mathbf{n}, \quad \varphi_0 = \varphi,$$

defined for time $t \in [0, T)$.

(This relies on nontrivial results regarding solutions to PDEs.)

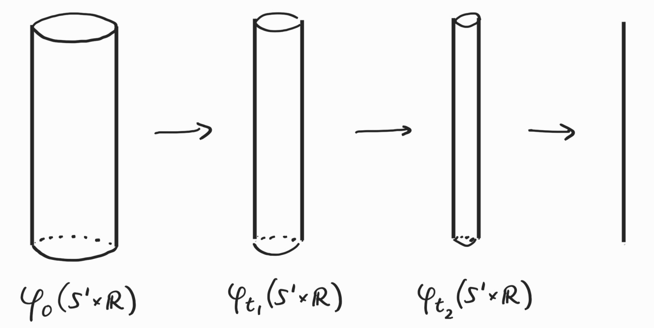
The shrinking sphere $\varphi_t : S^n \to \mathbb{R}^{n+1}$,

$$\varphi_t(\mathbf{x}) = \sqrt{1 - 2nt} \cdot \mathbf{x}, \quad t \in \left[0, \frac{1}{2n}\right).$$



The shrinking cylinder $\varphi_t : S^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n+1}$,

$$\varphi_t(\mathbf{x}, \mathbf{y}) = (\sqrt{1 - 2kt} \cdot \mathbf{x}, \mathbf{y}), \quad t \in \left[0, \frac{1}{2k}\right).$$



A dumbbell evolving under the mean curvature flow.

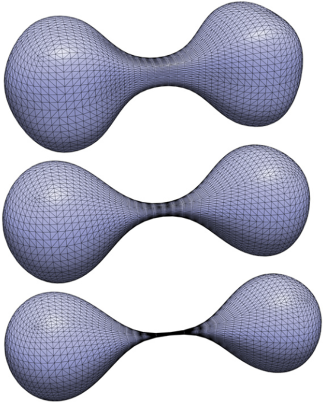
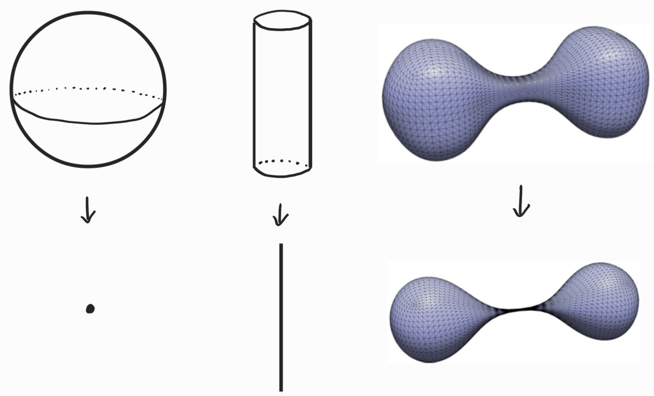


Image source: `http://geometry.cs.cmu.edu/ddgshortcourse/`

Singularities occur at the maximal time of existence, T.



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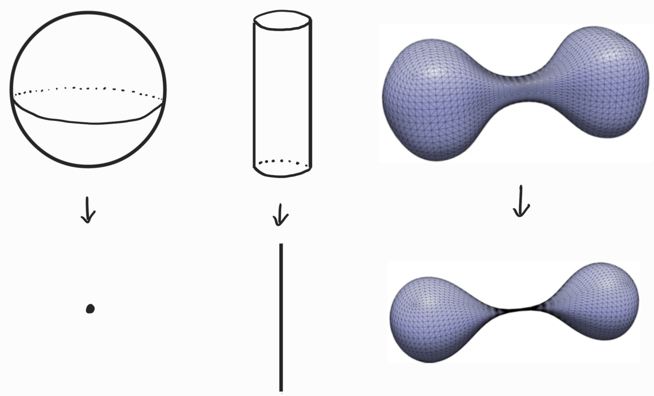

Different singularities look different. Can we classify them?

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If the singularity occurs at $\mathbf{x} \in \mathbb{R}^{n+1}$, consider

$$\widetilde{\varphi}_t : M \to \mathbb{R}^{n+1}, \quad \widetilde{\varphi}_t(q) = \frac{\varphi_t(q) - \mathbf{x}}{\sqrt{2(T-t)}}.$$

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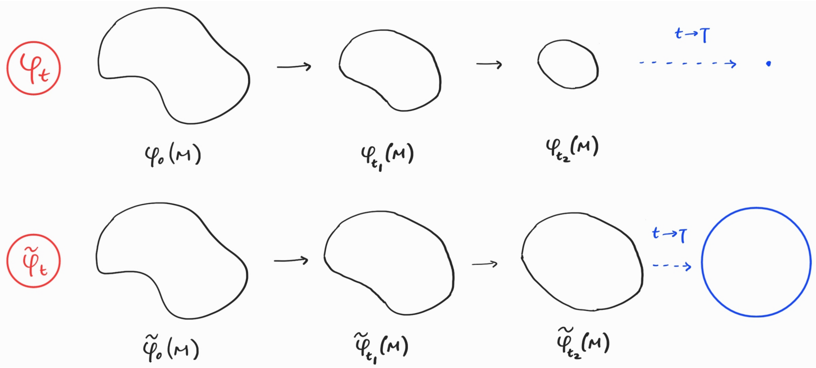

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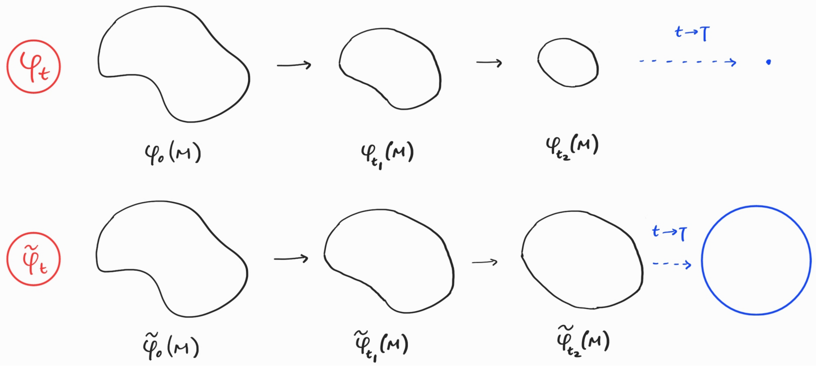

We classify singularities by the geometry of $\widetilde{\varphi}_t(M)$ as $t \to T$.

Theorem. [Huisken '90]
For every sequence of times $t_i \to T$, there is a subsequence of times such that the rescaled flow

$$\widetilde{\varphi}_{t_i}(q) = \frac{\varphi_{t_i}(q) - \mathbf{x}}{\sqrt{2(T - t_i)}}$$

converges (in an appropriate sense) to a limiting embedding $\widetilde{\varphi}_T : M \to \mathbb{R}^{n+1}$.

Lemma. For each ball $B_R(0) \subset \mathbb{R}^{n+1}$, a uniform 'local area bound' holds:[1]

$$\mathcal{H}^n(\widetilde{\varphi}_t(M) \cap B_R(0)) \leq Ce^{R^2/2}.$$

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Furthermore, the functions have equibounded derivatives $\Rightarrow$ Arzela-Ascoli.

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This limit better not depend on the subsequence chosen!

Otherwise, our classification of the singularity would depend on the choice of subsequence, defeating the purpose of having a classification.

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If one of the limits is a compact surface, then it is the unique limit.

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Invariably, these proofs employ so-called Łojasiewicz-type inequalities. (No time for this, unfortunately...)

There is still a lot to be discovered about the structure of singularities in mean curvature flow (and more generally the whole field itself!).