AMSI VACATION RESEARCH SCHOLARSHIPS 2020–21

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# Topological Phases in Quantum Systems with Quantum Group Symmetries


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**Supervised by Dr. Thomas Quella**

**University of Melbourne**



*Vacation Research Scholarships are funded jointly by the Department of Education, Skills and Employment and the Australian Mathematical Sciences Institute.*

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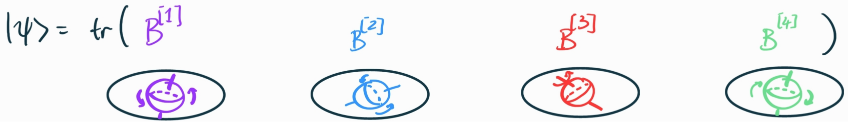
AMSI CONNECT

# Matrix Product States

- We analyse quantum systems on a 1D lattice.
- Consider states $|\psi\rangle$ of the form

$$|\psi\rangle = \mathrm{tr}(\mathcal{B}^{[1]}\mathcal{B}^{[2]} \ldots \mathcal{B}^{[L]})$$

where each $\mathcal{B}^{[i]}$ is a matrix with vector entries (rank-3 tensor) associated to site $i$.



# Matrix Product States

*Example:* **spin-1 AKLT ground state** [Affleck, Kennedy et al. '87]

$$\mathcal{B}^{[i]} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\left|0\right\rangle & \sqrt{2}\left|1\right\rangle \\ -\sqrt{2}\left|-1\right\rangle & \left|0\right\rangle \end{bmatrix} \qquad i = 1, 2, \ldots, L.$$

where $\left|1\right\rangle, \left|0\right\rangle, \left|-1\right\rangle \in V$, the one-site state space.

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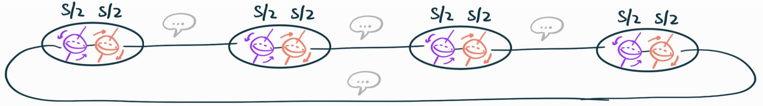
where $\left|1\right\rangle, \left|0\right\rangle, \left|-1\right\rangle \in V$, the one-site state space.

$$\begin{aligned} \left|\psi_{AKLT}\right\rangle &= \mathrm{tr}(\mathcal{B}^{[1]}\mathcal{B}^{[2]}\ldots\mathcal{B}^{[L]}) \\ &\overset{(L=2)}{=} \frac{2}{3}\big(\underbrace{\left|0\right\rangle \otimes \left|0\right\rangle} - \underbrace{\left|1\right\rangle \otimes \left|-1\right\rangle}_{\text{entanglement}} + \underbrace{\left|-1\right\rangle \otimes \left|1\right\rangle}\big). \end{aligned}$$

# Generalised AKLT states

- The underlying construction: [Affleck, Kennedy et al. '87]




- Uses representation theory to assert $\mathfrak{su}(2)$ (rotational) symmetry:

$$g\,|\psi_{AKLT}\rangle = |\psi_{AKLT}\rangle \quad \forall g \in \mathfrak{su}(2).$$

- Choice of dim $V \Rightarrow$ different AKLT states of 'spin' $S \in \mathbb{Z}^+$.

# Generalised AKLT states

$$S = 1: \quad \mathcal{B}_{(S=1)} = \frac{1}{\sqrt{3}} \begin{bmatrix} -|0\rangle & \sqrt{2}|1\rangle \\ -\sqrt{2}|-1\rangle & |0\rangle \end{bmatrix}$$

$$S = 2: \quad \mathcal{B}_{(S=2)} = \frac{1}{\sqrt{10}} \begin{bmatrix} |0\rangle & -\sqrt{3}|1\rangle & \sqrt{6}|2\rangle \\ \sqrt{3}|-1\rangle & -2|0\rangle & \sqrt{3}|1\rangle \\ \sqrt{6}|-2\rangle & -\sqrt{3}|-1\rangle & |0\rangle \end{bmatrix}$$

$$\vdots$$

$$\text{General } S: \quad \mathcal{B}_S = (\text{some } (S+1)\times(S+1) \text{ matrix})$$

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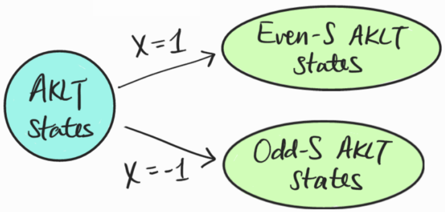
Are all these states ‘the same’ in some physical sense?

If not, how can we classify them?

# Topological phases in AKLT states




[Pollman, Berg et al. '09], [Wierschem, Beach '16]

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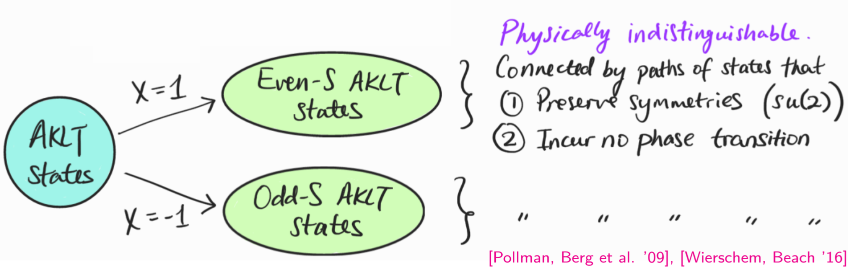


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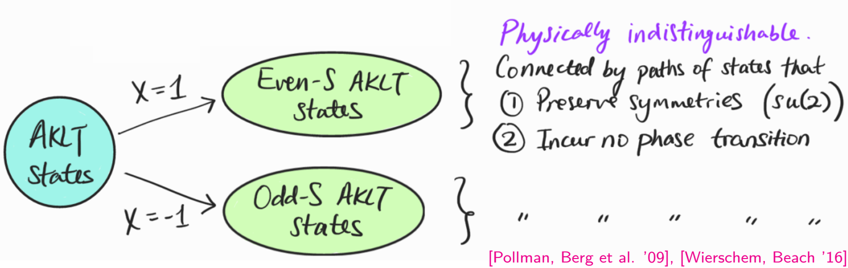


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- Use paths of tensors $\mathcal{B}_t, 0 \leq t \leq 1$.
- No phase transition $\Leftrightarrow$ dominant eigenvalue of the **transfer matrix** $\sum_\sigma \mathcal{B}_t^\sigma \otimes \overline{\mathcal{B}_t}^\sigma$ is unique ∀t.

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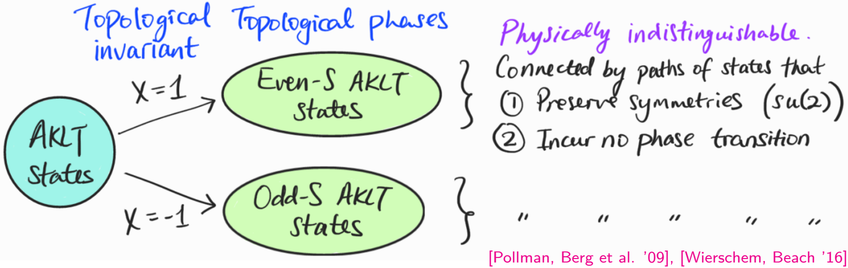


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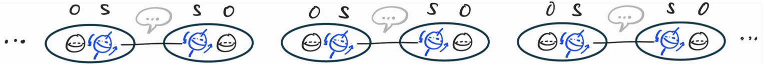
# Paths between even-$S$ AKLT states

[Pollman, Berg et al. '09]

Connect every even-$S$ AKLT state to a **trivial state**:



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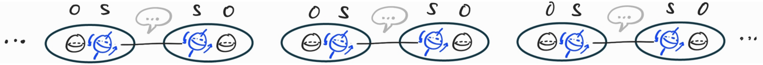


Path from the $S = 2$ AKLT state to a trivial state:

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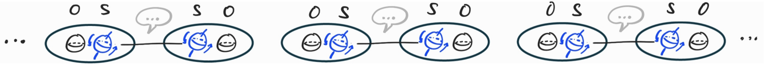


Path from the $S = 2$ AKLT state to a trivial state:

$$\mathcal{B}_t = \left[\begin{array}{c|c} (1-t)A & t(1-t)B \\ \hline 0 & tC \end{array}\right] \left[\begin{array}{c|c} (1-t)A & 0 \\ \hline t(1-t)D & tE \end{array}\right]$$

where $A \cdots E$ are derived using rep. theory of $\mathfrak{su}(2)$.

# Paths between even-$S$ AKLT states

[Pollman, Berg et al. '09]

$$A = \frac{1}{\sqrt{10}} \begin{bmatrix} |0\rangle & -\sqrt{3}|1\rangle & \sqrt{6}|2\rangle \\ \sqrt{3}|-1\rangle & -2|0\rangle & \sqrt{3}|1\rangle \\ \sqrt{6}|-2\rangle & -\sqrt{3}|-1\rangle & |0\rangle \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{1}{\sqrt{3}}|-1\rangle & \frac{1}{\sqrt{2}}|0\rangle & -\frac{1}{\sqrt{2}}|1\rangle & \frac{1}{\sqrt{3}}|2\rangle & 0 \\ -\sqrt{\frac{2}{3}}|-2\rangle & \frac{1}{\sqrt{6}}|-1\rangle & 0 & -\frac{1}{\sqrt{6}}|1\rangle & \sqrt{\frac{2}{3}}|2\rangle \\ 0 & -\frac{1}{\sqrt{3}}|-2\rangle & \frac{1}{\sqrt{2}}|-1\rangle & -\frac{1}{\sqrt{2}}|0\rangle & \frac{1}{\sqrt{3}}|1\rangle \end{bmatrix}$$

$$C = \frac{1}{\sqrt{5}} \begin{bmatrix} |-2\rangle \\ -|-1\rangle \\ |0\rangle \\ -|1\rangle \\ |2\rangle \end{bmatrix}^T \qquad D = \begin{bmatrix} \frac{1}{\sqrt{3}}|1\rangle & -\sqrt{\frac{2}{3}}|2\rangle & 0 \\ \frac{1}{\sqrt{2}}|0\rangle & -\frac{1}{\sqrt{6}}|1\rangle & -\frac{1}{\sqrt{3}}|2\rangle \\ \frac{1}{\sqrt{2}}|-1\rangle & 0 & -\frac{1}{\sqrt{2}}|1\rangle \\ \frac{1}{\sqrt{3}}|-2\rangle & \frac{1}{\sqrt{6}}|-1\rangle & -\frac{1}{\sqrt{2}}|0\rangle \\ 0 & \sqrt{\frac{2}{3}}|-2\rangle & -\frac{1}{\sqrt{3}}|-1\rangle \end{bmatrix} \qquad E = \begin{bmatrix} |2\rangle \\ |1\rangle \\ |0\rangle \\ |-1\rangle \\ |-2\rangle \end{bmatrix}$$

# Paths between even-$S$ AKLT states

[Pollman, Berg et al. '09]

The path $\{\mathcal{B}_t\}_{t\in[0,1]}$ must satisfy the following:

1. $\mathfrak{su}(2)$ symmetry $\forall$t
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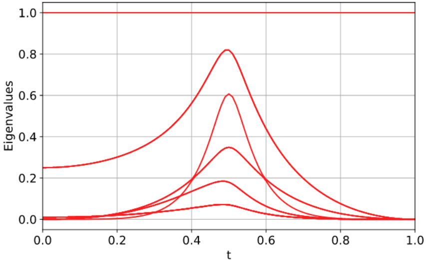


Similarly done for other even values of $S$.

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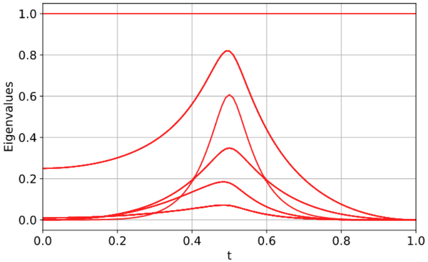


Can we prove this is true for all even $S$?

# What it looks like for $S = 2$

```
transfer = ConstantArray[0, {16, 16}];
For[i = 1, i ≤ 25, i++,
  kron = KroneckerProduct[MPS$MN[[i]], MPS$MN[[i]]]; kron = Simplify[kron]; transfer = transfer + kron;];
transfer = Simplify[transfer]; MatrixForm[transfer]
```

Out[37]//MatrixForm=

| | | | | | |
|---|---|---|---|---|---|
| $\frac{1}{450}(-1+t)^2(207-360t+685t^2-650t^3+325t^4)$ | $0$ | $0$ | $0$ | $0$ | $\frac{1}{900}(-1+t)^2(2$ |
| $0$ | $\frac{1}{900}(-1+t)^2(117-810t+1135t^2-650t^3+325t^4)$ | $0$ | $-\frac{(-1+t)^2 t(9-5t+5t^2)}{6\sqrt{6}}$ | $0$ | |
| $0$ | $0$ | $\frac{1}{300}(-1+t)^2(3-90t+265t^2-350t^3+175t^4)$ | $0$ | $0$ | |
| $0$ | $\frac{(-1+t)^2 t(9-5t+5t^2)}{6\sqrt{30}}$ | $0$ | $\frac{t(3-8t+10t^2-5t^3)}{3\sqrt{5}}$ | $0$ | |
| $0$ | $0$ | $0$ | $0$ | $\frac{1}{900}(-1+t)^2(117-810t+1135t^2-650t^3+325t^4)$ | |
| $\frac{1}{900}(-1+t)^2(297+90t+235t^2-650t^3+325t^4)$ | $0$ | $0$ | $0$ | $0$ | $\frac{1}{450}(-1+t)^2(1$ |
| $0$ | $-\frac{1}{225}(-1+t)^2(-27+135t-85t^2-100t^3+50t^4)$ | $0$ | $-\frac{(-1+t)^2 t(9-5t+5t^2)}{6\sqrt{6}}$ | $0$ | |
| $-\frac{(-1+t)^2 t(9-5t+5t^2)}{6\sqrt{30}}$ | $0$ | $0$ | $0$ | $0$ | |
| $0$ | $0$ | $0$ | $0$ | $0$ | |
| $0$ | $0$ | $0$ | $0$ | $-\frac{1}{225}(-1+t)^2(-27+135t-85t^2-100t^3+50t^4)$ | |
| $\frac{7}{300}(3+2t-10t^2+5t^3)^2$ | $0$ | $0$ | $0$ | $0$ | $\frac{1}{900}(-1+t)^2(2$ |
| $0$ | $0$ | $0$ | $0$ | $-\frac{(-1+t)^2 t(9-5t+5t^2)}{6\sqrt{30}}$ | |
| $0$ | $0$ | $0$ | $0$ | $\frac{(-1+t)^2 t(9-5t+5t^2)}{6\sqrt{30}}$ | |
| $-\frac{(-1+t)^2 t(9-5t+5t^2)}{6\sqrt{30}}$ | $0$ | $0$ | $0$ | $0$ | |
| $0$ | $-\frac{(-1+t)^2 t(9-5t+5t^2)}{6\sqrt{30}}$ | $0$ | $\frac{1}{3}\sqrt{5}(-1+t)^2 t^2$ | $0$ | |
| $\frac{1}{3}(-1+t)^2 t^2$ | $0$ | $0$ | $0$ | $0$ | |

# What it looks like for $S = 2$

```
In[40]:= eigs = Simplify[Eigenvalues[transfer]]
```

Out[40]=

$$\left\{-\frac{(-1+t)\,t}{\sqrt{5}},\ -\frac{(-1+t)\,t}{\sqrt{5}},\ -\frac{(-1+t)\,t}{\sqrt{5}},\ \frac{1}{300}(-1+t)^2\left(3-90t+265t^2-350t^3+175t^4\right),\ \frac{1}{300}(-1+t)^2\left(3-90t+265t^2-350t^3+175t^4\right),\right.$$

$$\frac{1}{300}(-1+t)^2\left(3-90t+265t^2-350t^3+175t^4\right),\ \frac{1}{300}(-1+t)^2\left(3-90t+265t^2-350t^3+175t^4\right),$$

$$\frac{1}{300}(-1+t)^2\left(3-90t+265t^2-350t^3+175t^4\right),\ \frac{1}{6}\left(3-6t+11t^2-20t^3+30t^4-20t^5+5t^6-\sqrt{9-36t+66t^2-180t^3+505t^4-920t^5+1330t^6-1700t^7+1810t^8-1400t^9+700t^{10}-200t^{11}+25t^{12}}\right),$$

$$\frac{1}{6}\left(3-6t+11t^2-20t^3+30t^4-20t^5+5t^6+\sqrt{9-36t+66t^2-180t^3+505t^4-920t^5+1330t^6-1700t^7+1810t^8-1400t^9+700t^{10}-200t^{11}+25t^{12}}\right),$$

$$\frac{1}{360}\left(45+36\left(-10+\sqrt{5}\right)t+\left(880-156\sqrt{5}\right)t^2+10\left(-91+24\sqrt{5}\right)t^3-60\left(-7+2\sqrt{5}\right)t^4-100t^5+25t^6-\sqrt{5}\sqrt{(-1+t)^2\left(405-162\left(35+4\sqrt{5}\right)t-9\left(-3479+120\sqrt{5}\right)t^2-72\left(1195+17\sqrt{5}\right)t^3+\left(131705+11112\sqrt{5}\right)t^4-30\left(3889+308\sqrt{5}\right)t^5+\left(58305-2520\sqrt{5}\right)t^6+1600\left(-10+3\sqrt{5}\right)t^7-75\left(-61+16\sqrt{5}\right)t^8-750t^9+125t^{10}\right)}\right),$$

$$\frac{1}{360}\left(45+36\left(-10+\sqrt{5}\right)t+\left(880-156\sqrt{5}\right)t^2+10\left(-91+24\sqrt{5}\right)t^3-60\left(-7+2\sqrt{5}\right)t^4-100t^5+25t^6-\sqrt{5}\sqrt{(-1+t)^2\left(405-162\left(35+4\sqrt{5}\right)t-9\left(-3479+120\sqrt{5}\right)t^2-72\left(1195+17\sqrt{5}\right)t^3+\left(131705+11112\sqrt{5}\right)t^4-30\left(3889+308\sqrt{5}\right)t^5+\left(58305-2520\sqrt{5}\right)t^6+1600\left(-10+3\sqrt{5}\right)t^7-75\left(-61+16\sqrt{5}\right)t^8-750t^9+125t^{10}\right)}\right),$$

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# Some semblance of success

Want exact forms of e-values of all transfer matrices

$$E_t := \sum_{\sigma} \mathcal{B}_t^{\sigma} \otimes \overline{\mathcal{B}_t}^{\sigma}, \quad t \in [0, 1].$$

1. Recognise that TM can be block-diagonalised by permuting rows/columns
2. Appeal to rep. theory of $\mathfrak{su}(2)$ and linear algebra to tame the problem (Schur's Lemma, C-G coefficients, Perron-Frobenius theorem...)

$\{$eigenvalues of TM$\}$ = $\{$eigenvalues of largest block$\}$!

# $q$-deformed AKLT states

- By modifying one of the relations defining $\mathfrak{su}(2)$,

$$\mathbb{S}^+\mathbb{S}^- - \mathbb{S}^-\mathbb{S}^+ = [2\mathbb{S}^z]_q = \frac{q^{2\mathbb{S}^z} - q^{-2\mathbb{S}^z}}{q - q^{-1}}$$

  we obtain the **quantum group** $\mathcal{U}_q[sl(2)]$.
- Construct the AKLT states exactly the same way.
- Do these $q$-deformed AKLT states also admit a binary classification?
- Numerical evidence: yes, at least for $q \in \mathbb{R}^+$.

# Summary

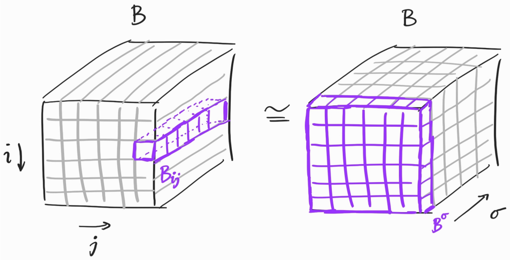
- Despite being constructed identically, the AKLT states exhibit different physical properties for different $S$.
- Existing numerical evidence supports a classification of AKLT states into 2 distinct phases of matter, depending on the parity of $S$.
- Our analytic results give way to a formal proof.
- In related news, our numerical evidence supports the same classification of $q$AKLT states.

# More on the transfer matrix

The **transfer matrix** is defined by $E = \sum_\sigma \mathcal{B}^\sigma \otimes \overline{\mathcal{B}}^\sigma$.

- View the object $\mathcal{B}$ as a tuple of (scalar) matrices

$$\mathcal{B} := (\mathcal{B}^{-S}, \mathcal{B}^{-S+1}, \dots, \mathcal{B}^{S}).$$



# More on the transfer matrix

- The symbol $\otimes$ denotes the Kronecker product

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}.$$

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