Multiplication of Large Integers

Michael Angel

6 May 2019

Overview

Using Fourier Transforms to multiply integers

- Theory
- Speed
- Code
- Demo

How?

Consider integers as polynomials of their radix/base

$$\begin{aligned}\mathbf{4092} &= \mathbf{2}(10)^0 + \mathbf{9}(10)^1 + \mathbf{0}(10)^2 + \mathbf{4}(10)^3 \\ \mathbf{373} &= \mathbf{3}(10)^0 + \mathbf{7}(10)^1 + \mathbf{3}(10)^2 + \mathbf{0}(10)^3\end{aligned}$$

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Polynomial multiplication is the convolution of coefficients

$a = ..0, 0, 2, 9, 0, 4, 0, 0..$
$b = ..0, 0, 3, 7, 3, 0, 0, 0, ..$
$c = a * b$

$$c_i = (a * b)_i = \sum_{k=-\infty}^{\infty} a_k b_{i-k}$$

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$$c_i = \sum_{j=0}^{n-1} a_j b_{i-j}$$

$$c_0 = a_0 b_0 + a_1 b_{-1} + a_2 b_{-2} + a_3 b_{-3} = 2 \cdot 3 + 9 \cdot 0 + 0 \cdot 0 + 4 \cdot 0 = 6$$

$$c_1 = a_0 b_1 + a_1 b_0 + a_2 b_{-1} + a_3 b_{-2} = 2 \cdot 7 + 9 \cdot 3 + 0 \cdot 0 + 4 \cdot 0 = 41$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0 + a_3 b_{-1} = 2 \cdot 3 + 9 \cdot 7 + 0 \cdot 3 + 4 \cdot 0 = 69$$

$$c_3 = \ldots$$

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Polynomial multiplication is the convolution of coefficients

$a = ..0, 0, 2, 9, 0, 4, 0, 0..$

$b = ..0, 0, 3, 7, 3, 0, 0, 0, ..$

$c = a * b = ..0, 0, 6, 41, 69, 39, 28, 12, 0, 0, ..$

$c = 6(10)^0+41(10)^1+69(10)^2+39(10)^3+28(10)^4+12(10)^5+0(10)^6$

carry the 10's

$c = 6(10)^0 + 1(10)^1 + 3(10)^2 + 6(10)^3 + 2(10)^4 + 5(10)^5 + 1(10)^6$

$\mathbf{4092} \cdot \mathbf{373} = 1526316$

The Schönhage-Strassen Algorithm

By Arnold Schönhage and Volker Strassen in 1971.

Summary

- $a = \mathcal{F}(a), b = \mathcal{F}(b)$
- $c = a \cdot b$
- $c = \mathcal{F}^{-1}(c)$
- carry(c), make every coefficient less than 10

The Schönhage-Strassen Algorithm

$\mathcal{O}(nlog(n))$ complex multiplications (Fast Fourier Transforms).
Extra time needed to compute roots of unity.

$\mathcal{O}(nlog(n) \cdot log(log(n)))$ total running time

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18 March, 2019 (seven weeks ago) David Harvey and Joris Van Der Hoeven published $\mathcal{O}(nlog(n))$ algorithm. [HVD]

Naïve Algorithm

```
   1234
 x 1234
     16
    120
    800
   4000
    120
    900
   6000
  30000
    800
   6000
  40000
 200000
   4000
  30000
 200000
1000000
```

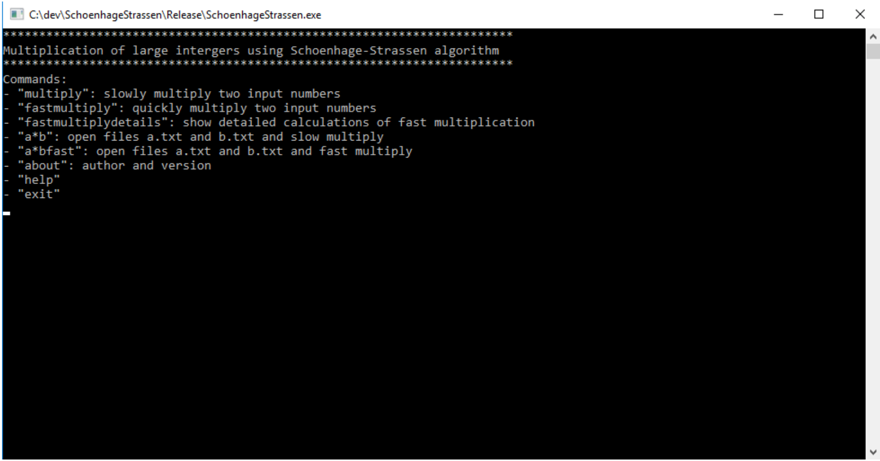
For each digit, multiply for each digit, then add.

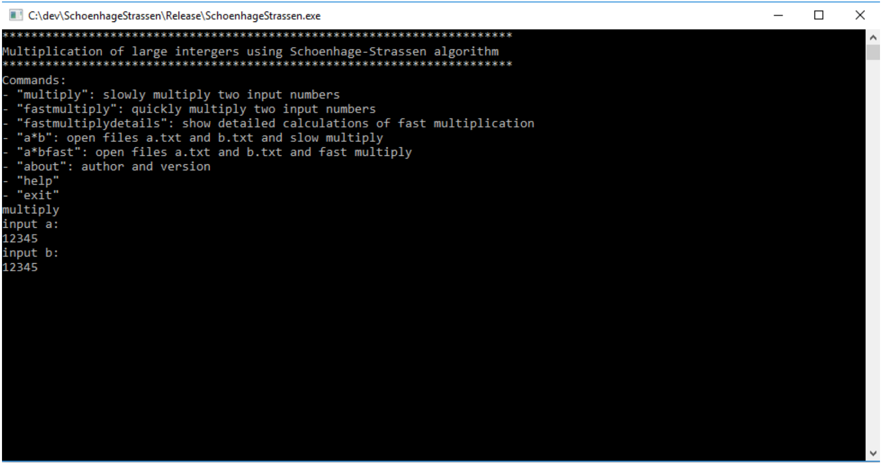
$\mathcal{O}(n^2)$ running time

```cpp
bigInt operator * (const bigInt &v) const {
    bigInt product;
    for (long i = 0; i < a.size(); ++i) {
        for (long j = 0; j < v.a.size(); ++j) {
            vector<int> p(i + j, 0);
            int h = a[i] * v.a[j];
            while (h) {
                p.insert(p.begin(), h % 10);
                h = int(h / 10);
            }

            for (auto x : p) cout << x;
            cout << endl;

            product = product + bigInt(p);
        }
    }
    return product;
}
```

```
valarray<int> multiply(PolynomialCoefficients a, PolynomialCoefficients b, bool show_details) {
    PolynomialCoefficients c;
    FFT(a);
    FFT(b);
    c = a * b;
    iFFT(c);

    valarray<int> res(a.size());
    for (long i = 0; i < a.size(); i++) {
        res[i] = round(real(c[i]));
    }

    for (long i = a.size() - 1; i > 0; i--) {
        res[i] += res[i-1] % 10;
        res[i - 1] = int(res[i - 1] / 10);
        if (res[i] > 9) res[i - 1] += int(res[i] / 10);
        res[i] %= 10;
    }

    return res;
```

```cpp
const double PI = 3.141592653589793238460;

void FFT(PolynomialCoefficients& a) {
    // Cooley-Tukey algorithm: radix-2 decimation-in-time Fast Fourier Transform
    const size_t N = a.size();
    if (N <= 1) return;

    PolynomialCoefficients even = a[slice(0, N / 2, 2)];
    PolynomialCoefficients odd = a[slice(1, N / 2, 2)];

    FFT(even);
    FFT(odd);

    for (size_t k = 0; k < N/2; ++k)
    {
        Complex t = polar(1.0, -2*PI*k/N) * odd[k];
        // polar(rho, theta) = rho(cos(theta) + isin(theta)) = rho*e^(i*theta)
        // polar(1.0, -2*PI*k/N) = e^i(-2*PI*k/N)
        a[k] = even[k] + t;
        a[k + N / 2] = even[k] - t;
    }
}
```


C:\dev\SchoenhageStrassen\Release\SchoenhageStrassen.exe
****
Multiplication of large intergers using Schoenhage-Strassen algorithm
****
Commands:
- "multiply": slowly multiply two input numbers
- "fastmultiply": quickly multiply two input numbers
- "fastmultiplydetails": show detailed calculations of fast multiplication
- "a*b": open files a.txt and b.txt and slow multiply
- "a*bfast": open files a.txt and b.txt and fast multiply
- "about": author and version
- "help"
- "exit"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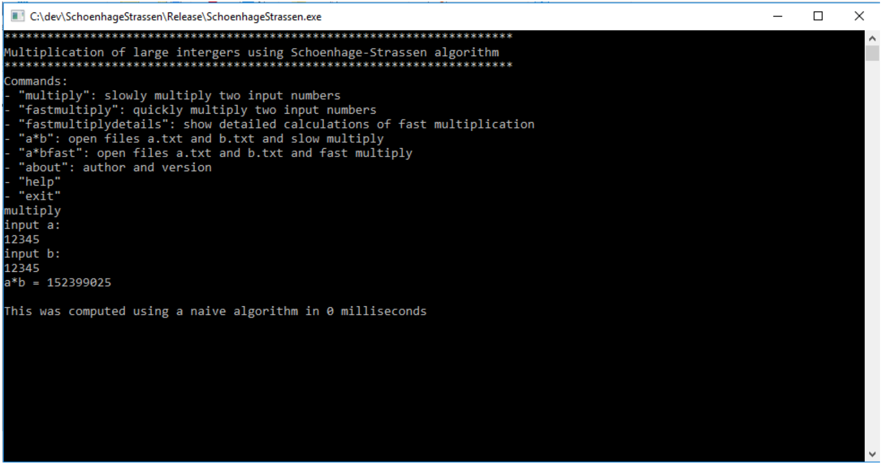
multiply
input a:
12345
input b:
12345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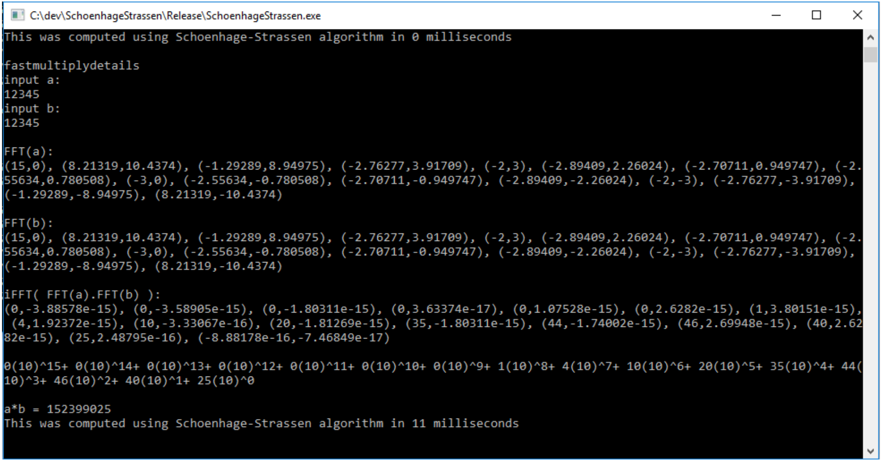
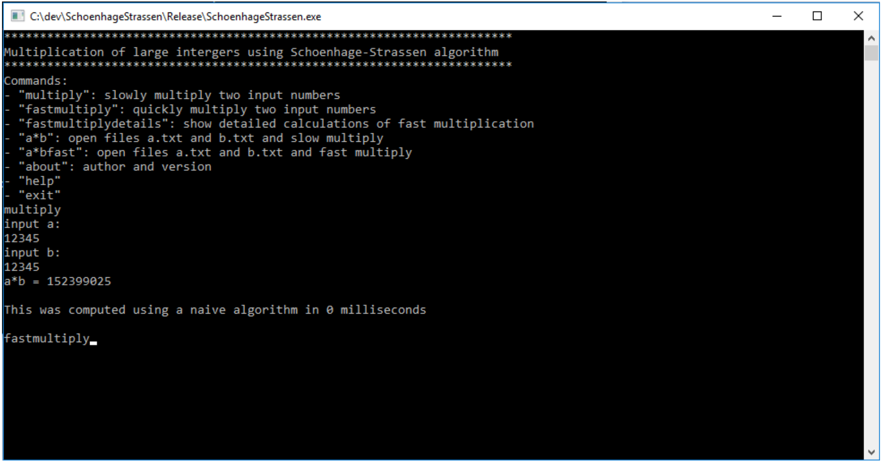
multiply
input a:
12345
input b:
12345
a*b = 152399025
This was computed using a naive algorithm in 0 milliseconds


```
C:\dev\SchoenhageStrassen\Release\SchoenhageStrassen.exe
This was computed using Schoenhage-Strassen algorithm in 0 milliseconds

fastmultiplydetails
input a:
12345
input b:
12345

FFT(a):
(15,0), (8.21319,10.4374), (-1.29289,8.94975), (-2.76277,3.91709), (-2,3), (-2.89409,2.26024), (-2.70711,0.949747), (-2.
55634,0.780508), (-3,0), (-2.55634,-0.780508), (-2.70711,-0.949747), (-2.89409,-2.26024), (-2,-3), (-2.76277,-3.91709),
(-1.29289,-8.94975), (8.21319,-10.4374)

FFT(b):
(15,0), (8.21319,10.4374), (-1.29289,8.94975), (-2.76277,3.91709), (-2,3), (-2.89409,2.26024), (-2.70711,0.949747), (-2.
55634,0.780508), (-3,0), (-2.55634,-0.780508), (-2.70711,-0.949747), (-2.89409,-2.26024), (-2,-3), (-2.76277,-3.91709),
(-1.29289,-8.94975), (8.21319,-10.4374)

iFFT( FFT(a).FFT(b) ):
(0,-3.88578e-15), (0,-3.58905e-15), (0,-1.80311e-15), (0,3.63374e-17), (0,1.07528e-15), (0,2.6282e-15), (1,3.80151e-15),
 (4,1.92372e-15), (10,-3.33067e-16), (20,-1.81269e-15), (35,-1.80311e-15), (44,-1.74002e-15), (46,2.69948e-15), (40,2.62
82e-15), (25,2.48795e-16), (-8.88178e-16,-7.46849e-17)

0(10)^15+ 0(10)^14+ 0(10)^13+ 0(10)^12+ 0(10)^11+ 0(10)^10+ 0(10)^9+ 1(10)^8+ 4(10)^7+ 10(10)^6+ 20(10)^5+ 35(10)^4+ 44(
10)^3+ 46(10)^2+ 40(10)^1+ 25(10)^0

a*b = 152399025
This was computed using Schoenhage-Strassen algorithm in 11 milliseconds
```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fastmultiply

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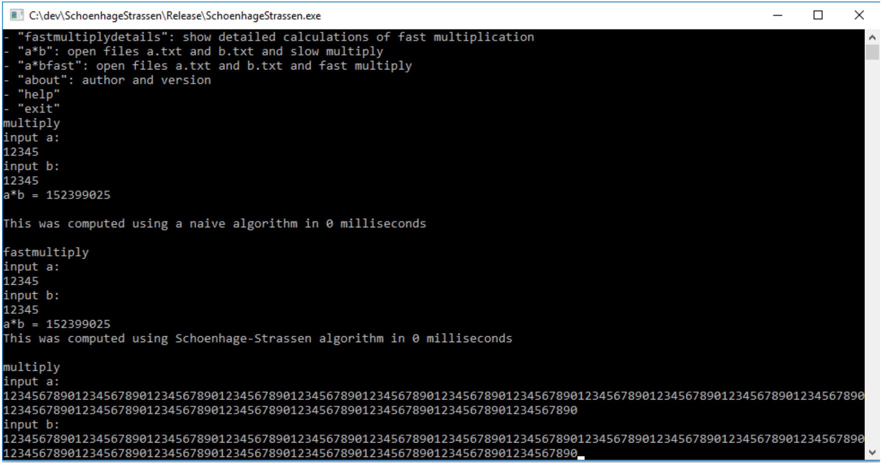
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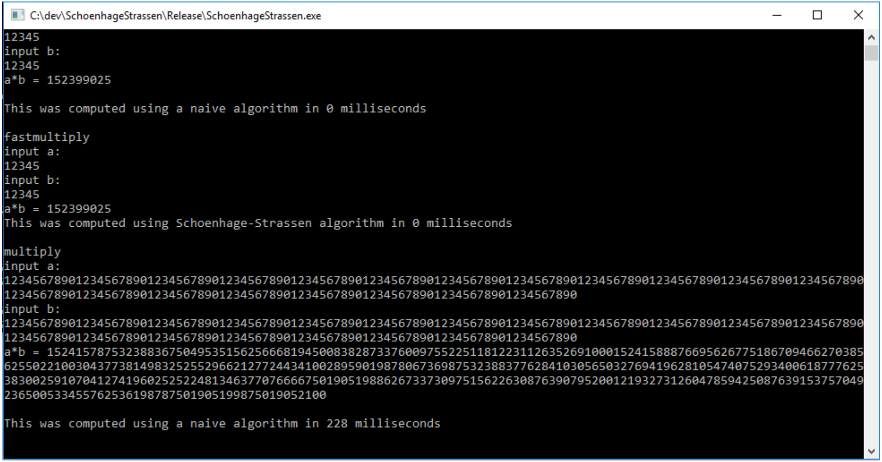
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890
input b:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890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12345678901234567890123456789012345678901234567890123456789012345678901234567890
input b:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890
a*b = 152415787532388367504953515625666819450083828733760097552251181223112635269100015241588876695626775186709466270385
625502210030437738149832525529662127724434100289590198780673698753238837762841030565032769419628105474075293400618777625
383002591070412741960252522481346377076666750190519886267337309751562263087639079520012193273126047859425087639153757049
236500533455762536198787501905199875019052100
This was computed using a naive algorithm in 228 milliseconds

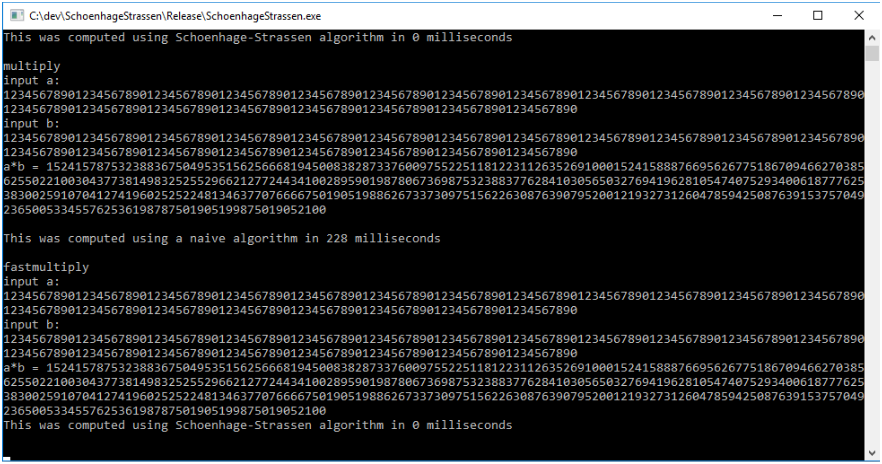
C:\dev\SchoenhageStrassen\Release\SchoenhageStrassen.exe

```
fastmultiply
input a:
12345
input b:
12345
a*b = 152399025
This was computed using Schoenhage-Strassen algorithm in 0 milliseconds

multiply
input a:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890
input b:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890
a*b = 152415787532388367504953515625666819450083828733760097552251181223112635269100015241588876695626775186709466270385
625502210030437738149832525529662127724434100289590198780673698753238837762841030565032769419628105474075293400618777625
383002591070412741960252522481346377076666750190519886267337309751562263087639079520012193273126047859425087639153757049
236500533455762536198787501905199875019052100

This was computed using a naive algorithm in 228 milliseconds

fastmultiply
input a:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890
input b:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890
```


C:\dev\SchoenhageStrassen\Release\SchoenhageStrassen.exe
This was computed using Schoenhage-Strassen algorithm in 0 milliseconds
multiply
input a:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890
input b:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
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383002591070412741960252522481346377076666750190519886267337309751562263087639079520012193273126047859425087639153757049
236500533455762536198787501905199875019052100
This was computed using a naive algorithm in 228 milliseconds
fastmultiply
input a:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890
input b:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890
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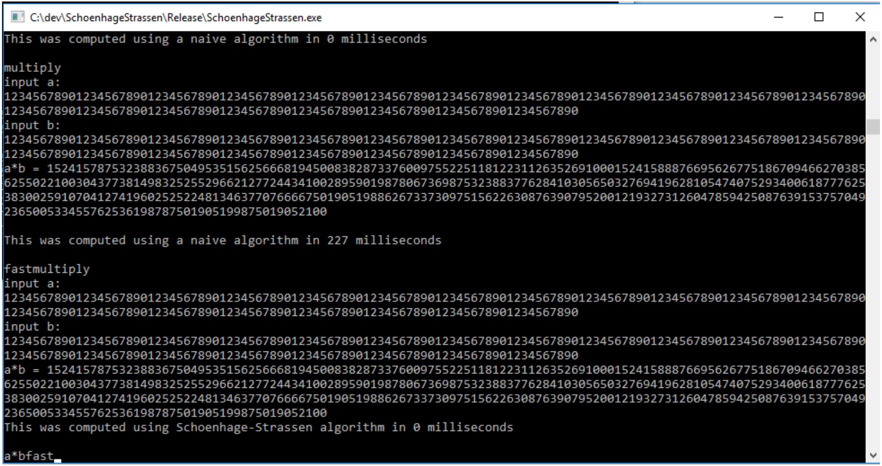
383002591070412741960252522481346377076666750190519886267337309751562263087639079520012193273126047859425087639153757049
236500533455762536198787501905199875019052100
This was computed using Schoenhage-Strassen algorithm in 0 milliseconds


C:\dev\SchoenhageStrassen\Release\SchoenhageStrassen.exe
This was computed using a naive algorithm in 0 milliseconds
multiply
input a:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890
input b:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
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383002591070412741960252522481346377076666750190519886267337309751562263087639079520012193273126047859425087639153757049
2365005334557625361987875019051998750190521OO
This was computed using a naive algorithm in 227 milliseconds
fastmultiply
input a:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890
input b:
123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890123456789012345678901234567890
12345678901234567890123456789012345678901234567890123456789012345678901234567890
a*b = 152415787532388367504953515625666819450083828733760097552251181223112635269100015241588876695626775186709466270385
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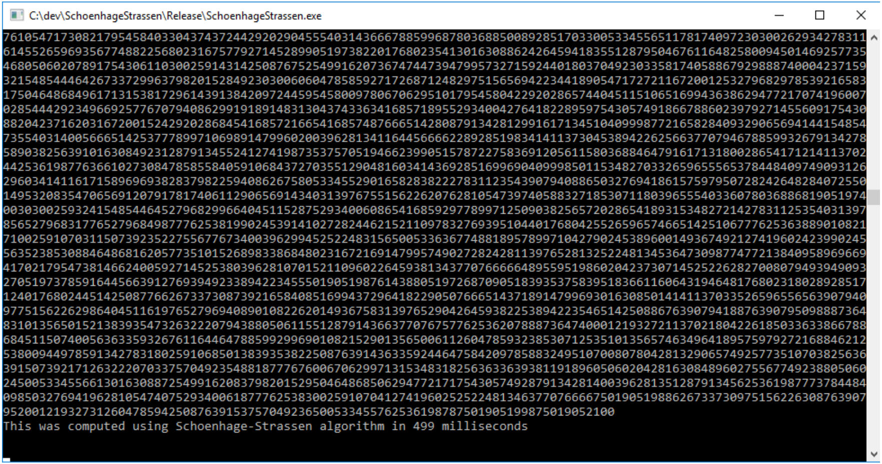
383002591070412741960252522481346377076666750190519886267337309751562263087639079520012193273126047859425087639153757049
236500533455762536198787501905199875019052100
This was computed using Schoenhage-Strassen algorithm in 0 milliseconds
a*bfast


C:\dev\SchoenhageStrassen\Release\SchoenhageStrassen.exe
7610547173082179545840330437437244292029045554031436667885996878036885008928517033005334556511781740972303002629342783311
6145526596935677488225680231675779271452899051973822017680235413016308862426459418355128795046761164825800945014692577735
4680506020789175430611030025914314250876752549916207367474473947995732715924401803704923033581740588679298887400042371559
3215485444642673372996379820152849230300606047858592717268712482975156569422344189054717272116720012532796829785392165883
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This was computed using Schoenhage-Strassen algorithm in 499 milliseconds

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