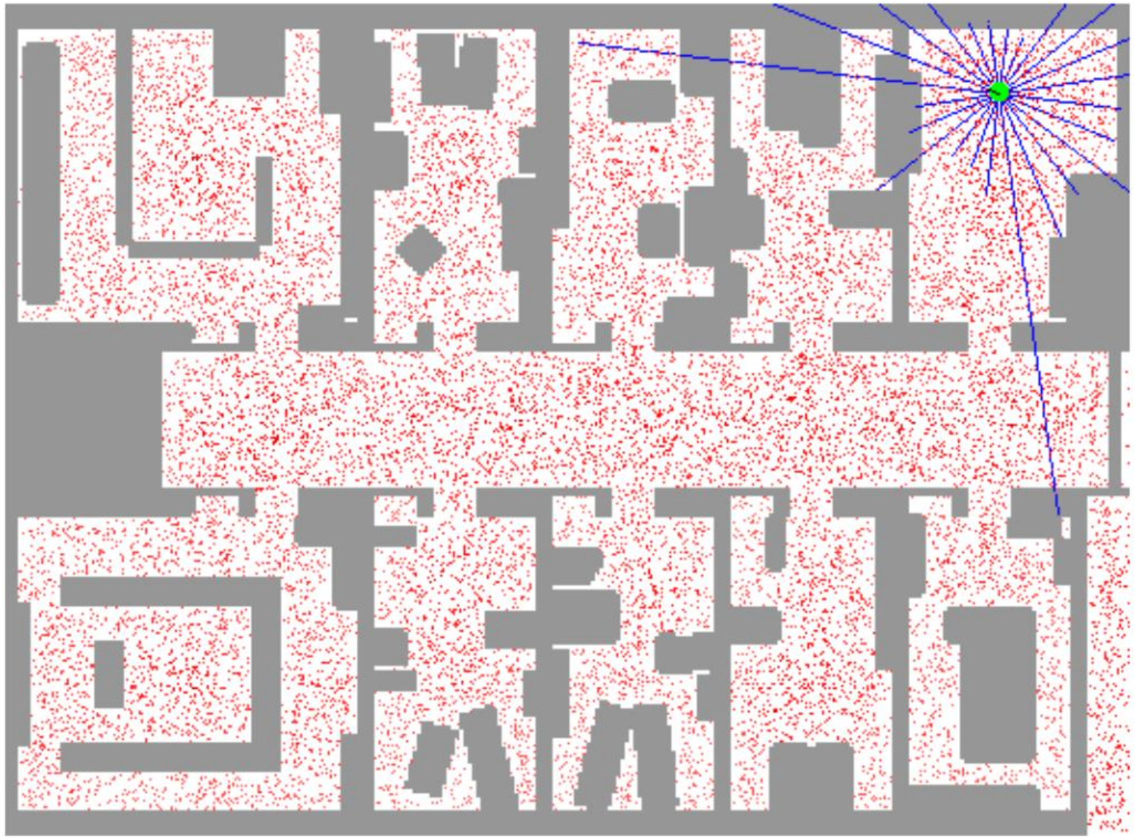
4 – Particle filter

Advanced Methods for Mapping and Self-localization in Robotics
MPC-MAP

Tomas Lazna
Brno University of Technology
2025

What can a convenient localization algorithm offer?

- Mutlimodality
- Continuity
- Intuitivness
- Efficiency
- Scalability


[1]

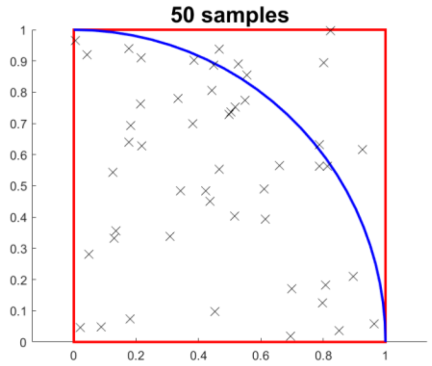
PDF = Probability Density Function

1. TRIEBEL, Rudolph. The Particle Filter. In: *Machine Learning for Computer Vision* [online]. Technische Universität München, 2017 [cit. 2021-02-19]. Available at:https://vision.in.tum.de/_media/teaching/ss2017/ml4cv/variationalinference.pdf

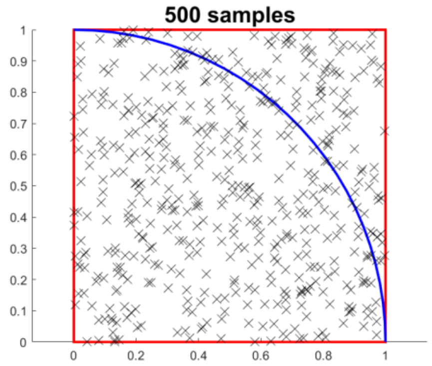
Monte Carlo methods

- Numerical methods based on random sampling
- Optimization, numerical intergration, drawing from PDFs, modeling
- Law of large numbers
- Estimating the π value (Buffon‘s needle)
- Particle filter = Sequential MC method

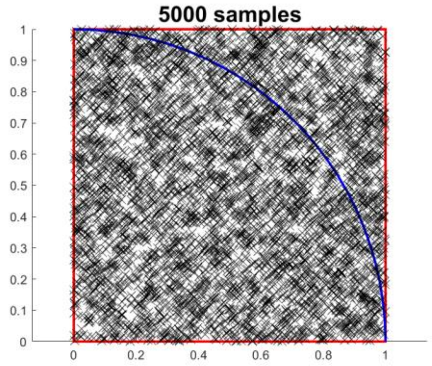
[1]



$\pi \approx 3.36$



$\pi \approx 3.10$



$\pi \approx 3.13$

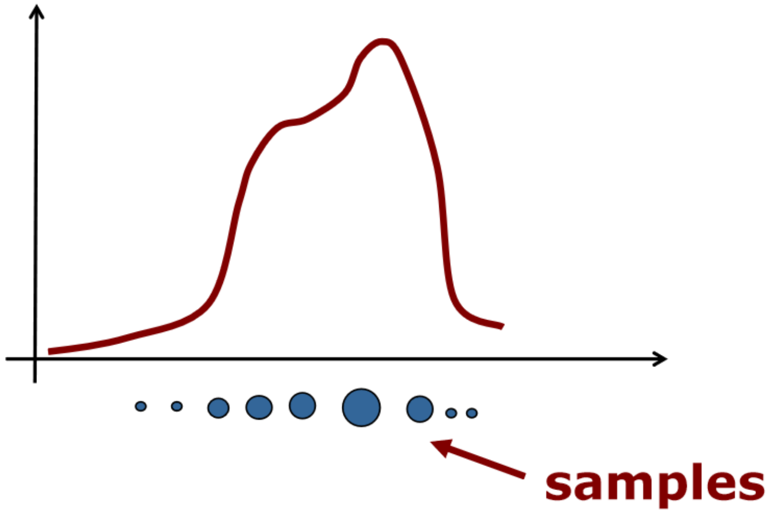
1. Ceasars Slots. Casino Dice Games: The Complete List. Note: Illustrative photo

Introduction

- Particle representation of a PDF

$$\chi = \{[\boldsymbol{x}^{(i)}, w^{(i)}]\}_{i=1,\dots,N} \qquad p(\boldsymbol{x}) = \sum_{i=1}^{N} w^{(i)} \delta_{\boldsymbol{x}^{(i)}}(\boldsymbol{x})$$

State hypothesis — Belief (weight) — Dirac delta function



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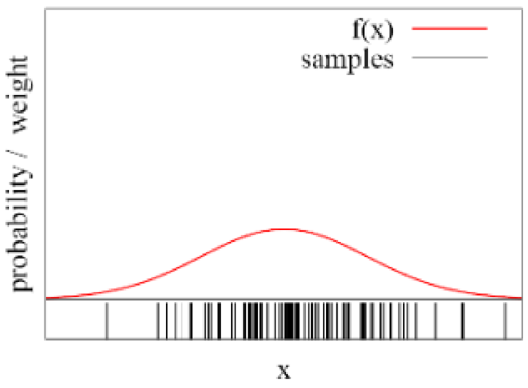
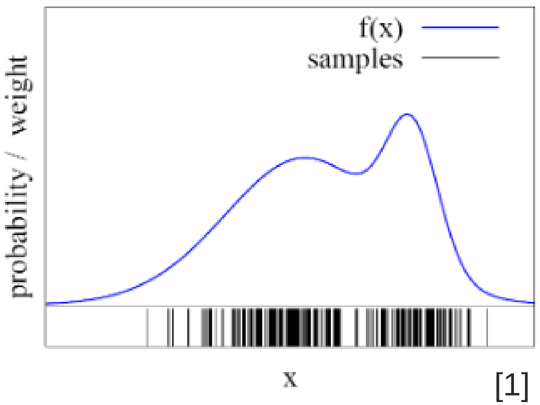
- Bayes‘ theorem

$$P(H|E) = \frac{P(E|H) \cdot P(H)}{P(E)}$$

$H \dots$ hypothesis

$E \dots$ evidence





[1]

- Particle filter applications:
  - Robot localization
  - Object tracking, computer vision
  - General estimation in nonlinear systems
- Original article (referred to as ‚bootstrap filter‘) [2]

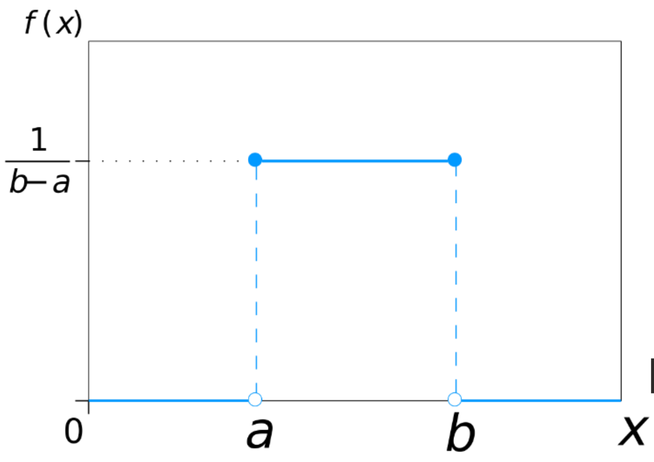
1. STACHNISS, Cyrill. Short Introduction to Particle Filters and Monte Carlo Localization [online]. Uni Freiburg, 2013 [cit. 2021-02-18]. Available at: http://ais.informatik.uni-freiburg.de/teaching/ws13/mapping/pdf/slam10-particle-filter-4.pdf
2. GORDON, N.J. et al. Novel approach to nonlinear/non-Gaussian Bayesian state estimation. *IEE Proceedings F Radar and Signal Processing* [online]. 1993, **140**(2) [cit. 2021-02-21]. DOI: 10.1049/ip-f-2.1993.0015

Drawing random samples

- Uniform distribution
  - Any random number generator
  - Usually a pseudorandom series
  - Can be utilized for other PDFs
- Normal distribution
  - Parameric function
    (mean μ, variance σ^2)
  - Approximation by UD:

$$x \leftarrow \sum_{i=1}^{12} \mathrm{rand}(-0.5\sigma, 0.5\sigma)$$

- Other distributions
  - Parametric
  - Non-parametric



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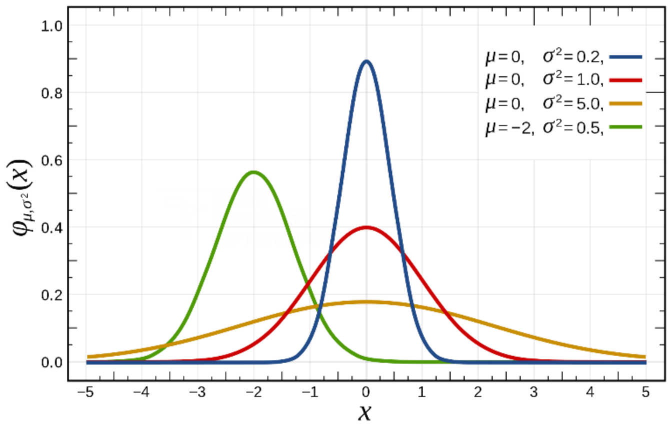
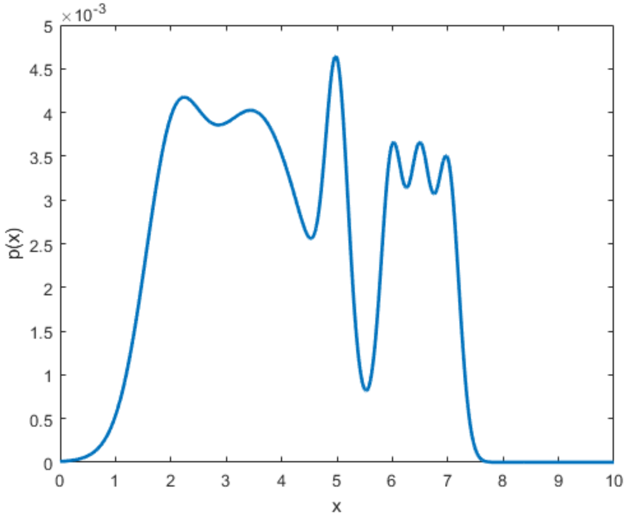
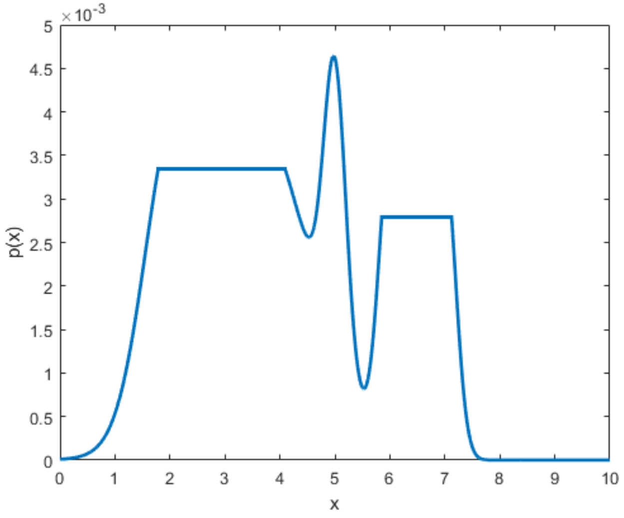


[2]

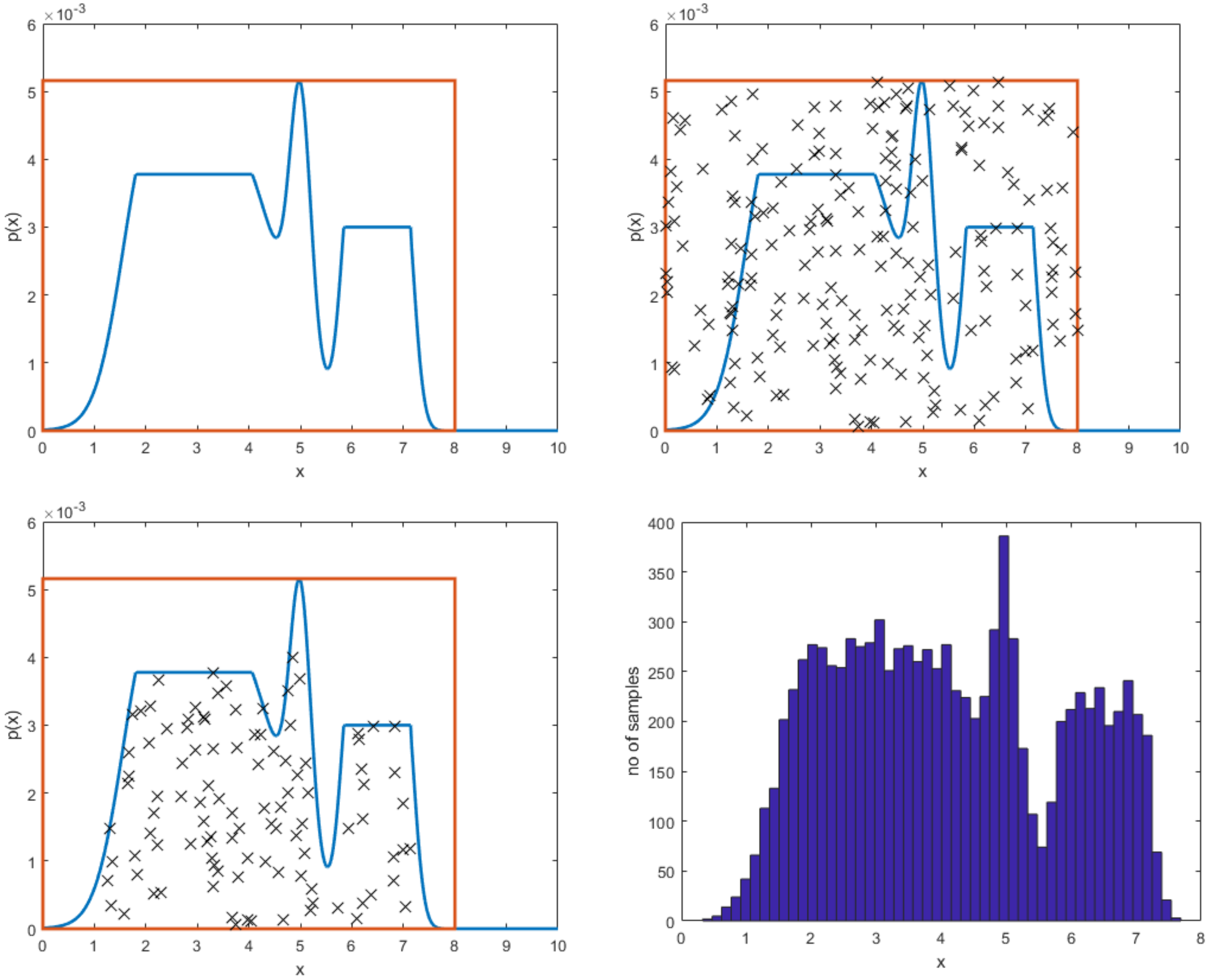




1. Continuous uniform distribution. Wikipedia, Wikimedia Foundation, 20 Dec 2020. Available at: https://en.wikipedia.org/wiki/Continuous_uniform_distribution
2. Normal distribution. Wikipedia, Wikimedia Foundation, 13 Feb 2021. Available at: https://en.wikipedia.org/wiki/Normal_distribution

Rejection sampling

- Generate random samples $[y,\ f(y)]$ in range:
  - $y \in \langle x_{\min};\ x_{\max}\rangle$
  - $f(y) \in (0; \max(p(x))\rangle$
- Reject sample if:
  - $f(y) > p(y)$
- The random variable Y is now distributed according to $p(x)$



Rejection sampling

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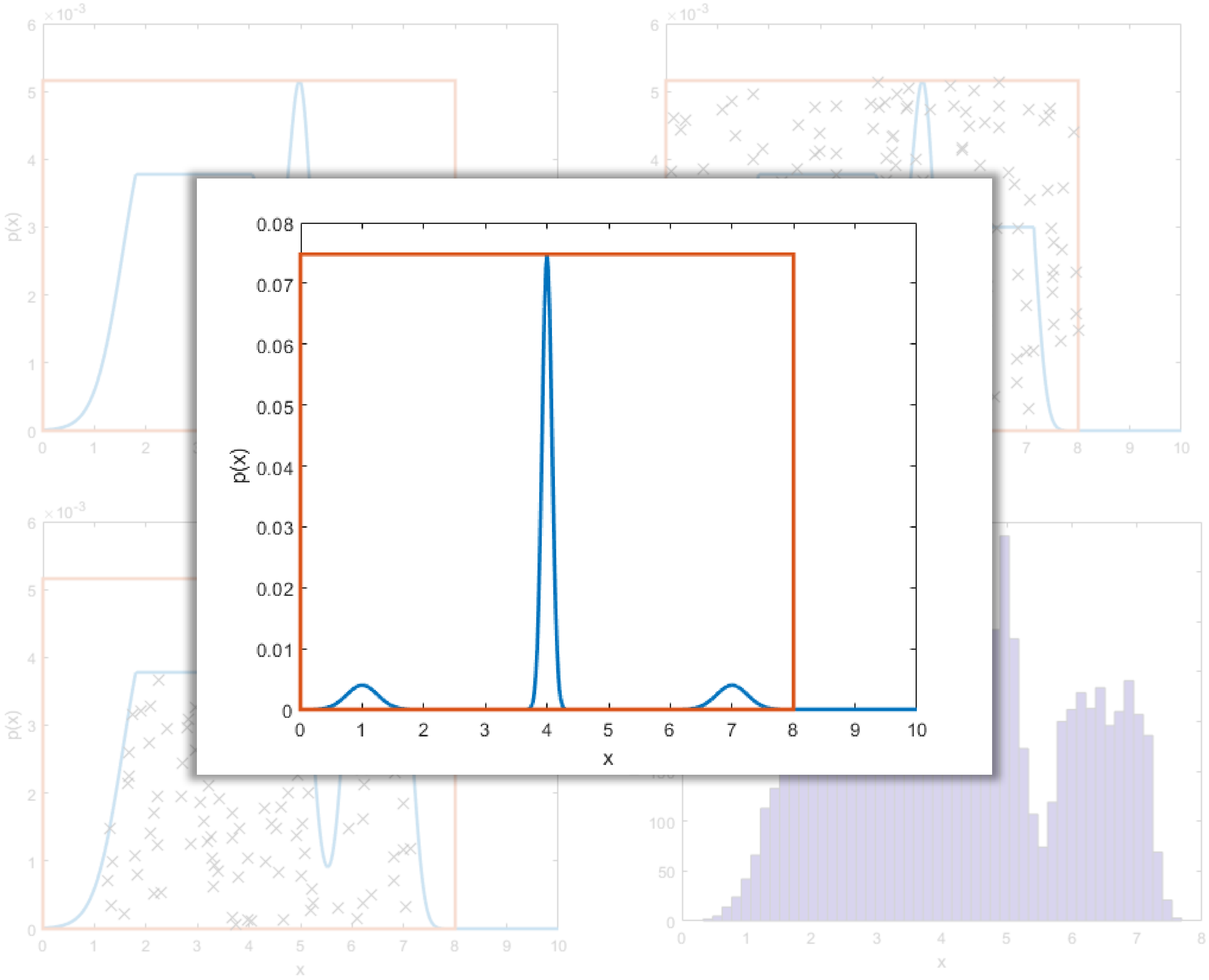
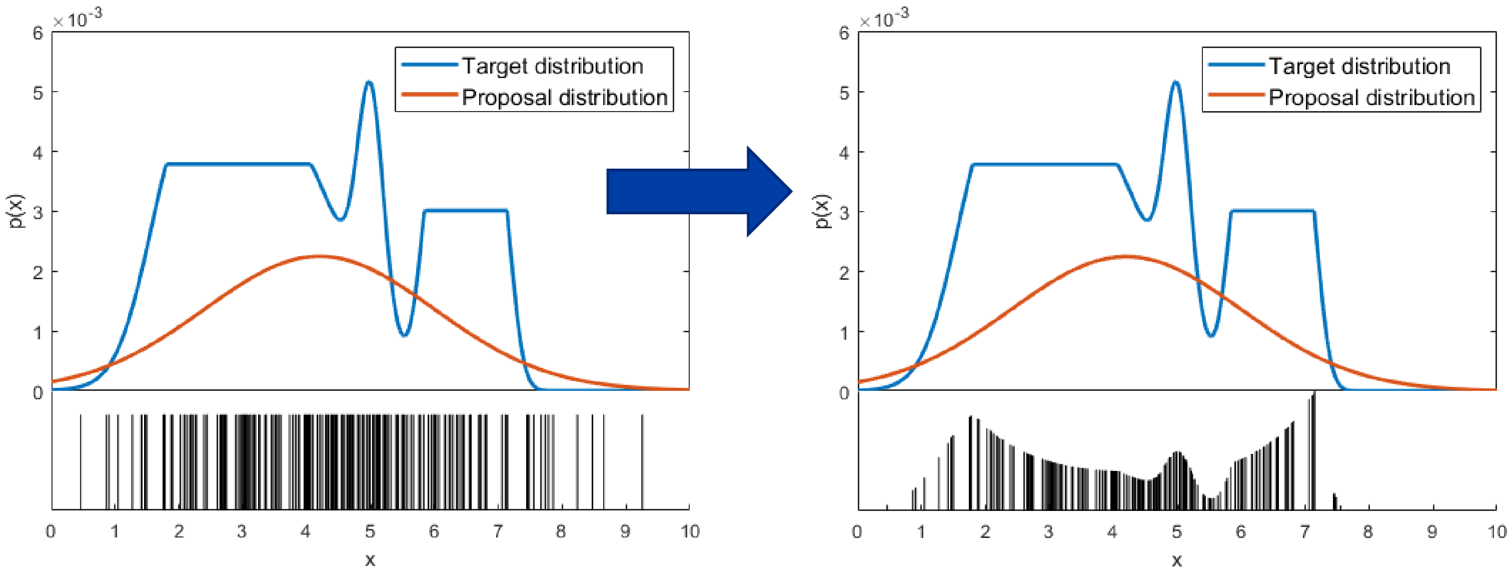
- Reject sample if:
  - $f(y) > p(y)$
- The random variable Y is now distributed according to $p(x)$
- Tends to be inefficient



Importance sampling

- Use other distribution π that is simple to draw from
- Correct the difference between the target distribution f and proposal π by assigning ‚weights‘ to random samples:

$$w(x) = \frac{f(x)}{\pi(x)}$$



- To ensure samples are drawn from the whole target distribution, following condition needs to be met:

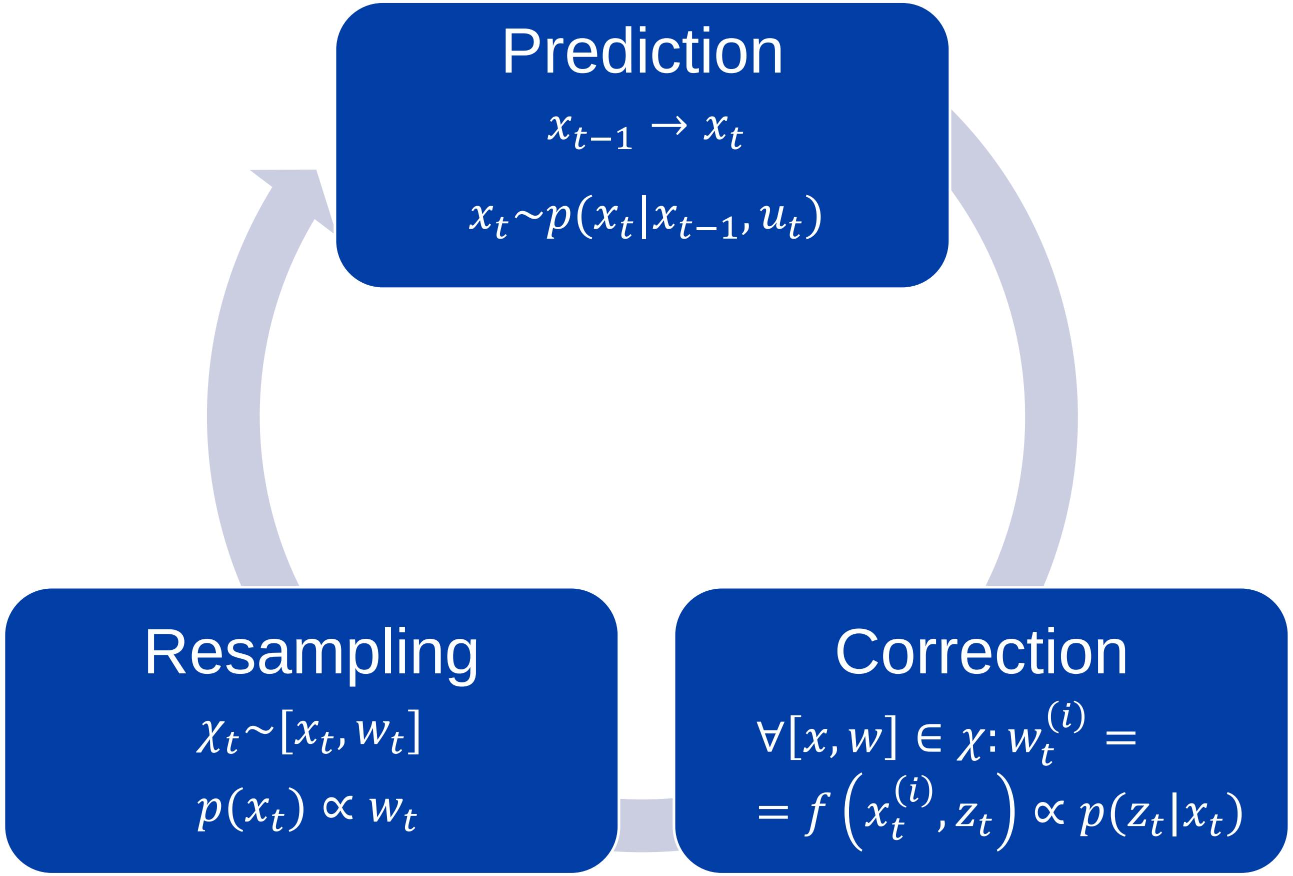
$$\forall x \in \mathbb{R}: f(x) > 0 \Rightarrow \pi(x) > 0$$

- Other methods for drawing random samples
  - Adaptive rejection sampling [1]
  - Markov chain Monte Carlo (MCMC) methods, e.g., Metropolis-Hastings algorithm [2]

1. MARTINO, Luca and Joaquín MÍGUEZ. A generalization of the adaptive rejection sampling algorithm. *Statistics and Computing* [online]. 2011, **21**(4). [cit. 2021-02-19]. DOI: 10.1007/s11222-010-9197-9
2. CHIB, Siddhartha and Edward GREENBERG. Understanding the Metropolis-Hastings Algorithm. *The American Statistician* [online]. 1995, **49**(4). [cit. 2021-02-19]. DOI: 10.1080/00031305.1995.10476177

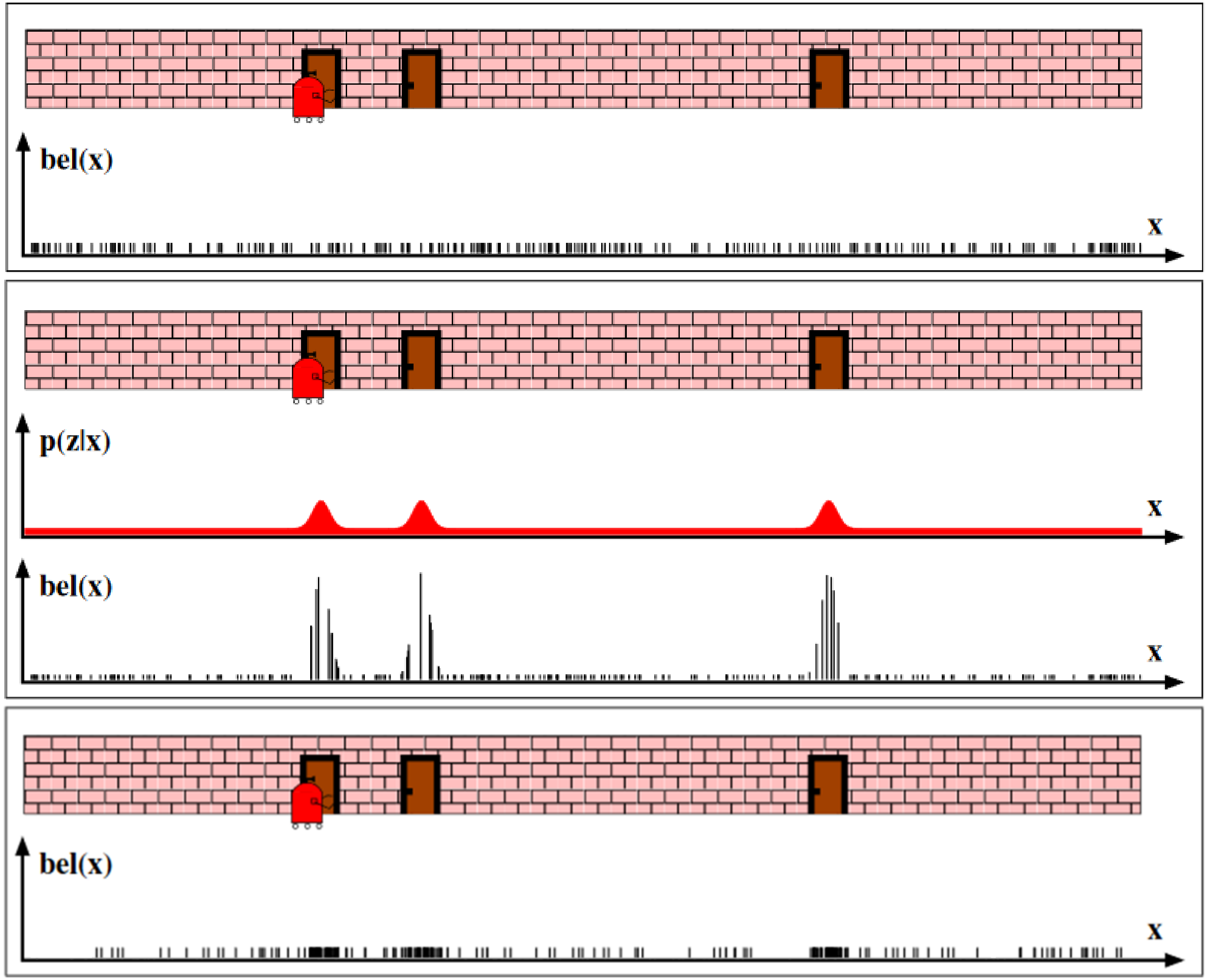
Particle filter principle

- Prediction step
  - State of all particles is updated in accordance with starting state and the input vector
  - Goal: Estimate the state transition
- Correction step
  - Particles‘ hypotheses are compared with actual measurement
  - Goal: Find out which particles are the fittest
- Resampling step
  - Draw random samples from the previous particle set with the probability given by weights
  - Goal: Increase the particle density in more probable parts of state space



Particle filter example in 1D

1. Initialization
2. Correction
3. Resampling

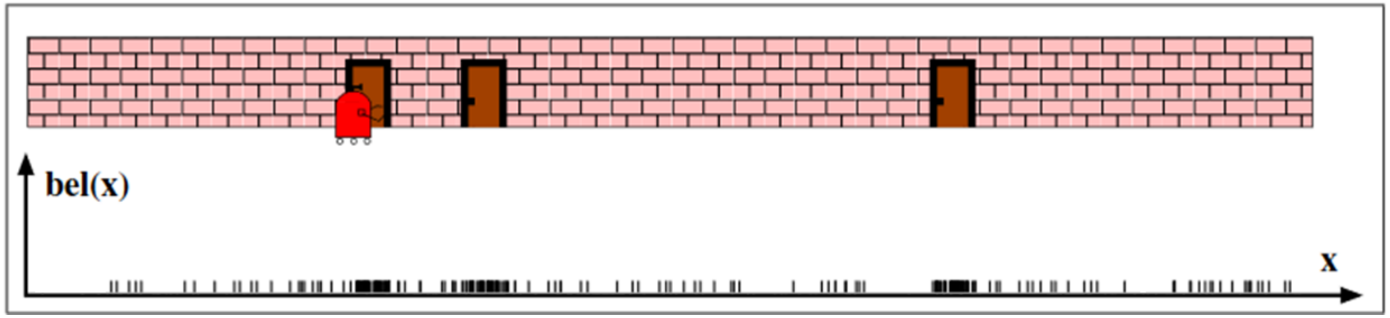


[1]

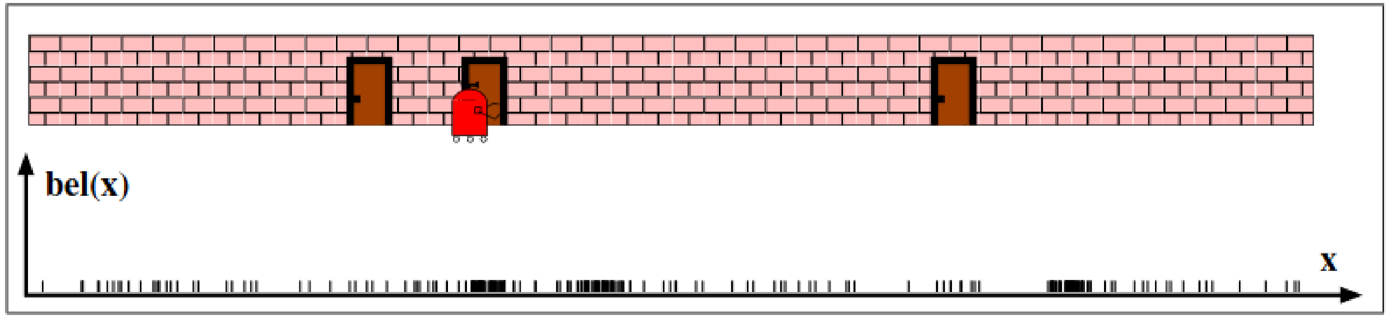
1. THRUN, Sebastian, Wolfram BURGARD and Dieter FOX. Probabilistic robotics. Massachusetts: MIT Press, 2006. ISBN 978-0-262-20162-9.

Particle filter example in 1D

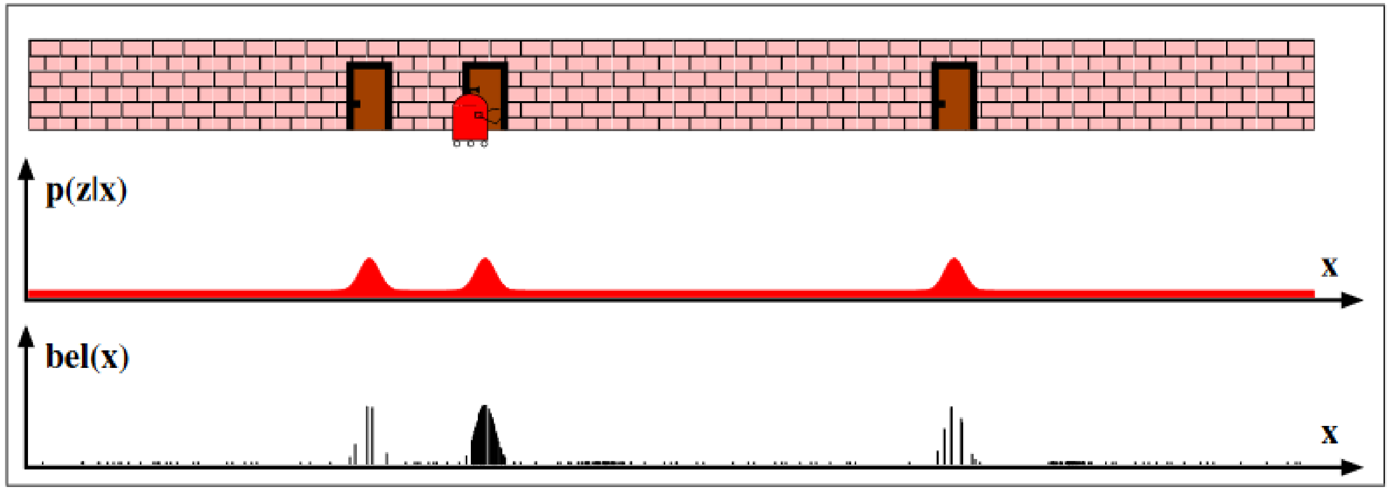
3. Resampling



4. Prediction



5. Correction



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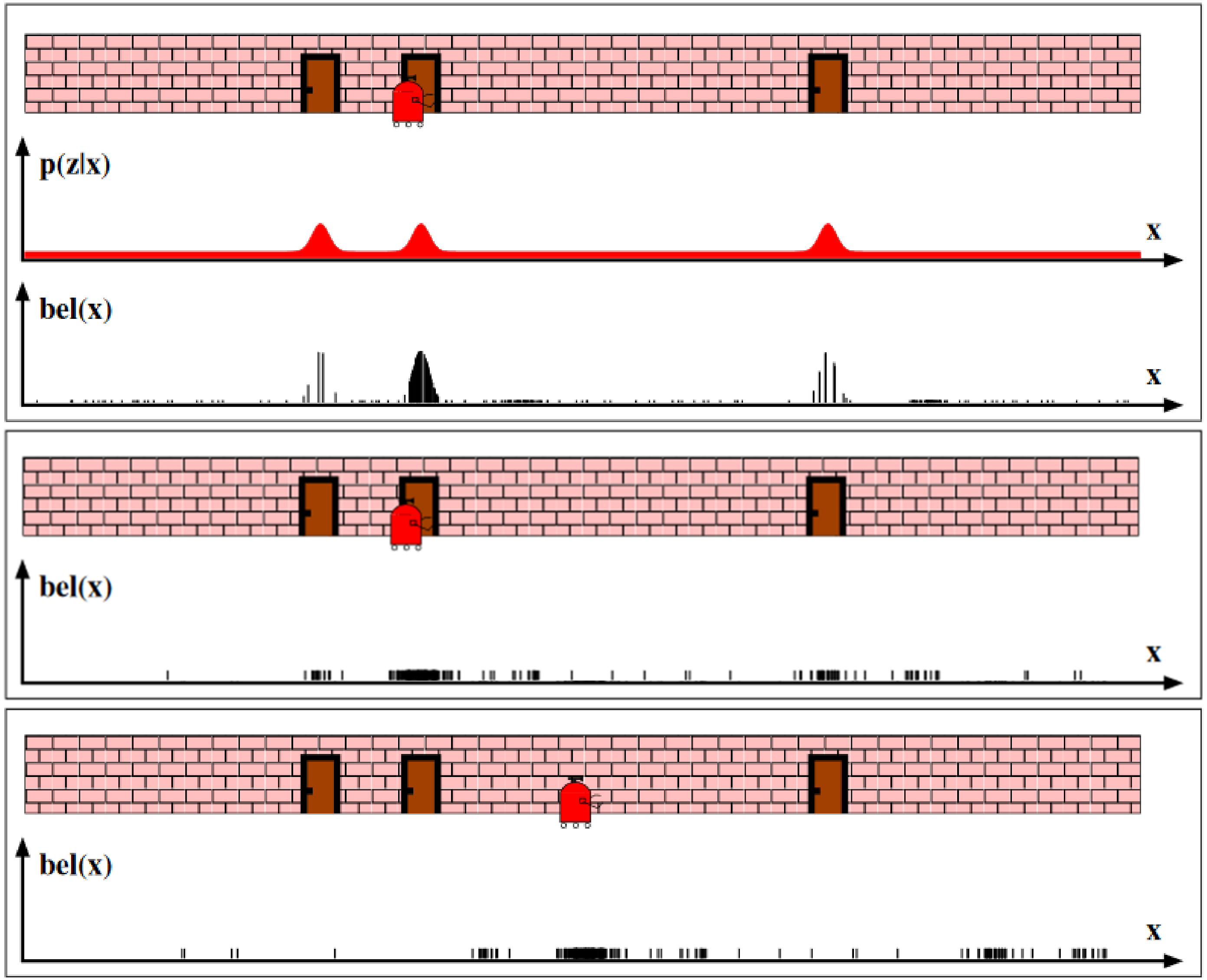
1. THRUN, Sebastian, Wolfram BURGARD and Dieter FOX. Probabilistic robotics. Massachusetts: MIT Press, 2006. ISBN 978-0-262-20162-9.

Particle filter example in 1D

5. Correction

6. Resampling

7. Prediction



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Prediction

- Predict state of the system after a control vector is applied
- Increase variance of particles
- Let us assume that the state transition depends on the previous state only (first order Markov Process)
- The noise can have arbitrary distribution

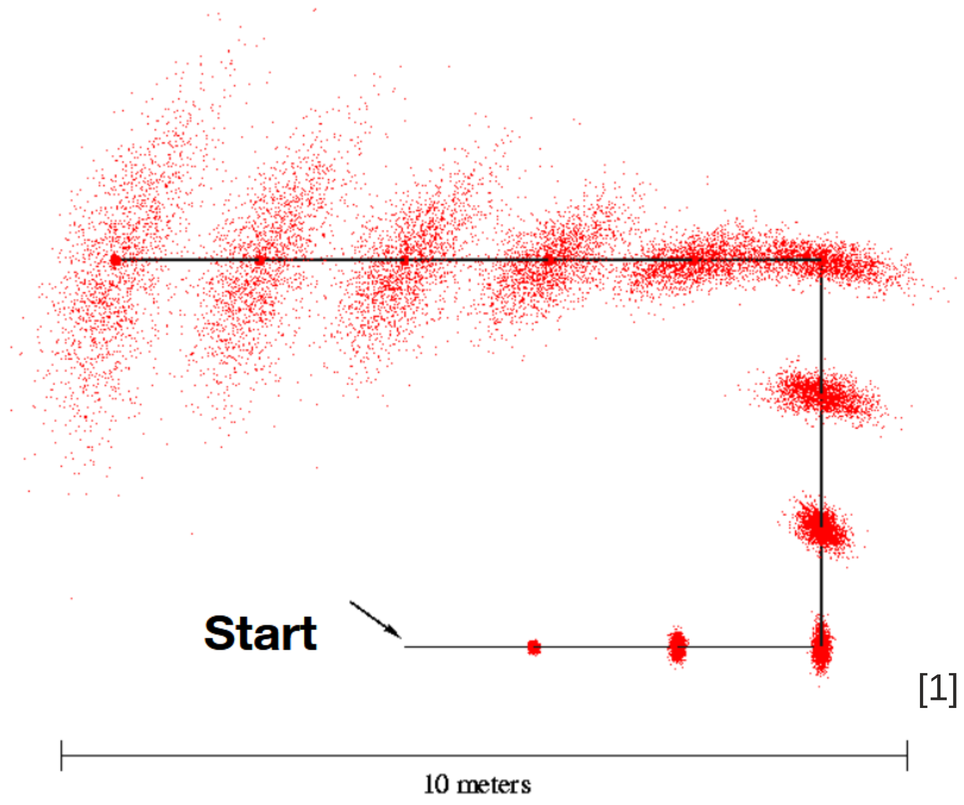
$$\boldsymbol{x}_t^{(i)} = f\left(\boldsymbol{x}_{t-1}^{(i)}, \boldsymbol{u}_t, \boldsymbol{Q}\right)$$

Predicted state

Original state

Control vector

Random variable – noise



[1]

- Example – omnidirectional drive

$$\begin{pmatrix} x_t \\ y_t \\ \theta_t \end{pmatrix} = \begin{pmatrix} x_{t-1} \\ y_{t-1} \\ \theta_{t-1} \end{pmatrix} + \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} \Delta t + \boldsymbol{Q}$$

Differential drive:

$$\boldsymbol{x}_t + \boldsymbol{Q} \neq f(\boldsymbol{x}_{t-1}, g(\boldsymbol{u}_t, \boldsymbol{Q}))$$

where $\boldsymbol{Q}$ has normal distribution

Prediction

$$x_{t-1} \rightarrow x_t$$

$$x_t \sim p(x_t | x_{t-1}, u_t)$$

1. TRIEBEL, Rudolph. The Particle Filter. In: *Machine Learning for Computer Vision* [online]. Technische Universität München, 2017 [cit. 2021-02-19]. Available at: https://vision.in.tum.de/_media/teaching/ss2017/ml4cv/variationalinference.pdf

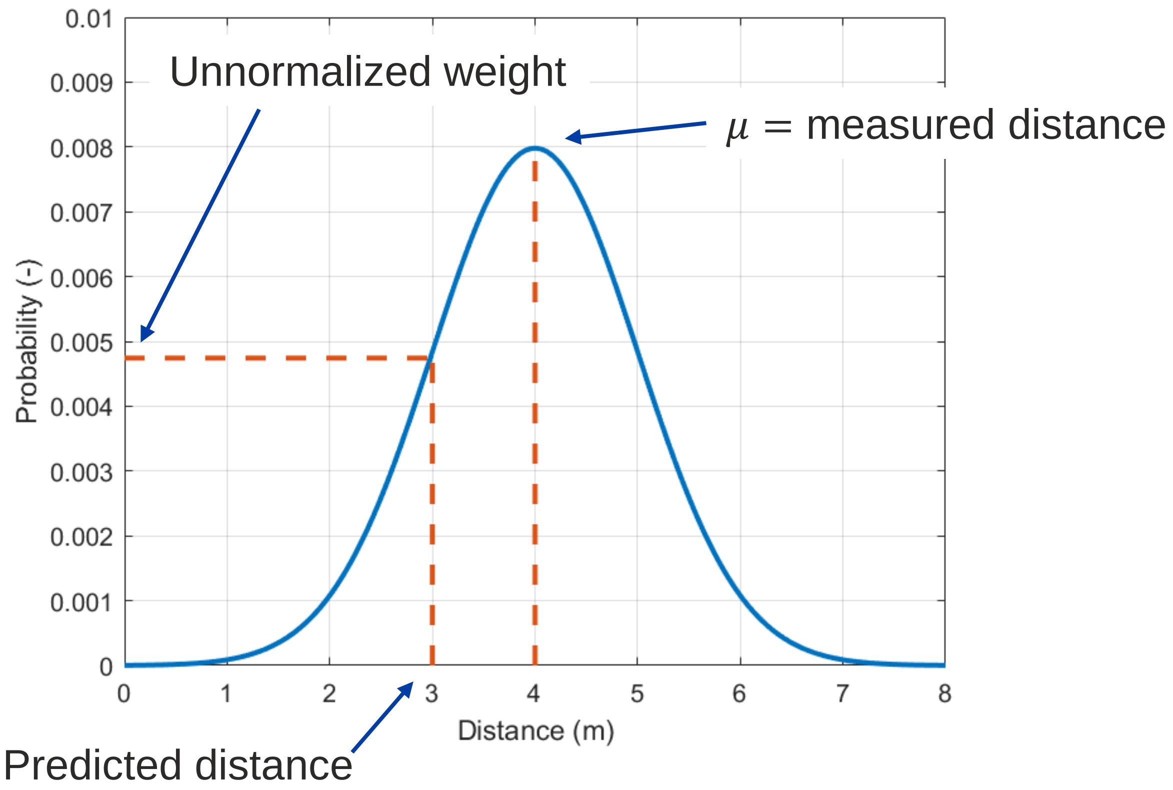
Correction

- Application of Bayes‘ rule
  - Prior: all particles have the same weight 1 / N
  - Posterior: proportional to the measurement model
- Weights should be normalized
- Examples for rangefinders:
  - Using normal distribution (σ)

$$w \propto \prod_{m=1}^{M} e^{-\frac{1}{2}\left(\frac{d_m - p_m}{\sigma}\right)^2}$$

  - Using Euclidean distance

$$w \propto \frac{1}{\sqrt{\sum_{m=1}^{M}(d_m - p_m)^2}}$$



d … measured distance
p … predicted distance
M … number of measurements

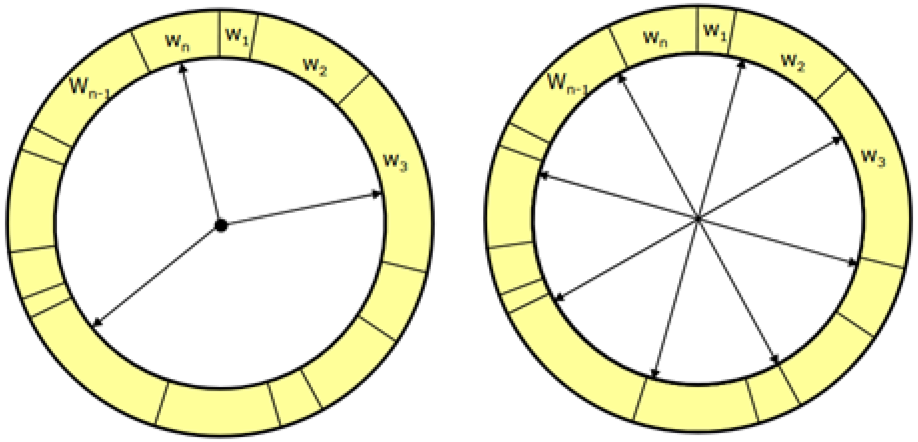
Correction

$$\forall [x, w] \in \chi : w_t^{(i)} = \\ = f\left(x_t^{(i)}, z_t\right) \propto p(z_t | x_t)$$

Resampling

- Increase density of particles in regions of high posterior probability and vice versa
- Needed in case of limited number of samples
- Draw N particles with the probability given by weights of original set
- General algorithm:
  1. Generate sorted set of N random numbers u_k in range $\langle 0, 1)$
  2. Compute cummulative sum of weights
  3. For each u_k pick particle x_i according to condition:

$$u_k \in \left\langle \sum_{s=1}^{i-1} w_s, \sum_{s=1}^{i} w_s \right)$$



Roulette wheel

Systematic

[1]

Resampling

$\chi_t \sim [x_t, w_t]$

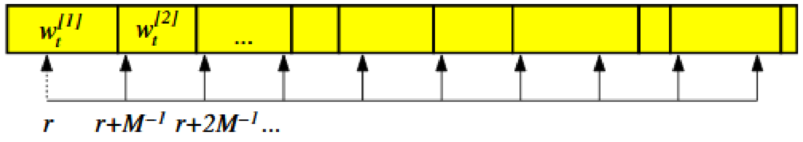
$p(x_t) \propto w_t$

1. STACHNISS, Cyrill. Short Introduction to Particle Filters and Monte Carlo Localization [online]. Uni Freiburg, 2013 [cit. 2021-02-18]. Available at: http://ais.informatik.uni-freiburg.de/teaching/ws13/mapping/pdf/slam10-particle-filter-4.pdf

Resampling – algorithms

- Low variance systematic resampling
  - Only one iteration through the weights set
  - Keeps particles of even weights alive
  - Generate random number $\tilde{u}$ in range $\left(0, \frac{1}{N}\right)$

$$u_k = \frac{(k-1) + \tilde{u}}{N}$$



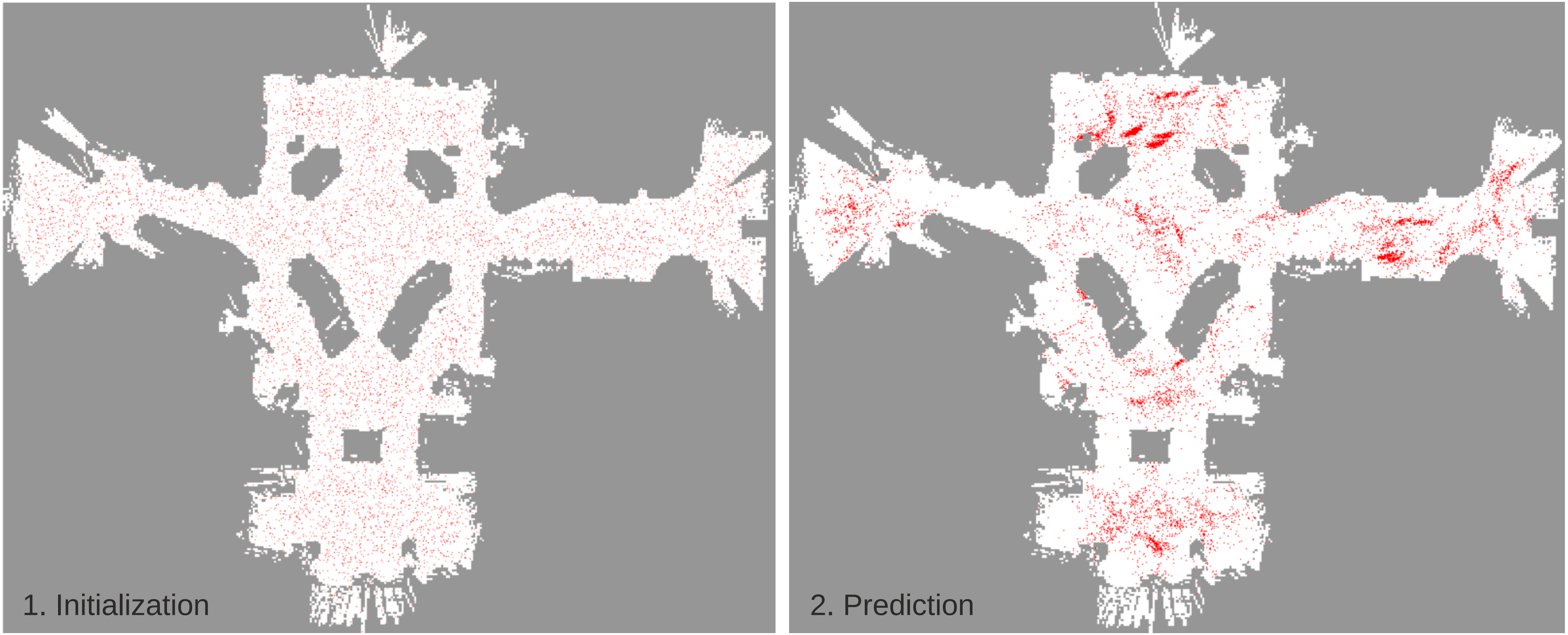
[1]

- Thrun's heuristic algorithm
  - Higher degree of randomness
  - Easy implementation

```
index = rand(0, N-1)
for i = 1 to N do
    beta = rand(0, 2w_max)
    while w[index] < beta
        beta = beta – w[index]
        index++
        if index > N
            index = 1
    new_particles[i] = particles[index]
```

1. THRUN, Sebastian, Wolfram BURGARD and Dieter FOX. Probabilistic robotics. Massachusetts: MIT Press, 2006. ISBN 978-0-262-20162-9.

Particle filter example in 2D



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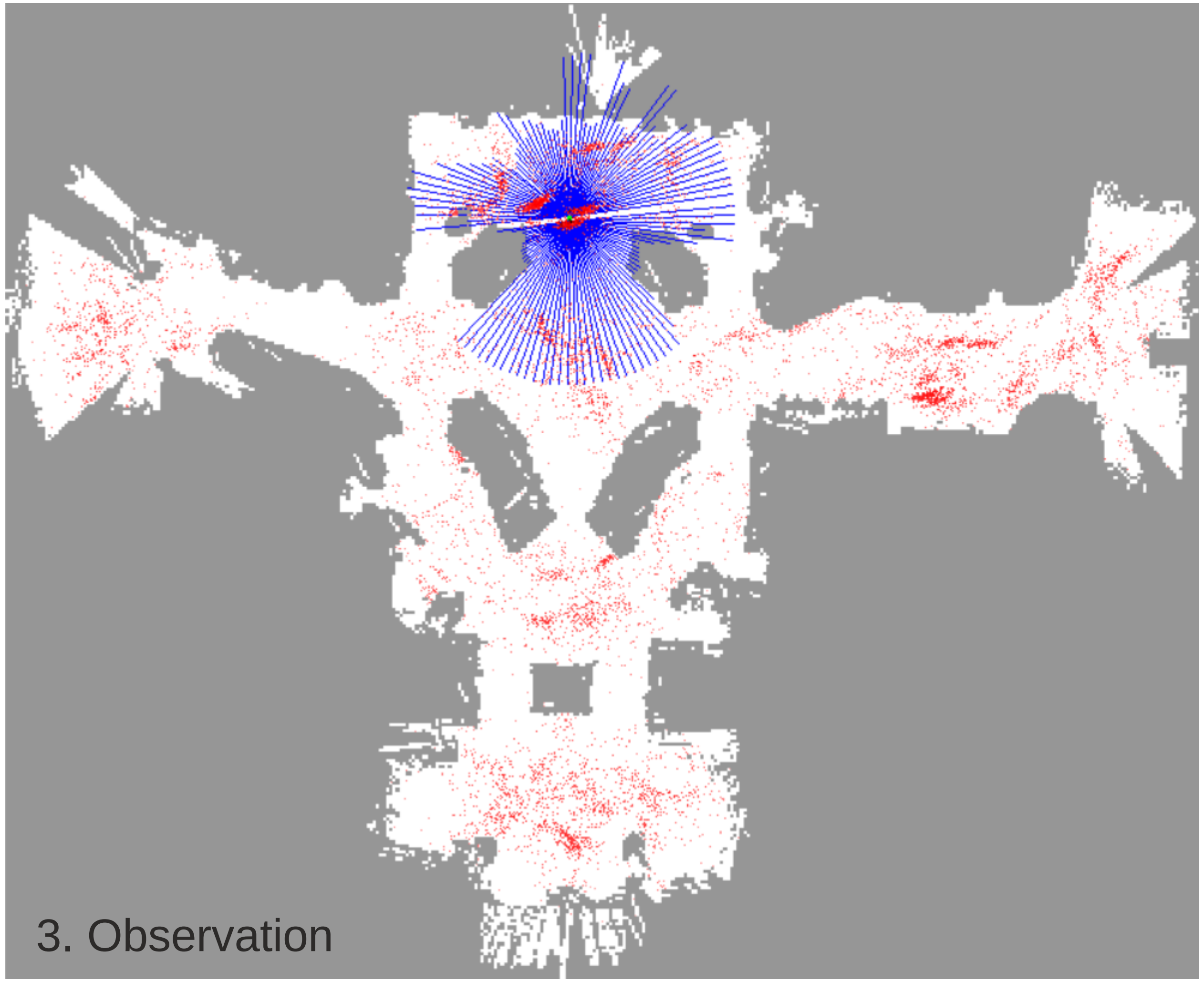

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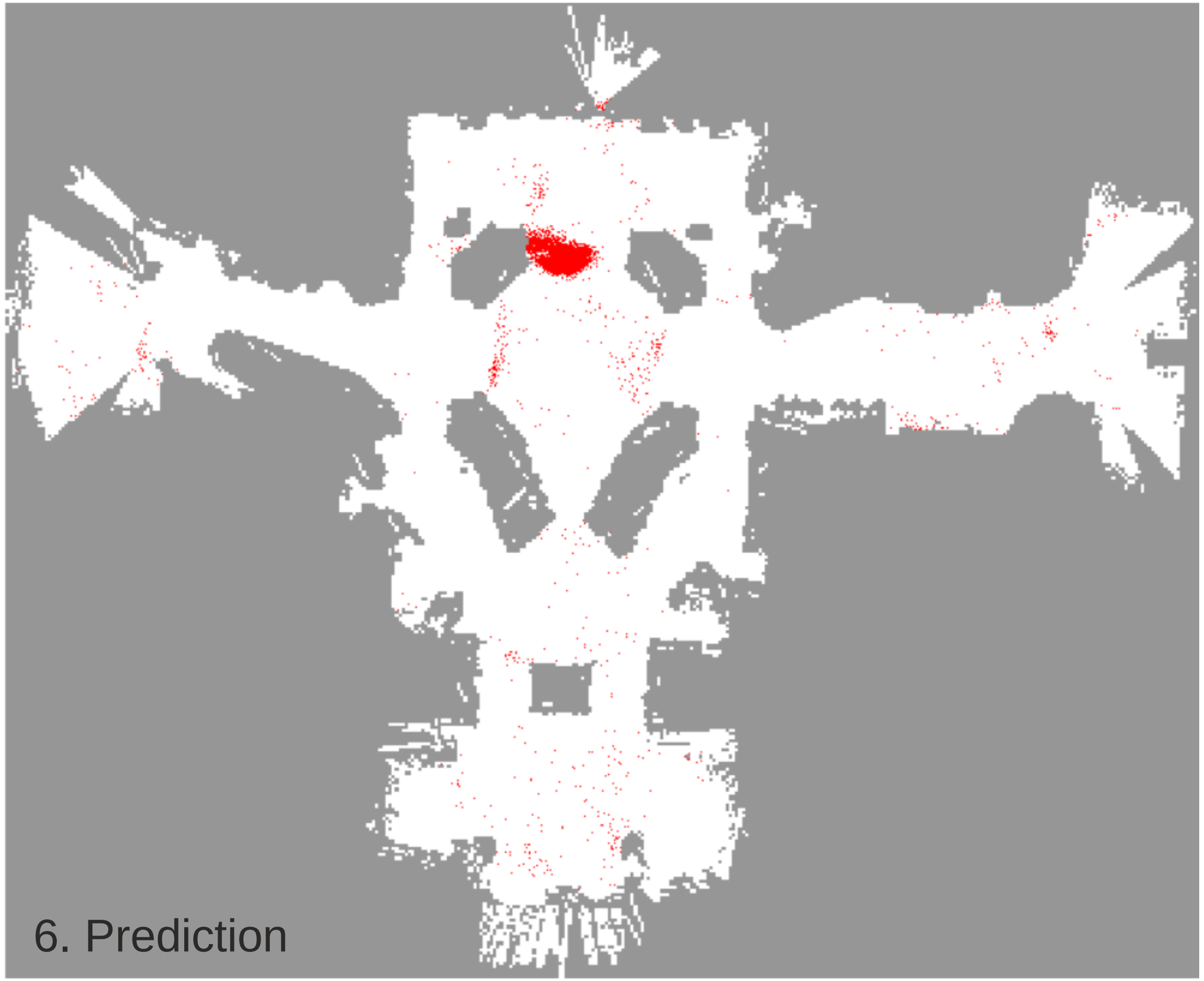

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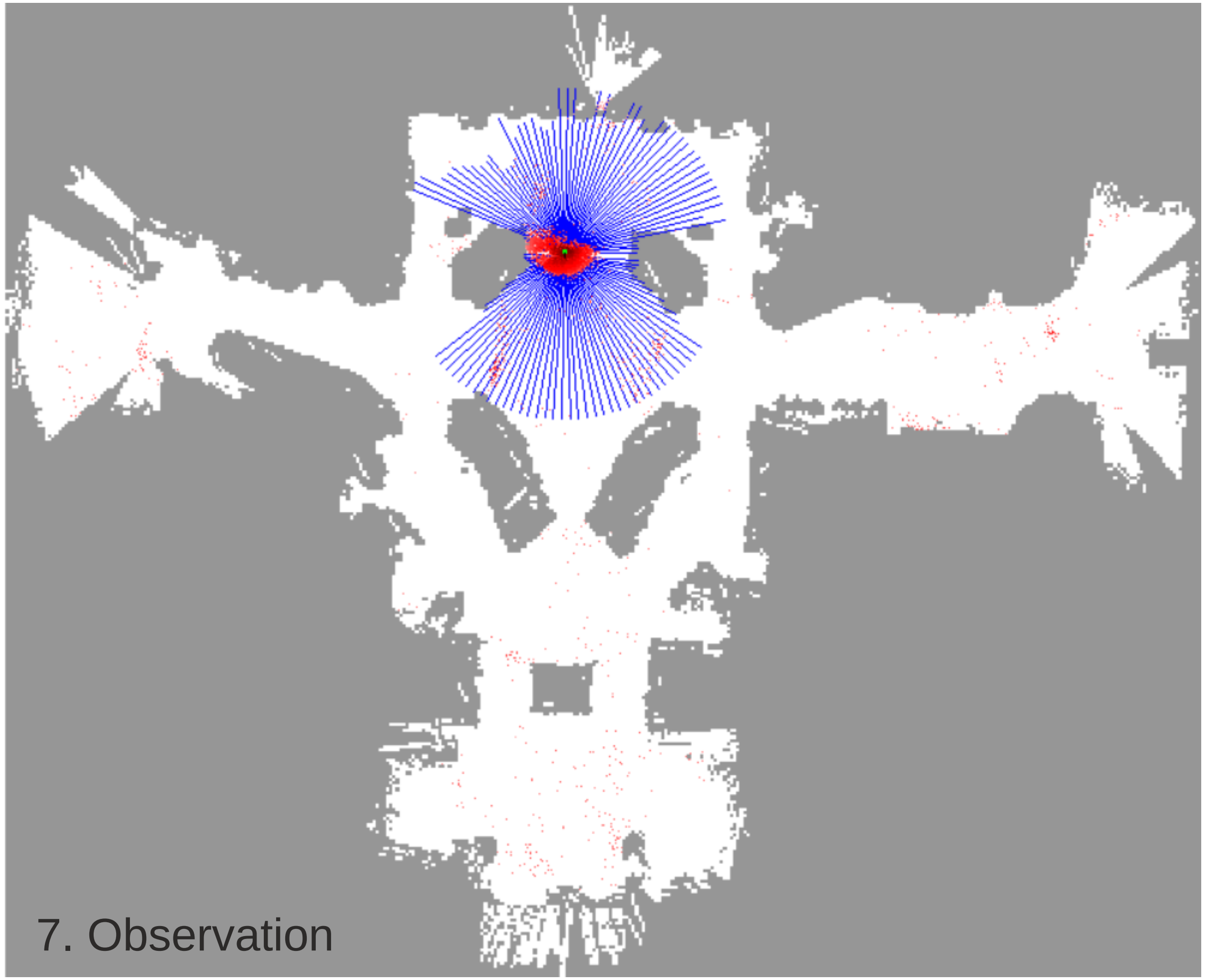

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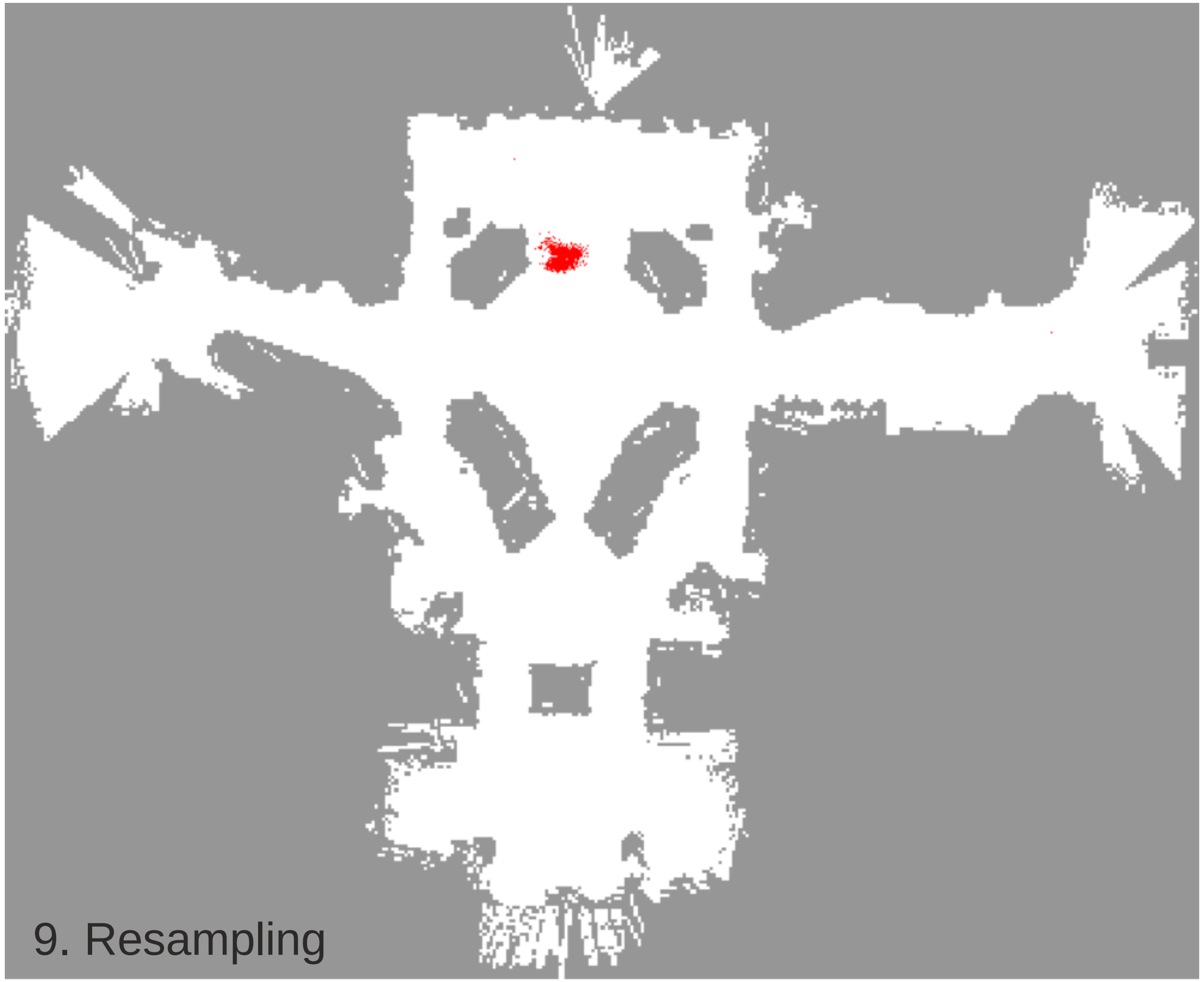

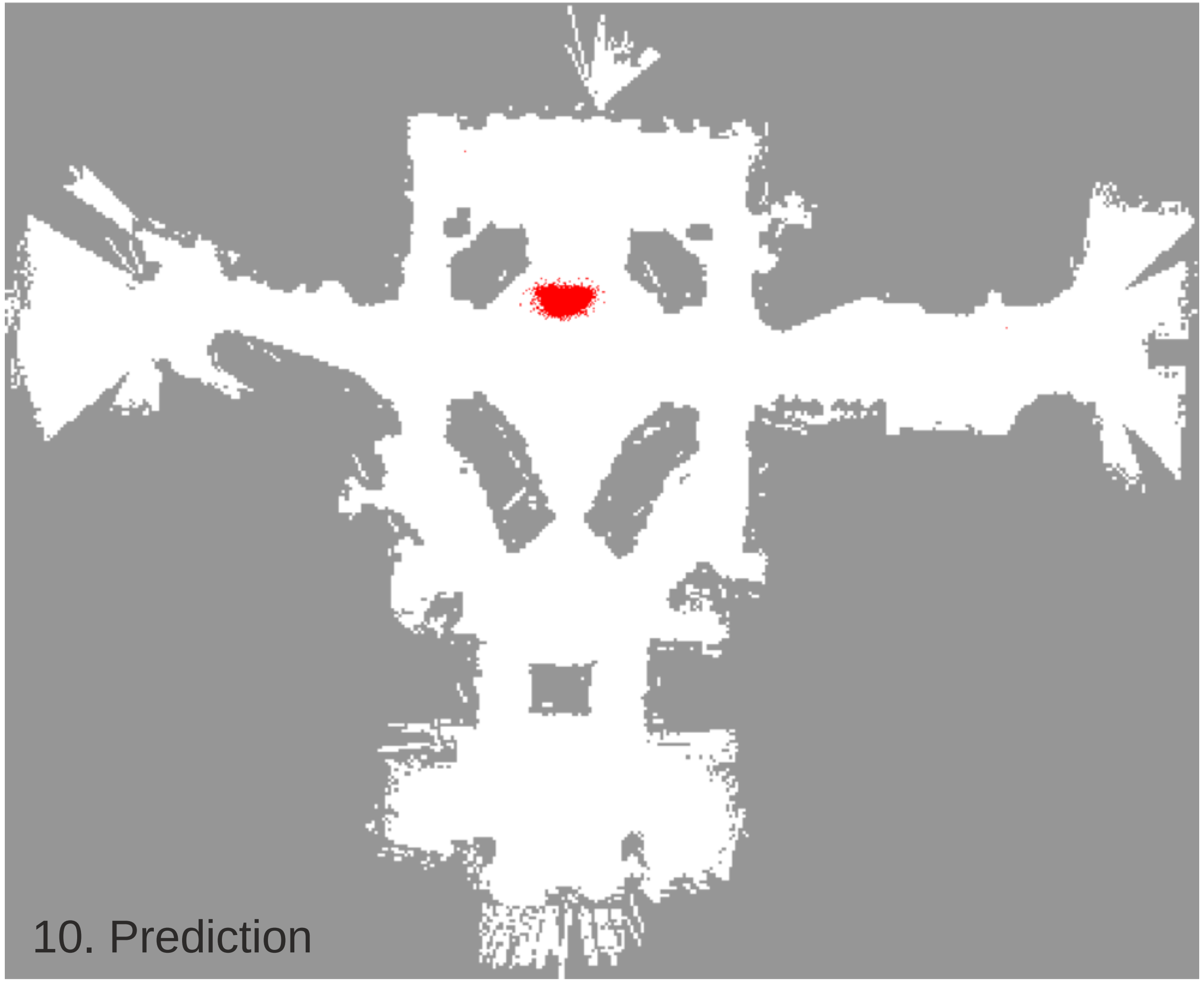

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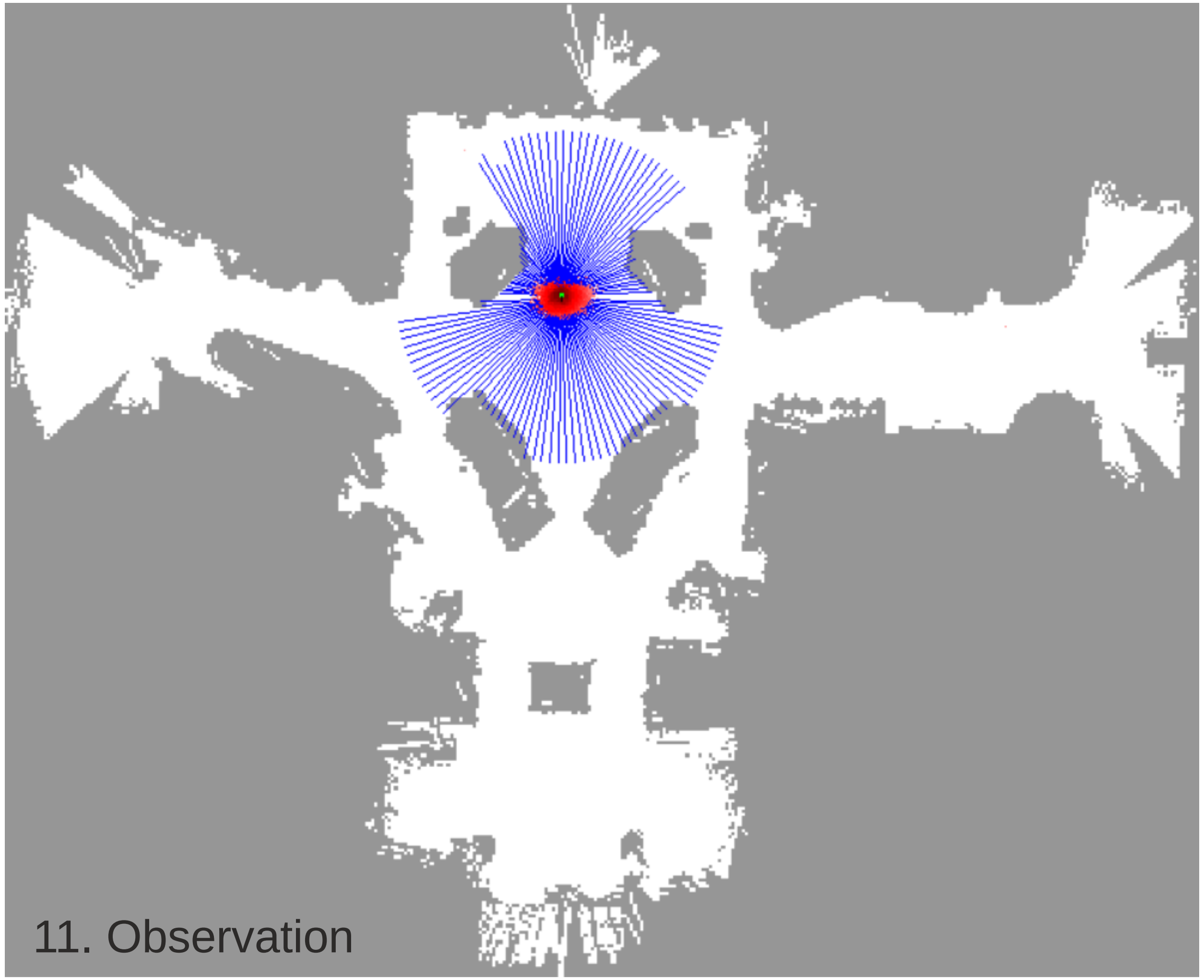

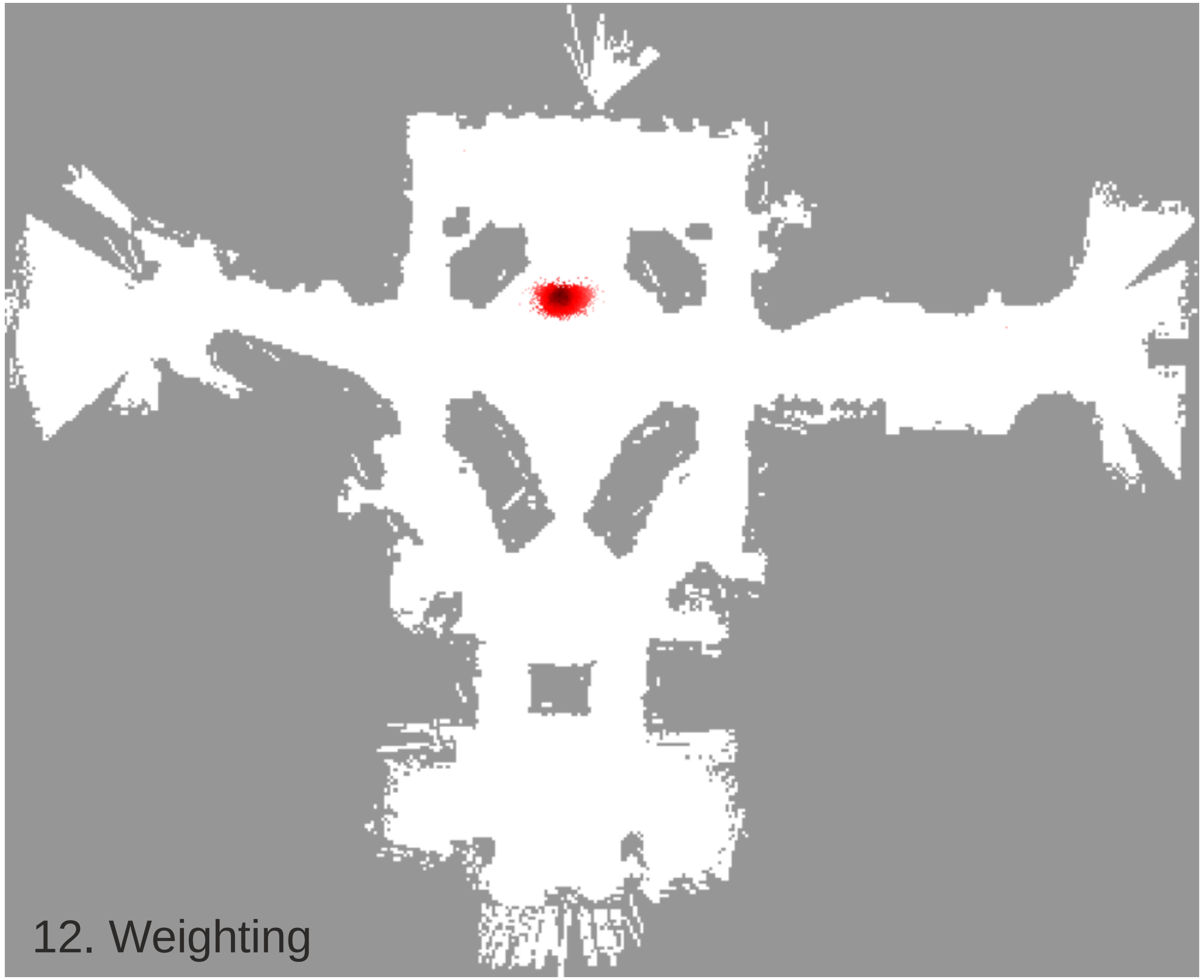

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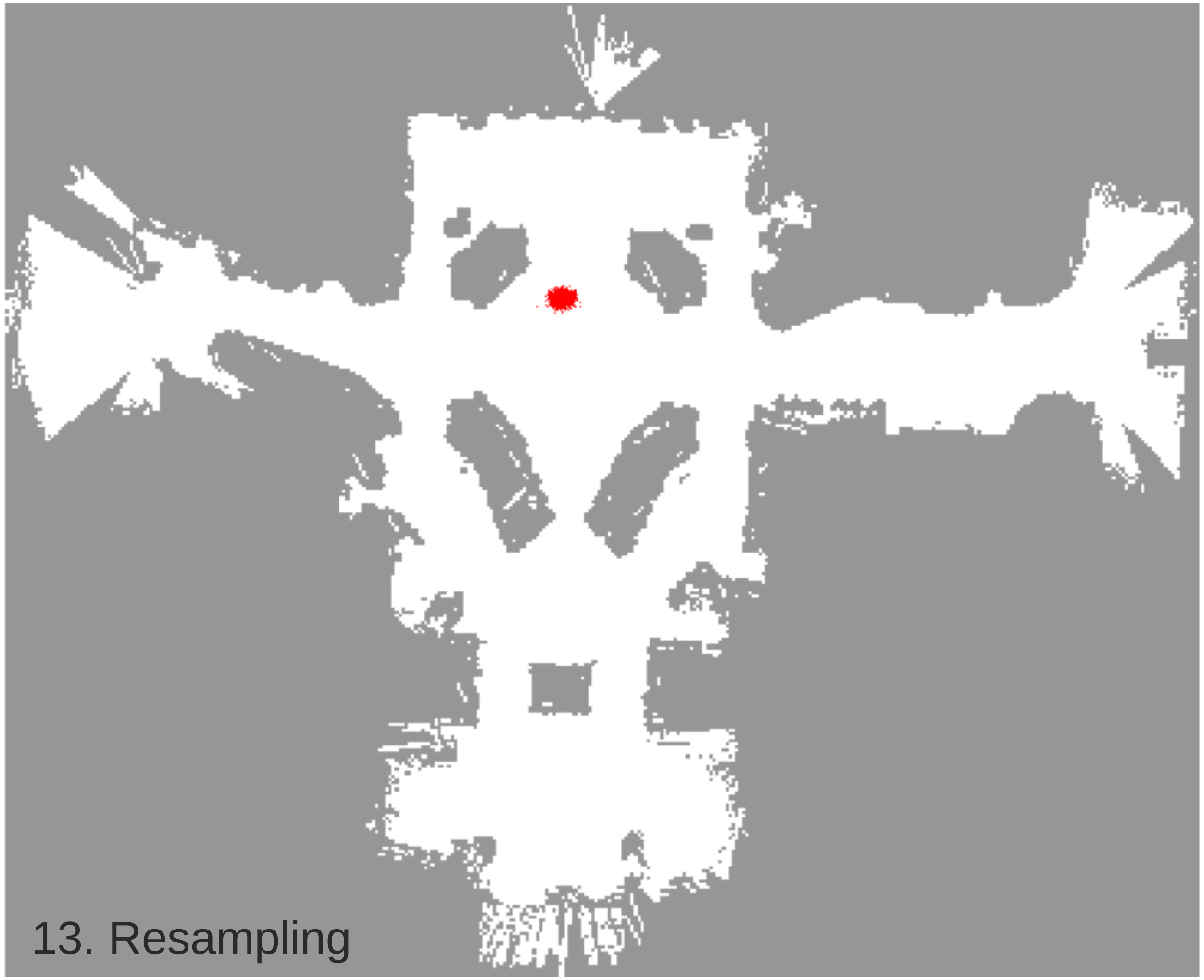

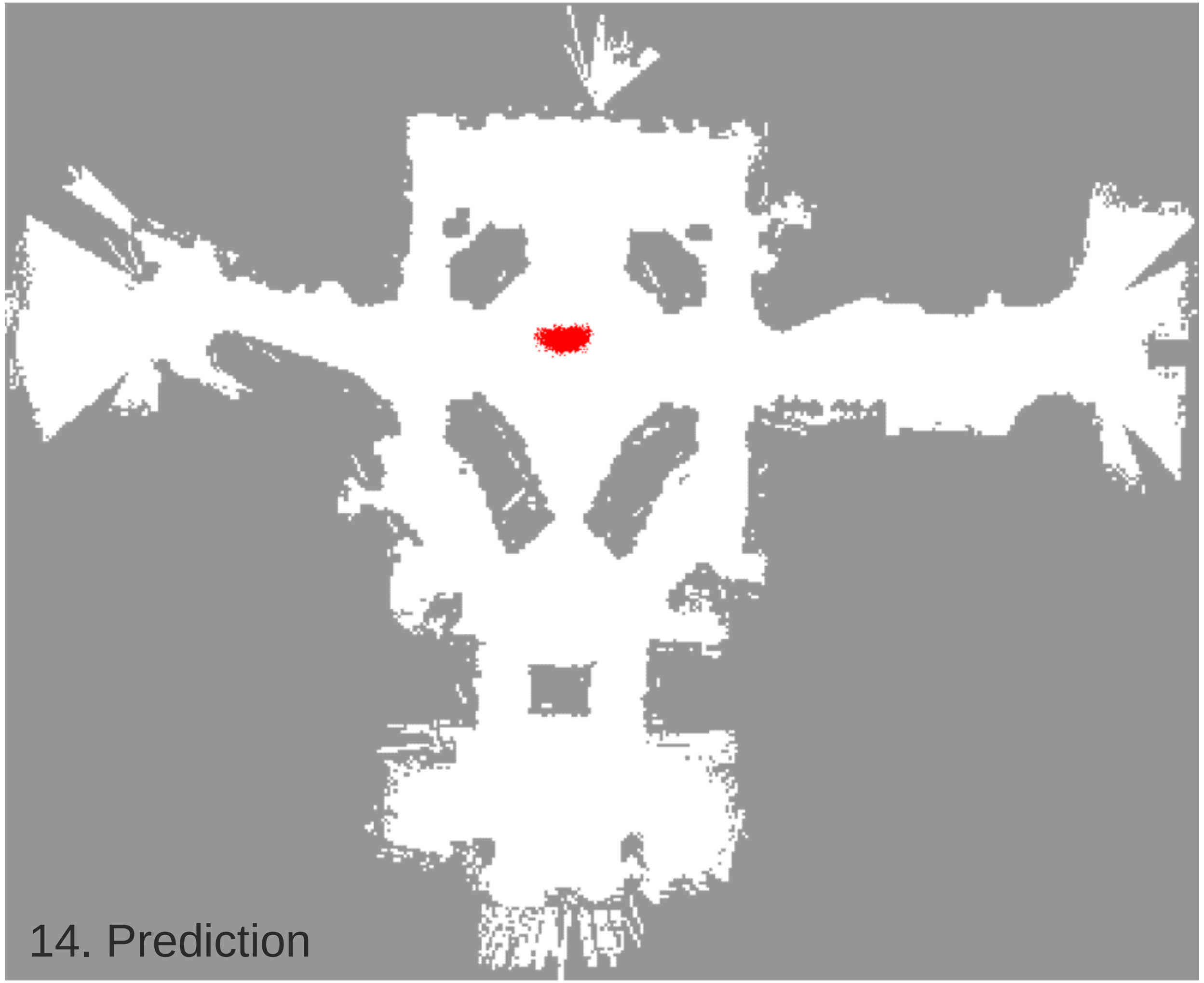

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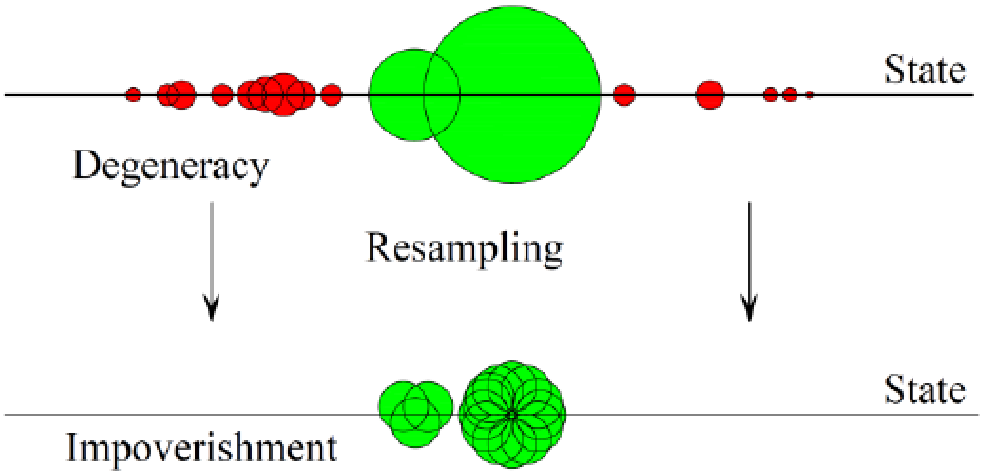
Particle filter in pseudocode

Particle_filter(χ_{t-1}, u_t, z_t)**:**	
$\bar{\chi}_t = \chi_t = \emptyset$	
for $n = 1$ **to** N **do**	
sample $x_t^{(n)} \sim p\left(x_t \mid u_t, x_{t-1}^{(n)}\right)$	prediction step
$w_t^{(m)} = p\left(z_t \mid x_t^{(n)}\right)$	compute weight (correction)
$\bar{\chi}_t \leftarrow \left[x_t^{(n)}, w_t^{(m)}\right]$	keep list of weighted original particles
r = rand(0, 1/N)	low variance resampling algorithm
$c = w_t^{(1)}, i = 1$	
for $n = 1$ **to** N **do**	
$u = r + (n-1)/N$	
while $u > c$	
$c = c + w_t^{(++i)}$	
$\chi_t \leftarrow x_t^{(i)}$	
return χ_t	

Particle degeneracy

- Each resampling step results in so-called particle degeneracy and impoverishment
- Particles with low probability are eliminated while particles with large weights are exist in too many copies
- How to address the degeneracy issue?
  - It is essential to increase the variance of the particle set
  - Intensify noise in the prediction step
  - Do not resample in each iteration
  - Add Gaussin noise in the resampling step (= regularized particle filters [2])



[1]

$$x_t^{(i)} = x_t^{(i)} + h\Gamma_t \varepsilon$$

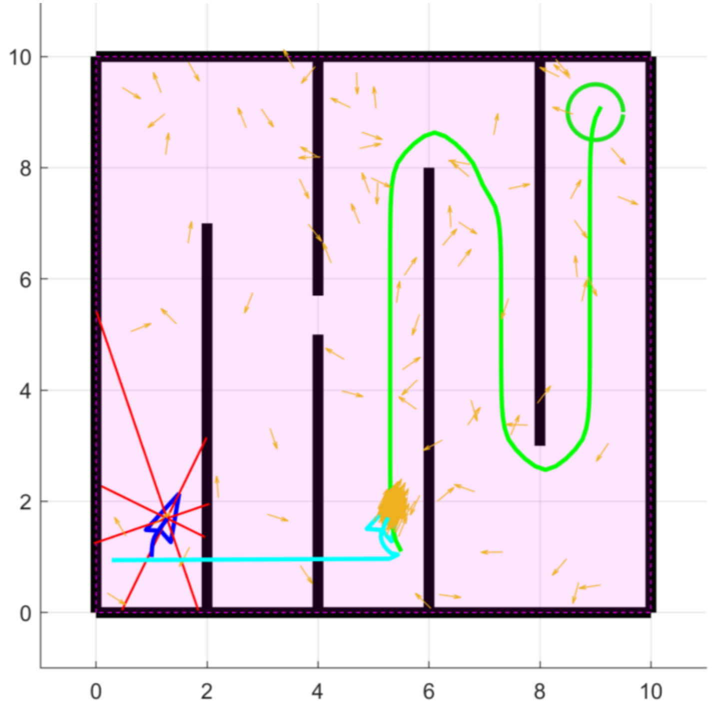
h … Bandwidth
Γ_t … Square root of empirical covariance matrix
ε … Random vector drawn from Gaussian kernel

1. LI, Tiancheng et al. Fight sample degeneracy and impoverishment in particle filters: A review of intelligent approaches. *Expert Systems with Applications* [online]. 2014, **41**(8) [cit. 2021-02-21]. DOI: 10.1016/j.eswa.2013.12.031
2. MUSSO, Christian et al. Improving Regularised Particle Filters. *Sequential Monte Carlo Methods in Practice* [online]. New York, NY: Springer New York, 2001 [cit. 2021-02-21]. DOI: 10.1007/978-1-4757-3437-9_12

Kidnapped robot problem

[1]

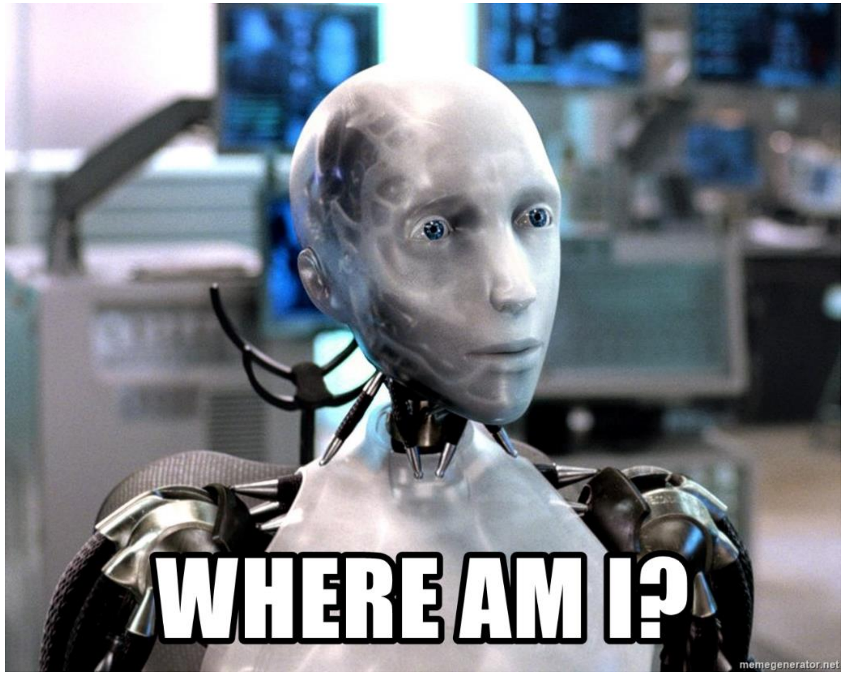
- Robot is relocated by some unpredictable intervention
- The localitazion algorithm converges to a wrong location due to similarity of different parts of a map
- Tests the robustness of the localization solution
- Possible approaches for the particle filter:
  - Assessing the quality of localization
    - Sample covariance matrix
    - Error function (unnormalized weights)
  - Reset the filter (re-initialize the particles)
  - Injection of random particles
    - Fixed rate
    - Adaptive rate



1. https://cz.pinterest.com/pin/497507090058081626/ [cit. 2024-02-20]

Summary

- Particle filter
  - Non-parametric recursive Bayes filter
  - Approximates the posterior by weighted samples
  - Can handle non-Gaussian PDFs and non-linear transitions
  - Basic principles: Importance sampling and Survival-of-the-fittest
- Monte Carlo localization (MCL)
  - Based on the particle filter
  - Prediction: Applying the motion model to particles
  - Correction: Likelihood of observations
  - Easy implementation
  - Accuracy and robustness depends on the quality of motion and measurement models
  - Standard for mobile robots localization



[1]

1. I, robot [film]. Directed by PROYAS, Alex. USA: 20th Century Fox, 2004.

Tomas Lazna

tomas.lazna@ceitec.vutbr.cz

Brno University of Technology
Faculty of Electrical Engineering and Communication
Department of Control and Instrumentation

Robotics and AI Research Group