



2 – Probabilistic robotics

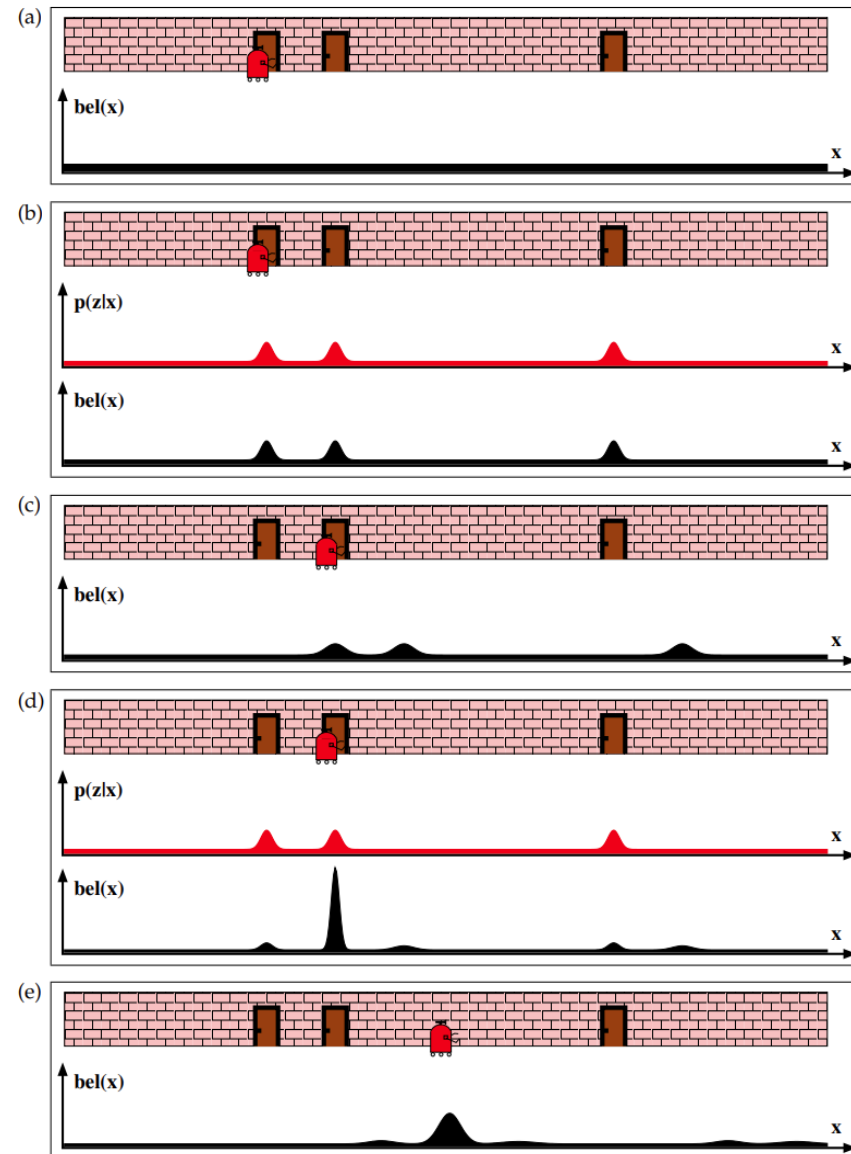
Advanced Methods for Mapping and Self-localization in Robotics
MPC-MAP

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Brno University of Technology
2025



Probabilistic robotics

Key ideas of the probabilistic approach

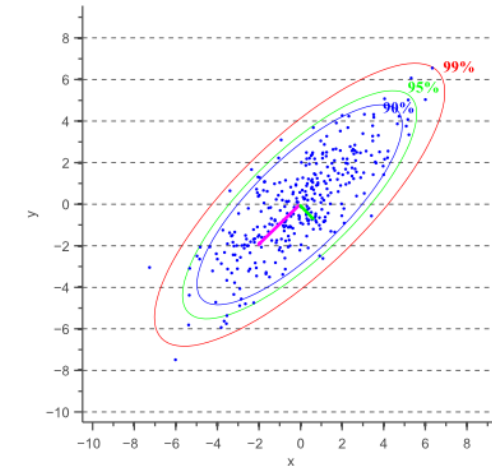


- ◦ ◦ I've a map, but I've absolutely no idea where I am.
- ◦ ◦ Let's use my door sensor! It says I'm probably at a door.
- ◦ ◦ Ok, I'm probably at a door.
- ◦ ◦ Let's go on. I'm getting lost, but I still have some ideas where I could be.
- ◦ ◦ Let's use the sensor again. Great, I'm at another door!
- ◦ ◦ Based on my previous guess and this new information I'm very likely at the second door.
- ◦ ◦ Let's go on. I'm getting little lost again, but I still know where I probably am.

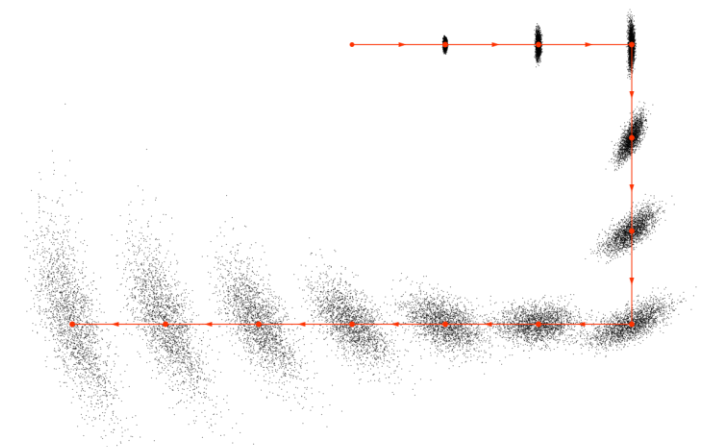
[1]



- **Probabilistic approach – nothing is deterministic, all phenomena are stochastic**
- Uncertainty is in both:
 - **Perception** – sensor/measured data are not 100 % reliable; data from all real sensors are more or less noisy
 - **Action** – robot actions/movements are not 100 % reliable; all actions taken differ to some extent from the action desired
- All uncertain phenomena are **represented statistically**



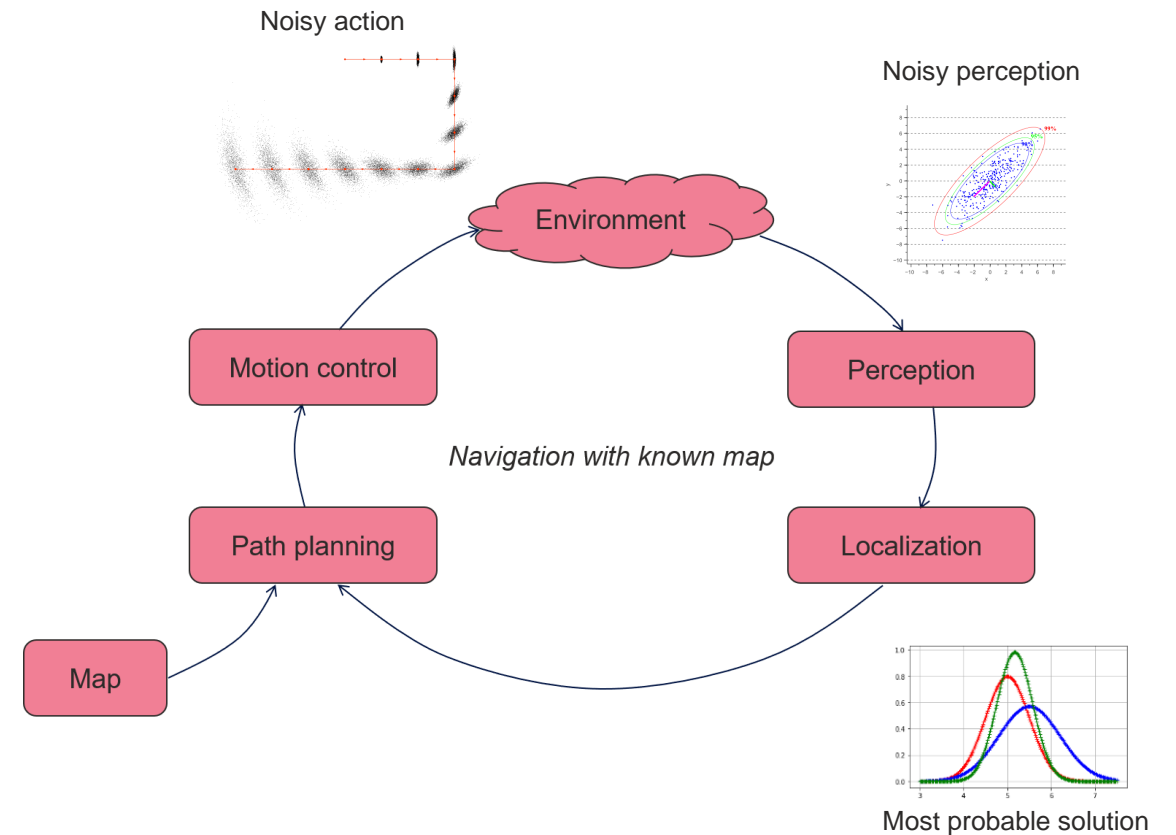
Noisy measurement (e.g., GNSS positioning data) [1]



Noisy motion - propagation of error [2]



- Robot **navigation** in the probabilistic concept **considers the uncertainties** to provide optimal solution.
- This is especially crucial for the **localization** task – what is the **most probable robot position** according **noisy sensor data**? And considering **known uncertain actions**?





Probability

Basic terms and distribution types



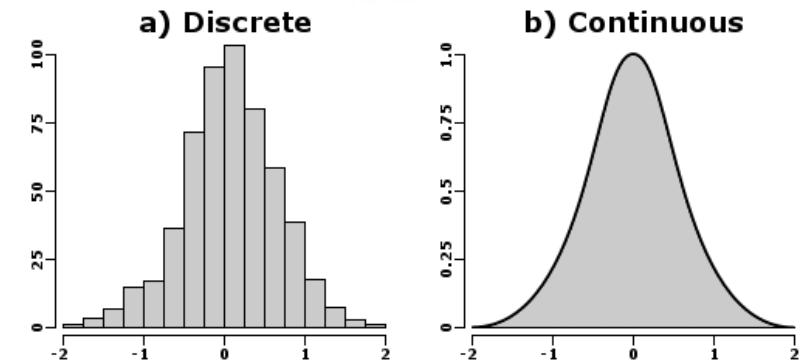
- **Probability** expresses **how likely** a random **event occurs**

- A number between **0 and 1**; higher value → higher probability

- **Probability distribution**

- „Mathematical **function** that gives the **probabilities of occurrence** of different possible **outcomes** for an experiment.“ [1]
 - „Mathematical **description of a random phenomenon** in terms of its **sample space** and the **probabilities** of events.“ [1]

- **Discrete** space → **probability mass function (PMF)**
 - **Continuous** space → **probability density function (PDF)** or cumulative distribution function (CDF)



PMF (left) and PDF (right) examples [1]



- **Probability mass function (PMF)** – probability that a **discrete random variable** is exactly equal to some value

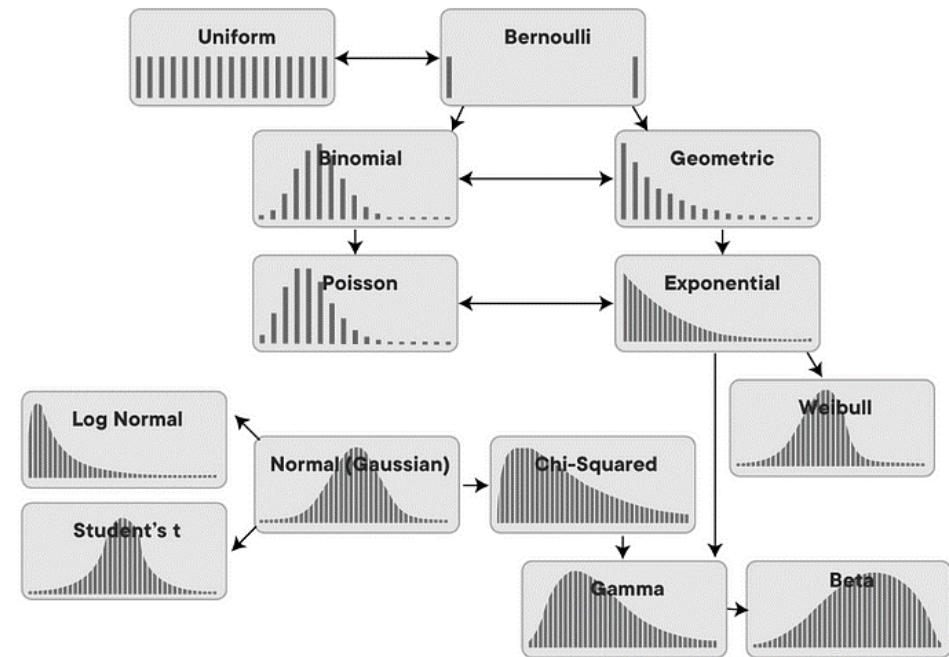
- $p_X(x) = P(X = x)$

- The probabilities sum up to 1

- $\sum_x p_X(x) = 1$

- Values are non-zero, positive

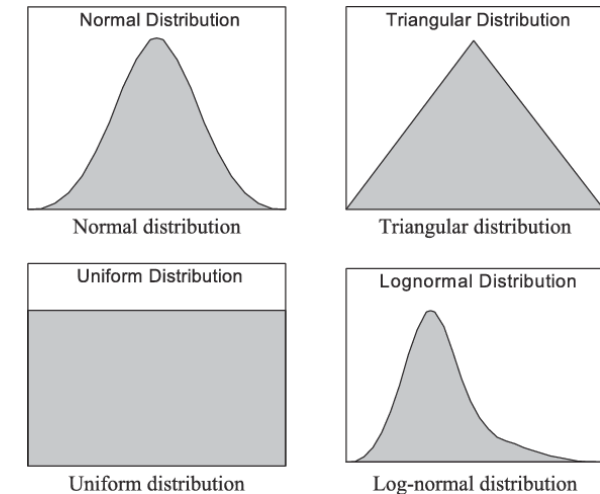
- $p_X(x) \geq 0$



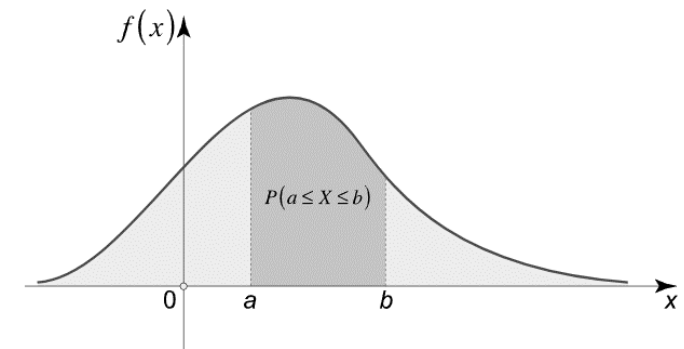
Various probability mass functions (PMF) [1]



- **Probability density function (PDF)** – a function f_X describing the infinitesimal probability of any given value of a **continuous random variable** X
- The **probability** P of a **particular value** equals **0**
 - $P(X) = f_X(x) = 0$
- The **probability** of a **particular range** of value equals to the **integral** of the PDF over the range
 - The area between the PDF and horizontal axis over the range
 - $P(a \leq X \leq b) = \int_a^b f_X(x) dx$
- PDF is nonnegative everywhere, integrates to 1
 - $P(-\infty \leq X \leq \infty) = \int_{-\infty}^{\infty} f_X(x) dx = 1$



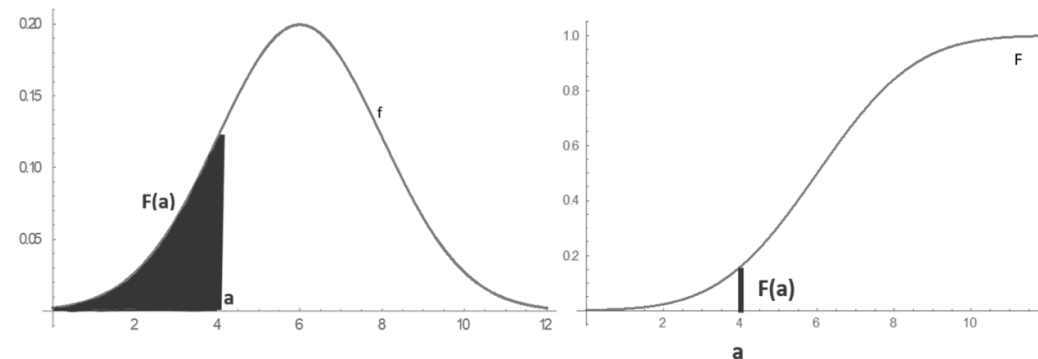
PDF examples [1]



Probability equals to the area under the PDF [2]



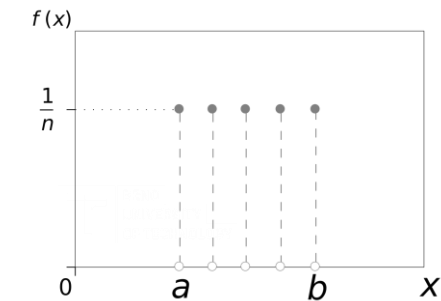
- **Cumulative distribution function (CDF)** – $F_X(x)$ represent a cumulative representation of the PDF
- Describes the probability that the random variable is no larger than a given value
 - The area under the PDF from $-\infty$ to x
 - $F_X(x) = \int_{-\infty}^x f_X(u) du$



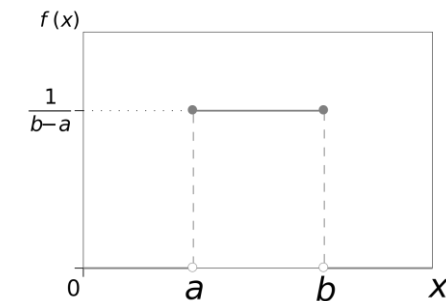
PDF and corresponding CDF [1]



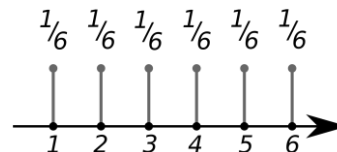
- **Uniform** distribution – all hypotheses will occur with **equal probability**
- **Discrete space** – a finite number of values has equal probability
 - $p(x) = \frac{1}{n} \quad n = b - a + 1 \quad k \in \{a, a + 1, \dots, b - 1, b\} \quad b \geq a$
- **Continuous space** – all intervals of the same length within the bounds are equally probable
 - $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases} \quad -\infty < a < b < \infty$
- Examples
 - Rolling the dice
 - **Initial guess** in the **global localization** task



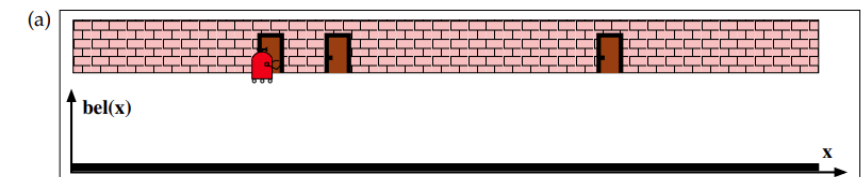
PMF of the discrete uniform distribution example [1]



PDF of the continuous uniform distribution example [2]



Dice rolling – typical discrete uniform distribution

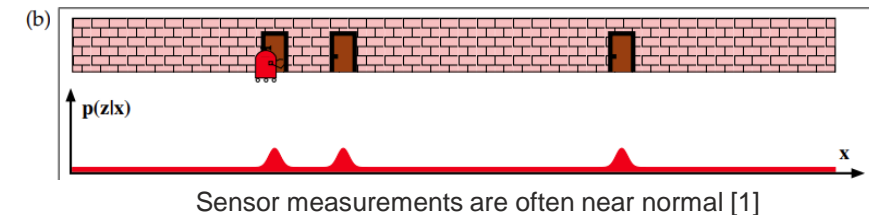
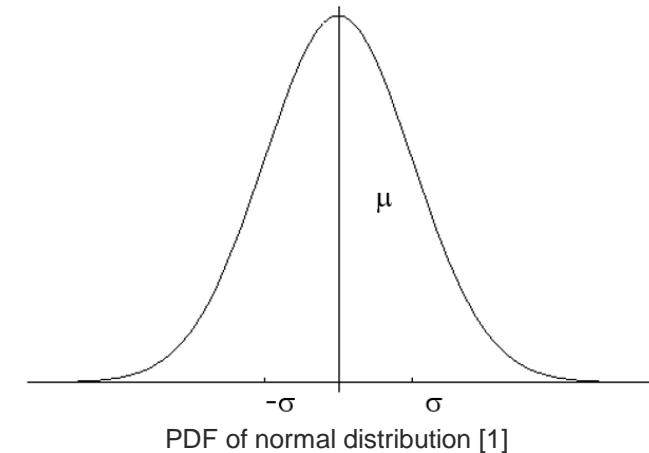


Initial guess with the uniform distribution [3]

1. https://en.wikipedia.org/wiki/Discrete_uniform_distribution
2. https://en.wikipedia.org/wiki/Continuous_uniform_distribution
3. <http://www.probablistic-robotics.org/>

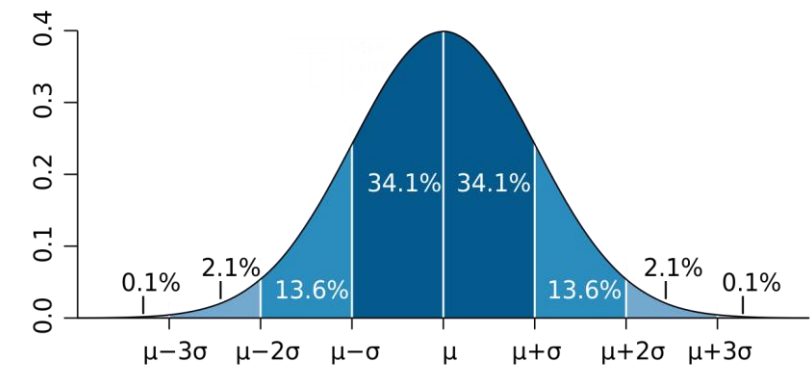


- **Normal** distribution = **Gaussian** distribution – type of **continuous** probability distribution
- Important in natural sciences – measurement errors caused by many independent processes posses nearly normal distribution
- Notation: $p(x) \sim \mathcal{N}(x; \mu, \sigma^2)$
- PDF: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$
- Parametric distribution defined by:
 - **Mean** (expectation) μ
 - **Standard deviation** σ or **variance** σ^2
- Examples
 - Natural random processes
 - Approximate distribution of many **sensors in robotics**





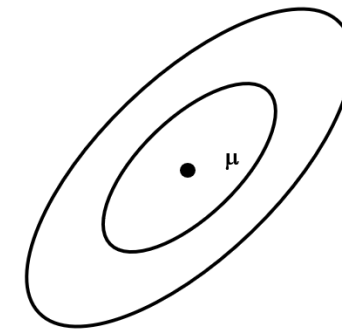
- In the measurement field the uncertainty is often expressed by an interval of possible values occurring with certain probability defined via standard deviation σ
- variable = value \pm interval typically meaning 1σ interval
- Well known probabilities for the normal distribution
 - $P(\mu - \sigma \leq X \leq \mu + \sigma) = 68,3 \%$
 - $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 95,4 \%$
 - $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = 99,7 \%$



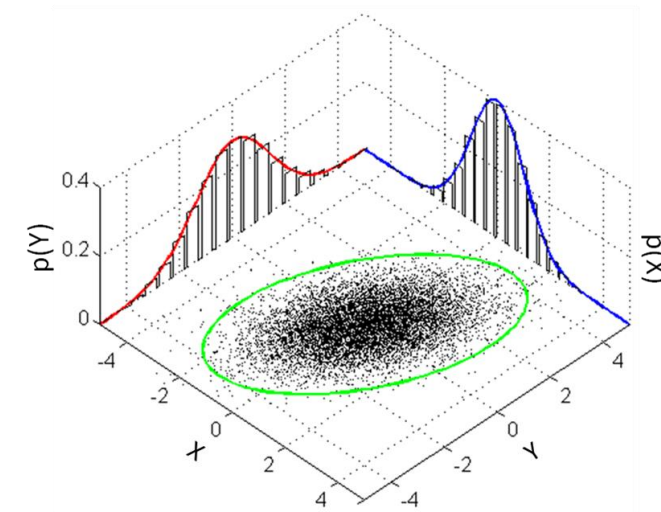
PDF of normal distribution with probabilities for various intervals defined with σ [1]



- Normal distribution over vectors is called ***multivariate*** (x is a vector)
- Often used in localization algorithms when the space vector is multidimensional
- Notation: $p(x) \sim \mathcal{N}(x; \mu, \Sigma)$
- PDF: $p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\{-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)\}$
- Parameters:
 - **Mean vector** (expectation vector) μ
 - **Covariance matrix** Σ



[1]

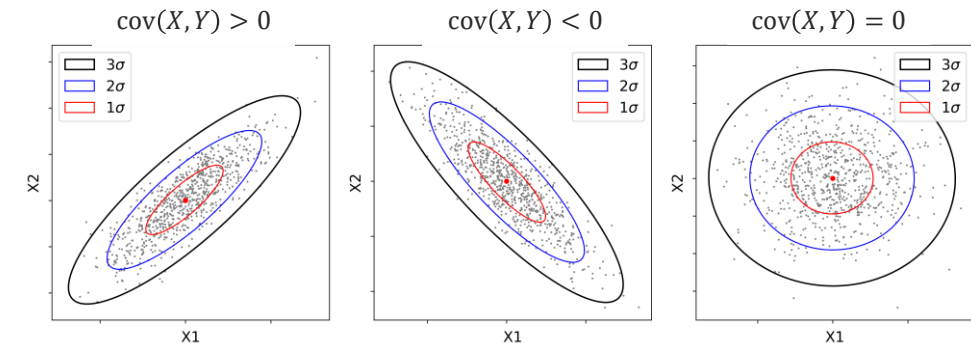


Normal distribution of two variables is often illustrated with ellipse in 2D space [2]



- **Covariance** is a measure of **the joint variability of two random variables**

- The mean of the product of X and Y random variable deviations from their means
- $\text{cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$ expectation $E[X] \approx \text{mean}(X)$
- Alternative notations: σ_{XY} or $\sigma(X, Y)$
- **Covariance with itself is variance**: $\text{cov}(X, X) = \text{var}(X) \equiv \sigma^2(X)$
- Zero covariance means **independent variables**
- Normalized covariance \rightarrow **correlation**



The covariance sign and the „shape“ of data/ellipses [1]

- **Covariance matrix** contains covariance between each pair of elements

- Matrix dimension: dimensionality of the state x squared
- Key matrix properties: square, symmetric, quadratic

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_n^2 \end{pmatrix}$$

- **Main diagonal contains variances**

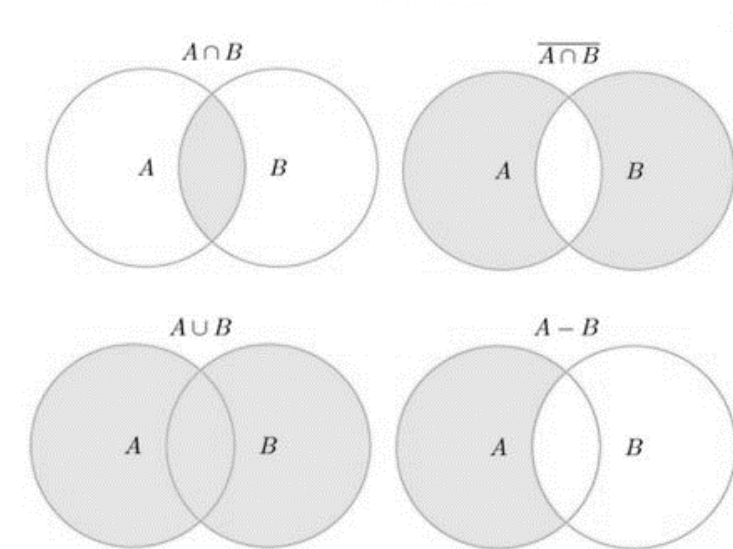
```
process_noise_covariance: [0.05, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0.05, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0.06, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0.03, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0.03, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0.06, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0.025, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0.025, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0.04, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0.01, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.01, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.02, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.01, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.01, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.015]
```

Covariance matrix describing noise characteristics of a process (robot motion) [2]



- The probabilities can be combined in different ways according to the phenomenon we are describing

Event	Probability
A	$P(A) \in [0, 1]$
not A	$P(A^c) = 1 - P(A)$
A or B	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$ $P(A \cup B) = P(A) + P(B)$ if A and B are mutually exclusive
A and B	$P(A \cap B) = P(A B)P(B) = P(B A)P(A)$ $P(A \cap B) = P(A)P(B)$ if A and B are independent
A given B	$P(A B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B A)P(A)}{P(B)}$



- The most important event in the **probabilistic robotics** is **A given B**



Bayes' theorem

The theorem and its application in robotics



- Statistical theory based on the Bayesian interpretation of probability
- **Probability** is based on **prior knowledge** of an event and related **evidence**
- The probability expresses a degree of **belief** in an event

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)} = \frac{P(E|H)P(H)}{P(E|H)P(H) + P(E|\neg H)P(\neg H)} = \eta P(E|H)P(H)$$

- H **hypothesis** whose probability may be affected by evidence
- E **evidence** (observation) – new data that were not used in computing the prior probability
- $P(H)$ **prior** probability – the estimate of the probability of the hypothesis H before the data E
- $P(H|E)$ **posterior** probability – is the probability of H after E is observed
- $P(E|H)$ **likelihood** – probability of observing E given H
- $P(E)$ model evidence, probability of seeing evidence independently on H



Likelihood

Prior probability

Posterior probability

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)}$$

Model evidence



Sensor model

Old estimate

New estimate

$$P(H|E) = \frac{P(E|H)P(H)}{P(E)}$$

Normalization

The diagram illustrates the components of Bayes' theorem. The equation $P(H|E) = \frac{P(E|H)P(H)}{P(E)}$ is centered. Four labels with dashed arrows point to its parts: 'Sensor model' points to $P(E|H)$, 'Old estimate' points to $P(H)$, 'Normalization' points to $P(E)$, and 'New estimate' points to the entire expression $P(H|E)$.



- **Bayes filter** – standard approach for the **state estimation** in robotics
- General algorithm for calculating **belief** from **measurement** z and **control data** u
- **Recursive**, includes the previous **state** x_{t-1}
- Each period includes **prediction** and **correction** step
- Bayes filter forms the basis for widely used algorithms:
 - **Kalman filter** – Gaussian filter, belief is represented via normal distribution
 - **Particle filter** – Monte Carlo filter, belief is represented by a set of particles

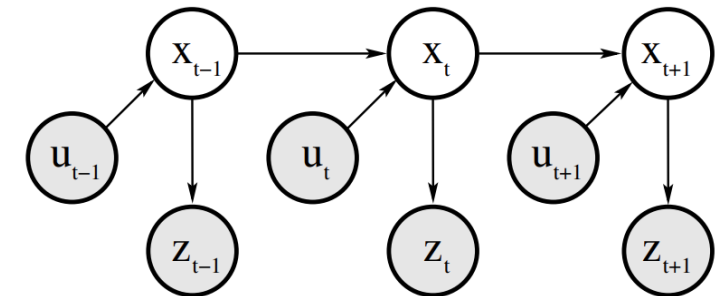
Prediction (control update)

```

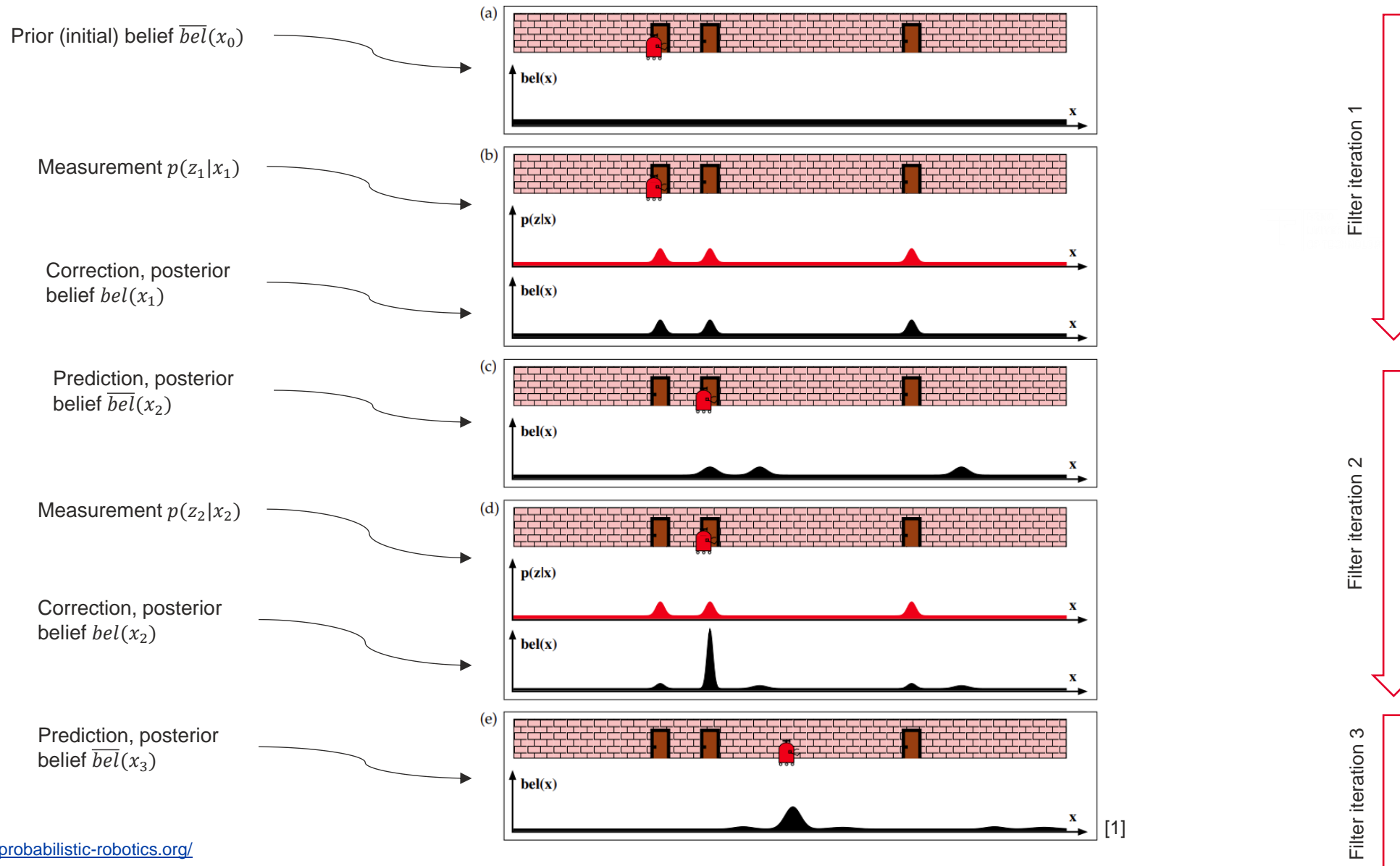
1: Algorithm Bayes_filter( $bel(x_{t-1}), u_t, z_t$ ):
2:   for all  $x_t$  do
3:      $\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx$ 
4:      $bel(x_t) = \eta p(z_t | x_t) \overline{bel}(x_t)$ 
5:   endfor
6:   return  $bel(x_t)$ 

```

Correction (measurement update)



The evolution of controls, states and measurements [1]





Sensors

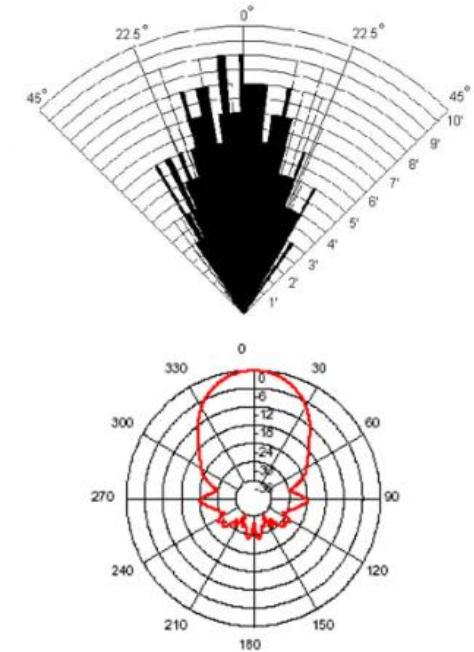
Range and pose sensors



- Provide information about the **distance** and the (eventually) **azimuth** of the object with respect to robot's coordinate system
- The most common types:
 - **LiDAR** – narrow laser beam, time-of-flight
 - Mapping, localization, obstacle detection
 - 1D, 2D and 3D
 - **Ultrasonic** – wide „beam“, time-of-flight
 - Obstacle detection
 - 1D only
 - **Radar** – wide „beam“, time-of-flight, Doppler
 - Obstacle detection, velocity measurement
 - 1D (2D, 3D)



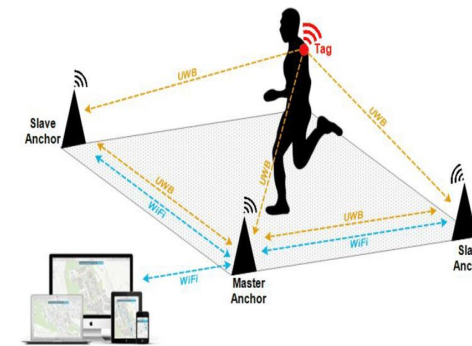
2D LiDAR scan



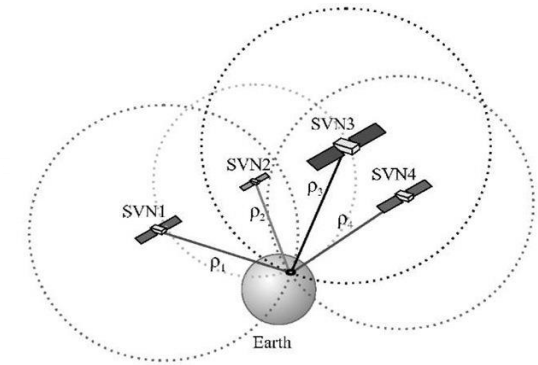
Ultrasonic sensor characteristics



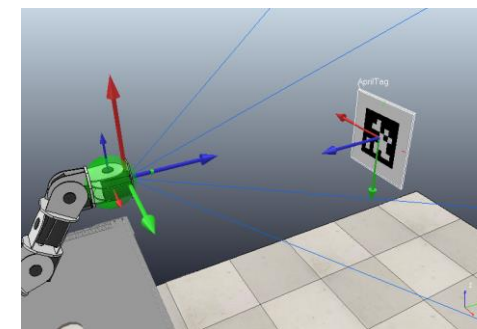
- Determines **position** or **orientation** or both position and orientation – **pose** within an external coordinate system
- The most common types:
 - **GNSS** – Global Navigation Satellite System
 - Global, world-wide localization
 - Position only, orientation – dual-antenna RTK systems
 - **Motion capture and beacon localization**
 - Local localization
 - Based on cameras and reflective markers or on radio beacons and receivers
 - **Fiducial markers**
 - Local localization
 - Relative pose of graphical marker and camera



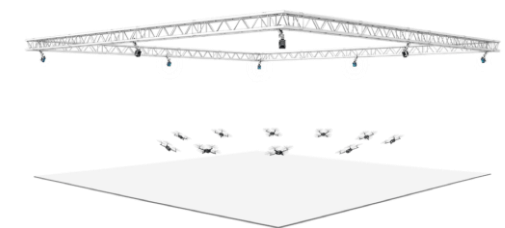
Beacon localization (UWB, BT, ...) [3]



GNSS [1]



Fiducial markers/tags [4]



Motion Capture [2]



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