Week 7 - Matrix analysis

Michel Cancalon - 325002

May 2, 2024

1 Exercise 1

1. Let's study the simple case where each couple produces 4 children.
    - Couple AA - AA : 4 children with genotype AA
    - Couple Aa - Aa : 1 child with genotype AA, 2 children with genotype Aa, 1 child with genotype aa
    - Couple aa - aa : 4 children with genotype aa

    As such we can see that the next generation will have:
    - 5 children with genotype AA: 4 from the AA population and 1 from the Aa population
    - 2 children with genotype Aa: 2 from the Aa population
    - 5 children with genotype aa: 4 from the aa population and 1 from the Aa population

    This leads us to the following matrix which represents the population increase after one generation:

    $$\begin{pmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{pmatrix}$$

    However, we want to know the population proportion after one generation. As such we need to normalize the matrix so that given $\alpha_n + \beta_n + \gamma_n = 1$, we must have the sum of each population of the next generation also equal to 1. This leads us to the following calculation:

    $$\frac{1}{N} \cdot \begin{pmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{pmatrix} = \frac{1}{N} \cdot \begin{pmatrix} 4\alpha_n + \beta_n \\ 2\beta_n \\ \beta_n + 4\gamma_n \end{pmatrix} \iff \frac{1}{N}(4\alpha_n + \beta_n + 2\beta_n + \beta_n + 4\gamma_n) = 1 \iff N = 4$$

    As such, the normalized matrix is:

    $$\boxed{M = \begin{pmatrix} 1 & 0.25 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0.25 & 1 \end{pmatrix}}$$

2. Given the recursive formula $x_n = Mx_{n-1}$, for n generations, we have:

    $$\boxed{x_n = M^n x_0}$$

3. Let's first find the eignevalues of the matrix M:

    $$\det(M - \lambda I) = \begin{vmatrix} 1-\lambda & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} - \lambda & 0 \\ 0 & \frac{1}{4} & 1-\lambda \end{vmatrix}$$
    $$= (1-\lambda)^2 \left(\frac{1}{2} - \lambda\right)$$

We find $\lambda_1 = 1$ and $\lambda_2 = \frac{1}{2}$.
Let's now find the eigenvectors associated with these eigenvalues:

- For $\lambda_1 = 1$:

$$(M - I)v_1 = 0$$
$$\begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix} v_1 = 0$$

  We find $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

- For $\lambda_2 = \frac{1}{2}$:

$$\left(M - \frac{1}{2}I\right) v_3 = 0$$
$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} v_3 = 0$$

  We find $v_3 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

4. We first notice that if we diagonalize the matrix by writting $M = PDP^{-1}$, we have:

$$x_n = PDP^{-1}PDP^{-1}PDP^{-1}...PDP^{-1}x_0 = PD^nP^{-1}x_0$$

Let's diagonalize the matrix M:

$$P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0.5 & 0 \end{pmatrix}$$

As such, we have:

$$D = P^{-1}MP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}$$

We can now easily calculate x_n:

$$\begin{aligned} x_n &= PD^nP^{-1}x_0 \\ &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 0.5^n \end{pmatrix} \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0.5 & 0 \end{pmatrix} x_0 \\ &= \begin{pmatrix} 1 & 0.5 \cdot (1 - 0.5^n) & 0 \\ 0 & 0.5^n & 0 \\ 0 & 0.5 \cdot (1 - 0.5^n) & 1 \end{pmatrix} \end{aligned}$$

For the case where $n = 10$, we have:

$$\boxed{x_{10} = \begin{pmatrix} 1 & 0.5 \cdot (1 - 0.5^{10}) & 0 \\ 0 & 0.5^{10} & 0 \\ 0 & 0.5 \cdot (1 - 0.5^{10}) & 1 \end{pmatrix} \begin{pmatrix} 1/8 \\ 3/4 \\ 1/8 \end{pmatrix} = \begin{pmatrix} \frac{1}{8}(1 + 3\frac{1023}{1024}) \\ \frac{3}{4 \cdot 1024} \\ \frac{1}{8}(1 + 3\frac{1023}{1024}) \end{pmatrix}}$$

5. As n goes to infinity, we have:

$$\lim_{n\to\infty} x_n = \begin{pmatrix} 1 & 0.5\cdot(1-0.5^n) & 0 \\ 0 & 0.5^n & 0 \\ 0 & 0.5\cdot(1-0.5^n) & 1 \end{pmatrix} \begin{pmatrix} 1/8 \\ 3/4 \\ 1/8 \end{pmatrix} = \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0.5 & 1 \end{pmatrix} \begin{pmatrix} 1/8 \\ 3/4 \\ 1/8 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0 \\ 0.5 \end{pmatrix}$$

   This makes sense because we saw that at the next generation, population β decreased while α and γ increased equally.

6.

2 Exercise 2

1. Given A with elements $a_{i,j}$ and the matrix D with elements $d_{i,j}$.
   Let's first write $L = D - A$, we see that L has the form:

$$l_{i,j} = \begin{cases} d_{i,i} = \text{Amount of edges connected to node } i & \text{if } i = j \\ -a_{i,j} = -1 & \text{if } i \neq j \text{ and there is an edge between node } i \text{ and } j \\ -a_{i,j} = 0 & \text{if } i \neq j \text{ and there is no edge between node } i \text{ and } j \end{cases}$$

   Now let's consider $\tilde{L} = BB^T$, the element $l_{i,j}$ is given by:

$$\tilde{l}_{i,j} = \sum_{m=1}^{|E|} b_{i,m} b_{j,m} \tag{1}$$

   For the matrix B, we see that each element $b_{i,j}$ with $i \in \{1, 2, ..., |V|\}$ and $j \in \{1, 2, ..., |E|\}$ describes whether the node i in relationship to a edge j is:

   - Not connected: 0
   - Connected and on the ass of the arrow: -1
   - Connected and on the pointy bit of the arrow: 1

   So, if we sum the absolute value of all elements of a line i in the matrix B, we get the total amount of edges connected to the node i.
   Looking at equation (1), we see that this is exactly what the elements of the diagonal achieves:

$$\tilde{l}_{i,i} = \sum_{m=1}^{|E|} b_{i,m} b_{i,m} = \text{Amount of edges connected node } i$$

   Let's now consider elements $\tilde{l}_{i,j}$ with $i \neq j$ (those who are not on the diagonal).
   We can see that $b_{i,m} b_{j,m}$ represents the statements:

   - Node i and j are not connected through edge m: 0
   - Node i goes into j or vice versa, through edge m: -1

   ($b_{i,m} b_{j,m} \neq 1$ because a edge can not be bidirectional)
   Now if we sum $b_{i,m} b_{j,m}$ i.e we calculate $\tilde{l}_{i,j} = \sum_{m=1}^{|E|} b_{i,m} b_{j,m}$, we see that $\tilde{l}_{i,j} \in \{-1, 0\}$ because for two nodes i and j, there can be at most one edge that passes between them.
   So we conclude that:

$$\tilde{l}_{i,j} = \begin{cases} \text{Amount of edges connected to node } i & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and there is an edge between node } i \text{ and } j \\ 0 & \text{if } i \neq j \text{ and there is no edge between node } i \text{ and } j \end{cases}$$

   We see that the expressions of the elements of L and $\tilde{L}$ match so $BB^T = D - A$.

2. Let's first show that $\lambda_1 = 0$ is an eigenvalue of L.
We know that for such a eigenvalue, the eigenvectors must satisfy the equation

$$Lv = \lambda_1 v_1 \iff (D - A)v_1 = 0$$

Let's consider elements $a_{i,j}$ of the matrix A, we know that $a_{i,j} = 1$ if there is an edge between node i and j and $a_{i,j} = 0$ otherwise. This means that if we sum all elements along line i in the matrix A, we get the total amount of edges connected to the node i.
Now lets consider the elements $d_{i,i}$ of the matrix D, we know that $d_{i,i}$ is the total amount of edges connected to the node i.
As such we deduce that if we sum all elements $l_{i,j} = d_{i,j} - a_{i,j}$ along a line i in the matrix L, the result is 0.
Summing all elements of along a line of L corresponds to the dot product of the line with a vector of ones. This means we respect the following equation:

$$(D - A)\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = \begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$$

This means we have found that the vector of ones is an eigenvector of L associated with the eigenvalue $\lambda_1 = 0$.

Given that $L = BB^T$, the following also holds true:

$$\sqrt{\lambda_1 L} = \sqrt{0 \cdot BB^T} = 0$$

This is exactly the definition of a singular value of B, so we have shown that $\lambda_1 = 0$ is also a singular value of B.

3. We know that L is of form:

$$L = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

Given that L is symetric by constrution, we know that it's eigenvectors are orthogonal to the one associated to $\lambda_1 = 0$.
This means that if λ_i $(i \neq 0)$ is another eigenvalue with eigen vector v_i, we have:

$$\sum_{j=1}^{n} v_{i,j} v_{1,j} = \sum_{j=1}^{n} v_{i,j} = 0$$

Now let's develop the eigenvalue equation for $Lv_i = \lambda_i v_i$:

$$\begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix} \begin{pmatrix} v_{i,1} \\ v_{i,2} \\ \vdots \\ v_{i,n} \end{pmatrix} = \lambda_i \begin{pmatrix} v_{i,1} \\ v_{i,2} \\ \vdots \\ v_{i,n} \end{pmatrix}$$

$$\iff$$

$$\begin{pmatrix} v_{i,1}(n-1) - \sum_{j=2}^{n-1} v_{i,j} \\ v_{i,2}(n-1) - \sum_{j=1, j\neq 2}^{n-1} v_{i,j} \\ \vdots \\ v_{i,n}(n-1) - \sum_{j=1}^{n-2} v_{i,j} \end{pmatrix} = \begin{pmatrix} \lambda_i v_{i,1} \\ \lambda_i v_{i,2} \\ \vdots \\ \lambda_i v_{i,n} \end{pmatrix}$$

We see that for a line k in the matrix, we have:

$$v_{i,k}(n-1) - \sum_{j=1, j\neq k}^{n-1} v_{i,j} = \lambda_i v_{i,k}$$