

# Week 7 - Matrix analysis

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## 1 Exercise 1

1. Let's study the simple case where each couple produces 4 children.

- Couple AA - AA : 4 children with genotype AA
- Couple Aa - Aa : 1 child with genotype AA, 2 children with genotype Aa, 1 child with genotype aa
- Couple aa - aa : 4 children with genotype aa

As such we can see that the next generation will have:

- 5 children with genotype AA: 4 from the AA population and 1 from the Aa population
- 2 children with genotype Aa: 2 from the Aa population
- 5 children with genotype aa: 4 from the aa population and 1 from the Aa population

This leads us to the following matrix which represents the population increase after one generation:

$$\begin{pmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{pmatrix}$$

However, we want to know the population proportion after one generation. As such we need to normalize the matrix so that given  $\alpha_n + \beta_n + \gamma_n = 1$ , we must have the sum of each population of the next generation also equal to 1. This leads us to the following calculation:

$$\frac{1}{N} \cdot \begin{pmatrix} 4 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} \alpha_n \\ \beta_n \\ \gamma_n \end{pmatrix} = \frac{1}{N} \cdot \begin{pmatrix} 4\alpha_n + \beta_n \\ 2\beta_n \\ \beta_n + 4\gamma_n \end{pmatrix} \iff \frac{1}{N}(4\alpha_n + \beta_n + 2\beta_n + \beta_n + 4\gamma_n) = 1 \iff N = 4$$

As such, the normalized matrix is:

$$M = \begin{pmatrix} 1 & 0.25 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0.25 & 1 \end{pmatrix}$$

2. Given the recursive formula  $x_n = Mx_{n-1}$ , for n generations, we have:

$$x_n = M^n x_0$$

3. Let's first find the eigenvalues of the matrix M:

$$\begin{aligned}\det(M - \lambda I) &= \begin{vmatrix} 1 - \lambda & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} - \lambda & 0 \\ 0 & \frac{1}{4} & 1 - \lambda \end{vmatrix} \\ &= (1 - \lambda)^2 \left( \frac{1}{2} - \lambda \right)\end{aligned}$$

We find  $\lambda_1 = 1$  and  $\lambda_2 = \frac{1}{2}$ .

Let's now find the eigenvectors associated with these eigenvalues:

- For  $\lambda_1 = 1$ :

$$\begin{aligned}(M - I)v_1 &= 0 \\ \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & 0 \end{pmatrix} v_1 &= 0\end{aligned}$$

We find  $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$  and  $v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

- For  $\lambda_2 = \frac{1}{2}$ :

$$\begin{aligned}\left(M - \frac{1}{2}I\right)v_3 &= 0 \\ \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} v_3 &= 0\end{aligned}$$

We find  $v_3 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

4. We first notice that if we diagonalize the matrix by writting  $M = PDP^{-1}$ , we have:

$$x_n = PDP^{-1}PDP^{-1}PDP^{-1}\dots PDP^{-1}x_0 = PD^nP^{-1}x_0$$

Let's diagonalize the matrix M:

$$P = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow P^{-1} = \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0.5 & 0 \end{pmatrix}$$

As such, we have:

$$D = P^{-1}MP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{pmatrix}$$

We can now easily calculate  $x_n$ :

$$\begin{aligned}x_n &= PD^nP^{-1}x_0 \\ &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 1^n & 0 \\ 0 & 0 & 0.5^n \end{pmatrix} \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 0.5 & 1 \\ 0 & 0.5 & 0 \end{pmatrix} x_0 \\ &= \begin{pmatrix} 1 & 0.5 \cdot (1 - 0.5^n) & 0 \\ 0 & 0.5^n & 0 \\ 0 & 0.5 \cdot (1 - 0.5^n) & 1 \end{pmatrix} x_0\end{aligned}$$

For the case where  $n = 10$ , we have:

$$x_{10} = \begin{pmatrix} 1 & 0.5 \cdot (1 - 0.5^{10}) & 0 \\ 0 & 0.5^{10} & 0 \\ 0 & 0.5 \cdot (1 - 0.5^{10}) & 1 \end{pmatrix} \begin{pmatrix} 1/8 \\ 3/4 \\ 1/8 \end{pmatrix} = \begin{pmatrix} \frac{1}{8}(1 + 3\frac{1023}{1024}) \\ \frac{3}{4\cdot 1024} \\ \frac{1}{8}(1 + 3\frac{1023}{1024}) \end{pmatrix}$$

5. As  $n$  goes to infinity, we have:

$$\lim_{n \rightarrow \infty} x_n = \begin{pmatrix} 1 & 0.5 \cdot (1 - 0.5^n) & 0 \\ 0 & 0.5^n & 0 \\ 0 & 0.5 \cdot (1 - 0.5^n) & 1 \end{pmatrix} \begin{pmatrix} 1/8 \\ 3/4 \\ 1/8 \end{pmatrix} = \begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0.5 & 1 \end{pmatrix} \begin{pmatrix} 1/8 \\ 3/4 \\ 1/8 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0 \\ 0.5 \end{pmatrix}$$

This makes sense because we saw that at the next generation, population  $\beta$  decreased while  $\alpha$  and  $\gamma$  increased equally.

6.

## 2 Exercise 2

1. Given  $A$  with elements  $a_{i,j}$  and the matrix  $D$  with elements  $d_{i,j}$ .

Let's first write  $L = D - A$ , we see that  $L$  has the form:

$$l_{i,j} = \begin{cases} d_{i,i} = \text{Amount of edges connected to node } i & \text{if } i = j \\ -a_{i,j} = -1 & \text{if } i \neq j \text{ and there is an edge between node } i \text{ and } j \\ -a_{i,j} = 0 & \text{if } i \neq j \text{ and there is no edge between node } i \text{ and } j \end{cases}$$

Now let's consider  $\tilde{L} = BB^T$ , the element  $l_{i,j}$  is given by:

$$\tilde{l}_{i,j} = \sum_{m=1}^{|E|} b_{i,m} b_{j,m} \quad (1)$$

For the matrix  $B$ , we see that each element  $b_{i,j}$  with  $i \in \{1, 2, \dots, |V|\}$  and  $j \in \{1, 2, \dots, |E|\}$  describes whether the node  $i$  in relationship to a edge  $j$  is:

- Not connected: 0
- Connected and on the ass of the arrow: -1
- Connected and on the pointy bit of the arrow: 1

So, if we sum the absolute value of all elements of a line  $i$  in the matrix  $B$ , we get the total amount of edges connected to the node  $i$ .

Looking at equation (1), we see that this is exactly what the elements of the diagonal achieves:

$$\tilde{l}_{i,i} = \sum_{m=1}^{|E|} b_{i,m} b_{i,m} = \text{Amount of edges connected node } i$$

Let's now consider elements  $\tilde{l}_{i,j}$  with  $i \neq j$  (those who are not on the diagonal).

We can see that  $b_{i,m} b_{j,m}$  represents the statements:

- Node  $i$  and  $j$  are not connected through edge  $m$ : 0
- Node  $i$  goes into  $j$  or vice versa, through edge  $m$ : -1

( $b_{i,m}b_{j,m} \neq 1$  because a edge can not be bidirectional)

Now if we sum  $b_{i,m}b_{j,m}$  i.e we calculate  $\tilde{l}_{i,j} = \sum_{m=1}^{|E|} b_{i,m}b_{j,m}$ , we see that  $\tilde{l}_{i,j} \in \{-1, 0\}$  because for two nodes  $i$  and  $j$ , there can be at most one edge that passes between them.

So we conclude that:

$$\tilde{l}_{i,j} = \begin{cases} \text{Amount of edges connected to node } i & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ and there is an edge between node } i \text{ and } j \\ 0 & \text{if } i \neq j \text{ and there is no edge between node } i \text{ and } j \end{cases}$$

We see that the expressions of the elements of  $L$  and  $\tilde{L}$  match so  $BB^T = D - A$ .

2. Let's first show that  $\lambda_1 = 0$  is an eigenvalue of  $L$ .

We know that for such a eigenvalue, the eigenvectors must satisfy the equation

$$Lv = \lambda_1 v \iff (D - A)v = 0$$

Let's consider elements  $a_{i,j}$  of the matrix  $A$ , we know that  $a_{i,j} = 1$  if there is an edge between node  $i$  and  $j$  and  $a_{i,j} = 0$  otherwise. This means that if we sum all elements along line  $i$  in the matrix  $A$ , we get the total amount of edges connected to the node  $i$ .

Now lets consider the elements  $d_{i,i}$  of the matrix  $D$ , we know that  $d_{i,i}$  is the total amount of edges connected to the node  $i$ .

As such we deduce that if we sum all elements  $l_{i,j} = d_{i,i} - a_{i,j}$  along a line  $i$  in the matrix  $L$ , the result is 0.

Summing all elements of along a line of  $L$  corresponds to the dot product of the line with a vector of ones. This means we respect the following equation:

$$(D - A) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This means we have found that the vector of ones is an eigenvector of  $L$  associated with the eigenvalue  $\lambda_1 = 0$ .

Given that  $L = BB^T$ , the following also holds true:

$$\sqrt{\lambda_1}L = \sqrt{0 \cdot BB^T} = 0$$

This is exactly the definition of a singular value of  $B$ , so we have shown that  $\lambda_1 = 0$  is also a singular value of  $B$ .

3. We know that  $L$  is of form:

$$L = \begin{pmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{pmatrix}$$

Given that  $L$  is symetric by constrution, we know that it's eigenvectors are orthogonal.