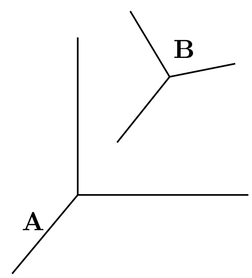
Pre- and Postmultiplication by Transformation Operators

For simplicity this discussion will be restricted to rotational operators, but translation and transformation (rotation + translation) operators behave in exactly the same manner. Consider a rotational *operator* designated simply by matrix $\mathbf{R}$. The columns and rows of $\mathbf{R}$ are unit vectors as we have seen before:

$$\mathbf{R} = \begin{bmatrix} \hat{\mathbf{X}} & \hat{\mathbf{Y}} & \hat{\mathbf{Z}} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{X}}^T \\ \hat{\mathbf{Y}}^T \\ \hat{\mathbf{Z}}^T \end{bmatrix}$$

Consider frames **A** and **B** as shown in the illustration below.



The matrix *describing* frame **B** relative to frame **A** is ${}^A_B\mathbf{R}$ whose three columns are ${}^A_B\mathbf{R} = \begin{bmatrix} {}^A\hat{\mathbf{X}}_B & {}^A\hat{\mathbf{Y}}_B & {}^A\hat{\mathbf{Y}}_B \end{bmatrix}$ and whose three rows are $\begin{bmatrix} {}^B\hat{\mathbf{X}}_A^T \\ {}^B\hat{\mathbf{Y}}_A^T \\ {}^B\hat{\mathbf{Z}}_A^T \end{bmatrix}$.

Premultiplying: First we will premultiply ${}^A_B\mathbf{R}$ by operator $\mathbf{R}$. This can be stated as

$$\mathbf{R}\,{}^A_B\mathbf{R} = {}_A\mathbf{R}\,{}^A_B\mathbf{R} = \underbrace{\begin{bmatrix} {}^A\hat{\mathbf{X}}^T \\ {}^A\hat{\mathbf{Y}}^T \\ {}^A\hat{\mathbf{Z}}^T \end{bmatrix}}_{\text{relative to A axes}} \begin{bmatrix} {}^A\hat{\mathbf{X}}_B & {}^A\hat{\mathbf{Y}}_B & {}^A\hat{\mathbf{Y}}_B \end{bmatrix} \tag{1}$$

In (1) the operator $\mathbf{R}$ has been given the leading superscript A for compatibility in multiplication. This implies that its operation has been relative to the frame A axes.

Postmultiplying: In the same manner consider postmultiplication, which can be stated as

$${}^A_B\mathbf{R}\,\mathbf{R} = {}^A_B\,\mathbf{R}\,{}^B\mathbf{R} = \begin{bmatrix} {}^B\hat{\mathbf{X}}_A^T \\ {}^B\hat{\mathbf{Y}}_A^T \\ {}^B\hat{\mathbf{Z}}_A^T \end{bmatrix} \underbrace{\begin{bmatrix} {}^B\hat{\mathbf{X}} & {}^B\hat{\mathbf{Y}} & {}^B\hat{\mathbf{Y}} \end{bmatrix}}_{\text{relative to B axes}} \tag{2}$$

Just like before, in (2) the operator $\mathbf{R}$ has been given the leading superscript B for compability in multiplication. This implies that its operation has been relative to the frame B axes.

On the next page is an example.

An Example.

Consider frame {A} and {B} as seen in Figure 1 below.

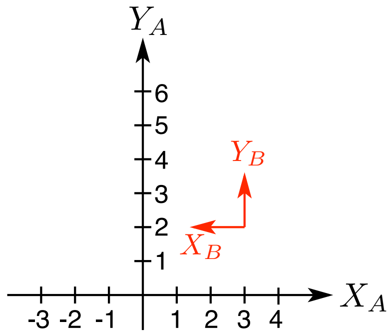


Figure 1: Frames {A} and {B}.

Frame Description. The matrix describing the pose of frame {B} relative to frame {A} is

$$ {}^{A}_{B}T = \begin{bmatrix} -1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

Transformation Operator. Consider a transformation operator consisting of (1) a rotation about the Z axis by $90°$, and a translation along the X axis by 2 units. Construct this transformation operator T by multiplying below:

$$T = R_Z(90°)D_X(2) = \underbrace{\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R_Z(90°)} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{D_X(2)} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

Now we will use this transformation operator T in both pre- and postmultiplication.

1. Premultiplication. When we premultiply ${}^{A}_{B}T$ by operator T we obtain the following result:

$$T\,{}^{A}_{B}T = \begin{bmatrix} 0 & -1 & 0 & -2 \\ -1 & 0 & 0 & 5 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

Both the original position of ${}^{A}_{B}T$ (dashed lines) and the new position of ${}^{A}_{B}T$ (solid lines) are shown in Figure 2 on the next page.

Since we are *premultiplying* by T, both operations will take place along the frame {A} (the "reference" frame) axes, and the first operation to take effect is the *rightmost* operator in equation (2), the X_A translation by 2 units. That is followed by the rotation around Z_A by $90°$. Figure 2 confirms that—frame {B} is first shifted 2 units to the right along the X_A axis, to point (5,2), then rotated $90°$ around the Z_A axis to (-2,5). If these operations had been reversed we would have obtained a quite different result.

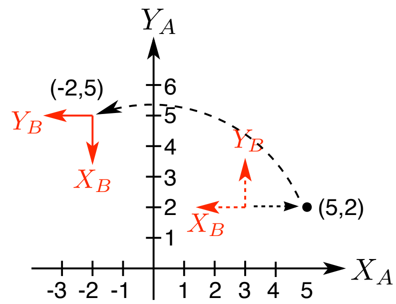


Figure 2: Transformation of ${}^A_BT$ by premultiplication.

2. Postmultiplication. When we postmultiply ${}^A_BT$ by operator T we obtain the following result:

$$
{}^A_BT\,T = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}
$$

Again, both the original position of ${}^A_BT$ (dashed lines) and the new position of ${}^A_BT$ (solid lines) are shown in Figure 3 below. Since we are *postmultiplying* by T, both operations will take place along the frame {B} axes, and the first

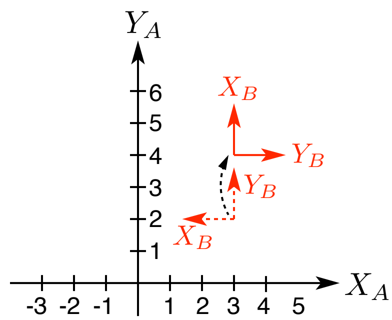


Figure 3: Transformation of ${}^A_BT$ by postmultiplication.

operation to take effect is the *leftmost* operator in equation (2), the Z_B rotation by 90°. That is followed by the X_B translation of 2 units. Examination of Figure 3 confirms this: frame {B} is first rotated CW by 90° "in place" so X_B points upward, then moved up 2 units along X_B.