Initialization & Library Loading

In[1]:=
```
Quiet[Remove["Global`*"]];
Quiet[Remove["SpecialRelativity`*"]];
SetDirectory[NotebookDirectory[]];
Get["SpecialRelativity`"]
```

Hyperbolic Motion Worldline

In[5]:=
```
SetAttributes[α, Constant];
$Assumptions = α > 0 && t > 0 && λ > 0 && τ > 0;

HyperbolicMotion[α_] := τ ↦ mkFourVector[1 / α * Sinh[α τ], 1 / α * (Cosh[α τ] - 1)];
```

In[8]:=
```
HyperbolicMotion[α][λ]
```

Out[8]= $\text{FourVector}\left[\frac{\text{Sinh}[\alpha\,\lambda]}{\alpha}, \frac{-1+\text{Cosh}[\alpha\,\lambda]}{\alpha}, 0, 0\right]$

Proper Time

The motion as defined is already parametrized by proper time

In[9]:=
```
Dtau[HyperbolicMotion[α]][λ] // Simplify
```

Out[9]= Dt[λ]

In[10]:=
```
ProperTime[HyperbolicMotion[α]][λ] // Simplify
```

Out[10]= λ

In[11]:=
```
ProperTimeToFrameTime[HyperbolicMotion[α]][τ] // Simplify
```

Out[11]= $\left\{\frac{\text{Sinh}[\alpha\,\tau]}{\alpha}\right\}$

In[12]:=
```
FrameTimeToProperTime[HyperbolicMotion[α]][t] // Simplify
```

Out[12]= $\left\{\frac{\text{ArcSinh}[t\,\alpha]}{\alpha}\right\}$

Frame Properties

In[13]:=
```
FrameVelocity[HyperbolicMotion[α]][τ]
```

Out[13]= {Tanh[α τ], 0, 0}

```
In[14]:= FrameAcceleration[HyperbolicMotion[α]][τ] // Simplify
```
Out[14]= $\{\alpha\,\mathrm{Sech}[\alpha\,\tau]^3,\ 0,\ 0\}$

Velocity and Acceleration

```
In[15]:= FourVelocity[HyperbolicMotion[α]][λ]
Out[15]= FourVector[Cosh[α λ], Sinh[α λ], 0, 0]

In[16]:= ProperVelocity[HyperbolicMotion[α]][λ] // Simplify
Out[16]= 1

In[17]:= FourAcceleration[HyperbolicMotion[α]][λ]
Out[17]= FourVector[α Sinh[α λ], α Cosh[α λ], 0, 0]

In[18]:= ProperAcceleration[HyperbolicMotion[α]][λ] // Simplify
Out[18]= ⅈ α
```

Frame Time Parametrization

```
In[19]:= FrameTimeHyperbolicMotion[α_] := t ⟼ FrameTimeParametrization[HyperbolicMotion[α]][t][[1]]

In[20]:= FrameTimeHyperbolicMotion[α][t]
```
Out[20]= $\mathrm{FourVector}\left[t,\ \frac{-1+\sqrt{1+t^2\,\alpha^2}}{\alpha},\ 0,\ 0\right]$

```
In[21]:= Dtau[FrameTimeHyperbolicMotion[α]][t]
```
Out[21]= $\sqrt{\frac{1}{1+t^2\,\alpha^2}}\ \mathrm{Dt}[t]$

```
In[22]:= ProperTime[FrameTimeHyperbolicMotion[α]][t]
```
Out[22]= $\frac{\mathrm{ArcSinh}[t\,\alpha]}{\alpha}$

Time relationship stays the same irrespective of parametrization :

```
In[23]:= ProperTimeToFrameTime[FrameTimeHyperbolicMotion[α]][τ] // Simplify
```
Out[23]= $\left\{\frac{\mathrm{Sinh}[\alpha\,\tau]}{\alpha}\right\}$

In[24]:=
```
FrameTimeToProperTime[FrameTimeHyperbolicMotion[α]][t] // Simplify
```

Out[24]= $\left\{\frac{\text{ArcSinh}[t\,\alpha]}{\alpha}\right\}$

In[25]:=
```
FourVelocity[FrameTimeHyperbolicMotion[α]][t] // Simplify
```

Out[25]= $\text{FourVector}\left[\sqrt{1+t^2\,\alpha^2},\ t\,\alpha,\ 0,\ 0\right]$

In[26]:=
```
FourAcceleration[FrameTimeHyperbolicMotion[α]][t]
```

Out[26]= $\text{FourVector}\left[t\,\alpha^2,\ \alpha\,\sqrt{1+t^2\,\alpha^2},\ 0,\ 0\right]$

In[27]:=
```
ProperAcceleration[FrameTimeHyperbolicMotion[α]][t] // Simplify
```

Out[27]= $i\,\alpha$

In[28]:=
```
ProperTimeParametrization[FrameTimeHyperbolicMotion[α]][τ] // Simplify
```

Out[28]= $\left\{\text{FourVector}\left[\frac{\text{Sinh}[\alpha\,\tau]}{\alpha},\ \frac{-1+\text{Cosh}[\alpha\,\tau]}{\alpha},\ 0,\ 0\right]\right\}$

Local Frame:

In[29]:=
```
FrameVelocity[FrameTimeHyperbolicMotion[α]][t]
```

Out[29]= $\left\{\frac{t\,\alpha}{\sqrt{1+t^2\,\alpha^2}},\ 0,\ 0\right\}$

In[30]:=
```
FrameAcceleration[FrameTimeHyperbolicMotion[α]][t] // Simplify
```

Out[30]= $\left\{\frac{\alpha}{\left(1+t^2\,\alpha^2\right)^{3/2}},\ 0,\ 0\right\}$

Hyperbolic Motion As Solution of Equation of Motion with Constant Acceleration

In[31]:=
```
Generic1DMotion[x_] := t ⟼ mkFourVector[t, x[t]]
```

In[32]:=
```
ProperAcceleration[Generic1DMotion[x]][t] // Simplify
```

Out[32]= $\sqrt{\frac{x''[t]^2}{\left(-1+x'[t]^2\right)^3}}$

```
In[33]:= sols = DSolve[{ProperAcceleration[Generic1DMotion[x]][t] == ⅈ α, x[0] == 0}, x[t], t]
```

Out[33]=

$$\left\{\left\{x[t] \to \frac{\sqrt{1-\mathbb{c}_1^2} - \sqrt{1+t^2\alpha^2 - 2\,ⅈ\,t\,\alpha\,\mathbb{c}_1 - \mathbb{c}_1^2}}{\alpha}\right\},\right.$$

$$\left\{x[t] \to \frac{-\sqrt{1-\mathbb{c}_1^2} + \sqrt{1+t^2\alpha^2 - 2\,ⅈ\,t\,\alpha\,\mathbb{c}_1 - \mathbb{c}_1^2}}{\alpha}\right\},$$

$$\left\{x[t] \to \frac{\sqrt{1-\mathbb{c}_1^2} - \sqrt{1+t^2\alpha^2 + 2\,ⅈ\,t\,\alpha\,\mathbb{c}_1 - \mathbb{c}_1^2}}{\alpha}\right\},$$

$$\left.\left\{x[t] \to \frac{-\sqrt{1-\mathbb{c}_1^2} + \sqrt{1+t^2\alpha^2 + 2\,ⅈ\,t\,\alpha\,\mathbb{c}_1 - \mathbb{c}_1^2}}{\alpha}\right\}\right\}$$

```
In[34]:= ConstantAccelerationSolutions =
           Function[t, (mkFourVector[t, x[t]]) /. # & /@ (sols /. {C[1] → 0})];

In[35]:= ConstantAccelerationSolutions[t]
```

Out[35]=

$$\left\{\text{FourVector}\left[t, \frac{1-\sqrt{1+t^2\alpha^2}}{\alpha}, 0, 0\right], \text{FourVector}\left[t, \frac{-1+\sqrt{1+t^2\alpha^2}}{\alpha}, 0, 0\right],\right.$$

$$\left.\text{FourVector}\left[t, \frac{1-\sqrt{1+t^2\alpha^2}}{\alpha}, 0, 0\right], \text{FourVector}\left[t, \frac{-1+\sqrt{1+t^2\alpha^2}}{\alpha}, 0, 0\right]\right\}$$