On the Non-Existence of Kervaire Invariant One Manifolds

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Main Result

Theorem (H.-Hopkins-Ravenel)

There are smooth Kervaire invariant one manifolds only in dimensions 2, 6, 14, 30, 62, *and maybe* 126.

Exemplars:

- 2: $S^1 \times S^1$
- 6: $SU(2) \times SU(2)$
- 14: $S(\mathbb{O}) \times S(\mathbb{O})$
- 30: (Bökstedt) Related to $E_6/\big(U(1) \times Spin(10)\big)$
- 62: Possibly a similar construction.

Geometry and History

1930s Pontryagin proves

$\{\text{framed } n-\text{manifolds}\}/\text{cobordism} \cong \pi_n^S$.

Tries to use surgery to reduce to spheres & misses an obstruction.

1950s Kervaire-Milnor show can always reduce to case of spheres

Except possibly in dimension $4k+2$, where there is an obstruction: Kervaire Invariant.

Adams Spectral Sequence

$$[X, Y] \rightsquigarrow \mathrm{Hom}_{\mathcal{A}}\big(H^*(Y), H^*(X)\big)$$

Have a SS with

$$E_2 = \mathrm{Ext}_{\mathcal{A}}\big(H^*(Y), H^*(X)\big)$$

and converging to $[X, Y]$.

- (Adem) $\mathrm{Ext}^1(\mathbb{F}_2, \mathbb{F}_2)$ is generated by classes h_i, $i \geq 0$.
- h_j survives the Adams SS if $\mathbb{R}^{2^j}$ admits a division algebra structure.

Browder's Reformulation

Theorem (Browder 1969)

1. *There are no smooth Kervaire invariant one manifolds in dimensions not of the form $2^{j+1}-2$.*
2. *There is such a manifold in dimension $2^{j+1}-2$ iff h_j^2 survives the Adams spectral sequence.*

Adams showed that h_j itself survives only if $j < 4$

$$d_2(h_{j+1}) = h_0 h_j^2.$$

Previous Progress

h_1^2, h_2^2, and h_3^2 classically exist.

Theorem (Mahowald-Tangora)

The class h_4^2 survives the Adams SS.

Theorem (Barratt-Jones-Mahowald)

The class h_5^2 survives the Adams SS.

Theorem (H.-Hopkins-Ravenel)

For $j \geq 7$, h_j^2 does not survive the Adams SS.

General Outline

There are four main steps

1. Reduce to a simpler homotopy computation which faithfully sees the Kervaire classes
2. Rigidify the problem to get more structure and less wiggle-room
3. Show homotopy is automatically zero in dimension -2
4. Show homotopy is periodic with period 2^8

Reduction to Simpler Cases

Adams-Novikov SS

"More initial"
More complicated Ext

HFP SS

Algebraically simple
$H^*(\mathbb{Z}/8; R)$

Adams SS

Contains our classes
$\mathrm{Ext}_{\mathcal{A}}(\mathbb{F}_2, \mathbb{F}_2)$

Benefits of Reduction

Reduction is purely algebraic!

1. Lifting from Adams to Adams-Novikov is well understood.
2. Reduction from Adams-Novikov to homotopy fixed points is formal deformation theory.

So good choice of R gives us something that is

- easily computable
- strong enough to detect the classes.

Why Go Equivariant?

- Homotopy fixed point spectral sequence is still too complicated.
- Simplify computation by adding extra structure: equivariance.
- Here have fixed points, rather than homotopy fixed points.
- And there are spheres for every real representation.

Example

If $G = \mathbb{Z}/2$, then have $S^{\rho_2} = \mathbb{C}^+$ and S^2.

Important Representations

Focus now on $G = \mathbb{Z}/8$.
$RO(\mathbb{Z}/8)$ is rank 5 over $\mathbb{Z}$, generated by 1-dim reps:

- trivial rep 1
- sign rep σ

and 2-dim reps: $L = \mathbb{C}, L^2, L^3$.
We care only about $\rho_8 = 1 \oplus \sigma \oplus L \oplus L^2 \oplus L^3$. Plus the regular reps for subgroups.

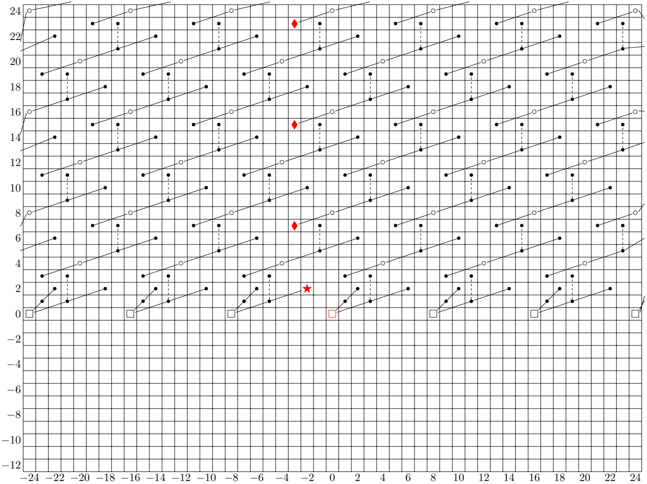
What is R?

1. Begin with MU with $\mathbb{Z}/2$ given by complex conjugation.
2. "induce" up to a $\mathbb{Z}/8$ spectrum:

$$\begin{tikzcd}
\mathrm{MU} \arrow[rr, bend right] & \otimes & \mathrm{MU} \arrow[rr, bend right] & \otimes & \mathrm{MU} \arrow[rr, bend right] & \otimes & \mathrm{MU} \arrow[llllll, bend right, "\overline{(-)}"']
\end{tikzcd}$$

3. The "fixed points" for the $\mathbb{Z}/8$-action is geometric.
4. Inverting an equivariant class Δ makes the fixed points and homotopy fixed points agree.

Advantages of the Slice SS



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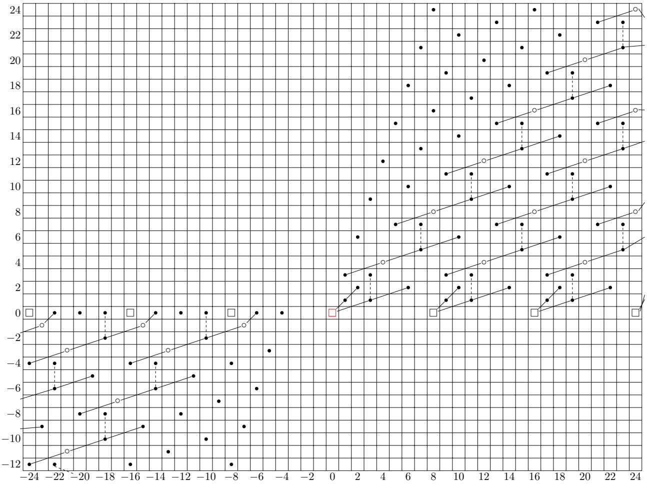


Basic Idea of Slices

Want to decompose X into computable pieces.
Similar to Postnikov tower.
Key difference: don't use all spheres!

Acceptable Ones

1. $S^{k\rho_8}$, $S^{k\rho_8 - 1}$
2. $\mathbb{Z}/8 \otimes_{\mathbb{Z}/4} S^{k\rho_4}$
3. $\mathbb{Z}/8 \otimes_{\mathbb{Z}/2} S^{k\rho_2}$
4. $\mathbb{Z}/8 \otimes S^k$

Unacceptable Ones

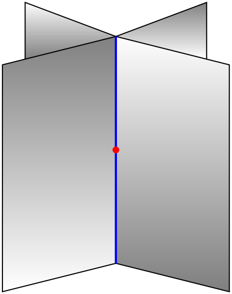
1. $S^{k\rho_8 - 2}$
2. $\mathbb{Z}/8 \otimes_{\mathbb{Z}/4} S^{\sigma}$
3. $\mathbb{Z}/8 \otimes_{\mathbb{Z}/2} S^{\sigma - 1}$
4. S^k

Computing with Slices

Key Fact

For spectra like MU, slices can be computed from equivariant simple chain complexes.

These algebraically describe the fixed points of the acceptable spheres.



Cellular Chains for $S^{\rho_4 - 1}$

Gives the chain complex

$$\mathbb{Z}^4 \to \mathbb{Z}^4 \to \mathbb{Z}^2 \to \mathbb{Z} = C_\bullet.$$

Maps determined by
$H_*(C_\bullet) = \tilde{H}_*(S^3)$.

Gaps

Theorem

For any non-trivial subgroup H of $\mathbb{Z}/8$ and for any slice sphere $\mathbb{Z}/8 \otimes_H S^{\rho_H}$,

$$H_{-2}(C_*^{\mathbb{Z}/8}) = 0$$

The proof is an easy direct computation:

1. If $k \geq 0$, then we are looking at something connected.
2. If $k \leq 0$, then we look at the associated *co*chain algebra.
3. In the relevant degrees, the complex is $\mathbb{Z} \to \mathbb{Z}^2$ by $1 \mapsto (1, 1)$.

Gap Theorem

Theorem

$\pi_{-2}(R) = 0.$

Proof.

- Slices of $MU \otimes MU \otimes MU \otimes MU$ are all of the form

$$H\mathbb{Z} \otimes (\mathbb{Z}/8 \otimes_H S^{k\rho_H}).$$

- Class we are inverting is carried by an $S^{k\rho_8}$.
- Inversion is a colimit and first steps show $\pi_{-2} = 0$. □

Take Home Message

1. Slices are easy to compute with
2. Things built from MU have easy, geometric slices.

Happy A_5 Birthday,
Bob and Ron!