TOPOLOGICAL HOCHSCHILD HOMOLOGY OF ℓ AND ko

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Abstract. We calculate the integral homotopy groups of $\mathrm{THH}(\ell)$ at any prime and of $\mathrm{THH}(ko)$ at $p = 2$, where ℓ is the Adams summand of the connective complex p-local K-theory spectrum and ko is the connective real K-theory spectrum.

1. Introduction

1.1. **Motivation.** Topological Hochschild homology is a generalization of Hochschild homology to the context of structured ring spectra. In analogy with Hochschild homology, it helps classifying deformations and extensions of structured ring spectra.

In addition, the work of numerous authors on the cyclotomic trace now gives machinery allowing the computation of the algebraic K-theory of connective ring spectra R. The first necessary input to these computations is the topological Hochschild homology $\mathrm{THH}(R)$.

After localization at a fixed prime p, the connective K-theory spectrum has a summand ℓ, known as the Adams summand. McClure and Staffeldt carried out the computation of the mod p homotopy of $\mathrm{THH}(\ell)$ at primes $p \geq 3$ [6]. Their method was to first compute the Adams spectral sequence converging to the mod p homotopy, and then use knowledge about the $K(1)$-homology to obtain necessary information about the differentials. The computations were extended by Rognes and the first author to the case $p = 2$ [1]. These methods lead to difficulties at the prime 2 because the mod 2 Moore spectrum is not a ring spectrum.

Ausoni and Rognes have computed the $V(1)$-homotopy of the topological cyclic homology $TC(\ell)$ and the algebraic K-theory $K(\ell)$ [3], beginning by computing the homotopy of $V(1) \wedge \mathrm{THH}(\ell)$. This leads to problems at the prime 3, where $V(1)$ is not a ring spectrum, and at the prime 2, where $V(1)$ does not exist, and so their computations were only valid for $p \geq 5$. Ausoni later extended these computations to $p = 3$ and to computations for ku, rather than for the Adams summand [2].

McClure and Staffeldt stated in [6] their intent to continue their project: "In the sequel we will investigate the integral homotopy groups of $\mathrm{THH}(\ell)$ using our present results as a starting point." While extensive computations were carried out, this sequel never appeared.

The aim of this paper is to use some of the recent advances in structured ring spectra to both simplify the previous computations of $\mathrm{THH}(\ell)$, in some cases removing the prime dependency, and to exhibit a complete integral computation of $\mathrm{THH}(\ell)$ as an ℓ_*-module. One finds that there is a weak equivalence between the spectrum $V(1) \wedge \mathrm{THH}(\ell)$, for p odd, and $\mathrm{THH}(\ell; H\mathbb{F}_p)$, and the latter is the realization of a simplicial commutative $H\mathbb{F}_p$-algebra. It should be noted that this method does not simplify the computations of topological cyclic homology and algebraic K-theory, as neither of these spectra inherit the structure of module spectra over ℓ.

In addition, there are Bockstein spectral sequences for computing the homotopy of $\mathrm{THH}(\ell; H\mathbb{Z}_{(p)})$ and that of $\mathrm{THH}(\ell; \ell/p)$ from $\mathrm{THH}_*(\ell; H\mathbb{F}_p)$, and for computing $\mathrm{THH}_*(\ell)$ from $\mathrm{THH}_*(\ell; \ell/p)$ or from $\mathrm{THH}_*(\ell; H\mathbb{Z}_{(p)})$. It happens that the integral computation of the homotopy groups of $\mathrm{THH}(\ell)$ is determined by the requirement that the two Bockstein spectral sequences converging to $\mathrm{THH}_*(\ell)$ agree. We highly recommend that the reader experiment with this method at $p = 2$ to gain insight into the final result.

Similarly, this "dueling Bockstein" method can be used to give a complete computation of $\mathrm{THH}(ko; ku)_{(2)}$. The results are strikingly similar to the computation of $\mathrm{THH}(ku)_{(2)}$. There is then a final η-Bockstein spectral sequence computing the homotopy of $\mathrm{THH}(ko)_{(2)}$ which is directly computable. One, perhaps unexpected, result of this computation is that η^2 acts by zero on the homotopy of $\overline{\mathrm{THH}}(ko)$, the summand of $\mathrm{THH}(ko)$ complementary to ko.

1.2. **Organization.** We begin in § 2 by summarizing the key tools we will need to start the computations and stating our main results. In § 3, we run the first two Bockstein spectral sequences. This provides the necessary input to allow us to run the last two spectral sequences. In § 4, we analyze the third Bockstein spectral sequence, using it to get information about the possible structure of the homotopy groups. The next section is a brief digression into topological Hochschild cohomology, and in it, we find elements in $\mathrm{THH}^*(\ell)$ that pair nicely with the generators we found in early sections. In § 6, we use the vanishing and cyclicity results we found in § 4 to find all of the differentials and extensions in the fourth Bockstein spectral sequence. This completes the computation of $\mathrm{THH}(\ell)$.

We round out our computations in § 7, where we calculate $\mathrm{THH}_*(ko)_{(2)}$. Finding $\mathrm{THH}_*(ko; ku)$ requires a similar analysis as that for $\mathrm{THH}_*(ku)$, and we pass from $\mathrm{THH}_*(ko; ku)$ to $\mathrm{THH}_*(ko)$ by analyzing the η-Bockstein spectral sequence and resolving hidden extensions.

2. Preliminary Remarks and Statement of Results

2.1. **Algebraic Preliminaries.** As a global piece of notation, we will write $a \doteq b$ when a is equal to b up to multiplication by a p-local unit.

We begin with a few lemmas which allow us to state the kinds of Bökstedt spectral sequences we will use. If R is an S-algebra and M is an R-bimodule, let $\mathrm{THH}(R; M)$ denote the derived smash product $M \wedge_{R \wedge R^{op}} R$ [4, § IX].

Lemma 2.1. *Suppose R is a commutative S-algebra and M is an R-module given the commutative bimodule structure. Then there is a weak equivalence*

$$\mathrm{THH}(R; M) \simeq M \wedge_R \mathrm{THH}(R).$$

Proof. We show a chain of weak equivalences whose composite is the one in question. By definition, we have

$$\mathrm{THH}(R; M) \simeq M \wedge_{R\wedge R^{op}} R \simeq (M \wedge_R R) \wedge_{R\wedge R^{op}} R,$$

and reassociating gives that this is equivalent to

$$M \wedge_R (R \wedge_{R\wedge R^{op}} R) \simeq M \wedge_R \mathrm{THH}(R). \qquad \square$$

Lemma 2.2. *Suppose $R \to Q$ is a map of S-algebras and M is a Q-R bimodule, given an R-R bimodule structure by pullback. Then there is a weak equivalence*

$$\mathrm{THH}(R; M) \simeq M \wedge_{Q\wedge R^{op}} Q.$$

Proof. Similarly to the previous lemma, this follows from the following chain of weak equivalences:

$$\mathrm{THH}(R; M) \simeq M \wedge_{R\wedge R^{op}} R \simeq (M \wedge_{Q\wedge R^{op}} Q \wedge R^{op}) \wedge_{R\wedge R^{op}} R,$$

and reassociating gives that this is equivalent to

$$M \wedge_{Q\wedge R^{op}} (Q \wedge R^{op} \wedge_{R\wedge R^{op}} R) \simeq M \wedge_{Q\wedge R^{op}} Q. \qquad \square$$

Corollary 2.3. *Under these circumstances, there is a Künneth spectral sequence with E_2-term*

$$\mathrm{Tor}_{**}^{Q_* R^{op}}(M_*, Q_*) \Rightarrow \mathrm{THH}_*(R; M).$$

This expression for topological Hochschild homology often leads to strictly simpler computations than are usually carried out by means of the Bökstedt spectral sequence. For instance, if $R = Q = H\mathbb{F}_p$, we obtain a spectral sequence starting from

$$\mathrm{Tor}_{**}^{A_*}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow \mathrm{THH}_*(\mathbb{F}_p).$$

Here A_* is the dual Steenrod algebra. This can be identified as the part of the Bökstedt spectral sequence consisting of the primitives under the A_*-comodule action.

There are dual statements for topological Hochschild cohomology that we will need in § 5. Let $\mathrm{THH}^\bullet(R; M)$ denote the derived function spectrum $F_{R\wedge R^{op}}(R, M)$.

Lemma 2.4. *Suppose $R \to Q$ is a map of S-algebras and M is a Q-R bimodule, given an R-R bimodule structure by pullback. Then there is a weak equivalence*

$$\mathrm{THH}^\bullet(R; M) \simeq F_{Q\wedge R^{op}}(Q, M).$$

Corollary 2.5. *Under these circumstances, there is a universal coefficient spectral sequence*

$$\mathrm{Ext}^{**}_{Q_* R^{op}}(Q_*, M_*) \Rightarrow \mathrm{THH}^*(R; M).$$

2.2. **Method and Main Results.** Computations using the Bökstedt spectral sequence or Corollary 2.3 allow us to see that

$$\mathrm{THH}_*(\ell; H\mathbb{F}_p) = E(\lambda_1, \lambda_2) \otimes \mathbb{F}_p[\mu],$$

where $|\lambda_i| = 2p^i - 1$, and $|\mu| = 2p^2$ [1, 6]. Moreover, since the bimodule $H\mathbb{F}_p$ is the quotient of ℓ by v_0 and v_1, we can find two intermediate ℓ-modules between ℓ and $H\mathbb{F}_p$, namely Morava $k(1) = \ell/p$ and $H\mathbb{Z}_{(p)}$. This allows us to go from $\mathrm{THH}(\ell)$ to $\mathrm{THH}(\ell; H\mathbb{F}_p)$ in two ways:

$$\begin{tikzcd} & \mathrm{THH}(\ell) \arrow[dl] \arrow[dr] & \\ \mathrm{THH}(\ell; H\mathbb{Z}_{(p)}) \arrow[dr] & & \mathrm{THH}(\ell; k(1)) \arrow[dl] \\ & \mathrm{THH}(\ell; H\mathbb{F}_p) & \end{tikzcd}$$

Each of the arrows in the above diagram gives a Bockstein spectral sequence going the other way. Thus we have the following spectral sequences:

$$\mathrm{THH}_*(\ell; H\mathbb{F}_p)[v_1] \implies \mathrm{THH}_*\big(\ell; k(1)\big); \tag{1}$$
$$\mathrm{THH}_*(\ell; H\mathbb{F}_p)[v_0] \implies \mathrm{THH}_*(\ell; H\mathbb{Z}_{(p)})^\wedge_p; \tag{2}$$
$$\mathrm{THH}_*\big(\ell; k(1)\big)[v_0] \implies \mathrm{THH}_*(\ell)^\wedge_p; \tag{3}$$
$$\mathrm{THH}_*(\ell; H\mathbb{Z}_{(p)})[v_1] \implies \mathrm{THH}_*(\ell). \tag{4}$$

We can understand the first two spectral sequences easily, and this gives us two spectral sequences which we can play against each other to understand $\mathrm{THH}_*(\ell)$. Moreover, since $\mathrm{THH}(\ell)$ is finitely generated, these results actually give p-local, rather than p-complete, information.

We recall that for a commutative S-algebra R with a module M given the commutative bimodule structure, there is a splitting in R-modules

$$\mathrm{THH}(R; M) \simeq M \vee \overline{\mathrm{THH}}(R; M).$$

For convenience, we will often perform computations with $\overline{\mathrm{THH}}(\ell)$ and exclude the factor of ℓ which splits off.

We will show how to completely understand $\mathrm{THH}_*(\ell)$ as an ℓ_*-module.

Theorem 2.6. *As an ℓ_*-module,*

$$\mathrm{THH}_*(\ell) = \ell_* \oplus \Sigma^{2p-1} F \oplus T,$$

where F is a torsion free summand and T is an infinite direct sum of torsion modules concentrated in even degrees.

2.3. **The Torsion Free Part.** Since rational homotopy is rational homology, we can easily run the Bökstedt spectral sequence computing the rational homotopy of $\mathrm{THH}(\ell)$. For degree reasons, the spectral sequence collapses with no possible extensions, and we have

$$\mathrm{THH}_*(\ell) \otimes \mathbb{Q} = \mathbb{Q}[\lambda_1, v_1]/\lambda_1^2,$$

where $|\lambda_1| = 2p-1$. This tells us exactly where all of the torsion free summands of $\mathrm{THH}_*(\ell)$ lie.

Theorem 2.7. *The torsion free summand of* $\overline{\mathrm{THH}}_*(\ell)$ *is* $\Sigma^{2p-1}F$*, where* F *the* ℓ_**-module*

$$F = \ell_* \left[\frac{v_1^{p^k+\cdots+p}}{p^k} \right] \subset \ell_* \otimes \mathbb{Q}.$$

Thus the classes $v_1^k \lambda_1$ become increasingly p-divisible as k gets large. We pause to mention the relation of this structure to known computations. At an odd prime McClure and Staffeldt already found ([6, Theorem 8.1]) that

$$\mathrm{THH}(L) \simeq v_1^{-1}\,\mathrm{THH}(\ell) \simeq L \vee \Sigma L_{\mathbb{Q}},$$

where L is the periodic Adams summand. This can be seen directly here at any prime: inverting v_1 in the homotopy of $\mathrm{THH}(\ell)$ leaves

$$L_* \oplus v_1^{-1}\Sigma^{2p-1}F \cong L_* \oplus \Sigma(L_{\mathbb{Q}})_*,$$

recovering their result for odd primes and extending it to $p = 2$.

2.4. **The Torsion Part.** The torsion is rather complicated, but it can also be understood. It is concentrated in even degrees, and follows a kind of tower-of-Hanoi picture with increasingly complicated, inductively built, components.

We define a sequence of torsion modules T_n for $n \geq 0$ as follows. As an ℓ_*-module, each T_n has generators g_w for all strings w on letters $0, \dots p-1$. We impose two kinds of relations. First, we require that $g_w = 0$ if $|w| > n$, where $|w|$ denotes the length of the string. Second, if we write $w \cdot w'$ for the concatenation of strings, we have

$$pg_w = \begin{cases} v_1^{p^{(n-|w|+2)}} g_{w'} + g_{w\cdot 0} & \text{if } w = w' \cdot (p-1), \\ g_{w\cdot 0} & \text{otherwise.} \end{cases}$$

One can show inductively that that these relations imply that

$$v_1^{p^{n-|w|+1}+\cdots+p} g_w = 0$$

for all w. The non-zero generators are graded by saying that if $w = a_1 \dots a_k$, then

$$|g_w| = 2p^2(a_1 p^{n-1} + \cdots + a_k p^{n-k}).$$

An easy example is that $T_0 = \ell_*/(p, v_1^p)$.

More generally, the modules T_n have only finitely many nonzero elements. The modules T_n are self-dual; the duality is given by

$$g_w \longleftrightarrow v_1^{p^{n-|w|+1}+\cdots+p-1} g_{\bar{w}},$$

where $\bar{w}$ is the string w with each digit a replaced by $p-1-a$.

The modules T_n are more easily viewed through a recursive construction. First there are inclusions of direct summands $\Sigma^{2kp^{n+1}}T_{n-1} \hookrightarrow T_n$ given by $g_w \mapsto g_{k\cdot w}$, for $1 \le k \le p-2$. There is also an inclusion of T_{n-1} given by $g_w \mapsto g_{0\cdot w}$ and a projection $T_n \to \Sigma^{2(p-1)p^{n+1}}T_{n-1}$ given by

$$g_w \mapsto \begin{cases} g_{w'} & w = (p-1)\cdot w' \\ 0 & \text{otherwise.} \end{cases}$$

The structure of T_n as a module is determined by having these summands, submodules, and quotient modules, together with a generator $g_\varnothing$ satisfying relations

$$\begin{aligned} p \cdot g_\varnothing &= g_0 \\ v_1^{p^{n+1}} g_\varnothing &= g_{(p-1)0} - p \cdot g_{(p-1)}. \end{aligned}$$

One can view T_n as being recursively constructed out of p copies of T_{n-1}, where we glue together the first and last copies along a v_1-tower of length $p^n + \cdots + p - 1$.

Theorem 2.8. *The torsion summand of the homotopy of* THH(ℓ)*, as an* ℓ_**-module, is isomorphic to*

$$\bigoplus_{n\ge 0}\bigoplus_{k=1}^{p-1} \Sigma^{2kp^{n+2}+2(p-1)}T_n.$$

In particular, for all $n \ge 1$ *and* $2 \le k \le p$*, the even dimensional homotopy between degrees* $2kp^{n+2} - 2p + 1$ *and* $2kp^{n+2} + 2p - 3$ *is zero.*

To facilitate understanding of the modules T_n, we have included a picture of the torsion for $p = 2$ starting in degrees 18 and 34 as Figure 1. These correspond to T_1 and T_2.

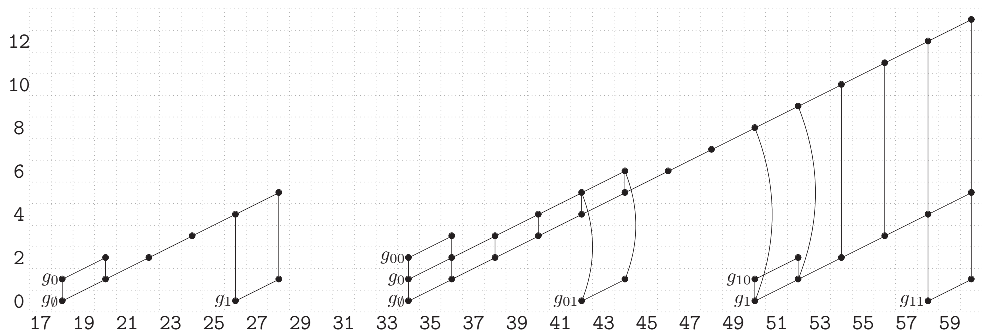

FIGURE 1. The torsion in degrees 18 through 60 for $p = 2$

3. The first two Bockstein spectral sequences

In this section, we compute the base case Bockstein spectral sequences, Spectral Sequences (1) and (2). As was mentioned above,

$$\mathrm{THH}_*(\ell; H\mathbb{F}_p) = E(\lambda_1, \lambda_2) \otimes \mathbb{F}_p[\mu],$$

where λ_1, λ_2 and μ are in degrees $2p-1$, $2p^2-1$ and $2p^2$, respectively [1].

3.1. **The $k(1)$-Bockstein spectral sequence.**

In [6], McClure and Staffeldt ran the first Bockstein spectral sequence at an odd prime, calculating the homotopy groups of $\mathrm{THH}(\ell; k(1))$, and in [1] Rognes and the first author extended the calculation to $p = 2$.

The calculation depends on the following observation. The $K(1)_*$-based Bökstedt spectral sequence collapses, so $K(1)_*\ell \cong K(1)_*\mathrm{THH}(\ell)$ and only the v_1-tower starting at the unit element 1 can survive. There is only one pattern of differentials compatible with this, and it produces v_1-towers of various length on $\mu^i\lambda_1$, $\mu^i\lambda_1\lambda_2$ and $\mu^{pi}\lambda_2$. For the reader's convenience we recall the result here.

Recursively define $r(n)$ by $r(1) = p$, $r(2) = p^2$ and $r(n) = p^n + r(n-2)$ for $n \geq 3$. Also define λ_n by $\lambda_n = \lambda_{n-2}\mu^{p^{n-3}(p-1)}$.

Theorem 3.1 ([1, 6]). *The homotopy of* $\mathrm{THH}(\ell; k(1))$ *is generated as a module over* $\mathbb{F}_p[v_1]$ *by* 1, $x_{n,m} = \lambda_n\mu^{p^{n-1}m}$ *and* $x'_{n,m} = \lambda_n\lambda_{n+1}\mu^{p^{n-1}m}$ *for* $n \geq 1$ *and* $m \geq 0$, $m \not\equiv p-1 \mod p$. *The relations are generated by*

$$v_1^{r(n)}x_{n,m} = v_1^{r(n)}x'_{n,m} = 0.$$

Figure 2 shows the homotopy of $\overline{\mathrm{THH}}\big(ku; k(1)\big)$ through dimension 33 at $p = 2$.

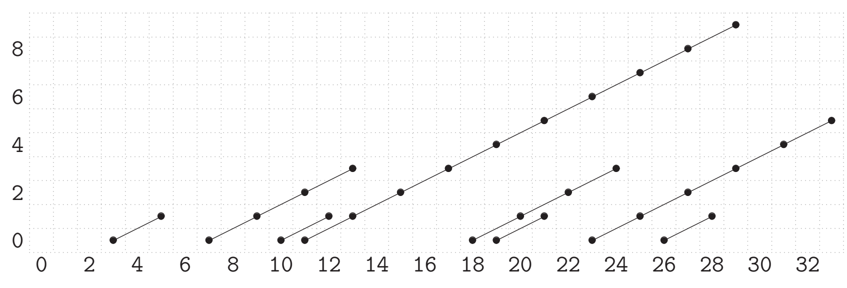

Figure 2. The homotopy of $\overline{\mathrm{THH}}\big(ku; k(1)\big)$

3.2. **The $H\mathbb{Z}$-Bockstein spectral sequence.** Next we run the Bockstein spectral sequence converging to $\mathrm{THH}_*(\ell; H\mathbb{Z}_{(p)})^\wedge_p$. There is an immediate differential $d_1(\mu) = v_0\lambda_2$, since the corresponding classes in the homology of $\mathrm{THH}(\ell)$ are connected by a Bockstein.

To get the remaining differentials, we use the following "Leibniz" rule for higher differentials in a Bockstein spectral sequence.

Proposition 3.2 ([5])**.** *If x supports a d_j differential, then*

$$d_{j+1}(x^p) = v_0 x^{p-1} d_j(x).$$

Using this result, we can conclude all remaining differentials.

Proposition 3.3. *We have differentials*

$$d_{i+1}(\mu^{p^i}) = v_0^{i+1}\mu^{p^i-1}\lambda_2.$$

Remark 3.4. *Just as we used the $K(1)$-based Bökstedt spectral sequence to get information about what happens when we "put v_1 back", we can use the $K(0) = H\mathbb{Q}$-based Bökstedt spectral sequence to get information about what happens when we put p back. The $H\mathbb{Q}$-based Bökstedt spectral sequence gives*

$$\mathrm{THH}_*(\ell; H\mathbb{Q}) \cong \mathbb{Q}[\lambda_1]/\lambda_1^2,$$

so we know that all the rest is torsion. This method alone tells us where all the differentials are, though not how long they are in this case.

Let $a_i = \mu^{i-1}\lambda_2$ and $b_i = \mu^{i-1}\lambda_1\lambda_2$, for $i \geq 1$. Then $|a_i| = 2p^2 i - 1$ and $|b_i| = 2p^2 i + 2(p-1)$, and they both have order p^{k+1}, where $k = \nu_p(i)$, the p-adic valuation of i. The above analysis shows the following proposition.

Proposition 3.5. *The homotopy of $\overline{\mathrm{THH}}(\ell; H\mathbb{Z}_{(p)})$ is a copy of $\mathbb{Z}_{(p)}$ generated by λ_1 plus torsion. The torsion is generated as a $\mathbb{Z}_{(p)}$-module by the elements a_i and b_i.*

4. The third Bockstein spectral sequence

In this section, we say as much as we can about the spectral sequence

$$\overline{\mathrm{THH}}_*\big(\ell; k(1)\big)[v_0] \Rightarrow \overline{\mathrm{THH}}_*(\ell)^\wedge_p.$$

The E_1-page through dimension 33 for $p = 2$ is depicted in Figure 3. Note that v_1 is really in filtration 0 here; we draw it in filtration 1 to reduce the clutter.

This spectral sequence allows us to conclude key facts about some of the homotopy groups, making upper bounds on the order of the groups or determining if they are cyclic. The following lemmas serve as the workhorses for all computations in Spectral Sequence (4).

Lemma 4.1.

(a) *The groups $\overline{\mathrm{THH}}_{2p^2(p^n)-2p}(\ell)$ for $n \geq 1$ are cyclic.*

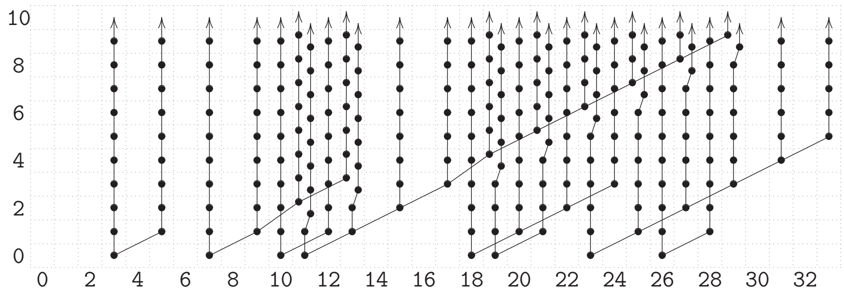

FIGURE 3. The v_0-Bockstein spectral sequence converging to $\overline{\mathrm{THH}}_*(ku)$

(b) *The groups* $\overline{\mathrm{THH}}_{2p^2(p^n)}(\ell)$ *and* $\overline{\mathrm{THH}}_{2p^2(p^n)-2}(\ell)$ *for* $n \geq 0$ *are 0.*

Proof. We show this by showing that in this Bockstein spectral sequence, E_1 is $\mathbb{F}_2[v_0]$ in degree $2p^2(p^n)-2p$ and is 0 in degrees $2p^2(p^n)-2$ and $2p^2p^n$.

The only even generators from Theorem 3.1 are the generators $x'_{n,m}$. A simple counting argument shows that

$$|x'_{n,m}| = 2(p^{n+1}-1) + 2p + 2mp^{n+1} = 2(p-1) + 2(m+1)p^{n+1}.$$

The element $x'_{n,m}$ supports a v_1-tower truncated at height $r(n)$, where $r(n)$ was recursively defined for Theorem 3.1 as well. We can now check by degree, arguing by p-adic expansion.

The proof of the results in the lemma are very similar. We must first find all of the elements of the form $x'_{j,m}v_1^k$ in dimension $2p^2(p^n)-2p$. In other words, we must find all triples (j,m,k) such that

$$2p^2(p^n) - 2p = 2(m+1)p^{j+1} + (1+k)2(p-1),$$

subject to the condition that $k < r(j)$. The integer $2(p^{n+2}-p)$ is divisible by $2(p-1)$, so we see that $m+1 = (p-1)\hat{m}$ for some integer $\hat{m}$. Dividing through by $2(p-1)$ leaves

$$\hat{m}p^{j+1} + 1 + k = p^{n+1} + \cdots + p.$$

In particular, we must have that

$$k \geq p^j + \cdots + p - 1.$$

If $j > 1$, then this quantity is bigger than $r(j)$ by induction. If $j = 1$, then there is a solution with $k < r(j)$, namely $k = p-1$, $\hat{m} = 1 + \cdots + p^{n-1}$. This means that modulo p, there is one generator in degree $2p^2(p^n)-2p$, namely $x'_{1,p^n-2}v_1^{p-1}$.

For degree $2p^2(p^n)-2$, we note that this degree is the degree just argued plus $|v_1|$. In that degree, the generating class was the largest allowed v_1 multiple of the class x'_{1,p^n-2}, so there can be no classes in $2p^2(p^n)-2$.

For degree $2p^2(p^n)$, we search for classes $x'_{j,m}v_1^k$ such that

$$|x'_{j,m}v_1^k| = 2(m+1)p^{j+1} + 2(p-1) + 2(p-1)k = 2p^{n+2}.$$

Combining terms and reducing modulo p^{j+1}, we see that k is at least $p^{j+1}-1$, which is in particular larger than $r(j)$. $\square$

Lemma 4.2. *The groups* $\overline{\mathrm{THH}}_{2p^2(p^n)-1}(\ell)$ *for* $n \geq 0$ *are* $\mathbb{Z}_{(p)}$.

Proof. The proof of this lemma is very similar. Here we consider the odd classes $x_{j,m}$, and the argument depends on the parity of j. A combinatorial check shows that

$$|x_{j,m}| = \begin{cases} 2(p-1)\big(p^{j-1}+p^{j-3}+\cdots+p\big)+2p-1+2mp^{n+1} & j \text{ even} \\ 2(p-1)\big(p^{j-1}+p^{j-3}+\cdots+1\big)+1+2mp^{n+1} & j \text{ odd.} \end{cases}$$

We must find those values of j, m, and k such that

$$|x_{j,m}v_1^k| = 2p^{n+2}-1.$$

Regardless of the parity of j, if we subtract 1 from both sides, then the righthand side is 0 modulo $2(p-1)$. This implies that again $m = \hat{m}(p-1)$. At this point, the argument does not depend on the parity in an essential way, so we spell out only the case of $j = 2i$. Dividing by $2(p-1)$ leaves

$$\hat{m}p^{2i+1}+p^{2i-1}+\cdots+p^3+p+1+k = p^{n+1}+\cdots+1.$$

If $n+1 > 2i-1$, then $k \geq p^{2i}+\cdots+p^2 = r(2i)$. On the other hand, if $n+1 = 2i-1$, then we can choose $k = r(2i-2)$, $\hat{m} = 0$. Thus we have for each n a single generator in degree $2p^{n+2}-1$, namely $x_{n+2,0}v_1^{r(n)}$.

To finish the lemma, we need only recall that the previous lemma ensured that the E_1 page was zero in degrees $2p^{n+2}$ and $2p^{n+2}-2$. Since differentials change the degree by 1, we conclude that the $\mathbb{F}_2[v_0]$ tower originating on $x_{n+2,0}v_1^{r(n)}$ neither supports nor is the target of any differentials. $\square$

Lemma 4.3. *The groups* $\overline{\mathrm{THH}}_{2p^2(p^n)+2p-3}(\ell)$ *for* $n \geq 0$ *are cyclic.*

Proof. The proof is the same as for the previous case. The dimension in question is that of the previous lemma plus $|v_1|$. Here we must choose $k = r(n)+1$, and the generating class is $x_{n+2,0}v_1^{r(n)+1}$. $\square$

5. Topological Hochschild cohomology

To finish the computations, we will use the "cap product" pairing of topological Hochschild cohomology with topological Hochschild homology:

$$\mathrm{THH}^n(\ell)\otimes\mathrm{THH}_m(\ell)\to\mathrm{THH}_{m-n}(\ell).$$

Many of the torsion patterns, both for the v_0 and v_1 Bockstein spectral sequences, arise from multiplication by powers of μ. While no power of μ survives the Bockstein spectral sequences, the μ^k translates of permanent cycles do survive. Using the pairing with $\mathrm{THH}^\bullet$, we can actually connect these elements on the E_∞ page.

We first note that certain topological Hochschild cohomology spectra inherits "Hopf algebra" type structures. It was proven in [1] that when R is commutative, $\mathrm{THH}(R)$ has the structure of a Hopf algebra spectrum over R, and hence the base extension $\mathrm{THH}(R;Q) \cong \mathrm{THH}(R) \wedge_R Q$ inherits a Hopf algebra spectrum structure over Q when Q is a commutative R-algebra.

Let D_Q denote the Q-Spanier-Whitehead dualization functor. The dual $D_Q(C)$ of a Hopf algebra spectrum C over Q inherits a Q-algebra structure from the coalgebra structure, and there are natural maps of Q-algebras

$$D_Q(C) \to D_Q(C \wedge_Q C) \leftarrow D_Q(C) \wedge_Q D_Q(C),$$

where the first map is induced by the multiplication. If the second map is a weak equivalence (such as when C is a finite cell object, or when Q is $H\mathbb{Z}_{(p)}$ or $H\mathbb{F}_p$ and C has finitely generated homotopy groups), there is an induced Hopf algebra spectrum structure on $D_Q(C)$ up to homotopy. In particular, $\mathrm{THH}^\bullet(\ell; H\mathbb{F}_p)$ and $\mathrm{THH}^\bullet(\ell; H\mathbb{Z}_{(p)})$ are both Hopf algebra spectra over $H\mathbb{F}_p$ and $H\mathbb{Z}_{(p)}$ respectively.

Since $\mathrm{THH}_*(\ell; H\mathbb{F}_p)$ is an $H\mathbb{F}_p$-algebra with finitely generated homotopy groups, we can dualize its homotopy groups directly to conclude that as a Hopf algebra,

$$\mathrm{THH}^*(\ell; H\mathbb{F}_p) = E(x_{2p-1}, x_{2p^2-1}) \otimes \Gamma(c_1),$$

where $\Gamma(c_1)$ denotes a divided power algebra on a class c_1 in degree $2p^2$, and the generators are all primitive. The divided power generator $c_k = \gamma_k(c_1)$ is dual to μ^k, so if it survives the Adams spectral sequence to give an element of $\mathrm{THH}^*(\ell)$, then capping with it will undo the multiplications by μ^k that were seen on the E_∞ page. However, since this module over the Steenrod algebra is negatively graded and not bounded below, there are convergence problems with the Adams spectral sequence. We instead compare with relative $\mathrm{THH}^\bullet$.

5.1. **Relative $\mathrm{THH}^\bullet$ of ℓ.** We write the remainder of the section with the assumption that BP is an E_∞ ring spectrum. If this is not the case, then we can replace BP with MU. Many of the key points are the same; the notation is slightly simpler in the BP case. To streamline notation further, we also let $\tau_i = \xi_{i+1}$ if $p = 2$.

Proposition 5.1. *As a ring,*

$$\pi_* H\mathbb{F}_p \wedge_{BP} \ell = E(\bar{\tau}_2, \bar{\tau}_3 \dots),$$

and the map from $H\mathbb{F}_{p_*}\ell$ induced by the unit $S^0 \to BP$ is the canonical quotient.

Proof. We use the equivalence in ring spectra

$$H\mathbb{F}_p \wedge_{BP} \ell \simeq H\mathbb{F}_p \wedge_{H\mathbb{F}_p \wedge BP} (H\mathbb{F}_p \wedge \ell).$$

The Künneth theorem then gives both parts of the theorem, since $H_*(\ell; \mathbb{F}_p)$ is free over $H_*(BP; \mathbb{F}_p)$. □

The universal coefficient spectral sequence on the above exterior algebra then collapses, telling us that

$$\mathrm{THH}^*_{BP}(\ell; H\mathbb{F}_p) = \mathbb{F}_p[e_1, e_2, \dots],$$

where e_i is the class in Ext corresponding to $\bar{\tau}_{i+1}$.

Since this is concentrated in even degrees, we conclude that the Bockstein spectral sequence taking us from $\mathrm{THH}_{BP}(\ell; H\mathbb{F}_p)$ to $\mathrm{THH}_{BP}(\ell)$ collapses, giving

$$\mathrm{THH}^*_{BP}(\ell) = \ell_*[\![e_1, \dots]\!].$$

The structure map $S^0 \to BP$ induces a commutative diagram

$$\begin{tikzcd} \mathrm{THH}_{BP}(\ell) \arrow[r] \arrow[d] & \mathrm{THH}_{BP}(\ell; H\mathbb{F}_p) \arrow[d] \\ \mathrm{THH}(\ell) \arrow[r] & \mathrm{THH}(\ell; H\mathbb{F}_p) \end{tikzcd}$$

We want to show that the elements c_k in $\mathrm{THH}^*(\ell; H\mathbb{F}_p)$ lift to $\mathrm{THH}^*(\ell)$. However, we can see this using the commutativity of the above diagram.

Proposition 5.2. *The map from $\mathrm{THH}^*_{BP}(\ell; H\mathbb{F}_p)$ to $\mathrm{THH}^*(\ell; H\mathbb{F}_p)$ sends e_k to $c_{p^{k-1}}$.*

Proof. This is immediate from our discussion of the map in homotopy

$$\pi_*(H\mathbb{F}_p \wedge \ell) \to \pi_*(H\mathbb{F}_p \wedge_{BP} \ell)$$

induced by the unit $S^0 \to BP$. The classical Bökstedt spectral sequence identifies the generators in $\mathrm{Ext}_{H_*(\ell)}$ coming from $\bar{\tau}_{i+2}$ with $\gamma_{p^k}(c_1)$. $\square$

Remark 5.3. *In order to use THH relative to MU rather than relative to BP, the following changes must be noted. The ring $\pi_*(H\mathbb{F}_p \wedge_{MU} \ell)$ is the ring previously calculated as $\pi_*(H\mathbb{F}_p \wedge_{BP} \ell)$ tensored with an exterior algebra on classes in odd degrees. The universal coefficient spectral sequence then shows that $\mathrm{THH}^*_{MU}(\ell; H\mathbb{F}_p)$ is the tensor product of the ring calculated as $\mathrm{THH}^*_{BP}(\ell; H\mathbb{F}_p)$ with a polynomial algebra on generators in even degrees.*

We can therefore conclude that in fact the elements c_k all survive to homotopy classes in $\mathrm{THH}^*(\ell)$. It should be noted, however, that they survive to classes which are possibly torsion. To see this, we analyze $\mathrm{THH}^*(\ell; H\mathbb{Z}_{(p)})$.

Theorem 5.4. *As a Hopf algebra,*

$$\mathrm{THH}^*(\ell; H\mathbb{Z}_{(p)}) = E(x_{2p-1}) \otimes \Gamma(c_1)/(pc_1),$$

where x_{2p-1} and c_1 are again primitive.

Remark 5.5. *As $\mathrm{THH}^*(\ell; H\mathbb{Z}_{(p)})$ is not flat over $\mathbb{Z}_{(p)}$, it is not immediate that the comultiplication on the topological Hochschild cohomology spectrum gives rise to a comultiplication on the level of homotopy groups. However, the classes in THH^* lie in degrees congruent to 0 and $2p-1$ mod $2p^2$, and hence the Tor-terms in the homotopy groups of $\mathrm{THH} \wedge_{H\mathbb{Z}_{(p)}} \mathrm{THH}$, which lie in degrees congruent to $1, 2p$, and $4p-1$ mod $2p^2$, cannot be in the image.*

Proof. We first note that $\mathrm{THH}(\ell; H\mathbb{Z}_{(p)})$ is a commutative Hopf algebra spectrum over $H\mathbb{Z}_{(p)}$, and the homotopy groups are finitely generated over $H\mathbb{Z}_{(p)}$ in each degree. The $H\mathbb{Z}_{(p)}$-dual, the spectrum $\mathrm{THH}(\ell; H\mathbb{Z}_{(p)})$, therefore has finitely generated homotopy groups in each degree, and hence the Bockstein spectral sequence

$$\mathrm{THH}^*(\ell; H\mathbb{F}/p)[v_0] \Rightarrow \mathrm{THH}^*(\ell; H\mathbb{Z}_{(p)})$$

is a convergent spectral sequence. Multiplication by p commutes with the comultiplication, and hence this Bockstein spectral sequence is a spectral sequence of Hopf algebras. In order for the result to be $H\mathbb{Z}_{(p)}$-dual to $\mathrm{THH}(\ell; H\mathbb{Z}_{(p)})$, the differentials are generated by those of the form

$$d_{i+1}(c_{p^i-1}x_{2p^2-1}) \doteq v_0^{i+1}c_{p^i}$$

for $i \geq 0$, where we use the convention that $c_0 = 1$. $\square$

This theorem allows us to compute the cap product

$$\mathrm{THH}^k(\ell; H\mathbb{Z}_{(p)}) \otimes \mathrm{THH}_m(\ell; H\mathbb{Z}_{(p)}) \to \mathrm{THH}_{m-k}(\ell; H\mathbb{Z}_{(p)}).$$

Corollary 5.6. *For $k < n$, the cap product satisfies the following formulae*

$$c_k \frown a_n \doteq \binom{n-1}{k} a_{n-k} \text{ and } c_k \frown b_n \doteq \binom{n-1}{k} b_{n-k}.$$

We see that c_{p^k} is p^{k+1} torsion in $\mathrm{THH}^*(\ell; H\mathbb{Z}_{(p)})$, which means that in $\mathrm{THH}^*(\ell)$, c_{p^k} is at worst p^{k+1} torsion. However, the Adams spectral sequence for $\mathrm{THH}^*(\ell)$ suggests that in fact these classes are torsion free.

Proposition 5.7. *As an ℓ_*-module,*

$$\overline{\mathrm{THH}}^*(\ell) = \left(\lim_{\leftarrow} \Sigma^{-2(p-1)p^{n+1}} T_n\right) \oplus \Sigma^{2p-1}\mathbb{Z}_p/\mathbb{Z}_{(p)}[v_1],$$

where the structure maps $\Sigma^{-2(p-1)p^{n+2}} T_{n+1} \to T_n$ in the limit are the desuspensions of the quotients of T_{n+1} to T_n described in the recursive construction of T_n.

Naturality of the cap product moreover implies that it commutes with the differentials in the Bockstein spectral sequences. We will exploit both of these remarks to compute the differentials and extensions in the remaining spectral sequence.

6. The last Bockstein spectral sequence and $\overline{\mathrm{THH}}_*(\ell)$

The v_1-Bockstein spectral sequence is pictured for $p = 2$ through dimension 35 in Figure 4. Here multiplication by 2 preserves the filtration, though we have drawn it as increasing the filtration by 1 to reduce clutter.

We can now get all the differentials and extensions in this spectral sequence. Recall that in § 3.2, we defined torsion elements $a_i = \mu^{i-1}\lambda_2$ and $b_i = \lambda_1 a_i$. Since these and λ_1 were all of the generators of $\overline{\mathrm{THH}}_*(\ell; H\mathbb{Z}_{(p)})$, we can immediately conclude the following proposition for degree reasons.

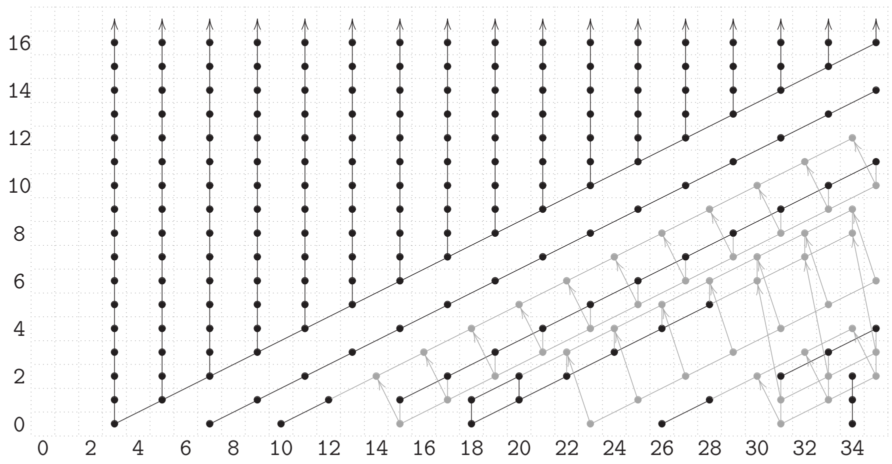

FIGURE 4. The v_1-Bockstein spectral sequence converging to $\overline{\mathrm{THH}}_*(ku)$

Proposition 6.1. *The v_1 tower on λ_1 survives to E_∞.*

Lemma 6.2. *The classes b_{p^i} are permanent cycles for all i.*

Proof. The classes b_{p^i} are represented in the third spectral sequence by $x'_{i+1,0}$. The possible targets for a differential on this class are in degree

$$2p^2p^i + 2(p-1) - 1 = |v_1^{p^{i+1}+\cdots+1}\lambda_1|,$$

which, by the previous proposition and Lemma 4.3, cannot be the target of any differentials. Thus the class $x'_{i+1,0}$ is a permanent cycle in the third spectral sequence, and hence in this one. $\square$

Capping allows us to bootstrap from this to a much stronger statement.

Lemma 6.3. *The classes b_i are permanent cycles for all i.*

Proof. Pick j such that $p^j \geq i$, and let $k = p^j - i$. If we consider the p-adic expansion of $p^j - 1$, then Lucas' lemma for binomial coefficients modulo p shows that the binomial coefficient in

$$c_k \frown b_{p^j} = \binom{p^j-1}{k} b_i$$

is a p-adic unit. Naturality of the cup product then ensures that for all m,

$$d_m(b_i) \doteq d_m(c_k \frown b_{p^j}) = c_k \frown d_m(b_{p^j}) = 0. \qquad \square$$

6.1. **The differentials.** The second part of Lemma 4.1 shows that there are no classes in degree $2p^2 \cdot p^i - 2$. However, there are a great many classes in the spectral sequence there. All of these classes must be killed, and there is only one pattern of differentials that achieves this.

Theorem 6.4. *The differentials are determined by a family of differentials*

(a) $$d_{p^n+\cdots+p}(p^{n-1}a_{p^n}) \doteq v_1^{p^n+\cdots+p} b_{(p-1)p^{n-1}}.$$

These give two types of differentials: differentials of the form

(b) $$d_{p^n+\cdots+p}(p^{n-1}a_{kp^{n-1}+jp^n}) \doteq v_1^{p^n+\cdots+p}b_{(k-1)p^{n-1}+jp^n},$$

for $2 \leq k \leq p$ and $j \geq 0$, and differentials of the form

(c) $$d_{p^n+\cdots+p}(p^{n-1}a_{p^{n-1}+jp^n}) \doteq p^{\nu_p(j)+1}v_1^{p^n+\cdots+p}b_{jp^n},$$

for $j \geq 1$, where $\nu_p(j)$ is the p-adic valuation of j.

Proof. We begin by showing how the first family of differentials implies the remaining ones, then we will show that this is the only possible pattern of differentials. For ease of readability, let $m = p^n + \cdots + p$.

We first show we can use the cap product to get differentials on the classes $a_{p^n+jp^n}$, $j \geq 0$. Naturality of the cap product with respect to the Bockstein differentials shows that

$$c_{jp^n} \frown d_m(p^{n-1}a_{p^n+jp^n}) = d_m(c_{jp^n} \frown p^{n-1}a_{p^n+jp^n}).$$

We know that

$$c_{jp^n} \frown p^{n-1}a_{p^n+jp^n} = \binom{jp^n + p^n - 1}{jp^n} p^{n-1}a_{p^n}.$$

This binomial coefficient is a p-adic unit, so we see that

$$d_m(p^{n-1}a_{p^n+jp^n}) \doteq v_1^m b_{(p-1)p^{n-1}+jp^n}.$$

Next we find that

$$d_m(c_k \frown p^{n-1}a_{p^n+jp^n}) = c_k \frown d_m(p^{n-1}a_{p^n+jp^n}) \doteq v_1^m c_k \frown b_{(p-1)p^{n-1}+jp^n},$$

for all k and $j \geq 0$. Since the only classes on which p^{n-1} multiplication is non-trivial are those of the form $a_{p^{n-1}k'}$ for some k', we only need consider the effect of capping with $c_{kp^{n-1}}$. Lucas' lemma for computing binomial coefficients modulo p using a p-adic expansion shows us that if $k \leq p-1$, the binomial coefficient of

$$c_{kp^{n-1}} \frown a_{p^n+jp^n} = \binom{jp^n + p^n - 1}{kp^{n-1}} a_{(p-k)p^{n-1}+jp^n}$$

is non-zero modulo p. In particular, this makes it a p-adic unit, and therefore invertible in $\mathbb{Z}/p^n$. This lets us compute the d_m differentials on $a_{(p-k)p^{n-1}+jp^n}$ for all $0 \leq k \leq p-1$, $j \geq 0$. To complete the first part of the implications, we need only consider the effect of capping on the target.

If $k \leq p-2$, then Lucas' lemma again shows that the binomial coefficient of

$$c_{kp^{n-1}} \frown b_{(p-1)p^{n-1}+jp^n} = \binom{jp^n + (p-1)p^{n-1} - 1}{kp^{n-1}} b_{(p-k-1)p^{n-1}+jp^n}$$

is non-zero modulo p and hence also invertible in $\mathbb{Z}/p^n$. We can therefore conclude that for $k \leq p-2$,

$$d_m(p^{n-1}a_{(p-k)p^{n-1}+jp^n}) \doteq v_1^m b_{(p-k-1)p^{n-1}},$$

and replacing $p-k$ with k shows that the differentials of Equation (a) imply those of Equation (b).

If $k = p-1$, then we no longer have that this binomial coefficient in $c_{kp^{n-1}} \frown b_{(p-1)p^{n-1}+jp^n}$ non-zero modulo p. If $j=0$, then the cap product takes us to $b_0 = 0$, so and so $p^{n-1}a_{(p-1)p^{n-1}}$ is a d_m-cycle. If $j \geq 1$, then a counting argument based on the p-adic valuation of binomial coefficients, using standard results from p-adic analysis, shows that

$$\binom{jp^n + (p-1)p^{n-1} - 1}{(p-1)p^{n-1}} b_{jp^n} \doteq p^{\nu_p(j)+1} b_{jp^n},$$

showing that Equation (a) implies Equation (c).

To show that the differentials of Equation (a) are the correct differentials, we argue by induction on n using again the result of Lemma 4.1 that the $2p^2 \cdot p^n - 2$ stem is zero in $\overline{\mathrm{THH}}_*(\ell)$. The base case is clear, since there is visibly only one generator in dimension $2p^3 - 2$, namely $v_1^p b_{(p-1)}$. Now we assume that for all $m < n$, we have differentials of the desired form on the classes a_i.

To show the inductive step, we first identify all classes in this stem on E_1. The only even generators are the classes b_i, so we are looking for those values of k and i such that

$$|v_1^k b_i| = 2(p-1)k + 2p^2 i + 2p - 2 = 2p^{n+2} - 2.$$

Reducing modulo $p-1$ or p shows that $k = p\hat{k}$ and $i = (p-1)\hat{\imath}$ for some integers $\hat{k}$ and $\hat{\imath}$. In other words, we are looking at pairs of positive integers $(\hat{k}, \hat{\imath})$ such that

$$\hat{k} + p\hat{\imath} = p^n + \cdots + 1.$$

This breaks the problem into two cases: either $\hat{\imath} = p^{n-1}$ or $\hat{\imath} < p^{n-1}$. The former corresponds to $b_{(p-1)p^{n-1}}$ and is the only value we want to show remains by $E_{p^n+\cdots+p}$. We want to rule out the latter, so assume that $\hat{\imath} < p^{n-1}$. If we let $m = \nu_p(\hat{\imath})$, then the inductive hypothesis tells us that $v_1^{p^{m+1}+\cdots+p} b_i = 0$ on $E_{p^n+\cdots+p}$. However, we also know that

$$\hat{k} = p^n + \cdots + 1 - p\hat{\imath} \geq p^m + \cdots + 1$$

by the above analysis, so $v_1^k b_i = v_1^{p\hat{k}} b_i = 0$, as required. This means there is only one class remaining in degree $2p^2 \cdot p^n - 2$, namely $v_1^{p^n+\cdots+p} b_{(p-1)p^{n-1}}$. Since $b_{(p-1)p^{n-1}}$ are cycles, this must be the target of a differential.

In degree $2p^2 \cdot p^n - 1$, there are many classes, namely various v_1-multiples of $p^k a_{p^k}$ for $k < n$ and $p^{n-1} a_{p^n}$. By induction, the classes $p^k a_{p^k}$ are the only cycles (since there are no classes in the degree immediately preceeding theirs), and thus we must have the required differential on $p^{n-1} a_{p^n}$. $\square$

An easy corollary of this is that there are gaps in the even dimensional homotopy.

Lemma 6.5. *For all $n \geq 0$ and $2 \leq k \leq p$, there is a gap in the even dimensional homotopy between degrees $2kp^{n+2} - 2p + 1$ and $2kp^{n+2} + 2p - 3$.*

Proof. On the E_∞ page, the classes b_j support a v_1 tower of height $p^i + \cdots + p - 1$, where $i = \nu_p(j) + 1$. Thus if there are classes in the desired range, then they originate as v_1 multiples of classes $b_{kp^n - m}$ for some m. The p-adic valuation of the subscript is determined by that of m, and it is clear that we need only check the largest integers $kp^n - m$ for any given p-adic valuation, namely $kp^n - p^m$ for $0 \leq m \leq n$. However, the top v_1 multiples of each of these classes lie in the same dimension: $2kp^{n+2} - 2p$, proving the result. $\square$

We also note that the proof of Theorem 6.4 shows that the torsion pattern generated by pb_{p^k} is the same as the one generated by $b_{p^{k-1}}$.

6.2. **The Torsion Free Extensions.** We adopt the notation in this section that $v_0 x$ is the image of multiplication by p on the E_1 page of the spectral sequence. We begin with the torsion free part, proving a restatement of Theorem 2.7.

Theorem 6.6. *We have additive extensions*

$$p \cdot a_1 \doteq v_1^p \lambda_1$$

and for $k \geq 1$,

$$p \cdot v_0^k a_{p^k} \doteq v_1^{p^{k+1}} v_0^{k-1} a_{p^{k-1}}.$$

Proof. We saw in Lemma 4.2 that

$$\pi_{2p^2(p^i)-1} \overline{\mathrm{THH}}(\ell) = \mathbb{Z}_{(p)}.$$

However, since the elements a_{p^j} are p^{j+1}-torsion, and the differentials above only involve multiples of a_{p^j} up to $v_0^{j-1} a_{p^j}$, in the $2p^2(p^i) - 1$ stem, there are elements $v_0^i a_{p^i}$ and $v_1^{p^{i+1} + \cdots + p^{k+2}} v_0^k a_{p^k}$ for $0 \leq k < i$ in this degree. Since the group must be cyclic, we have non-trivial additive extensions, and for degree reasons, we must have

$$p \cdot (v_1^{p^{i+1} + \cdots + p^{k+2}} v_0^k a_{p^k}) \doteq v_1^{p^{i+1} + \cdots + p^{k+1}} v_0^{k-1} a_{p^{k-1}}.$$

Comparing powers of v_1 provides the desired result. $\square$

6.3. **The Torsion Extensions.** Understanding the extensions in the torsion patterns is much harder. We begin by isolating repeating patterns in the torsion. For $n \geq 0$ and $1 \leq k \leq p - 1$, let $T_{n,k}$ be the submodule of $\overline{\mathrm{THH}}_*(\ell)$ generated by all classes b_i, $kp^n \leq i \leq (k+1)p^n - 1$, which degrees shifted down so that the lowest class is in degree 0. Lemma 6.5 shows that the torsion of $\overline{\mathrm{THH}}_*(\ell)$ is the direct sum of shifts of these modules.

Theorem 6.7. *The submodule $T_{n,k}$ is independent of k.*

Proof. This is another capping argument. We will cap down from the case $k = p-1$. To get the lower torsion submodules, we will cap with classes c_{jp^n}, $1 \le j \le p-2$. The generators of the examined submodule are those b_i with $(p-1)p^n \le i \le p^{n+1}-1$. When we cap with c_{jp^n}, we get

$$c_{jp^n} \frown b_i = \binom{i-1}{jp^n} b_{i-jp^n}.$$

However, for all i in the desired range, the p-adic expansion of $i-1$ begins with at most $(p-2)p^n$, this binomial coefficient is a p-adic unit and hence an isomorphism onto. $\square$

Corollary 6.8. *The torsion submodule of* $\mathrm{THH}_*(\ell)$ *splits as a direct sum*

$$\bigoplus_{n\ge 0} \bigoplus_{k=1}^{p-1} \Sigma^{2kp^{n+2}+2(p-1)} T_{n,k}$$

It remains only to determine the structure of $T_{n,k}$ for some k.

We now exploit the first part of Lemma 4.1: $\pi_{2p^2(p^n)-2p}\,\overline{\mathrm{THH}}(\ell)$ is a cyclic group. On the E_∞ page of the spectral sequence, there are only the elements

$$v_1^{(p^{k+1}+\cdots+p)-1} b_{p^n-p^k}, \quad 0 \le k \le n-1,$$

since the other v_1 multiples of the intermediate elements b_j are all killed by differentials. This means that there must be extensions linking these elements.

Proposition 6.9. *There are choices of lifts of the generators in this spectral sequence so that we have hidden additive extensions*

$$p \cdot b_{p^{n+1}-p^k} = v_0 b_{p^{n+1}-p^k} + v_1^{p^{k+2}} b_{p^{n+1}-p^{k+1}},$$

where $v_0 b_{p^{n+1}-p^k}$ *is the image of* p *times* $b_{p^{n+1}-p^k}$ *on the* E_1 *page.*

Proof. We show the extensions by increasing degree. We first note that the convergence of the Bockstein spectral sequence ensures that we can find lifts of the classes $v_0 b_{p^{n+1}-p^k}$ which have v_1 order exactly what is seen on the E_∞ page. Since the order of all elements of higher filtration in the same degree is larger than that of $v_0 b_{p^{n+1}-p^k}$, this choice is unique. Moreover, this implies that p^m times this lift is a lift of $v_0^{m+1} b_{p^{n+1}-p^k}$ which has the same v_1 order as the image in E_∞. We choose this unit so that $p(b_{p^{n+1}-p^k}) = v_0 b_{p^{n+1}-p^k}$ modulo elements of higher filtration.

Since we must have the extensions at the end of the v_1 torsion patterns generated by the $b_{p^{n+1}-p^k}$, there must be extensions linking the generating classes. The argument is now one of decreasing induction on k. There is only one class of higher filtration than $b_{p^{n+1}-p^{n-1}}$, $v_1^{p^{n+1}} b_{p^{n+1}-p^n}$. We must therefore have an extension

$$p \cdot b_{p^{n+1}-p^{n-1}} - v_0 b_{p^{n+1}-p^{n-1}} \doteq v_1^{p^{n+1}} b_{p^{n+1}-p^n}.$$

By changing both $b_{p^{n+1}-p^{n-1}}$ and $v_0 b_{p^{n+1}-p^{n-1}}$ by the same unit, we can ensure actual equality.

The inductive step is identical. There must be a non-trivial extension from $b_{p^{n+1}-p^k}$ to the subgroup generated by elements of higher filtration which realizes this element as the generator of a cyclic group. The subgroup of elements of higher filtration is generated by $v_1^{p^{k+2}} b_{p^{n+1}-p^{k+1}}$, so we must have an extension of the form

$$p \cdot b_{p^{n+1}-p^k} - v_0 b_{p^{n+1}-p^k} \doteq v_1^{p^{k+2}} b_{p^{n+1}-p^{k+1}}.$$

Simultaneously changing the lifts of $b_{p^{n+1}-p^k}$ and $v_0 b_{p^{n+1}-p^k}$ by a unit allows us to produce an equality. $\square$

Combined with Theorem 6.7, this gives hidden extensions of the form

$$p \cdot b_{2p^n-p^k} \doteq v_0 b_{2p^n-p^k} + v_1^{p^{k+2}} b_{2p^n-p^{k+1}}.$$

We will use this to find $T_{n,1}$. After changing the choices of generators by multiplication by units if necessary, we find the following.

Theorem 6.10. *There are lifts of the generators $v_0^i b_j$ to $T_{n,1}$ as follows. Let $m = p^n + a_1 p^{n-1} + \cdots + a_k p^{n-k} + a_{k+j+1} p^{n-k-j-1}$, where $0 \leq a_i \leq p-1$ for $i < k$, and $1 \leq a_i \leq p-1$ for $i \geq k$. Then*

$$p \cdot b_m = \begin{cases} v_0 b_m + v_1^{p^{n-k-j+1}} v_0^j b_{m-a_{k+j+1}p^{n-k-j-1}} & a_{k+j+1} = p-1 \\ v_0 b_m & \textit{otherwise.} \end{cases}$$

Proof. Since the p-adic valuation of m is $n-k-j-1$, we will show this by capping down from $b_{2p^n-p^{n-k-j-1}}$. This class caps down to all classes b_i, up to multiplication by a unit, with the same p-adic valuation for $p^n \leq i \leq 2p^n - 1$, again by Lucas' lemma for computing binomial coefficients modulo p. For an integer a between 0 and $p-1$, let $\bar{a}$ denote $p-1-a$. To reach b_m, we cap with the class c_s, where

$$s = \bar{a}_1 p^{n-1} + \cdots + \bar{a}_k p^{n-k} + p^{n-k-1} - p^{n-k-j} + \bar{a}_{k+j+1} p^{n-k-j-1}.$$

Naturality of the cap product ensures that when we pull back $p \cdot b_{2p^n-p^{n-k-j-1}}$ via capping with c_s, we get p times b_m. In other words, we must analyze

$$v_1^{p^{k+2}} c_s \frown b_{2p^n-p^{n-k-j}} = v_1^{p^{k+2}} \binom{2p^n - p^{n-k-j} - 1}{s} b_{2p^n-p^{n-k-j}-s}.$$

We consider first the case $a_{k+j+1} = p-1$. This forces $\bar{a}_{k+j+1} = 0$. We care only about the p-adic valuation of the binomial coefficient that occurs in the cap product, and since we are working p-adically, this is the same as the number of "carries" when we add s to $2p^n - p^{n-k-j} - s$, p-adically. Since $a_k > 0$, $\bar{a}_k < p-1$. This means that the p-adic valuation of the binomial coefficient above is that same as that of

$$\binom{(p-1)p^{n-k} + \cdots + (p-1)p^{n-k-j+1} + (p-2)p^{n-k-j}}{\bar{a}_k p^{n-k} + (p-1)p^{n-k-1} + \cdots + (p-1)p^{n-k-j}}.$$

Since this visibly has j carries, we have proved the first part.

The case $a_{k+j+1} < p-1$ is substantially easier. The integer $2p^n - p^{n-k-j} - s$ has p-adic valuation $n-k-j-1$. This means that

$$v_1^{p^{n-k-j+2}} b_{2p^n - p^{n-k-j}-s} = 0,$$

and that part of $p \cdot b_m$ pulls back to zero, giving the second part. $\square$

This result completes our analysis of $T_{n,1}$. We can now provide a dictionary linking $T_{n,1}$ with the module T_n defined in § 2. We define a bijection between strings of length at most n and v_0 multiples of classes b_i, $p^n \leq i \leq 2p^n - 1$, via

$$a_1 \dots a_k \underbrace{0 \dots 0}_{j} \longleftrightarrow v_0^j b_{p^n + a_1 p^{n-1} + \cdots + a_k p^{n-k}}.$$

The restriction on the lengths of the strings reflects both the fact that we only consider classes in a prescribed range and the fact that the order of a class b_i is $p^{\nu_p(i)+1}$. The previous theorem then shows that all of the relations from § 2 are satisfied.

7. Topological Hochschild homology of ko

We can follow the same program as for $\mathrm{THH}(ku)$ to 2-locally calculate $\mathrm{THH}_*(ko)$. As a starting point, the first author and John Rognes [1] used the Bökstedt spectral sequence to conclude that

$$\mathrm{THH}_*(ko; H\mathbb{F}_2) = E(\lambda_1', \lambda_2) \otimes P(\mu),$$

where this time $|\lambda_1'| = 5$, $|\lambda_2| = 7$ and $|\mu| = 8$.

This serves as the starting point for a chain of spectral sequences, just as before. In this case, however, we have an η-Bockstein spectral sequence in addition to the four spectral sequences analogous to the ku case.

7.1. **Statement of results.** The homotopy groups of $\overline{\mathrm{THH}}(ko)$ sit as an extension of two parts, one from the torsion free part of $\overline{\mathrm{THH}}_*(ko; ku)$ (though this part will contain torsion) and the other from the torsion part of $\overline{\mathrm{THH}}_*(ko; ku)$.

Define a ko_*-module F^{ko} as follows. Additively,

$$F^{ko} = \bigoplus_{i \neq 2^n - 2} \Sigma^{4i} \mathbb{Z}[\eta]/(2\eta, \eta^2) \oplus \bigoplus_{n \geq 1} \Sigma^{4(2^n-2)} \mathbb{Z}.$$

Multiplication by v_1^2 sends the $\mathbb{Z}$ in degree $4i$ isomorphically to the $\mathbb{Z}$ in degree $4i+4$, except when $i = 2^n - 2$, in which case multiplication by v_1^2 sends the $\mathbb{Z}$ to $2\mathbb{Z}$. (Since $v_1^2 \notin ko_*$ we should instead say that multiplication with $2v_1^2 \in ko_*$ acts as multiplication by 2 when $i \neq 2^n - 2$ and as multiplication by 4 when $i = 2^n - 2$.) In all figures that follow, multiplication by v_1^2 we be denoted by a dashed line.

We next define T_n^{ko} inductively as follows. As a ko_*-module, $T_1^{ko} = \mathbb{Z}/2 \oplus \Sigma^6 \mathbb{Z}/2$. We build T_n^{ko} out two copies of T_{n-1}^{ko} and two v_1^2-towers. To be precise, the two copies of T_{n-1}^{ko} are in degree 0 and 2^{n+2}, and we add a copy of $\mathbb{Z}/2[v_1^2]/((v_1^2)^{2^n-1})$ to the bottom of the copy of T^{ko}_{n-1} in degree 0 and another copy of $\mathbb{Z}/2[v_1^2]/((v_1^2)^{2^n-1})$ to the top of the copy of T^{ko}_{n-1} in degree 2^{n+1}. Each T^{ko}_n is self-dual, and η acts trivially on T^{ko}_n.

Theorem 7.1. *There is a short exact sequence of ko_*-modules*

$$0 \to \Sigma^5 F^{ko} \to \overline{\mathrm{THH}}_*(ko) \to \bigoplus_{n\geq 1} \Sigma^{8(2^n)+4} T^{ko}_n \to 0.$$

The additive extensions are trivial except in degrees congruent to 2 modulo 4 between degrees $2^{n+3}+2^{n+2}+2$ and $2^{n+4}-6$ for $n \geq 1$ where the group is cyclic.

Corollary 7.2. *On $\overline{\mathrm{THH}}(ko)$, η^2 acts trivially.*

Proof. The groups T^{ko}_n are all concentrated in even degrees, and multiplication by η^2 kills all elements in $\Sigma^5 F^{ko}$. □

Conjecture 7.3. *The ko-module summand $\overline{\mathrm{THH}}(ko)$ admits the structure of a module over kc, the connective, self-conjugate K-theory spectrum.*

Figure 5 shows the homotopy of $\overline{\mathrm{THH}}(ko)$ through degree 34, while Figures 6 and 7 show some of the torsion in $\overline{\mathrm{THH}}_*(ko)$ coming from the T^{ko}_n for $n = 1$, 2, and 3.

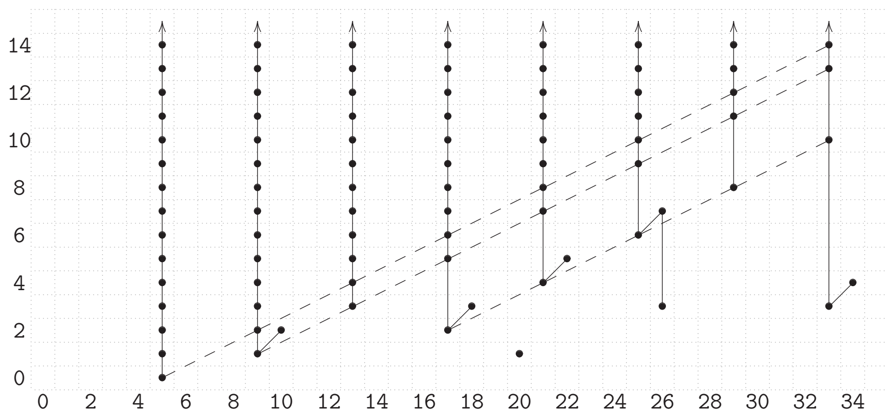

FIGURE 5. $\overline{\mathrm{THH}}_*(ko)$ through degree 34

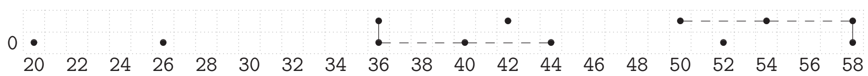

FIGURE 6. The ko_*-modules T^{ko}_1 and T^{ko}_2

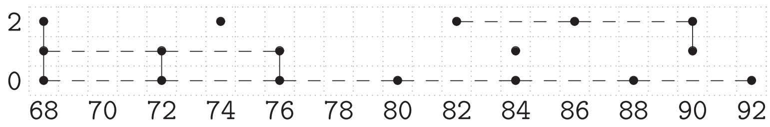

FIGURE 7. The first half of the ko_*-module T_3^{ko}

7.2. **Computing** $\mathrm{THH}_*(ko; ku)$**.** This part is very similar to the story for $\mathrm{THH}(ku)$. Let a_i' denote $\lambda_2\mu^{i-1}$ and let b_i' be $\lambda_1' a_i'$. These classes play the roles of the classes a_i and b_i in this story. We have the following results:

Theorem 7.4. *The torsion free summand of* $\overline{\mathrm{THH}}_*(ko; ku)$ *is* $\Sigma^5 F'$*, where* F' *is the* ku_**-module*

$$F' = ku_*\left[\frac{v_1^{2^{k+1}-3}}{2^k}\right] \subset ku_* \otimes \mathbb{Q}.$$

The torsion is also similar. Define $T_1' = ku_*/(2, v_1)$, and define T_n' recursively by gluing together two copies of T_{n-1}' along a v_1-tower of length $2^{n+1} - 3$. Alternatively, define T_n' as $v_1 T_n \subset T_n$.

Theorem 7.5. *The torsion summand of* $\overline{\mathrm{THH}}_*(ko; ku)$ *is, as a* ku_**-module, 2-locally isomorphic to the following direct sum:*

$$\bigoplus_{n\geq 0} \Sigma^{4(2^n)+4} T_n'.$$

These results actually follow from those of § 6. The natural map from ko to ku induces maps of the four Bockstein spectral sequences, and this will allow us to conclude the result. As initial input, the Bökstedt spectral sequence shows that the natural map from $\mathrm{THH}_*(ko; H\mathbb{F}_2)$ to $\mathrm{THH}_*(ku; H\mathbb{F}_2)$ sends λ_1' to 0 and λ_2 and μ to themselves. This means that a_1', being represented by λ_2, maps to a_1. This in turn forces λ_1' in $\mathrm{THH}_*(ko)$ to map to $v_1\lambda_1$ in $\mathrm{THH}_*(ku)$. Using this observation, we can conclude that the differentials and extensions are almost the same as for the ku case.

Proposition 7.6. *The homotopy of* $\overline{\mathrm{THH}}(ko; ku)$ *sits as the* ku_**-submodule of* $\overline{\mathrm{THH}}_*(ku)$ *generated by* $v_1\lambda_1$*, the classes* $p^k a_{p^k}$*, and* v_1 *times the torsion patterns* T_n*.*

Considering the effect of inverting v_1 allows us to conclude the following Corollary.

Corollary 7.7. *The canonical map* $\mathrm{THH}(KO; KU) \to \mathrm{THH}(KU)$ *is a weak equivalence.*

In particular, a homotopy-fixed point argument allows us to determine $\mathrm{THH}(KO)$.

Corollary 7.8. *As a* KO*-module,*

$$\mathrm{THH}(KO) = KO \vee \Sigma KO_{\mathbb{Q}}.$$

The homotopy of $\overline{\mathrm{THH}}(ko; ku)$ is depicted through degree 35 in Figure 8.

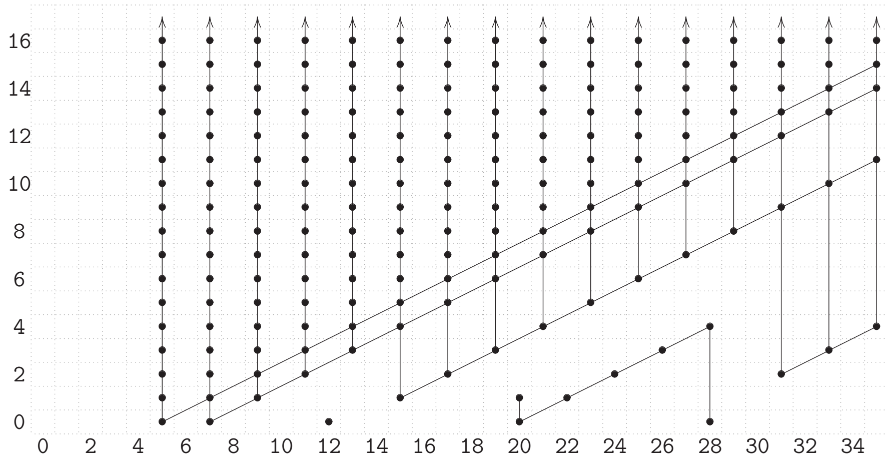

FIGURE 8. The homotopy of $\overline{\mathrm{THH}}(ko; ku)$

7.3. **The η-Bockstein spectral sequence.** It remains only to run the η-Bockstein spectral sequence

$$E_1 = \overline{\mathrm{THH}}_*(ko; ku)[\eta] \Rightarrow \overline{\mathrm{THH}}_*(ko) \tag{5}$$

coming from the cofiber sequence $\Sigma ko \xrightarrow{\eta} ko \to ku$. Because $\eta^3 = 0$ in ko_*, this spectral sequence collapses at the E_4-term at the latest. The surprise is that η^2 acts trivially on $\overline{\mathrm{THH}}_*(ko)$, so in fact the spectral sequence collapses at the E_3-term.

It is instructive to first review how the η-Bockstein spectral sequence $ku_*[\eta] \Rightarrow ko_*$ works. In this case we have differentials $d_1(v_1) = 2\eta$ and $d_3(v_1^2) = \eta^3$. In particular, the E_2-term has infinite η-towers starting at v_1^{2i} for $i \geq 0$.

Let us write Spectral Sequence (5) as

$$E_1 = \Sigma^5 E_1(F') \oplus \bigoplus_{n\geq 1} \Sigma^{4(2^n)+4} E_1(T'_n),$$

where $E_1(F') = F'[\eta]$ and $E_1(T'_n) = T'_n[\eta]$. This is depicted through dimension 35 in Figure 9. In this and subsequent Bockstein spectral sequences, the possible targets of a differential are depicted in gray, while possible sources are in black. The η towers orginating on gray classes are not shown to prevent clutter.

There can be no d_1-differentials connecting the different summands for degree reasons, so we can study the d_1-differential on each summand separately. Note that E_r in this spectral sequence is a differential graded module over $E_r(ku)$, so while multiplication by v_1^2 does not technically make sense on the abutment, it does make sense on the E_1 and E_2-term.

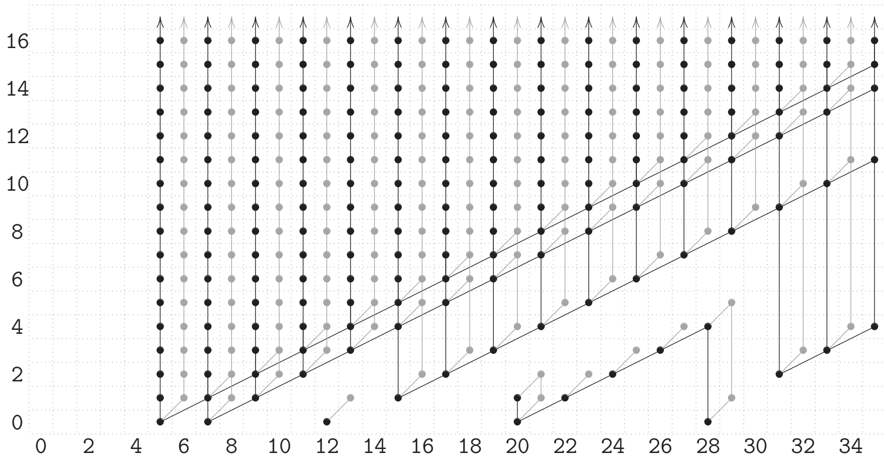

FIGURE 9. The η-Bockstein E_1 Page

Proposition 7.9. *On $E_1(F')$ we have $d_1(v_1) = 2\eta$, and*

$$E_2(F') = \bigoplus_{i \neq 2^n - 2} \Sigma^{4i}\mathbb{Z}[\eta]/(2\eta) \oplus \bigoplus_{n \geq 1} \Sigma^{4(2^n-2)}\mathbb{Z}$$

additively, with v_1^2-multiplication as in F^{ko}.

Proof. It is clear that we get the same answer as for $E_2(ku)$, except that when the v_1-tower becomes more 2-divisible we kill η instead of 2η. $\square$

Note that there can be no d_2 or d_3-differential on the η-tower starting in degree $4(2^n)+1$ in E_2 coming from degree $4(2^n - 1)$ in $E_2(F')$. We shall see that this forces all of the higher differentials.

Proposition 7.10. *The d_1-differentials on $E_1(T'_n)$ are generated by*

$$d_1(b'_{a2^k}) = 2^{\nu_2(a-1)-1}\eta v_1^{4(2^k)-1} b'_{(a-1)2^k}$$

for $a \neq 1$ odd and $k \geq 0$ as an $E_1(ku)$-module. $E_2(T'_n)$ is an extension

$$0 \to T_n^{ko} \to E_2(T'_n) \to \mathbb{Z}/2[v_1^2, \eta]/((v_1^2)^{2^n-1}) \to 0.$$

Proof. Many of the differentials follow by considering $E_1(T'_n)$ as a differential graded module over $E_1(ku)$. In particular, this shows that there are differentials

$$d_1(v_1^{2i+1}b'_{2^n}) = 2\eta v_1^{2i} b'_{2^n}.$$

This leaves the classes $v_1^{2i}b'_{2^n}$ and, for each $1 \leq k \leq n$ and $0 \leq i \leq 2^k - 2$, the classes $2^{n-k}v_1^{2^{k+1}+2i-1}b'_{2^n}$.

The classes $2^{n-k}v_1^{2^{k+1}-1}b'_{2^n}$ are the troubling ones. For degree reasons, there are no possible d_i on this class for $i \leq 4$ (the smallest possible differential would take this class to η^4 on a v_1 multiple of the generator of $4(2^n - 1)$). This implies that it cannot support any differentials, and hence some η multiple of it must be a boundary. This in turn implies that $v_1^{2^{k+1}-4}$

on this class is also a d_i cycle for all i. For degree reasons, it must be a d_1 boundary. The possible sources for such a differential are manifold. The source must lie in the cyclic group generated by $b'_{2^n+2^k-1}$.

The only possible source for a differential targeting $2^{n-k}\eta v_1^{2^{k+1}-1}b'_{2^n}$ is b'_{2^n+i}, where $i = 2^{k-1}$, for degree reasons. If $d_1(b'_{2^n+i}) = 0$, then we have differentials of the form

$$d_1(v_1^{2m+1}b'_{2^n+i}) = 2\eta v_1^{2m}b'_{2^n+i} = \eta v_1^{2m}(v_0 b'_{2^n+i} + v_0^{n-k}v_1^{2^{k+1}}b'_{2^n}).$$

In particular, we have a differential

$$d_1(v_1^{2^{k+1}-5}b'_{2^n+i}) = \eta 2^{n-k}v_1^{2^{k+2}-6}b'_{2^n} = d_1(2^{n-k-1}v_1^{2^{k+2}-5}b'_{2^n}).$$

This means that in addition to the cycle $2^{n-k}v_1^{2^{k+2}-5}b'_{2^n}$, we have a d_1 cycle

$$v_1^{2^{k+1}-5}b'_{2^n+i} + 2^{n-k-1}v_1^{2^{k+2}-5}b'_{2^n}.$$

The same degree based argument from before show that this must be a d_1 boundary. Again, the only possible sources for this differential are in the cyclic group generated by $b'_{2^n+2^k-1}$. At this point, we arrive at our contradiction. This group must support d_1 differentials hitting both of the aforementioned cycles. However, on E_1, two times each of these cycles is 0, meaning that they cannot both be annihilated by the cyclic group. We must therefore instead have that

$$d_1(b_{2^n+i}) = 2^{n-k}\eta v_1^{2^{k+1}-1}b'_{2^n}.$$

To complete the proof, we note that the argument is an inductive one on the length of the 2-adic word representing the index of an element b'_j. Assume that the differentials of the desired form for all integers 2^n+i where the length of i is at most r. In particular, there are cycles

$$2^{m-k}v_1^{2^{k+1}-1+2i}b'_j,$$

where $j = 2^n + j'$ has 2-adic length exactly r, $m = \nu_2(j)$, $1 \leq k \leq n$ and $0 \leq i \leq 2^k-2$. In other words, we see exactly the picture for $b'_{2^{\nu_2(j)}}$ described above. The degrees of the elements are again such that these classes must be d_i cycles for all $i \leq 4$ (in fact, the inductive hypothesis ensures that the only possible targets are in fact the classes coming from $E_1(F')$). At this point, the argument from above (again, essentially the one for $b'_{2^{\nu_2(j)}}$) repeats, allowing us to see that

$$d_1(b'_{j+2^{k-1}}) = 2^{m-k}\eta v_1^{2^{k+1}-1+2i}b'_j.$$

□

We present the E_2 page through dimension 35 as Figure 10.

To finish the proof of Theorem 7.1, we show that there are only d_2 differentials remaining.

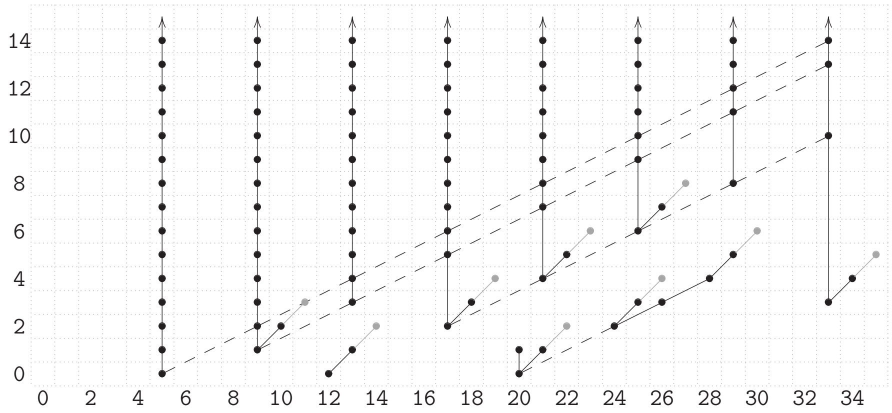

FIGURE 10. The η-Bockstein E_2 Page

Proposition 7.11. *We have d_2 differentials of the form*

$$d_2(b'_{2^n}) = \eta^2 v_1 A_n,$$

where A_n is the generator of the piece of $E_2(F')$ in degree $4(2^n-1)$.

Proof. The class A_n is a permanent cycle for degree reasons. However, since $v_1^{4(2^{n-1}-1)+2}A_n$ does not support η-multiplication, we cannot have the expected d_3 differential on $v_1^2 A_n$. This means that $\eta^2 A_n$ must be killed, and that forces the required differential. □

In particular, all classes in the spectral sequence are either η or η^2 torsion.

7.4. **Resolving the Extensions.** The results about differentials show that $\overline{\mathrm{THH}}(ko)$ is some extension of the direct sum of all of the torsion modules with $\Sigma^5 F^{ko}$. It remains only to solve this extension problem. We begin with an observation based on degree.

Lemma 7.12. *The only possible additive extensions coming from T_n^{ko} arise from the summand generated by the top v_1^2 tower.*

Proof. For degree reasons, the additive extensions must take classes in T_n^{ko} to classes in $\eta\Sigma^5 F^{ko}$. The v_1^4-order of all classes in $\eta\Sigma^5 F^{ko}$ is larger than that of all of the classes in T_n^{ko} with the exception of those arising as v_1^4-multiples of the top v_1^2 tower. These classes have the same v_1^4-order as the classes in the same degree in $\eta\Sigma^5 F^{ko}$, so these are the only possible sources of extensions. □

Actually showing that there are extensions is somewhat trickier. We break the computation into two pieces, first showing that there is an extension of the right form in degree $2^{n+3}-6$ for all $n > 1$ and then showing that there is a similar extension in degree $2^{n+3}-10$ for $n > 2$. Multiplication by v_1^4 then forces extensions on all previous classes in that v_1^2 tower. If $\overline{\mathrm{THH}}(ko)$ admits the structure of a kc-module, then the second argument is superfluous, since in this setting, v_1^2 is an honest operation.

We use a technique similar to that of the ku story: using $\overline{\mathrm{THH}}_*(ko; ko/2)$, we determine the structure of certain groups near the end of the torsion patterns. To compute this, we will use the η-Bockstein spectral sequence

$$\overline{\mathrm{THH}}_*(ko; ku/2)[\eta] \Longrightarrow \overline{\mathrm{THH}}_*(ko; ko/2). \tag{6}$$

However, since $ko/2$ is not a ring spectrum, this is not a spectral sequence of algebras. We will again only need that E_r is a differential graded module over the E_r page of the ordinary η-Bockstein spectral sequence.

The computation of $\overline{\mathrm{THH}}_*(ko; ku/2)$ is exactly like in the ku story. Recursively define $r'(n)$ by $r'(1) = 1$, $r'(2) = 4$ and $r'(n) = 2^n + r'(n-2)$ for $n \geq 3$. Also define λ_n by $\lambda_n = \lambda_{n-2}\mu^{2^{n-3}}$. Here we abusively let λ_1 denote the class previously denoted λ_1'.

Proposition 7.13. *([1]) The homotopy of $\overline{\mathrm{THH}}(ko; ku/2)$ is generated as a module over $\mathbb{F}_2[v_1]$ by $x_{n,m} = \lambda_n\mu^{2^n m}$ and $x'_{n,m} = \lambda_n\lambda_{n+1}\mu^{2^n m}$ for $n \geq 1$ and $m \geq 0$. The relations are generated by*

$$v_1^{r'(n)}x_{n,m} = v_1^{r'(n)}x'_{n,m} = 0.$$

The map in homotopy induced by the natural map from $\overline{\mathrm{THH}}(ko; ku/2)$ to $\overline{\mathrm{THH}}(ku; ku/2)$ is defined on generators by

$$\begin{pmatrix} x_{2n,m} \\ x_{2n+1,m} \end{pmatrix} \mapsto \begin{pmatrix} x_{2n,m} \\ v_1 x_{2n+1,m} \end{pmatrix}.$$

This map is almost an injection; only the top v_1-multiple of $x'_{2k,m}$ is lost. The relevant degrees for the extension questions are very close to the degrees considered in § 4, this allows us to apply those results to this situation.

We gain an immediate corollary to Lemma 4.2.

Corollary 7.14. *In degree $2^{n+3} - 1$, the group $\overline{\mathrm{THH}}(ko; ku/2)$ is $\mathbb{F}_2$, generated by $v_1^{r'(n)}x_{n+2,0}$.*

A more careful analysis of the previous degrees allows us to determine the structure of the groups in degrees $2^{n+3} - 16$ through $2^{n+3} - 1$. The proof is quite lengthy and will be postponed.

Lemma 7.15. *As a $ku/2_*$-module, $\overline{\mathrm{THH}}_*(ko; ku/2)$ between degrees $2^{n+3} - 16$ and $2^{n+3} - 1$ is generated in odd degrees by the classes*

$$v_1^{r'(n+1)-7}x_{n+1,0},\ v_1^{r'(n)-7}x_{n+2,0},\ \text{and } x_{1,2^{n-1}-1}$$

and in even degrees by the classes

$$x'_{1,2^{n-1}-1},\ x'_{2,2^{n-2}-1},\ \text{and } v_1^6 x'_{3,2^{n-3}-1}.$$

We present a picture of the E_1 page of Spectral Sequence 6 in the range $2^{n+3} - 16$ to $2^{n+3} - 1$ as Figure 11. Only powers of η through η^3 are depicted, and the horizontal degrees are to be read modulo 2^{n+3}. Horizontal lines depict v_1-multiplication. The vertical direction is used only to reduce clutter and has no relation to filtrations. Additionally, classes are labeled by the element that starts the v_1-tower to which the depicted elements belong. In particular, elements in degrees $2^{n+3}-15$ and $2^{n+3}-16$ are relatively deep in their respective v_1 towers.

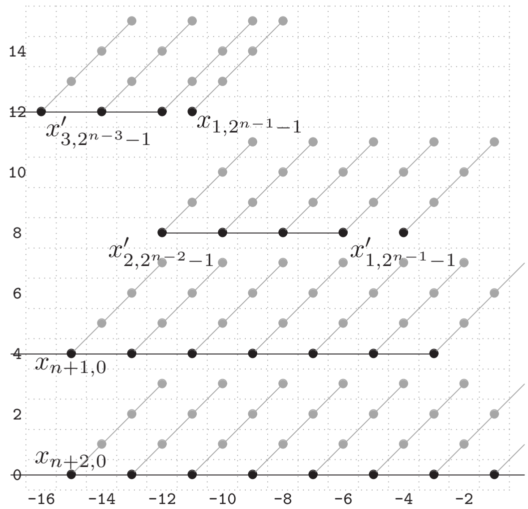

FIGURE 11. The η-Bockstein E_1 Page for $\overline{\mathrm{THH}}(ko; ko/2)$

Analyzing the associated graded of $\overline{\mathrm{THH}}_*(ko)$ allows us to readily see gaps in the mod 2 homotopy in a similar range.

Proposition 7.16. *In degrees $2^{n+3}-2^k$, $0 \le k \le 4$, $\overline{\mathrm{THH}}_*(ko; ko/2)$ is 0.*

This proposition, Corollary 7.14, and Lemma 7.15 imply results about the differentials in Spectral Sequence 6. Since v_1-multiplication makes sense on the E_1 page and since $2 = 0$ in $ku/2$, multiplication by v_1 is a well defined operation on E_1 that commutes with the differentials.

Lemma 7.17. *We have a d_1-differential of the form*

$$d_1(v_1^{r'(n)} x_{n+2,0}) = \eta v_1^{r'(n+1)-1} x_{n+1,0}.$$

Proof. Corollary 7.14 shows that on the E_1 page in Spectral Sequence 6 there is the class $v_1^{r'(n)} x_{n+2,0}$ in degree $2^{n+3}-1$. Proposition 7.16 shows that this group must be 0, so this class must support a differential. This proposition also shows that the generator of degree $2^{n+3}-3$ neither supports η multiplication nor lies in the image of η multiplication. This forces the desired differential. $\square$

This lemma rules out most of the classes occuring degree $2^{n+3}-6$. With it and with the previous proposition, we can show the first extension.

Theorem 7.18. *In degree $2^{n+3}-6$, for $n > 1$, $\overline{\mathrm{THH}}(ko)$ is $\mathbb{Z}/2^n$.*

Proof. The group in question is an extension of $\mathbb{Z}/2^{n-1}$ by $\mathbb{Z}/2$. The mod two reduction is therefore either $\mathbb{F}_2 \oplus \mathbb{F}_2$ or $\mathbb{F}_2$. Since $\overline{\mathrm{THH}}(ko)$ is torsion free in degree $2^{n+3}-7$, we can conclude that degree $2^{n+3}-6$ is cyclic if we can show that only one class in that degree survives Spectral Sequence 6. Moreover, since this is an η-Bockstein spectral sequence converging to a ko_*-module, the largest possible power of η that can survive the spectral sequence is η^2. This means we need only determine the structure of the spectral sequence between degrees $2^{n+3}-8$ and $2^{n+3}-6$.

Lemma 7.15 shows that in this range of degrees, we have generators

$$v_1^2 x'_{2,2^{n-2}-1},\ v_1^3 x'_{2,2^{n-2}-1},\ v_1^{r'(n+1)-3} x_{n+1,0},\ \text{and}\ v_1^{r'(n)-3} x_{n+2,0},$$

together with their η multiples. Lemma 7.17 shows that the last pair of generators completely annihilates their η towers with a d_1-differential. In particular, these generators contribute nothing to classes in even degree. Proposition 7.16 shows that in degree $2^{n+3}-8$, there can be no classes which survive the spectral sequence. This implies that the class $v_1^2 x'_{2,2^{n-2}-1}$ must support a differential, killing its η tower. In particular, we see that there is only one class remaining in degree $2^{n+3}-6$, namely $v_1^3 x_{2,2^{n-2}-1}$. □

A slightly more complicated argument shows the extension in degree $2^{n+3}-10$ for $n>2$.

Theorem 7.19. *We have differentials*

$$d_2(v_1^2 x'_{2,2^{n-2}-1}) = \eta^2 x_{1,2^{n-1}-1},$$

for $n>1$, *and*

$$d_1(v_1 x'_{2,2^{n-2}-1}) = \eta v_1^8 x_{3,2^{n-3}-1},$$

for $n>2$.

In degree $2^{n+3}-10$, $\overline{\mathrm{THH}}(ko;ko/2)$ *is* $\mathbb{Z}/2^{n-1}$.

Proof. We saw in the proof of Theorem 7.18 that $v_1^2 x'_{2,2^{n-2}-1}$ must support a differential and that $v_1^3 x'_{2,2^{n-2}-1}$ is a permanent cycle. In degree $2^{n+3}-8$, we also have η^2 times the class $v_1 x'_{2,2^{n-2}-1}$. By Proposition 7.16, this class cannot survive to E_∞. For degree reasons, there are no possible sources for differentials targeting either η or η^2 on $v_1 x'_{2,2^{n-2}-1}$. We conclude that it must support a differential. For degree reasons, this differential must be either

$$d_1(v_1 x'_{2,2^{n-2}-1}) = \eta v_1^8 x'_{3,2^{n-3}-1} \text{ or } d_3(v_1 x'_{2,2^{n-2}-1}) = \eta^3 v_1^7 x'_{3,2^{n-3}-1}.$$

This observation alone allows us to compute the first differential. We first show that $x_{1,2^{n-1}-1}$ is a permanent cycle. For degree reasons, if this is the case, then the only possible way to kill the η tower on it is to have the desired d_2 differential on $v_1^2 x'_{2,2^{n-2}-1}$. Again for degree reasons, $x_{1,2^{n-1}-1}$ could only support a d_2 differential, hitting $\eta^2 v_1^7 x'_{3,2^{n-3}-1}$. This in turn would force the differential on $v_1 x'_{2,2^{n-2}-1}$ to be a d_1-differential, and since d_1-differentials commute with v_1 multiplication, we would have a d_1 differential

$$d_1(x'_{2,2^{n-2}-1}) = \eta v_1^7 x'_{3,2^{n-3}-1}.$$

In particular, this provides a contradiction, since now the target of our initial d_2 differential on $x_{1,2^{n-1}-1}$ is not present on E_2. We must therefore conclude that this class is a permanent cycle, and we have the required d_2 differential.

It remains to resolve whether the differential on $v_1 x'_{2,2^{n-2}-1}$ is a d_1 or a d_3. Since $v_1^2 x'_{2,2^{n-2}-1}$ supported a d_2 differential, targeting a class which is not v_1^2 divisible, and since v_1^2-multiplication commutes with d_2 differentials, we conclude that $x'_{2,2^{n-2}-1}$ must support a d_1 differential. The only possible target is $v_1^7 x'_{3,2^{n-3}-1}$, and this also forces the d_1 differential on $v_1 x'_{2,2^{n-2}-1}$.

Since the odd dimensional classes coming from $x_{n+1,0}$ and $x_{n+2,0}$ still annihilate each others η towers, the result of all of these differentials is a single class in dimension $2^{n+3}-10$, namely $\eta x_{1,2^{n-1}-1}$. Thus $\overline{\mathrm{THH}}(ko)$ is cyclic in this degree as well. □

We now repay a combinatorial debt.

Proof of Lemma 7.15. In all cases, the proof is very similar. We expand the integer $2^{n+3}-k$ 2-adically and then consider the possible contributions of classes $x_{j,m}$ and $x'_{j,m}$. In both cases, the argument breaks up into two pieces depending on k: the case of large j and a few exceptional, small cases.

We begin with the even case: $k = 2k'$ in the range. We recall that

$$|x'_{j,m} v_1^s| = 4 + 2^{j+2} + m2^{j+3} + 2s.$$

Reducing our problem modulo 16, we can rule out the existence of any instance of $x'_{1,m}$ except in degree $2^{n+3}-4$. In this case, we have $x'_{1,2^{n-1}-1}$, and we will henceforth assume $j > 1$. Both the degree of $x'_{j,m} v_1^s$ and $2^{n+3}-k$ are even, so we may rearrange and divide by two. Our problem is then one of counting triples (j, m, s) such that

$$2^{j+1} + m2^{j+2} + s = 2^{n+2} - k' - 2.$$

It is obvious that $r'(j) < 2^{j+1}$, so since we only want those $s < r'(j)$, we see that the binary digits of the three terms on the left hand side do not interact. We can conclude that $m = 2^{n-j}-1$, and $s = 2^{j+1} - k' - 2$. If we solve the recursive equation defining $r'(j)$, we can get a closed form for $r'(j)$ in terms of j:

$$r'(j) = \begin{cases} \frac{1}{3}(2^{j+2}-5) & j \text{ odd} \\ \frac{1}{3}(2^{j+2}-4) & j \text{ even.} \end{cases}$$

We can then rewrite the condition $s < r'(j)$ as

$$k' > \begin{cases} \frac{1}{3}(2^{j+1}-1) & j \text{ odd} \\ \frac{1}{3}(2^{j+1}-2) & j \text{ even.} \end{cases}$$

At this point, the argument is an elementary case-by-case check for the values of k' between 1 and 8. In particular, we immediately rule out $j \geq 4$. On the other hand, we do have $j = 3$ for $k' > 6$, $j = 2$ for $k' > 2$, and $j = 1$ for $k' > 1$. Since we have already found the corresponding values of m and s for a given value of j, we have completed the even case.

The odd case is easier to handle. The only subtlety is that the dyadic expansion of the degree of $x_{j,m}$ depends on the parity of j. For the reader's convenience, we recall the relevant degrees:

$$|v_1^s x_{j,m}| = \begin{cases} 2^j + 2^{j-2} + \cdots + 4 + 3 + 2^{j+3}m + 2s & j \text{ even} \\ 2^j + 2^{j-2} + \cdots + 2 + 3 + 2^{j+3}m + 2s & j \text{ odd.} \end{cases}$$

Write $k = 2k' - 1$, $2 \leq k' \leq 8$. Rearranging the equality $|v_1^s x_{j,m}| = 2^{n+3} - k$ allows us to see that we are counting the number of triples (j, m, s) such that

$$s + 2^{j+2}m + 2^{j-1} + 2^{j-3} + \cdots = 2^{n+2} - k' - 1.$$

Just as before, reducing modulo 16 shows that the only class with $j = 1$ in this range of degrees is $x_{1,2^{n-1}-1}$.

We start by linking j and n. It is clear that $n + 1 \geq j - 1$. We need a complementary bound. Assume that $n + 1 > j$. If we reduce modulo 2^{j+2}, then we see that $s \geq r'(j) + (2^{j+1} - k' + 1)$, and since we have already ruled out the possibility of $j = 1$, this is at least as large as $r'(j)$. We conclude that we can only have $j - 1 = n + 1$ or $j = n + 1$, and hence, $m = 0$. As long as $2 \leq k' \leq 8$, we can find both equalities represented as classes. Classes with $j = n + 2$ occur in the v_1 tower on $x_{n+2,0}$ between $v_1^{r'(n)-7}$ and $v_1^{r'(n)-1}$, while classes with $j = n + 1$ occur in the v_1 tower on $x_{n+1,0}$ between $v_1^{r'(n+1)-7}$ and $v_1^{r'(n+1)-1}$. This completes our analysis of the odd degrees. $\square$

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