The twisted K-homology of simple Lie groups

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In this talk, we illustrate how to compute the twisted K-homology of the simple, connected, simply connected compact Lie groups, largely following Douglas' computation thereof [3]. Since $H^3(G;\mathbb{Z}) = \mathbb{Z}$ and the cohomology and K-theory of these spaces are nice, the simple, simply connected Lie groups form a useful starting point for computations. Our goal is to prove part of one of the main results of [3]: If G is a simple, simply connected, compact Lie group of rank n and $\tau \in H^3(G)$ is non-zero, then

$$K_*^\tau(G) = E(y_1, \dots, y_{n-1}) \otimes \mathbb{Z}/c(G,\tau),$$

where $E(y_1, \dots, y_{n-1})$ denotes the exterior algebra on classes y_1 through y_{n-1} and $c(G,\tau)$ is an integer depending on G and τ.

We begin rationally, running the twisted Atiyah-Hirzebruch spectral sequence to compute the rational twisted K-theory of a space [1]. We know that $H(G;\mathbb{Q})$ is an exterior algebra on a class in dimension 3 tensored with other exterior factors. If our chosen twisting $\tau \in H^3(G)$ is non-zero, then the analysis of the d_3 differential in [1] shows that the Atiyah-Hirzebruch E_2 term is an acyclic complex, since rationally, $d_3(x) = -x \smile \tau$. This implies that if we choose a non-zero twisting class, then the twisted K-theory of G is all torsion for G of our form.

For technical reasons, it will be easier for us to compute the twisted K-homology of G, rather than the twisted K-cohomology. To define twisted K-homology, we consider bundles over a space X with fiber the K-theory spectrum, satisfying certain technical conditions. The homology is then naturally defined as the homotopy of the total space of the bundle, relative to the base, and this is a homology theory on the category $\mathcal{K}$ of pairs (X,E), where $E \to X$ is a bundle of the desired form, and maps $(X,E) \to (Y,F)$ are bundle maps that are fiberwise equivalences [3]. Since our notion of twisted homology is a homology theory, if $S_\bullet$ is a simplicial object in $\mathcal{K}$, we have a spectral sequence of the form

$$E_2 = H_*(E_*(S)) \Rightarrow E_*(|S|).$$

Here $E_*(S)_\bullet$ is the simplicial graded abelian group that in position p is just $E_*(S_p)$.

We apply this machinery to a special, computable case. Fix a twisting $\tau \in H^3(G) = H^2(\Omega G)$, where ΩG is the loop space of G. Let $S_\bullet = B(*, \Omega G, *_\tau)_\bullet$ be the simplicial object in $\mathcal{K}$ where

$$S_n = \big((\Omega G)^{\times n}, (\Omega G)^{\times n} \times K\big).$$

The simplicial maps are somewhat trickier. The maps on the underlying base spaces are just the ordinary maps in the two-sided bar complex $B(*, \Omega G, *)_\bullet$. On the bundles, all of the face maps but the last are just the obvious lifts of the maps on spaces to the trivial K-bundle over them. The last face map is twisted by the composite

$$\Omega G \times K \xrightarrow{\tau \times \mathrm{Id}} K(\mathbb{Z},2) \times K \to K,$$

where the last map is the action of $K(\mathbb{Z},2)$ on K. This construction is entirely analogous to Brown's simplicial construction of the Serre spectral sequence, building fibrations as twisted products in a way that the twisting is controlled simplicially [2]. The geometric realization of this simplicial object in $\mathcal{K}$ is therefore just the twisted K bundle over $B\Omega G = G$ determined by the twisting class τ.

Since ΩG has only even cells, we know that

$$K_*(\Omega G^{\times n}) = K_*(\Omega G)^{\otimes n}.$$

This lets us identify the spectral sequence defined above as a twisted form of the Rothenberg-Steenrod / homology Eilenberg-Moore spectral sequence:

$$E_2 = \mathrm{Tor}^{K_*(\Omega G)}(K_*, K_*^\tau) \Rightarrow K_*^\tau(G),$$

where K_*^τ is the $K_*(\Omega G)$-module induced by the last face map. The computation now varies depending on the group.

If G is $SU(n+1)$ or $Sp(n)$, then the homology of ΩG is polynomial on n generators of even degree. This implies that the Atiyah-Hirzebruch spectral sequence computing K-homology also collapses, and we see that $K_*(\Omega G)$ is polynomial on n generators. Since this is acting on K_*, we see that for the module K_*^τ, the generator x_k acts as multiplication by a number c_k. At this point, we deviate slightly from the presentation in [3]. For these classical groups (and indeed for $Spin(n)$ as well), we use the fibration in topological groups

$$\Omega S^{2n+1} \to SU(n) \to SU(n+1)$$

and the analogue for $Sp(n)$. If we consider the two sided bar complex in $\mathcal{K}$ given by $B(*, \Omega S^{2n+1}, SU(n)_\tau)$, where ΩS^{2n+1} and $*$ have the trivial bundle associated to them and $SU(n)_\tau$ is $SU(n)$ with the twisted K-bundle over it corresponding to the twisting τ, then the realization is $SU(n+1)_\tau$. This gives us a spectral sequence

$$E_2 = Tor^{K_*(\Omega S^{2n+1})}(K_*, K_*^\tau(SU(n))) \Rightarrow K_*^\tau(SU(n+1)).$$

As a ring, $K_*(\Omega S^{2n+1})$ is polynomial on one generator that corresponds to the last polynomial generator of $K_*(\Omega SU(n+1))$. In other words, this slight recasting allows us to isolate the contribution of each generator of $K_*(\Omega SU(n+1))$. The Tor groups are easy to compute, since we consider only a polynomial generator, and for degree reasons, we conclude that the spectral sequence collapses with no possible extensions. This shows Douglas' theorem for $SU(n)$ and $Sp(n+1)$.

For the exceptional groups, we return to Douglas' initial formulation. We need to understand the K-homology of ΩG. Here we use the fact that the map $G \to K(\mathbb{Z},3)$ inducing $1 \in H^3(G)$ is an equivalence through a range that increases with the rank of G. We can use this, together with the computation of $K_*(\mathbb{C}P^\infty)$, to compute the K-homology of ΩG as a ring [4]. Feeding this into the twisted Rothenberg-Steenrod spectral sequence allows us to see Douglas' aforementioned theorem for the exceptional groups.

References

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