Recent Computational work on EO_n

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(joint work with Michael J. Hopkins, Douglas C. Ravenel)

The Hopkins-Miller theorem ensures that there is an action of the the Morava stabilizer group S_n on the Lubin-Tate spectrum E_n by E_∞ ring maps. If n is divisible by $p-1$, then S_n has infinite cohomological dimension arising from p-torsion elements in the group. Since the $K(n)$-local sphere, $L_{K(n)}S^0$, is the homotopy fixed points of S_n acting on E_n [1], it is hoped that by restricting attention to finite subgroup (which carry the bulk of the higher cohomology), we can understand computationally the homotopy of $L_{K(n)}S^0$. Indeed, this was successfully done by Adams-Baird and Ravenel for $n = 1$ [2] and by Goerss-Henn-Mahowald-Rezk, and Behrens for $n = 2$, $p = 3$ [3, 4]. The algebraic approximation to $\pi_* L_{K(n)}S^0$ by a finite subgroup G can be made rigid by considering the homotopy fixed point spectra $EO_n(G) = E_n^{hG}$ of Hopkins and Miller.

The computations of the homotopy of EO_{p-1} by Hopkins and Miller allowed Nave to demonstrate quite strong results about the non-existence of Smith-Toda complexes [5]. This computation relied on an understanding of the homotopy of the Lubin-Tate spectrum E_{p-1} as an algebra over $\mathbb{Z}/p$ and was facilitated by a reinterpretation of this algebra using judiciously chosen invariant elements in the mod p homotopy. This talk focused on generalizations of this computation to higher heights divisible by $p-1$.

Using formal group machinery, Devinatz and Hopkins computed the action of S_n on the homotopy groups of E_n [6]. While complete, their description was difficult to apply to computations. Hopkins conjectured that there is a more natural collection of generators of E_{n*} for which the action of finite subgroups is especially simple. We begin by recalling that S_n is the group of units in the maximal order $\mathcal{O}_n$ of the division algebra D_n over $\mathbb{Q}_p$ of Hasse invariant $\frac{1}{n}$. The natural left action of S_n on $\mathcal{O}_n$ commutes with the right action of $\mathbb{Z}_{p^n}$, the Witt vectors for $\mathbb{F}_{p^n}$, and this makes $\mathcal{O}_n$ into a $\mathbb{Z}_{p^n}[S_n]$-module, the Dieudonné module M_n.

Conjecture 1 (Hopkins). *If $G \subset S_n$ is a finite subgroup, then there is a G-equivariant isomorphism*

$$E_{n*} \cong S_{\mathbb{Z}_{p^n}}(M_n)[\Delta^{-1}]^{\wedge}_I,$$

where S denotes the symmetric algebra functor, M_n is placed in degree -2, Δ is a trivial representation corresponding to the multiplicative norm over the group on M_n, and I is an ideal in degree 0.

The conjecture is most important when $p-1$ divides n and p divides the order of G, as here an obstruction theory argument reduces the proof of this conjecture to verifying it for $\mathbb{Z}/p \subset G$. By using the theory of formal A-modules, Hopkins, Ravenel, and I have made significant headway in proving this conjecture.

Let $A = \mathbb{Z}_p[\zeta]$, where ζ is a p^{th} root of unity. There is an inclusion of A into $\mathcal{O}_n$, and this induces a formal A-module structure on F_n, the Honda formal group of height n. If we write $n = (p-1)f$, then as a formal A-module, F_n has height f, and there is a Lubin-Tate deformation theory of formal A-modules similar to that of formal groups, corepresented by a ring E^A_{f*}. Since A has a p^{th} root of unity, we can easily describe the action of $\mathbb{Z}/p$ on E^A_{f*}. Moreover, if we forget down to formal groups, then we get a natural $\mathbb{Z}/p$-equivariant map of corepresenting rings:

$$E_{(p-1)f*} \to E^A_{f*}.$$

This map is surjective, and it therefore produces a spectral sequence computing the cohomology of $\mathbb{Z}/p$ with coefficients in $E_{(p-1)f*}$ from the cohomology of $\mathbb{Z}/p$ with coefficients in E^A_{f*}. This spectral sequence has the advantage of having a well understood algebraic model, and by mirroring Devinatz and Hopkins original arguments, we have been able to show that through a large range, these spectral sequences coincide.

Assuming Hopkins' conjecture, we have also been able to describe the E_2 term of and compute the differentials in the homotopy fixed point spectral sequence of $\pi_* EO_{(p-1)f}(\mathbb{Z}/p)$. The differentials generalize those found by Hopkins and Miller in their original analysis of $\pi_* EO_{p-1}(\mathbb{Z}/p)$, and their construction is very similar.

Proposition 1. *As an algebra, the E_2 term of the homotopy fixed point spectral sequence for $EO_{(p-1)f}(\mathbb{Z}/p)$ is*

$$E(\alpha_1, \dots, \alpha_f) \otimes P(\beta) \otimes P(\delta_1, \dots, \delta_f^{\pm 1}) \oplus \mathit{Free},$$

where the bidegrees of the elements, written as $(t-s, s)$ are $|\alpha_i| = (-3, 1)$, $|\beta| = (-2, 0)$, and $|\delta_i| = -2p$.

The elements referred to as "Free" arise from free summands of E_{n*} and pair trivially with all elements of higher filtration. They also lie in the image of the transfer map from $EO_{(p-1)f}(\{1\})$, making them permanent cycles. The element β is the periodicity generator of $\mathbb{Z}/p$ cohomology, and the elements α_i, β, and δ_i are related by the power operation

$$\beta \mathcal{P}^0(\alpha_i) = \langle \alpha_i, \dots, \alpha_i \rangle = \beta \delta_i.$$

Using formal group arguments lifted from the analogous story for E^A_f, we can relate the elements α_i to the elements $h_{i,0}$, appropriately translated by powers of δ_f, and the corresponding elements $b_{i,0}$ are similarly related to the classes labeled δ_i and β. These relations allow us to understand differentials that arise on norm classes in the homotopy fixed point spectral sequence.

Proposition 2. *The differentials are algebraically determined by the following properties.*

1. *There are differentials $d_{1+2(p^i-1)}(\delta_f^{p^{i-1}}) = \delta_f^{p^{i-1}} h_{i,0} \beta^{p^i - 1}$.*
2. *There are corresponding $d_{1+2(p-1)(p^i-1)}$ Toda style differentials truncating the β towers on δ_i.*
3. *The classes $\Delta_i = \delta_i / \delta_f$ are permanent cycles.*

(4) *The class* $\delta_f^{p^f}$ *is a permanent cycle, and these describe all of the differentials.*

With the exception of the final statement, these results are all proved in essentially the same way: there exist universal examples for certain differentials in homotopy fixed point spectral sequences. Let g be a generator of $\mathbb{Z}/p$. If $u\colon S^k \to E_n$, then let

$$Nu = u \cdot gu \ldots g^{p-1}u\colon S^{pk} \to E_n.$$

Since S_n acts on E_n by E_∞ maps, this map is $\mathbb{Z}/p$-equivariant and therefore descends to homotopy fixed points. The spectrum $(S^{pk})^{h\mathbb{Z}/p}$ is the Spanier-Whitehead dual of a Thom spectrum, and the attaching maps of the top cell determine differentials on the class represented by Nu. The classes $\delta_f^{p^i}$ and the classes Δ_i are of the form Nu for appropriately chosen u, and this general argument produces the described differentials.

References

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