Economics 144
Economic Forecasting

Lecture 7
Characterizing Cycles
Autoregressive Models

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Today's Class

- Cycles
- Autoregressive (AR) Models
  - The AR(1) Process
  - The AR(2) Process
  - The AR(p) Process
- Chain Rule for Forecasting
- R Example

Cycles 1 of 3

- Def: Cycle = Time series pattern of periodic fluctuations.

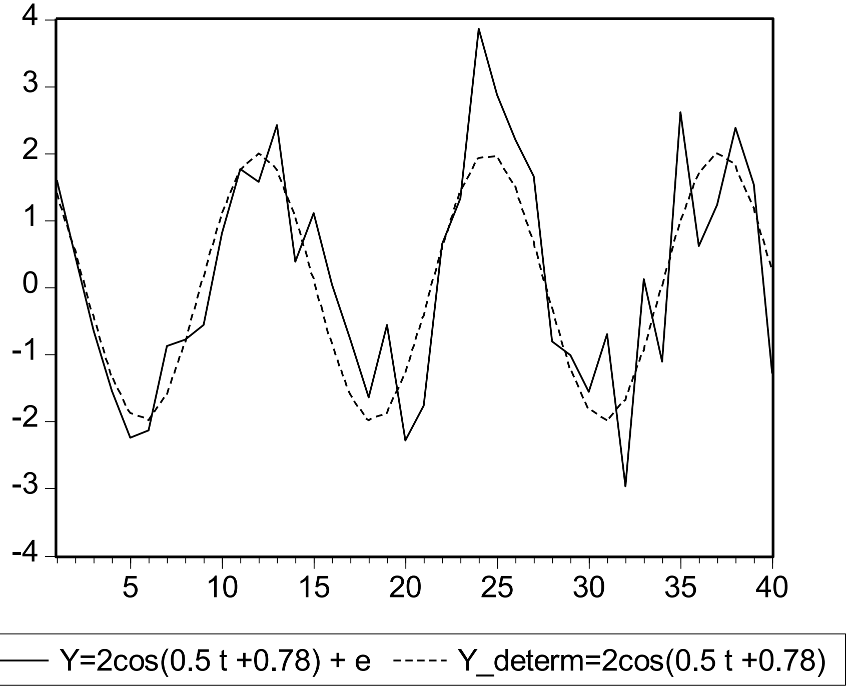
  – Deterministic: Common in engineering, physical sciences, etc. Example: $Y_t = 2\cos(0.5t+0.8)+\varepsilon_t$

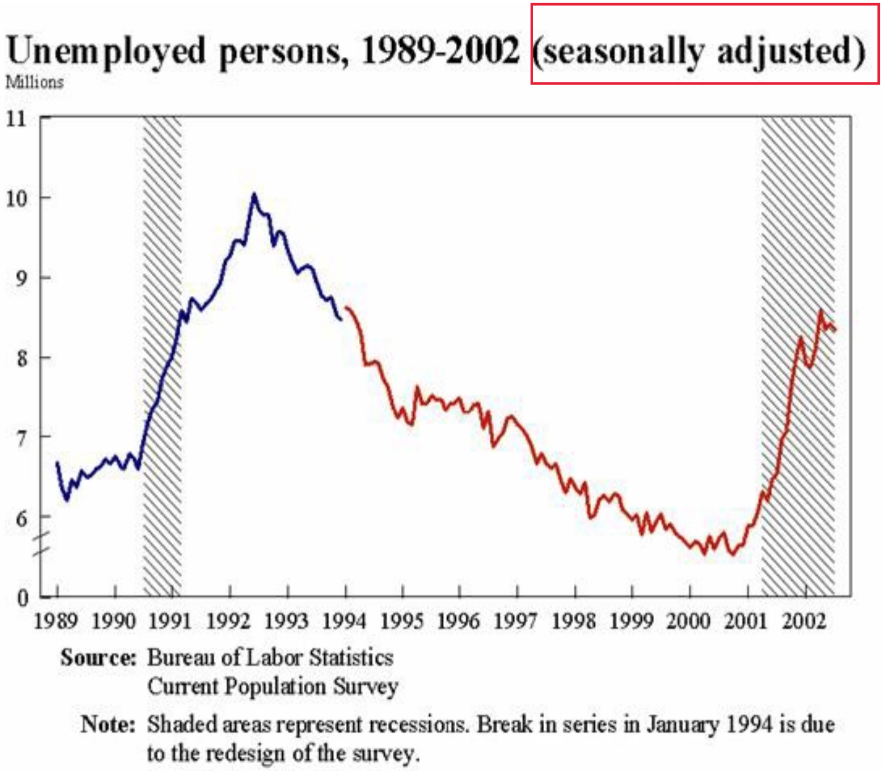
  – Stochastic: Common in economics, business, etc. Example: $Y_t = 0.5Y_{t-1} + 0.3Y_{t-2} + \varepsilon_t$

estimate

Cycles 2 of 3

Persistence
upstate and bottomstate





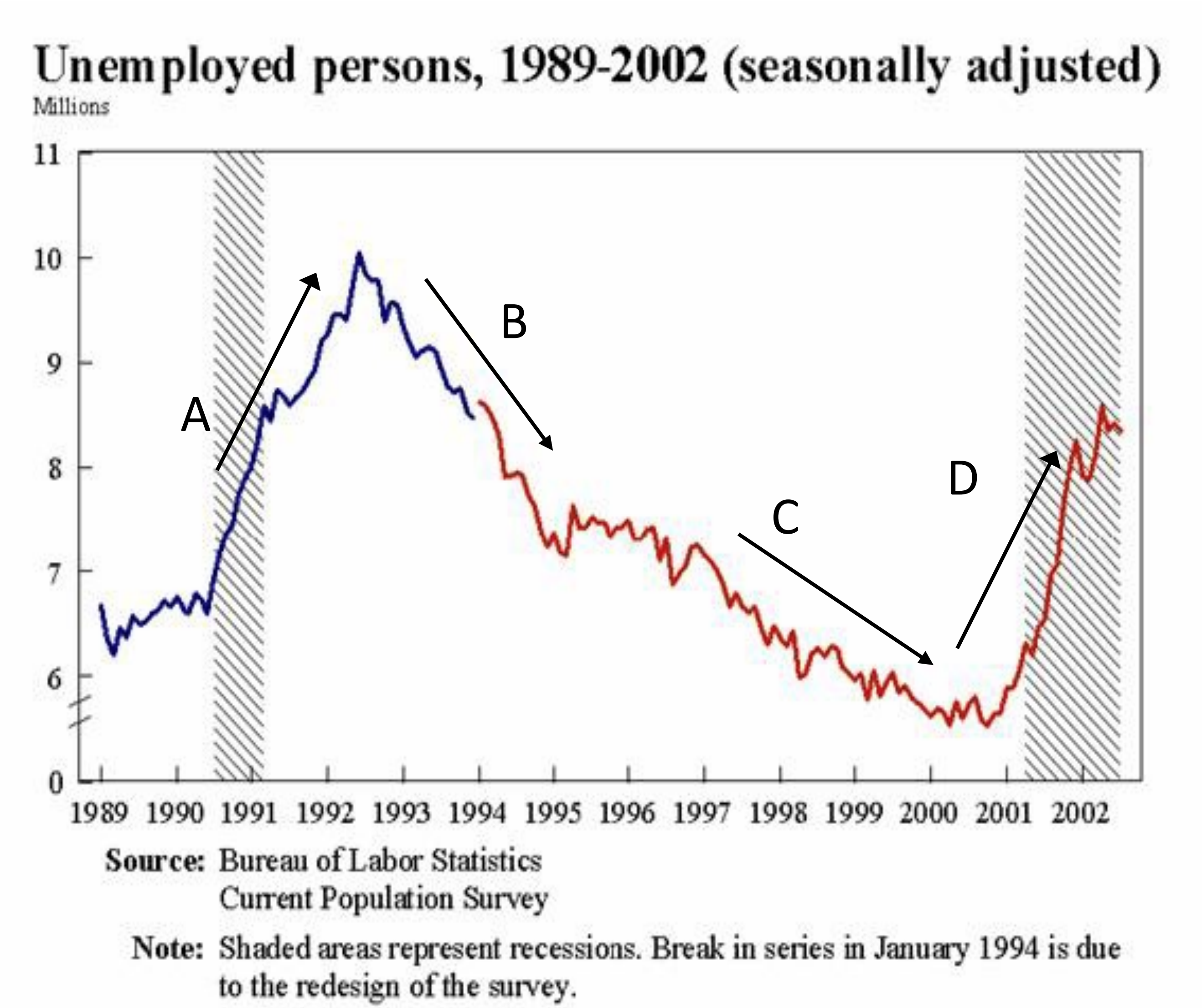
Deterministic

$$Y_t = 2\cos(0.5t+0.8)+\varepsilon_t$$

Stochastic

$$Y_t = 0.5Y_{t-1} + 0.3Y_{t-2} + \varepsilon_t$$

Cycles 3 of 3


Unemployed persons, 1989-2002 (seasonally adjusted)
Millions
11
10
9
8
7
6
0
1989
1990
1991
1992
1993
1994
1995
1996
1997
1998
1999
2000
2001
2002
A
B
C
D
Source: Bureau of Labor Statistics
Current Population Survey
Note: Shaded areas represent recessions. Break in series in January 1994 is due to the redesign of the survey.

Autoregressive Models

- Def: AR(p) = Autoregressive process of order p≥0: $Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \varepsilon_t$, where $\varepsilon_t \sim (0, \sigma^2)$, and φ = persistence parameter.

the influence past observations exert on the current
see textbook a

- Examples:
  - AR(1): $Y_t = c + \phi_1 Y_{t-1} + \varepsilon_t$
  - AR(2): $Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$
  - AR(6): $Y_t = c + \phi_6 Y_{t-6} + \varepsilon_t$

Autoregressive Models

For every AR(p) process, we need to address the following 3 questions:

1. What does a time series of an AR process look like?
2. What do the corresponding ACFs and PACFs look like?
3. What is the optimal forecast?

Autoregressive Models
Example: AR(1) Process

- (1) What does a time series of an AR process look like?

- Consider the following AR(1) process:

$$Y_t = c + \phi\, Y_{t-1} + \varepsilon_t$$

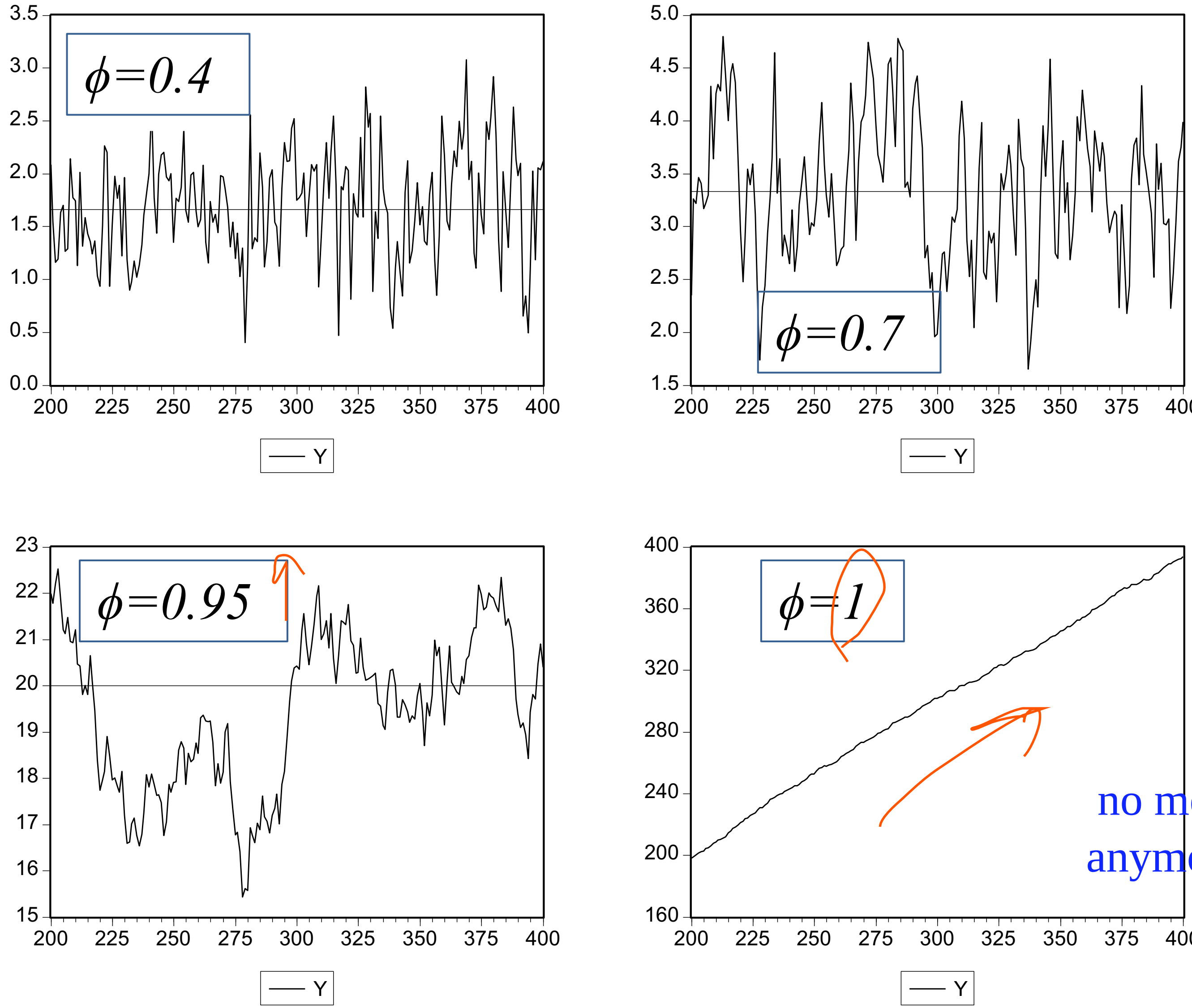
the AR mean reverts slowly, as ϕ goes up, harder to mean reverts, eventually goes up when $\phi = 1$ (see figures next slide

- We can plot this process for c =1, and different values of φ, e.g., for φ=0.4, φ=0.7, φ=0.95, and φ=1.

- We can show that $E(Y_t)=c+\varphi\mu$, and $\sigma^2(Y_t) = \sigma^2_\varepsilon / (1-\varphi^2)$

Note: Since $E(Y_t)=\mu \rightarrow \mu = c+\varphi\mu \rightarrow \mu = c/(1-\varphi)$

AR(1): $Y_t = c + \phi Y_{t-1} + \varepsilon_t$



no mean reverting anymore

Autoregressive Models
Example: AR(1) Process

- (2) What do the corresponding ACFs and PACFs look like?

- ACF:
  - We would expect to see that $\rho_1 = p_1 = \varphi$, and all others decay to zero according to $\rho_k = \varphi^k$.

- PACF:
  - We would expect to see only 1 spike different from zero, i.e., $p_1 = \varphi$, and all others equal to zero ($p_k=0$, $k>1$).

Actually there is a function in R that directly tells AR(?)

Autocorrelation Functions of Simulated AR(1) Processes

Sample: 2 1000
Included observations: 999

Autocorrelation	Partial Correlation		AC	PAC
		1	0.334	0.334
		2	0.082	-0.034
		3	0.018	0.002
		4	0.043	0.044
		5	-0.022	-0.056
		6	0.006	0.033
		7	0.060	0.058
		8	0.016	-0.031
		9	0.041	0.053
		10	0.051	0.024
		11	0.028	-0.006
		12	0.001	-0.001
		13	0.036	0.037
		14	0.008	-0.023
		15	-0.026	-0.023

$\boxed{\phi = 0.4}$

Small persistence
→faster decay to 0

Sample: 2 1000
Included observations: 999

Autocorrelation	Partial Correlation		AC	PAC
		1	0.732	0.732
		2	0.546	0.021
		3	0.400	-0.015
		4	0.279	-0.030
		5	0.168	-0.058
		6	0.085	-0.025
		7	0.032	-0.003
		8	-0.026	-0.057
		9	-0.071	-0.033
		10	-0.084	0.011
		11	-0.060	0.057
		12	-0.048	-0.010
		13	-0.060	-0.053
		14	-0.063	-0.013
		15	-0.054	0.006

$\boxed{\phi = 0.7}$

$$Y_t = c + \phi Y_{t-1} + \varepsilon_t$$

Sample: 2 1000
Included observations: 999

Autocorrelation	Partial Correlation		AC	PAC
		1	0.951	0.951
		2	0.904	-0.006
		3	0.861	0.009
		4	0.817	-0.019
		5	0.774	-0.016
		6	0.734	0.005
		7	0.696	-0.002
		8	0.658	-0.022
		9	0.622	0.013
		10	0.590	0.010
		11	0.561	0.011
		12	0.534	0.017
		13	0.508	-0.014
		14	0.481	-0.016
		15	0.457	0.007

$\boxed{\phi = 0.95}$

Large persistence
→slower decay to 0

For all three cases $\boxed{\hat{\rho}_1 = \hat{p}_1 = \hat{\phi}}$

Autoregressive Models
Example: AR(1) Process

- A useful Property for AR Processes:

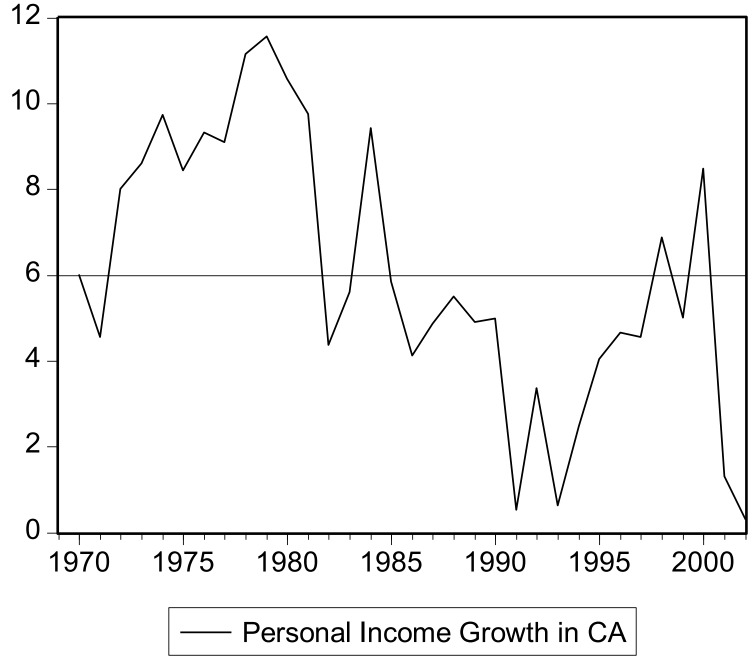
IFF

A necessary and sufficient condition for an AR(1) process $Y_t = c + \phi Y_{t-1} + \varepsilon_t$ to be covariance stationary is that $|\phi|<1$.

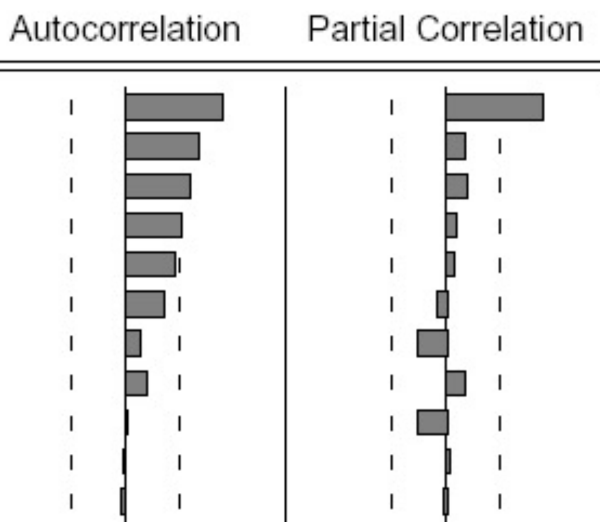
if | ϕ| >= 1, then mean does not revert, variance is not constant, assumptions about covariance stationary is broken (why we need to extract trend and seasonality and apply ARMA to remainder)

Autoregressive Models

Example: AR(1) Process



Sample: 1969 2002
Included observations: 33



	AC	PAC
1	$\boxed{0.629}$	$\boxed{0.629}$
2	0.471	0.125
3	0.417	0.134
4	0.365	0.059
5	0.327	0.051
6	0.247	-0.050
7	0.098	-0.180
8	0.135	0.126
9	0.024	-0.179
10	-0.009	0.021
11	-0.021	-0.006

Q: Does this series look like an AR(1) process?

A: Yes, since $\rho_1 = p_1 = \varphi$ and $\rho_k \rightarrow 0$ for $k>1$.

$$\hat{\rho}_1 = \hat{p}_1 = \hat{\phi} = 0.63$$

Autoregressive Models

Forecasting in an AR(1) Process

- Consider first the 1-step-ahead forecast, h = 1:
  AR(1): $Y_t = c + \phi Y_{t-1} + \varepsilon_t \rightarrow Y_{t+1} = c + \phi Y_t + \varepsilon_{t+1}$
- Optimal Point Forecast: $f_{t,1} = \mathrm{E}(Y_{t+1}|I_t) = c + \phi Y_t$
- One-period-ahead Forecast Error: $e_{t,1} = Y_{t+1} - f_{t,1} = \varepsilon_{t+1}$
- Uncertainty of the Forecast: $\sigma^2_{t+1|t} = var(Y_{t+1}|I_t) = \sigma^2_\varepsilon$
- Density Forecast: $f(Y_{t+1}|I_t) \sim N(c + \phi Y_t\, , \sigma^2_\varepsilon)$
  - Note: We can compute the confidence intervals from the density forecast.

Please go over the steps for h=1 and h=2 (Section 7.2[a])

Autoregressive Models
Forecasting in an AR(1) Process

- Consider the k-step-ahead forecast, h = k for the AR(1): $Y_t = c + \phi Y_{t-1} + \varepsilon_t$ process:

- Optimal Point Forecast: $f_{t,k}$ = E($Y_{t+k}|I_t) = c(1+\phi+\ldots+\phi^{k-1}) + \phi^k Y_t$

- k-period-ahead Forecast Error:
  $e_{t,k} = Y_{t+k} - f_{t,k} = \varepsilon_{t+k} + \phi\varepsilon_{t+k-1} + \ldots + \phi^{k-1}\varepsilon_{t+1}$

- Uncertainty of the Forecast:
  $\sigma^2_{t+k|t} = var(Y_{t+k}|I_t) = \sigma^2_\varepsilon(1+\phi^2+\phi^4+\ldots+\phi^{2(k-1)})$

- Density Forecast: $f(Y_{t+k}|I_t) \sim N(\mu_{t+k|t}, \sigma^2_{t+k|t})$
  - Note: We can compute the confidence intervals from the density forecast.

Autoregressive Models
Forecasting in an AR(1) Process

- Recall that for covariance-stationary processes, $|\varphi|<1$ and $\varphi^k \rightarrow 0$ for large values of k ($k \rightarrow \infty$). Therefore, the optimal forecast does not depend on the information set, and thus has a 'short-term memory'.

$$
\begin{cases}
f_{t,k} = c(1+\phi+\phi^2+\ldots) = c/(1-\phi) \\
\sigma^2_{t+1|t} = \sigma^2_{\varepsilon}(1+\phi^2+\phi^4+\ldots) = = \sigma^2_{\varepsilon}/(1-\phi^2)
\end{cases}
$$

Autoregressive Models
Example: AR(2) Process

- **Example**: $Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \varepsilon_t$

- A useful Property for an AR(2) Process:
  The necessary conditions for an AR(2) process to be covariance stationary are that $-1<\phi_2<1$ and $-2<\phi_1<2$, and the sufficient conditions are that $\phi_1+\phi_2<1$ and $\phi_2-\phi_1<1$.

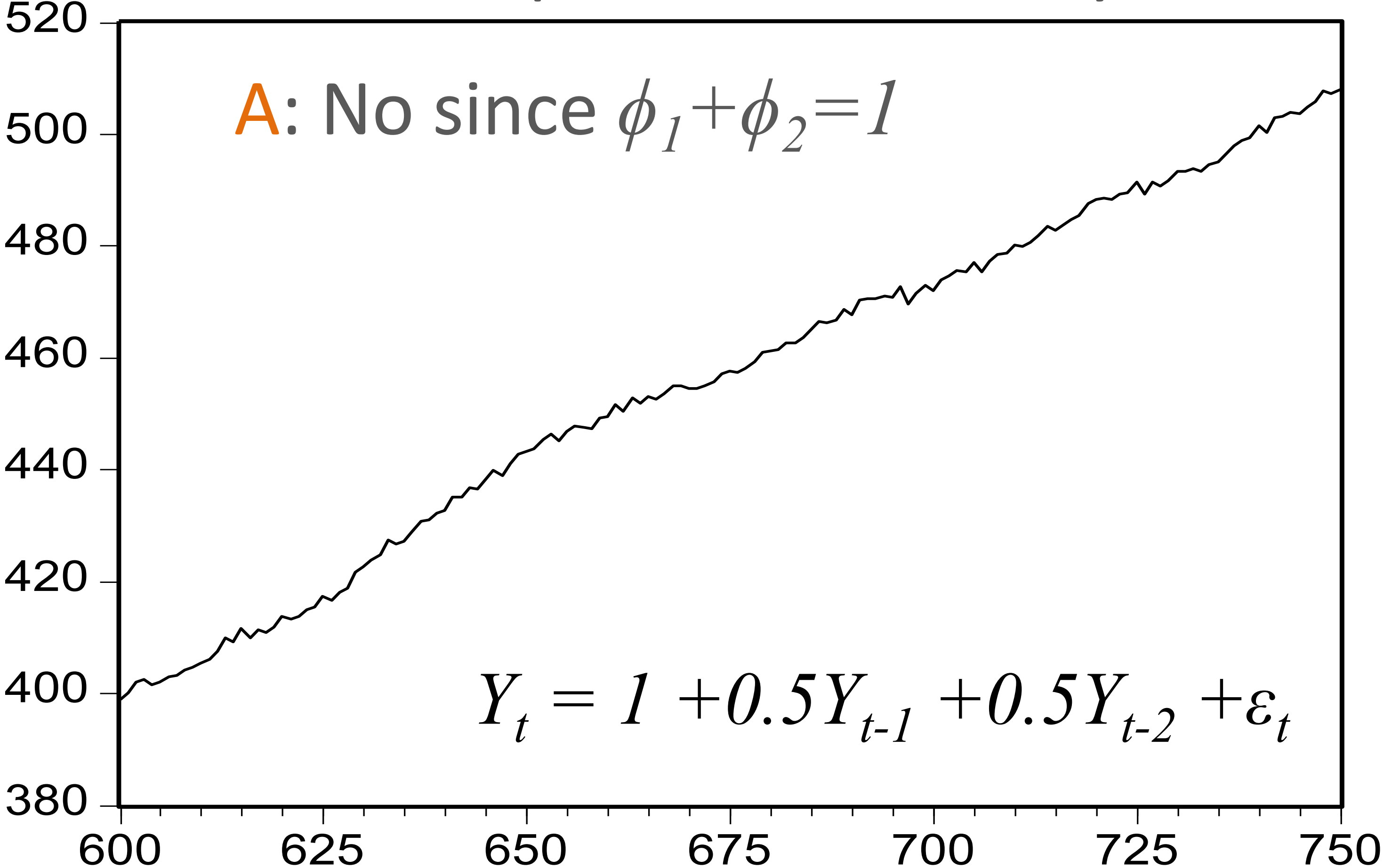
- **Note**: For an AR(2) the respective unconditional mean is given by:

$$\mu = \frac{c}{1 - \phi_1 - \phi_2}$$

Autoregressive Models
Example: AR(2) Process

Q: Is this process stationary?

A: No since $\phi_1+\phi_2=1$

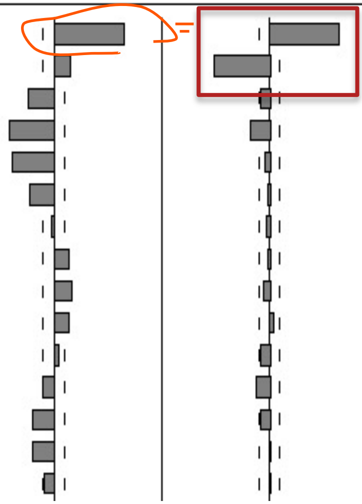


$$Y_t = 1 + 0.5Y_{t-1} + 0.5Y_{t-2} + \varepsilon_t$$

Autoregressive Models
Example: AR(2) Process

$$Y_t = 1 + Y_{t-1} - 0.5Y_{t-2} + \varepsilon_t$$

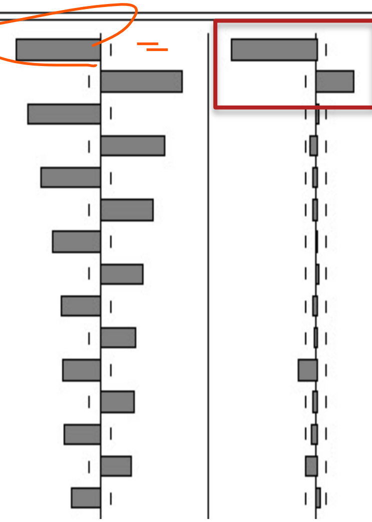
Sample: 300 700
Included observations: 401



	AC	PAC
1	0.668	0.668
2	0.148	-0.537
3	-0.241	-0.085
4	-0.430	-0.181
5	-0.409	-0.038
6	-0.230	-0.004
7	-0.017	-0.026
8	0.136	-0.003
9	0.173	-0.055
10	0.138	0.041
11	0.041	-0.084
12	-0.102	-0.119
13	-0.211	-0.074
14	-0.203	0.009
15	-0.091	0.011

$$Y_t = 1 - 0.5Y_{t-1} + 0.4Y_{t-2} + \varepsilon_t$$

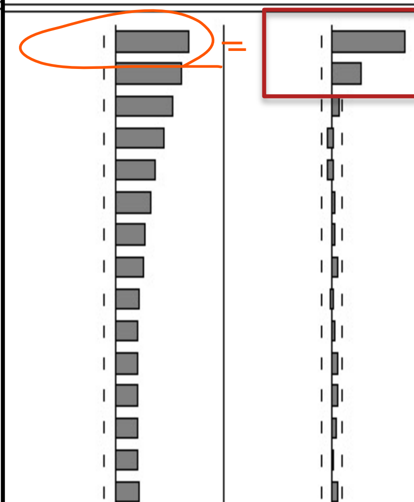
Sample: 300 700
Included observations: 401



	AC	PAC
1	-0.810	-0.810
2	0.782	0.365
3	-0.692	0.023
4	0.622	-0.052
5	-0.566	-0.020
6	0.500	-0.025
7	-0.451	0.003
8	0.408	0.021
9	-0.373	-0.024
10	0.336	-0.011
11	-0.355	-0.166
12	0.323	-0.017
13	-0.336	-0.039
14	0.291	-0.103
15	-0.274	0.033

$$Y_t = 1 + 0.5Y_{t-1} + 0.3Y_{t-2} + \varepsilon_t$$

Sample: 300 700
Included observations: 401



	AC	PAC
1	0.701	0.701
2	0.637	0.286
3	0.553	0.073
4	0.459	-0.035
5	0.378	-0.042
6	0.329	0.023
7	0.283	0.019
8	0.263	0.049
9	0.223	-0.014
10	0.212	0.027
11	0.212	0.051
12	0.213	0.044
13	0.215	0.033
14	0.211	0.005
15	0.223	0.044

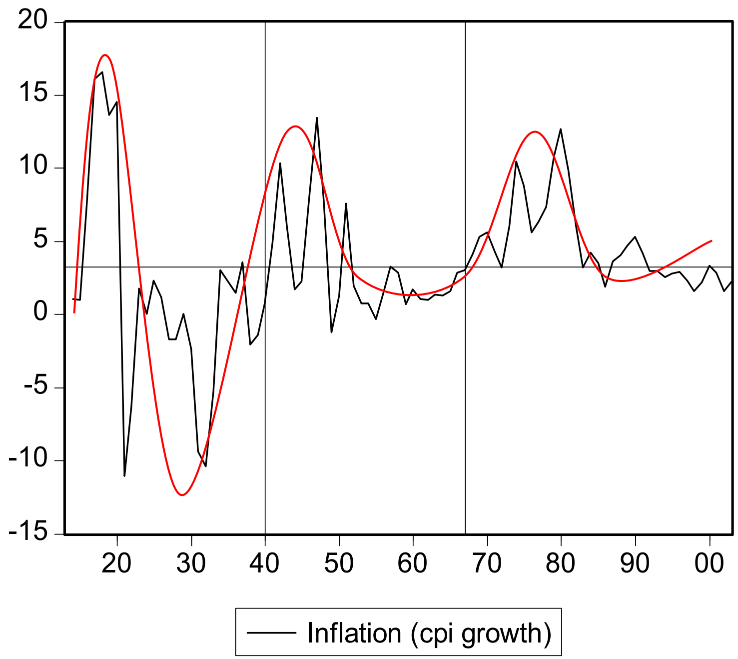
All three AR(2) Processes have in common:

ρ_1 = p_1, p_2 = ϕ_2, and $p_1 \neq 0$ and $p_2 \neq 0$, all other p_k =0 (k>2)

Autoregressive Models

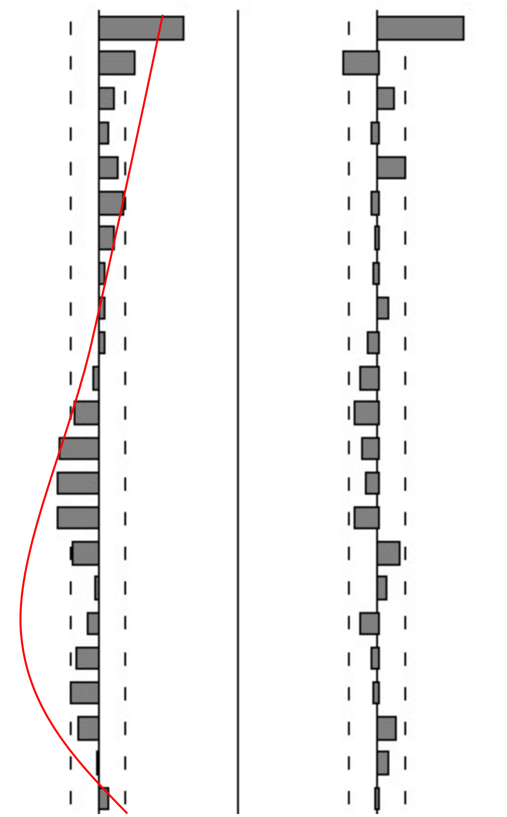
Example: AR(2) Process

1913-2003 Inflation Rate



Sample: 1913 2003
Included observations: 90

Autocorrelation | Partial Correlation



	AC	PAC
1	0.639	0.639
2	0.259	-0.252
3	0.117	0.128
4	0.066	-0.038
5	0.144	0.210
6	0.181	-0.039
7	0.115	-0.016
8	0.039	-0.032
9	0.039	0.089
10	0.035	-0.067
11	-0.047	-0.128
12	-0.174	-0.162
13	-0.280	-0.114
14	-0.303	-0.081
15	-0.306	-0.172
16	-0.184	0.156
17	-0.032	0.073
18	-0.075	-0.125
19	-0.161	-0.034
20	-0.208	-0.031
21	-0.155	0.137
22	-0.008	0.081
23	0.073	-0.005

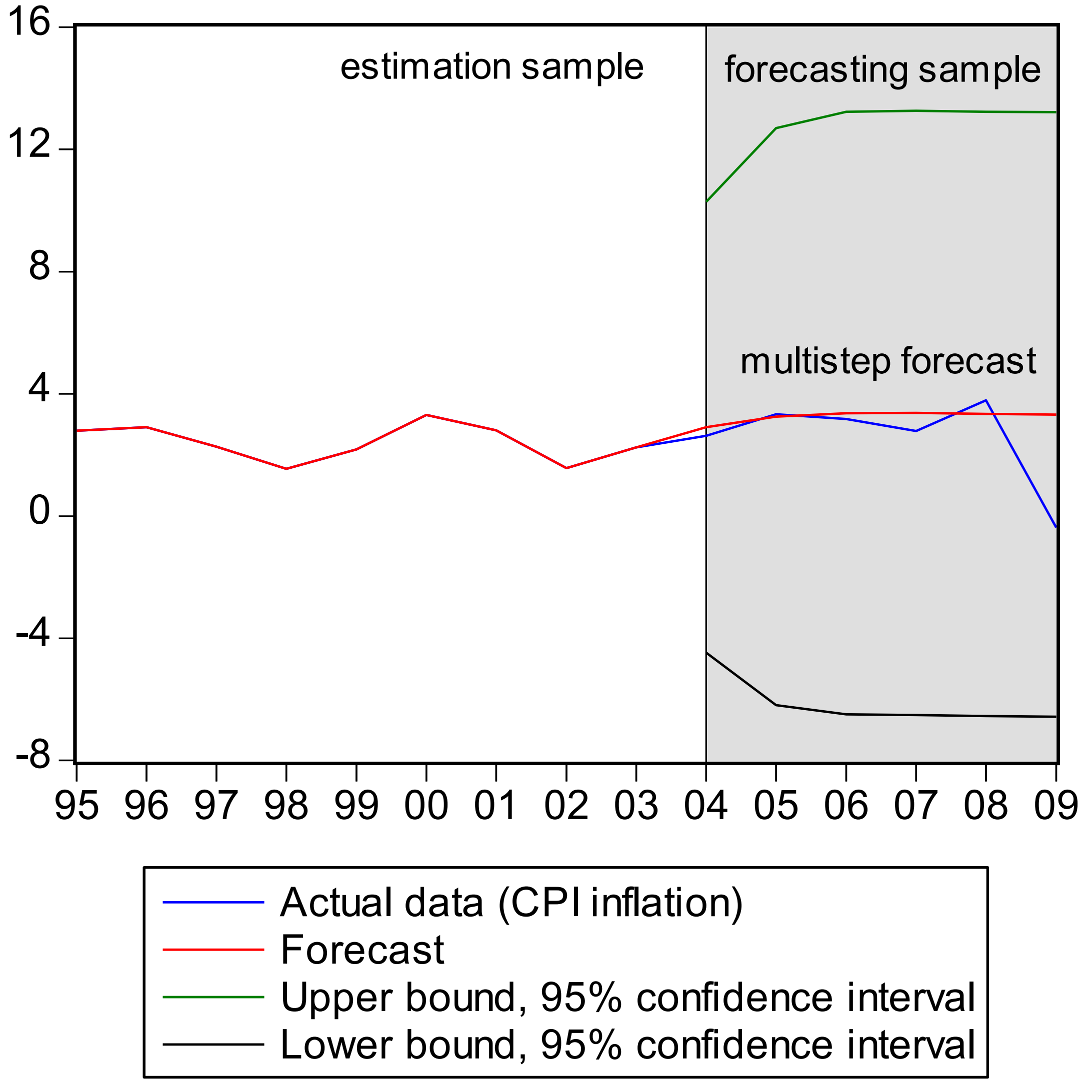
Q: What model would you suggest?

A: An AR(2) with $\phi 2 = -0.25$

Multistep Forecast of the U.S. Inflation Rate

$h=1$ 2004	$f_{t,1} = \hat{c} + \hat{\phi}_1 Y_t + \hat{\phi}_2 Y_{t-1} =$ $= 1.49 + 0.79 \times 2.25 - 0.25 \times 1.56 =$ ≈ 2.90	$\sigma^2_{t+1\|t} = \hat{\sigma}^2_\varepsilon = 3.74^2$	$f(Y_{t+1} \mid I_t) \to N(\mu_{t+1\|t}, \sigma^2_{t+1\|t})$ $= N(2.90, 3.74^2)$
$h=2$ 2005	$f_{t,2} = \hat{c} + \hat{\phi}_1 f_{t,1} + \hat{\phi}_2 Y_t =$ $= 1.49 + 0.79 \times 2.90 - 0.25 \times 2.25 =$ ≈ 3.25	$\sigma^2_{t+2\|t} = \hat{\sigma}^2_\varepsilon (1 + \hat{\phi}_1^{\ 2}) =$ $= 3.74^2 (1 + 0.79^2) =$ $\approx 4.81^2$	$f(Y_{t+2} \mid I_t) \to N(3.25, 4.81^2)$
$h=3$ 2006	$f_{t,3} = \hat{c} + \hat{\phi}_1 f_{t,2} + \hat{\phi}_2 f_{t,1} =$ $= 1.49 + 0.79 \times 3.25 - 0.25 \times 2.90 =$ ≈ 3.36	$\sigma^2_{t+3\|t} = \hat{\sigma}^2_\varepsilon (1 + \hat{\phi}_1^2 +$ $+ (\hat{\phi}_2 + \hat{\phi}_1^2)^2) =$ $\approx 5.03^2$	$f(Y_{t+3} \mid I_t) \to N(3.36, 5.03^2)$
$h=4$ 2007	$f_{t,4} = \hat{c} + \hat{\phi}_1 f_{t,3} + \hat{\phi}_2 f_{t,2} =$ $= 1.49 + 0.79 \times 3.36 - 0.25 \times 3.25 =$ ≈ 3.37	$\sigma^2_{t+4\|t} \approx 5.04^2$	$f(Y_{t+4} \mid I_t) \to N(3.37, 5.04^2)$
$h=5$ 2008	$f_{t,5} = \hat{c} + \hat{\phi}_1 f_{t,4} + \hat{\phi}_2 f_{t,3} =$ $= 1.49 + 0.79 \times 3.37 - 0.25 \times 3.36 =$ ≈ 3.34	$\sigma^2_{t+5\|t} \approx 5.04^2$	$f(Y_{t+5} \mid I_t) \to N(3.34, 5.04^2)$
$h=6$ 2009	$f_{t,6} = \hat{c} + \hat{\phi}_1 f_{t,5} + \hat{\phi}_2 f_{t,4} =$ $= 1.49 + 0.79 \times 3.34 - 0.25 \times 3.37 =$ ≈ 3.32	$\sigma^2_{t+6\|t} \approx 5.04^2$	$f(Y_{t+6} \mid I_t) \to N(3.32, 5.04^2)$

Autoregressive Models
Example: AR(2) Process



Autoregressive Models
AR(P) Process

- In general, for an AR(p) process:

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-1} + \ldots + \phi_p Y_{t-p} + \varepsilon_t$$

- We can show that
  - $\rho_1 = p_1,$ ρk=pk for k<=p
  - The speed of the ACF's decay depends on the persistence of $\phi_1 + \phi_2 + \ldots + \phi_p$
  - $p_1 \neq 0$, $p_2 \neq 0$, $\ldots p_p \neq 0$, and $p_p = 0$ for $k > p$

Autoregressive Models

Forecasting in an AR(p) Process

- Consider the k-step-ahead forecast, h = k for the AR(p): $Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + \varepsilon_t$ process:
- Optimal Point Forecast:

  $f_{t,k} = \mathrm{E}(Y_{t+k}|I_t) = c + \phi_1 f_{t,k-1} + \ldots + \phi_p f_{t,k-p}$

- k-period-ahead Forecast Error:

  $e_{t,k} = Y_{t+k} - f_{t,k} = \varepsilon_{t+k} + \phi_1 e_{t,k-1} + \ldots + \phi_p e_{t,k-p}$

- Uncertainty of the Forecast:

  $$\sigma^2_{t+1|t} = \sigma_\varepsilon^2 + \sum_{i=1}^{p} \phi_i^2 var(e_{t,k-i}) + 2 \sum_{i \neq j} \phi_i \phi_j cov(e_{t,k-i}, e_{t,k-j})$$

- Density Forecast: $f(Y_{t+k}|I_t) \sim N(\mu_{t+k|t}, \sigma^2_{t+k|t})$

Autoregressive Models
Forecasting in an AR(p) Process

- Chain Rule for Forecasting

$$Y_t = c + \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \dots + \phi_p Y_{t-p} + \varepsilon_t$$

– Step1 (change t→t+1): Compute $f_{t,1} = Y_{t+1}$

$$f_{t,1} = c + \phi_1 Y_t + \phi_2 Y_{t-1} + \dots + \phi_p Y_{t+1-p}$$

– Step2 (repeat Step 1): Compute $f_{t,2}$

$$f_{t,2} = c + \phi_1 \boldsymbol{Y_{t+1}} + \phi_2 Y_t + \dots + \phi_p Y_{t+2-p}$$

$$= c + \boldsymbol{f_{t,1}} + \phi_2 Y_t + \dots + \phi_p Y_{t+2-p}$$

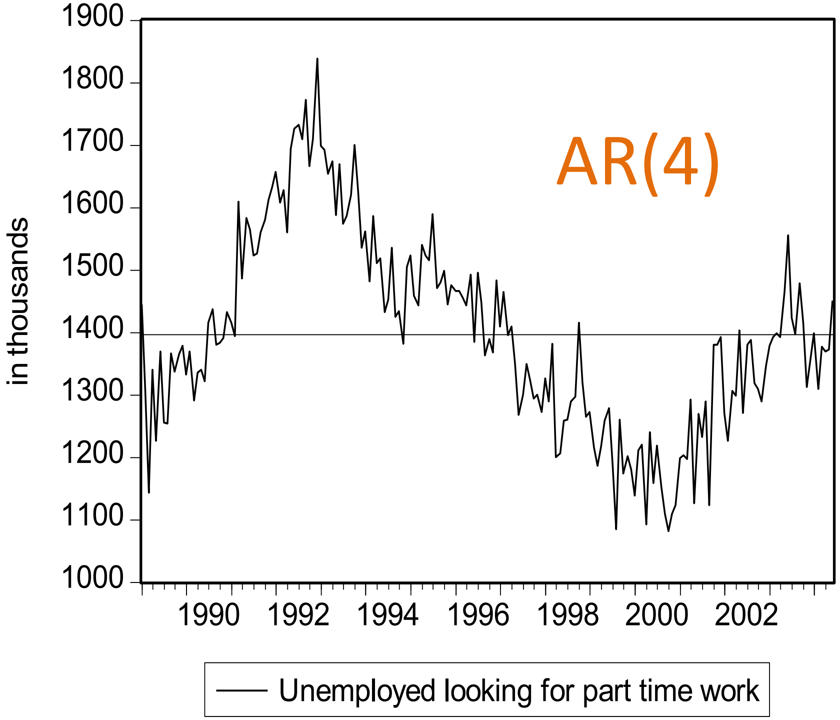
...

$$f_{t,k} = c + f_{t,k-1} + \phi_2 f_{t,k-2} + \dots + \phi_p f_{t,k-p}$$

Autoregressive Models

Example: AR(p) Process

Q: What type of a process is this?



Sample: 1989:01 2004:06
Included observations: 186

	AC	PAC
1	0.882	0.882
2	0.874	0.434
3	0.851	0.176
4	0.841	0.136
5	0.829	0.094
6	0.803	-0.030
7	0.796	0.040
8	0.775	-0.009
9	0.766	0.024
10	0.737	-0.072
11	0.712	-0.084
12	0.672	-0.158
13	0.676	0.098
14	0.638	-0.076
15	0.623	0.002
16	0.586	-0.086
17	0.553	-0.096
18	0.539	0.009
19	0.508	0.002
20	0.482	-0.041

In general, for any AR(p) process, $p_k = 0$ for any $k>p$.

For Next Class

- Readings about today's class:
  Chapter 7[a], 8[b]

- Review Exercises / Problems:
  Chapter 8[b]: 1, 2
  Chapter 8[b]: 2, 3, 4, 9

- Readings for next class:
  Chapter 7[a] (Section 7.3), Chapter 9[b]