Economics 144
Economic Forecasting

Lecture 8
Characterizing Cycles
ARMA Models

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Today's Class

- Review of the MA(q) Process
- Review of the AR(p) Process
- Seasonal AR(p) Model: S-AR(p)
- Seasonal MA(q) Model: S-MA(q)
- Rational Distributed Lags
- Autoregressive Moving Average (ARMA) Models
- R Example

Moving Average (MA) Models

- Moving-average (MA) models are always weakly stationary because they are finite linear combinations of a white noise sequence for which the first two moments are time invariant.

- Moving average processes are useful in describing phenomena in which events produce an immediate effect that only lasts for short periods of time.

- The MA model is a simple extension of the white noise series.

Moving Average (MA) Models
The MA(1) Process 1 of 4

- The first-order moving average process, MA(1) is:

$$y_t = \varepsilon_t + \theta\varepsilon_{t-1} = (1 + \theta L)\varepsilon_t$$

$$\varepsilon_t \sim WN(0, \sigma^2)$$

- Unconditional mean and variance are:

$$E(y_t) = 0 \quad \text{and} \quad var(y_t) = \sigma^2(1 + \theta^2)$$

- Conditional mean and variance are:

$$E(y_t|\Omega_{t-1}) = \theta\varepsilon_{t-1} \quad \text{and} \quad var(y_t|\Omega_{t-1}) = \sigma^2$$

One-period memory of the MA(1) process.

Moving Average (MA) Models
The MA(1) Process

- The autocovariance function for the MA(1) process is:

$$\gamma(\tau) = \begin{cases} \theta\sigma^2, & \tau = 1 \\ 0, & \text{otherwise} \end{cases}$$

- The autocorrelation function for the MA(1) process is:

$$\rho(\tau) = \begin{cases} \theta/(1+\theta^2), & \tau = 1 \\ 0, & \text{otherwise} \end{cases}$$

Sharp cutoff beyond displacement 1.

Moving Average (MA) Models
The MA(1) Process 3 of 4

- Invertible MA(1) process: If $|\theta| < 1$, then can 'invert' the MA(1) process. The inverted series is referred to as an autoregressive representation.
- Example: Autoregressive representation of the MA(1) process. $y_t = \varepsilon_t + \theta\varepsilon_{t-1}$ and $\varepsilon_t \sim WN(0, \sigma^2)$
- Solve for ε_t →

$$\varepsilon_t = y_t - \theta\varepsilon_{t-1}$$

$$\varepsilon_{t-1} = y_{t-1} - \theta\varepsilon_{t-2}$$

$$\varepsilon_{t-2} = y_{t-2} - \theta\varepsilon_{t-3}$$

...

Moving Average (MA) Models
The MA(1) Process

- After backward substitution:

$$y_t = \varepsilon_t + \theta y_{t-1} - \theta^2 y_{t-2} + \theta^3 y_{t-3} - \cdots$$

- In lag operator notation: $\frac{1}{1+\theta L} y_t = \varepsilon_t$ $\left\{ \text{autoregressive representation} \right\}$

- The lag operator polynomial has one root, which is obtained by solving for L from: $1 + \theta L = 0$

  $\rightarrow L = -1/\theta$

- Note: The inverse will be less than 1 in absolute value if $|\theta|<1$.

Moving Average (MA) Models
The MA(q) Process

- Consider the finite-order moving average process of order q, MA(q):

$$y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} = \Theta(L)\varepsilon_t$$

$$\varepsilon_t \sim WN(0, \sigma^2)$$

where $\Theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$

- The higher order terms in the MA(q) process can capture more complex dynamic patterns.

- The MA(q) process is invertible provided the inverses of all of the roots are inside the unit circle.

$$\left.\begin{aligned} &\frac{1}{\Theta(L)} y_t = \varepsilon_t \end{aligned}\right\} \text{autoregressive representation}$$

$$\rightarrow y_t = \Theta(L)\varepsilon_t$$

The MA(q) approximates an infinite moving average with a finite-order moving average.

Autoregressive (AR) Models

- Autoregressive Models (AR) models are always invertible. However, to be stationary, the roots of $\Phi\ (L)y_t=\varepsilon_t$ must lie outside the unit circle.

- Autoregressive processes are useful in describing situations in which the present value of a time series depends on its preceding values plus a random shock.

- AR processes are stochastic difference equations. They are used for modeling discrete-time stochastic dynamic processes (among others).

Autoregressive (AR) Models
The AR(1) Process 1 of 3

- The first-order autoregressive process, AR(1) is:

$$y_t = \varphi y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim WN(0, \sigma^2)$$

- Unconditional mean and variance are:

$$E(y_t) = 0 \quad \text{and} \quad var(y_t) = \frac{\sigma^2}{1-\varphi^2}$$

- Conditional mean and variance are:

$$E(y_t|y_{t-1}) = \varphi y_{t-1} \text{ and } var(y_t|y_{t-1}) = \sigma^2$$

Autoregressive (AR) Models
The AR(1) Process 2 of 3

- Yule-Walker Equation: $\gamma(\tau) = \varphi\, \gamma(\tau\text{-}1)$ (recursive relation)

- The autocovariance function for the AR(1) process is:

$$\gamma(\tau) = \varphi^{\tau} \frac{\sigma^2}{1-\varphi^2}\,, \quad \tau = 0, 1, 2, \ldots$$

- The autocorrelation function for the AR(1) process is:

$$\rho(\tau) = \varphi^{\tau}, \quad \tau = 0, 1, 2, \ldots$$

- The partial autocorrelation function for the AR(1) process is:

$$p(\tau) = \begin{cases} \varphi, & \tau = 0 \\ 0, & \tau > 1 \end{cases}$$

Autoregressive (AR) Models
The AR(1) Process 3 of 3

- After backward substitution:

$$y_t = \varepsilon_t + \varphi\varepsilon_{t-1} + \varphi^2\varepsilon^2_{t-2} + \cdots$$

- In lag operator notation: $y_t = \dfrac{1}{1-\varphi L}\varepsilon_t$

- The moving average representation for y is convergent if and only if |φ|<1 (covariance stationary condition for the AR(1) process).

Autoregressive (AR) Models
The AR(p) Process

- The general pth order autoregressive process, AR(p) is:

$$y_t = \varphi_1 y_{t-1} + \varphi_2 y_{t-2} + \cdots + \varphi_p y_{t-p} + \varepsilon_t$$
$$\varepsilon_t \sim WN(0, \sigma^2)$$

- In lag operator form:

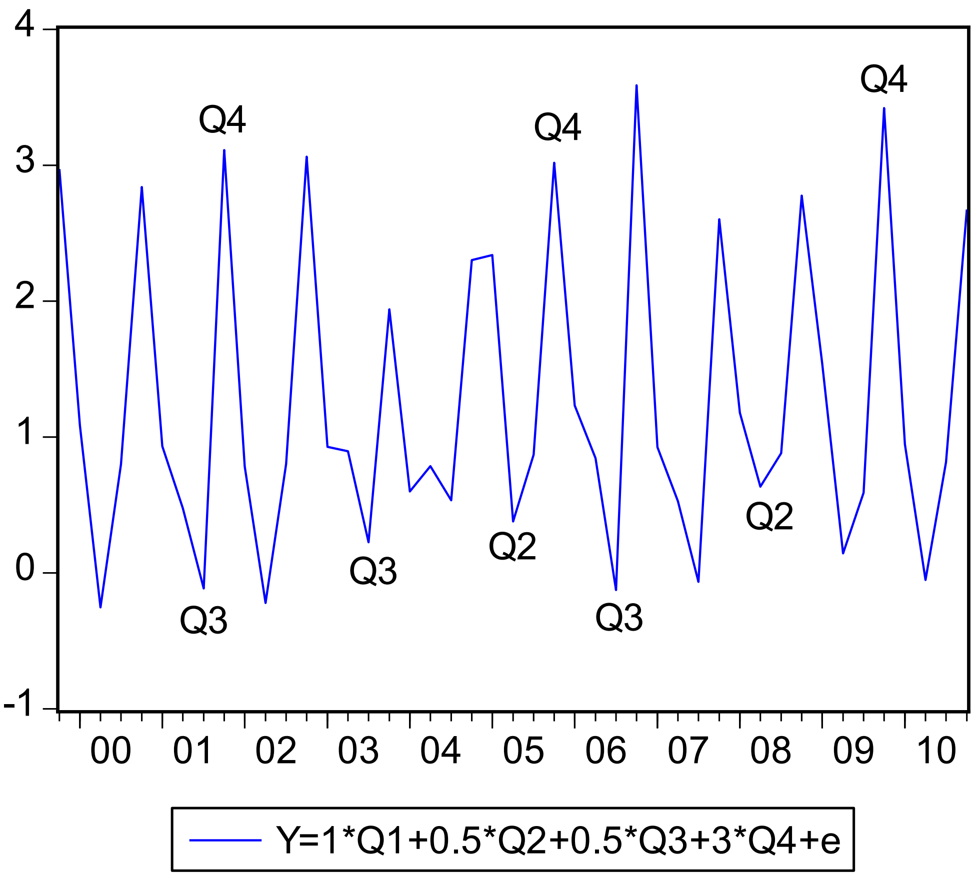
$$\Phi(L)y_t = (1 - \varphi_1 L - \varphi_2 L^2 - \cdots - \varphi_p L^p)y_t = \varepsilon_p$$

- The AR(p) process is covariance stationary if and only if the inverses of all roots of $\Phi(L)$ are inside the unit circle ($\Sigma\varphi<1$).

$$y_t = \frac{1}{\Phi(L)}\varepsilon_t \left\{\text{convergent infinite moving average}\right\}$$

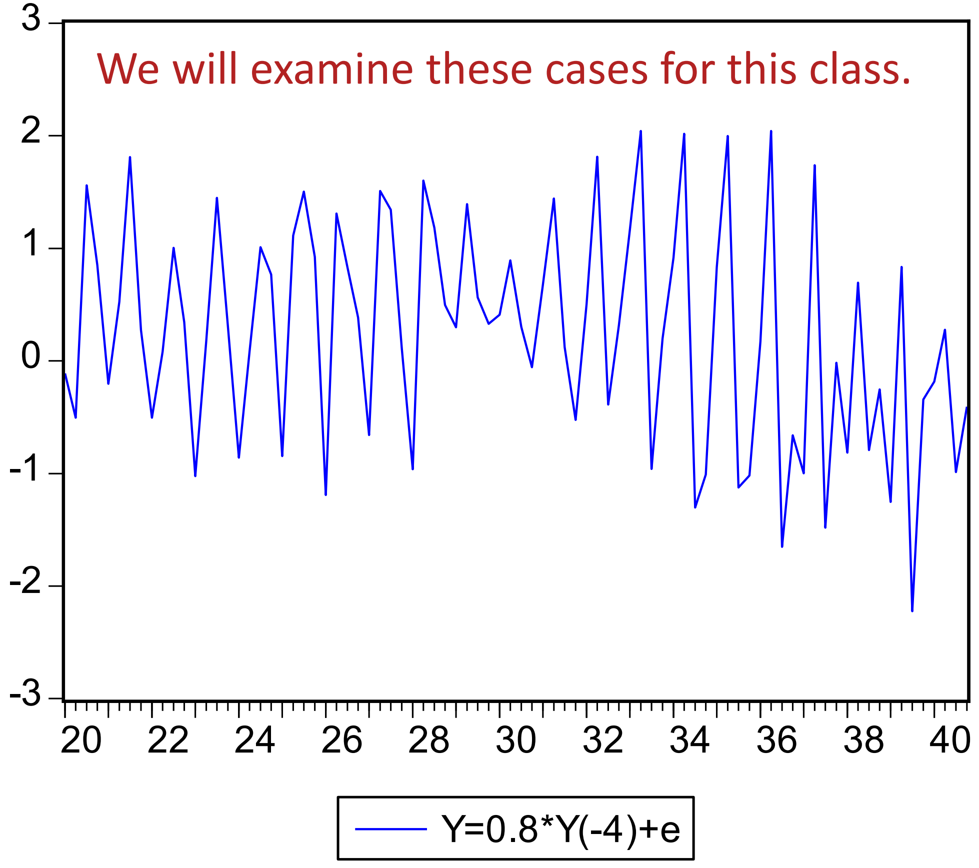
Seasonal Autoregressive Models: S-AR(p)

Deterministic Seasonal Cycle



Seasonal Autoregressive Models: S-AR(p)

Stochastic Seasonal Cycle



Seasonal Autoregressive Models: S-AR(p)

- S-AR Model: $Y_t = c + \phi_s \, Y_{t-s} + \varepsilon_t$, where s = frequency.

- Convention:
  - quarterly → s = 4
  - monthly → s = 12
  - daily → s = 7 or s = 5 (weekdays only)

- Example: S-AR(1) with s=4 (quarterly data)

$$Y_t = c + \phi_4 \, Y_{t-4} + \varepsilon_t$$

Seasonal Autoregressive Models: S-AR(p)

- Def: S-AR(p) Model = Seasonal AR model of order p.
- $Y_t = c + \phi_{1s}\, Y_{t-1s} + \phi_{2s}\, Y_{t-2s} + \ldots + \phi_{ps}\, Y_{t-ps} + \varepsilon_t$

p=order s=frequency

- We can also express the S-AR(p) model in lag-operator form as:

$$(1-\phi_{1s}L^{s} + \phi_{2s}\, L^{2s} + \ldots + \phi_{ps}\, L^{ps}\,)\; Y_t = c+\varepsilon_t$$

Seasonal Autoregressive Models: S-AR(p)

- Example 1: Identify the correct S-AR(p) model for the process: $Y_t = c + \phi_4 Y_{t-4} + \varepsilon_t$

  We can express it as: $Y_t = c + \phi_{1\times 4} Y_{t-1\times 4} + \varepsilon_t \rightarrow p=1,\ s=4$

  → S-AR(1) with s=4 (Quarterly)

- Example 2: Identify the correct S-AR(p) model for the process: $Y_t = c + \phi_{12} Y_{t-12} + \varepsilon_t$

  We can express it as: $Y_t = c + \phi_{1\times 12} Y_{t-1\times 12} + \varepsilon_t \rightarrow p=1,\ s=12$

  → S-AR(1) with s=12 (Monthly)

Seasonal Autoregressive Models: S-AR(p)

- Example 3: Identify the correct S-AR(p) model for the process: $Y_t = c + \phi_4 \, Y_{t-4} + \phi_8 \, Y_{t-8} + \varepsilon_t$

  We can express it as: $Y_t = c + \phi_{1\times 4} \, Y_{t-1\times 4} + \phi_{2\times 4} \, Y_{t-2\times 4} + \varepsilon_t$

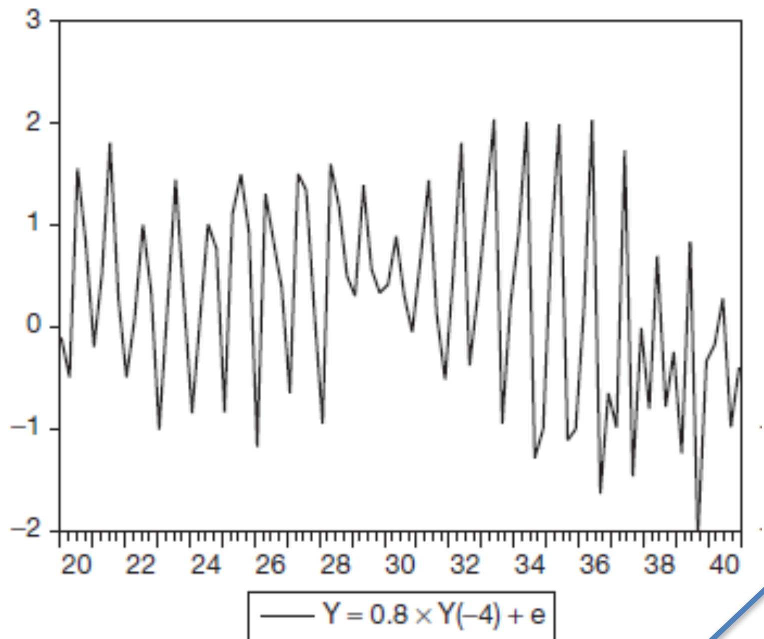
  $\rightarrow p=2,\ s=4$ →S-AR(2) with s=4 (Quarterly)

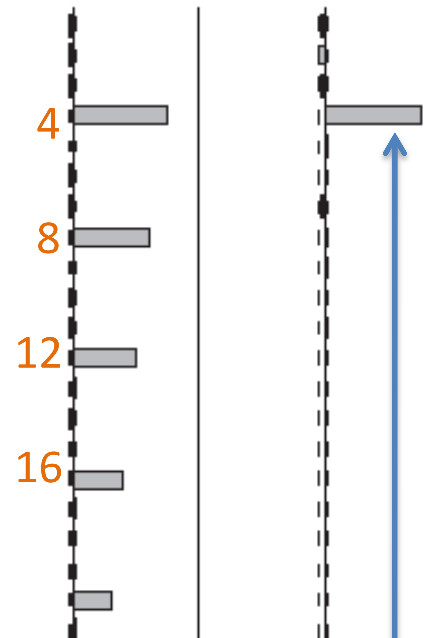
- Example 4: Identify the correct S-AR(p) model for the process: $Y_t = c + \phi_{12} \, Y_{t-12} + + \phi_{24} \, Y_{t-24} + \varepsilon_t$

  We can express it as: $Y_t = c + \phi_{1\times 12} \, Y_{t-1\times 12} + \phi_{2\times 12} \, Y_{t-2\times 12} + \varepsilon_t$

  $\rightarrow p=2,\ s=12$→S-AR(2) with s=12 (Monthly)

Seasonal AR(1)





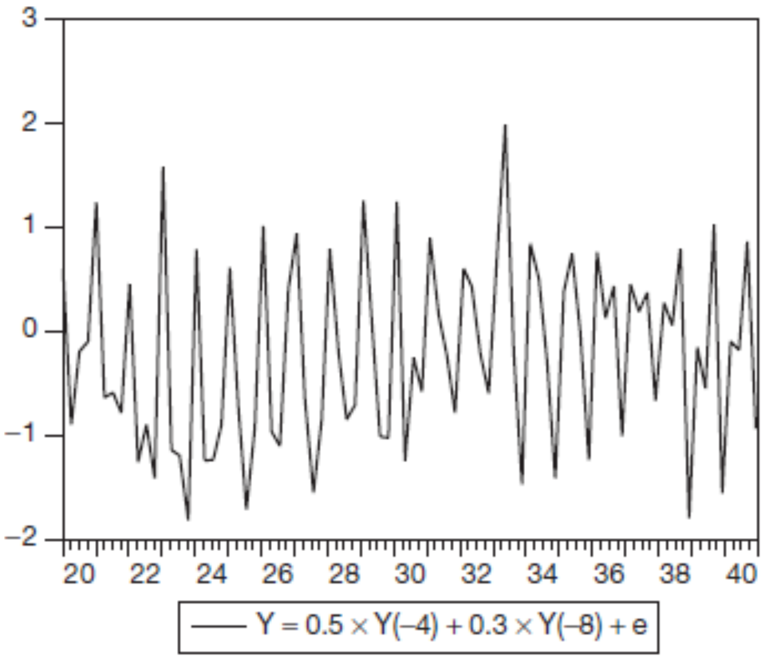
Included observations: 4201

	AC	PAC
1	-0.020	-0.020
2	-0.027	-0.027
3	-0.022	-0.023
4	0.796	0.796
5	-0.007	0.030
6	-0.024	0.004
7	-0.025	-0.009
8	0.641	0.021
9	0.012	0.027
10	-0.017	0.014
11	-0.014	0.036
12	0.515	-0.001
13	0.021	-0.005
14	-0.011	0.004
15	-0.001	0.015
16	0.414	0.002
17	0.025	-0.001
18	-0.004	0.006
19	0.004	-0.007
20	0.337	0.012
21	0.028	0.004

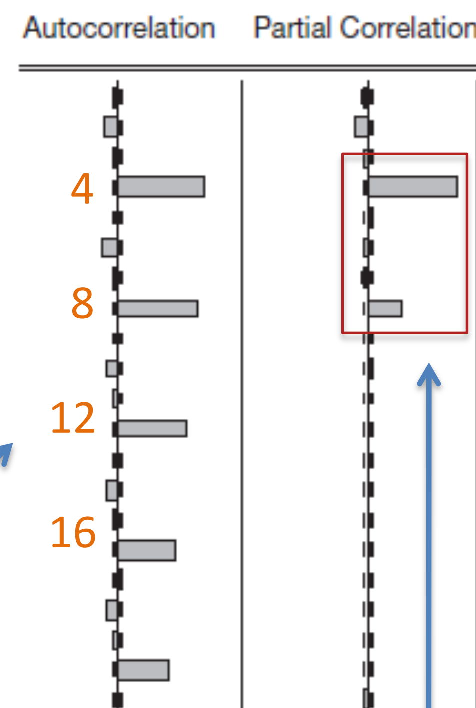
ACF: spikes at 1s, 2s,…
Then decays to zero.

PACF: 1-spike → AR(1)
Lag =4 → s=4(quarterly)

Seasonal AR(2)



Included observations: 4197

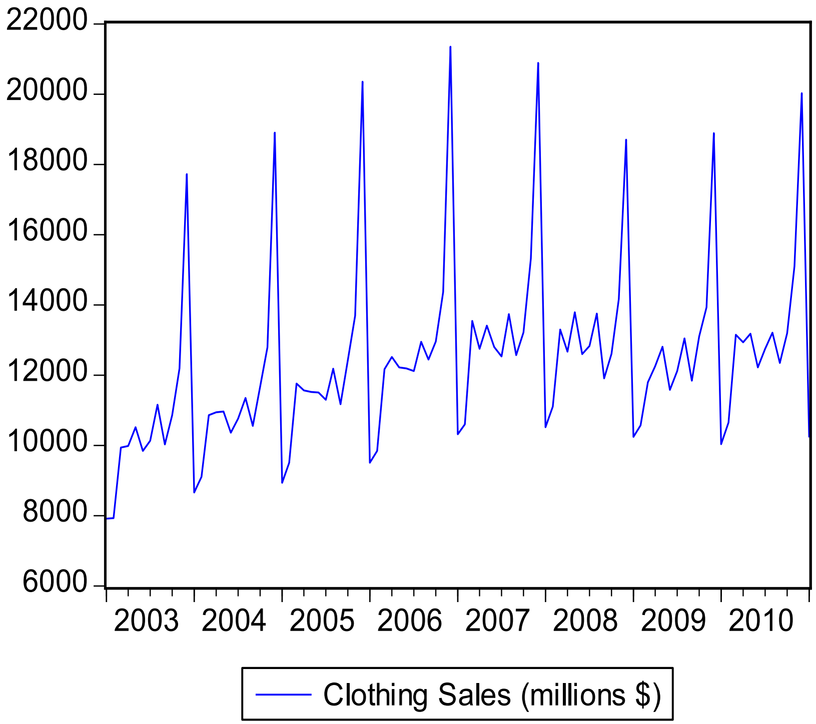


	AC	PAC
1	-0.018	-0.018
2	-0.109	-0.109
3	-0.022	-0.026
4	0.729	0.725
5	-0.003	0.022
6	-0.116	-0.033
7	-0.032	-0.021
8	0.668	0.290
9	0.005	0.010
10	-0.097	0.033
11	-0.035	0.000
12	0.560	0.005
13	0.016	0.008
14	-0.097	-0.005
15	-0.033	0.014
16	0.485	0.002
17	0.021	0.003
18	-0.088	-0.003
19	-0.037	-0.006
20	0.417	0.005
21	0.009	-0.040

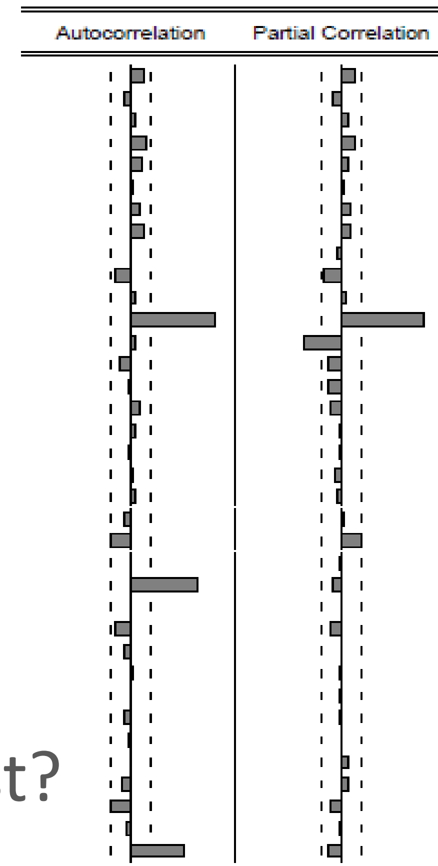
ACF: spikes at 1s, 2s,…
Then decays to zero.

PACF: 2-spikes→AR(2)
Lag =4→ s=4(quarterly)

Seasonal AR(1)
Example: Monthly Clothing Sales



Sample: 2003M01 2011M01
Included observations: 97



	AC	PAC
1	0.132	0.132
2	-0.069	-0.088
3	0.047	0.070
4	0.157	0.138
5	0.106	0.077
6	0.029	0.026
7	0.095	0.093
8	0.125	0.082
9	-0.001	-0.041
10	-0.150	-0.161
11	0.051	0.051
12	0.826	0.827
13	0.054	-0.375
14	-0.109	-0.132
15	-0.012	-0.119
16	0.088	-0.111
17	0.044	-0.024
18	-0.026	-0.019
19	0.030	-0.066
20	0.055	-0.033
21	-0.059	0.034
22	-0.182	0.197
23	-0.002	-0.024
24	0.672	-0.084
25	0.001	0.006
26	-0.147	-0.112
27	-0.058	-0.004
28	0.031	-0.024
29	0.000	-0.018
30	-0.063	-0.018
31	-0.014	0.004
32	0.011	0.071
33	-0.083	0.059
34	-0.202	-0.096
35	-0.038	-0.011
36	0.536	-0.119

Q: What model would you suggest?

→S-AR(1) with s=12

Seasonal Autoregressive Models: S-MA(q)

- Def: S-MA(q) Model = Seasonal MA model of order q.
- $Y_t = \mu + \theta_{1s}\, \varepsilon_{t-1s} + \theta_{2s}\, \varepsilon_{t-2s} + \ldots + \theta_{qs}\, \varepsilon_{t-qs} + \varepsilon_t$

q=order s=frequency

- We can also express the S-MA(q) model in lag-operator form as:

$$Y_t = \mu + (1 + \theta_{1s} L^s + \theta_{2s} L^{2s} + \ldots + \theta_{qs} L^{qs})\varepsilon_t$$

Seasonal Autoregressive Models: S-MA(q)

- Example 1: Identify the correct S-MA(q) model for the process: $Y_t = \mu + \theta_4 \varepsilon_{t-4} + \varepsilon_t$

  We can express it as: $Y_t = \mu + \theta_{1\times 4} \varepsilon_{t-1\times 4} + \varepsilon_t \rightarrow q=1,\ s=4$

  → S-MA(1) with s=4 (Quarterly)

- Example 2: Identify the correct S-MA(q) model for the process: $Y_t = \mu + \theta_{12} \varepsilon_{t-12} + \varepsilon_t$

  We can express it as: $Y_t = \mu + \theta_{1\times 12} \varepsilon_{t-1\times 12} + \varepsilon_t \rightarrow q=1,\ s=12$

  → S-MA(1) with s=12 (Monthly)

Seasonal Autoregressive Models: S-MA(q)

- Example 1: Identify the correct S-MA(q) model for the process: $Y_t = \mu + \theta_4\, \varepsilon_{t-4} + \theta_8\, \varepsilon_{t-8} + \varepsilon_t$

  We can express it as: $Y_t = \mu + \theta_{2\times 4}\, \varepsilon_{t-2\times 4} + \varepsilon_t \rightarrow q=2,\ s=4$

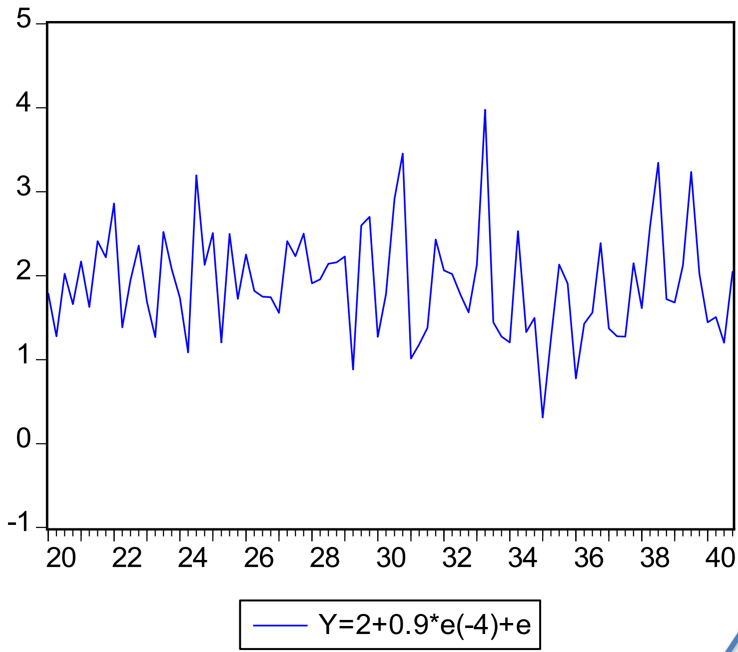
  →S-MA(2) with s=4 (Quarterly)

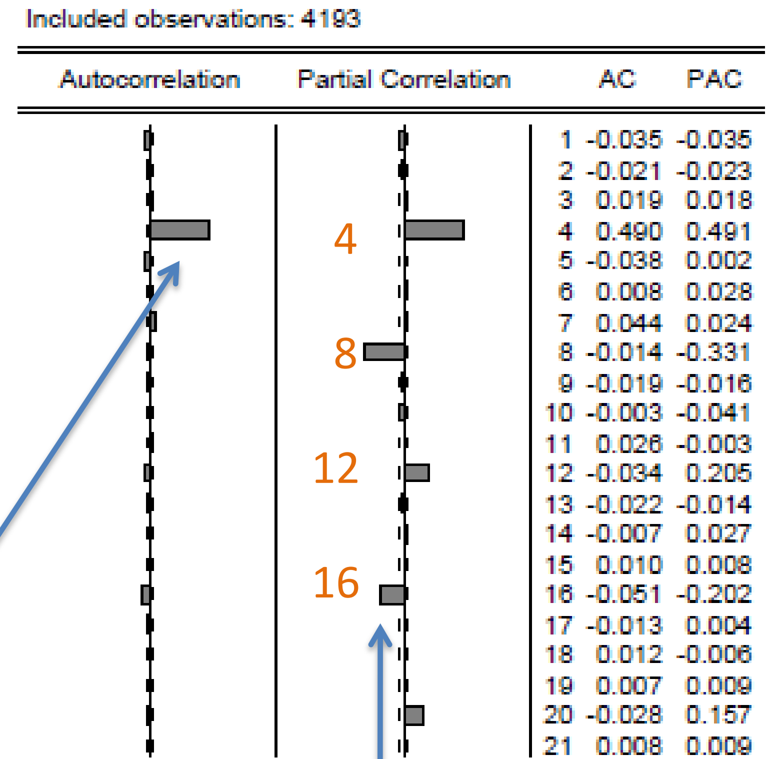
- Example 2: Identify the correct S-MA(q) model for the process: $Y_t = \mu + \theta_{12}\, \varepsilon_{t-12} + \theta_{24}\, \varepsilon_{t-24} + \varepsilon_t$

  We can express it as: $Y_t = \mu + \theta_{2\times 12}\, \varepsilon_{t-2\times 12} + \varepsilon_t \rightarrow q=2,\ s=12$

  →S-MA(2) with s=12 (Monthly)

Seasonal MA(1)



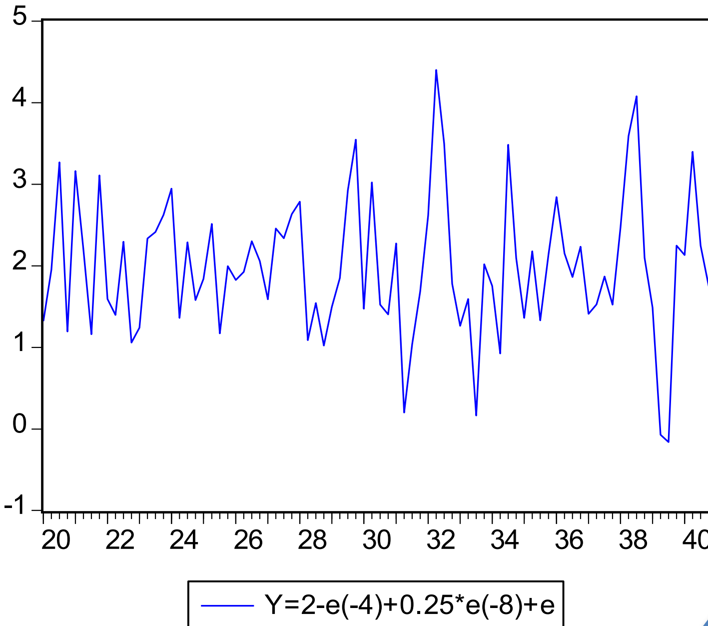

Included observations: 4193

Autocorrelation	Partial Correlation		AC	PAC
		1	-0.035	-0.035
		2	-0.021	-0.023
		3	0.019	0.018
		4	0.490	0.491
		5	-0.038	0.002
		6	0.008	0.028
		7	0.044	0.024
		8	-0.014	-0.331
		9	-0.019	-0.016
		10	-0.003	-0.041
		11	0.026	-0.003
		12	-0.034	0.205
		13	-0.022	-0.014
		14	-0.007	0.027
		15	0.010	0.008
		16	-0.051	-0.202
		17	-0.013	0.004
		18	0.012	-0.006
		19	0.007	0.009
		20	-0.028	0.157
		21	0.008	0.009

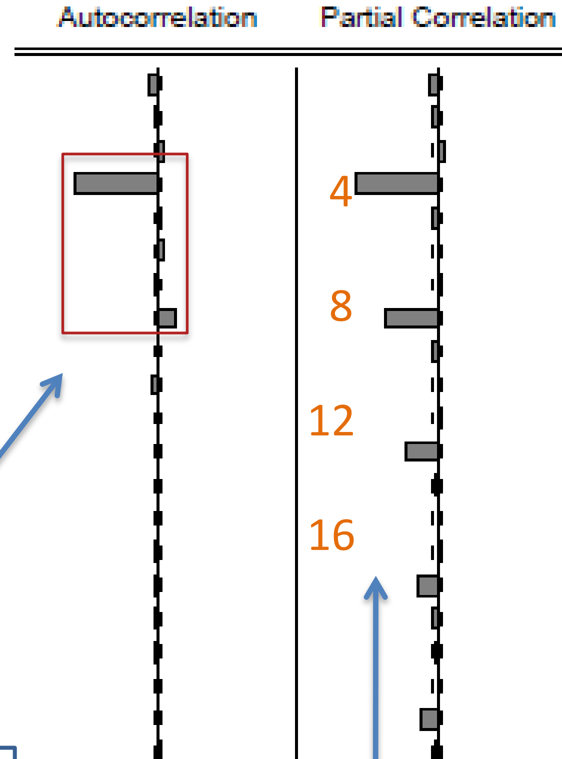
ACF: 1-spike→MA(1)
Lag =4→ s=4(quarterly)

PACF: spikes at 1s, 2s,…
Then decays to zero.

Seasonal MA(2)



Included observations: 4189



	AC	PAC
1	-0.056	-0.056
2	-0.023	-0.027
3	0.047	0.044
4	-0.613	-0.612
5	0.026	-0.040
6	0.047	0.012
7	-0.016	0.034
8	0.134	-0.383
9	-0.005	-0.031
10	-0.034	0.002
11	0.011	0.040
12	-0.000	-0.229
13	0.006	-0.010
14	0.013	0.001
15	-0.012	0.025
16	-0.016	-0.155
17	-0.019	-0.034
18	-0.010	-0.020
19	0.001	0.003
20	0.003	-0.117
21	0.022	-0.020

ACF: 2-spikes→MA(2)
Lag =4→ s=4(quarterly)

PACF: spikes at 1s, 2s,…
Then decays to zero.

Rational Distributed Lags

- Rational Distributed Lags:

$$B(L) = \frac{\Theta(L)}{\Phi(L)}$$

$$\Theta(L) = \sum_{i=0}^{q} \theta_i L^i$$ polynomial of degree q

$$\Phi(L) = \sum_{i=0}^{p} \phi_i L^i$$ polynomial of degree p

(In practice use an approximation)

- Rational distributed lags produce models of cycles that economize on parameters.

- If p and q are small (e.g., 0, 1, or 2), then estimation of $B(L)$ is easy.

Autoregressive Moving Average (ARMA) Models→ARMA(p,q)

- Recall from Wold's approximation that:

$$B(L) = \frac{\Theta(L)}{\Phi(L)} \text{ and } y_t = B(L)\varepsilon_t \longrightarrow y_t = \frac{\Theta(L)}{\Phi(L)}\varepsilon_t$$

$$\longrightarrow \underbrace{\Phi(L)y_t}_{\text{AR(p)}} = \underbrace{\Theta(L)\varepsilon_t}_{\text{MA(q)}}$$

ARMA Models are often highly accurate and highly parsimonious.

The ARMA model combines the ideas of AR and MA models into a compact form so that the number of parameters used is kept small, achieving parsimony in parameterization.

Autoregressive Moving Average (ARMA) Models→ARMA(p,q)

- The ARMA(1,1) Process: $Y_t = \phi Y_{t\text{-}1} + \theta\varepsilon_{t\text{-}1} + \varepsilon_t$

- In lag operator form: $(1\text{-}\phi L)Y_t = (1+\theta L)\,\varepsilon_t$

  where $|\phi|$ <1 for stationarity and |θ| <1 for invertibility.

$$Y_t = \frac{(1+\theta L)}{(1-\phi L)}\varepsilon_t$$

(if stationary)

$$\frac{(1-\phi L)}{(1+\theta L)}Y_t = \varepsilon_t$$

(if invertible)

Autoregressive Moving Average (ARMA) Models→ARMA(p,q)

- The ARMA(p,q) Process:

$$Y_t = \phi_1 Y_{t-1} + \cdots + \phi_p Y_{t-p} + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

$$\varepsilon_t \sim WN(0, \sigma^2)$$

→ $\Phi(L) Y_t = \Theta(L) \varepsilon_t$

where $\Phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$

and

$$\Theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$$

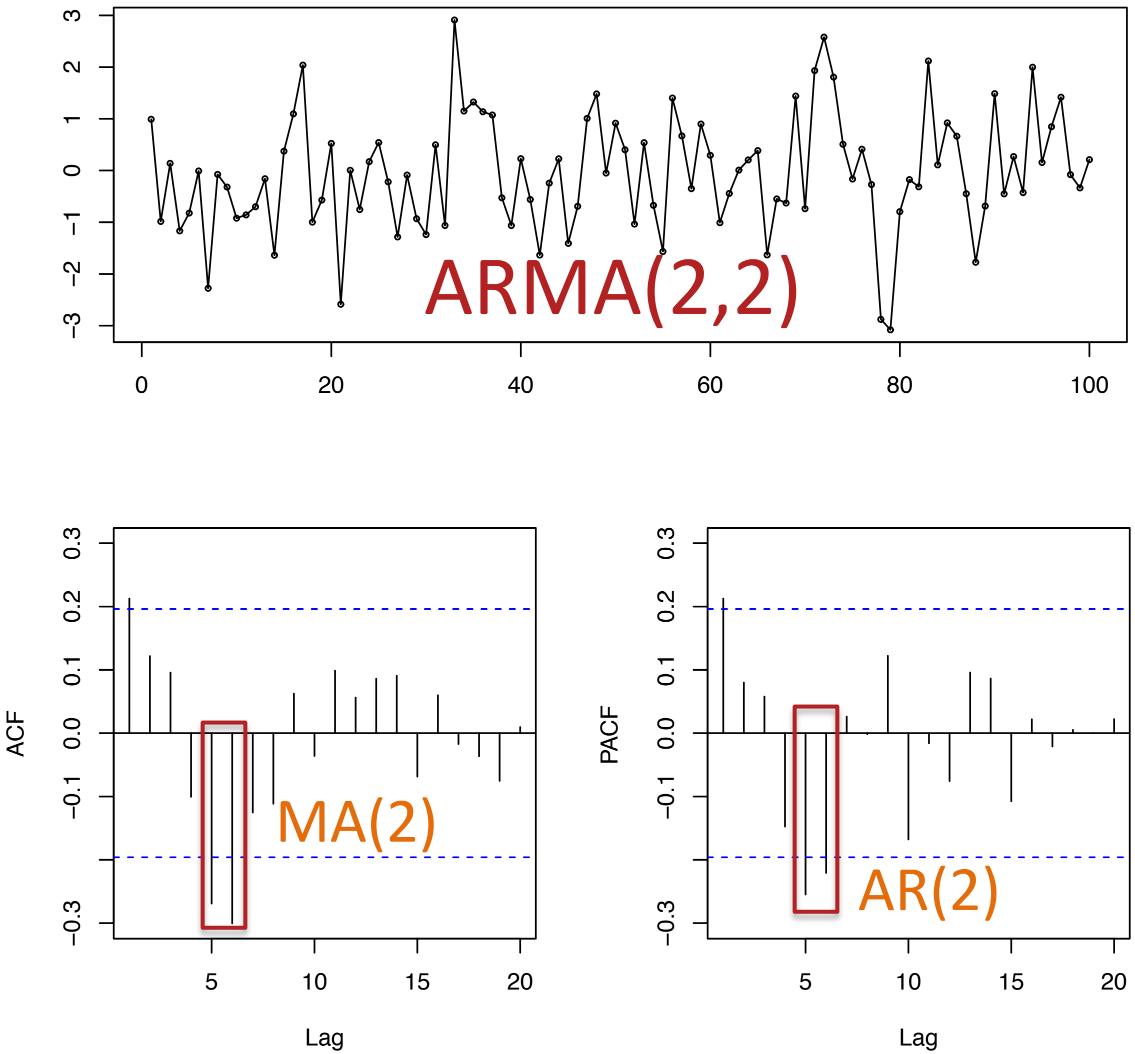
$$\boxed{Y_t = \frac{\Theta(L)}{\Phi(L)} \varepsilon_t}$$

(if stationary)

$$\boxed{\frac{\Phi(L)}{\Theta(L)} Y_t = \varepsilon_t}$$

(if invertible)

Mystery Process ☺



Summary

- AR(p): Current value of Y_t can be found from past values, plus a random shock ε_t.
  - Y_t is regressed on past values of Y_t.

- MA(q): Current value of Y_t can be found from past shocks, plus a new shock/error ε_t.
  - The time series is regarded as a moving average (unevenly weighted, because of different coefficients) of a random shock series ε_t.

Summary 2 of 3

- For MA models, ACF is useful in specifying the order because ACF cuts off at lag q for an MA(q) series.

- For AR models, PACF is useful in order determination because PACF cuts off at lag p for an AR(p) process.

- For an ARMA(p,q) process, lower-order models are better. For example, ARMA(1,1) is better than AR(3).

Summary 3 of 3

- Full Model: $Y_t = T + S + C$
  - Trend: $T = \alpha + \beta t$
  - Seasonal: $S = \sum_{i=1}^{s} \gamma_i D_i$
  - Cycle: $\Phi(L)R_t = \Theta(L)\varepsilon_t$