Economics 144
Economic Forecasting

Lecture 9
A Forecasting Model with Trend, Seasonal, and Cyclical Components

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Today’s Class

- Full model: Trend + Seasonal + Cycle(s)
- R Example
- Recursive Estimation Procedures

Full Model 1 of 2

- The full model includes a trend, seasonal dummies, and cyclical dynamics.

$$y_t = \underbrace{T_t(\theta)}_{\text{Trend}} + \underbrace{\sum_{i=1}^{s} \gamma_i D_{it} + \sum_{i=1}^{v_1} \delta_i^{HD} HDV_{it} + \sum_{i=1}^{v_1} \delta_i^{TD} TDV_{it} + \varepsilon_t}_{\text{Seasonality}}$$

cannot be done with arma

can be done with arma

$$\Phi(L) R_t = \Theta(L) \varepsilon_t$$ Cycles

can be done with arma

Need a model that can model everything

$$\Phi(L) = 1 - \phi_1 L - \phi_2 L^2 - \cdots - \phi_p L^p$$

$$\Theta(L) = 1 + \theta_1 L + \theta_2 L^2 + \cdots + \theta_q L^q$$

$$\underbrace{\varepsilon_t \sim WN(0, \sigma^2)}_{\text{Innovation}}$$

Full Model 2 of 2

We can now construct the h-step-ahead point forecast at time T, $y_{T+h,T}$.

$$y_{T+h} = T_{T+h}(\theta) + \sum_{i=1}^{s} \gamma_i D_{i,T+h} + \sum_{i=1}^{v_1} \delta_i^{HD} HDV_{i,T+h} + \sum_{i=1}^{v_2} \delta_i^{TD} TDV_{i,T+h} + \varepsilon_{T+h}$$

Project the r.h.s variables on Ω_t.

$$y_{T+h,T} = T_{T+h}(\theta) + \sum_{i=1}^{s} \gamma_i D_{i,T+h} + \sum_{i=1}^{v_1} \delta_i^{HD} HDV_{i,T+h} + \sum_{i=1}^{v_2} \delta_i^{TD} TDV_{i,T+h} + \varepsilon_{T+h,T}$$

Make the point forecast operational by replacing unknown parameters with estimates.

$$\hat{y}_{T+h,T} = T_{T+h}(\hat{\theta}) + \sum_{i=1}^{s} \hat{\gamma}_i D_{i,T+h} + \sum_{i=1}^{v_1} \hat{\delta}_i^{HD} HDV_{i,T+h} + \sum_{i=1}^{v_2} \delta_i^{TD} TDV_{i,T+h} + \hat{\varepsilon}_{T+h,T}$$

$$\hat{y}_{T+h,T} \pm z\hat{\sigma}_h$$

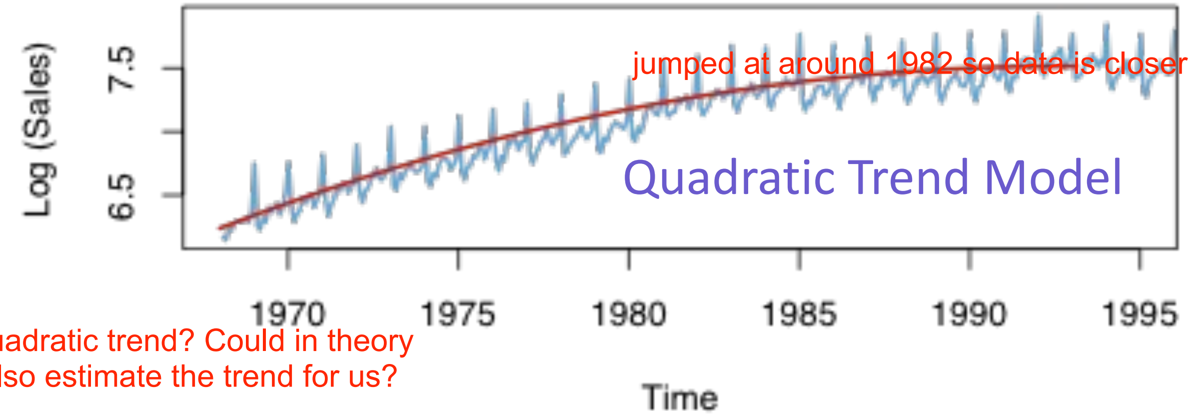
Confidence Interval

$$\mathcal{N}(\hat{y}_{T+h,T}, \hat{\sigma}_h^2)$$

Density Forecast

Liquor Sales Example

Step: startup with data - do some trend
residuals - tsdisplay - spot seaonality/cycles
Seasonal adjust our data - look at our residual - tsdiaplay residuals to analyze model - done.

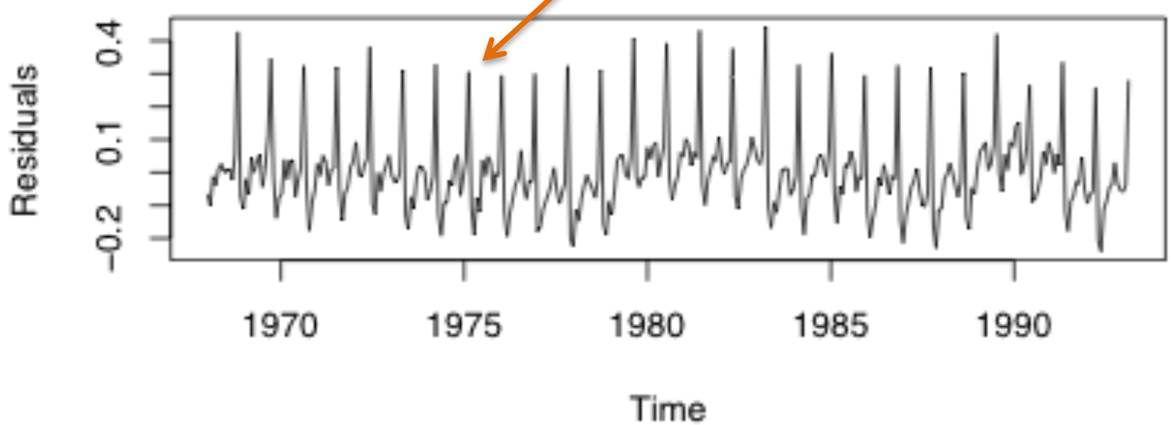


Adjust for trends

Why quadratic trend? Could in theory arma also estimate the trend for us?

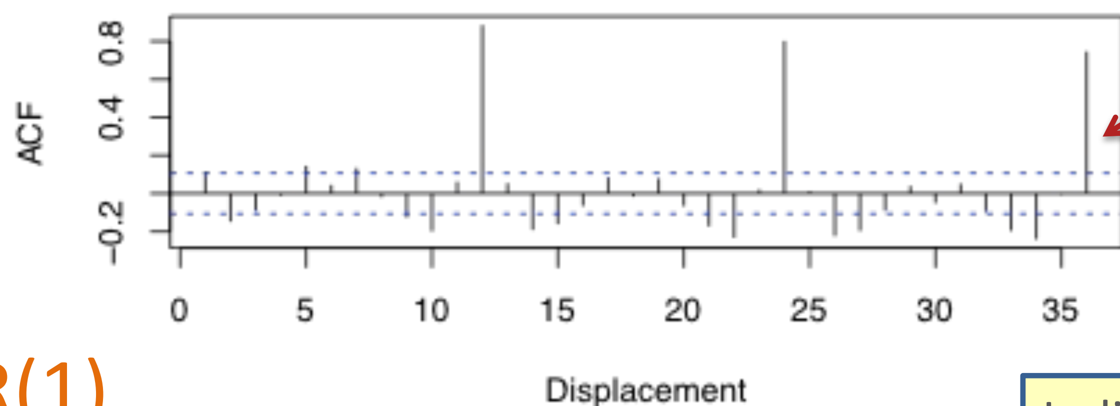
Strong residual seasonality (maybe even cycles)

REcursive cusum test - great test of sensitivity



Liquor Sales Example 2 of 7

Residual Sample Partial Autocorrelations



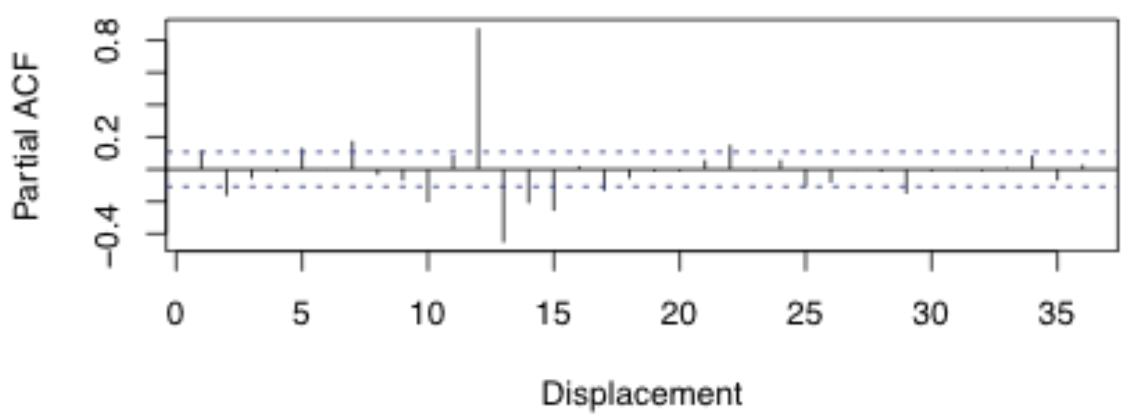
Bartlett Bands

S-AR(1) with s =12

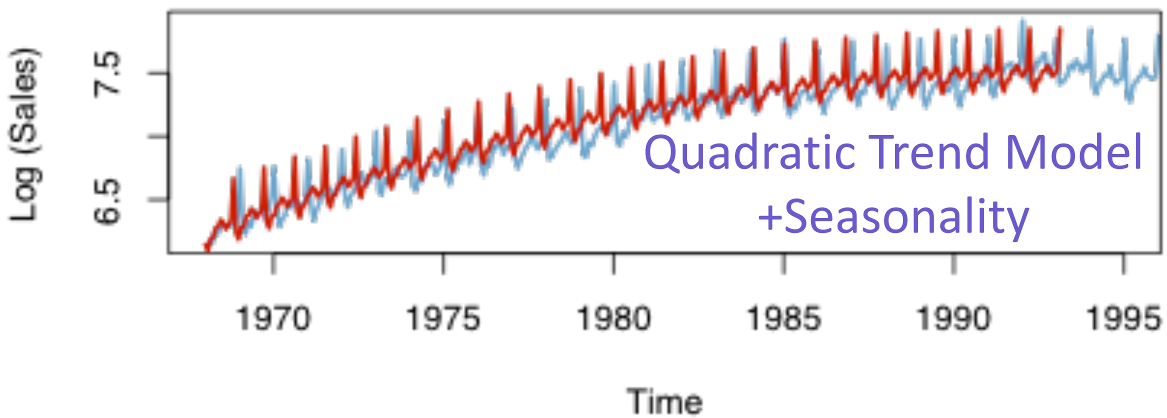
Indicate the presence of cyclical dynamics.

Residual Sample Autocorrelations

seasonality so strong that it overpowers cycle - what to do: seasonally adjust our data to see the spikes of cycles

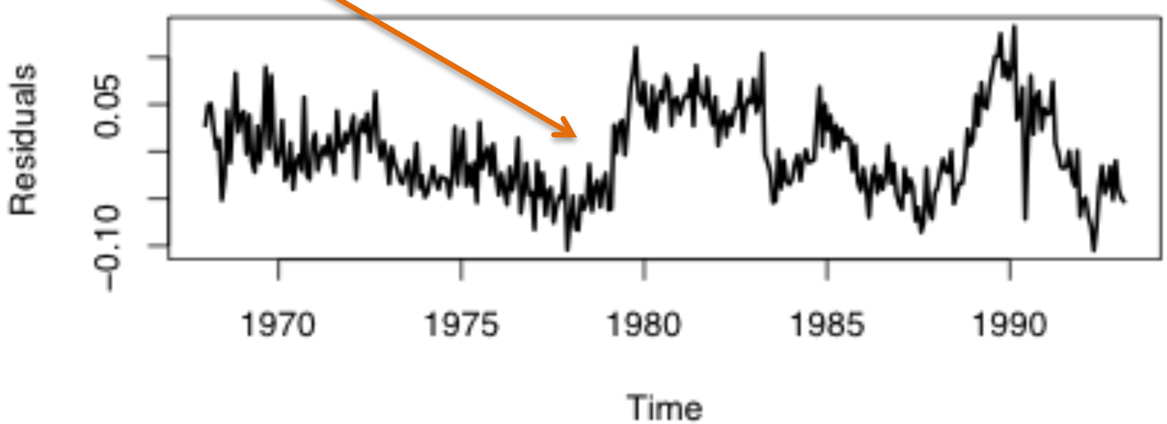


Liquor Sales Example



Good quadratic fit

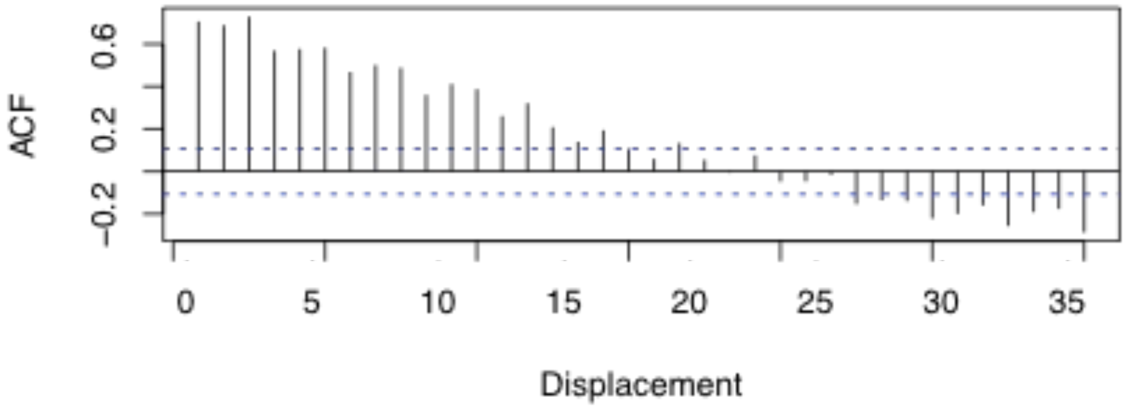
No seasonality but strong serial correlation → Highly Predictable!



AR with quite a bit of persistence

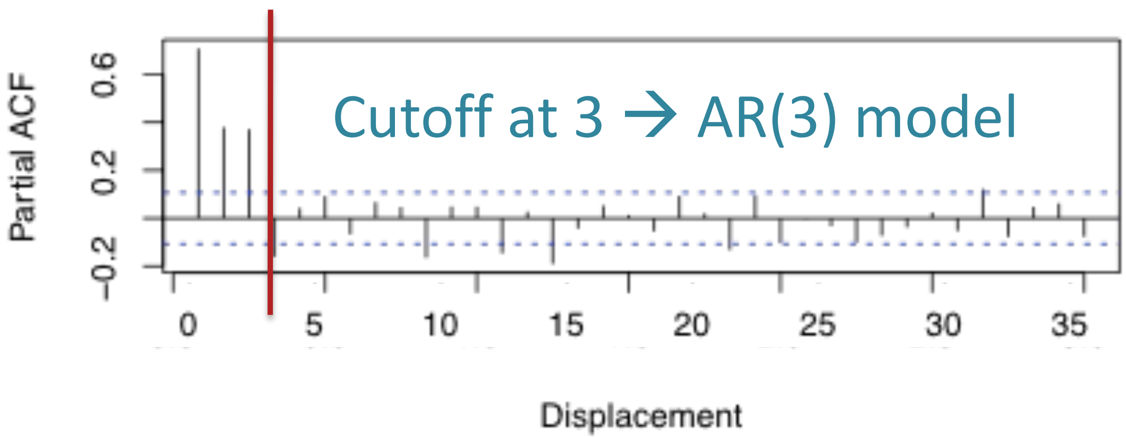
Liquor Sales Example

Residual Sample Autocorrelations

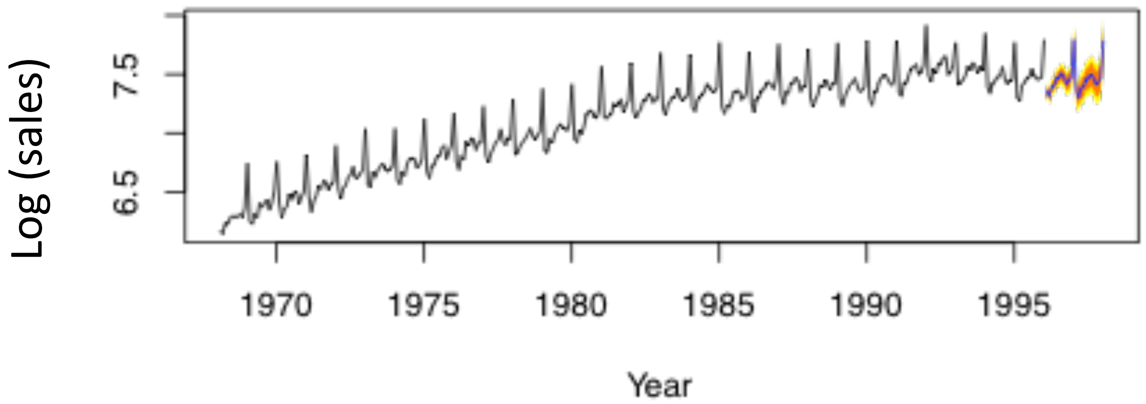


Cutoff at 3 -> AR(3) Model for the cycles

Residual Sample Partial Autocorrelations

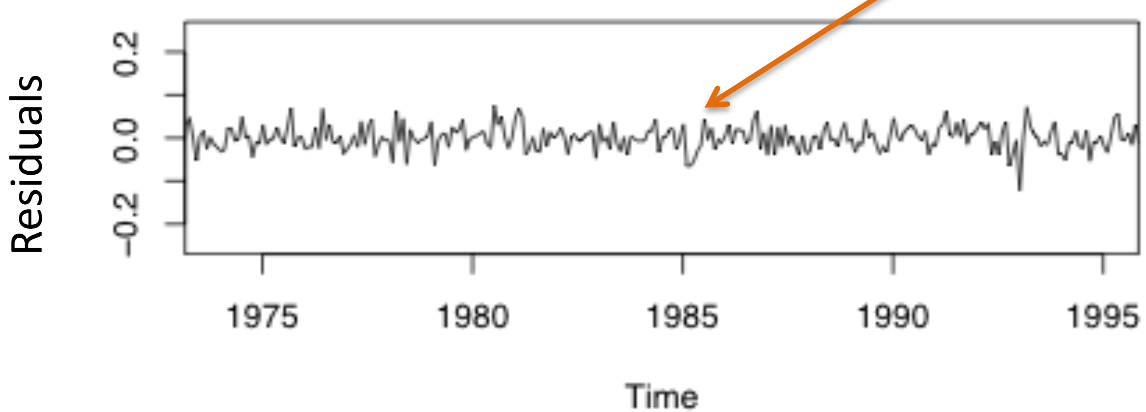


Liquor Sales Example 5 of 7



Trend+ seasonlaity+ cycles model forecast

No structure left in the residuals!



Recursive Estimation Procedures

Another way to test our model

- Strategy:

Train - test split

1. Start with a small sample of data

Cross validation: problem in time sereis - time series data have to be ordered
- Cross validation gonna partition data into five folds and predict
- TS: want data to be preserved for prediction.
- Book: UP - flat - down- randomly mix and match - script in book

2. Estimate a model
3. Add an observation
4. Re-estimate the model

Recursive scheme estimation

5. Continue until all data are used

- Useful for forecasting, stability assessment, and model selection.

Recursive Parameter Estimation and Recursive Residuals 1 of 2

- Recursive estimates provide information about parameter stability.

- Linear Model: $$y_t = \sum_{i=1}^{k} \beta_i x_{i,t} + \varepsilon_t, \;\; \varepsilon_t \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$$

Regression - simple to see than before - just about the estimate of parameter Bi,t

- If the model contains k parameters, start with first k obs and estimate the model, the $k+1$,…
  → recursive parameter estimates $\hat{\beta}_{i,t}$
  for $t=k, \ldots T$ and $i=1, \ldots, k$.

Recursive Parameter Estimation and Recursive Residuals 2 of 2

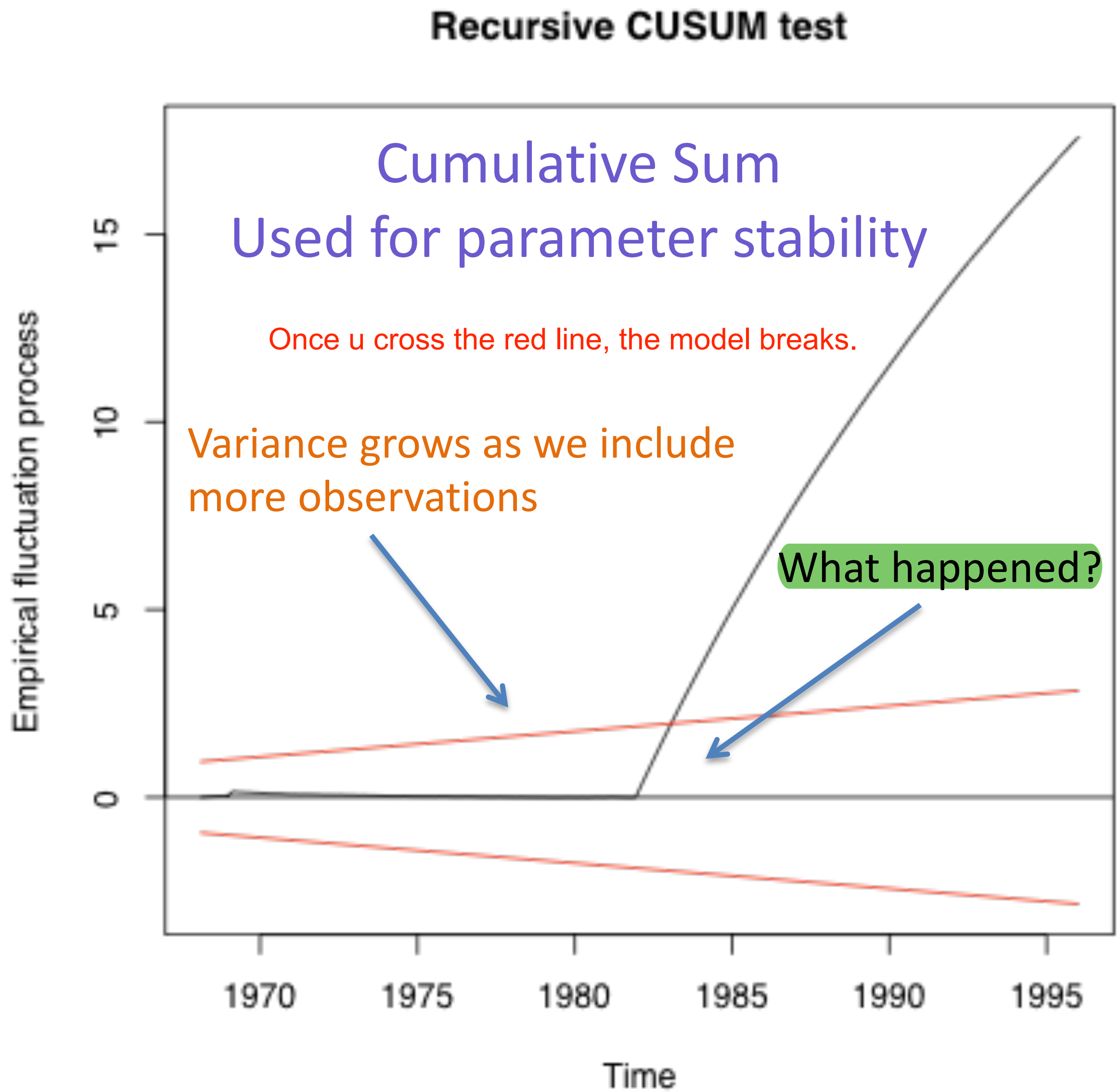
- 1-step-ahead-forecast: $\hat{y}_{t+1} = \sum_{i=1}^{k} \hat{\beta}_{i,t} x_{i,t+1}$

Regression model: Get t+h residuals - every residual u get, we're going to normalize it and add all the erros.

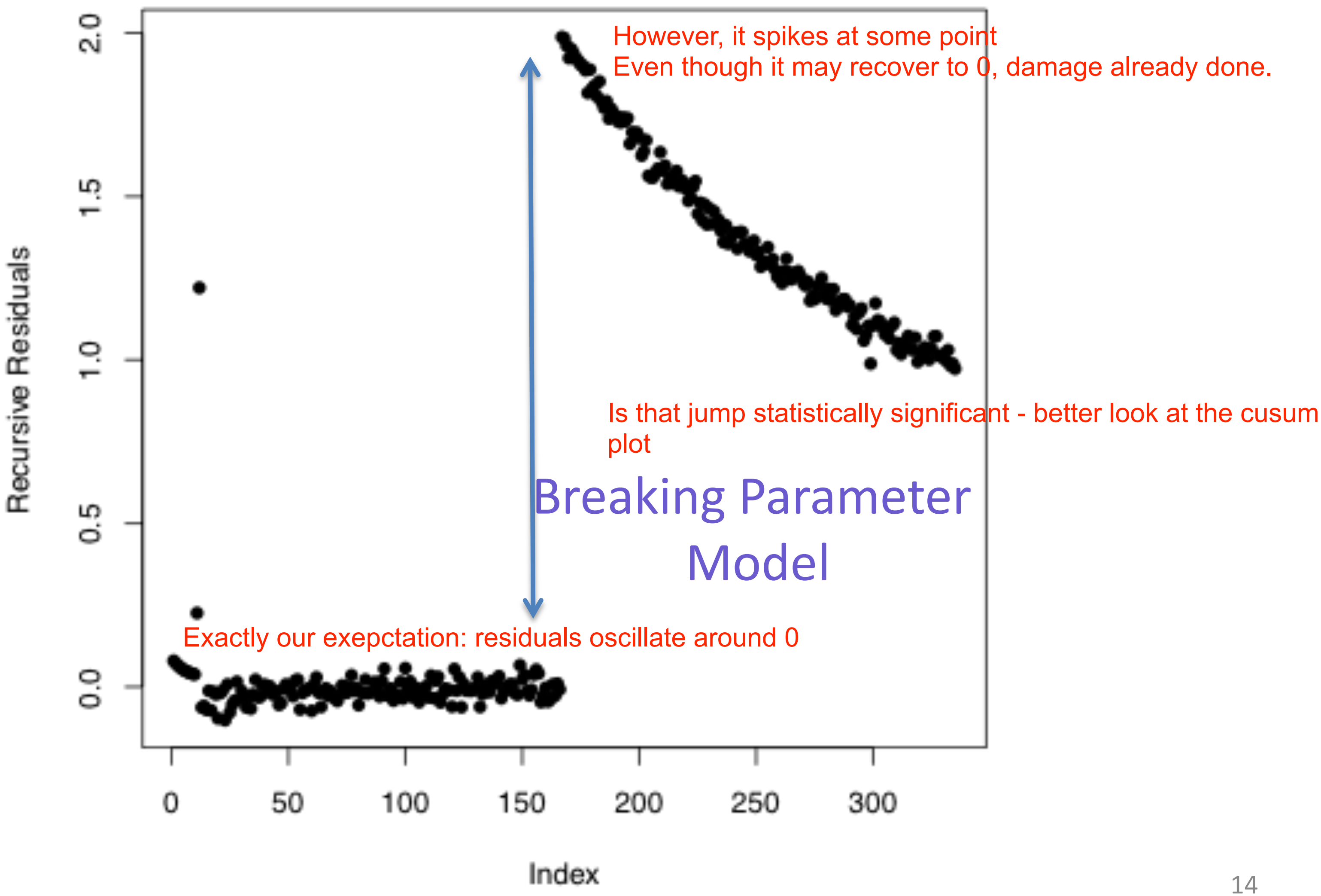
- Recursive residuals: $\hat{e}_{t+1,t} \sim \mathcal{N}(0, \sigma^2 r_t)$ where $r_t > 1$ for all t. Note: r_t depends on the data.
- Standardized Recursive Residuals: $w_{t+1,t} \equiv \dfrac{\hat{e}_{t+1,t}}{\sigma\sqrt{r_t}}$ where $w_{t+1,t} \overset{iid}{\sim} \mathcal{N}(0,1)$
- Cumulative Sum: $CUSUM_t = \sum_{\tau=k}^{t} w_{t+1,t}$ where $t = k, \ldots, T\text{-}1$

Should be fluctuating around zero if model is running normally
if there is a spike, then yhat is either too large or too small.

Liquor Sales Example



Liquor Sales Example



Model Selection Based on Simulated Forecasting Performance 1 of 2

- Cross Validation: Select among N forecasting models.
  1. Start with model 1
  2. Estimate it using all data observations except the first
  3. Use it to forecast the first observation
  4. Compute the squared forecast error.
     - Continue estimating the model with one observation deleted and then using the estimated model to forecast the deleted observation until each observation has been sequentially deleted.
  5. Average the squared errors
  6. Repeat procedure for model $n = 1, \ldots, N$
  7. Select the model with the smallest squared error.

Model Selection Based on Simulated Forecasting Performance 2 of 2

Recursive Cross Validation:

Same observation used for training + test

- Let the initial estimation sample run from $t = 1, \ldots, T^*$
- Let the `holdout sample' run from $t = T^*+1, \ldots, T$

Proper way to stress-test model

- For each model:
  - Estimate the model using observations $t = 1, \ldots, T^*$.
  - Use the model to forecast observation T^*+1.
  - Compute the associated squared error.
  - Update the sample by 1 observation (T^*+1)
  - Estimate the model using the updated sample $t = 1, \ldots, T^*+1$.
  - Forecast observation T*+2, computed associated squared error.
  - Repeat previous steps until sample is exhausted.
- Average the squared errors in predicting observations T^*+1 through T.
- Select the model with the smallest squared forecast error.

Stochastic Trends and Forecasting

- Often in Economics we encounter many nonstationary series (a.k.a. unit-root nonstationary), e.g., interest rates, foreign exchange rates, and the price series of an asset of interest.
- Consider an ARMA(p,q) process where one of the p roots of its autoregressive lag operator polynomial is 1 (unit root).

origina
l data

factor AR

$$\Phi(L)y_t = \Theta(L)\varepsilon_t \longrightarrow \Phi(L) = \Phi'(L)(1-L)$$

Δy_t is covariance stationary. $\longrightarrow$ $\Phi'(L)\Delta y_t = \Theta(L)\varepsilon_t$

Detrended the data by taking the difference

Sometimes may need diff of diff

> A nonstationary series is integrated if its nonstationarity is undone by differencing.

Diff allow modelling of trend

Stochastic Trends and Forecasting -Random Walk 1 of 2

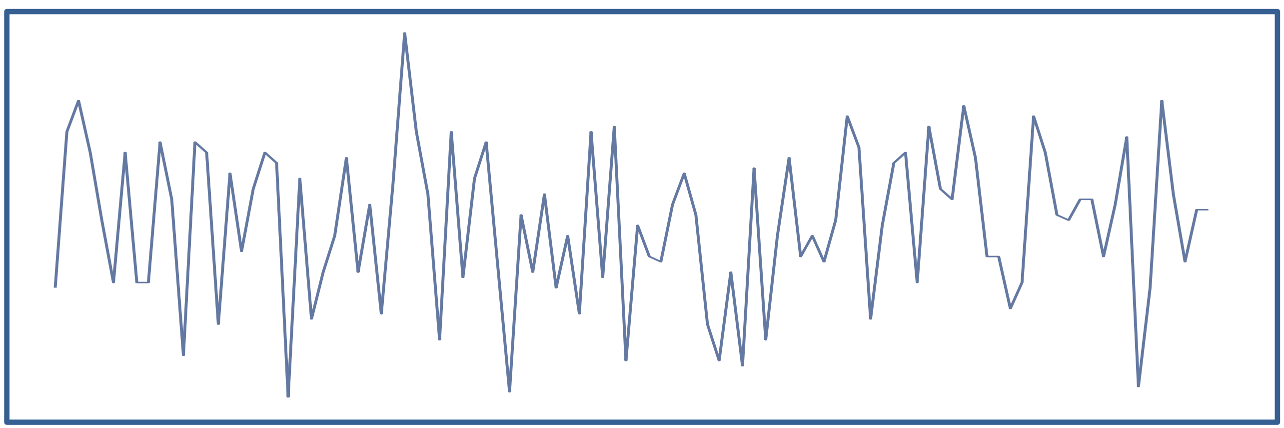
- If only one difference is required, the series is said to be integrated of order 1, $I(1)$. In general, for d differences, we have $I(d)$ where the number of differences equals the number of unit roots.

inverse of taking difference is integrate it

integrate it of order d

cleaniest of the series

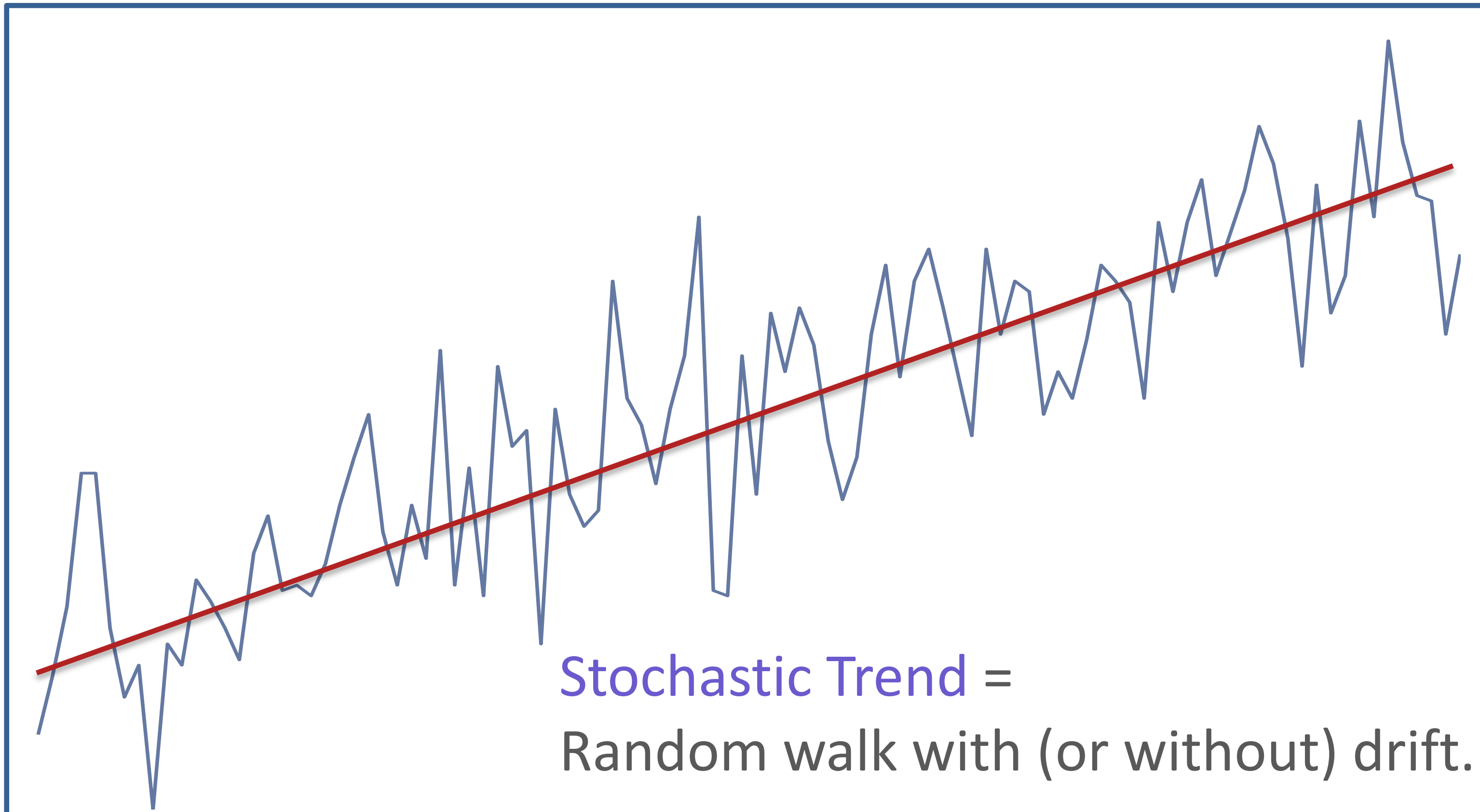
- Random Walk: Is an AR(1) process with unit coefficient → $y_t = y_{t-1} + \varepsilon_t$ and $\varepsilon_t \sim WN(0,\sigma^2)$.



Random walk = the cumulative sum of white noise changes.

Stochastic Trends and Forecasting -Random Walk 2 of 2

- Random Walk with Drift: Is an AR(1) process with unit coefficient → $y_t = \delta t + y_{t-1} + \varepsilon_t$ and $\varepsilon_t \sim WN(0,\sigma^2)$.



Stochastic Trend =
Random walk with (or without) drift.

Stochastic Trends and Forecasting -ARIMA(p,1,q)

- ARIMA: Autoregressive integrated moving average.

Model Trend, seasonality, and cycles

- The ARIMA$(p, 1, q)$ process is a stationary and invertible ARMA(p,q) process in first differences: $\Phi(L)\underbrace{(1-L)}_{\text{Trend}}y_t = c + \underbrace{\Theta(L)}_{\text{Cycles}}\varepsilon_t$

$$\longrightarrow (1-L)y_t = c\Phi(L)^{-1} + \Phi(L)^{-1}\Theta(L)\varepsilon_t$$

where
$$\begin{cases} \Phi(L) = 1 - \Phi_1 L - \cdots - \Phi_p L^p \\ \Theta(L) = 1 - \Theta_1 L - \cdots - \Theta_p L^p \end{cases}$$

Stochastic Trends and Forecasting -ARIMA(p,d,q)

- In general, for the ARIMA(p,d,q) model

d - order of differencing - keep differencing until u effectively do away with trend

$$\Phi(L)(1-L)^d y_t = c + \Theta(L)\varepsilon_t$$

$$\longrightarrow (1-L)^d y_t = c\Phi^{-1}(1) + \Phi^{-1}(L)\Theta(L)\varepsilon_t$$

$$\text{where} \begin{cases} \Phi(L) = 1 - \Phi_1 L - \cdots - \Phi_p L^p \\ \Theta(L) = 1 - \Theta_1 L - \cdots - \Theta_p L^p \end{cases}$$

- The ARIMA(p,d,q) process is a stationary and invertible ARMA(p,q) after differencing d times.

Trend, Seasonality and Cycles -ARIMA(p,d,q)(P,D,Q)

Differencing in the seasonal scale
Most powerful of the models we covered - looks like we have a 1 size fits all model

- $\text{ARIMA}\overbrace{\underbrace{(p,d,q)}_{\text{Cycles via ARMA(p,q)}}}^{\text{Trend via I(d)}}\ \overbrace{\underbrace{(P,D,Q)}_{\text{Seasonality via S-ARMA(P,Q)}}}^{\text{Seasonal Trend via S-I(D)}}$

Both trend and seasonal trend

- The ARIMA(p,d,q) process is a stationary and invertible ARMA(p,q) after differencing d times.