A Dissertation

entitled

Ultraproducts of Tannakian Categories and Generic Representation Theory of

Unipotent Algebraic Groups

by

Michael Crumley

Submitted to the Graduate Faculty as partial fulfillment of the requirements

for the Doctor of Philosophy Degree in Mathematics

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Let G be an affine group scheme defined over a field k, and denote by $\mathrm{Rep}_k G$ the category of finite dimensional representations of G over k. The principle of tannakian duality states that any neutral tannakian category is tensorially equivalent to $\mathrm{Rep}_k G$ for some affine group scheme G and field k, and conversely.

Originally motivated by an attempt to find a first-order explanation for generic cohomology of algebraic groups, we study neutral tannakian categories as abstract first-order structures and, in particular, ultraproducts of them. One of the main theorems of this dissertation is that certain naturally definable subcategories of these ultraproducts are themselves neutral tannakian categories, hence tensorially equivalent to Comod_A for some Hopf algebra A over a field k. We are able to give a fairly tidy description of the representing Hopf algebras of these categories, and explicitly compute them in several examples. The work done in this vein constitutes roughly half of this dissertation.

The second half is much less abstract in nature, as we turn our attention to working out the representation theories of certain unipotent algebraic groups, namely the additive group G_a and the Heisenberg group H_1. The results we obtain for these groups in characteristic zero are not at all new or surprising, but in positive characteristic they perhaps are. In both cases we obtain that, for a given dimension

n, if p is large enough with respect to n, all n-dimensional modules for these groups in characteristic p are given by commuting products of representations, with the constituent factors resembling representations of the same group in characteristic zero. This has led us to define the 'height-restricted ultraproduct' of the categories $\operatorname{Rep}_{k_i} G$ for a sequence of fields k_i of increasing positive characteristic, and the above result can be summarized by saying that these height-restricted ultraproducts are tensorially equivalent to $\operatorname{Rep}_k G^n$, where G^n denotes a direct product of copies of G and k is a certain field of characteristic zero. We later use these results to extrapolate some generic cohomology results for these particular unipotent groups.

To Sarah

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Contents

Chapter 1

Introduction

Consider the following two theorems:

Theorem 1.0.1. *(see Corollary 3.4 of [8]) Let G be a simple, simply connected algebraic group defined and split over $\mathbb{F}_p$, and λ a dominant weight. If p is sufficiently large with respect to G, λ and n, then the dimension of $H^n(G(\mathbb{F}_p), S(\lambda))$ is independent of p.*

Theorem 1.0.2. *Let ϕ be a first-order statement in the language of fields such that ϕ is true for every characteristic zero field. Then ϕ is true for all fields of sufficiently large positive characteristic.*

The first is a classic generic cohomology theorem; if you can assume such and such a thing to be large (in this case, characteristic), cohomology stabilizes. The second is a textbook exercise in model theory, an easy consequence of the compactness theorem for first-order logic. The analogy between these two statements has been the broad motivation for the following: is there a first-order explanation for the phenomenon of generic cohomology?

Our investigations into this question have, as fate would have it, led us far astray from our original objective. The majority of this dissertation is devoted to the study of *neutral tannakian categories* as abstract first-order structures (roughly speaking,

the categories which can in some sense be thought of as $\mathrm{Rep}_k G$ for some affine group scheme G and field k), and in particular, ultraproducts of them. To this end we identify certain subcategories of these ultraproducts which themselves are neutral tannakian categories, hence tensorially equivalent to Comod_A for some Hopf algebra A. We are able to provide a general formula for A, and explicitly compute it in several examples.

For the remainder we turn away from ultraproducts, and instead to the study of the concrete representation theories of certain unipotent algebraic groups, namely the additive group G_a and the Heisenberg group H_1. For both groups we obtain a certain 'generic representation theory' result: that while the characteristic $p > 0$ and characteristic zero theories of both can, by and large, be expected to bear little resemblance to one another, if instead one is content to keep positive characteristic large with respect to dimension, there is in fact a very strong correspondence between the two. These results are later codified by considering the 'height-restricted ultraproduct' of these groups for increasing characteristic, and from them we are able to generate some modest, 'height-restricted' generic cohomology results for these groups.

1.1   Preliminaries

For an algebraic group G defined over $\mathbb{Z}$ and a field k, $\mathrm{Rep}_k G$ is the category of finite dimensional representations of G over k. This category is tensorially equivalent to $\mathrm{Comod}_{A \otimes k}$, where A is the representing Hopf algebra of G over $\mathbb{Z}$, and we generally prefer to think of it as the latter. If k_i is a collection of fields indexed by I and $\mathcal{U}$ a non-principal ultrafilter over I, we consider the ultraproduct of the categories Comod_{A_i} with respect to $\mathcal{U}$, with $A_i = A \otimes k_i$, which we denote as $\prod_{\mathcal{U}} \mathrm{Comod}_{A_i}$.

The language over which these categories are realized as first-order structures, which we call the 'language of abelian tensor categories' (section 4.1), includes sym-

bols denoting an element being an object or morphism, composition of morphisms, addition of morphisms, morphisms pointing from one object to another, and notably, a symbol for the tensor product (of objects and morphisms). It also includes symbols denoting certain natural transformations on the category, necessary to describe certain regularity properties of the tensor product, e.g. being naturally associative and commutative. The primary reason we have chosen these symbols is

Theorem 1.1.1. *(see chapter 4) In the language of abelian tensor categories, the statement "is a tannakian category" is a first-order sentence.*

Chapter 3 is devoted to giving an explicit definition of a tannakian category. Suffice it to say for the moment, it is an abelian category $\mathcal{C}$, endowed with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, which satisfies a plethora of regularity conditions, e.g. being naturally associative and possessing internal Homs. We say that a tannakian category $\mathcal{C}$ is *neutral* (see definition 3.2.6) if it comes equipped with a *fibre functor*, i.e. an exact, faithful, k-linear tensor preserving functor ω from $\mathcal{C}$ to Vec_k (the category of finite dimensional vector spaces over k, where k is the field $\mathrm{End}_{\mathcal{C}}(\underline{1})$, and $\underline{1}$ denotes the identity object of $\mathcal{C}$). The motivation for the definition of a neutral tannakian category is the following theorem.

Theorem 1.1.2. *(see theorem 2.11 of [5]) Let $\mathcal{C}$ be a neutral tannakian category over the field k with fibre functor ω. Then*

1. *The functor $\mathrm{Aut}^{\otimes}(\omega)$ on k-algebras is representable by an affine group scheme G*

2. *ω defines an equivalence of tensor categories between $\mathcal{C}$ and $\mathrm{Rep}_k G$*

The moral: a neutral tannakian category is (tensorially equivalent to) the category of finite dimensional representations of an affine group scheme over a field, and vice versa. (Section 3.3 is devoted to describing how one goes about, in principle, recovering the representing Hopf algebra of a neutral tannakian category.)

1.2 The Restricted Ultraproduct of Neutral Tannakian Categories

As is argued in section 15.1, the basic concepts of cohomology of modules (at least in the case of Ext^1) are quite naturally expressible in the language of abelian tensor categories. Therefore, to study cohomology over a particular group G and field k, a reasonable object of study is the category $\mathrm{Rep}_k G$ as a first-order structure in this language. But further, we are interested in studying *generic* cohomology; that is, for a fixed group G and sequence of fields k_i, we would like to know if a particular cohomological computation eventually stabilizes for large enough i. We are then drawn to the study of not the single category $\mathrm{Rep}_k G$ for fixed k, but rather the infinite sequence of the categories $\mathrm{Rep}_{k_i} G$. And as ultraproducts of relational structures, by design, tend to preserve only those first-order properties which are true 'almost all of the time', it is for this reason that we have chosen to study ultraproducts of categories of the form $\mathrm{Rep}_{k_i} G$, which we denote by $\prod_{\mathcal{U}} \mathrm{Rep}_{k_i} G$.

While being a tannakian category is a first-order concept, the property of being endowed with a fibre functor, so far as we can tell, is not. If $\mathcal{C}_i$ is a sequence of tannakian categories neutralized by the fibre functors ω_i, the natural attempt to endow $\prod_{\mathcal{U}} \mathcal{C}_i$ with a fibre functor would go as follows. Define a functor ω on $\prod_{\mathcal{U}} \mathcal{C}_i$ which takes an object $[X_i] \in \prod_{\mathcal{U}} \mathcal{C}_i$ to $\prod_{\mathcal{U}} \omega_i(X_i)$ (ultraproduct of vector spaces; see section 6.2), and similarly for a morphism $[\phi_i]$ (ultraproduct of linear maps; see section 6.2.1). But this will not do; $\omega([X_i])$ will in general be infinite dimensional (proposition 6.2.4), specifically disallowed by the definition of a fibre functor. Further, for any collection of vector spaces V_i and W_i over the fields k_i, we have a natural injective map $\prod_{\mathcal{U}} V_i \otimes \prod_{\mathcal{U}} W_i \rightarrow \prod_{\mathcal{U}} V_i \otimes W_i$ (section 6.2.2). But unless at least one of the collections is boundedly finite dimensional, this will not be an isomorphism; thus ω will not be tensor preserving in general. We therefore make the following compromise:

Definition 1.2.1. The **restricted ultraproduct** of the $\mathcal{C}_i$, denoted $\prod_R \mathcal{C}_i$, is the full subcategory of $\prod_{\mathcal{U}} \mathcal{C}_i$ consisting of those objects $[X_i]$ such that the dimension of $\omega_i(X_i)$ is bounded.

Then we indeed have

Theorem 1.2.1. *(see theorems 7.2.3 and 7.2.4)* $\prod_R \mathcal{C}_i$ *is a tannakian category, neutralized over the field* $k = \prod_{\mathcal{U}} k_i$ *by the functor* ω *described above.*

Thus, $\prod_R \mathcal{C}_i$ is tensorially equivalent to $\mathrm{Comod}_{A_\infty}$ for some Hopf algebra A_∞ over the field $k = \prod_{\mathcal{U}} k_i$. The question then: what is A_∞?

The obvious first guess, that it is the ultraproduct of the Hopf algebras A_i representing each of the $\mathcal{C}_i$, is not correct; problem being, this is not a Hopf algebra at all. We start by defining a map Δ on $\prod_{\mathcal{U}} A_i$ by the formula $\prod_{\mathcal{U}} A_i \xrightarrow{[\Delta_i]} \prod_{\mathcal{U}} A_i \otimes A_i$ (the ultraproduct of the maps Δ_i). But again, unless the A_i are boundedly finite dimensional, we cannot expect this Δ to point to $\prod_{\mathcal{U}} A_i \otimes \prod_{\mathcal{U}} A_i \subset \prod_{\mathcal{U}} A_i \otimes A_i$ in general. So we make another compromise:

Definition 1.2.2. (see section 9.1) The **restricted ultraproduct** of the Hopf algebras A_i, denoted A_R, is the collection of all $[a_i] \in \prod_{\mathcal{U}} A_i$ such that the rank of a_i is bounded.

Defining exactly what "rank" means here takes some doing, so we defer it; suffice it to say, A_R can indeed be given the structure of a coalgebra, under the definition of Δ given above. We are able to prove 9

Theorem 1.2.2. *The representing Hopf algebra of the restricted ultraproduct of the categories* Comod_{A_i} *is isomorphic to the restricted ultraproduct of the Hopf algebras* A_i.

In section 9.5 we explicitly work out A_∞ for a few examples. If G is a finite group defined over $\mathbb{Z}$ with representing Hopf algebra A, and if k_i is any collection of

fields, then the Hopf algebras $A \otimes k_i$ are constantly finite dimensional, whence the full ultraproduct $\prod_{\mathcal{U}} A \otimes k_i$ *is* in fact a Hopf algebra. In this case A_∞ can be identified with $A \otimes \prod_{\mathcal{U}} k_i$, whence $\prod_R \mathrm{Rep}_{k_i} G \simeq \mathrm{Rep}_{\prod_{\mathcal{U}} k_i} G$.

For non-finite groups, the situation becomes considerably more delicate. As an example, consider the multiplicative group $G = G_m$ (subsection 9.5.2) and let k_i be any collection of fields. For a fixed ultrafilter $\mathcal{U}$, let $\prod_{\mathcal{U}} \mathbb{Z}$ denote the ultrapower of the integers. Then we can identify A_∞ as the $k = \prod_{\mathcal{U}} k_i$-span of the formal symbols $x^{[z_i]}$, $[z_i] \in \prod_{\mathcal{U}} \mathbb{Z}$, with Δ and mult defined by

$$A_\infty = \mathrm{span}_k(x^{[z_i]} : [z_i] \in \prod_{\mathcal{U}} \mathbb{Z})$$

$$\Delta : x^{[z_i]} \mapsto x^{[z_i]} \otimes x^{[z_i]}$$

$$\mathrm{mult} : x^{[z_i]} \otimes x^{[w_i]} \mapsto x^{[z_i + w_i]}$$

We also note here that chapter 8 contains an interesting theorem about finite dimensional subcoalgebras of Hopf algebras which was necessary to prove theorem 1.2.2, but is certainly of interest in its own right, and requires no understanding of ultraproducts.

1.3   From Ultraproducts to Generic Cohomology

The reason we chose to study these categories in the first place is because cohomology of modules (at least in the Ext^1 case) is a naturally expressible concept in the language of abelian tensor categories. That is (see section 15.1)

Proposition 1.3.1. *For fixed n, the statement $\phi(M, N) \stackrel{def}{=}$ "$Ext^1(M, N)$ has dimension n" is a first-order formula in the language of abelian tensor categories.*

Here we have adopted the view that $\mathrm{Ext}^1(M, N)$, relative to a given abelian category, consists of equivalence classes of module extensions of M by N, as opposed to

the more standard definition via injective or projective resolutions; necessary, since the category $\mathrm{Rep}_k G$ will in general not have enough injective or projective objects (due to it consisting of only *finite* dimensional representations of G over k). Suppose then that M and N are modules for G over $\mathbb{Z}$, and that k_i is a collection of fields. We wish to discover whether the quantity

$$\dim \mathrm{Ext}^1_{G(k_i)}(M, N)$$

stabilizes for large i. We have a criterion for this to be true.

Theorem 1.3.2. *Let M_i and N_i denote the images inside $Rep_{k_i}(G)$ of the modules M and N, and let $[M_i], [N_i]$ denote the images of the tuples $(M_i), (N_i)$ inside the category $\prod_R Rep_{k_i}(G)$. Then if the computation $\dim Ext^1([M_i], [N_i])$ is both finite and the same inside the category $\prod_R Rep_{k_i}(G)$ for* every *choice of non-principal ultrafilter, the computation $\dim Ext^1_{G(k_i)}(M_i, N_i)$ is the same for all but finitely many i.*

Proof. The key fact (and the reason we restrict to Ext^1 in the first place) is that computing Ext^1 in the restricted ultracategory $\prod_R \mathcal{C}_i$ is the same as doing so in the full ultracategory $\prod_{\mathcal{U}} \mathcal{C}_i$, since the extension module of any 1-fold extension of $[M_i]$ by $[N_i]$ has bounded dimension $\dim(M_i) + \dim(N_i)$. First-order statements that are true in $\prod_{\mathcal{U}} \mathrm{Rep}_{k_i}(G)$ for *every* choice of non-principal ultrafilter correspond to statements that are true for all but finitely many of the categories $\prod_{\mathcal{U}} \mathrm{Rep}_{k_i}(G)$, namely the statement "$\dim \mathrm{Ext}^1_{G(k_i)}(M_i, N_i) = n$". $\qquad \square$

Our attempts to extend these results to the case of Ext^n, $n > 1$, have so far met with resistance; for more on this see section 15.3.

1.4 Generic Representation Theory of Unipotent Groups

Beginning in chapter 10 we take a break from working with ultraproducts, and instead focus on the concrete representation theory of two unipotent algebraic groups, both in zero and positive characteristic. Starting with the additive group G_a (chapter 12) we prove

Theorem 1.4.1. *(see theorem 12.3.6)*

1. *Let k have characteristic zero. Then every n-dimensional representation of G_a over k is given by an $n \times n$ nilpotent matrix N over k according to the formula*

$$e^{xN}$$

2. *Let k have positive characteristic p. Then if $p >> n$, every n-dimensional representation of G_a over k is given by a finite ordered sequence N_i of $n \times n$ commuting nilpotent matrices over k according to the formula*

$$e^{xN_0} e^{x^p N_1} \ldots e^{x^{p^m} N_m}$$

This is our first indication of a connection between the characteristic zero theory of a unipotent group and its positive characteristic theory for p large with respect to dimension. In chapter 13 we obtain an identical result for the Heisenberg group H_1:

Theorem 1.4.2. *(see theorems 13.5.4 and 13.5.5)*

1. *Let k have characteristic zero. Then every n-dimensional representation of H_1 is given by a triple X, Y, Z of $n \times n$ nilpotent matrices over k satisfying $Z = [X, Y]$*

and $[X, Z] = [Y, Z] = 0$, according to the formula

$$e^{xX + yY + (z - xy/2)Z}$$

2. Let k have positive characteristic p. Then if $p >> n$, every n-dimensional representation of H_1 over k is given by a sequence $X_0, Y_0, Z_0, X_1, Y_1, Z_1 \ldots, X_m, Y_m, Z_m$ of $n \times n$ nilpotent matrices over k satisfying $Z_i = [X_i, Y_i]$, $[X_i, Z_i] = [Y_i, Z_i] = 0$, and whenever $i \neq j$, X_i, Y_i, Z_i all commute with X_j, Y_j, Z_j, according to the formula

$$e^{xX_0 + yY_0 + (z - xy/2)Z_0} e^{x^p X_1 + y^p Y_1 + (z^p - x^p y^p/2)Z_1} \ldots e^{x^{p^m} X_m + y^{p^m} Y_m + (z^{p^m} - x^{p^m} y^{p^m}/2)Z_m}$$

We see then that for p sufficiently larger than dimension, characteristic p representations for these unipotent groups are simply commuting products of representations, each of which 'look like' a characteristic zero representation, with each factor accounting for one of its 'Frobenius layers'. It is this phenomenon to which the phrase 'generic representation theory' in the title refers.

Note that this is only a theorem about $p >>$ dimension; for any positive characteristic field k, once dimension becomes too large in the category $\text{Rep}_k H_1$, the analogy completely breaks down, and representations of H_1 over k can be expected to bear no resemblance to representations in characteristic zero.

1.5 The Height-Restricted Ultraproduct

Let G be either of the two above discussed unipotent groups, k a field of characteristic $p > 0$, V a representation of G over k. Suppose that V is of the form described in part 2. of the two preceding theorems. Then we define the **height** of V to be $m + 1$, that is, the number of Frobenius layers in the representation. For

instance, in the case of G_a, height is simply the largest m such that $x^{p^{m-1}}$ occurs in the matrix formula of the representation.

Now let k_i be a sequence of fields of strictly increasing positive characteristic, and let $\mathcal{C}_i = \operatorname{Rep}_{k_i} G$. For an object $[X_i]$ of the restricted ultraproduct $\prod_R \mathcal{C}_i$ (i.e. where the X_i have bounded dimension), by the above two theorems, for large enough i, X_i will be of the aforementioned form, so that for all but finitely many i, the height of X_i is well-defined. We therefore define the **height** of $[X_i]$ as the essential supremum of $\{\operatorname{height}(X_i) : i \in I\}$. Note that the height of a given object $[X_i]$ of $\prod_R \mathcal{C}_i$ might well be infinite. In case it is not, we define

Definition 1.5.1. (see definition 14.0.2) The **height-restricted ultraproduct** of the categories $\mathcal{C}_i = \operatorname{Rep}_{k_i} G$, for k_i of increasing positive characteristic, is the full subcategory of the restricted ultraproduct $\prod_R \mathcal{C}_i$ consisting of those objects $[X_i]$ of finite height. We denote this category by $\prod_H \mathcal{C}_i$. For $n \in \mathbb{N}$, we denote by $\prod_{H \leq n} \mathcal{C}_i$ the full subcategory of $\prod_H \mathcal{C}_i$ consisting of those objects of height no greater than n.

We have seen already that, for p sufficiently large with respect to dimension, representations of G in characteristic p resemble representations of G^n in characteristic zero. We shall also see later that this resemblance is functorial, in the sense that the analogy carries over to morphisms between the representations, and to various other constructions, e.g. direct sums and tensor products. The most compact way to express this is

Theorem 1.5.1. *(see theorem 14.0.6) If k_i is a sequence of fields of strictly increasing positive characteristic, the category $\prod_H \operatorname{Rep}_{k_i} G$ is tensorially equivalent to $\operatorname{Rep}_k G^\infty$, where G^∞ denotes a countable direct power of G, and k is the ultraproduct of the fields k_i. Similarly, for any $n \in \mathbb{N}$, $\prod_{H \leq n} \operatorname{Rep}_{k_i} G$ is tensorially equivalent to $\operatorname{Rep}_k G^n$.*

Note in particular that the group G^n obtained is independent of the choice of non-principal ultrafilter, and while the field k does vary, it will in all cases have

characteristic zero.

1.6 Height-Restricted Generic Cohomology

This last result will allow us to derive some generic cohomology theorems for the unipotent algebraic groups discussed above, at least for the case of Ext^1. Rather than state the theorem precisely here it will be much more illuminating to illustrate with an example, which is worked out in more detail in section 15.2.1.

Let $G = G_a$ and let k have characteristic $p > 0$. Then direct computation shows (using theorem 12.3.1) that a basis for $\mathrm{Ext}^1_{G_a(k)}(k, k)$ is given by the sequence of linearly independent extensions

$$\xi_m : 0 \to k \to \begin{pmatrix} 1 & x^{p^m} \\ & 1 \end{pmatrix} \to k \to 0$$

for $m = 0, 1, \ldots$. This is obviously infinite dimensional, so we ask the more interesting question: what happens when we restrict the height of the extension module? Specifically, let $\mathrm{Ext}^{1,h}_{G_a(k)}(k, k)$ denote the space of equivalence classes of extensions whose extension module has height no greater than h. Then of course $\{\xi_m : m = 0, 1, \ldots, h-1\}$ forms a basis for it, and its dimension is h.

Now assume k has characteristic zero and compute $\mathrm{Ext}^1_{G^h(k)}(k, k)$. Then similarly (using theorems 12.2.1 and 11.0.3) we have the basis

$$\xi_m : 0 \to k \to \begin{pmatrix} 1 & x_m \\ & 1 \end{pmatrix} \to k \to 0$$

for $m = 0, 1, \ldots, h - 1$, where x_m denotes the mth free variable of G^h. Thus we see

that, for k_i of sufficiently large positive characteristic

$$\dim \operatorname{Ext}^{1,h}_{G_a(k_i)}(k_i, k_i) = \dim \operatorname{Ext}^1_{G_a^h(\Pi_\mathcal{U} k_i)}(\prod_\mathcal{U} k_i, \prod_\mathcal{U} k_i)$$

This example is misleading in that the above equality holds for all primes p (due to the small dimension of the extension modules of the extensions); in general we merely claim that the above holds for sufficiently large characteristic. We shall prove in section 15.2 that this is quite a general phenomenon.

1.7   Notational Conventions

Throughout this dissertation our convention for expressing composition of maps is as follows. If $\phi : X \to Y$ and $\psi : Y \to Z$ are maps, then the composition $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ shall be expressed as

$$\phi \circ \psi$$

However, if x is an element of X, then the element $z \in Z$ gotten by evaluating $\phi \circ \psi$ at x shall be expressed as

$$\psi(\phi(x))$$

In other words, when we are expressing a composition with no inputs we shall write functions on the right, and when we are evaluating a composition at an input we shall write functions on the left. The reader need only remember that whenever he sees the symbol $\circ$ as in $\phi \circ \psi$, we are writing functions on the right, and when he instead sees the parenthetical notation $\psi(\phi(x))$ we are writing functions on the left. In particular, we shall never write $(x)(\phi \circ \psi)$, $(\phi \circ \psi)(x)$, $((x)\phi)\psi$, $\phi(\psi(x))$, or $\psi \circ \phi$.

A matrix, if we wish to emphasize what its entries are, is generally written in the notation (a_{ij}), suppressing mention of its dimensions. If $\phi : U \to V$ is a linear map

between finite dimensional vector spaces, and if (k_{ij}) is the matrix representing ϕ in certain bases, then it is understood to be doing so be passing column vectors to the right of (k_{ij}). In particular, if $\phi : U \to V$, $\psi : V \to W$ are linear maps, and if (k_{ij}) represents ϕ and (l_{ij}) represents ψ, then the matrix product $(l_{ij})(k_{ij})$ represents the map $\phi \circ \psi$.

If M_i is a collection of relational structures in a common signature indexed by the set I, and if $\mathcal{U}$ is a non-principal ultrafilter over I, we denote by $\prod_{\mathcal{U}} M_i$ the ultraproduct of these structures with respect to $\mathcal{U}$. The reader may consult the appendix for a review of ultrafilters and ultraproducts in general. In several instances in this dissertation we shall be considering certain substructures of ultraproducts, e.g. the restricted ultraproduct of the neutral tannakian categories $\mathcal{C}_i$, which in this case we denote by $\prod_R \mathcal{C}_i$. Note that, in this case and in several others, to avoid using a double subscript, we have dropped reference to the particular non-principal ultrafilter being applied; as it will always be assumed to be fixed but arbitrary, no confusion should result.

The reader is encouraged to consult the index for a more complete list of commonly used symbols.

Chapter 2

Algebraic Groups, Hopf Algebras, Modules, Comodules and Cohomology

Here we review the basic facts concerning the duality between algebraic groups and their Hopf algebras, modules for algebraic groups vs. comodules for Hopf algebras, and the definitions concerning cohomology of modules and comodules. We shall also define the equivalent categories $\mathrm{Rep}_k G$ and Comod_A, where A is the representing Hopf algebra of G, and the important constructions within them. We shall mostly be content to recording definitions and important theorems, only rarely supplying proofs. The reader may consult [13], [4], and [16] for a more thorough and excellent account of what follows. We particularly recommend the first several chapters of [16] for those less accustomed to the 'functorial' view of algebraic groups we shall be adopting. [12] and [2] are also excellent references, but with much less of an emphasis on this functorial view.

2.1 Algebraic Groups, Coalgebras and Hopf Algebras

In this dissertation, "algebraic group" over a ring k shall always mean a particular kind of affine group scheme. It is at this level of generality in which we will operate throughout. If k is any commutative ring with identity (and all rings will assumed to be so) a k-**algebra** shall always mean a commutative k-algebra with identity.

Definition 2.1.1. (see section 1.2 of [16]) An **affine group scheme** over a commutative ring k with identity is a representable covariant functor from the category of all k-algebras to the category of groups. We say it is an **algebraic group** if the representing object of the functor is finitely generated as a k-algebra, and a **finite group** if it is finitely generated as a k-module.

Definition 2.1.2. (see section 1.1 of [4]) Let k be a ring, C a k-module. C is called a k-**coalgebra** if it comes equipped with k-linear maps $\Delta : C \to C \otimes C$ (comultiplication) and $\varepsilon : C \to k$ (co-unit) making the following two diagrams commute:

$$
\begin{array}{ccc}
C & \xrightarrow{\ \Delta\ } & C \otimes C \\
{\scriptstyle \Delta}\big\downarrow & & \big\downarrow{\scriptstyle 1 \otimes \Delta} \\
C \otimes C & \xrightarrow[\scriptstyle \Delta \otimes 1]{} & C \otimes C \otimes C
\end{array}
\qquad (2.1.1)
$$

$$
\begin{array}{ccccc}
& & C & & \\
& {\scriptstyle \simeq}\swarrow & \big\downarrow{\scriptstyle \Delta} & \searrow{\scriptstyle \simeq} & \\
k \otimes C & & & & C \otimes k \\
& {\scriptstyle \varepsilon \otimes 1}\nwarrow & \big\downarrow & \nearrow{\scriptstyle 1 \otimes \varepsilon} & \\
& & C \otimes C & &
\end{array}
\qquad (2.1.2)
$$

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A k-**bialgebra** is a k-module C which is simultaneously a k-algebra and a k-coalgebra in such a way that Δ and ε are algebra maps. A bialgebra C is called a **Hopf algebra** if it comes equipped with a k-algebra map $S : C \to C$ (co-inverse or antipode) making the following commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\ \Delta\ } & A \otimes A \\
& & \\
& & \downarrow{\scriptstyle S \otimes 1} \\
\varepsilon \downarrow & & A \otimes A \\
& & \downarrow{\scriptstyle \mathrm{mult}} \\
k & \longrightarrow & A
\end{array}
\qquad (2.1.3)
$$

A **morphism** between the k-coalgebras (C, Δ) and (C', Δ') is a k-linear map $\phi : C \to C'$ such that the following diagram commutes:

$$
\begin{array}{ccc}
C & \xrightarrow{\ \phi\ } & C' \\
\Delta \downarrow & & \downarrow \Delta' \\
C \otimes C & \xrightarrow{\ \phi \otimes \phi\ } & C' \otimes C'
\end{array}
$$

Theorem 2.1.1. *(see section* 1.4 *of [16]) The representing object of an affine group scheme over k is a Hopf algebra over k. Conversely, any Hopf algebra over k defines an affine group scheme over k.*

Let $G(\_) = \mathrm{Hom}_k(A, \_)$ be an affine group scheme over k represented by the Hopf algebra A. As the names suggest, the co-multiplication, co-unit, and co-inverse maps attached to a Hopf algebra encode the group multiplication, identity, and inversion, respectively. If R is a k-algebra then an element of $G(R)$ is by definition a k-homomorphism $\phi : A \to R$. Then the map Δ tells us how to multiply elements of

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$G(R)$; given $\phi, \psi : A \to R$ their product, call it $\phi * \psi$, is defined to be the unique map making the following diagram commute:

$$
\begin{array}{ccc}
A & \xrightarrow{\;\;\Delta\;\;} & A \otimes A \\
\downarrow{\scriptstyle \phi*\psi} & & \downarrow{\scriptstyle \phi\otimes\psi} \\
R & \xleftarrow[\text{mult}]{} & R \otimes R
\end{array}
$$

Similarly the inverse of the element ϕ, call it $\mathrm{inv}(\phi)$, is the unique map making

$$
\begin{array}{ccc}
A & \xrightarrow{\;\;S\;\;} & A \\
\downarrow{\scriptstyle \mathrm{inv}(\phi)} & \swarrow{\scriptstyle \phi} & \\
R & &
\end{array}
$$

commute. Finally, the identity element e of $G(R)$ is defined by the commutativity of the diagram.

$$
\begin{array}{ccc}
A & \xrightarrow{\;\;\varepsilon\;\;} & k \\
\downarrow{\scriptstyle e} & \swarrow & \\
R & &
\end{array}
$$

With this in mind, diagrams 2.1.1, 2.1.2 and 2.1.3 are really not mysterious at all. 2.1.1 merely encodes the fact that the multiplication defined by Δ is associative, 2.1.2 corresponds to the statement that $A \xrightarrow{\varepsilon} k \to R$ is always the identity element of $G(R)$, and one can probably guess what fact about groups 2.1.3 represents.

When we refer to a Hopf algebra we shall often write it as (A, Δ, ε), emphasizing the fact that these are often the only pieces of information we require for a given purpose. Besides, just as inverses can be discovered by looking at the multiplication table of a group, so also is the map S completely determined by the map Δ (and the

same can be said for the map ε).

Essential to the study of representable functors in any category is the so-called Yoneda lemma, which tells us that natural transformations from representable functors to other functors are quite easy to describe.

Lemma 2.1.2. *(Yoneda lemma; see section* 1.3 *of [16]) Let $\mathcal{C}$ be any locally small category (so that Hom-sets are actually sets), $G, H : \mathcal{C} \to$ Sets any two (covariant) set valued functors, and suppose that G is representable by the object $A \in \mathcal{C}$ (so that $G(\_) = Hom_{\mathcal{C}}(A, \_)$). Let Φ be a natural transformation from the functor G to H. Then for any object X of $\mathcal{C}$ and element ϕ of $Hom_{\mathcal{C}}(A, X)$, $\Phi_X(\phi) = (H\phi)(\Phi_A(1_A))$.*

Proof. Given $\phi : A \to X$, consider the commutative diagram

$$
\begin{array}{ccc}
G(A) & \xrightarrow{\Phi_A} & H(A) \\
\Big\downarrow{\scriptstyle G\phi} & & \Big\downarrow{\scriptstyle H\phi} \\
G(X) & \xrightarrow[\Phi_X]{} & H(X)
\end{array}
$$

Here $G\phi$ refers to the map that sends $\psi : A \to A$ to $\psi \circ \phi$, and $H\phi$ refers to whatever map H sends ϕ to. Start with $1_A \in G(A)$ in the upper left corner, chase it around both paths to $H(X)$, and you get the equation claimed. $\qquad\square$

As the names suggest, there is a duality between k-algebras and k-coalgebras.

Definition 2.1.3. (see section 1.3 of [4]) Let k be a field, (C, Δ, ε) a k-coalgebra. The **dual algebra** to C is the set C^* of linear functionals on C endowed with the following multiplication: if $\phi, \psi \in C^*$, then $\text{mult}(\phi \otimes \psi)$ is the k-linear map given by the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\phi \otimes \psi} k \otimes k \simeq k$$

If A is an infinite dimensional algebra, it is generally not possible to introduce a

coalgebra structure on the entire dual space of A. We can however introduce one on a certain subspace.

Definition 2.1.4. (see section 1.5 of [16]) Let k be a field, A a k-algebra. The **finite dual coalgebra** of A, denoted A°, is the subspace of A^* consisting of those linear functionals on A which kill an ideal of A of finite codimension. For $\alpha \in A^\circ$, we define $\Delta(\alpha)$ as follows: let α act on $A \otimes A$ by the composition

$$A \otimes A \xrightarrow{\text{mult}} A \xrightarrow{\alpha} k$$

and then pass this composition to the isomorphism $(A \otimes A)^\circ \simeq A^\circ \otimes A^\circ$. We define $\varepsilon : A^\circ \to k$ by $\varepsilon(\alpha) = \alpha(1)$.

If A is finite dimensional, then A° is all of A^*. In this case there is a natural isomorphism $A \simeq A^{\circ*}$ of algebras, and likewise a natural isomorphism $C \simeq C^{*\circ}$ of coalgebras.

This duality is functorial. Given an algebra map $\phi : A \to B$ we get a coalgebra map $\phi^\circ : B^\circ \to A^\circ$ defined by, for $\beta \in B^\circ$, $\phi^\circ(\beta)$ is the composition

$$A \xrightarrow{\phi} B \xrightarrow{\beta} k$$

In a similar fashion we get algebra maps from coalgebra maps.

2.2 G-modules and A-comodules

Definition 2.2.1. (see section 3.1 of [16]) Let G be an affine group scheme over the ring k, and V a k-module. A **linear representation** of G on V is a natural transformation from the functor $G(\_)$ to the functor $GL_V(\_)$, where $\mathrm{GL}_V(R) \overset{def}{=} \mathrm{Aut}_R(V \otimes R)$. We say also then that V is a G-**module**.

The concept of a linear representation, put this way, is a bit intimidating. However, it is a consequence of (the proof of) the Yoneda lemma that linear representations correspond to very concrete things, called comodules.

Definition 2.2.2. (see definition 2.1.3 of [4]) Let (C, Δ, ε) be a coalgebra over the ring k, V a k-module. V is called a (right) C-**comodule** if it comes equipped with a k-linear map $\rho : V \to V \otimes C$ such that the following two diagrams commute:

$$
\begin{array}{ccc}
V & \xrightarrow{\ \rho\ } & V \otimes C \\
{\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\
V \otimes C & \xrightarrow[\rho\otimes 1]{} & V \otimes C \otimes C
\end{array}
\tag{2.2.1}
$$

$$
\begin{array}{ccc}
V & \xrightarrow{\ \rho\ } & V \otimes C \\
 & {\scriptstyle \simeq}\searrow & \downarrow{\scriptstyle 1\otimes\varepsilon} \\
 & & V \otimes k
\end{array}
\tag{2.2.2}
$$

Theorem 2.2.1. *(see section 3.2 of [16]) If G is an affine group scheme represented by the Hopf algebra A, then linear representations of G on V correspond to A-comodule structures on V.*

It is for this reason that, in this dissertation, we shall quite often confuse the notions of G-module and A-comodule, and shall sometimes speak glibly of A-modules, representations for A, comodules for G, etc.

Here is the correspondence. Given an A-comodule (V, ρ), we get a representation Φ of G on V as follows: if $g \in G(R)$, then g is by definition a k-homomorphism from A to R. Define g to act on $V \otimes R$ via the composition

$$
V \xrightarrow{\ \rho\ } V \otimes A \xrightarrow{\ 1\otimes g\ } V \otimes R
\tag{2.2.3}
$$

and then extend to $V \otimes R$ by R-linearity. Conversely, let $\Phi : G \to \mathrm{GL}_V$ be a representation. Then we may ask how $\mathrm{id}_A \in G(A)$ acts on V, that is, what is the map

$$\Phi_A(\mathrm{id}_A) : V \otimes A \to V \otimes A$$

As we demand this map to be A-linear it is necessarily determined by its restriction to $V \simeq V \otimes 1$, and it is this map, call it ρ, which gives V the structure of an A-comodule. A Yoneda lemma type argument guarantees that, for any $g \in G(R)$, $\Phi_R(g)$ is given by equation 2.2.3.

Let us be more explicit. Suppose k is a field and (V, ρ) a finite dimensional A-comodule. Fix a basis $e_1, \ldots, e_n$ of V and write

$$\rho : e_j \mapsto \sum_{i=1}^{n} e_i \otimes a_{ij}$$

Then the matrix (a_{ij}) is the 'formula' for the representation of G on V. That is, for $g \in G(R)$, g acts on $V \otimes R$ via the matrix $(g(a_{ij}))$ in the given basis, and then extending by R-linearity.

Comparing equations 2.1.1 and 2.1.2 with equations 2.2.1 and 2.2.2, we see that (A, Δ) itself qualifies as a (usually infinite dimensional) A-comodule, and we call it the **regular representation**. Among other reasons, this is an important representation because

Theorem 2.2.2. *(see section 3.5 of [16]) If (V, ρ) is an n-dimensional A-comodule, then V is embeddable in the n-fold direct sum of the regular representation.*

Over fields, we have the following elementary yet eminently useful results.

Theorem 2.2.3. *(Fundamental theorem of coalgebras; see theorem 1.4.7 of [4]) Let k be a field, C a k-coalgebra. Then C is the directed union of its finite dimensional subcoalgebras.*

Theorem 2.2.4. *(Fundamental theorem of comodules; see theorem 2.1.7 of [4]) Let k be a field, C a k-coalgebra, V a C-comodule. Then V is the directed union of its finite dimensional subcomodules.*

2.3 The Categories $\text{Rep}_k G$ and Comod_A

Let k be a field, G an affine group scheme over k, A its representing Hopf algebra.

Definition 2.3.1. Comod_A is the category whose objects are finite dimensional A-comodules, and whose morphisms between (V, ρ) and (W, μ) are those k-linear maps $\phi : V \to W$ making the following commute:

$$
\begin{array}{ccc}
V & \xrightarrow{\ \ \phi\ \ } & W \\
\Big\downarrow{\scriptstyle \rho} & & \Big\downarrow{\scriptstyle \mu} \\
V \otimes A & \xrightarrow[\ \phi \otimes 1\ ]{} & W \otimes A
\end{array}
$$

Definition 2.3.2. $\text{Rep}_k G$ is the category whose objects are finite dimensional vector spaces V with a prescribed G-module structure $\Phi : G \to \text{GL}_V$, and whose morphisms between $(V, \Phi), (W, \Psi)$ are those linear maps $\phi : V \to W$ such that, for every k-algebra R and element $g \in G(R)$, the following commutes:

$$
\begin{array}{ccc}
V \otimes R & \xrightarrow{\ \phi \otimes 1\ } & W \otimes R \\
\Big\downarrow{\scriptstyle \Phi_R(g)} & & \Big\downarrow{\scriptstyle \Psi_R(g)} \\
V \otimes R & \xrightarrow[\ \phi \otimes 1\ ]{} & W \otimes R
\end{array}
$$

Note well that whenever we write Comod_A or $\text{Rep}_k G$ it consists of only *finite dimensional* modules/comodules, in contrast to the notation of some other authors.

The correspondence between G-modules and A-comodules given in theorem 2.2.1

is actually a functorial one (and it would be of no use otherwise).

Theorem 2.3.1. *The categories $Rep_k G$ and $Comod_A$ are equivalent as k-linear abelian tensor categories.*

This theorem is merely the assertion that a linear map $\phi : V \to W$ is a morphism for V and W as G-modules if and only if it is for V and W as A-comodules, and that the tensor structure on Comod_A (defined below) induces the usual tensor structure we expect to see in $\text{Rep}_k G$.

Definition 2.3.3. Let A be a Hopf algebra, (V, ρ), (W, μ) finite dimensional A-comodules.

1. Their **direct sum** is the A-comodule with underlying vector space $V \oplus W$ and comodule map given by the composition

$$V \oplus W \xrightarrow{\rho \oplus \mu} (V \otimes A) \oplus (W \otimes A) \simeq (V \oplus W) \otimes A$$

2. Their **tensor product** is the A-comodule with underlying vector space $V \otimes W$ and comodule map given by the composition

$$V \otimes W \xrightarrow{\rho \otimes \mu} (V \otimes A) \otimes (W \otimes A) \xrightarrow{1 \otimes \text{Twist} \otimes 1} V \otimes W \otimes A \otimes A \xrightarrow{1 \otimes 1 \otimes \text{mult}} V \otimes W \otimes A$$

3. The **tensor product** of two morphisms $\phi : V \to X$, $\psi : W \to Y$ is the usual tensor product of linear maps $\phi \otimes \psi : V \otimes W \to X \otimes Y$

4. The **trivial A-comodule**, or **trivial representation**, is the A-comodule having underlying vector space k and comodule map $\rho : k \to k \otimes A$ given by $\rho : 1 \mapsto 1 \otimes 1$

5. The **dual** of (V, ρ) is the A-comodule with underlying vector space V^* and comodule map $\mu : V^* \to V^* \otimes A$ defined as follows. Let $\rho^* : V^* \to \text{Hom}_k(V, A)$

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be the map that sends the functional $\phi : V \to k$ to the composition $V \xrightarrow{\rho}$ $V \otimes A \xrightarrow{\phi \otimes 1} k \otimes A \simeq A$. Then μ is the composition

$$V^* \xrightarrow{\rho^*} \mathrm{Hom}_k(V, A) \simeq V^* \otimes A \xrightarrow{1 \otimes S} V^* \otimes A$$

where S denotes the antipode of A

6. The **internal Hom** of V and W, denoted $\underline{\mathrm{Hom}}(V, W)$, is the tensor product of the dual of V with W.

An alternative, basis dependent definition of the dual (V^*, μ) of the comodule (V, ρ) is as follows. Pick a basis $\{e_1, \ldots, e_n\}$ of V, and let $\{\alpha_1, \ldots, \alpha_n\}$ be the dual basis of V^*. Write $\rho : e_j \mapsto \sum_i e_i \otimes a_{ij}$, and set $\bar{\rho} : \alpha_j \mapsto \sum_i \alpha_i \otimes a_{ji}$ (note the transpose being applied). Then μ is the composition

$$V^* \xrightarrow{\bar{\rho}} V^* \otimes A \xrightarrow{1 \otimes S} V^* \otimes A$$

In other words, if (a_{ij}) is the matrix formula for a representation on V in a given basis, then the dual representation on the dual space V^*, in the dual basis, is the inverse of the transpose of (a_{ij}).

2.4 Cohomology of Comodules

The relevant definitions for cohomology in Comod_A make sense in any k-linear abelian category, so we state them at this level of generality. The reader may consult [17] or [1] for a more thorough introduction to these matters.

Definition 2.4.1. Let $\mathcal{C}$ be a k-linear abelian category, M, N objects of $\mathcal{C}$. An n-fold **extension** of M by N is an exact sequence

$$\xi : 0 \to N \to X_{n-1} \to \ldots \to X_0 \to M \to 0$$

Two extensions are **equivalent** if there exist morphisms $\phi_i : X_i \to Y_i$ such that

commutes. The set of equivalence classes (with respect to the equivalence relation generated by the relation 'being equivalent') of n-fold extensions of M by N is denoted $\mathrm{Ext}^n(M, N)$.

In case $n > 1$, the term 'equivalent' is abusive; it is a not necessarily symmetric or transitive relation. Nonetheless, this relation generates a unique equivalence relation, and it is this relation with respect to which $\mathrm{Ext}^n(M, N)$ is defined.

On the other hand, this relation *is* a bona fide equivalence relation on $\mathrm{Ext}^1(M, N)$. This is because, as can be shown, the map ϕ_0 given by the definition of equivalence must necessarily be an isomorphism.

Let

$$\xi : 0 \to N \to X_{n-1} \to \ldots \to X_0 \to M \to 0$$

$$\chi : 0 \to N \to Y_{n-1} \to \ldots \to Y_0 \to M \to 0$$

be two n-fold extensions of M by N with $n > 1$. The **Baer sum** of ξ and χ, denoted $\xi \oplus \chi$, is the extension gotten as follows. Let Γ and Ω be the pullback/pushout

respectively of the following two diagrams

$$\begin{array}{ccc}
\Omega & \longrightarrow & Y_0 \\
\downarrow & & \downarrow \\
X_0 & \longrightarrow & M
\end{array}
\qquad
\begin{array}{ccc}
\Gamma & \longleftarrow & Y_{n-1} \\
\uparrow & & \uparrow \\
X_{n-1} & \longleftarrow & N
\end{array}$$

Then $\xi \oplus \chi$ is the extension

$$0 \to N \to \Gamma \to X_{n-2} \oplus Y_{n-2} \to \ldots \to X_1 \oplus Y_1 \to \Omega \to M \to 0$$

For $n = 1$, the Baer sum is defined slightly differently. Let

$$\xi : 0 \to N \xrightarrow{\phi_1} X_1 \xrightarrow{\psi_1} M \to 0$$

$$\chi : 0 \to N \xrightarrow{\phi_2} X_2 \xrightarrow{\psi_2} M \to 0$$

be two 1-fold extensions of M by N. Let X be the pullback of X_1 and X_2 under M, and $\phi', \bar{\phi}$ the unique maps making

commute. Let $X \xrightarrow{\tau} Y$ be the cokernel of ϕ', and ψ the unique map making

$$N \xrightarrow{\phi'} X \xrightarrow{\pi_1 \psi_1} M$$

with $\tau: X \to Y$ and $\psi: Y \to M$

commute. Set $\phi = \bar{\phi}\tau$. Then the Baer sum $\xi \oplus \chi$ is the extension

$$0 \to N \xrightarrow{\phi} Y \xrightarrow{\psi} M \to 0$$

Let $a \neq 0$ be a scalar, ξ an n-fold extension as above, and let $M \xrightarrow{\phi} X_{n-1}$ be the first map in the extension. Then we define the **scalar multiplication** of a on ξ to be the extension

$$a\xi : 0 \to M \xrightarrow{\frac{1}{a}\phi} X_{n-1} \to \ldots \to X_0 \to N \to 0$$

with all of the other maps and objects staying the same. For $a = 0$, we define 0ξ to be the trivial extension (defined below).

Theorem 2.4.1. *For any two objects M and N of a k-linear abelian category, Baer sum and scalar multiplication respect equivalence classes, and $Ext^n(M, N)$ is a vector space under those operations.*

The additive identity of $\text{Ext}^n(M, N)$ is called the **trivial** or **split** extension. In the case of Ext^1, it can be identified as the equivalence class of the extension $0 \to N \to N \oplus M \to M \to 0$.

Chapter 3

Tannakian Duality

Here we present the basic facts concerning the correspondence between categories of finite dimensional representations of affine group schemes over a field, and so-called neutral tannakian categories. We have no intention of giving a full account, especially concerning proofs; we shall mostly content ourselves with giving the definition of a neutral tannakian category, and, following the proof given in [5], describing a method for recovering the representing Hopf algebra of such a category. The reader may consult [15] or [3], but the development given here follows almost exclusively that of [5]. The reader may also consult [7] for an excellent introduction to the theory of abelian categories, and [14] for a good account of tensor categories in the abstract (there referred to as *monoidal* categories, but without many of the assumptions we shall be placing on them).

The theory of tannakian categories, while having broader implications than what we will be discussing is, as far as we are concerned, a successful attempt to answer three natural questions about the category $\mathrm{Rep}_k G$ (equivalently, Comod_A, where A is the representing Hopf algebra of G). Firstly, to what extent does purely categorical information about the category $\mathrm{Rep}_k G$ determine the group G? The answer is, completely, if one allows for one piece of external information (called a *fibre functor*).

Secondly, can we recover in some constructive way the Hopf algebra A from Comod_A? The answer again is yes, and it is to this that we will devoting most of our time. Finally, is there a set of axioms one can write down which are equivalent to a category being $\mathrm{Rep}_k G$ for some k and G? The answer again is yes, and these axioms serve as the definition for a neutral tannakian category.

3.1   A Motivating Example

To motivate the definitions given in the next section it will help to keep in mind the simplest and yet most important example, the category Vec_k consisting of all finite dimensional vector spaces over a field k, with morphisms being k-linear maps between the vector spaces (or, if you like, $\mathrm{Rep}_k G_0$, where G_0 is the trivial group represented by the Hopf algebra k). Firstly, Vec_k is a k-linear abelian category. This means, among other things, that the Hom-sets themselves have a k-linear structure on them, and composition of morphisms is bilinear with respect to this structure. It also means that Vec_k satisfies some nice regularity conditions, and that certain desirable constructions are always possible within it: finite biproducts always exist (in the form of the usual direct sum of vector spaces), kernels and cokernels always exist, and all monomorphisms and epimorphisms are normal (every injective map is the kernel of its cokernel, and every surjective map is the cokernel of its kernel).

As it happens, this is not quite enough to recover fully the fact that Vec_k is indeed Vec_k. Enter the tensor product. To every pair of vector spaces V and W we assign an object called $V \otimes W$, and to every pair of morphisms $V \xrightarrow{\phi} X$ and $W \xrightarrow{\psi} Y$ we assign a morphism, denoted $V \otimes W \xrightarrow{\phi \otimes \psi} X \otimes Y$. We also have that, for every composable pair ϕ, ψ and composable pair a, b, $(\phi \otimes a) \circ (\psi \otimes b) = (\phi \circ \psi) \otimes (a \circ b)$. This amounts to the assertion that $\otimes$ is a bifunctor on Vec_k. This bifunctor $\otimes$ is a bilinear functor in the sense that, for any $c \in k$, $(c\phi + \psi) \otimes \eta = c(\phi \otimes \eta) + (\psi \otimes \eta)$, and similarly for

the other slot.

We know also that $\otimes$ is a commutative operation. That is, for every pair of vector spaces A and B there is a natural isomorphism $A \otimes B \overset{\text{comm}_{A,B}}{\simeq} B \otimes A$ (namely the map $a \otimes b \mapsto b \otimes a$). Naturality here means that for every pair of maps $A \overset{\phi}{\to} X$ and $B \overset{\psi}{\to} Y$ the following commutes:

$$
\begin{array}{ccc}
A \otimes B & \xrightarrow{\text{comm}_{A,B}} & B \otimes A \\
{\scriptstyle \phi \otimes \psi} \downarrow & & \downarrow {\scriptstyle \psi \otimes \phi} \\
X \otimes Y & \xrightarrow[\text{comm}_{X,Y}]{} & Y \otimes X
\end{array}
$$

Similarly, $\otimes$ is naturally associative, given by the natural isomorphism $(A \otimes B) \otimes C \overset{\text{assoc}_{A,B,C}}{\simeq} A \otimes (B \otimes C)$ (namely $(a \otimes b) \otimes c \mapsto a \otimes (b \otimes c)$).

Vec_k also has an identity object for $\otimes$, namely the vector space k. This means that to every vector space V, there is a natural isomorphism $V \overset{\text{unit}_V}{\simeq} k \otimes V$ (namely $v \mapsto 1 \otimes v$), satisfying diagrams analogous to the above. In the context of abstract tannakian categories this identity object is denoted as $\underline{1}$.

We also mention that the isomorphisms comm and assoc satisfy some coherence conditions with one another. These are expressed by the so-called pentagon and hexagon axioms, to be discussed in the next section.

Recall the universal bilinear mapping property of the tensor product. For every V and W there is a bilinear map $\otimes : V \times W \to V \otimes W$ with the following property: to every bilinear map $V \times W \overset{\phi}{\to} Z$ there is a unique linear map $V \otimes W \overset{\psi}{\to} Z$ such that the following diagram commutes:

$$
\begin{array}{ccc}
V \otimes W & \xrightarrow{\psi} & Z \\
{\scriptstyle \otimes} \uparrow & \nearrow {\scriptstyle \phi} & \\
V \times W & &
\end{array}
$$

What this gives us is a natural isomorphism between linear maps on $V \otimes W$ and bilinear maps on $V \times W$. But a bilinear map $V \times W \to Z$ is just another name for a linear map $V \to \text{Hom}(W, Z)$. Thus we have an isomorphism

$$\text{Hom}(V \otimes W, Z) \simeq \text{Hom}(V, \text{Hom}(W, Z))$$

Another way of stating this is that $\text{Hom}(\_\_ \otimes W, Z)$ is a representable functor, and its representing object is $\text{Hom}(W, Z)$. Vec_k enjoys the property that, for any objects V and W, $\text{Hom}(V, W)$ is also an object of Vec_k. For a category in which this is not exactly the case, we have a different name, when it exists, for the representing object of $\text{Hom}(\_\_ \otimes W, Z)$: we call it *internal Hom*, and denote it by $\underline{\text{Hom}}(W, Z)$. The above discussion can thus be summed up as saying that, in Vec_k, internal Homs always exist.

Of special interest then is, for any vector space V, the object $\underline{\text{Hom}}(V, \underline{1})$, which in Vec_k can be identified as the space of linear functionals on V. We denote this object by $V^\vee$. It is well known that any vector space V is naturally isomorphic to $V^{\vee\vee}$ via the map $v \mapsto \text{ev}_v$, where ev_v is the map that evaluates any functional $V \to k$ at the element v. We say then that all objects of Vec_k are *reflexive* in the sense that $v \mapsto \text{ev}_v$ is always an isomorphism.

For any vector spaces X_1, X_2, Y_1, Y_2, there is an obvious isomorphism $\underline{\text{Hom}}(X_1, Y_1) \otimes \underline{\text{Hom}}(X_2, Y_2) \simeq \underline{\text{Hom}}(X_1 \otimes X_2, Y_1 \otimes Y_2)$, namely the map that sends the element $\phi \otimes \psi$ to the map of the same name. This isomorphism can be thought of as expressing the fact that the tensor product, acting on $\underline{\text{Hom}}(X, Y)$ as an object, is compatible with its action on it as a Hom-set.

In Vec_k, $\text{End}(\underline{1}) = \text{End}(k)$ can be identified with the field k itself, if we take addition to be addition of maps and multiplication to be composition of morphisms. Thus we can say that $\text{End}(\underline{1})$ is a field. With this in mind, and given everything else

we've done, we can actually define the k-linear structure on Vec_k without assuming it. If $c : k \to k$ is the linear map given by multiplication by the constant $c \in k$ and $\phi : V \to W$ any linear map, we can define $c\phi$ as the composition

$$V \xrightarrow{\mathrm{unit}_V} k \otimes V \xrightarrow{c \otimes \phi} k \otimes W \xrightarrow{\mathrm{unit}_W^{-1}} W$$

Finally, there is a so-called *fibre functor* on Vec_k, i.e. an exact, k-linear tensor preserving functor from Vec_k to $\mathrm{Vec}_{\mathrm{End}(\underline{1})}$. Here, we can simply take this functor to be the identity $\mathrm{Vec}_k \to \mathrm{Vec}_k$.

The preceding discussion amounts to the assertion that Vec_k is a neutral tannakian category, which we formally define now.

3.2 Definition of a Neutral Tannakian Category

Definition 3.2.1. An **abelian category** is a category $\mathcal{C}$ with the following properties:

1. For all objects $A, B \in \mathcal{C}$, $\mathrm{Hom}(A, B)$ is endowed with the structure of an abelian group, and composition of morphisms is bilinear with respect to this structure

2. Every pair of objects in $\mathcal{C}$ has a biproduct

3. Every morphism has a kernel and a cokernel

4. Every monomorphism is the kernel of some morphism, and every epimorphism is the cokernel of some morphism

Definition 3.2.2. Let $\mathcal{C}$ be a category endowed with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and denote by, for objects A, B and morphisms ϕ, ψ, $A \otimes B \overset{\mathrm{def}}{=} \otimes(A, B)$ and $\phi \otimes \psi \overset{\mathrm{def}}{=} \otimes(\phi, \psi)$. Then $\otimes$ is called a **tensor product**, and $\mathcal{C}$ is called a **tensor category**, if the following hold:

1. There is a functorial isomorphism $\mathrm{assoc}_{X,Y,Z} : X \otimes (Y \otimes Z) \simeq (X \otimes Y) \otimes Z$

2. There is a functorial isomorphism $\mathrm{comm}_{X,Y} : X \otimes Y \simeq Y \otimes X$

3. There is an **identity object**, denoted $\underline{1}$, and a functorial isomorphism $\mathrm{unit}_V :$ $V \simeq \underline{1} \otimes V$ inducing an equivalence of categories $\mathcal{C} \to \mathcal{C}$

4. (pentagon axiom) For all objects X, Y, Z and T, the following commutes:

$$
\begin{array}{ccc}
 & (X \otimes Y) \otimes (Z \otimes T) & \\
 \nearrow^{\mathrm{assoc}} & & \searrow^{\mathrm{assoc}} \\
X \otimes (Y \otimes (Z \otimes T)) & & ((X \otimes Y) \otimes Z) \otimes T \\
\downarrow^{1 \otimes \mathrm{assoc}} & & \downarrow^{\mathrm{assoc} \otimes 1} \\
X \otimes ((Y \otimes Z) \otimes T) & \xrightarrow{\mathrm{assoc}} & (X \otimes (Y \otimes Z)) \otimes T
\end{array}
$$

(the obvious sub-scripts on assoc have been omitted)

5. (hexagon axiom) For all objects X, Y and Z, the following commutes:

$$
\begin{array}{ccc}
 & (X \otimes Y) \otimes Z & \\
\nearrow^{\mathrm{assoc}} & & \searrow^{\mathrm{comm}} \\
X \otimes (Y \otimes Z) & & Z \otimes (X \otimes Y) \\
\downarrow^{1 \otimes \mathrm{comm}} & & \downarrow^{\mathrm{assoc}} \\
X \otimes (Z \otimes Y) & & (Z \otimes X) \otimes Y \\
\searrow^{\mathrm{assoc}} & & \nearrow^{\mathrm{comm} \otimes 1} \\
 & (X \otimes Z) \otimes Y &
\end{array}
$$

6. For all objects X and Y, the following commute:

$$
\begin{array}{ccc}
X \otimes Y & \xrightarrow{\text{unit}} & \underline{1} \otimes (X \otimes Y) \\
& \searrow{\scriptstyle\text{unit}\otimes 1} & \downarrow{\scriptstyle\text{assoc}} \\
& & (\underline{1} \otimes X) \otimes Y
\end{array}
\qquad
\begin{array}{ccc}
X \otimes Y & \xrightarrow{\text{unit}\otimes 1} & (\underline{1} \otimes X) \otimes Y \\
\downarrow{\scriptstyle 1\otimes\text{unit}} & & \downarrow{\scriptstyle\text{comm}\otimes 1} \\
X \otimes (\underline{1} \otimes Y) & \xrightarrow[\text{assoc}]{} & (X \otimes \underline{1}) \otimes Y
\end{array}
$$

For reasons of convenience our definition of a tensor category is a slight deviation from that given on page 105 of [5]. There conditions 3. and 6. above are replaced with the seemingly weaker requirement that there exist an identity object U and isomorphism $u : U \to U \otimes U$ such that $X \mapsto U \otimes X$ is an equivalence of categories. However, proposition 1.3 on that same page makes it clear that Deligne's definition implies ours, so we have not changed anything.

What we call a tensor category others might call a *monoidal* category, and our demand that it be, e.g., naturally commutative is quite often not assumed by other authors. Saavedra in [15] would in fact call this an ACU tensor category, indicating that is associative, commutative, and unital. We shall have no occasion to consider any tensor categories but this kind, so we call them simply tensor categories.

The significance of the pentagon and hexagon axioms is that, loosely speaking, they introduce enough constraints to ensure that any diagram that should commute, does commute. The reader should see [14] and [5] for more on this.

We also note that the identity object $\underline{1}$ and the isomorphism unit are not demanded to be unique. However, any two such are isomorphic up to a unique isomorphism commuting with the unit isomorphisms, so it is unique for all intents and purposes; see proposition 1.3 of [5].

We define an **abelian tensor category** to be an abelian category equipped with a tensor product in such a way that $\otimes$ is a bi-additive functor, i.e. $(\phi + \psi) \otimes a =$

$\phi \otimes a + \psi \otimes a$, and similarly for the other slot.

Our goal now is to define what is called a *rigid abelian tensor category.* This is defined to be an abelian tensor category in which all internal Homs exist, every object is reflexive, and for all objects X_1, X_2, Y_1, Y_2, a certain natural map

$$\underline{\mathrm{Hom}}(X_1, Y_1) \otimes \underline{\mathrm{Hom}}(X_2, Y_2) \to \underline{\mathrm{Hom}}(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

is always an isomorphism. We define now what these things mean in purely categorical terms.

Let X and Y be objects of the tensor category $\mathcal{C}$ and consider the contravariant functor $\mathrm{Hom}(\underline{} \otimes X, Y)$ from $\mathcal{C}$ to the category of sets. It sends any object T to $\mathrm{Hom}(T \otimes X, Y)$ and any morphism $T \overset{\phi}{\to} W$ to the morphism $\mathrm{Hom}(W \otimes X, Y) \overset{\hat{\phi}}{\to} \mathrm{Hom}(T \otimes X, Y)$ defined by, for $W \otimes X \overset{\psi}{\to} Y$, the image of ψ under $\hat{\phi}$ is the composition

$$T \otimes X \overset{\phi \otimes 1}{\longrightarrow} W \otimes X \overset{\psi}{\longrightarrow} Y$$

Suppose that, for fixed X and Y, this functor is representable, and call the representing object $\underline{\mathrm{Hom}}(X, Y)$. $\underline{\mathrm{Hom}}(X, Y)$ is by definition an **internal Hom** object for X and Y (so called because, e.g. in Vec_k, internal Hom is just Hom). Then we have a natural isomorphism Ω going from the functor $\mathrm{Hom}(\underline{}, \underline{\mathrm{Hom}}(X, Y))$ to the functor $\mathrm{Hom}(\underline{} \otimes X, Y)$. If we plug in the object $\underline{\mathrm{Hom}}(X, Y)$ to each slot and apply $\Omega_{\underline{\mathrm{Hom}}(X,Y)}$ to the element $\mathrm{id} \in \mathrm{Hom}(\underline{\mathrm{Hom}}(X, Y), \underline{\mathrm{Hom}}(X, Y))$, we get a map $\underline{\mathrm{Hom}}(X, Y) \otimes X \to Y$, which by definition we call $\mathrm{ev}_{X,Y}$ (so called because, in Vec_k, $\mathrm{ev}_{X,Y}$ is the evaluation map $\phi \otimes x \mapsto \phi(x)$).

The significance of the map $\mathrm{ev}_{X,Y}$ is that, analyzing the situation in light of the Yoneda lemma (page 18), we find that for any morphism ϕ in $\mathrm{Hom}(T, \underline{\mathrm{Hom}}(X, Y))$,

applying the isomorphism Ω_T yields

$$\Omega_T(\phi) = (\phi \otimes 1) \circ \mathrm{ev}_{X,Y}$$

which is to say, the following is always commutative:

We suppose now that for any objects X and Y, $\underline{\mathrm{Hom}}(X, Y)$ exists. We define the **dual** of X, denoted $X^\vee$, to be $\underline{\mathrm{Hom}}(X, \underline{1})$, where $\underline{1}$ is the identity object for our tensor category, and we simply write ev_X for $\mathrm{ev}_{X,\underline{1}}$, which is a map $X^\vee \otimes X \to \underline{1}$. In Vec_k, ev_X is the familiar map $\phi \otimes x \mapsto \phi(x)$.

We now proceed to define a map $\iota_X : X \to X^{\vee\vee}$, which in Vec_k will correspond to the usual evaluation map $x \mapsto (\phi \mapsto \phi(x))$. We have, for any X, an isomorphism Ω_X between

$$\mathrm{Hom}(X, X^{\vee\vee}) \xrightarrow{\Omega_X} \mathrm{Hom}(X \otimes X^\vee, \underline{1})$$

Define $\iota_X : X \to X^{\vee\vee}$ to be the map on the left hand side of the above isomorphism which corresponds to the composition $X \otimes X^\vee \xrightarrow{\mathrm{comm}} X^\vee \otimes X \xrightarrow{\mathrm{ev}_X} \underline{1}$ on the right hand

side. In other words, ι_X is the unique map making the following commute:

$$
\begin{array}{ccc}
X \otimes X^\vee & & \\
\downarrow^{\iota_X \otimes 1} & \searrow^{\text{comm}} & \\
& X^\vee \otimes X & \\
& \downarrow^{\text{ev}_X} & \\
X^{\vee\vee} \otimes X^\vee & \xrightarrow{\text{ev}_{X^\vee}} & \underline{1}
\end{array}
\tag{3.2.1}
$$

We define an object X of $\mathcal{C}$ to be **reflexive** if the map ι_X just constructed is an isomorphism.

Consider the following composition:

$$
(\underline{\text{Hom}}(X_1, Y_1) \otimes \underline{\text{Hom}}(X_2, Y_2)) \otimes (X_1 \otimes X_2) \xrightarrow{\simeq}
\tag{3.2.2}
$$

$$
(\underline{\text{Hom}}(X_1, Y_1) \otimes X_1) \otimes (\underline{\text{Hom}}(X_2, Y_2) \otimes X_2)) \xrightarrow{\text{ev} \otimes \text{ev}} Y_1 \otimes Y_2
$$

where the first isomorphism is the obvious one built by application of the comm and assoc isomorphisms. Call the above composition Ψ. Then there is a unique map, call it Φ, making the following commute:

$$
\begin{array}{ccc}
(\underline{\text{Hom}}(X_1, Y_1) \otimes \underline{\text{Hom}}(X_2, Y_2)) \otimes (X_1 \otimes X_2) & & \\
\downarrow^{\Phi \otimes 1} & \searrow^{\Psi} & \\
\underline{\text{Hom}}(X_1 \otimes X_2, Y_1 \otimes Y_2) \otimes (X_1 \otimes X_2) & \xrightarrow{\text{ev}} & Y_1 \otimes Y_2
\end{array}
\tag{3.2.3}
$$

Definition 3.2.3. An abelian tensor category is called **rigid** if $\underline{\text{Hom}}(X, Y)$ exists for all X and Y, all objects are reflexive, and for any quadruple X_1, X_2, Y_1, Y_2, the map Φ just constructed is an isomorphism.

Next consider the Hom-set $\text{End}(\underline{1})$. As composition is demanded to be bilinear with respect to the additive structure on Hom-sets, $\text{End}(\underline{1})$ is a ring with unity under addition and composition. We say then

Definition 3.2.4. A rigid abelian tensor category is called a **tannakian category** if the ring $\text{End}(\underline{1})$ is a field.

Assume now that $\mathcal{C}$ is tannakian and let k be the field $\text{End}(\underline{1})$. Then $\mathcal{C}$ has a k-linear structure forced upon it as follows. If c is an element of k (that is, a morphism $\underline{1} \xrightarrow{c} \underline{1}$) and ϕ is a morphism from V to W, then we define the scalar multiplication $c\phi$ to be the composition

$$V \xrightarrow{\text{unit}_V} \underline{1} \otimes V \xrightarrow{c \otimes \phi} \underline{1} \otimes W \xrightarrow{\text{unit}_W^{-1}} W$$

As $\otimes$ acts bilinearly on morphisms, we have $c(\phi + \psi) = c\phi + c\psi$ for all $c \in \text{End}(\underline{1})$, $\phi, \psi \in \text{Hom}(V, W)$. Thus, $\text{Hom}(V, W)$ is a vector space over k.

If $\mathcal{C}$ is any category, then being tannakian is not quite enough for us to conclude that it is the category of representations of some affine group scheme over the field $\text{End}(\underline{1})$. We need the additional fact that objects of $\mathcal{C}$ can, in some sense, be 'thought of' as concrete finite dimensional vector spaces, and morphisms as concrete linear maps between them. This is the role fulfilled by a *fibre functor*, a certain kind of functor from $\mathcal{C}$ to Vec_k, where k is the field $\text{End}(\underline{1})$.

We need to define first what is meant by a *tensor functor* $F : \mathcal{C} \to \mathcal{D}$, where $\mathcal{C}$ and $\mathcal{D}$ are tensor categories. We denote with the same symbol $\otimes$ the tensor product in both categories. We denote by assoc the requisite associativity isomorphism in $\mathcal{C}$, and by assoc$'$ that in $\mathcal{D}$; similarly for the natural isomorphisms comm, unit, and the identity object $\underline{1}$.

Definition 3.2.5. Let $\mathcal{C}$ and $\mathcal{D}$ be tensor categories, and $F : \mathcal{C} \to \mathcal{D}$ a functor. F is a **tensor functor** if there is a functorial isomorphism $c_{X,Y} : F(X) \otimes F(Y) \xrightarrow{\simeq} F(X \otimes Y)$

satisfying

1. For all objects X, Y, Z of $\mathcal{C}$, the following commutes:

$$F(X) \otimes F(Y \otimes Z)$$

$$1 \otimes c \qquad\qquad c$$

$$F(X) \otimes (F(Y) \otimes F(Z)) \qquad\qquad F(X \otimes (Y \otimes Z))$$

$$\mathrm{assoc}' \qquad\qquad F(\mathrm{assoc})$$

$$(F(X) \otimes F(Y)) \otimes F(Z) \qquad\qquad F((X \otimes Y) \otimes Z)$$

$$c \otimes 1 \qquad\qquad c$$

$$F(X \otimes Y) \otimes F(Z)$$

2. For all objects X and Y of $\mathcal{C}$, the following commutes:

$$F(X) \otimes F(Y) \xrightarrow{c} F(X \otimes Y)$$

$$\mathrm{comm}' \qquad\qquad F(\mathrm{comm})$$

$$F(Y) \otimes F(X) \xrightarrow{c} F(Y \otimes X)$$

3. If $(\underline{1}, \mathrm{unit})$ is an identity object of $\mathcal{C}$, then there is an identity object $(\underline{1}', \mathrm{unit}')$ of $\mathcal{D}$ such that $F(\underline{1}) = \underline{1}'$, and for every object X of $\mathcal{C}$, $\mathrm{unit}'_{F(x)} = F(\mathrm{unit}_X) \circ c_{X,\underline{1}}$.

Again, condition 3. appears stronger than condition (c) on page 114 of [5], but they are actually equivalent, again by proposition 1.3 of [5].

If $\mathcal{C}$ is any tannakian category over the field $k = \mathrm{End}(\underline{1})$ then we define a **fibre functor** on $\mathcal{C}$, usually denoted ω, to be any exact, faithful, k-linear tensor functor from $\mathcal{C}$ to Vec_k. We can finally define

Definition 3.2.6. A **neutral tannakian category** is a tannakian category equipped with a fibre functor.

3.3 Recovering an Algebraic Group from a Neutral Tannakian Category

Let G be an affine group scheme over the field k and let $\omega : \mathrm{Rep}_k G \to \mathrm{Vec}_k$ be the forgetful functor, i.e. the functor which sends every representation of G to its underlying k-vector space, and every map to itself. If R is a k-algebra, we define $\mathrm{Aut}^{\otimes}(\omega)(R)$ to be the collection of tensor preserving automorphisms of the functor $\omega^R : \mathrm{Rep}_k G \to \mathrm{Mod}_R$. Here ω^R is the functor which sends any object X of $\mathrm{Rep}_k G$ to the R-module $X \otimes R$ (strictly speaking we should write $\omega(X) \otimes R$, but we deliberately confuse X with its underlying vector space $\omega(X)$ to keep the notation simple), and sends any morphism $X \xrightarrow{\phi} Y$ to the R-linear map $X \otimes R \xrightarrow{\phi \otimes 1} Y \otimes R$. To be more explicit

Definition 3.3.1. An element of $\mathrm{Aut}^{\otimes}(\omega)(R)$ is a family $(\lambda_X : X \in \mathrm{Rep}_k G)$, where each λ_X is an R-linear automorphism of $X \otimes R$, subject to

1. $\lambda_{\underline{1}}$ is the identity map on $R \simeq k \otimes R$

2. $\lambda_{X \otimes Y} = \lambda_X \otimes \lambda_Y$ for all $X, Y \in \mathrm{Rep}_k G$

3. For all morphisms $X \xrightarrow{\phi} Y$ in $\mathrm{Rep}_k G$, the following commutes:

$$
\begin{array}{ccc}
X \otimes R & \xrightarrow{\lambda_X} & X \otimes R \\
{\scriptstyle \phi \otimes 1}\downarrow & & \downarrow{\scriptstyle \phi \otimes 1} \\
Y \otimes R & \xrightarrow[\lambda_Y]{} & Y \otimes R
\end{array}
$$

40

If g is an element of the group $G(R)$ then it is trivial to verify that g defines an element of $\mathrm{Aut}^{\otimes}(\omega)(R)$, just by working through the definitions. If, for $X \in \mathrm{Rep}_k G$, we write g_X for the automorphism $X \otimes R \to X \otimes R$ defined by the representation $G \to \mathrm{GL}(X)$, then the above three requirements are all satisfied. $\underline{1}$ is of course the trivial representation in $\mathrm{Rep}_k G$, so g acts identically on $\underline{1}$ by definition. The action of $g_{X \otimes Y}$ on $X \otimes Y$ is defined by the equation $g_{X \otimes Y} = g_X \otimes g_Y$, giving us 2., and 3. is true since morphisms in $\mathrm{Rep}_k G$ must by definition commute with the action of g.

Thus, we have a natural map from the functor G to the functor $\mathrm{Aut}^{\otimes}(\omega)$. We can now state one half of the principle of tannakian duality:

Theorem 3.3.1. *(proposition 2.8 of [5]) If G is an affine group scheme over the field k, then the natural map of functors $G \to \mathrm{Aut}^{\otimes}(\omega)$ is an isomorphism.*

Stated more plainly: the only tensor preserving automorphisms of the functor ω^R are ones that are given by elements of $G(R)$. We see then that the category $\mathrm{Rep}_k G$, along with the forgetful functor ω, completely determines the group G: it can be recovered as the affine group scheme $\mathrm{Aut}^{\otimes}(\omega)$.

This first half of our main theorem points the way to the second half. Starting with an abstract neutral tannakian category $\mathcal{C}$ with fibre functor ω, the functor $G = \mathrm{Aut}^{\otimes}(\omega)$ is itself an affine group scheme such that $\mathcal{C}$ is tensorially equivalent to $\mathrm{Rep}_k G$. That is:

Theorem 3.3.2. *(theorem 2.11 of [5]) Let $\mathcal{C}$ be a neutral tannakian category over the field k with fibre functor ω. Then*

1. *The functor $\mathrm{Aut}^{\otimes}(\omega)$ on k-algebras is representable by an affine group scheme G*

2. *ω defines an equivalence of tensor categories between $\mathcal{C}$ and $\mathrm{Rep}_k G$*

This is the principle of tannakian duality.

The remainder of this section is devoted to, following the proof of the above theorem given in [5], giving an 'algorithm' of sorts for recovering the representing Hopf algebra A from an abstract neutral tannakian category. We shall not justify most of the steps taken; the interested reader should see the actual proof for this. For the remainder, $\mathcal{C}$ will denote a fixed neutral tannakian category over the field k with fibre functor ω. We denote the image of the object X under the functor ω as $\omega(X)$, and the image of a morphism ϕ under ω simply as ϕ. The usual names for $\underline{1}$, comm, assoc, $\otimes$, etc. also hold here.

Definition 3.3.2. (see page 133 of [5]) For an object X of $\mathcal{C}$, $\langle X \rangle$, the **principal subcategory** generated by X, is the full subcategory of $\mathcal{C}$ consisting of those objects which are isomorphic to a subobject of a quotient of $X^n = X \oplus \ldots \oplus X$ for some n.

Note firstly that $\langle X \rangle$ is not itself a tannakian category, in general not being closed under the tensor product; it is however a k-linear abelian category. Note that $Y \in \langle X \rangle$ if and only if $\langle Y \rangle \subset \langle X \rangle$. We can say then that $\mathcal{C}$ is the direct limit of its principal subcategories, with the direct system being the inclusions $\langle Y \rangle \subset \langle X \rangle$ when applicable.

Definition 3.3.3. (see lemma 2.13 of [5]) For X an object of $\mathcal{C}$, we define $\mathrm{End}(\omega|\langle X \rangle)$, the collection of all **endomorphisms of the fibre functor** ω restricted to $\langle X \rangle$, to consist of families $\lambda = (\lambda_Y : Y \in \langle X \rangle)$ such that $\lambda_Y : \omega(Y) \to \omega(Y)$ is a k-linear map, and for every $\mathcal{C}$-morphism $Y \overset{\phi}{\to} Z$, the following commutes:

$$
\begin{array}{ccc}
\omega(Y) & \overset{\lambda_Y}{\longrightarrow} & \omega(Y) \\
\phi \downarrow & & \downarrow \phi \\
\omega(Z) & \underset{\lambda_Z}{\longrightarrow} & \omega(Z)
\end{array}
$$

An important point, used often in this dissertation, is the fact that every $\lambda \in$

$\mathrm{End}(\omega|\langle X \rangle)$ is determined by λ_X. If $\iota_i : X \to X^n$ is the ith inclusion map then

$$
\begin{array}{ccc}
\omega(X) & \xrightarrow{\ \iota_i\ } & \omega(X^n) \\
\downarrow{\scriptstyle \lambda_X} & & \downarrow{\scriptstyle \lambda_{X^n}} \\
\omega(X) & \xrightarrow[\ \iota_i\ ]{} & \omega(X^n)
\end{array}
$$

must commute for every i, which clearly forces $\lambda_{X^n} = \lambda_X^n$. If Y is a subobject of some X^n with $Y \xrightarrow{\ \iota\ } X^n$ injective, then λ_Y must commute with

$$
\begin{array}{ccc}
\omega(Y) & \xhookrightarrow{\ \iota\ } & \omega(X^n) \\
\downarrow{\scriptstyle \lambda_Y} & & \downarrow{\scriptstyle \lambda_X^n} \\
\omega(Y) & \xhookrightarrow{\ \iota\ } & \omega(X^n)
\end{array}
$$

and since ι is injective, λ_Y is unique in this respect. Finally, if Z is a quotient of some $Y \in \langle X \rangle$ with Y a subobject of X^n, then we have a surjective map $Y \xrightarrow{\ \pi\ } Z$ and the commutative diagram

$$
\begin{array}{ccccc}
\omega(\ker(\pi)) & \xhookrightarrow{\ \iota\ } & \omega(Y) & \xtwoheadrightarrow{\ \pi\ } & \omega(Z) \\
\downarrow{\scriptstyle \lambda_{\ker(\pi)}} & & \downarrow{\scriptstyle \lambda_Y} & & \downarrow{\scriptstyle \lambda_Z} \\
\omega(\ker(\pi)) & \xhookrightarrow{\ \iota\ } & \omega(Y) & \xtwoheadrightarrow{\ \pi\ } & \omega(Z)
\end{array}
$$

with ι the inclusion of the kernel of π into Y. By commutativity of the left square λ_Y must stabilize $\omega(\ker(\pi))$. This shows that there is at most one λ_Z making this diagram commute, hence λ_Z is determined by λ_X as well.

Therefore it does no harm to confuse $\mathrm{End}(\omega|\langle X \rangle)$ with $\{\lambda_X : \lambda \in \mathrm{End}(\omega|\langle X \rangle)\}$, its image in $\mathrm{End}(\omega(X))$; we refer to this subalgebra of $\mathrm{End}(\omega(X))$ as L_X.

Now suppose that $X \in \langle Y \rangle$, which is the same as saying $\langle X \rangle \subset \langle Y \rangle$. If $\lambda \in$

$\text{End}(\omega|\langle Y\rangle)$ it is straightforward to check that $\lambda_X \in L_X$. This gives, for every such X and Y, a canonical map from L_Y to L_X, denoted $T_{X,Y}$; we call this the **transition mapping** from L_Y to L_X. It is clear from the definition that, for $X \in \langle Y\rangle$ and $Y \in \langle Z\rangle$, then also $X \in \langle Z\rangle$, and the diagram

$$
\begin{array}{ccc}
L_Z & \xrightarrow{\ T_{Y,Z}\ } & L_Y \\
& \searrow^{T_{X,Z}} & \downarrow{T_{X,Y}} \\
& & L_X
\end{array}
$$

commutes, which give the $L_X, X \in \mathcal{C}$ the structure of an inverse system.

For each $X \in \mathcal{C}$ let B_X be the dual coalgebra to L_X. Then from the k-algebra maps $L_Y \xrightarrow{T_{X,Y}} L_X$ we get k-coalgebra maps $B_X \xrightarrow{T^{\circ}_{X,Y}} B_Y$. Thus, for objects X, Y and Z of $\mathcal{C}$ with $X \in \langle Y\rangle$ and $Y \in \langle Z\rangle$, the diagram

$$
\begin{array}{ccc}
B_Z & \xleftarrow{\ T^{\circ}_{Y,Z}\ } & B_Y \\
& \nwarrow^{T^{\circ}_{X,Z}} & \uparrow{T^{\circ}_{X,Y}} \\
& & B_X
\end{array}
$$

commutes, giving the $B_X, X \in \mathcal{C}$ the structure of a direct system. Then let

$$
B = \varinjlim_{X \in \mathcal{C}} B_X
$$

be its direct limit; this B is the underlying coalgebra of our eventual Hopf algebra.

We now define an equivalence of categories $F : \mathcal{C} \to \text{Comod}_B$ which carries the fibre functor ω into the forgetful functor $\text{Comod}_B \to \text{Vec}_k$, that is, such that the

diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\ F\ } & \mathrm{Comod}_B \\
& \searrow{\omega} & \downarrow{\text{forget}} \\
& & \mathrm{Vec}_k
\end{array}$$

commutes. Let A be a finite dimensional k-algebra, A° its dual coalgebra, and V a finite dimensional k-vector space. We define a map $\mathrm{Hom}_k(A\otimes V, V) \xrightarrow{\ \Phi\ } \mathrm{Hom}_k(V, V\otimes A^\circ)$ as follows. For $\rho \in \mathrm{Hom}(A\otimes V, V)$, $\Phi(\rho)$ is the composition

$$V \xrightarrow{\simeq} k\otimes V \xrightarrow{\mathrm{diag}\otimes\mathrm{Id}} \mathrm{End}_k(A^\circ)\otimes V \xrightarrow{\simeq\otimes\mathrm{Id}} A^{\circ *}\otimes A^\circ \otimes V$$

$$\xrightarrow{\mathrm{Id}\otimes\mathrm{Twist}} A^{\circ *}\otimes V\otimes A^\circ \xrightarrow{\simeq\otimes\mathrm{Id}\otimes\mathrm{Id}} A\otimes V\otimes A^\circ \xrightarrow{\rho\otimes\mathrm{Id}} V\otimes A^\circ$$

Reading from left to right, the various maps occurring in this composition are defined as: $\simeq$ is the canonical isomorphism $V\simeq k\otimes V$, diag is the map that sends $1\in k$ to $\mathrm{Id}\in\mathrm{End}_k(A^\circ)$, $\simeq$ is the canonical isomorphism $\mathrm{End}_k(A^\circ)\simeq A^{\circ *}\otimes A^\circ$, Twist is the obvious commutativity isomorphism, and $\simeq$ is the canonical isomorphism $A^{\circ *}\simeq A$.

Lemma 3.3.3. *Let A be a finite dimensional k-algebra, A° its dual coalgebra, and V a finite dimensional k-vector space. Then the map $\Phi : \mathrm{Hom}_k(A\otimes V, V) \longrightarrow \mathrm{Hom}_k(V, V\otimes A^\circ)$ just defined is a bijection. Further, $\rho\in\mathrm{Hom}_k(A\otimes V, V)$ defines a valid A-module structure on V if and only if $\Phi(\rho)\in\mathrm{Hom}_k(V, V\otimes A^\circ)$ defines a valid A°-comodule structure on V.*

The reader should see proposition 2.2.1 of [4] for a proof of this fact. However, be aware that our map Φ is actually the inverse of the map they consider, and we have replaced the coalgebra C and dual algebra C^* with the coalgebra A° and algebra $A^{\circ *}\simeq A$.

For any $X\in\mathcal{C}$ the vector space $\omega(X)$ is in the obvious way a module for the

k-algebra L_X. Then according to the previous lemma $\omega(X)$ carries with it also the structure of a comodule over B_X, call it ρ_X. Then if $B_X \xrightarrow{\phi_X} B$ is the canonical map given by the definition of B as a direct limit, we get a B-comodule structure on $\omega(X)$, call it ρ, via the composition

$$\rho : \omega(X) \xrightarrow{\rho_X} \omega(X) \otimes B_X \xrightarrow{1 \otimes \phi_X} \omega(X) \otimes B$$

If $X \in \langle Y \rangle$ for some Y, then similarly $\omega(X)$ is a module over L_Y (via the transition mapping $L_Y \xrightarrow{T_{X,Y}} L_X$), hence a comodule over B_Y, and yet again over B. The various commutativities of the relevant diagrams ensure that we will get the same B-comodule structure on $\omega(X)$ no matter which principal subcategory we consider it to be an object of. We therefore define the image of the object X under the functor F to be

$$F(X) = (\omega(X), \rho)$$

That is, the B-comodule with underlying vector space $\omega(X)$ and comodule map $\rho : \omega(X) \to \omega(X) \otimes B$ just defined.

It is tedious but straightforward to argue that, if $X \xrightarrow{\phi} Y$ is a morphism in the category $\mathcal{C}$, then working through the definitions of L_X, B_X and B, (the image under the fibre functor of) ϕ is actually a map of B-comodules. We therefore define the image of a morphism ϕ under F to be the same map between the vector spaces $\omega(X)$ and $\omega(Y)$.

Theorem 3.3.4. *The functor $F : \mathcal{C} \to Comod_B$ just defined is an equivalence of categories.*

That F is a faithful functor is clear from the fact that ω is as well. The claim that F is both full and essentially surjective is however by no means obvious; see proposition 2.14 of [5] for a proof of this.

We have thus far recovered a k-coalgebra B and an equivalence between our abstract neutral tannakian category $\mathcal{C}$ and Comod_B. What remains is to recover the multiplication on B. As the usual tensor product on comodules over a Hopf algebra is defined in terms of its multiplication, it is not surprising that we should turn the process around to recover the multiplication from the tensor product.

Let B be a k-coalgebra and $u : B \otimes_k B \to B$ be any k-homomorphism. Then we can define a bifunctor $\phi^u : \mathrm{Comod}_B \times \mathrm{Comod}_B \to \mathrm{Comod}_B$ as follows: it sends the pair of comodules $(X, \rho), (Y, \mu)$ to the comodule $\phi^u(X, Y)$ having underlying vector space $X \otimes_k Y$ and comodule map given by the composition

$$X \otimes Y \xrightarrow{\rho \otimes \mu} X \otimes B \otimes Y \otimes B \xrightarrow{1 \otimes \mathrm{Twist} \otimes 1} X \otimes Y \otimes B \otimes B \xrightarrow{1 \otimes 1 \otimes u} X \otimes Y \otimes B$$

(In case B is a Hopf algebra and u is mult, then this is by definition the tensor product on Comod_B.) What is not quite obvious is that in fact all such bifunctors arise in this fashion.

Proposition 3.3.5. *(see proposition 2.16 of [5]) For any k-coalgebra B, the map $u \mapsto \phi^u$ defines a bijective correspondence between the set of all k-homomorphisms $u : B \otimes B \to B$, and the set of all bifunctors $F : \mathrm{Comod}_B \times \mathrm{Comod}_B \to \mathrm{Comod}_B$ having the property that the underlying vector space of $F(X, Y)$ is the tensor product of the underlying vector spaces of X and Y.*

So then, let us define a bifunctor on Comod_B which for the moment we call $\square$. F is an equivalence, so it has an 'inverse' functor, call it F^{-1}. Then for two B-comodules S and T, we set

$$S \square T \stackrel{\mathrm{def}}{=} F(F^{-1}(S) \otimes F^{-1}(T))$$

where $\otimes$ on the right refers to the given tensor structure on $\mathcal{C}$. We define $\square$ to act

on morphisms in an analogous fashion. As the diagram

$$\mathcal{C} \xrightarrow{\ F\ } \mathrm{Comod}_B$$

with ω and forget mapping to Vec_k

commutes, it is easy to see that $\square$ as a bifunctor satisfies the hypothesis of the previous proposition; hence, $\square$ is uniquely of the form ϕ^u for some $u : B \otimes B \to B$. This u, call it now mult, is the recovered multiplication on B, finally giving B the structure of a Hopf algebra.

We close by mentioning that the necessary conditions needed for B to be a commutative Hopf algebra follow from certain properties assumed about $\otimes$ on $\mathcal{C}$. For instance, it is the existence of the natural isomorphisms comm and assoc which guarantees that mult should be a commutative and associative operation, and the existence of an identity element for mult follows from the existence of the identity object $\underline{1}$ for $\otimes$. The interested reader should see pg. 137 of [5] for more on this.

3.4 Recovering a Hopf Algebra in Practice

Here we record some results which will later be useful in computing the representing Hopf algebra for a given neutral tannakian category, according to the method outlined in the previous section.

3.4.1 A Categorical Lemma

Much of the work done in this dissertation entails the computing of direct/inverse limits over very large and unwieldy collections of objects. The following lemma will allow us at times to drastically simplify our computations.

If $(I, \leq)$ is a directed set, we say that $(J, \leq)$ is a **sub-directed set** if J is a subset of I, J is directed, and whenever $j \leq k$ in J, $j \leq k$ in I. Note that we do not demand the converse to hold; in case it does, we call J **full**. We say the sub-directed set J is **essential** in I if, for every $i \in I$, there is a $j \in J$ such that $i \leq j$ in I.

Lemma 3.4.1. *Let $\mathcal{C}$ be any category, I a directed set, $\{X_i\}$ a collection of objects indexed over I, and $\{X_i \xrightarrow{\phi_{ij}} X_j\}$ a direct system for the X_i over I, and let*

be the direct limit of this system. Let $J \subset I$ be a (not necessarily full) sub-directed set, and let

be the direct limit over J. Then if J is essential in I, these two direct limits are isomorphic, via a unique isomorphism commuting with the canonical injections.

Proof. It is well known that any two direct limits for the same system are isomorphic in the above mentioned way. Thus, we will prove the theorem by showing that the X_J, the direct limit over J, can also be made into a direct limit object for the X_i over all of I. For any $i \in I$, define a map $X_i \xrightarrow{\rho_i} X_J$ as ψ_i if $i \in J$, and in case $i \notin J$, as the composition

$$X_i \xrightarrow{\phi_{ij}} X_j \xrightarrow{\psi_j} X_J$$

where j is any member of J such that $i \leq j$. This is well-defined: if $k \in J$ is any other such that $i \leq k$, let l be an upper bound for j and k in J. Then every sub-triangle of the diagram

$$
\begin{array}{ccc}
& X_J & \\
\psi_j \nearrow \; \psi_l \uparrow \; \nwarrow \psi_k & & \\
X_l & & \\
\phi_{jl} \; \phi_{il} \; \phi_{kl} & & \\
X_j \qquad X_i \qquad X_k & & \\
\phi_{ij} \qquad \phi_{ik} & &
\end{array}
$$

commutes, and thus so does the outermost diamond.

We claim that with these $X_i \xrightarrow{\rho_i} X_J$, X_J is a direct limit for the X_i over all of I. Let Y be any object with morphisms $X_i \xrightarrow{t_i} Y$ such that, for every $i \leq j \in I$, the following commutes:
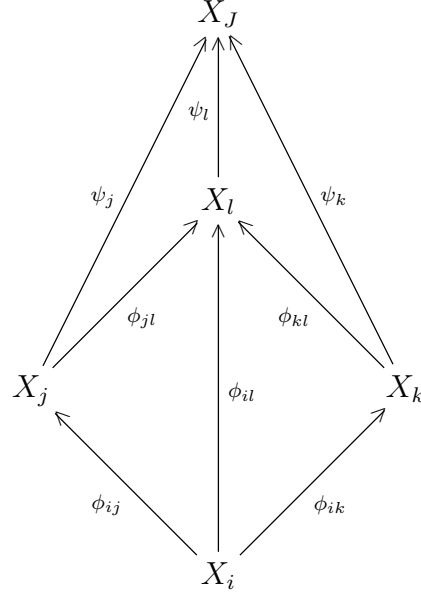
$$
\begin{array}{ccc}
& Y & \\
t_i \nearrow & & \nwarrow t_j \\
X_i & \xrightarrow{\phi_{ij}} & X_j
\end{array}
$$
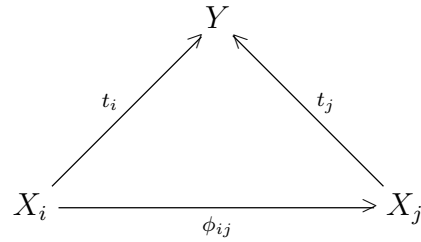
Then this diagram obviously commutes for every $i \leq j \in J$, and the universal property

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of X_J guarantees a unique map $X_J \xrightarrow{t} Y$ making

$$
\begin{array}{c}
Y \\
\nearrow \uparrow \nwarrow \\
t_j \quad t \quad t_k \\
X_J \\
\psi_j \quad \psi_k \\
X_j \xrightarrow{\phi_{jk}} X_k
\end{array}
$$
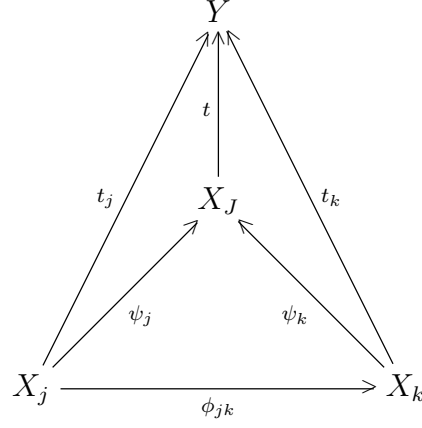
commute for every $j, k \in J$. But this map t also satisfies the universal property required for X_J to be a direct limit over all of I. For if $i, l \in I$ with $i \leq l$, then let $j, k \in J$ with $i \leq j$, $l \leq k$, and $j \leq k$, and the following also commutes:

$$
\begin{array}{c}
Y \\
\nearrow \uparrow \nwarrow \\
t_j \quad t \quad t_k \\
X_J \\
\psi_j \quad \psi_k \\
X_j \xrightarrow{\phi_{jk}} X_k \\
\phi_{ij} \uparrow \qquad \uparrow \phi_{lk} \\
X_i \xrightarrow{\phi_{il}} X_l
\end{array}
$$
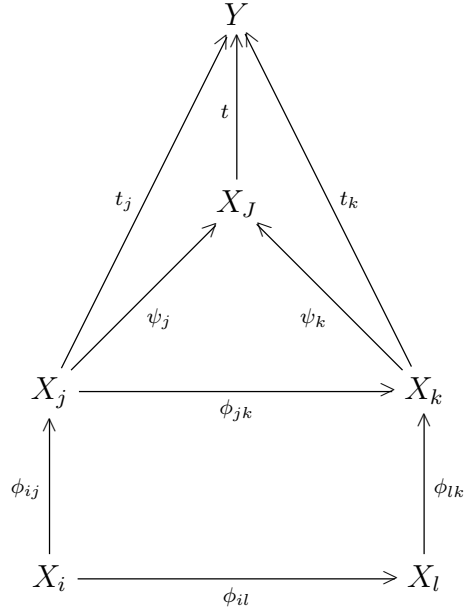
and hence so does
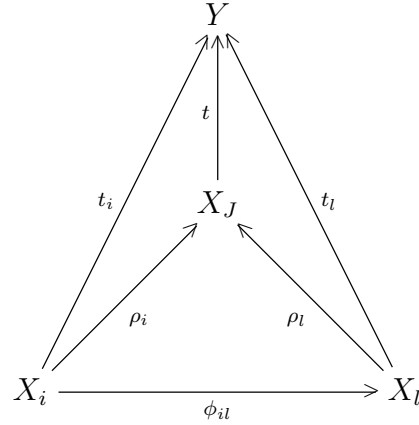
This map t is still unique, since satisfying universality over all of I is clearly a more stringent requirement than doing so over all of J.

□

There is an obvious analogue to this lemma as concerns inverse limits which we state but do not prove.

Lemma 3.4.2. *Let $\mathcal{C}$ be any category, I a directed set, $\{X_i\}$ a collection of objects indexed over I, and $\{X_i \xleftarrow{\tau_{ij}} X_j\}$ an inverse system for the X_i over I, and let*
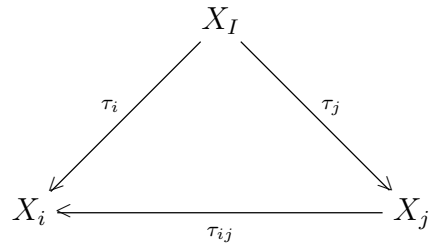
be the inverse limit of this system. Let $J \subset I$ be a (not necessarily full) sub-directed set, and let
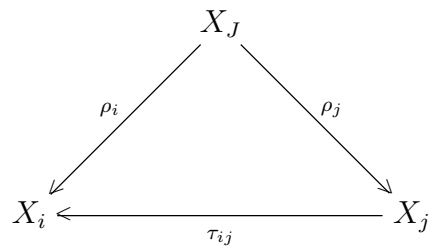
be the inverse limit over J. Then if J is essential in I, these two inverse limits are isomorphic, via a unique isomorphism commuting with the canonical projections.

3.4.2 Computing $\mathrm{End}(\omega|\langle X\rangle)$

Recall from page 43 that if X is an object of $\mathcal{C}$, we define L_X to be the subalgebra of $\mathrm{End}_k(\omega(X))$ consisting of those linear maps which are 'starting points' for a full endomorphism of the fibre functor restricted to $\langle X\rangle$. Here we describe a practical method for computing L_X, which is gleaned from pages 132, 133 of [5]. The definition of L_X given, *a priori*, seems to require that we look at arbitrarily large powers of X to discover if a given transformation of $\omega(X)$ is or is not in L_X, but the method described here shows that we need only look inside a fixed power of X ($X^{\dim(\omega(X))}$ in fact).

Let $n = \dim(\omega(X))$ and write $X^n = X_1 \oplus X_2 \oplus \ldots \oplus X_n$, where each X_i is simply a labelled copy of X. If $Y \xrightarrow{\psi} X^n$ is any embedding then we can write $\psi = \psi_1 \oplus \ldots \oplus \psi_n$, where $Y \xrightarrow{\psi_i} X_i$ is the ith component of ψ.

As $\omega(X)$ is n-dimensional, so is $\omega(X)^*$, so fix an isomorphism $\alpha : k^n \to \omega(X)^*$, and let $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$ be the standard basis of k^n. From this ψ and α we define a linear map $\psi_\alpha : \omega(Y) \to \omega(X)^* \otimes \omega(X)$ as follows: for a vector $y \in \omega(Y)$,

$$\psi_\alpha(y) = \sum_{i=1}^n \alpha(e_i) \otimes \psi_i(y)$$

If we identify $\omega(X)^* \otimes \omega(X)$ with $\mathrm{End}_k(\omega(X))$ in the usual fashion, we may speak of whether the image of ψ_α does or does not contain the element $\mathrm{id} : \omega(X) \to \omega(X)$. Further, exactness and faithfulness of the functor ω imply that, just as in Vec_k, the concept of a "smallest" object having a given property make sense. So we define

Definition 3.4.1. For an object X and fixed isomorphism $\alpha : k^n \to \omega(X)^*$, P_X^α is the smallest subobject of X^n having the property that the image of $\omega(P_X^\alpha)$ under ψ_α

contains id $: \omega(X) \to \omega(X)$, where ψ is the embedding $P_X^\alpha \to X^n$.

It is a completely non-obvious fact that

Theorem 3.4.3. *(see lemmas 2.12 and 2.13 of [5]) For any α, the image of $\omega(P_X^\alpha)$ under ψ_α is the algebra L_X.*

We have no intention of justifying this, although we do mention the reason that α can be chosen arbitrarily. Consider the subobject X^n itself of X^n, with the embedding being $\psi = \text{id}$. Certainly if one chooses a different β then the subobjects P_X^α and P_X^β will be different, but their images under ψ_α and ψ_β respectively will not change. Obviously ψ_α and ψ_β are isomorphisms of vector spaces, and thus we have a commutative diagram

$$
\begin{array}{ccc}
\omega(X^n) & \xrightarrow{\ \phi_{\beta,\alpha}\ } & \omega(X^n) \\
\ \downarrow{\scriptstyle \psi_\alpha} & \swarrow{\scriptstyle \psi_\beta} & \\
\omega(X)^\vee \otimes \omega(X) & &
\end{array}
$$

where $\phi_{\beta,\alpha}$ is a linear isomorphism. But it can in fact be shown $\phi_{\beta,\alpha}$ must in fact be (the image of) an actual isomorphism between the object X^n and itself in the original category. Such a $\phi_{\beta,\alpha}$ must then preserve the notion 'smallest subobject', and so the computation will always yield the same subspace L_X of $\text{End}(\omega(X))$.

Example: consider the following module X for the additive group G_a over a field k, with matrix formula

$$
\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}
$$

in the basis f_1, f_2 for $\omega(X)$. We will compute $\text{End}(\omega|\langle X \rangle)$ using the method outlined above. This is a 2-dimensional module, so we consider the module X^2, with matrix

formula

$$\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & x \\ & & & 1 \end{pmatrix}$$

in the basis $f_{1,1}, f_{1,2}, f_{2,1}, f_{2,2}$ for $\omega(X^2)$. Next we consider an arbitrary subobject of X^2. As any subobject factors through the identity mapping $X^2 \to X^2$, it does no harm to choose $\psi = 1$. Then the coordinate maps $\psi_1, \psi_2 : \omega(X^2) \to \omega(X)$, using the usual canonical injections $X \to X^2$, are

$$\psi_1 : f_{11} \mapsto f_1, \quad f_{12} \mapsto f_2, \quad f_{21} \mapsto 0, \quad f_{22} \mapsto 0$$

$$\psi_2 : f_{11} \mapsto 0, \quad f_{12} \mapsto 0, \quad f_{21} \mapsto f_1, \quad f_{22} \mapsto f_2$$

For the isomorphism $\alpha : k^2 \to \omega(X)^*$, let's keep life simple and choose $\alpha : (1,0) \mapsto f_1^*, (0,1) \mapsto f_2^*$, where f_1^*, f_2^* is the dual basis for $\omega(X)^*$. From this ψ and α we compute $\psi_\alpha : \omega(X^2) \to \omega(X)^* \otimes \omega(X)$, which has formula

$$\psi_\alpha(x) = \alpha((1,0)) \otimes \psi_1(x) + \alpha((0,1)) \otimes \psi_2(x)$$

and thus

$$\psi_\alpha : f_{11} \mapsto f_1^* \otimes f_1$$
$$f_{12} \mapsto f_1^* \otimes f_2$$
$$f_{21} \mapsto f_2^* \otimes f_1$$
$$f_{22} \mapsto f_2^* \otimes f_2$$

Now, to compute P_X^α, we ask: what is the smallest subobject of X^2 such that, under the map ψ_α, contains the identity map $X \to X$? We identify the identity map of course as the element $f_1^* \otimes f_1 + f_2^* \otimes f_2 \in \omega(X)^* \otimes \omega(X)$. This projects back

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to, under ψ_α, the element $f_{11} + f_{22}$ of $\omega(X^2)$, which, in the given bases, corresponds to the vector $(1, 0, 0, 1)$. A quick computation shows that the smallest subspace of $\omega(X)^2$ stable under X and containing $f_{11} + f_{22}$ is

$$\mathrm{span}(f_{11} + f_{22}, f_{21})$$

which, under ψ_α and then the isomorphism $\omega(X)^* \otimes \omega(X) \simeq \mathrm{End}_k(\omega(X))$, maps to the span of the transformations

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where we have written these transformations as matrices in the bases f_1, f_2 for $\omega(X)$. Thus, $\mathrm{End}(\omega|\langle X \rangle)$ can be identified with the algebra of all 2×2 matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

for arbitrary a and b.

Another Example: Consider the module

$$\begin{pmatrix} x & & \\ & x^2 & \\ & & x^3 \end{pmatrix}$$

for the multiplicative group G_m. Skipping all the mumbo-jumbo with ψ and α, all

we really have to do is find the invariant subspace of

$$
\begin{pmatrix}
x & & & & & & & & \\
& x^2 & & & & & & & \\
& & x^3 & & & & & & \\
& & & x & & & & & \\
& & & & x^2 & & & & \\
& & & & & x^3 & & & \\
& & & & & & x & & \\
& & & & & & & x^2 & \\
& & & & & & & & x^3
\end{pmatrix}
$$

generated by the vector $(1, 0, 0, 0, 1, 0, 0, 0, 1)$. This we compute to be the span of the vectors $(1, 0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 1, 0, 0, 0, 0),$ and $(0, 0, 0, 0, 0, 0, 0, 0, 1)$. These in turn project to the span of the matrices

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

That is, we can identify $\mathrm{End}(\omega|\langle X \rangle)$ with the collection of all diagonal transformations on $\omega(X)$.

Chapter 4

First-Order Definability of Tannakian Categories

The goal of this chapter is to prove that, in a certain appropriately chosen language, the sentence "is a tannakian category" is first-order. Note that we do not claim that the property of 'being neutral' is necessarily first-order.

4.1 The Language of Abelian Tensor Categories

The title of this section is a misnomer for two reasons. First, the article "the" implies there is only one such language, and this is certainly not the case. It is however, as far as the author can tell, the most natural and minimal choice for our purposes. Secondly, not all structures in the 'language of abelian tensor categories' are abelian tensor categories, and we may as well have called it the 'language of tannakian categories'. But the name seems natural enough.

Our language is purely relational; it has no function or constant symbols. The

primitive symbols are

$$\_\_ \in \mathrm{Mor} \qquad \_\_ \in \mathrm{Ob} \qquad \_\_ : \_\_ \to \_\_ \qquad \_\_ \circ \_\_ \doteq \_\_ \qquad \_\_ + \_\_ \doteq \_\_$$

$$\_\_ \otimes \_\_ \doteq \_\_ \qquad \mathrm{assoc}_{\_\_,\_\_,\_\_} \doteq \_\_ \qquad \mathrm{comm}_{\_\_,\_\_} \doteq \_\_ \qquad \mathrm{unit}_{\_\_} \doteq \_\_$$

The intended interpretation of the symbols are as follows. $x \in \mathrm{Mor}$ expresses that x is a morphism in the category, $x \in \mathrm{Ob}$ that x is an object. $x : y \to z$ expresses that the morphism x points from the object y to z, $x \circ y \doteq z$ expresses that the morphism x composed with y is equal to z, and $x + y \doteq z$ expresses that the morphism x added to y is equal to z. $x \otimes y \doteq z$ means that the tensor product of x and y is equal to z, and this could either mean tensor product of objects or tensor product of morphisms.

The symbols $\mathrm{assoc}, \mathrm{comm}$, and unit stand for the requisite natural isomorphisms present in an abelian tensor category. For instance, $\mathrm{assoc}_{x,y,z} \doteq t$ expresses that the associativity isomorphism $(x \otimes y) \otimes z \simeq x \otimes (y \otimes z)$ attached to the objects x, y and z is equal to t, and similarly for $\mathrm{comm}_{x,y} \doteq t$. $\mathrm{unit}_x \doteq y$ expresses that y is the natural isomorphism between the object x and $x \otimes \underline{1}$, where $\underline{1}$ is an identity object for the tensor category.

The reader should take care not to automatically identify the symbol $\doteq$ occurring as a sub-symbol of the above symbols with actual equality of elements; $\doteq$ is purely formal in this context. For a random structure in the above signature it is entirely possible to have four elements a, b, c, d such that $a \circ b \doteq c$, $a \circ b \doteq d$, but not $c = d$, where this last equation is actual equality of elements. Of course, we chose the symbol $\doteq$ because, modulo the theory we are going to write down, $\doteq$ does in fact behave like equality.

To make anything we are about to do manageable, we must find a way to treat certain of the relational symbols in our language as if they were functional right from the start, and hence to treat expressions such as $x \circ y$ as if they were terms. For

instance, we would like to be able to write the sentence

$$(\forall x, y)(x \circ y = y \circ x)$$

and attach the intended meaning to it. But as it stands, this is not a sentence in our language. Here is how this can be remedied. In the case of $\circ$ we can treat the symbol $x \circ y$ as a term as follows. If $\Phi(z)$ is any formula, x and y variables, then we define $\Phi(x \circ y)$ to be the formula

$$(\forall t)(x \circ y \doteq t \implies \Phi(t))$$

where t is some variable not occurring in Φ and not equal to x or y. By iteration of this process we can in fact treat any 'meaningful composition' of variables as a term. By meaningful composition we mean: any variable x is a meaningful composition, and if Ψ, Σ are meaningful compositions, then so is $(\Psi) \circ (\Sigma)$ (for instance, $((x \circ y) \circ z) \circ (s \circ x)$ is a meaningful composition). Then for a meaningful composition $(\Psi) \circ (\Sigma)$ and formula $\Phi(x)$, we define $\Phi((\Psi) \circ (\Sigma))$ by induction on the length of the composition to be

$$(\forall s, t, r)((s = \Psi \wedge t = \Sigma \wedge s \circ t \doteq r) \implies \Phi(r))$$

where s, t and r are some not already being used variables. This formula is well-defined, since the formulas $s = \Psi$, $t = \Sigma$, and $\Phi(r)$ are by induction. For example then, the formula $x \circ y = y \circ x$ literally translates to

$$(\forall s)(y \circ x \doteq s \implies (\forall r)(x \circ y \doteq r \implies r = s))$$

The same trick can obviously be applied to the symbols involving $+, \otimes, \text{assoc}, \text{comm},$

and unit. Thus we can be confident in the meaning of something like

$$(\forall x, y, z)((x + y) \circ z = (x \circ z) + (y \circ z))$$

Let us agree on some abbreviations. All capital English letter variables $(A, B, X, Y,$ etc.) are understood to range over objects, lower case Greek letters $(\phi, \psi, \alpha, \beta,$ etc.) over morphisms, and if we wish to be nonspecific we will use lower case English letters $(a, b, x, y,$ etc.). So if $\Phi(x)$ is a formula, we define $(\forall X)\Phi(X)$ to mean $(\forall x)(x \in \mathrm{Ob} \implies \Phi(x))$, and $(\forall \psi)\Phi(\psi)$ means $(\forall x)(x \in \mathrm{Mor} \implies \Phi(x))$. The formula $(\exists X)\Phi(X)$ stands for $(\exists x)(x \in \mathrm{Ob} \wedge \Phi(x))$, and similarly for $(\exists \psi)\Phi(\psi)$. $(\forall x)\Phi(x)$ and $(\exists x)\Phi(x)$ mean exactly what they say.

If $a_1, \ldots, a_n, x, y$ are variables then $a_1, \ldots, a_n : x \to y$ is shorthand for $a_1 : x \to y \wedge \ldots \wedge a_n : x \to y$. $(\forall a_1, \ldots, a_n : x \to y)\Phi(a_1, \ldots, a_n)$ is shorthand for $(\forall a_1, \ldots, a_n)(a_1, \ldots, a_n : x \to y \implies \Phi(a_1, \ldots, a_n))$, and $(\exists a_1, \ldots, a_n : x \to y)\Phi(a_1, \ldots, a_n)$ is shorthand for $(\exists a_1, \ldots, a_n)(a_1, \ldots, a_n : x \to y \wedge \Phi(a_1, \ldots, a_n))$. We make identical definitions for the expressions $x_1, \ldots, x_n \in \mathrm{Ob}$ and $x_1, \ldots, x_n \in \mathrm{Mor}$.

If x and y are variables, we define the formula $\mathrm{Dom}(x) \doteq y$ to mean $(\exists z)(x : y \to z)$, and we make an analogous definition for $\mathrm{Codom}(x) \doteq y$. We can treat Dom and Codom as if they were functions by declaring: if $\Phi(x)$ is a formula, we define $\Phi(\mathrm{Dom}(x))$ to mean $(\forall y)(\mathrm{Dom}(x) \doteq y \implies \Phi(y))$, and similarly for Codom.

The remainder of this chapter is devoted to proving, piecemeal, that the statement "is a tannakian category" is expressible by a first-order sentence in the language of abelian tensor categories.

4.2 Axioms for a Category

1. Every element of $\mathcal{C}$ is either an object or a morphism, but not both:

$$(\forall x)((x \in \mathrm{Ob} \lor x \in \mathrm{Mor}) \land \neg(x \in \mathrm{Ob} \land x \in \mathrm{Mor}))$$

2. All arrows are morphisms, and all vertices are objects:

$$(\forall x, y, z)(x : y \to z \implies (x \in \mathrm{Mor} \land y \in \mathrm{Ob} \land z \in \mathrm{Ob}))$$

3. Every morphism points to and from exactly one object:

$$(\forall \phi)(\exists! X, Y)(\phi : X \to Y)$$

4. Composition only makes sense on morphisms:

$$(\forall x, y, z)(x \circ y \doteq z \implies x, y, z \in \mathrm{Mor})$$

5. Composition only makes sense between composable morphisms, and the composition points where it should:

$$(\forall \phi, \psi, \eta)(\phi \circ \psi \doteq \eta \implies (\mathrm{Codom}(\phi) = \mathrm{Dom}(\psi)$$
$$\land \eta : \mathrm{Dom}(\phi) \to \mathrm{Codom}(\psi)))$$

6. Composition is a function on composable arrows:

$$(\forall \phi, \psi)(\mathrm{Codom}(\phi) = \mathrm{Dom}(\psi) \implies (\exists! \eta)(\phi \circ \psi \doteq \eta))$$

7. Composition is associative:

$$(\forall x, y, z)((\exists t)((x \circ y) \circ z = t) \implies (x \circ (y \circ z) = (x \circ y) \circ z))$$

We define the formula $x \doteq 1_y$ to mean that x is a two-sided identity morphism for y, i.e. as the formula $(x : y \to y) \wedge (\forall z)((x \circ z = t \vee z \circ x = t) \implies z = t)$. We write $x \doteq 1$ to mean that x is an identity morphism for some object, i.e. $(\exists X)(x \doteq 1_X)$. If $\Phi(x)$ is a formula, we define $\Phi(1_x)$ to mean $(\forall y)(y \doteq 1_x \implies \Phi(y))$, and similarly for 1.

8. Every object has an identity morphism:

$$(\forall X)(\exists \phi)(\phi \doteq 1_X)$$

4.3 Axioms for an Abelian Category

In this section we build axioms amounting to the statement that a given category is abelian, using definition 3.2.1 as our guide.

1. Addition is only defined on addable morphisms, and their sum points where it should:

$$(\forall x, y, z)(x + y \doteq z \implies (x, y, z \in \mathrm{Mor} \wedge \mathrm{Dom}(x) = \mathrm{Dom}(y) \wedge \mathrm{Dom}(y)$$
$$= \mathrm{Dom}(z) \wedge \mathrm{Codom}(x) = \mathrm{Codom}(y) \wedge \mathrm{Codom}(y) = \mathrm{Codom}(z)))$$

2. Addition is a function on addable morphisms:

$$(\forall x, y)((\mathrm{Dom}(x) = \mathrm{Dom}(y) \wedge \mathrm{Codom}(x) = \mathrm{Codom}(y)) \implies (\exists! z)(x + y = z))$$

3. Addition is associative and commutative:

$$(\forall x, y, z)((\exists t)((x+y)+z = t) \implies (x+y = y+x \wedge (x+y)+z = x+(y+z)))$$

Define the formula $x \doteq 0_{y,z}$ to mean x is an additive identity for $\operatorname{Hom}(x, y)$. That is, $(x : y \to z) \wedge (\forall t : y \to z)(x + t = t)$. Define $x \doteq 0$ to be $(\exists X, Y)(x = 0_{X,Y})$. If $\Phi(x)$ is a formula, $\Phi(0_{x,y})$ means $(\forall z)(z = 0_{x,y} \implies \Phi(z))$, and similarly for 0.

4. Existence of zero morphisms for addition:

$$(\forall A, B)(\exists \phi)(x \doteq 0_{A,B})$$

Define the formula $x \doteq -y$ to mean x is an additive inverse for y. That is, $x+y = 0$. For a formula $\Phi(x)$, $\Phi(-y)$ is shorthand for the formula $(\forall x)(x \doteq -y \implies \Phi(x))$.

5. Existence of additive inverses:

$$(\forall \phi)(\exists x)(x = -\phi)$$

6. Bilinearity of composition over addition:

$$(\forall A, B, C, D)(\forall \eta, \phi, \psi, \nu)((\eta : A \to B \wedge \phi, \psi : B \to C \wedge \nu : C \to D) \implies$$

$$(\eta \circ (\phi + \psi) = \eta \circ \phi + \eta \circ \psi \wedge (\phi + \psi) \circ \eta = \phi \circ \eta + \psi \circ \eta))$$

The definition of an abelian category calls for the existence of pair-wise biproducts, kernels and cokernels, and normality of monomorphisms and epimorphisms. Here we give first-order definitions of these concepts.

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Let A and B be objects. Then a biproduct, which we denote as $A \oplus B$, is a diagram
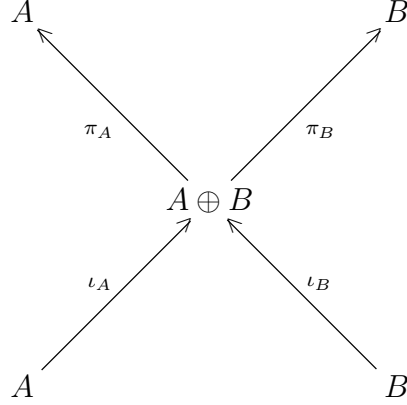
with the following properties: $\pi_A \circ \iota_A + \pi_B \circ \iota_B = 1_{A \oplus B}$, $\iota_A \circ \pi_A = 1_A$, $\iota_B \circ \pi_B = 1_B$, $\iota_A \circ \pi_B = 0$, and $\iota_B \circ \pi_A = 0$. One could clearly write down these conditions as a first-order formula. Thus, for objects A and B we define the formula $\mathrm{Sum}(Z; A, B)$ to mean that there exists maps $\iota_A, \iota_B, \pi_A, \pi_B$ satisfying all the above criteria.

Let $\psi : A \to B$ be a morphism. A kernel for ψ is by definition a map k pointing from some object K to A such that $k \circ \psi = 0_{K,B}$, and for any object C and map $\rho : C \to A$ with $\rho \circ \psi = 0_{C,B}$ there is a unique map $\hat{\rho} : C \to K$ such that $\hat{\rho} \circ k = \rho$. Again, this is clearly first-order. Thus, we define: $\ker(k; \psi)$ means that the morphism k is a kernel for ψ. The same obviously holds for the dual concept of cokernel, so we define $\mathrm{coker}(c; \psi)$ in like fashion.

In the language of categories saying that a morphism is an epimorphism is to say that it is right-cancellative. That is, $\psi : A \to B$ is an epimorphism if for any maps $\eta, \nu : B \to C$, $\psi \circ \eta = \psi \circ \nu$ implies that $\eta = \nu$. Again, this is clearly a first-order concept, so we define the formula $\mathrm{epic}(\phi)$ to mean ϕ is an epimorphism, and likewise $\mathrm{monic}(\phi)$ that ϕ is a monomorphism.

7. Every pair of objects has a biproduct:

$$(\forall A, B)(\exists Z)(\mathrm{Sum}(Z; A, B))$$

8. Every morphism has a kernel and a cokernel:

$$(\forall \phi)(\exists k, c)(\ker(k; \phi) \wedge \mathrm{coker}(c; \phi))$$

9. Every monomorphism is normal:

$$(\forall \phi)(\mathrm{monic}(\phi) \implies (\exists \psi)(\ker(\phi; \psi)))$$

10. Every epimorphism is normal:

$$(\forall \phi)(\mathrm{epic}(\phi) \implies (\exists \psi)(\mathrm{coker}(\phi; \psi)))$$

4.4 Axioms for an Abelian Tensor Category

Here we build axioms asserting that given abelian tensor category is an abelian tensor category, per definition 3.2.2. Our first task is to assert that $\otimes$ is a bi-additive functor.

1. Every pair of morphisms and objects has a unique tensor product:

$$(\forall X, Y)(\exists! Z)(X \otimes Y \doteq Z) \wedge (\forall \phi, \psi)(\exists! \eta)(\phi \otimes \psi \doteq \eta)$$

2. The tensor product of objects is an object, that of morphisms is a morphism, and there's no such thing as a tensor product of an object and a morphism:

$$(\forall X, Y)(X \otimes Y \in \mathrm{Ob}) \wedge (\forall \phi, \psi)(\phi \otimes \psi \in \mathrm{Mor}) \wedge (\forall X, \psi)(\nexists x)(X \oplus \psi \doteq x \vee \psi \oplus X \doteq x)$$

3. The tensor product of morphisms points where it should:

$$(\forall\phi,\psi)(\forall A,B,X,Y)((\phi:A\to B\wedge\psi:X\to Y)\implies(\phi\otimes\psi:A\otimes X\to B\otimes Y))$$

4. The tensor product preserves composition:

$$(\forall\phi,\psi,\eta,\nu)(\forall A,B,X,Y,S,T)((\phi:A\to X\wedge\psi:B\to Y\wedge\eta:X\to S\wedge\nu:Y\to T)$$
$$\implies((\phi\circ\eta)\otimes(\psi\circ\nu))=(\phi\otimes\psi)\circ(\eta\otimes\nu)))$$

5. The tensor product preserves identity:

$$(\forall A,B)((1_A\otimes 1_B)=1_{A\otimes B})$$

6. The tensor product is a bi-additive functor:

$$(\forall A,B,X,Y)(\forall\phi,\psi:A\to B)(\forall\eta:X\to Y)$$
$$(((\phi+\psi)\otimes\eta=\phi\otimes\eta+\psi\otimes\eta)\wedge(\eta\otimes(\phi+\psi)=\eta\otimes\phi+\eta\otimes\psi))$$

Next we assert that the natural isomorphisms assoc, comm, and unit are doing the job we need them to. We start with assoc.

7. assoc accepts objects and returns morphisms:

$$(\forall x,y,z,t)(\mathrm{assoc}_{x,y,z}\doteq t\implies(x,y,z\in\mathrm{Ob}\wedge t\in\mathrm{Mor}))$$

8. assoc is a function on triples of objects:

$$(\forall X,Y,Z)(\exists!\phi)(\mathrm{assoc}_{X,Y,Z}\doteq\phi)$$

If $\Phi(x)$ is a formula and a, b, c variables, by $\Phi(\text{assoc}_{a,b,c})$ we mean the formula $(\forall t)(\text{assoc}_{a,b,c} \doteq t \implies \Phi(t))$.

9. assoc points where it should:

$$(\forall X, Y, Z)(\text{assoc}_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z)$$

We define a formula $\text{iso}(\phi)$ to mean that the morphism ϕ is an isomorphism: $(\forall A, B)(\phi : A \to B \implies (\exists \psi : B \to A)(\phi \circ \psi = 1_A \wedge \psi \circ \phi = 1_B))$.

10. assoc is always an isomorphism:

$$(\forall X, Y, Z)(\text{iso}(\text{assoc}_{X,Y,Z}))$$

11. assoc is a natural transformation:

$$(\forall X, Y, Z, R, S, T)(\forall \phi : X \to R, \psi : Y \to S, \eta : Z \to T)(\text{the following commutes:}$$

$$
\begin{array}{ccc}
X \otimes (Y \otimes Z) & \xrightarrow{\text{assoc}_{X,Y,Z}} & (X \otimes Y) \otimes Z \\
\downarrow{\scriptstyle \phi \otimes (\psi \otimes \eta)} & & \downarrow{\scriptstyle (\phi \otimes \psi) \otimes \eta} \\
R \otimes (S \otimes T) & \xrightarrow[\text{assoc}_{R,S,T}]{} & (R \otimes S) \otimes T
\end{array}
$$

We make the necessary assertions and definitions for comm and unit in like fashion.

12. comm accepts objects and returns morphisms:

$$(\forall x, y, z)(\text{comm}_{x,y} \doteq z \implies (x, y \in \text{Ob} \wedge z \in \text{Mor}))$$

13. comm is a function on pairs of objects:

$$(\forall X, Y)(\exists! \phi)(\text{comm}_{X,Y} \doteq \phi)$$

If $\Phi(x)$ is a formula and a, b variables, by $\Phi(\text{comm}_{a,b})$ we mean the formula $(\forall t)(\text{assoc}_{a,b} \doteq t \implies \Phi(t))$.

14. comm points where it should:

$$(\forall X, Y)(\text{comm}_{X,Y} : X \otimes Y \to Y \otimes X)$$

15. comm is always an isomorphism:

$$(\forall X, Y)(\text{iso}(\text{comm}_{X,Y}))$$

16. comm is a natural transformation:

$$(\forall X, Y, R, S)(\forall \phi : X \to R, \psi : Y \to S)(\text{the following commutes:}$$

$$
\begin{array}{ccc}
X \otimes Y & \xrightarrow{\text{comm}_{X,Y}} & Y \otimes X \\
\phi \otimes \psi \downarrow & & \downarrow \psi \otimes \phi \\
R \otimes S & \xrightarrow[\text{comm}_{R,S}]{} & S \otimes R
\end{array}
$$

17. unit accepts objects and returns morphisms:

$$(\forall x, y)(\text{unit}_x \doteq y \implies (x \in \text{Ob} \wedge y \in \text{Mor}))$$

18. unit is a function on objects:

$$(\forall X)(\exists! \phi)(\text{unit}_X \doteq \phi)$$

If $\Phi(x)$ is a formula and a a variable, by $\Phi(\text{unit}_a)$ we mean the formula $(\forall t)(\text{unit}_a \doteq t \implies \Phi(t))$.

19. unit is always an isomorphism:

$$(\forall X)(\text{iso}(\text{unit}_X))$$

For unit, we must make the additional assertion that there exists an identity object for $\otimes$.

20. unit has an identity object associated to it:

$$(\exists U)(\forall X)(\text{unit}_X : X \to U \otimes X)$$

We define the formula $\text{id}^{\otimes}(u)$ to mean that u is an identity object associated to unit. That is, $(\forall X)(\text{unit}_X : X \to u \otimes X)$.

21. unit is a natural transformation:

$$(\forall X, Y)(\forall \phi : X \to Y)(\forall U)(\text{id}^{\otimes}(U) \implies \text{ the following commutes:}$$

$$
\begin{array}{ccc}
X & \xrightarrow{\text{unit}_X} & U \otimes X \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle 1_U \otimes \phi} \\
Y & \xrightarrow[\text{unit}_Y]{} & U \otimes Y
\end{array}
$$

We must now assert that the functor $X \mapsto \mathrm{id}^{\otimes}(u) \otimes X$ is an equivalence, which is to say that it is full, faithful, and essentially surjective. Essential surjectivity is already asserted by previous axioms: for every X, X is isomorphic to $\mathrm{id}^{\otimes}(u) \otimes X$. Thus we must assert that it is full and faithful.

22. The functor $X \mapsto \mathrm{id}^{\otimes}(u) \otimes X$ is full and faithful:

$$(\forall \phi)(\forall X, Y, U)((\mathrm{id}^{\otimes}(U) \wedge \phi : U \otimes X \to U \otimes Y) \implies (\exists! \psi)(\psi : X \to X \wedge 1_X \otimes \psi = \phi))$$

All that is left then is to assert the various coherence conditions among assoc, comm, and unit, conditions 4, 5, and 6 in definition 3.2.2. All of these statements are of the form, for some fixed n, "for all objects $X_1, \ldots, X_n$, the following diagram commutes." These are plainly first-order, so we do not repeat them.

4.5   Axioms for a Tannakian Category

Here we must assert the rigidity of the abelian tensor category $\mathcal{C}$ (definition 3.2.3) and that $\mathrm{End}(\underline{1})$ is a field, modulo all of our previous axioms. The first condition for rigidity is the existence of an internal Hom object for every pair of objects (see page 35). This is by definition an object $\underline{\mathrm{Hom}}(X, Y)$ such that the functors $\mathrm{Hom}(\_\_, X \otimes Y)$ and $\mathrm{Hom}(\_\_, \underline{\mathrm{Hom}}(X, Y))$ are naturally isomorphic. Here is how we can define this in a first-order fashion.

Suppose that Z is an internal Hom object for X and Y. Then we have a natural isomorphism of functors $\mathrm{Hom}(\_\_, Z) \xrightarrow{\Phi} \mathrm{Hom}(\_\_ \otimes X, Y)$. The Yoneda lemma (page 18) guarantees that this map Φ must take the following form: for an object T and

map $T \xrightarrow{\phi} Z$, $\Phi_T(\phi)$ is the unique map making the diagram

$$
\begin{array}{ccc}
T \otimes X & & \\
\phi \otimes 1 \downarrow & \searrow^{\Phi_T(\phi)} & \\
Z \otimes X & \xrightarrow[\text{ev}]{} & Y
\end{array}
$$

commute, where we have given the name ev to the element $\Phi_Z(1_Z) \in \mathrm{Hom}(Z \otimes X, Y)$. And in fact, any such map $Z \otimes X \to Y$ gives you a natural transformation between the functors $\mathrm{Hom}(\_, Z)$ and $\mathrm{Hom}(\_ \otimes X, Y)$. Thus, to assert the existence of a natural isomorphism, we need only assert the existence of a map $\mathrm{ev} : Z \otimes X \to Y$ such that the natural transformation Φ it defines gives a bijection $\mathrm{Hom}(T, Z) \xrightarrow{\Phi_T} \mathrm{Hom}(T \otimes X, Y)$ for every T. We therefore define the formula $\underline{\mathrm{Hom}}(Z, \mathrm{ev}; X, Y)$ to mean that the object Z and morphism ev form an internal Hom pair for X and Y:

$$(\mathrm{ev} : Z \otimes X \to Y) \wedge (\forall T)(\forall \psi : T \otimes X \to Y)(\exists! \phi : T \to Z)(\text{the following commutes:}$$

$$
\begin{array}{ccc}
T \otimes X & & \\
\phi \otimes 1 \downarrow & \searrow^{\psi} & \\
Z \otimes X & \xrightarrow[\text{ev}]{} & Y
\end{array}
$$

1. Every pair of objects has an internal Hom:

$$(\forall X, Y)(\exists Z)(\exists \phi)(\underline{\mathrm{Hom}}(Z, \phi; X, Y))$$

Recall now the definition of reflexivity of the object X (see page 37); it is the assertion that a certain map $\iota_X : X \to X^{\vee\vee}$ is an isomorphism. This map is defined

by the property that it uniquely makes

$$
\begin{array}{ccc}
X \otimes X^\vee & & \\
\downarrow \scriptstyle{\iota_X \otimes 1} & \searrow \scriptstyle{\text{comm}} & \\
& X^\vee \otimes X & \\
& & \searrow \scriptstyle{\text{ev}_X} \\
X^{\vee\vee} \otimes X^\vee & \xrightarrow[\text{ev}_{X^\vee}]{} & \underline{1}
\end{array}
$$

commute. But of course ι_X is not really unique, since neither is e.g. $X^\vee$, since there are in general many (mutually isomorphic) choices for internal Hom. But for a fixed choice of the various internal Hom objects referenced in this diagram, it is unique. We therefore define $\mathrm{incl}(\iota; X)$ to mean that ι qualifies as one of these maps:

$$
(\exists T, R, U)(\exists \phi, \psi)(\mathrm{id}^\otimes(U) \wedge \underline{\mathrm{Hom}}(T, \phi; X, U) \wedge \underline{\mathrm{Hom}}(R, \psi; T, U)
$$

$$
\wedge\, \iota : X \to R \wedge \text{ the following commutes:}
$$

$$
\begin{array}{ccc}
X \otimes T & & \\
\downarrow \scriptstyle{\iota \otimes 1_T} & \searrow \scriptstyle{\text{comm}_{X,T}} & \\
& T \otimes X & \\
& & \searrow \scriptstyle{\phi} \\
R \otimes T & \xrightarrow[\psi]{} & U
\end{array}
$$

2. All objects are reflexive:

$$
(\forall X)(\forall \iota)(\mathrm{incl}(\iota; X) \implies \mathrm{iso}(\iota))
$$

Our last task in defining rigidity is to assert that the map Φ referenced in diagram 3.2.3 is an isomorphism. First, given objects R, S, T and U, let us define the isomorphism $(R \otimes S) \otimes (T \otimes U) \simeq (R \otimes T) \otimes (S \otimes U)$ referenced in diagram 3.2.2. This is gotten by composition of the following sequence of commutativity and associativity isomorphisms (the subscripts of which we suppress):

$$(R \otimes S) \otimes (T \otimes U) \xrightarrow{\text{assoc}} R \otimes (S \otimes (T \otimes U)) \xrightarrow{1 \otimes \text{assoc}} R \otimes ((S \otimes T) \otimes U)$$

$$\xrightarrow{1 \otimes (\text{comm} \otimes 1)} R \otimes ((T \otimes S) \otimes U) \xrightarrow{\text{assoc}} (R \otimes (T \otimes S)) \otimes U \xrightarrow{\text{assoc} \otimes 1} ((R \otimes T) \otimes S) \otimes U$$

$$\xrightarrow{\text{assoc}} (R \otimes T) \otimes (S \otimes U)$$

Define the formula $\text{ISO}(R, S, T, U, \Psi)$ to mean that Ψ is the above composition with respect to the objects R, S, T and U. Next we must define the map Φ defined by the commutativity of diagram 3.2.3. This is done using a similar strategy to that used to define reflexivity. Define the formula $\text{QUAD}(X_1, X_2, Y_1, Y_2, \Phi)$ to mean that Φ qualifies as one of the maps referenced in diagram 3.2.3 with respect to the objects X_1, X_2, Y_1, Y_2:

$$(\exists Z_1, Z_2, Z)(\exists \text{ev}_1, \text{ev}_2, \text{ev}, \Psi)(\underline{\text{Hom}}(Z_1, \text{ev}_1; X_1, Y_1) \wedge \underline{\text{Hom}}(Z_2, \text{ev}_2; X_2, Y_2)$$

$$\wedge \underline{\text{Hom}}(Z, \text{ev}; X_1 \otimes X_2, Y_1 \otimes Y_2) \wedge \text{ISO}(Z_1, Z_2, X_1, X_2, \Psi) \wedge \text{ the following commutes:}$$

$$
\begin{array}{ccc}
(Z_1 \otimes Z_2) \otimes (X_1 \otimes X_2) & \xrightarrow{\Psi} & (Z_1 \otimes X_1) \otimes (Z_2 \otimes X_2) \\
\downarrow{\scriptstyle \Phi \otimes 1} & & \downarrow{\scriptstyle \text{ev}_1 \otimes \text{ev}_2} \\
Z \otimes (X_1 \otimes X_2) & \xrightarrow{\text{ev}} & Y_1 \otimes Y_2
\end{array}
$$

3. For all objects X_1, X_2, Y_1, Y_2, the map Φ in diagram 3.2.3 is an isomorphism:

$$(\forall X_1, X_2, Y_1, Y_2)(\forall \Phi)(\text{QUAD}(X_1, X_2, Y_1, Y_2, \Phi) \implies \Phi \text{ is an isomorphism})$$

Finally, we have to assert that the ring $\mathrm{End}(\underline{1})$ is a field. Given any object X, our previous axioms already assert that $\mathrm{End}(X)$ is a ring with unity, so let $\mathrm{field}(X)$ be the assertion that $\mathrm{End}(X)$ is a commutative ring with inverses:

$$(\forall \phi, \psi : X \to X)(\phi \circ \psi = \psi \circ \phi) \wedge (\forall \phi : X \to X)(\neg(\phi = 0) \implies (\exists \psi)(\phi \circ \psi = 1_X))$$

4. $\mathrm{End}(\underline{1})$ is a field:
$$(\forall U)(\mathrm{id}^\otimes(U) \implies \mathrm{field}(U))$$

We have proved that the statement "is a tannakian category" is expressible by a first-order sentence in the language of abelian tensor categories.

Chapter 5

Subcategories of Tannakian Categories

In this chapter we record some results on (abelian, tannakian) subcategories of (abelian, tannakian) categories which will be needed later. For instance we will prove useful criteria which allow us to conclude that a given subcategory of a neutral tannakian category is also neutral tannakian.

Proposition 5.0.1. *Let $\mathcal{C}$ be a full subcategory of the tannakian category $\mathcal{D}$. Then $\mathcal{C}$ is tannakian if it is closed under the taking of biproducts, subobjects, quotients, tensor products, duals, and contains an identity object.*

Proof. We first show that $\mathcal{C}$ is abelian. $\mathcal{C}$ is a full subcategory of $\mathcal{D}$, and so obviously Hom-sets still have the structure of an abelian group, and composition is still bilinear. Consequently, being a zero or identity morphism in $\mathcal{C}$ is coincident with being one in

$\mathcal{D}$. If $A \oplus B$ is the $\mathcal{D}$-biproduct of the $\mathcal{C}$-objects A and B, then we have a $\mathcal{D}$-diagram

such that $\pi_A \circ \iota_A + \pi_B \circ \iota_B = 1_{A \oplus B}$, $\iota_A \circ \pi_A = 1_A$, $\iota_B \circ \pi_B = 1_B$, $\iota_A \circ \pi_B = 0$, and $\iota_B \circ \pi_A = 0$. But all these maps exist in $\mathcal{C}$ as well, along with the given relations, so this diagram constitutes a $\mathcal{C}$-biproduct for A and B.

Let $A \xrightarrow{\phi} B$ be a $\mathcal{C}$-morphism, and $K \xrightarrow{k} A$ its $\mathcal{D}$-kernel; this map exists in $\mathcal{C}$ as well, since K is a subobject of A. If $L \xrightarrow{\psi} A$ is any $\mathcal{C}$-morphism with $\psi \circ \phi = 0$ in $\mathcal{C}$, then this composition is zero in $\mathcal{D}$ as well; consequently, there is a unique morphism $L \xrightarrow{\bar{\psi}} K$ such that $\bar{\psi} \circ k = \psi$. As $\mathcal{C}$ is full, this map $\bar{\psi}$ exists also in $\mathcal{C}$, and is clearly still unique. Thus k is a $\mathcal{C}$-kernel for ϕ as well, which shows that all kernels exist in $\mathcal{C}$. An analogous proof holds for the existence of cokernels, using the fact that $\mathcal{C}$ is closed under quotients.

Let $A \xrightarrow{\phi} B$ be a $\mathcal{C}$-monomorphism; we claim that it is also a monomorphism in $\mathcal{D}$. Let $X \xrightarrow{\psi} A$ be any $\mathcal{D}$-morphism such that $\psi \circ \phi = 0$; we wish to show that $\psi = 0$. As every morphism in an abelian category factors through an epimorphism and a monomorphism (page 199 of [14]), we have a commutative diagram

where π and ι are a $\mathcal{D}$-epimorphism/monomorphism respectively, giving $\pi \circ \iota \circ \phi = 0$. As π is epic, we have $\iota \circ \phi = 0$. But C is a subobject of A, hence a member of $\mathcal{C}$, and so by the $\mathcal{C}$-monomorphic property of ϕ, $\iota = 0$. Thus $\psi = \pi \circ \iota$ equals 0 as well, and we have shown that ϕ is a $\mathcal{D}$-monomorphism.

So if $A \xrightarrow{\phi} B$ is any $\mathcal{C}$-monomorphism, it is also a $\mathcal{D}$-monomorphism, thus normal in $\mathcal{D}$. Then let $B \xrightarrow{\psi} C$ be the $\mathcal{D}$-morphism for which ϕ is the $\mathcal{D}$-kernel. Again ψ factors through an epimorphism and a monomorphism, and we have a commutative diagram

$$
\begin{array}{ccc}
 & & X \\
 & \nearrow \pi & \searrow \iota \\
A \xrightarrow{\phi} & B \xrightarrow{\psi} & C
\end{array}
$$

The map $B \xrightarrow{\pi} X$ exists in $\mathcal{C}$, X being a quotient of B. We claim that ϕ is a $\mathcal{C}$-kernel for π. If $L \xrightarrow{\eta} B$ is any $\mathcal{C}$-map such that $\eta \circ \pi = 0$, then also $\eta \circ \psi = \eta \circ \pi \circ \iota = 0$; as ϕ is a kernel for ψ, there is a unique map $L \xrightarrow{\bar{\eta}} A$ such that $\bar{\eta} \circ \phi = \eta$, which satisfies the universal property of ϕ being a $\mathcal{C}$-kernel for π. Therefore all monomorphisms are normal in $\mathcal{C}$. An analogous argument shows that all epimorphisms in $\mathcal{C}$ are normal. Therefore $\mathcal{C}$ is an abelian category.

Since the tensor product of two objects in $\mathcal{C}$ is also in $\mathcal{C}$, so also is the tensor product of two morphisms, since $\mathcal{C}$ is full. For objects A, B and C of $\mathcal{C}$, the associativity map $(A \otimes B) \otimes C \xrightarrow{\text{assoc}_{A,B,C}} A \otimes (B \otimes C)$ exists in $\mathcal{C}$, and is clearly still natural. Just as 'monomorphic' and 'epimorphic' are identical concepts in $\mathcal{C}$ and $\mathcal{D}$, so is 'isomorphic', and thus assoc is a natural isomorphism in $\mathcal{C}$. Analogous statements hold for the requisite isomorphisms comm and unit, the latter existing in $\mathcal{C}$ since the identity element of $\mathcal{D}$ is stipulated to exist in $\mathcal{C}$. The coherence conditions 4., 5. and 6. of definition 3.2.2 clearly also still hold, as well as the bilinearity of $\otimes$.

In any tensor category, one can in fact identify the object $\underline{\mathrm{Hom}}(A, B)$ with $A^\vee \otimes B$; as duals are assumed exist in $\mathcal{C}$, so also do all internal Homs, as well as the requisite 'ev' maps since $\mathcal{C}$ is full. The remaining conditions of definition 3.2.3 merely stipulate that certain maps must be isomorphisms; as $\mathcal{C}$ is full, these maps also exist in $\mathcal{C}$, and are isomorphisms since they are in $\mathcal{D}$. And of course, $\mathrm{End}(\underline{1})$ is still a field. This completes the proof.

$\square$

Lemma 5.0.2. *Let $\mathcal{C}$ be a full abelian subcategory of the abelian category $\mathcal{D}$ which is closed under the taking of biproducts, subobjects, and quotients.*

1. *The $\mathcal{C}$-diagram*

$$
\begin{array}{ccc}
X & \xrightarrow{\pi_1} & X_1 \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \phi_1} \\
X_2 & \xrightarrow{\phi_2} & Z
\end{array}
$$

   is a $\mathcal{C}$-pullback for $X_2 \xrightarrow{\phi_2} Z \xleftarrow{\phi_1} X_1$ if and only if it is also a $\mathcal{D}$-pullback for $X_2 \xrightarrow{\phi_2} Z \xleftarrow{\phi_1} X_1$

2. *The $\mathcal{C}$-diagram*

$$
\begin{array}{ccc}
X & \xleftarrow{\iota_1} & X_1 \\
\uparrow{\scriptstyle \iota_2} & & \uparrow{\scriptstyle \psi_1} \\
X_2 & \xleftarrow{\psi_2} & Z
\end{array}
$$

   is a $\mathcal{C}$-pushout for $X_2 \xleftarrow{\psi_2} Z \xrightarrow{\psi_1} X_1$ if and only if it is also a $\mathcal{D}$-pushout for $X_2 \xleftarrow{\psi_2} Z \xrightarrow{\psi_1} X_1$

Proof. We will prove this for pullbacks, leaving the pushout case to the reader. Suppose first that

$$
\begin{array}{ccc}
X & \xrightarrow{\ \pi_1\ } & X_1 \\
\big\downarrow{\scriptstyle\pi_2} & & \big\downarrow{\scriptstyle\phi_1} \\
X_2 & \xrightarrow[\ \phi_2\ ]{} & Z
\end{array}
$$

is a $\mathcal{D}$-pullback diagram. Let T be an object of $\mathcal{C}$, and suppose we have a commutative diagram

$$
\begin{array}{ccc}
T & \xrightarrow{\ \rho_1\ } & X_1 \\
\big\downarrow{\scriptstyle\rho_2} & & \big\downarrow{\scriptstyle\phi_1} \\
X_2 & \xrightarrow[\ \phi_2\ ]{} & Z
\end{array}
$$

Then by the universal property of being a $\mathcal{D}$-pullback, there is a unique $\mathcal{D}$-map $\rho : T \to X$ such that $\rho \circ \pi_1 = \rho_1$ and $\rho \circ \pi_2 = \rho_2$. As $\mathcal{C}$ is full, this map ρ exists in $\mathcal{C}$ as well, satisfies these relations, and is clearly still unique. Thus this diagram constitutes a $\mathcal{C}$-pullback as well.

Conversely, suppose the above is a $\mathcal{C}$-pullback diagram. As $\mathcal{D}$ is abelian, we know that $X_1 \xrightarrow{\phi_1} Z \xleftarrow{\phi_2} X_2$ *has* a $\mathcal{D}$-pullback, say

$$
\begin{array}{ccc}
U & \xrightarrow{\ \mu_1\ } & X_1 \\
\big\downarrow{\scriptstyle\mu_2} & & \big\downarrow{\scriptstyle\phi_1} \\
X_2 & \xrightarrow[\ \phi_2\ ]{} & Z
\end{array}
$$

The proof of theorem 2.15 of [7] shows that U can always be taken to be, up to isomorphism, a certain subobject of $X_1 \oplus X_2$. As $\mathcal{C}$ is closed under the taking of

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biproducts and subobjects, U is an object of $\mathcal{C}$, and as $\mathcal{C}$ is full, μ_1 and μ_2 are morphisms in $\mathcal{C}$; thus, this diagram belongs to $\mathcal{C}$. Then by the above, as this diagram is a $\mathcal{D}$-pullback, so also is it a $\mathcal{C}$-pullback. As any two pullbacks in an abelian category are isomorphic up to a unique isomorphism, we must have that

$$
\begin{array}{ccc}
T & \xrightarrow{\ \rho_1\ } & X_1 \\
\downarrow{\scriptstyle\rho_2} & & \downarrow{\scriptstyle\phi_1} \\
X_2 & \xrightarrow[\ \phi_2\ ]{} & Z
\end{array}
$$

is a $\mathcal{D}$-pullback as well. $\qquad\square$

Lemma 5.0.3. *Let $\mathcal{D}$ be an abelian category, $\mathcal{C}$ a non-empty full abelian subcategory of $\mathcal{D}$. Then exact sequences in $\mathcal{C}$ are also in $\mathcal{D}$ if and only if for every morphism $A \xrightarrow{\ \phi\ } B$ in $\mathcal{C}$, there is a $\mathcal{D}$-kernel of ϕ, $\mathcal{D}$-cokernel of ϕ, and $\mathcal{D}$-direct sum which all lie in $\mathcal{C}$.*

Proof. See theorem 3.41 of [7]. $\qquad\square$

In the case of 1-fold extensions in a k-linear abelian category, we shall need something slightly stronger.

Proposition 5.0.4. *Let $\mathcal{D}$ be a k-linear abelian category, $\mathcal{C}$ a non-empty full k-linear abelian subcategory of $\mathcal{D}$. Let M and N be objects of $\mathcal{C}$ and let $\xi_1, \ldots, \xi_m$ be a sequence of 1-fold extensions of N by M in $\mathcal{C}$. Then the ξ_j are linearly independent in $\mathcal{C}$ if and only if they are linearly independent in $\mathcal{D}$.*

Proof. By 'linearly independent', we mean with respect to the k-vector space structure defined by the Baer sum on $\mathrm{Ext}^1(M, N)$ (see section 2.4).

Denote by $\mathrm{EXT}^1_{\mathcal{C}}(M, N)$ the collection of all 1-fold extensions of M by N in $\mathcal{C}$, and define similarly $\mathrm{EXT}^1_{\mathcal{D}}(M, N)$ (this is different from $\mathrm{Ext}^1_{\mathcal{C}}(M, N)$, which is the

collection of all *equivalence classes* of 1-fold extensions). By the previous lemma, for every $\xi \in \text{EXT}^1_{\mathcal{C}}(M, N)$, ξ is also a member of $\text{EXT}^1_{\mathcal{D}}(M, N)$, whence we have a map $\text{EXT}^1_{\mathcal{C}}(M, N) \to \text{EXT}^1_{\mathcal{D}}(M, N)$.

We claim firstly that this map respects equivalence of extensions. If $\xi : 0 \to N \to X \to M \to 0$, $\chi : 0 \to N \to Y \to M \to 0$ are two $\mathcal{C}$-equivalent extensions, then we have a $\mathcal{C}$-isomorphism $\phi : X \to Y$ making

$$
\begin{array}{ccccccccc}
\xi : 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \phi} & & \| & & \\
\chi : 0 & \longrightarrow & N & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

commute. But as ϕ is a $\mathcal{D}$-isomorphism as well, ξ and χ are also $\mathcal{D}$-equivalent. Thus, our map $\text{EXT}^1_{\mathcal{C}}(M, N) \to \text{EXT}^1_{\mathcal{D}}(M, N)$ is actually a map $\text{Ext}^1_{\mathcal{C}}(M, N) \to \text{Ext}^1_{\mathcal{D}}(M, N)$. This map is injective, for if ξ and χ are $\mathcal{D}$-equivalent according to the above diagram, then they are also $\mathcal{C}$-equivalent, since the map ϕ exists in $\mathcal{C}$.

What is left then is to verify that this map is linear. Let k be a scalar and $\xi : 0 \to N \xrightarrow{\phi} X \xrightarrow{\psi} M \to 0$ a $\mathcal{C}$-extension of M by N. Then the scalar multiplication of k, when $k \neq 0$, is defined to be

$$
k\xi : 0 \to N \xrightarrow{k^{-1}\phi} X \xrightarrow{\psi} M \to 0
$$

and in case $k = 0$, as the trivial extension. As scalar multiplication of morphisms and trivial extensions are defined the same way in $\mathcal{C}$ as in $\mathcal{D}$, so also is scalar multiplication of extensions.

Let ξ, χ be two extensions. To compute the Baer sum $\xi \oplus \chi$, we are asked to compute a certain pullback, to compute a pair of unique maps pushing through a pullback, to compute a certain cokernel of one of these maps, to compute a unique map pushing through a cokernel, and finally to compute a certain composition. Lemma

5.0.2, as well as the proof of lemma 5.0.1, show that all of these constructions lead to the same answer whether done in $\mathcal{C}$ and $\mathcal{D}$. We conclude that if $\xi \oplus \chi = \eta$ in $\mathcal{C}$, so also does this equation hold in $\mathcal{D}$. This completes the proof. $\qquad\square$

Lemma 5.0.5. *A full tannakian subcategory of a neutral tannakian category is also neutral (over the same field and via the restriction of the same fibre functor).*

Proof. We are given a neutral tannakian category $\mathcal{D}$ with fibre functor $\omega : \mathcal{D} \to \mathrm{Vec}_k$, where k is the field $\mathrm{End}(\underline{1})$. We want to show that this functor restricted to $\mathcal{C}$, which we still call ω, qualifies as a fibre functor on $\mathcal{C}$. Looking at the conditions of definition 3.2.5 it is easy to verify that ω restricted to $\mathcal{C}$ is still a tensor functor. The requisite isomorphism $c_{X,Y} : \omega(A \otimes B) \simeq \omega(A) \otimes \omega(B)$ still exists, is natural, and still satisfies the relevant diagrams, since e.g. assoc in $\mathcal{C}$ is the same as assoc in $\mathcal{D}$. Faithfulness and k-linearity are also clearly still satisfied; all that remains to check is exactness. By proposition 5.0.1, $\mathcal{C}$ and $\mathcal{D}$ satisfy the hypothesis of lemma 5.0.3, and thus exact sequences in $\mathcal{C}$ are also in $\mathcal{D}$. As ω preserves exact sequences in $\mathcal{D}$, so must it also when restricted to $\mathcal{C}$. $\qquad\square$

Chapter 6

Some Ultraproduct Constructions

In the next chapter we shall be studying ultraproducts of tannakian categories. In this chapter we define several ultraproduct constructions and record several results on them that will soon be necessary; the learned reader may wish to treat this chapter merely as a reference. The reader may also consult the appendix for a review of ultrafilters and ultraproducts in general.

In this dissertation, if M_i is a collection of relational structures in a common first-order signature, indexed by I, and if $\mathcal{U}$ is a non-principal ultrafilter on I, we denote by $\prod_{\mathcal{U}} M_i$ the ultraproduct of those structures with respect to $\mathcal{U}$. For a tuple of elements (x_i) from the M_i, we denote by $[x_i]$ its equivalence class, that is, its image as an element of $\prod_{\mathcal{U}} M_i$. When we make statements like "the ultraproduct of vector spaces is a vector space over the ultraproduct of the fields", it will always be the case that these ultraproducts, both for fields and vector spaces, are being taken with respect to the same fixed ultrafilter (as indeed it makes no sense to assume otherwise).

6.1 Fields

Let k_i be a sequence of fields indexed by I. We treat the k_i as structures in the language $+, *, -, 0, 1$ with the obvious interpretation. For brevity we write the term

$x * y$ as the juxtaposition xy. As always we fix a non-principal ultrafilter $\mathcal{U}$ on I throughout.

Proposition 6.1.1. $\prod_{\mathcal{U}} k_i$ *is a field.*

Proof. Simply realize that the axioms for a field are first-order sentences in this language:

1. $(\forall x, y, z)((x + y) + z = x + (y + z) \wedge x(yz) = (xy)z)$

2. $(\forall x, y)(x + y = y + x \wedge xy = yx)$

3. $(\forall x)(1x = x \wedge 0 + x = x)$

4. $(\forall x)(x + (-x) = 0)$

5. $(\forall x, y, z)(x(y + z) = xy + xz)$

6. $(\forall x)(\neg(x = 0) \implies (\exists y)(xy = 1))$

7. $\neg(1 = 0)$

Now apply corollary C.0.16.

$\square$

Proposition 6.1.2. *Suppose that k_i is a sequence of fields of strictly increasing positive characteristic. Then $\prod_{\mathcal{U}} k_i$ has characteristic zero.*

Proof. For a fixed prime p, let char$_p$ be the statement $1 + 1 + \ldots + 1 = 0$ (p-occurrences of 1). As the k_i have strictly increasing characteristic, for fixed p, char$_p$ is false in all but finitely many of them. Thus $\neg$char$_p$ holds on a cofinite set, which is always large, and so $\neg$char$_p$ holds in the ultraproduct. This goes for every p, which is equivalent to $\prod_{\mathcal{U}} k_i$ having characteristic zero. $\square$

6.2   Vector Spaces

Let V_i be an indexed collection of vector spaces over the fields k_i. We treat the V_i simply as structures in the signature $+, 0$, i.e. as abelian groups, forgetting for the moment the scalar multiplication. Then $\prod_{\mathcal{U}} V_i$ is also an abelian group in this signature, and the addition is of course given by $[v_i] + [w_i] \overset{\text{def}}{=} [v_i + w_i]$.

Let $k = \prod_{\mathcal{U}} k_i$ be the ultraproduct of the fields k_i as in the previous section. We assume as always that both of these ultraproducts are being taken with respect to the same fixed non-principal ultrafilter.

Theorem 6.2.1. *$\prod_{\mathcal{U}} V_i$ is a vector space over $\prod_{\mathcal{U}} k_i$, under the scalar multiplication*

$$[a_i][v_i] \overset{\text{def}}{=} [a_i v_i]$$

Proof. The given multiplication is well-defined: If $(a_i), (b_i)$ are equal on the large set J, and if $(v_i), (w_i)$ are equal on the large set K, then $(a_i v_i)$ and $(b_i w_i)$ are equal on at least the large set $J \cap K$. It is routine to verify that this definition satisfies the axioms of a vector space. $\square$

Proposition 6.2.2. *The finite collection of linear equations of the form*

$$[a_i]_1 [v_i]_1 + \ldots + [a_i]_n [v_i]_n = [0]$$

is true in $\prod_{\mathcal{U}} V_i$ if and only if the corresponding collection of linear equations

$$a_{i,1} v_{i,1} + \ldots + a_{i,n} v_{i,n} = 0$$

is true for almost every i.

Proof. For the forward implication, the claim is obvious in the case of a single equation, since the first equation is equivalent to $[a_{i,1} v_{i,1} + \ldots + a_{i,n} v_{i,n}] = [0]$. For a

finite set of equations, each individual equation holds on a large set, and taking the finite intersection of these large sets, we see that there is a large set on which all the equations hold. The reverse implication is obvious. $\qquad\square$

Proposition 6.2.3. *A finite set of vectors $[e_i]_1, \ldots, [e_i]_n$ is a basis for $\prod_{\mathcal{U}} V_i$ over $\prod_{\mathcal{U}} k_i$ if and only if, for almost every i, the set of vectors $e_{i,1}, \ldots, e_{i,n}$ is a basis for V_i over k_i.*

Proof. Given a linear dependence $[a_i]_1[e_i]_1 + \ldots + [a_i]_n[e_i]_n = [0]$ in $\prod_{\mathcal{U}} V_i$, we get a linear dependence for almost every i, by the previous proposition. If $[a_i]_j \neq [0]$, then $a_{i,j} \neq 0$ for almost every i, and by taking another intersection we get a non-trivial dependence in almost every i. Conversely, if we have a non-trivial dependence $a_{i,1}e_{i,1} + \ldots + a_{i,n}e_{i,n} = 0$ in almost every i, the equation $[a_i]_1[e_i]_1 + \ldots + [a_i]_n[e_i]_n = [0]$ holds in $\prod_{\mathcal{U}} V_i$. By lemma B.0.9, at least one of the $[a_i]_j$ must be non-zero.

If $e_{i,j}$ span V_i for almost every i, then for every $[v_i] \in \prod_{\mathcal{U}} V_i$, we have an almost everywhere valid equation $a_{i,1}e_{i,1} + \ldots + a_{i,n}e_{i,n} = v_i$ in V_i, which projects to an equation $[a_i]_1[e_i]_1 + \ldots + [a_i]_n[e_i]_n = [v_i]$, showing that the $[e_i]_j$ span $\prod_{\mathcal{U}} V_i$. Conversely, if the $e_{i,j}$, almost everywhere, do not span V_i, choose v_i for each of those slots which are not in the span of the $e_{i,j}$. Then neither can $[v_i]$ be in the span of the $[e_i]_j$, lest we project back to an almost everywhere linear combination for v_i in terms of the $e_{i,j}$. $\qquad\square$

Proposition 6.2.4. *For a fixed non-negative integer n, $\prod_{\mathcal{U}} V_i$ has dimension n over $\prod_{\mathcal{U}} k_i$ if and only if almost every V_i has dimension n over k_i. $\prod_{\mathcal{U}} V_i$ is infinite dimensional over $\prod_{\mathcal{U}} k_i$ if and only if, for every n, almost every V_i does not have dimension n over k_i.*

Proof. Apply the previous proposition. $\qquad\square$

We see then that if V_i is almost everywhere of dimension $n < \infty$, it does no harm to assume that it has dimension n everywhere. We call such collections **constantly**

finite dimensional or sometimes **boundedly finite dimensional**. With few exceptions it is these types of collections of vector spaces we will be concerning ourselves with.

6.2.1 Linear Transformations and Matrices

For a collection of vector spaces V_i, W_i and linear maps $\phi_i : V_i \to W_i$, denote by $[\phi_i]$ the linear map $\prod_{\mathcal{U}} V_i \to \prod_{\mathcal{U}} W_i$ defined by

$$[\phi_i]([v_i]) = [\phi_i(v_i)] \tag{6.2.1}$$

Denote by $\prod_{\mathcal{U}} \mathrm{Hom}_{k_i}(V_i, W_i)$ the collection of all such transformations of the form $[\phi_i]$. So long as the V_i are constantly finite dimensional, we are justified in using this notation because

Proposition 6.2.5. *If the V_i are of constant finite dimension, $[\phi_i] = [\psi_i]$ as linear transformations if and only if, for almost every i, $\phi_i = \psi_i$ as linear transformations. Further, $[\phi_i] \circ [\psi_i] = [\phi_i \circ \psi_i]$, $[\phi_i] + [\psi_i] = [\phi_i + \psi_i]$, and for an element $[a_i]$ of $\prod_{\mathcal{U}} k_i$, $[a_i][\phi_i] = [a_i \phi_i]$.*

Proof. The 'if' direction is obvious. For the converse, let $[e_i]_1, \ldots, [e_i]_n$ be a basis for $\prod_{\mathcal{U}} V_i$, whence, for almost every i, $e_{i,1}, \ldots, e_{i,n}$ is a basis for V_i. $[\phi_i] = [\psi_i]$ if and only if they agree on this basis, so let $J_m, m = 1 \ldots n$ be the large set on which $\phi_i(e_i^m) = \psi_i(e_i^m)$. Then the finite intersection of these J_m, on which ϕ_i and ψ_i agree on every basis element, thus on which $\phi_i = \psi_i$, is large. The last three claims of the proposition are now obvious. $\square$

If the V_i are of unbounded dimensionality, the theorem does not hold; the proof falls apart when we try to take the intersection of the J_m, which in this case may well be an infinite intersection, and not guaranteed to be large.

Proposition 6.2.6. *If V_i, W_i are constantly finite dimensional collections of vector spaces, then*

$$Hom_{\prod_\mathcal{U} k_i}(\prod_\mathcal{U} V_i, \prod_\mathcal{U} W_i) \simeq \prod_\mathcal{U} Hom_{k_i}(V_i, W_i)$$

Proof. It is easy to verify that the right hand side of the above claimed isomorphism is always included in the left hand side (even if the V_i or W_i are not boundedly finite dimensional). For the other inclusion, pick bases $[e_i^1], \ldots, [e_i^n]$ and $[f_i^1], \ldots, [f_i^m]$ of $\prod_\mathcal{U} V_i$ and $\prod_\mathcal{U} W_i$ respectively. For a linear transformation ϕ in the left hand side of the claimed isomorphism, write it as an $n \times m$ matrix

$$\begin{pmatrix} [a_i^{1,1}] & \cdots & [a_i^{1,n}] \\ \vdots & & \vdots \\ [a_i^{m,1}] & \cdots & [a_i^{m,n}] \end{pmatrix}$$

in the given bases. Then one can verify by hand that ϕ is of the form $[\phi_i]$, where each ϕ_i is the transformation $V_i \to W_i$ given by the matrix

$$\begin{pmatrix} a_i^{1,1} & \cdots & a_i^{1,n} \\ \vdots & & \vdots \\ a_i^{m,1} & \cdots & a_i^{n,m} \end{pmatrix}$$

in the bases $e_i^1, \ldots, e_i^n$, $f_i^1, \ldots, f_i^n$. $\qquad\square$

We can therefore always assume that a linear transformation $\prod_\mathcal{U} V_i \to \prod_\mathcal{U} W_i$ is uniquely of the form $[\phi_i]$, so long as the V_i and W_i are of constant finite dimension. This theorem is not true if the V_i and W_i are not both boundedly finite dimensional; the forward inclusion fails.

Definition 6.2.1. Let M_i be a sequence of $n \times m$ matrices over the fields k_i, given

by

$$\begin{pmatrix} a_i^{1,1} & \dots & a_i^{1,n} \\ \vdots & & \vdots \\ a_i^{m,1} & \dots & a_i^{m,n} \end{pmatrix}$$

Then we define the ultraproduct of these matrices, denoted $[M_i]$, to be the $n \times m$ matrix over the field $\prod_{\mathcal{U}} k_i$ given by

$$\begin{pmatrix} [a_i^{1,1}] & \dots & [a_i^{1,n}] \\ \vdots & & \vdots \\ [a_i^{m,1}] & \dots & [a_i^{m,n}] \end{pmatrix}$$

Proposition 6.2.7. *Let V_i, W_i have constant dimension n and m, with bases $e_i^1, \dots, e_i^n$, $f_i^1, \dots, f_i^m$ respectively. If $\phi_i : V_i \to W_i$ is represented by the $n \times m$ matrix M_i in the given bases, then $[\phi_i]$ is represented by the matrix $[M_i]$ in the bases $[e_i^1], \dots, [e_i^n], [f_i^1], \dots, [f_i^m]$.*

Proof. Obvious. □

Applying propositions 6.2.6 and 6.2.7 together yield

Corollary 6.2.8. *For integers m and n and fields k_i,*

$$Mat_{n,m}(\prod_{\mathcal{U}} k_i) \simeq \prod_{\mathcal{U}} Mat_{n,m}(k_i)$$

where the latter stands for all matrices of the form $[M_i]$, $M_i \in Mat_{n,m}(k_i)$.

As we have seen, an ultraproduct of linear transformations preserves composition, addition, and scalar multiplication. By induction on complexity, it thus also preserves any equation involving a finite combination of these three operations, and by considering a finite intersection of large sets, the same is true for any finite collection of such equations. We state this as a theorem.

Theorem 6.2.9. *Let $[\phi_i]_1, \ldots, [\phi_i]_n$ be a finite collection of linear transformations, all between ultraproducts of constantly finite dimensional vector spaces. Then a finite collection of equations among the $[\phi_i]_j$ involving addition of maps, composition, and scalar multiplication is valid if and only if the corresponding collection of equations among the $\phi_{i,1}, \ldots, \phi_{i,n}$ is valid almost everywhere.*

By proposition 6.2.7, the same is true for matrices:

Corollary 6.2.10. *The same is true for a finite collection $[M_i]_1, \ldots, [M_i]_k$ of matrices over $\prod_{\mathcal{U}} k_i$, if we replace 'addition of maps', 'composition', and 'scalar multiplication of maps' with 'addition of matrices', 'multiplication of matrices', and 'scalar multiplication of matrices'.*

Proposition 6.2.11. *Over collections of constantly finite dimensional vector spaces, ultraproducts preserve injectivity, surjectivity, kernels and cokernels.*

Proof. Let $V_i \xrightarrow{\phi_i} W_i$ be a collection of linear maps. Suppose first that almost every ϕ_i is injective. Then if $[v_i] \neq [0]$, $v_i \neq 0$ for almost every i, and taking the intersection of these two large sets, $\phi_i(v_i) \neq 0$ for almost every i, which is the same as saying $[\phi_i]([v_i]) \neq [0]$. Thus $[\phi_i]$ is injective.

For the converse, we must use the constant finite dimensionality of the V_i. Let $[e_i]_1, \ldots, [e_i]_n$ be a basis for $\prod_{\mathcal{U}} V_i$, so that $e_{i,1}, \ldots, e_{i,n}$ is a basis for almost every V_i, say on the large set J. Suppose that almost every ϕ_i is not injective, say on the large set K, and for each $i \in K$ let $v_i = a_{i,1}e_{i,1} + \ldots + a_{i,n}e_{i,n}$ be a non-zero vector such that $\phi_i(v_i) = 0$. Then at least one of the $a_{i,j}$ is non-zero for each i. By lemma B.0.9, at least one of $[a_i]_j$ is non-zero in $\prod_{\mathcal{U}} k_i$. Then we see that $[\phi_i]([a_i]_1[e_i]_1 + \ldots [a_i]_n[e_i]_n) = [0]$, but $[a_i]_m \neq 0$; hence $[\phi_i]$ is not injective.

The proof of surjectivity is similarly proved, using instead the constant dimensionality of the W_i.

Suppose that, for almost every i, ϕ_i is a kernel map for ψ_i. This is the assertion that ϕ_i is injective, that everything in the image of ϕ_i is killed by ψ_i, and that nothing outside the image of ϕ_i is killed by ψ_i. That $[\phi_i]$ is also injective has already been proved. To say that $[v_i]$ is in the image of $[\phi_i]$ is equivalent to saying that v_i is in the image of ϕ_i for almost every i, and to say that $[\psi_i]$ kills $[w_i]$ is equivalent to saying that ψ_i kills w_i for almost every i; the same goes for their negations.

The case of cokernels is proved similarly, using instead the fact that surjectivity is preserved. $\qquad\square$

Proposition 6.2.12. *Over collections of constantly finite dimensional vector spaces, the collection of diagrams*

$$0 \to X_i^1 \xrightarrow{\phi_i^1} \ldots \xrightarrow{\phi_i^n} X_i^{n+1} \to 0$$

is almost everywhere exact if and only if the corresponding sequence

$$[0] \to [X_i]_1 \xrightarrow{[\phi_i]_1} \ldots \xrightarrow{[\phi_i]_n} [X_i]_{n+1} \to [0]$$

is exact.

Proof. The assertion that the sequence

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$$

is exact amounts to the assertion that ϕ is a kernel for ψ, minus the requirement that ϕ be injective, which (the proof of) proposition 6.2.11 shows to be preserved by ultraproducts. Checking that the above sequences are exact amounts to checking finitely many sub-sequences of this form, which is preserved by ultraproducts. $\quad\square$

6.2.2 Tensor Products

Proposition 6.2.13. *Let V_i, W_i be (not necessarily boundedly finite dimensional) collections of vector spaces. Then there is a natural injective map*

$$\prod_{\mathcal{U}} V_i \otimes \prod_{\mathcal{U}} W_i \xrightarrow{\ \Phi\ } \prod_{\mathcal{U}} V_i \otimes W_i$$

given by $[v_i] \otimes [w_i] \mapsto [v_i \otimes w_i]$. The image of Φ consists exactly of those elements having bounded tensor length. The map Φ is an isomorphism if and only if at least one of the collections V_i or W_i are of bounded finite dimension.

Proof. That the map Φ is well-defined is easy to verify, remembering of course that the same ultrafilter applies to all ultraproducts under consideration. Injectivity is likewise easy to verify. Any element on the left hand side, by the very definition of tensor product, has bounded tensor length, and hence so must its image on the right hand side. Conversely, if $[\sum_{j=1}^{n} v_{ij} \otimes w_{ij}]$ is an element of bounded tensor length on the right hand side, then $\sum_{j=1}^{n} [v_i]_j \otimes [w_i]_j$ is a pre-image for it on the left.

To prove the isomorphism claim: if V and W are vector spaces of finite dimension n and m respectively, then the maximum tensor length of any element of $V \otimes W$ is $\min(n, m)$ (see lemma 9.1.4). Then if say V_i is of bounded finite dimension n, any $[x_i] \in \prod_{\mathcal{U}} V_i \otimes W_i$ is almost everywhere a sum of no more than n simple tensors, and is in the image of Φ.

Conversely, suppose neither of V_i or W_i are of bounded dimension. For each i, choose $x_i \in V_i \otimes W_i$ such that x_i is of maximum possible tensor length; we claim that $[x_i] \in \prod_{\mathcal{U}} V_i \otimes W_i$ is of unbounded tensor length, hence not in the image of Φ. If not, say the tensor length of x_i is almost everywhere bounded by n. This gives a large set on which the statement "at least one of V_i or W_i has dimension n" is true. This large set is covered by the union of $\{i \in I : V_i \text{ has dimension } n\}$ and $\{i \in I : W_i \text{ has dimension } n\}$, so by lemma B.0.9, at least one of them must be large.

This gives a large set on which at least one of V_i or W_i is of bounded dimension, a contradiction. $\qquad\square$

This next proposition justifies our calling the map Φ 'natural'.

Proposition 6.2.14. *If V_i, W_i, X_i, Y_i are collection of vector spaces, and $\phi_i : V_i \to X_i$, $\psi_i : W_i \to Y_i$ linear maps, then the following commutes:*

$$
\begin{array}{ccc}
\prod_{\mathcal{U}} V_i \otimes \prod_{\mathcal{U}} W_i & \overset{\Phi}{\longrightarrow} & \prod_{\mathcal{U}} V_i \otimes W_i \\
{\scriptstyle [\phi_i] \otimes [\psi_i]} \downarrow & & \downarrow {\scriptstyle [\phi_i \otimes \psi_i]} \\
\prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} Y_i & \underset{\Phi}{\longrightarrow} & \prod_{\mathcal{U}} X_i \otimes Y_i
\end{array}
\tag{6.2.2}
$$

Proof. Let $[v_i] \otimes [w_i]$ be a simple tensor in $\prod_{\mathcal{U}} V_i \otimes \prod_{\mathcal{U}} W_i$. Chasing it both ways gives the same result:

$$
\begin{array}{ccc}
[v_i] \otimes [w_i] & \overset{\Phi}{\longrightarrow} & [v_i \otimes w_i] \\
{\scriptstyle [\phi_i] \otimes [\psi_i]} \downarrow & & \downarrow {\scriptstyle [\phi_i \otimes w_i]} \\
[\phi_i(v_i)] \otimes [\psi_i(w_i)] & \underset{\Phi}{\longrightarrow} & [\phi(v_i) \otimes \psi_i(w_i)]
\end{array}
$$

$\qquad\square$

6.3 Algebras and Coalgebras

If (L_i, mult_i) is a collection of algebras over the fields k_i then it is easy to verify that $\prod_{\mathcal{U}} L_i$ is an algebra over the field $\prod_{\mathcal{U}} k_i$, under the obvious definitions of addition, multiplication, and scalar multiplication. The multiplication on $\prod_{\mathcal{U}} L_i$ is in particular defined as the composition

$$
\mathrm{mult} : \prod_{\mathcal{U}} L_i \otimes \prod_{\mathcal{U}} L_i \overset{\Phi}{\longrightarrow} \prod_{\mathcal{U}} L_i \otimes L_i \overset{[\mathrm{mult}_i]}{\longrightarrow} \prod_{\mathcal{U}} L_i
$$

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Alas, for coalgebras, things are not so easy. Here is what can go wrong. Suppose $(C_i, \Delta_i, \varepsilon_i)$ is a collection of coalgebras over the fields k_i. Then $\prod_{\mathcal{U}} C_i$ is at least a vector space over $\prod_{\mathcal{U}} k_i$. Now let us try to define a co-multiplication map Δ on $\prod_{\mathcal{U}} C_i$. We start by writing

$$\Delta : \prod_{\mathcal{U}} C_i \xrightarrow{[\Delta_i]} \prod_{\mathcal{U}} C_i \otimes C_i$$

But as it stands, this won't suffice; we need Δ to point to $\prod_{\mathcal{U}} C_i \otimes \prod_{\mathcal{U}} C_i$. As is shown in proposition 6.2.13, unless the C_i are of boundedly finite dimension, we only have an inclusion $\prod_{\mathcal{U}} C_i \otimes \prod_{\mathcal{U}} C_i \xrightarrow{\Phi} \prod_{\mathcal{U}} C_i \otimes C_i$, whose image consists of those elements of bounded tensor length, and for a typical collection C_i of coalgebras it is usually a simple matter to come up with an element $[c_i] \in \prod_{\mathcal{U}} C_i$ such that $[\Delta_i(c_i)]$ has unbounded tensor length. Thus, the Δ constructed above cannot be expected to point to $\prod_{\mathcal{U}} C_i \otimes \prod_{\mathcal{U}} C_i$ in general (this problem is dealt with at length in section 9.1).

Nonetheless, if the C_i *are* of boundedly finite dimension, then the map Φ is an isomorphism, whence we can define

$$\Delta : \prod_{\mathcal{U}} C_i \xrightarrow{[\Delta_i]} \prod_{\mathcal{U}} C_i \otimes C_i \xrightarrow{\Phi^{-1}} \prod_{\mathcal{U}} C_i \otimes \prod_{\mathcal{U}} C_i$$

Likewise, we define a co-unit map by

$$\varepsilon : \prod_{\mathcal{U}} C_i \xrightarrow{[\varepsilon_i]} \prod_{\mathcal{U}} k_i$$

and we have

Proposition 6.3.1. *If $(C_i, \Delta_i, \varepsilon_i)$ is a collection of boundedly finite dimensional coalgebras over the fields k_i, then $\prod_{\mathcal{U}} C_i$ is a coalgebra over the field $\prod_{\mathcal{U}} k_i$, under the definitions of Δ and ε given above.*

Proof. We must verify diagrams 2.1.1 and 2.1.2 of definition 2.1.2. Consider

$$
\begin{array}{ccccc}
& & \prod_{\mathcal{U}} C_i & & \\
\end{array}
$$

Commutativity of the top middle triangle follows from the everywhere commutativity of it, which is diagram 2.1.2 applied to each C_i. The rest of the subpolygons are easy to verify, whence we have commutativity of the outermost, which is diagram 2.1.2. Diagram 2.1.1 can be proved in a similar fashion. □

Proposition 6.3.2. *Let L_i be a collection of boundedly finite dimensional algebras over the fields k_i. Then there is a natural isomorphism of coalgebras*

$$
\prod_{\mathcal{U}} L_i^\circ \simeq \left(\prod_{\mathcal{U}} L_i \right)^\circ
$$

which sends the tuple of functionals $[\phi_i : L_i \to k_i]$ on the left to the functional $[\phi_i] : \prod_{\mathcal{U}} L_i \to \prod_{\mathcal{U}} k_i$ on the right.

Proof. Call the claimed isomorphism Ψ. That it is an isomorphism of vector spaces is clear from proposition 6.2.6. To see that it is a map of coalgebras we must verify

commutativity of

$$\prod_{\mathcal{U}} L_i^\circ \xrightarrow{\ \ \Psi\ \ } \left(\prod_{\mathcal{U}} L_i\right)^\circ$$

$$\downarrow {\scriptstyle [\Delta_i]} \qquad\qquad\qquad\qquad \downarrow {\scriptstyle \Delta}$$

$$\prod_{\mathcal{U}} L_i^\circ \otimes L_i^\circ$$

$$\downarrow {\scriptstyle \Phi^{-1}}$$

$$\prod_{\mathcal{U}} L_i^\circ \otimes \prod_{\mathcal{U}} L_i^\circ \xrightarrow[\Psi\otimes\Psi]{} \left(\prod_{\mathcal{U}} L_i\right)^\circ \otimes \left(\prod_{\mathcal{U}} L_i\right)^\circ$$

where Δ_i denotes the coalgebra structure on L_i° and Δ that on $(\prod_{\mathcal{U}} L_i)^\circ$. Let mult_i be the multiplication on the algebra L_i and mult that on $\prod_{\mathcal{U}} L_i$, so by definition $\mathrm{mult} = \Phi \circ [\mathrm{mult}_i]$.

Let $[\alpha_i : L_i \to k_i]$ be an arbitrary element of $\prod_{\mathcal{U}} L_i^\circ$ and let us chase it both ways. Working downward first, we ask how $\Phi^{-1}([\Delta_i]([\alpha_i]))$ acts on $\prod_{\mathcal{U}} L_i \otimes \prod_{\mathcal{U}} L_i$; it does so by the composition

$$\prod_{\mathcal{U}} L_i \otimes \prod_{\mathcal{U}} L_i \xrightarrow{\ \Phi\ } \prod_{\mathcal{U}} L_i \otimes L_i \xrightarrow{[\mathrm{mult}_i]} \prod_{\mathcal{U}} L_i \xrightarrow{[\alpha_i]} \prod_{\mathcal{U}} k_i$$

Next we ask how $\Delta(\Psi([\alpha_i]))$ acts on $\prod_{\mathcal{U}} L_i \otimes \prod_{\mathcal{U}} L_i$. It does so by the composition

$$\prod_{\mathcal{U}} L_i \otimes \prod_{\mathcal{U}} L_i \xrightarrow{\ \mathrm{mult}\ } \prod_{\mathcal{U}} L_i \xrightarrow{[\alpha_i]} \prod_{\mathcal{U}} k_i$$

But mult is *defined* to be $\Phi \circ [\mathrm{mult}_i]$, and so these actions are equal. This completes the proof. $\qquad\square$

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Chapter 7

The Restricted Ultraproduct of Neutral Tannakian Categories

In this chapter we prove one of the main theorems of this dissertation, namely that a certain natural subcategory of an ultraproduct of neutral tannakian categories is also neutral tannakian.

7.1 Smallness of the Category $\mathrm{Rep}_k G$

Before beginning in earnest, we pause in this section to address a subtle but important point. If one wishes to consider an ultraproduct of a collection of 'things', those things must be sets; in particular, they must be relational structures. Thus, if one wishes to consider the ultraproduct of a collection of categories, then those categories must be small categories, and the necessary (abelian, tensor, etc.) structure on the categories must be realized as actual relations and functions on that set. This forces the question: for an affine group scheme G and field k, can $\mathrm{Rep}_k G$ be taken to be small, up to tensorial equivalence?

This question is not fully addressed in this dissertation, but we shall at least give here some arguments that lead us to believe that this is a fair assumption. In

particular, we shall argue why we believe that the category Vec_k can be taken to be small, up to tensorial equivalence. Similar arguments we believe should apply to the category $\mathrm{Rep}_k G$ for arbitrary k and G.

For the remainder of this section, we shall use the term *small* and *tensorially small* in an abusive sense; the category $\mathcal{C}$ shall be said to be *small* if it is equivalent to a small category (even though she itself may not be), and the tensor category $\mathcal{C}$ shall be said to be *tensorially small* if there is a tensor preserving equivalence between $\mathcal{C}$ and a small tensor category (see definitions 3.2.2 and 3.2.5).

Consider the category Vec_k of finite dimensional vector spaces over a field k, which can be identified as $\mathrm{Rep}_k G_0$, the category of finite dimensional representations of the trivial group G_0. Denote further by VEC_k the category of *all* vector spaces over k (finite dimensional or not). We observe first that the category VEC_k should by no means be assumed to be small. Even her skeleton would consist of objects of every possible dimension over k, and hence of sets of every possible cardinality. If this skeleton were indeed a set, we could take the union of all objects contained in that set, and therefore arrive at a set of cardinality greater than that of any other set. This is anathema according to the basic tenets of set theory.

But the category Vec_k is indeed small. To see this, we shall follow page 93 of [14] in observing that any category is equivalent (though not necessarily *tensorially* equivalent) to its skeleton. To realize the skeleton of Vec_k as a small category, for each n we take the set $V_n = k^n$, i.e. the collection of all formal linear combinations of k over the set $\{1, 2, \ldots, n\}$ with the obvious k-vector space structure. Likewise define $\mathrm{Hom}(V_n, V_m)$ to be the set of all functions from V_n to V_m which qualify as k-linear maps under the given vector space structures. Then the V_n and $\mathrm{Hom}(V_n, V_m)$, themselves being a collection of sets indexed by the sets $\mathbb{N}$ and $\mathbb{N}^2$, can indeed be collected into a single set. Thus Vec_k is a small category.

What is far less obvious is that Vec_k is *tensorially* small. To illustrate the problem,

we again direct the reader to page 164 of [14], where the author shows that the skeleton of the category of sets cannot be given the structure of a tensor category (in this case defined as the usual cartesian product of sets). This is the reason, after all, that we bother with the assoc, comm, and unit isomorphisms in a tannakian category; demanding, for example, that $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ (strict equality) is simply too stringent. For similar reasons we do not believe that it suffices to endow the skeleton of Vec_k with the structure of a tensor category in the usual sense.

Here is what we believe is a possible approach to remedying this. Denote by $\mathcal{C}_0$ the skeleton of Vec_k as defined above, and for each $n \in \mathbb{N}$, denote by $\mathcal{C}_n$ the following category. The objects of $\mathcal{C}_n$ are the objects of $\mathcal{C}_{n-1}$, along with all pairwise tensor products of objects in $\mathcal{C}_{n-1}$ (via whatever standard construction one likes, e.g. as a certain quotient of the free vector space on $V \times W$; see section 1.7 of [9]). For objects $V, W \in \mathcal{C}_n$, we let $\mathrm{Hom}_{\mathcal{C}_n}(V, W)$ be $\mathrm{Hom}_{\mathcal{C}_{n-1}}(V, W)$ if V and W are in $\mathcal{C}_{n-1}$, and if not, as the collection of all functions from V to W which qualify as k-linear maps. Finally, we define $\mathcal{C}$ to be the union of the categories $\mathcal{C}_n$ for $n = 0, 1, 2, \ldots$.

Note that, since $\mathcal{C}_0$ already contains an isomorphic copy of every finite dimensional vector space over k, $\mathcal{C}$ contains no new objects up to isomorphism. The whole point of bothering with these new objects is so as not to encounter any paradoxes similar to that described on page 164 of [14]. Proving rigorously that this category satasfies the axioms of a tannakian category would no doubt require significant effort, but we believe that it could be done.

We would define the primitive relations of the language of abelian tensor categories on this structure in the obvious manner. For instance, the relation $\phi \circ \psi \doteq \eta$ would hold precisely when η is the composition of ϕ and ψ in the usual sense, and similarly for $\phi + \psi \doteq \eta$. Importantly, we would define the relation $X \otimes Y \doteq Z$ to hold when Z is the unique object of $\mathcal{C}_{n+1}$ such that $X \otimes Y = Z$, where n is the least integer such that X and Y are both objects of $\mathcal{C}_n$. The relation $\mathrm{assoc}_{X,Y,Z} \doteq \phi$ would hold when

ϕ is the unique map $(X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ such that ϕ sends $(x \otimes y) \otimes z$ to $x \otimes (y \otimes z)$, and similarly for $\mathrm{comm}_{X,Y} \doteq \phi$. As for the unit relation, denote by $\underline{1}$ the unique 1-dimensional vector space in the skeleton $\mathcal{C}_0$, and define $\mathrm{unit}_X \doteq \phi$ to hold when ϕ is the unique map $X \to \underline{1} \otimes X$ such that $\phi : x \mapsto 1 \otimes x$.

Given that one could verify that this category $\mathcal{C}$ satisfies the axioms of a tannakian category, showing it to be tensorially equivalent to the usual Vec_k, and hence small, should be straightforward; simply define $F : \mathcal{C} \to \mathrm{Vec}_k$ to be the inclusion functor. This functor is clearly full, faithful, and essentially surjective, hence an equivalence. Showing F to be tensor preserving (see definition 3.2.5) would likewise be straight-forward. Finally, apply definition 1.10 and proposition 1.11 of [5] to see that if F is an equivalence, and if it is a tensor functor, then it is also a *tensor equivalence*, in the sense that its inverse can also be taken to be tensor preserving. Thus F is an equivalence of abelian tensor categories, and Vec_k is a tensorially small category.

7.1.1   A Quotient Category Approach

Here we mention briefly a possible alternative to the ultraproduct approach taken in this dissertation, one which replaces the ultraproduct with a certain quotient category.

Let G_i be a collection of affine group schemes over the fields k_i, and let $\mathcal{C}_i$ be the category $\mathrm{Rep}_{k_i} G_i$. Denote by $\prod_{i \in I} \mathcal{C}_i$ the *product* of the categories $\mathcal{C}_i$; that is, the category whose objects are all possible tuples of objects $(X_i : i \in I)$, and whose morphisms are all possible tuples of morphisms $(\phi_i : i \in I)$, with the obvious definitions of morphism composition, addition of morphisms, tensor product of objects, etc. Fix a non-principal ultrafilter $\mathcal{U}$ on I, and for objects $(X_i), (Y_i) \in \prod_{i \in I} \mathcal{C}_i$, define the following congruence relation $\sim$ on $\mathrm{Hom}((X_i), (Y_i))$: $(\phi_i) \sim (\psi_i)$ if and only if the subset of I on which $\phi_i = \psi_i$ is large. Note that, if $(\phi_i), (\psi_i) : (X_i) \to (Y_i)$ with $(\phi_i) \sim (\psi_i)$, and if $(\rho_i), (\mu_i) : (Y_i) \to (Z_i)$ with $(\rho_i) \sim (\mu_i)$, then $(\phi_i \circ \rho_i) \sim (\psi_i \circ \mu_i)$, as

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the intersection of two large sets is also large (see definition B.0.1). Thus $\sim$ is indeed a congruence relation.

Let $\mathcal{C}'$ denote the quotient category of $\prod_{i \in I} \mathcal{C}_i$ with respect to $\sim$. Then under the assumption of the previous section, namely that any $\mathrm{Rep}_k G$ is tensorially equivalent to a small category, we have

Theorem 7.1.1. *The quotient category $\mathcal{C}'$ and the ultraproduct category $\prod_{\mathcal{U}} \mathcal{C}_i$ are equivalent as abelian tensor categories.*

Proof. For each i, denote by $\overline{\mathcal{C}}_i$ the posited small category, and by $F_i : \overline{\mathcal{C}}_i \to \mathcal{C}_i$ the posited tensor equivalence (i.e., the inclusion functor), of the previous section. Then by proposition 1.11 of [5], let G_i be the tensor preserving 'inverse' of F_i. Define a functor $G : \mathcal{C}' \to \prod_{\mathcal{U}} \overline{\mathcal{C}}_i$ as follows: for objects, $G((X_i)) = [G_i(X_i)]$, and for morphisms, $G([\phi_i]) = [G_i(\phi_i)]$. That G is essentially surjective is clear from the essential surjectivity of each G_i, and similarly the fullness and faithfulness of G follows. Thus G is an equivalence.

By hypothesis each G_i is a tensor equivalence, and so comes equipped with a functorial isomorphism $c^i_{X_i,Y_i} : G_i(X_i) \otimes G(Y_i) \xrightarrow{\simeq} G_i(X_i \otimes Y_i)$ for all $X_i, Y_i \in \mathcal{C}_i$. Then define a functorial isomorphism c as follows. For each pair of objects $(X_i), (Y_i)$ in the quotient category $\mathcal{C}'$, define $c_{(X_i),(Y_i)} : [G_i(X_i)] \otimes [G_i(Y_i)] \to [G_i(X_i \otimes Y_i)]$ as the composition

$$c_{(X_i),(Y_i)} : [G_i(X_i)] \otimes [G_i(Y_i)] \xrightarrow{\Phi} [G_i(X_i) \otimes G_i(Y_i)] \xrightarrow{[c^i_{X_i,Y_i}]} [G_i(X_i \otimes Y_i)]$$

where Φ is the natural injection defined in proposition 6.2.13 , and $[c^i_{X_i,Y_i}]$ is the ultraproduct of the linear maps $c^i_{X_i,Y_i}$ (see equation 6.2.1). So equipped with c, we believe G can now be shown to be tensor preserving according to definition 3.2.5. Apply again proposition 1.11 of [5] to see that G is indeed a tensor equivalence. $\square$

The chief disadvantage of ultraproducts, as highlighted, is that the constituent

categories of the ultraproduct must be shown to be small relational structures; the quotient category approach does away with this requirement. On the other hand, the chief advantage of the ultraproduct approach is that one immediately has Łos' theorem, which allows us to pass immediately from first-order statements on the factors to first-order statements in the ultraproduct. In particular, we may conclude immediately that $\prod_{\mathcal{U}}\mathcal{C}_i$ is a tannakian category, simply by virtue of the fact that 'being tannakian' is a first-order sentence in the language of abelian tensor categories. Of course, given the tensorial smallness of $\mathrm{Rep}_k G$, since the ultraproduct and quotient categories are tensorially equivalent, we conclude that a Łos' theorem-type result must indeed hold for the quotient category as well; but this can no longer be assumed, and must be proven, and is the chief disadvantage of the quotient category approach.

7.2   The Restricted Ultraproduct

For the remainder of this dissertation, by a (abelian, tensor, etc.) category, we shall always mean a small category realized as a structure in the language of abelian tensor categories, and by a tannakian category, we shall mean a structure satisfying the axioms given in chapter 4.

Theorem 7.2.1. *Let $\mathcal{C}_i$ be a sequence of tannakian categories indexed by I, $\mathcal{U}$ an ultrafilter on I. Then $\prod_{\mathcal{U}}\mathcal{C}_i$ is a tannakian category.*

Proof. The property of being tannakian, by the work done in chapter 4, is expressible by a first-order sentence in the language of these structures. By Łos's theorem (corollary C.0.16), the same sentence is true in the ultraproduct. □

A word or two about what $\prod_{\mathcal{U}}\mathcal{C}_i$ actually looks like. If $[x_i]$ is an element of $\prod_{\mathcal{U}}\mathcal{C}_i$, then $[x_i]$ is an object or a morphism of $\prod_{\mathcal{U}}\mathcal{C}_i$ according to whether the set on which $x_i \in \mathrm{Ob}$ or $x_i \in \mathrm{Mor}$ is large. The axioms of a category state that exactly one of these

statements hold in every slot, and in an ultrafilter, exactly one of a subset of I or its complement is large. It thus does no harm to think of every element $[x_i]$ of $\prod_{\mathcal{U}} \mathcal{C}_i$ as being represented by a tuple (x_i) consisting either entirely of objects or entirely of morphisms, since it is necessarily equivalent to a tuple (many in fact) of one of these forms.

If $[\phi_i], [\psi_i], [\eta_i]$ are morphisms of $\prod_{\mathcal{U}} \mathcal{C}_i$, then the relation $[\phi_i] \circ [\psi_i] \doteq [\eta_i]$ holds if and only if the relation $\phi_i \circ \psi_i \doteq \eta_i$ hold for almost every i. Similarly the tensor product of the objects $[X_i]$ and $[Y_i]$ is $[Z_i]$, where Z_i denotes the unique object such that $X_i \otimes Y_i \doteq Z_i$. In short, $[\phi_i] \circ [\psi_i] = [\phi_i \circ \psi_i]$, and $[X_i] \otimes [Y_i] = [X_i \otimes Y_i]$.

By Łos's theorem, the same is true for anything that can be expressed as a first-order concept in our language. For example, since 'being internal Hom' is first-order (see page 72) we conclude immediately that an internal Hom object for $[X_i]$ and $[Y_i]$ is necessarily an object $[Z_i]$, where almost every Z_i is an internal Hom object for X_i and Y_i. Importantly, an identity object for $\prod_{\mathcal{U}} \mathcal{C}_i$ is a tuple $[U_i]$ such that U_i is an identity object for almost every i, and an endomorphism of $[U_i]$ is an element $[\phi_i]$ consisting of morphisms which point from U_i to itself almost everywhere.

One should take care however not to be hasty in concluding that a given categorical concept is inherited by $\prod_{\mathcal{U}} \mathcal{C}_i$ from the $\mathcal{C}_i$, if you do not know beforehand that the concept is first-order. The following example illustrates this.

Take $\mathcal{C}_i = \mathrm{Vec}_k$ for a fixed field k, indexed by $I = \mathbb{N}$. For objects A and B of Vec_k, consider the (non first-order) categorical statement "A is isomorphic to an n-fold direct sum of B for some n". Now the following are both first-order: "X is isomorphic to Y" and for *fixed* n, "A is an n-fold direct sum of B". This means we can identify $[X_i]^n$ with $[X_i^n]$, and for fixed n, objects $[Y_i]$ that are isomorphic to $[X_i]^n$ with tuples of objects (Y_i) which are almost everywhere isomorphic to (X_i^n). So let $V_i \in \mathcal{C}_i$ be an i-dimensional vector space, and $W_i \in \mathcal{C}_i$ a 1-dimensional vector space. Then the statement "V is isomorphic to an n-fold direct sum of W for some n" is

true of every V_i and W_i; but the statement is clearly *not* true of the elements $[V_i]$ and $[W_i]$ inside the category $\prod_{\mathcal{U}} \mathcal{C}_i$. This observation in fact *proves* that the statement "A is isomorphic to an n-fold direct sum of B for some n" is not first-order.

In what follows we fix, for each i, an identity object for $\mathcal{C}_i$, and denote it by $\underline{1}_i$. We denote simply by $\underline{1}$ the object $[\underline{1}_i]$ of $\prod_{\mathcal{U}} \mathcal{C}_i$.

Proposition 7.2.2. *Let $\mathcal{C}_i$ be a sequence of tannakian categories, and denote by k_i the field $\mathrm{End}(\underline{1}_i)$. Then $\mathrm{End}(\underline{1})$ can be identified with $k = \prod_{\mathcal{U}} k_i$, the ultraproduct of the fields k_i.*

Proof. As mentioned, $\mathrm{End}(\underline{1})$ consists exactly of those elements $[\phi_i]$ such that ϕ_i is almost everywhere an endomorphism of $\underline{1}_i$. But this is exactly the description of $\prod_{\mathcal{U}} k_i$ (see section 6.1), if we identify $k_i = \mathrm{End}(\underline{1}_i)$. The multiplication and addition in $\mathrm{End}(\underline{1})$ and $\prod_{\mathcal{U}} k_i$ are clearly compatible with this identification, since multiplication is composition of maps, and addition is addition of morphisms. □

Now assume that $\mathcal{C}_i$ is a sequence of *neutral* tannakian categories, and denote by ω_i the fibre functor on each $\mathcal{C}_i$. As mentioned in the introduction, we see no way to endow $\prod_{\mathcal{U}} \mathcal{C}_i$ with a fibre functor, at least not one that is compatible with the each of the ω_i ($\prod_{\mathcal{U}} \mathcal{C}_i$ might thus be an interesting example of a non-neutral tannakian category, but that is not investigated in this dissertation). Instead we look to a certain subcategory of $\prod_{\mathcal{U}} \mathcal{C}_i$.

Definition 7.2.1. For a sequence of neutral tannakian categories $\mathcal{C}_i$, the **restricted ultraproduct** of the $\mathcal{C}_i$, denoted $\prod_R \mathcal{C}_i$, is the full subcategory of $\prod_{\mathcal{U}} \mathcal{C}_i$ consisting of those objects $[X_i]$ such that $\dim(\omega_i(X_i))$ is almost everywhere bounded.

To avoid the use of a double subscript, the notation $\prod_R \mathcal{C}_i$ makes no mention of the particular ultrafilter $\mathcal{U}$ being applied. As $\mathcal{U}$ is always assumed to be fixed but arbitrary, no confusion should result.

If $[X_i]$ has almost everywhere bounded dimension, then we may as well take it to have *everywhere* bounded dimension. And if $[X_i]$ is everywhere bounded, X_i takes on only finitely many values for its dimension; by lemma B.0.10, there is exactly one dimension m such that the set on which $\dim(\omega_i(X_i)) = m$ is large. Thus, it does no harm to think of $\prod_R \mathcal{C}_i$ as the full subcategory consisting of those (equivalence classes of) tuples having *constant* dimension.

Theorem 7.2.3. $\prod_R \mathcal{C}_i$ *is a tannakian subcategory of* $\prod_\mathcal{U} \mathcal{C}_i$.

Proof. By lemma 5.0.1 it is enough to show that $\prod_R \mathcal{C}_i$ is closed under the taking of biproducts, subobjects, quotients, tensor products, duals, and contains the (an) identity object.

Each ω_i is k_i-linear, hence additive, and theorem 3.11 of [7] ensures that ω_i carries direct sums into direct sums, and hence biproducts into biproducts. If $[X_i]$, $[Y_i]$ have constant dimension, then certainly so do the vector spaces $\omega_i(X_i) \oplus \omega_i(Y_i) \simeq \omega_i(X_i \oplus Y_i)$. Thus $\prod_R \mathcal{C}_i$ is closed under the taking of biproducts.

As each ω_i is exact, it certainly preserves injectivity of maps, i.e. subobjects. Then if $[X_i]$ has bounded dimension and $[Y_i]$ is a subobject of $[X_i]$, likewise $[Y_i]$ must have bounded dimension, since a vector space has larger dimension than any of its subobjects. A similar argument holds for quotients; thus $\prod_R \mathcal{C}_i$ is closed under the taking of quotients and subobjects.

That $\prod_R \mathcal{C}_i$ is closed under the taking of tensor products is evident from the definition of a tensor functor; if $[X_i]$ and $[Y_i]$ have constant dimension m and n respectively, then $\omega_i(X_i \otimes Y_i) \simeq \omega_i(X_i) \otimes \omega_i(Y_i)$ has constant dimension mn.

That $\prod_R \mathcal{C}_i$ has an identity object is similarly proved; tensor functors by definition carry identity objects to identity objects, and the only identity objects in Vec_k are 1-dimensional vector spaces.

Finally, we must show that the dual of an object $[X_i]$ of $\prod_R \mathcal{C}_i$ also has constant dimension. But this is evident from proposition 1.9 of [5], which says that ω_i carries

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dual objects to dual objects, and the dual of any vector space has dimension equal to itself.

$\square$

Now define a functor ω from $\prod_R \mathcal{C}_i$ to Vec_k as follows. For an object $[X_i]$ of $\prod_R \mathcal{C}_i$, we define $\omega([X_i]) \stackrel{\mathrm{def}}{=} \prod_{\mathcal{U}} \omega_i(X_i)$ (ultraproduct of vector spaces; see section 6.2), and for a morphism $[\phi_i]$, we define $\omega([\phi_i]) \stackrel{\mathrm{def}}{=} [\omega_i(\phi_i)]$ (ultraproduct of linear maps; see page 88).

Since $[X_i] \in \prod_R \mathcal{C}_i$ is assumed to have bounded dimension, proposition 6.2.4 guarantees that ω carries $[X_i]$ into a finite dimensional vector space (hence the reason we restrict to $\prod_R \mathcal{C}_i$ in the first place). As the ultraproduct of maps preserves composition, and since $1_{\prod_{\mathcal{U}} V_i} = [1 : V_i \to V_i]$ (proposition 6.2.5), ω is evidently a functor.

Theorem 7.2.4. ω *is a fibre functor on* $\prod_R \mathcal{C}_i$.

Proof. We prove first that ω is a tensor functor. For two objects $[X_i]$, $[Y_i]$ of $\prod_R \mathcal{C}_i$, we define the requisite natural isomorphism $c_{[X_i],[Y_i]}$ of definition 3.2.5 to be the composition

$$\prod_{\mathcal{U}} \omega_i(X_i) \otimes \prod_{\mathcal{U}} \omega_i(Y_i) \xrightarrow{\Phi} \prod_{\mathcal{U}} \omega_i(X_i) \otimes \omega_i(Y_i) \xrightarrow{[c_{X_i,Y_i}]} \prod_{\mathcal{U}} \omega_i(X_i \otimes Y_i)$$

where c_{X_i,Y_i} denotes the given requisite isomorphism in each individual category, and Φ is the natural isomorphism defined in proposition 6.2.13. We need to verify that

the three conditions of definition 3.2.5 are satisfied. Condition 1. translates to

$$\omega([X_i]) \otimes \omega([Y_i] \otimes [Z_i])$$

with arrows:
- $1 \otimes c$ from $\omega([X_i]) \otimes (\omega([Y_i]) \otimes \omega([Z_i]))$
- c to $\omega([X_i] \otimes ([Y_i] \otimes [Z_i]))$

$$\omega([X_i]) \otimes (\omega([Y_i]) \otimes \omega([Z_i])) \qquad \omega([X_i] \otimes ([Y_i] \otimes [Z_i]))$$

- assoc' down to $(\omega([X_i]) \otimes \omega([Y_i])) \otimes \omega([Z_i])$
- $\omega(\text{assoc})$ down to $\omega(([X_i] \otimes [Y_i]) \otimes [Z_i])$

$$(\omega([X_i]) \otimes \omega([Y_i])) \otimes \omega([Z_i]) \qquad \omega(([X_i] \otimes [Y_i]) \otimes [Z_i])$$

- $c \otimes 1$ to $\omega([X_i] \otimes [Y_i]) \otimes \omega([Z_i])$
- c to $\omega(([X_i] \otimes [Y_i]) \otimes [Z_i])$

$$\omega([X_i] \otimes [Y_i]) \otimes \omega([Z_i])$$

where assoc' denotes the usual associativity isomorphism in Vec_k, and we have dropped the obvious subscripts on c. The expanded form of this diagram is

$$\prod_{\mathcal{U}}\omega_i(X_i) \otimes \prod_{\mathcal{U}}\omega_i(Y_i \otimes Z_i) \xrightarrow{\ \Phi\ } \prod_{\mathcal{U}}\omega_i(X_i) \otimes \omega_i(Y_i \otimes Z_i)$$

- $[1_i] \otimes [c_{Y_i,Z_i}]$ up, $[c_{X_i,Y_i\otimes Z_i}]$ down

$$\prod_{\mathcal{U}}\omega_i(X_i) \otimes \left(\prod_{\mathcal{U}}\omega_i(Y_i) \otimes \omega_i(Z_i)\right) \qquad \prod_{\mathcal{U}}\omega_i(X_i \otimes (Y_i \otimes Z_i))$$

- $[1_i] \otimes \Phi$ up, $[\omega_i(\text{assoc}_i)]$ down

$$\prod_{\mathcal{U}}\omega_i(X_i) \otimes \left(\prod_{\mathcal{U}}\omega_i(Y_i) \otimes \prod_{\mathcal{U}}\omega_i(Z_i)\right) \qquad \prod_{\mathcal{U}}\omega_i((X_i \otimes Y_i) \otimes Z_i)$$

- assoc' down, $[c_{X_i\otimes Y_i,Z_i}]$ up

$$\left(\prod_{\mathcal{U}}\omega_i(X_i) \otimes \prod_{\mathcal{U}}\omega_i(Y_i)\right) \otimes \prod_{\mathcal{U}}\omega_i(Z_i) \qquad \prod_{\mathcal{U}}\omega_i(X_i \otimes Y_i) \otimes \omega_i(Z_i)$$

- $\Phi \otimes [1_i]$ down, Φ up

$$\left(\prod_{\mathcal{U}}\omega_i(X_i) \otimes \omega_i(Y_i)\right) \otimes \prod_{\mathcal{U}}\omega_i(Z_i) \xrightarrow[\ [c_{X_i,Y_i}] \otimes [1_i]\ ]{} \left(\prod_{\mathcal{U}}\omega_i(X_i \otimes Y_i)\right) \otimes \prod_{\mathcal{U}}\omega_i(Z_i)$$

Now consider the diagram

$$\prod_{\mathcal{U}}\omega_i(X_i) \otimes (\omega_i(Y_i) \otimes \omega_i(Z_i))$$

$$\xleftarrow{\hspace{2cm}}$$

$$\xrightarrow{[1_i \otimes c_{Y_i,Z_i}]} \prod_{\mathcal{U}}\omega_i(X_i) \otimes \omega_i(Y_i \otimes Z_i)$$

$$\Phi \uparrow$$

$$\prod_{\mathcal{U}}\omega_i(X_i) \otimes \left(\prod_{\mathcal{U}}\omega_i(Y_i) \otimes \omega_i(Z_i)\right) \xrightarrow{([1_i]\otimes[c_{Y_i,Z_i}])\circ\Phi}$$

$$[1_i]\otimes\Phi \uparrow \qquad\qquad [c_{X_i,Y_i\otimes Z_i}] \uparrow$$

$$\prod_{\mathcal{U}}\omega_i(X_i) \otimes \left(\prod_{\mathcal{U}}\omega_i(Y_i) \otimes \prod_{\mathcal{U}}\omega_i(Z_i)\right) \qquad \prod_{\mathcal{U}}\omega_i(X_i \otimes (Y_i \otimes Z_i))$$

$$[\mathrm{assoc}'_i] \qquad \mathrm{assoc}'\circ(\Phi\otimes[1_i]) \qquad [\omega_i(\mathrm{assoc}_i)] \uparrow$$

$$\left(\prod_{\mathcal{U}}\omega_i(X_i) \otimes \omega_i(Y_i)\right) \otimes \prod_{\mathcal{U}}\omega_i(Z_i) \qquad \prod_{\mathcal{U}}\omega_i((X_i \otimes Y_i) \otimes Z_i)$$

$$\xrightarrow{([c_{X_i,Y_i}]\otimes[1_i])\circ\Phi} \qquad [c_{X_i\otimes Y_i,Z_i}] \uparrow$$

$$\Phi \downarrow \qquad \prod_{\mathcal{U}}\omega_i(X_i \otimes Y_i) \otimes \omega_i(Z_i)$$

$$\prod_{\mathcal{U}}(\omega_i(X_i) \otimes \omega_i(Y_i)) \otimes \omega_i(Z_i) \xrightarrow{[c_{X_i,Y_i}\otimes 1_i]}$$

where assoc'_i is the associativity isomorphism in the category Vec_{k_i}. This diagram has four simple subpolygons. The middle polygon consisting of seven vertices is a contracted version of the previous diagram, and is what we are trying to prove commutes. Commutativity of the top and bottom triangles follow directly from the naturality of the isomorphism Φ (see diagram 6.2.2 on page 94), and the left-most polygon can be verified directly by hand. And since all of the maps are isomorphisms, some diagram chasing shows that if the outermost six vertex polygon can be shown to commute, so also does the simple seven vertex polygon. But the outermost polygon

is

$$\prod_{\mathcal{U}}\omega_i(X_i) \otimes \omega_i(Y_i \otimes Z_i)$$

$$[1_i \otimes c_{Y_i,Z_i}] \qquad\qquad [c_{X_i,Y_i \otimes Z_i}]$$

$$\prod_{\mathcal{U}}\omega_i(X_i) \otimes (\omega_i(Y_i) \otimes \omega_i(Z_i)) \qquad\qquad \prod_{\mathcal{U}}\omega_i(X_i \otimes (Y_i \otimes Z_i))$$

$$[\mathrm{assoc}'_i] \qquad\qquad\qquad [\omega_i(\mathrm{assoc}_i)]$$

$$\prod_{\mathcal{U}}(\omega_i(X_i) \otimes \omega_i(Y_i)) \otimes \omega_i(Z_i) \qquad\qquad \prod_{\mathcal{U}}\omega_i((X_i \otimes Y_i) \otimes Z_i)$$

$$[c_{X_i,Y_i} \otimes 1_i] \qquad\qquad [c_{X_i \otimes Y_i,Z_i}]$$

$$\prod_{\mathcal{U}}\omega_i(X_i \otimes Y_i) \otimes \omega_i(Z_i)$$

and by theorem 6.2.9 commutativity of this diagram is equivalent to the almost everywhere commutativity of

$$\omega_i(X_i) \otimes \omega_i(Y_i \otimes Z_i)$$

$$1_i \otimes c_{Y_i,Z_i} \qquad\qquad c_{X_i,Y_i \otimes Z_i}$$

$$\omega_i(X_i) \otimes (\omega_i(Y_i) \otimes \omega_i(Z_i)) \qquad\qquad \omega_i(X_i \otimes (Y_i \otimes Z_i))$$

$$\mathrm{assoc}'_i \qquad\qquad\qquad \omega_i(\mathrm{assoc}_i)$$

$$(\omega_i(X_i) \otimes \omega_i(Y_i)) \otimes \omega_i(Z_i) \qquad\qquad \omega_i((X_i \otimes Y_i) \otimes Z_i)$$

$$c_{X_i,Y_i} \otimes 1_i \qquad\qquad c_{X_i \otimes Y_i,Z_i}$$

$$\omega_i(X_i \otimes Y_i) \otimes \omega_i(Z_i)$$

But this is commutative everywhere, as it is merely condition 1. of definition 3.2.5, by virtue of each ω_i being a tensor functor. ω thus satisfies condition 1.

Condition 2. is proved similarly; consider the diagram

$$\prod_{\mathcal{U}} \omega_i(X_i) \otimes \omega_i(Y_i)$$

with morphisms Φ, $[c_{X_i, Y_i}]$,

$$\prod_{\mathcal{U}} \omega_i(X_i) \otimes \prod_{\mathcal{U}} \omega_i(Y_i) \qquad \prod_{\mathcal{U}} \omega_i(X_i \otimes Y_i)$$

with morphisms comm$'$, $[\text{comm}'_i]$, $[\omega_i(\text{comm}_i)]$,

$$\prod_{\mathcal{U}} \omega_i(Y_i) \otimes \prod_{\mathcal{U}} \omega_i(X_i) \qquad \prod_{\mathcal{U}} \omega_i(Y_i \otimes X_i)$$

with morphisms Φ, $[c_{Y_i, X_i}]$,

$$\prod_{\mathcal{U}} \omega_i(Y_i) \otimes \omega_i(X_i)$$

The outermost hexagon is our expanded version of condition 2., and is what we must prove. Commutativity of the left trapezoid can be verified directly by hand. And again by theorem 6.2.9, commutativity of the right trapezoid is equivalent to the almost everywhere commutativity of

$$
\begin{array}{ccc}
\omega_i(X_i) \otimes \omega_i(Y_i) & \xrightarrow{c_{X_i, Y_i}} & \omega_i(X_i \otimes Y_i) \\
\downarrow{\scriptstyle \text{comm}'_i} & & \downarrow{\scriptstyle \omega_i(\text{comm}_i)} \\
\omega_i(Y_i) \otimes \omega_i(X_i) & \xrightarrow[c_{Y_i, X_i}]{} & \omega_i(Y_i \otimes X_i)
\end{array}
$$

But this is condition 2. applied to each individual ω_i, which commutes by assumption.

For the purposes of this proof we shall replace condition 3. of definition 3.2.5 with the seemingly weaker but equivalent condition given in definition 1.8 of [5]: that whenever $[U_i]$ is an identity object of $\prod_R \mathcal{C}_i$ and $[u_i] : [U_i] \to [U_i] \otimes [U_i]$ an isomorphism, then so is $\omega([U_i])$ and $\omega([u_i])$. Since any two identity objects of a tensor category are naturally isomorphic via a unique isomorphism commuting with the

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unit maps (proposition 1.3 of [5]), we need only verify this for a single identity object, namely the pair $[1_i]$ and $[\text{unit}_{i,1_i}]$. As each ω_i is a tensor functor, it sends 1_i to an identity object in Vec_{k_i}, and we know of course that the only identity objects of Vec_{k_i} are 1-dimensional. Thus $\omega([1_i]) = \prod_{\mathcal{U}} \omega_i(1_i)$ is 1-dimensional (proposition 6.2.4), thus $\omega([1_i])$ is an identity object of Vec_k. And again, as each ω_i is a tensor functor, it sends $\text{unit}_{i,1_i}$ to an isomorphism $\omega_i(\underline{1}_i) \to \omega_i(\underline{1}_i) \otimes \omega_i(\underline{1}_i)$, whence ω sends $[\text{unit}_{i,1_i}]$ to an isomorphism as well.

ω is k-linear by the k_i-linearity of each ω_i and proposition 6.2.5:

$$\omega([a_i][\phi_i] + [\psi_i]) = \omega([a_i\phi_i + \psi_i])$$
$$= [\omega_i(a_i\phi_i + \psi_i)]$$
$$= [a_i\omega_i(\phi_i) + \omega_i(\psi_i)]$$
$$= [a_i][\omega_i(\phi_i)] + [\omega_i(\psi_i)]$$
$$= [a_i]\omega([\phi_i]) + \omega([\psi_i])$$

ω is faithful: if $[\phi_i]$ and $[\psi_i]$ are different morphisms, then (ϕ_i) and (ψ_i) differ on a large set. By faithfulness of each ω_i, so do $(\omega_i(\phi_i))$ and $(\omega_i(\psi_i))$, and by proposition 6.2.5, $[\omega_i(\phi_i)]$ and $[\omega_i(\psi_i)]$ are different linear maps.

ω is exact by the exactness of each ω_i, proposition 6.2.12, and the fact that "is an exact sequence" is a first-order concept. The sequence

$$[0] \to [X_i] \xrightarrow{[\phi_i]} [Y_i] \xrightarrow{[\psi_i]} [Z_i] \to [0]$$

in $\prod_R \mathcal{C}_i$ is exact if and only if the constituent sequences

$$0 \to X_i \xrightarrow{\phi_i} Y_i \xrightarrow{\psi_i} Z_i \to 0$$

are almost everywhere exact, in which case ω_i of these sequences is almost everywhere

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exact, in which case ω of the first sequence is exact. This completes the proof.

$\square$

Corollary 7.2.5. *If G_i is a sequence of affine group schemes defined over the fields k_i, then $\prod_R Rep_{k_i} G_i$ is (tensorially equivalent to) $Rep_{\prod_{\mathcal{U}} k_i} G$ for some affine group scheme G.*

Proof. By theorems 7.2.3 and 7.2.4, $\prod_R \text{Rep}_{k_i} G_i$ is a neutral tannakian category over the field $\prod_{\mathcal{U}} k_i$. Apply theorem 3.3.2. $\square$

Chapter 8

Finite Dimensional Subcoalgebras of Hopf Algebras

In this chapter we take a break entirely from working with ultraproducts; no understanding of them is required here whatsoever. The main theorem of this chapter is perhaps of interest in its own right, but for our purposes mostly serves as an invaluable lemma with which to prove the main theorem of the next chapter.

Here we investigate the special case of when a finite dimensional comodule C over a Hopf algebra (A, Δ, ε) over a field k is actually a sub-co*algebra* of A; our intent is to show that these satisfy some very nice regularity properties in terms of how they sit inside the category Comod_A. To say that $C \subset A$ is a subcoalgebra is to simply say that the image of the map $\Delta : A \to A \otimes A$, when restricted to C, is contained inside $C \otimes C \subset A \otimes A$, and that we are regarding the map Δ as the (left or right, depending) A-comodule structure for C. Throughout we will use the same symbols Δ and ε for their restrictions to C.

In a sense though, the case of a finite dimensional A-comodule being a subcoalgebra of A is really not that special. The fundamental theorem of coalgebras (theorem 2.2.3) states that any coalgebra (and hence Hopf algebra) is a directed union of finite

dimensional subcoalgebras. Further, theorem 2.2.2 states that any A-comodule can be embedded in some n-fold direct sum of the regular representation. Thus every finite dimensional A-comodule can be embedded in C^n, where C is some finite dimensional subcoalgebra of A. We see then that the entire category Comod_A can be realized as a direct limit of the principal subcategories $\langle C \rangle$, where C ranges over all finite dimensional subcoalgebras of A. Anything categorical we can say in general about these subcoalgebras of A must surely then (and will) be of value.

We would also like to mention that, so far as we can tell, these results are valid for *any* coalgebra A, Hopf algebra or not. Nonetheless, as all of our applications of these results will be toward Hopf algebras, we leave them as stated.

Let C be a subcoalgebra of (A, Δ, ε). Since Δ restricts to $C \otimes C$ on C, we can think of C as both a left and a right comodule over A. That is

$$\Delta : C \to C \otimes C \subset C \otimes A$$

gives a right A-comodule structure for C, and

$$\Delta : C \to C \otimes C \subset A \otimes C$$

gives a left A-comodule structure for C. Unless C is co-commutative we can expect these structures in general to be quite different.

For the remainder of this chapter denote by $\mathcal{C}_R$ the category of finite dimensional right A-comodules, and denote by ω_R the fibre (i.e. forgetful) functor $\mathcal{C}_R \to \mathrm{Vec}_k$. Define similarly $\mathcal{C}_L$ and ω_L. For a finite dimensional subcoalgebra C of A denote by $\mathrm{End}_{\mathcal{C}_R}(C)$ the algebra of all endomorphisms on C, where we consider C as an object in the category $\mathcal{C}_R$, as defined above; make a similar definition for $\mathrm{End}_{\mathcal{C}_L}(C)$. Denote as usual by $\mathrm{End}(\omega_R | \langle C \rangle)$ the collection of all natural transformations of the fibre functor ω_R restricted to the principal subcategory $\langle C \rangle$ (see definition 3.3.3), similarly

for $\text{End}(\omega_L|\langle C \rangle)$

The remainder of this chapter is devoted to proving, piecemeal, the following:

Theorem 8.0.6. *Let C be a finite dimensional subcoalgebra of the Hopf algebra A over the field k. Then*

1. *$\text{End}_{\mathcal{C}_R}(C) = \text{End}(\omega_L|\langle C \rangle) = \text{the centralizer of } \text{End}_{\mathcal{C}_L}(C)$, and $\text{End}_{\mathcal{C}_L}(C) = \text{End}(\omega_R|\langle C \rangle) = \text{the centralizer of } \text{End}_{\mathcal{C}_R}(C)$*

2. *All of the above are canonically isomorphic to the dual algebra of the coalgebra C.*

It is clear from the remarks on page 135 of [5], combined with lemma 2.13 of the same text, that the author is quite aware that the algebra C^* is isomorphic to both $\text{End}(\omega_R|\langle C \rangle)$ and $\text{End}(\omega_L|\langle C \rangle)$. This is not surprising; we shall argue at the end of this chapter that this theorem in fact proves that the 'algorithm' given in section 3.3 for recovering the Hopf algebra A from the category Comod_A does in fact give the correct answer. As to the other assertions of theorem 8.0.6, we are unable to locate any specific occurrence of them in the literature.

In the statement of the theorem we have deliberately confused (as we may, by the discussion on page 43) $\text{End}(\omega_R|\langle C \rangle)$ with its image inside $\text{End}_{\text{Vec}_k}(\omega_R(C))$. Note that these are equalities given in 1. above, not just isomorphisms.

We will prove first that C^*, the dual algebra to the coalgebra C, is isomorphic to $\text{End}_{\mathcal{C}_R}(C)$. We define maps

$$C^* \xrightarrow{\ \Omega\ } \text{End}_{\mathcal{C}_R}(C)$$
$$C^* \xleftarrow{\ \Gamma\ } \text{End}_{\mathcal{C}_R}(C)$$

as follows. For $\alpha \in C^*$, $\Omega(\alpha)$ is the composition

$$C \xrightarrow{\ \Delta\ } C \otimes C \xrightarrow{\ \alpha \otimes 1\ } k \otimes C \simeq C$$

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and for $\phi \in \text{End}_{\mathcal{C}_R}(C)$, $\Gamma(\phi)$ is the composition

$$C \xrightarrow{\phi} C \xrightarrow{\varepsilon} k$$

Theorem 8.0.7. *The maps Ω and Γ are well-defined algebra maps, and are left and right-sided inverses for one another, making them both isomorphisms of algebras.*

Proof. We need to prove first the non-obvious fact that, for any $\alpha \in C^*$, $\Omega(\alpha)$ is an endomorphism on C as a right A-comodule, that is, that the diagram

$$
\begin{array}{ccc}
C & \xrightarrow{\Omega(\alpha)} & C \\
\Delta \downarrow & & \downarrow \Delta \\
C \otimes C & \xrightarrow{\Omega(\alpha) \otimes 1} & C \otimes C
\end{array}
$$

commutes. Consider the diagram

$$
\begin{array}{ccccccc}
C & \xrightarrow{\Delta} & C \otimes C & \xrightarrow{\alpha \otimes 1} & k \otimes C & \xrightarrow{\simeq} & C \\
\Delta \downarrow & & 1 \otimes \Delta \downarrow & & 1 \otimes \Delta \downarrow & & \downarrow \Delta \\
C \otimes C & \xrightarrow{\Delta \otimes 1} & C \otimes C \otimes C & \xrightarrow{\alpha \otimes 1 \otimes 1} & k \otimes C \otimes C & \xrightarrow{\simeq} & C \otimes C
\end{array}
$$

The outermost rectangle is an expanded version of the previous diagram, and is what we are trying to prove commutes. Commutativity of the right-most simple rectangle follows directly from the naturality of $\simeq$, commutativity of the middle rectangle is obvious, and the left-most rectangle is a coalgebra identity. Thus the outermost rectangle commutes, and $\Omega(\alpha)$ is indeed an endomorphism of C as a right A-comodule.

We argue now that $\Omega \circ \Gamma$ and $\Gamma \circ \Omega$ are both the identity. Let $\alpha \in C^*$, and consider

$$
\begin{array}{ccccccccc}
C & \xrightarrow{\Delta} & C \otimes C & \xrightarrow{\alpha \otimes 1} & k \otimes C & \xrightarrow{\simeq} & C & \xrightarrow{\varepsilon} & k \\
 & \searrow{\simeq} & \downarrow{1 \otimes \varepsilon} & & \downarrow{1 \otimes \varepsilon} & & \swarrow{\simeq} & & \\
 & & C \otimes k & \xrightarrow{\alpha \otimes 1} & k \otimes k & & & &
\end{array}
$$

The top line is the map $\Gamma(\Omega(\alpha))$. We would like to see that this is equal to α, and α is clearly equal to the bottom three-map composition; thus we seek to prove commutativity of the outermost polygon. Commutativity of the right-most simple polygon follows again from the naturality of $\simeq$, commutativity of the middle square is obvious, and the left-most triangle is again a coalgebra identity; thus $\Gamma(\Omega(\alpha)) = \alpha$.

Now let $\phi \in \mathrm{End}_{\mathcal{C}_R}(C)$. Consider

$$
\begin{array}{ccccccc}
C & \xrightarrow{\Delta} & C \otimes C & \xrightarrow{(\phi \circ \varepsilon) \otimes 1} & k \otimes C & \xrightarrow{\simeq} & C \\
\downarrow{\phi} & & \downarrow{\phi \otimes 1} & \nearrow{\varepsilon \otimes 1} & & & \\
C & \xrightarrow{\Delta} & C \otimes C & & & &
\end{array}
$$

The top line is the map $\Omega(\Gamma(\phi))$, which we would like to see is equal to ϕ. Commutativity of the left-most square is the assertion that ϕ is an endomorphism of C as a right A-comodule, and commutativity of the middle triangle is obvious. Thus the outermost polygon commutes, giving us

$$
\Omega(\Gamma(\phi)) = \phi \circ (\Delta \circ (\varepsilon \otimes 1) \circ \simeq)
$$

But $(\Delta \circ (\varepsilon \otimes 1) \circ \simeq) = 1$ is coalgebra identity, and hence the right hand side is equal to ϕ, proving the claim.

We must finally prove that Γ is a k-algebra map. Recall the multiplication on C^*;

it sends the pair of functionals $\alpha, \beta : C \to k$ to the functional

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\alpha \otimes \beta} k \otimes k \xrightarrow{\simeq} k$$

Let $\phi, \psi \in \mathrm{End}_R(C)$, and consider the diagram

$$C \otimes C \xrightarrow{\phi \otimes 1} C \otimes C \xrightarrow{\varepsilon \otimes 1} k \otimes C \xrightarrow{1 \otimes \psi} k \otimes C \xrightarrow{1 \otimes \varepsilon} k \otimes k \xrightarrow{\simeq} k$$

$$\begin{array}{ccccccc}
\Delta \uparrow & & \Delta \uparrow & & \simeq \uparrow & & \simeq \uparrow \\
C & \xrightarrow{\phi} & C & \!\!=\!\!\!=\!\!\!=\!\! & C & \xrightarrow{\psi} & C
\end{array}$$

The composition that starts at the bottom left hand corner, goes up, and then all the way across, is an expanded version of the map $\Gamma(\phi) * \Gamma(\psi)$, where $*$ denotes the multiplication in the algebra C^*. The one that starts at the bottom left hand corner, goes across, and then diagonally up, is the map $\Gamma(\phi \circ \psi)$; we want to see of course that these are equal. It is enough to show then that all of the simple polygons commute. Starting from the left: commutativity of the first is the assertion that ϕ is an endomorphism of C as a right A-comodule, the second is a coalgebra identity, and commutativity of the third and fourth follow directly from the naturality of $\simeq$. Therefore Γ is a multiplicative map, and is obviously k-linear, since composition with ε (or any linear map) is so. Therefore Γ is an isomorphism of k-algebras. The same is true of Ω, since it is the inverse of such a map. $\qquad\square$

We claim also that C^* is in much the same way isomorphic to $\mathrm{End}_{\mathcal{C}_L}(C)$, the endomorphism algebra of C as a *left* A-comodule. This time we define a map

$$C^* \xrightarrow{\Theta} \mathrm{End}_{\mathcal{C}_L}(C)$$

as, for $\alpha \in C^*$, $\Theta(\alpha)$ is the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{1 \otimes \alpha} C \otimes k \simeq C$$

(notice the switching of the slots on which 1 and α act). We define a map Λ : $\mathrm{End}_{\mathcal{C}_L}(C) \to C^*$ the same way as before: for an endomorphism ϕ of C in the category $\mathcal{C}_L$, $\Lambda(\phi)$ is the composition

$$C \xrightarrow{\phi} C \xrightarrow{\varepsilon} k$$

An proof almost identical to that of the previous theorem shows again that Θ and Λ are isomorphisms of k-algebras, which we will not repeat.

Lemma 8.0.8. *Let V be a finite dimensional vector space over a field k, and let $r, s \in V \otimes V$ such that $r \neq s$. Then there exists a linear functional $\alpha : V \to k$ such that the composition*

$$V \otimes V \xrightarrow{\alpha \otimes 1} k \otimes V \simeq V$$

sends r and s to different things.

Proof. Let $e_1, \ldots, e_n$ be a basis for V. Then we can write

$$r = \sum_{i,j} c_{ij} e_i \otimes e_j$$

$$s = \sum_{i,j} d_{ij} e_i \otimes e_j$$

If $r \neq s$, then $c_{ij} \neq d_{ij}$ for some i and j. Then let α be the functional which sends e_i to 1 and all other e_j to 0. Then the composition above sends r to

$$c_{i,1} e_1 + c_{i,2} e_2 + \ldots + c_{i,j} e_j + \ldots + c_{i,n} e_n$$

while it sends s to

$$d_{i,1}e_1 + d_{i,2}e_2 + \ldots + d_{i,j}e_j + \ldots + d_{i,n}e_n$$

and these are clearly not equal, since $c_{i,j} \neq d_{i,j}$. $\qquad\square$

Theorem 8.0.9. *Let C be a finite dimensional subcoalgebra of the Hopf algebra A. Then $\mathrm{End}_{\mathcal{C}_L}(C) = $ the centralizer of $\mathrm{End}_{\mathcal{C}_R}(C)$.*

Proof. Let $\phi \in \mathrm{End}_{\mathcal{C}_L}(C)$ and $\psi \in \mathrm{End}_{\mathcal{C}_R}(C)$, and consider the diagram

$$
\begin{array}{ccccccc}
C & \xrightarrow{\phi} & C & \xrightarrow{\psi} & C & \xrightarrow{\phi} & C \\
\Delta\downarrow & & \Delta\downarrow & & \Delta\downarrow & & \Delta\downarrow \\
C \otimes C & \xrightarrow{1\otimes\phi} & C \otimes C & \xrightarrow{\psi\otimes 1} & C \otimes C & \xrightarrow{1\otimes\phi} & C \otimes C
\end{array}
$$

Commutativity of all three of the simple squares are merely the assertions that ϕ and ψ are morphisms in the categories $\mathcal{C}_L$ and $\mathcal{C}_R$ respectively; thus this entire diagram commutes. If we look at the rectangle consisting of the two left squares we obtain $\phi \circ \psi \circ \Delta = \Delta \circ (\psi \otimes \phi)$, and looking at the rectangle consisting of the two right squares we obtain $\psi \circ \phi \circ \Delta = \Delta \circ (\psi \otimes \phi)$. Thus we have

$$\phi \circ \psi \circ \Delta = \psi \circ \phi \circ \Delta$$

But Δ is injective, and hence $\phi \circ \psi = \psi \circ \phi$. This shows that $\mathrm{End}_{\mathcal{C}_L}(C)$ is contained in the centralizer of $\mathrm{End}_{\mathcal{C}_R}(C)$.

Now let ϕ be any linear map $C \to C$, and suppose that it is not a member of

$\text{End}_{\mathcal{C}_L}(C)$, which is to say that the diagram

$$C \xrightarrow{\phi} C$$

with vertical maps Δ on both sides, bottom arrow $C \otimes C \xrightarrow{1 \otimes \phi} C \otimes C$

does *not* commute; we claim there exists a member of $\text{End}_{\mathcal{C}_R}(C)$ with which ϕ does not commute. Recall from theorem 8.0.7 that for any linear functional $\alpha : C \to k$, the composition

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\alpha \otimes 1} k \otimes C \simeq C$$

belongs to $\text{End}_{\mathcal{C}_R}(C)$; our job is then to find an α so that ϕ does not commute with this map. Consider

$$C \xrightarrow{\Delta} C \otimes C \xrightarrow{\alpha \otimes 1} k \otimes C \xrightarrow{\simeq} C$$

with vertical maps ϕ, $1 \otimes \phi$, $1 \otimes \phi$, ϕ, bottom row $C \xrightarrow{\Delta} C \otimes C \xrightarrow{\alpha \otimes 1} k \otimes C \xrightarrow{\simeq} C$

All of the simple squares of this diagram commute, except for the left most one, which does not by assumption. We want to show that there is an α such that the outermost rectangle does not commute. Pick $v \in C$ such that commutativity of the left square fails, let $r \in C \otimes C$ be its image when chasing it one way, and $s \in C \otimes C$ its image when chasing it the other way. By the previous lemma, pick α such that the composition

$$C \otimes C \xrightarrow{\alpha \otimes 1} k \otimes C \xrightarrow{\simeq} C$$

sends r and s to different things, let's say $m \neq l$. Then if we chase v around one path

of the outermost rectangle we arrive at m, and the other way, we arrive at l; thus the outermost rectangle does not commute. This gives a member of $\mathrm{End}_{\mathcal{C}_R}(C)$ with which ϕ does not commute, and the theorem is proved. $\qquad\square$

An identical proof shows that $\mathrm{End}_{\mathcal{C}_R}(C)$ is the centralizer of $\mathrm{End}_{\mathcal{C}_L}(C)$, which we do not repeat.

Our last task is to show that $\mathrm{End}_{\mathcal{C}_L}(C)$ is equal to $\mathrm{End}(\omega_R|\langle C \rangle)$. Any member $\phi : C \to C$ of the latter must at the least make diagrams of the form

$$
\begin{array}{ccc}
C & \xrightarrow{\ \psi\ } & C \\
\phi \downarrow & & \downarrow \phi \\
C & \xrightarrow{\ \psi\ } & C
\end{array}
$$

commute, where ψ is an arbitrary element of $\mathrm{End}_{\mathcal{C}_R}(C)$. As $\mathrm{End}_{\mathcal{C}_L}(C)$ is equal to the commutator of $\mathrm{End}_{\mathcal{C}_R}(C)$, we already have the forward inclusion $\mathrm{End}(\omega_R|\langle C \rangle) \subset \mathrm{End}_{\mathcal{C}_L}(C)$; it remains to show the reverse.

Lemma 8.0.10. *Let $(V, \rho), (W, \mu)$ be finite dimensional comodules over the Hopf algebra A. Fix bases $e_1, \ldots, e_n$, $f_1, \ldots, f_m$ for V and W respectively, and write*

$$
\rho : e_j \mapsto \sum_i e_i \otimes a_{ij}
$$

$$
\mu : f_j \mapsto \sum_i f_i \otimes b_{ij}
$$

1. If V is a subobject of W, then each a_{ij} is a linear combination of the b_{ij}.

2. If W is a quotient object of V, then each b_{ij} is a linear combination of the a_{ij}.

Proof. This is a simple fact from linear algebra if we think of (a_{ij}) and (b_{ij}) as matrices. If $\phi : V \to W$ is a linear map, write it as the matrix (c_{ij}) in the relevant

bases. Then ϕ being a morphism of A-comodules is equivalent to the matrix equality

$$(c_{ij})(a_{ij}) = (b_{ij})(c_{ij})$$

In case (c_{ij}) is injective it has a right-sided inverse, given say by the matrix (d_{ij}). Then $(a_{ij}) = (d_{ij})(b_{ij})(c_{ij})$, and clearly every entry of the right hand side is a linear combination of the b_{ij}; this proves 1. If (c_{ij}) is surjective it has a left-sided inverse, again call it (d_{ij}). Then we have $(c_{ij})(a_{ij})(d_{ij}) = (b_{ij})$. This proves 2.

$\square$

Lemma 8.0.11. *If $(V, \rho : V \to V \otimes A)$ is a right A-comodule, and if it belongs to the principal subcategory $\langle C \rangle$, then the image of ρ is contained in $V \otimes C$.*

Proof. The claim is obvious in case V is direct sum of copies of C, since then its comodule map is the composition

$$\Delta^n : C^n \xrightarrow{\Delta^n} (C \otimes C)^n \simeq C^n \otimes C \subset C^n \otimes A$$

Suppose then (X, ρ) is a quotient of some C^n. Then choose (a_{ij}) for X and (b_{ij}) for C^n as in the previous lemma. Each b_{ij} is in C by assumption, and thus so is each a_{ij}, being a linear combination of the b_{ij}. A similar argument holds if we consider a subobject of the quotient object (X, ρ).

$\square$

Thus, for any object $(X, \rho) \in \langle C \rangle$, we can write $\rho : X \to X \otimes C$ instead of $\rho : X \to X \otimes A$.

Let $\phi : C \to C$ be any linear map. Then for $n \in \mathbb{N}$ we define a map $\phi_{C^n} : C^n \to C^n$ as ϕ^n, that is, the unique linear map commuting with all of the canonical injections $C \xrightarrow{\iota_i} C^n$. Now let (X, ρ_X) be any object of $\langle C \rangle$. Then there is an object (Y, ρ_Y) of

$\langle C \rangle$ and a commutative diagram

$$
\begin{array}{ccccc}
X & \xhookrightarrow{\iota} & Y & \xleftarrow{\pi} & C^n \\
\downarrow{\scriptstyle \rho_X} & & \downarrow{\scriptstyle \rho_Y} & & \downarrow{\scriptstyle \Delta^n} \\
X \otimes C & \xhookrightarrow{\iota \otimes 1} & Y \otimes C & \xleftarrow{\pi \otimes 1} & C^n \otimes C
\end{array}
$$

Now, if ϕ defines an element of $\operatorname{End}(\omega_R|\langle C \rangle)$, there exist unique linear maps ϕ_X and ϕ_Y making

$$
\begin{array}{ccccc}
X & \xhookrightarrow{\iota} & Y & \xleftarrow{\pi} & C^n \\
\downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi_Y} & & \downarrow{\scriptstyle \phi^n} \\
X & \xhookrightarrow{\iota} & Y & \xleftarrow{\pi} & C^n
\end{array}
$$

commute. But, unless we know *a priori* that ϕ defines an element of $\operatorname{End}(\omega_R|\langle C \rangle)$, all we can say is that ϕ_Y exists, but may not be unique, and that ϕ_X is unique, but may not exist. Further, if we choose another such n, (Y, ρ_Y), π and ι, one cannot expect to obtain the same linear map ϕ_X, again unless we know that $\phi \in \operatorname{End}(\omega_R|\langle C \rangle)$. For the moment then, we make the following deliberately ambiguous definition.

Definition 8.0.2. Let X be an object of $\langle C \rangle$, $\phi : C \to C$ any linear map. Then we define $\phi_X : X \to X$ to be any linear map satisfying any one of the following conditions.

1. If $X = C^n$, then $\phi_X = \phi^n$.

2. There exists an n and a surjective map $C^n \xrightarrow{\pi} X$ such that $\pi \circ \phi_X = \phi^n \circ \pi$.

3. There exists a quotient object Y of C^n such that ϕ_Y exists and satisfies condition 2. above, and there is an injective map $X \xrightarrow{\iota} Y$ such that $\phi_X \circ \iota = \iota \circ \phi_Y$.

So, when we prove theorems about "the" map ϕ_X, it is understood to apply to any ϕ_X satisfying any one of the above conditions, and under the assumption that it exists in the first place. It will only later be a consequence of these theorems that ϕ_X is well-defined; that it always exists, and is always unique.

Lemma 8.0.12. *Let (X, ρ_X) be an object of $\langle C \rangle$, ϕ an element of $End_{\mathcal{C}_L}(C)$. Then the following is always commutative:*

$$
\begin{array}{ccc}
X & \xrightarrow{\ \phi_X\ } & X \\
{\scriptstyle \rho_X}\downarrow & & \downarrow{\scriptstyle \rho_X} \\
X \otimes C & \xrightarrow[1 \otimes \phi]{} & X \otimes C
\end{array}
$$

Proof. The claim is obvious if $X = C$, since then the above diagram is the definition of ϕ being an endomorphism of $\mathcal{C}$ as a left A-comodule. If $X = C^n$, consider

$$
\begin{array}{ccc}
C^n & \xrightarrow{\ \phi^n\ } & C^n \\
{\scriptstyle \Delta^n}\downarrow & & \downarrow{\scriptstyle \Delta^n} \\
(C \otimes C)^n & \xrightarrow{\ (1 \otimes \phi)^n\ } & (C \otimes C)^n \\
{\scriptstyle \simeq}\downarrow & & \downarrow{\scriptstyle \simeq} \\
C^n \otimes C & \xrightarrow[1 \otimes \phi]{} & C^n \otimes C
\end{array}
$$

Commutativity of each of the squares is obvious, and commutativity of the outermost rectangle is what is desired.

Now suppose $C^n \xrightarrow{\ \pi\ } X$ is a quotient of C^n and that ϕ_X satisfies condition 2. of

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definition 8.0.2. Consider

$$
\begin{array}{ccccccc}
 & & & & \xrightarrow{\;\;\pi\otimes 1\;\;} & & \\
C^n & \xleftarrow{\;\Delta^n\;} & C^n & \xrightarrow{\;\pi\;} & X & \xrightarrow{\;\rho_X\;} & X\otimes C \\
{\scriptstyle 1\otimes\phi}\downarrow & & {\scriptstyle \phi^n}\downarrow & & {\scriptstyle \phi_X}\downarrow & & {\scriptstyle 1\otimes\phi}\downarrow \\
C^n\otimes C & \xleftarrow{\;\Delta^n\;} & C^n & \xrightarrow{\;\pi\;} & X & \xrightarrow{\;\rho_X\;} & X\otimes C \\
 & & & & \xrightarrow{\;\;\pi\otimes 1\;\;} & & \\
\end{array}
$$

We seek to prove commutativity of the right-most square. Commutativity of the left most square has been proved, the middle square commutes by definition, the top and bottom rectangles commute because π is a map of right comodules, and commutativity of the outermost polygon is obvious. If one starts at the second occurrence of C^n at the top and does some diagram chasing, he eventually obtains $\pi \circ (\rho_X \circ (1 \otimes \phi)) = \pi \circ (\phi_X \circ \rho_X)$. But π is surjective, and thus we have $\rho_X \circ (1 \otimes \phi) = \phi_X \circ \rho_X$, and the claim is proved.

Now suppose (X, ρ_X) is a subobject of the quotient object Y via the map $X \xrightarrow{\;\iota\;} Y$, so that ϕ_X satisfies condition 3. of definition 8.0.2, and that ϕ_Y satisfies condition 2.

Consider

$$
\begin{array}{ccccccc}
 & \xleftarrow{\hspace{3cm}\iota \otimes 1\hspace{3cm}} & & & & & \\
\downarrow & & & & & & \uparrow \\
Y \otimes C & \xleftarrow{\rho_Y} & Y & \xleftarrow{\iota} & X & \xrightarrow{\rho_X} & X \otimes C \\
{\scriptstyle 1 \otimes \phi}\downarrow & & {\scriptstyle \phi_Y}\downarrow & & {\scriptstyle \phi_X}\downarrow & & \downarrow{\scriptstyle 1 \otimes \phi} \\
Y \otimes C & \xleftarrow{\rho_Y} & Y & \xleftarrow{\iota} & X & \xrightarrow{\rho_X} & X \otimes C \\
\uparrow & & & & & & \downarrow \\
 & \xleftarrow{\hspace{3cm}\iota \otimes 1\hspace{3cm}} & & & & &
\end{array}
$$

Commutativity of the right most square is again what we seek to prove. We have proved commutativity of the left square, the middle commutes by definition, the top and bottom rectangles commute since ι is a map of comodules, and commutativity of the outermost rectangle is obvious. Starting at X on the top line, some diagram chasing shows that $\rho_X \circ (1 \otimes \phi) \circ (\iota \otimes 1) = \phi_X \circ \rho_X \circ (\iota \otimes 1)$. But $\iota \otimes 1$ is injective, whence we have $\rho_X \circ (1 \otimes \phi) = \phi_X \circ \rho_X$. The lemma is proved. $\qquad\square$

Proposition 8.0.13. *If (X, ρ_X), (Y, ρ_Y) are any objects of $\langle C \rangle$, $\psi : X \to Y$ any morphism in $\mathcal{C}_R$, and ϕ an element of $End_{\mathcal{C}_L}(C)$, then the following commutes:*

$$
\begin{array}{ccc}
X & \xrightarrow{\psi} & Y \\
{\scriptstyle \phi_X}\downarrow & & \downarrow{\scriptstyle \phi_Y} \\
X & \xrightarrow{\psi} & Y
\end{array}
$$

Proof. Consider

$$X \xrightarrow{\phi_X} X \xrightarrow{\psi} Y \xrightarrow{\phi_Y} Y$$

with vertical maps ρ_X, ρ_X, ρ_Y, ρ_Y to

$$X \otimes C \xrightarrow{1 \otimes \phi} X \otimes C \xrightarrow{\psi \otimes 1} Y \otimes C \xrightarrow{1 \otimes \phi} Y \otimes C$$

Commutativity of the right and left squares follow from lemma 8.0.12 and the middle square commutes because ψ is a morphism in $\mathcal{C}_R$. Thus this entire diagram commutes. Looking at the left two-square rectangle we have

$$\phi_X \circ \psi \circ \rho_Y = \rho_X \circ (\psi \otimes \phi)$$

and at the right we have

$$\psi \circ \phi_Y \circ \rho_Y = \rho_X \circ (\psi \otimes \phi)$$

Thus $(\phi_X \circ \psi) \circ \rho_Y = (\psi \otimes \phi_Y) \circ \rho_Y$. But ρ_Y is injective (as all comodule maps are), hence $\phi_X \circ \psi = \psi \circ \phi_Y$, and the proposition is proved.

$\square$

All that is left to show is that the map ϕ_X, for $\phi \in \mathrm{End}_{\mathcal{C}_L}(C)$ and $X \in \langle C \rangle$, actually exists and is unique.

Uniqueness is immediate: if ϕ_X and ϕ'_X are any two maps satisfying definition 8.0.2, then they both satisfy the hypotheses of the previous proposition. As the

identity map $1 : X \to X$ is a morphism in $\mathcal{C}_R$, the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ 1\ \ } & X \\
\downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi'_X} \\
X & \xrightarrow[\ \ 1\ \ ]{} & X
\end{array}
$$

commutes, showing ϕ_X and ϕ'_X to be equal.

For existence, suppose first that X is a quotient of C^n under π. Then ϕ_X certainly exists; if e_i is a basis for X, pull each e_i back through π^{-1}, down through ϕ^n, and back through π. But we know that ϕ_X is unique, and the only reason that it *would* be unique is because ϕ^n stabilizes the kernel of π; otherwise there would be many ϕ_X satisfying condition 2. of definition 8.0.2. This observation applies to any surjective map on C^n. As every subobject of C^n is necessarily the kernel of some surjective map we have proved

Proposition 8.0.14. *If $\phi \in End_{\mathcal{C}_L}(C)$, ϕ^n stabilizes all subobjects of C^n.*

Now let Y be a subobject of the quotient object X, with $X \xrightarrow{\ \pi\ } X/Y$ the canonical projection. Then there exists a map $\phi_{X/Y}$ making

$$
\begin{array}{ccc}
X & \xrightarrow{\ \ \pi\ \ } & X/Y \\
\downarrow{\scriptstyle \phi_X} & & \downarrow{\scriptstyle \phi_{X/Y}} \\
X & \xrightarrow[\ \ \pi\ \ ]{} & X/Y
\end{array}
$$

commute. But this $\phi_{X/Y}$ also makes

$$
\begin{CD}
C^n @>\pi'>> X @>\pi>> X/Y \\
@V\phi^n VV @V\phi_X VV @VV\phi_{X/Y}V \\
C^n @>\pi'>> X @>\pi>> X/Y
\end{CD}
$$

commute, in particular the outermost rectangle. Thus $\phi_{X/Y}$ satisfies condition 2. of definition 8.0.2, and is hence unique; but once again, the only reason this would be true is if ϕ_X stabilized the kernel of π, namely Y. We have proved

Proposition 8.0.15. *If X is a quotient object of C^n and ϕ_X satisfies condition 2. of definition 8.0.2, then ϕ_X stabilizes all subobjects of X.*

Finally, if Y is a subobject of the quotient object X, then ϕ_X stabilizes Y, whence there is a map ϕ_Y making

$$
\begin{CD}
Y @>\iota>> X \\
@V\phi_Y VV @VV\phi_X V \\
Y @>\iota>> X
\end{CD}
$$

In all cases then ϕ_X exists and is unique. We have proved

Theorem 8.0.16. *For a subcoalgebra C of A, $End_{\mathcal{C}_L}(C) = End(\omega_R|\langle C \rangle)$.*

8.1   Corollaries

Here we record some results based on the above which will be used later. We will prove the results for the category $\mathcal{C}_R$; we leave it to the reader to formulate the obvious analogues for $\mathcal{C}_L$.

Proposition 8.1.1. *Let C and D be finite dimensional subcoalgebras of the Hopf algebra A with $C \subset D$. Then the transition mapping $End(\omega_R|\langle D \rangle) \to End(\omega_R|\langle C \rangle)$ is dual to the inclusion map $C \to D$ via the isomorphism $End(\omega_R|\langle C \rangle) = End_{\mathcal{C}_L}(C) \simeq C^*$. In particular, this transition mapping is surjective.*

Proof. Let $C \xrightarrow{\iota} D$ be the inclusion mapping. Recall from page 120 that we have an isomorphism $\Lambda : End(\omega_R|\langle D \rangle) \longrightarrow D^*$ given by, for $\phi : D \to D$, $\Lambda(\phi)$ is the composition

$$D \xrightarrow{\phi} D \xrightarrow{\varepsilon} k$$

and that Λ has an inverse Θ which sends a linear functional α to

$$D \xrightarrow{\Delta} D \otimes D \xrightarrow{1 \otimes \alpha} D \otimes k \simeq D$$

As ι is a map of coalgebras ι^* of algebras, and we have a commutative diagram

$$
\begin{array}{ccc}
D^* & \xrightarrow{\quad \iota^* \quad} & C^* \\
{\Lambda}\big\uparrow & & \big\downarrow{\Theta} \\
End(\omega_R|\langle D \rangle) & \underset{T}{\longrightarrow} & End(\omega_R|\langle C \rangle)
\end{array}
$$

where T is some as of yet unidentified algebra map. Note that since ι is injective, ι^* is surjective, and hence so is T. We claim that T is the usual transition mapping.

Let $\phi \in \mathrm{End}(\omega_R|\langle D\rangle)$, and consider

$$
\begin{array}{ccccccccc}
C & \xrightarrow{\Delta} & C \otimes C & \xrightarrow{1\otimes\iota} & C \otimes D & \xrightarrow{1\otimes\phi} & C \otimes D & \xrightarrow{1\otimes\varepsilon} & C \otimes k & \xrightarrow{\simeq} & C \\
\downarrow^{\iota} & & & \searrow^{\iota\otimes\iota} & \downarrow^{\iota\otimes 1} & & \downarrow^{\iota\otimes 1} & & \downarrow^{\iota\otimes 1} & & \downarrow^{\iota} \\
D & \xrightarrow{\Delta} & & & D \otimes D & \xrightarrow{1\otimes\phi} & D \otimes D & \xrightarrow{1\otimes\varepsilon} & D \otimes k & \xrightarrow{\simeq} & D \\
\downarrow^{\phi} & & & & & & \uparrow^{\Delta} & & \nearrow_{1} & & \\
D & =\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!= & & & & & D & & & &
\end{array}
$$

Commutativity of the top left most simple polygon is the assertion that ι is a map of coalgebras, and commutativity of all the other simple polygons in the top row is trivial. The first polygon in the bottom row is the assertion that ϕ is an endomorphism on D as a left comodule (which it is, by the equality $\mathrm{End}(\omega_R|\langle D\rangle) = \mathrm{End}_{\mathcal{C}_L}(D)$) and the last polygon is a coalgebra identity; thus this entire diagram commutes. Now the composition comprising the entire top line is exactly the map $T(\phi) \overset{\mathrm{def}}{=} \Theta(\iota^*(\Lambda(\phi)))$, so the outermost polygon of this diagram is

$$
\begin{array}{ccc}
C & \xrightarrow{\iota} & D \\
\downarrow^{T(\phi)} & & \downarrow^{\phi} \\
C & \xrightarrow{\iota} & D
\end{array}
$$

But, the image of ϕ under the transition mapping $\mathrm{End}(\omega_R|\langle D\rangle) \to \mathrm{End}(\omega_R|\langle C\rangle)$ is by *definition* the unique linear map $\phi_C : C \to C$ that makes this diagram commute. Hence $\phi_C = T(\phi)$, and the proposition is proved. $\qquad\square$

Theorem 8.1.2. *Let D be a finite dimensional subcoalgebra of A, $\mathrm{End}(\omega_R|\langle D\rangle) \xrightarrow{\pi} L$ any surjective mapping of algebras. Then there exists a subcoalgebra C of D such that*

L is isomorphic to $\mathrm{End}(\omega_R|\langle C\rangle)$ and

$$
\begin{array}{ccc}
\mathrm{End}(\omega_R|\langle D\rangle) & \xrightarrow{\ \pi\ } & L \\
 & \searrow{\scriptstyle T} & \downarrow{\scriptstyle \simeq} \\
 & & \mathrm{End}(\omega_R|\langle C\rangle)
\end{array}
$$

commutes, where T is the transition mapping.

Proof. As $\mathrm{End}(\omega_R|\langle D\rangle) \xrightarrow{\Lambda} D^*$ is an isomorphism of algebras, any quotient of the former gives rise to a quotient of the latter via

$$
\begin{array}{ccc}
D^* & \xrightarrow{\ \pi'\ } & C^* \\
\uparrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle \simeq} \\
\mathrm{End}(\omega_R|\langle D\rangle) & \xrightarrow[\ \pi\ ]{} & L
\end{array}
$$

Here we have skipped a step and denoted this algebra by C^*, since, being finite dimensional, it is necessarily the dual algebra to a unique coalgebra C. Note that C is a subcoalgebra of D via the map $(\pi')^*$ and the natural isomorphism $C^{*\circ} \simeq C$. This means that we can take π' to be dual to the inclusion mapping $C \xrightarrow{\iota} D$; let us then replace π' with ι^*.

We need to verify that L can be identified with $\mathrm{End}(\omega_R|\langle C\rangle)$. Consider

$$
\begin{array}{ccc}
D^* & \xrightarrow{\ \iota^*\ } & C^* \\
\uparrow{\scriptstyle \Lambda} & & \downarrow{\scriptstyle \simeq}\ \ \searrow{\scriptstyle \Theta} \\
\mathrm{End}(\omega_R|\langle D\rangle) & \xrightarrow[\ \pi\ ]{} & L \\
 & \searrow{\scriptstyle T} & \ \ \ \vdots{\scriptstyle \Sigma} \\
 & & \mathrm{End}(\omega_R|\langle C\rangle)
\end{array}
$$

Commutativity of the outermost polygon was proved in proposition 8.1.1, and commutativity of the square is given. Define the map Σ to pass back up through the isomorphism with C^* and then down through Θ, i.e. $\Sigma = \simeq^{-1} \circ \Theta$. This Σ is the isomorphism we seek; the theorem is proved. $\qquad\square$

There is an obvious analogue to this theorem as regards subalgebras of $\text{End}(\omega_R|\langle D \rangle)$ as opposed to quotients, which we will state but not prove.

Theorem 8.1.3. *Let D be a finite dimensional subcoalgebra of A, and $L \xrightarrow{\iota} \text{End}(\omega_R|\langle D \rangle)$ any injective mapping of algebras. Then there exists a quotient coalgebra C of D such that L is isomorphic to $\text{End}(\omega_R|\langle C \rangle)$ and*

$$
\begin{array}{ccc}
L & \xrightarrow{\ \iota\ } & \text{End}(\omega_R|\langle D \rangle) \\
 & \searrow{\scriptstyle\simeq} & \downarrow{\scriptstyle T} \\
 & & \text{End}(\omega_R|\langle C \rangle)
\end{array}
$$

commutes, where T is the transition mapping.

As promised at the beginning of this chapter, these theorems actually prove that the 'algorithm' described in section 3.3 for recovering the Hopf algebra A from Comod_A does in fact give the correct answer. (This does *not* prove the general principle of tannakian duality; here we are assuming from the outset that the category we are looking at is Comod_A for some Hopf algebra A).

Theorem 8.1.4. *Let $\mathcal{C} = \text{Comod}_A$ for some Hopf algebra A. Then A can be recovered as the direct limit of the finite dimensional coalgebras $\text{End}(\omega|\langle X \rangle)^\circ$, with the direct system being the maps $\text{End}(\omega|\langle X \rangle)^\circ \xrightarrow{T_{X,Y}^\circ} \text{End}(\omega|\langle Y \rangle)^\circ$ whenever $X \in \langle Y \rangle$, where $\text{End}(\omega|\langle Y \rangle) \xrightarrow{T_{X,Y}} \text{End}(\omega|\langle X \rangle)$ is the transition mapping.*

Proof. As argued at the beginning of this chapter, the entire category Comod_A can be recovered as the direct limit of the principal subcategories $\langle C \rangle$, where C ranges over

all finite dimensional subcoalgebras of A, with the direct system being the inclusions $C \subset D$. Lemma 3.4.2 tells that, for purposes of computing the direct limit, we are justified in disregarding all objects but these subcoalgebras and all maps but these inclusions. But theorem 8.0.6 tells us that $\mathrm{End}(\omega|\langle C \rangle)^\circ \simeq C$, and proposition 8.1.1 tells us that under this isomorphism, the map $\mathrm{End}(\omega|\langle C \rangle)^\circ \xrightarrow{T^\circ_{X,Y}} \mathrm{End}(\omega|\langle D \rangle)^\circ$ can be identified with the inclusion map $C \subset D$. Apply theorem 2.2.3 to see that the direct limit of these is exactly the Hopf algebra A. $\qquad\square$

Chapter 9

The Representing Hopf Algebra of a Restricted Ultraproduct

Let k_i be an indexed collection of fields, $(A_i, \Delta_i, \varepsilon_i)$ a collection of Hopf algebras over those fields, and $\mathcal{C}_i$ the category Comod_{A_i}. The work done in chapter 7 tells us that the restricted ultraproduct $\prod_R \mathcal{C}_i$ is itself a neutral tannakian category over the field $k = \prod_{\mathcal{U}} k_i$, hence tensorially equivalent to $\mathrm{Comod}_{A_\infty}$ where A_∞ is some Hopf algebra over k. The question then: what is A_∞?

Before starting in earnest, let us examine an obvious first guess: A_∞ is the ultraproduct of the Hopf algebras A_i. But an ultraproduct of Hopf algebras is not, in general, a Hopf algebra. An ultraproduct of algebras over the fields k_i is indeed an algebra over the field k_i, with the obvious definitions of addition, multiplication, and scalar multiplication. The problem comes when we try to give it the structure of a coalgebra, consistent with the coalgebra structures on each A_i. We start by writing

$$\Delta : \prod_{\mathcal{U}} A_i \xrightarrow{[\Delta_i]} \prod_{\mathcal{U}} A_i \otimes A_i$$

But as it stands, this does not suffice; we need Δ to point to $\prod_{\mathcal{U}} A_i \otimes \prod_{\mathcal{U}} A_i$. Recall

from proposition 6.2.13 that there is a natural injective map

$$\prod_{\mathcal{U}} A_i \otimes \prod_{\mathcal{U}} A_i \xrightarrow{\Phi} \prod_{\mathcal{U}} A_i \otimes A_i$$

and that, unless the A_i have boundedly finite dimensionality, it is not surjective. The image of Φ consists exactly of those elements $[v_i]$ which have bounded tensor length, and for a given collection A_i of non-boundedly finite dimension, it is a relatively simple matter to come up with an element $[a_i] \in \prod_{\mathcal{U}} A_i$ such that $[\Delta_i(a_i)]$ has unbounded tensor length. Thus we cannot expect the image of Δ constructed above to be contained in $\prod_{\mathcal{U}} A_i \otimes \prod_{\mathcal{U}} A_i$ in general.

The next section is devoted to identifying a certain subset of $\prod_{\mathcal{U}} A_i$ which can indeed be given the structure of a coalgebra, using the definition of Δ given above. Thereafter we will show that this coalgebra is in fact a Hopf algebra, indeed equal to the A_∞ we seek.

9.1   The Restricted Ultraproduct of Hopf Algebras

To allay some of the suspense, we give the following definition, whose meaning will not be clear until later in this section.

Definition 9.1.1. The **restricted ultraproduct** of the Hopf algebras A_i, denoted A_R, is the subset of $\prod_{\mathcal{U}} A_i$ consisting of those elements $[a_i]$ such that $\mathrm{rank}(a_i)$ is bounded.

Our goal in this section is to define exactly what "rank" means in this context, and to show that A_R can be given the structure of a coalgebra. Note that the notation A_R makes no mention of the particular ultrafilter $\mathcal{U}$ being applied. As $\mathcal{U}$ is always understood to be fixed but arbitrary, no confusion should result.

Let A be a Hopf algebra, $\mathcal{C} = \mathrm{Comod}_A$. For each $X \in \mathcal{C}$ let L_X be the image

of $\mathrm{End}(\omega|\langle X\rangle)$ inside $\mathrm{End}(\omega(X))$ (see page 43), and $L_X \xleftarrow{T_{X,Y}} L_Y$ the usual transition mapping for $X \in \langle Y \rangle$. Then we have an inverse system of algebras, and from it obtain an inverse limit

$$
\begin{array}{ccc}
 & \varprojlim_{\mathcal{C}} L_X & \\
T_Y \swarrow & & \searrow T_Z \\
L_Y \xleftarrow{\quad T_{Y,Z} \quad} & & L_Z
\end{array}
$$

Lemma 9.1.1. *If C is a subcoalgebra of A, then $\varprojlim L_X \xrightarrow{T_C} L_C$ is surjective.*

Proof. By the discussion on page 114 the entire category $\mathcal{C}$ is generated by the principal subcategories $\langle C \rangle$, where C ranges over all finite dimensional subcoalgebras of A, with the direct system being inclusion mappings $C \subset D$ when applicable. We can then apply lemma 3.4.2 to see that we might as well have recovered $\varprojlim L_X$ with respect to the sub-inverse system consisting of all subcoalgebras of A under the inclusion mappings, that is as

$$
\begin{array}{ccc}
 & \varprojlim_{\mathcal{C}} L_X & \\
T_C \swarrow & & \searrow T_D \\
L_C \xleftarrow{\quad T_{C,D} \quad} & & L_D
\end{array}
$$

where C and D range over all subcoalgebras of A, and $T_{C,D}$ defined when $C \subset D$. Proposition 8.1.1 tells us that $T_{C,D}$ is always surjective, and it is a standard fact about inverse limits that if this is the case, T_C is always surjective.

$\square$

Now apply the finite dual operation to the above inverse system diagram to obtain

a diagram of coalgebras:

$$
\begin{array}{ccc}
 & \left(\varprojlim_{\mathcal{C}} L_X\right)^{\circ} & \\
{\scriptstyle T_Y^{\circ}} \nearrow & & \nwarrow {\scriptstyle T_Z^{\circ}} \\
L_Y^{\circ} & \xrightarrow[\quad T_{Y,Z}^{\circ} \quad]{} & L_Z^{\circ}
\end{array}
$$

The maps $L_Y^{\circ} \xrightarrow{T_{Y,Z}^{\circ}} L_Z^{\circ}$ therefore form a direct system; recall from page 44 that this is exactly the direct system from which A can be recovered as its direct limit. Let us rename $T_{Y,Z}^{\circ}$ as $\phi_{Y,Z}$, and so we have the direct limit diagram

$$
\begin{array}{ccc}
 & \varinjlim_{\mathcal{C}} L_X^{\circ} & \\
{\scriptstyle \phi_Y} \nearrow & & \nwarrow {\scriptstyle \phi_Z} \\
L_Y^{\circ} & \xrightarrow[\quad \phi_{Y,Z} \quad]{} & L_Z^{\circ}
\end{array}
$$

with $A = \varinjlim_{\mathcal{C}} L_X^{\circ}$. By the universal property of direct limits there is a unique map of coalgebras, call it $\phi : A \to \left(\varprojlim L_X\right)^{\circ}$, making the following diagram commute for all $Y \in \langle Z \rangle$:

$$
\begin{array}{ccc}
 & \left(\varprojlim_{\mathcal{C}} L_X\right)^{\circ} & \\
 & \phi \big\uparrow & \\
{\scriptstyle T_Y^{\circ}} \nearrow & \varinjlim_{\mathcal{C}} L_X^{\circ} & \nwarrow {\scriptstyle T_Z^{\circ}} \\
 & {\scriptstyle \phi_Y} \nearrow \quad \nwarrow {\scriptstyle \phi_Z} & \\
L_Y^{\circ} & \xrightarrow[\quad \phi_{Y,Z} \quad]{} & L_Z^{\circ}
\end{array}
\tag{9.1.1}
$$

Proposition 9.1.2. *The map ϕ is injective.*

Proof. Recall the concrete definition of a direct limit of algebraic objects; its underlying set consists of equivalence classes $[a]$ where a is some element of some L_Y°,

with $a \in L_Y$, $b \in L_Z$ equivalent when there is some L_T with $Y, Z \in \langle T \rangle$ such that $\phi_{Y,T}(a) = \phi_{Z,T}(a)$. As is discussed on page 114, for any object Y, there is a subcoalgebra C of A such that $Y \in \langle C \rangle$; this shows that any element a of any L_Y° is equivalent to some element c of L_C° for some subcoalgebra C of A. Thus, every element of $A = \varinjlim L_X^\circ$ can be written as $[c]$, for some $c \in L_C^\circ$, C a coalgebra. Further, given elements $[c]$ and $[d]$ of $\varinjlim L_X^\circ$, we can clearly choose subcoalgebras C and D of A so that $c \in L_C^\circ$, $d \in L_D^\circ$, and $C \subset D$ (just enlarge D to be a finite dimensional coalgebra containing both C and D).

So let $[c], [d] \in \varinjlim L_X^\circ$, with $c \in L_C^\circ$, $d \in L_D^\circ$, and $C \subset D$, and suppose that ϕ maps $[c]$ and $[d]$ to the same element. Consider the diagram

To say that $\phi([c]) = \phi([d])$ is the same as saying that $T_C^\circ(c) = T_D^\circ(d)$. But T_D is surjective by lemma 9.1.1, thus T_D° is injective. By commutativity of

we see that we must have $T_{C,D}^\circ(c) = d$; this means that $[c] = [d]$. $\square$

The map ϕ is by no means generally surjective; the author verified this with a

counterexample which he will not burden you with.

We pause for a moment to see what the map ϕ actually looks like. A typical element of $A = \varinjlim L_X^\circ$ is an equivalence class $[\alpha : L_Y \to k]$, where Y is some object of $\mathcal{C}$. Passing this element through ϕ is the same as pulling it back through ϕ_Y, and passing back up through T_Y°. Thus $\phi([\alpha : Y \to k])$ is the composition

$$\varprojlim_X L_X \xrightarrow{T_Y} L_Y \xrightarrow{\alpha} k$$

Recall the definition of the finite dual L° of the algebra L; it consists of those linear functionals $\alpha : L \to k$ which happen to kill an ideal of L having finite codimension. We therefore define the **rank** of an element of L° as the minimum m such that α kills an ideal of codimension m.

Definition 9.1.2. The **rank** of an element of $a \in A$ is the rank of $\phi(a) \in (\varprojlim L_X)^\circ$.

Proposition 9.1.3. *Let $(L, mult)$ be any algebra, (L°, Δ) its finite dual. Then if $\alpha \in L^\circ$ has rank m, $\Delta(\alpha)$ can be written as a sum of no more than m simple tensors. Further, we can write*

$$\Delta(\alpha) = \sum_{i=1}^{m} \beta_i \otimes \gamma_i$$

where each β_i and γ_i themselves have rank no larger than m.

Proof. Let $I \triangleleft L$ be an ideal of codimension m such that $\alpha(I) = 0$, and let $e_1, \ldots, e_m \in L$ be such that $e_1 + I, \ldots, e_m + I$ is a basis for L/I, and extend the e_i to a basis $e_1, \ldots, e_m, f_1, f_2, \ldots$ for all of L. Recall the definition of Δ in terms of mult; it sends the functional α to the map

$$L \otimes L \xrightarrow{\text{mult}} L \xrightarrow{\alpha} k$$

and then passes to the isomorphism $(L \otimes L)^\circ \simeq L^\circ \otimes L^\circ$. We have a basis for $L \otimes L$, namely those tensors of the form $e_i \otimes f_j, f_i \otimes e_j, f_i \otimes f_j, e_i \otimes e_j$. Since I is an ideal, the only of these basis elements that might *not* get sent to I under mult are those of

the form $e_i \otimes e_j$, $i, j \leq m$. Then since α kills I, $\Delta(\alpha)$ kills all but the $e_i \otimes e_j$. Letting γ_i be the functional that sends e_i to 1 and everything else to zero, we can write

$$\Delta(\alpha) = \sum_{i,j} c_{ij} \gamma_i \otimes \gamma_j = \sum_i \gamma_i \otimes \left(\sum_j c_{ij} \gamma_j \right)$$

for some scalars c_{ij}, which is a sum of no more than m simple tensors. As each γ_i still kills I, $\mathrm{rank}(\gamma_i) \leq m$ for each i, and the last claim is proved as well. $\square$

We shall need the following lemma from linear algebra.

Lemma 9.1.4. *Let V, W be vector spaces over some field, and let $\sum_{i=1}^n v_i \otimes w_i \in V \otimes W$. Then this expression is of minimal tensor length if and only if the vectors v_i are linearly independent, and the w_i are linearly independent.*

Proof. Suppose that one of the collections v_i or w_i are not linearly independent, let's say the v_i. Then say v_n is in the span of $v_1, \ldots, v_{n-1}$, and write

$$v_n = a_1 v_1 + \ldots + a_{n-1} v_{n-1}$$

Then

$$\sum_{i=1}^n v_i \otimes w_i = \sum_{i=1}^{n-1} v_i \otimes w_i + v_n \otimes w_n$$
$$= \sum_{i=1}^{n-1} v_i \otimes w_i + \left(\sum_{i=1}^{n-1} a_i v_i \right) \otimes w_n$$
$$= \sum_{i=1}^{n-1} v_i \otimes (w_i + a_i w_n)$$

which is a sum of less than n simple tensors. Therefore the expression is not of minimal tensor length.

Conversely, suppose that $\sum_{i=1}^n v_i \otimes w_i$ can be reduced in tensor length, and that both the collections v_i and w_i are linearly independent; we shall force a contradiction.

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Suppose we have a tensor length reduction given by the equation

$$\sum_{i=1}^{n} v_i \otimes w_i = \sum_{i=1}^{n} v_i' \otimes w_i' \tag{9.1.2}$$

where we let $v_n' = w_n' = 0$. The v_i, v_i' and w_i, w_i' span finite dimensional subspaces of V and W respectively; fix bases $\{e_1, \ldots, e_m\}$, $\{f_1, \ldots, f_l\}$ for these subspaces. Then write

$$v_i = \sum_{j=1}^{m} c_{ij} e_j \qquad w_i = \sum_{k=1}^{l} d_{ik} f_k$$

$$v_i' = \sum_{j=1}^{m} c_{ij}' e_j \qquad w_i' = \sum_{k=1}^{l} d_{ik}' f_k$$

If we plug these expressions into equation 9.1.2 above and rearrange the summations a bit, we obtain

$$\sum_{\substack{j=1\ldots m \\ k=1\ldots l}} \left(\sum_{i=1}^{n} c_{ij} d_{ik} \right) e_j \otimes f_k = \sum_{\substack{j=1\ldots m \\ k=1\ldots l}} \left(\sum_{i=1}^{n} c_{ij}' d_{ik}' \right) e_j \otimes f_k$$

By matching coefficients on the linearly independent simple tensors $e_j \otimes f_k$, we obtain, for every j and k, $\sum_{i=1}^{n} c_{ij} d_{ik} = \sum_{i=1}^{n} c_{ij}' d_{ik}'$. But the left hand side is the $(j, k)^{\text{th}}$ entry of the matrix $(c_{ij})^T (d_{ij})$, and the right hand side the $(j, k)^{\text{th}}$ entry of $(c_{ij}')^T (d_{ij}')$. Thus, equation 9.1.2 is equivalent to the matrix equation

$$(c_{ij})^T (d_{ij}) = (c_{ij}')^T (d_{ij}')$$

Now since the v_i sit in an m-dimensional space and the w_i sit in an l-dimensional space, and since we are assuming both the v_i, w_i to be linearly independent, we conclude that n is no bigger than either m or l. Further, the linear independence of the v_i is equivalent to the linear independence of the row vectors of the matrix (c_{ij}), i.e. the column vectors of $(c_{ij})^T$. Similarly the row vectors of (d_{ij}) are linearly

independent. This means that the matrix $(c_{ij})^T$ has fullest possible rank, namely n; for the same reason (d_{ij}) has rank n. Viewing the product $(c_{ij})^T(d_{ij})$ as a linear transformation $k^l \to k^n \to k^m$, we see that this product also has rank n.

But we claim that $(c'_{ij})^T(d'_{ij})$ has rank less than n. The condition that $v'_n = w'_n = 0$ forces the matrix $(c'_{ij})^T$ to have a column of zeroes at the far right, and (d'_{ij}) to have a row of zeroes at the bottom. Then $(c'_{ij})^T$ has less than n non-zero column vectors, and so has rank less than n; similarly (d_{ij}) has rank less than n. Then clearly also must their product.

We conclude then that $(c_{ij})^T(d_{ij}) = (c'_{ij})^T(d'_{ij})$ have different rank, a contradiction. This completes the proof. $\qquad\square$

Corollary 9.1.5. *Let $\psi : V \to W$ be an injective mapping of vector spaces, and let $v \in V \otimes V$. Then if $(\psi \otimes \psi)(v) \in W \otimes W$ can be written as a sum of no more than m simple tensors, so can v.*

Proof. Write $v = \sum_{i=1}^n v_i \otimes v'_i$ as a minimal sum of simple tensors, so by lemma 9.1.4 the v_i and the v'_i are linearly independent. As ψ is injective, the collections $\psi(v_i)$ and $\psi(v'_i)$ are also linearly independent. Then the expression $(\psi \otimes \psi)(v) = \sum_{i=1}^n \psi(v_i) \otimes \psi(v'_i)$, again by lemma 9.1.4, is of minimal tensor length in $W \otimes W$. $\qquad\square$

We can now prove the key fact which allows us to define a natural coalgebra structure on A_R.

Proposition 9.1.6. *If $a \in A = \varinjlim L_X^\circ$ has rank no greater than m, then $\Delta(a)$ can be written as*

$$\Delta(a) = \sum_{i=1}^m b_i \otimes c_i$$

where each b_i and c_i themselves have rank no greater than m.

Proof. Let Δ be the coalgebra structure on $\varinjlim L_X^\circ$, Δ' that on $(\varprojlim L_X)^\circ$, and let $\phi(a) = \alpha$. As α has rank no greater than m, proposition 9.1.3 tells us that $\Delta'(\alpha)$ can

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be written as a sum of no more than m simple tensors. As $\phi : \varinjlim L_X^\circ \to (\varprojlim L_X)^\circ$ is injective, and since ϕ is a map of coalgebras, we have

$$(\phi \otimes \phi)(\Delta(a)) = \Delta'(\alpha)$$

which, by the previous corollary, shows that $\Delta(a)$ is a sum of no more than m simple tensors. Then write

$$\Delta(a) = \sum_{i=1}^{n} b_i \otimes c_i$$

where n is minimal, and in particular so that the b_i and the c_i are linearly independent, and so that $n \leq m$. We claim that all of the b_i and c_i have rank no greater than m.

Suppose not, and say b_1 has rank greater than m. Let $I \lhd \varprojlim L_X$ be an ideal of codimension m killed by α. Then $\phi(b_1)$ cannot kill all of I; lets say it doesn't kill $f \in I$.

Now since the c_i are linearly independent, so are the $\phi(c_i)$. Then we can find linearly independent vectors $v_1, \ldots, v_n \in \varprojlim L_X$ such that $\phi(c_i)(v_j) = \delta_{ij}$ (lemma 1.5.8 of [4]). Then (under the isomorphism $(\varinjlim L_X)^\circ \otimes (\varinjlim L_X)^\circ \simeq (\varinjlim L_X \otimes \varinjlim L_X)^\circ$), $\sum_i \phi(b_i) \otimes \phi(c_i)$ does not kill the element $f \otimes v_1$.

But $\Delta'(\alpha)$ *does* kill $f \otimes v_1$; its action is given by the composition

$$\varinjlim L_X \otimes \varinjlim L_X \xrightarrow{\text{mult}} \varinjlim L_X \xrightarrow{\alpha} k$$

and as I is an ideal, $f \otimes v_1$ gets mapped into I under mult, and as α kills I, $\Delta'(\alpha)$ kills $f \otimes v_1$. But this is absurd, since $\Delta'(\alpha) = \sum_i \phi(a_i) \otimes \phi(b_i)$.

Thus $\phi(b_1)$ cannot have rank greater than m, and the same argument obviously applies to all of the b_i and c_i. This completes the proof. $\square$

Now let $(A_i, \Delta_i, \varepsilon_i)$ be an indexed collection of Hopf algebras over the fields k_i, $\mathcal{U}$ an ultrafilter on I, $\prod_{\mathcal{U}} A_i$ the ultraproduct (as a vector space) of the A_i, and

let $\phi_i : A_i \to (\varprojlim L_X)^\circ$ be the map defined by diagram 9.1.1 for each A_i. Define the subset $A_R \subset \prod_{\mathcal{U}} A_i$ to consist exactly of those elements $[a_i] \in \prod_{\mathcal{U}} A_i$ such that rank(a_i) (defined by each ϕ_i) is bounded (equivalently, constant); we call this subset the **restricted ultraproduct** of the Hopf algebras A_i. We show now that A_R can be given the structure of a coalgebra.

Consider

$$
\begin{array}{ccccc}
A_R & \subset & \prod_{\mathcal{U}} A_i & \xrightarrow{[\Delta_i]} & \prod_{\mathcal{U}} A_i \otimes A_i \\
& & & & \big\uparrow{\scriptstyle \Phi} \\
& {\scriptstyle \Delta} & & & \prod_{\mathcal{U}} A_i \otimes \prod_{\mathcal{U}} A_i \\
& & & & \big\uparrow{\scriptstyle \subset} \\
& & & & A_R \otimes A_R
\end{array}
$$

We want to show that there is a Δ making this diagram commute, which is simply the assertion that the image of $[\Delta_i]$, when restricted to A_R, is contained inside $A_R \otimes A_R$. For $[a_i] \in A_R$, say with rank m, write

$$[\Delta_i(a_i)] = [\sum_{j=1}^{m} b_{ij} \otimes c_{ij}]$$

where we can take m to be constant over i by proposition 9.1.6; this element is in the image of Φ. Pass it down through Φ^{-1} to

$$\sum_{j=1}^{m} [b_{ij}] \otimes [c_{ij}]$$

Again by proposition 9.1.6 we can take all of the b_{ij} and c_{ij} to have rank $\leq m$, showing that this expression is in fact contained in $A_R \otimes A_R$. Thus Δ does indeed exist.

We define a co-unit map ε from A_R to $\prod_{\mathcal{U}} k_i$ in the obvious manner, as the com-

position

$$\varepsilon : A_R \subset \prod_{\mathcal{U}} A_i \xrightarrow{[\varepsilon_i]} \prod_{\mathcal{U}} k_i$$

We forego the proof that $(A_R, \Delta, \varepsilon)$ satisfy the relevant diagrams making it a coalgebra. There are two diagrams to check, namely diagrams 2.1.1 and 2.1.2; but these follow from the almost everywhere commutativity of these diagrams with respect to each $(A_i, \Delta_i, \varepsilon_i)$, by application of the 'if' direction of proposition 6.2.5 (which holds even in the non-boundedly finite dimensional case), and the naturality of Φ.

We have not yet proved that A_R is a Hopf algebra, that it is closed under multiplication and can be given an antipode map. This will follow later as we show that A_R is coalgebra isomorphic to a Hopf algebra, namely A_∞, the representing Hopf algebra of $\prod_R \mathrm{Comod}_{A_i}$.

9.2 The Map from A_∞ to A_R

Here we define the map of coalgebras from our representing Hopf algebra for $\prod_R \mathrm{Comod}_{A_i}$, called A_∞, to the coalgebra A_R defined in the previous section, which we will show in the next section is an isomorphism.

Let us begin by fixing some notation. I is an indexing set, k_i is a collection of fields indexed by I, k is the ultraproduct of those fields, $(A_i, \Delta_i, \varepsilon_i)$ is a collection of Hopf algebras over those fields, $\mathcal{C}_i$ is the category Comod_{A_i}, ω_i is the fibre (i.e. forgetful) functor on each $\mathcal{C}_i$, $\mathcal{C}$ is the restricted ultraproduct of the categories $\mathcal{C}_i$, and ω is the fibre functor (as defined in theorem 7.2.4) on $\mathcal{C}$. For an object X of $\mathcal{C}_i$, L_X is the image of $\mathrm{End}(\omega_i | \langle X \rangle)$ inside $\mathrm{End}(\omega_i(X))$, and similarly for an object $[X_i]$ of $\mathcal{C}$, $L_{[X_i]}$ is the image of $\mathrm{End}(\omega | \langle [X_i] \rangle)$ inside $\mathrm{End}(\omega([X_i]))$.

Each A_i can be recovered as

$$A_i = \varinjlim_{X_i \in \mathcal{C}_i} L_{X_i}^\circ$$

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and we write the corresponding direct limit diagram as

$$\varinjlim_{\mathcal{C}_i} L^\circ_{X_i}$$

$$\phi_{Y_i} \qquad \phi_{Z_i}$$

$$L^\circ_{Y_i} \xrightarrow{\phi_{Y_i,Z_i}} L^\circ_{Z_i}$$

$$(9.2.1)$$

Note that we have used the same symbol $\phi\_\_$ for the several such existing in each category; no confusion should result.

We also have, in each category, the inverse limit diagram

$$\varprojlim_{\mathcal{C}_i} L_{X_i}$$

$$T_{Y_i} \qquad T_{Z_i}$$

$$L_{Y_i} \xleftarrow{T_{Y_i,Z_i}} L_{Z_i}$$

and again we have used T to stand for the transition maps in all of the categories. We also have the unique map ϕ_i making

$$(\varprojlim_{\mathcal{C}_i} L_{X_i})^\circ$$

$$\phi_i$$

$$T^\circ_{Y_i} \qquad \varinjlim_{\mathcal{C}_i} L^\circ_{X_i} \qquad T^\circ_{Z_i}$$

$$\phi_{Y_i} \qquad \phi_{Z_i}$$

$$L^\circ_{Y_i} \xrightarrow{\phi_{Y_i,Z_i}} L^\circ_{Z_i}$$

commute, as defined by diagram 9.1.1. As A_∞ is by definition the representing Hopf

algebra of $\mathcal{C}$, it can be recovered as a direct limit according to the diagram

$$
\begin{array}{ccc}
 & \varinjlim_{\mathcal{C}} L^{\circ}_{[X_i]} & \\
 {\scriptstyle \phi_{[Y_i]}}\nearrow & & \nwarrow {\scriptstyle \phi_{[Z_i]}} \\
 L^{\circ}_{[Y_i]} & \xrightarrow{\ \phi_{[Y_i],[Z_i]}\ } & L^{\circ}_{[Z_i]}
\end{array}
\qquad (9.2.2)
$$

where the direct system is, as usual, the objects of the category $\mathcal{C}$, with $[X_i] \le [Y_i]$ meaning that $[X_i] \in \langle [Y_i] \rangle$, and the $\phi_{[Y_i],[Z_i]}$ being dual to the transition mappings $T_{[Y_i],[Z_i]}$.

Proposition 9.2.1. *Let $[X_i], [Y_i]$ be objects of $\mathcal{C}$ with $[Y_i] \in \langle [X_i] \rangle$.*

1. *$L_{[X_i]} = \prod_{\mathcal{U}} L_{X_i}$ under the isomorphism $End_k(\omega([X_i])) \simeq \prod_{\mathcal{U}} End_{k_i}(\omega_i(X_i))$*

2. *$L^{\circ}_{[X_i]} = \prod_{\mathcal{U}} L^{\circ}_{X_i}$*

3. *The transition mapping $T_{[Y_i],[X_i]} : L_{[X_i]} \to L_{[Y_i]}$ can be identified with the ultra-product of the transition mappings, $[T_{X_i,Y_i}] : \prod_{\mathcal{U}} L_{X_i} \to \prod_{\mathcal{U}} L_{Y_i}$*

4. *The natural $L_{[X_i]}$-module structure on $\omega([X_i])$ can be identified with the ultra-product of the L_{X_i} module structures on $\omega_i(X_i)$*

5. *The natural $L^{\circ}_{[X_i]}$-comodule structure on $\omega([X_i])$ can be identified with the ultra-product of the $L^{\circ}_{X_i}$-comodule structures on $\omega_i(X_i)$*

Proof. Let $[X_i]$ have dimension n. To prove the first claim, we work through the characterization of $L_{[X_i]}$ given by theorem 3.4.3. We start by fixing an isomorphism $\alpha : k^n \to \omega([X_i])^*$. As $\omega([X_i])^* = (\prod_{\mathcal{U}} \omega_i(X_i))^*$ can be identified with $\prod_{\mathcal{U}} \omega_i(X_i)^*$, α can be uniquely written as $[\alpha_i : k_i^n \to \omega_i(X_i)^*]$. For our ψ we may as well choose the identity map $[X_i]^n \to [X_i]^n$, since any subobject factors through it; then ψ is the ultraproduct of the identity maps $[\psi^i = 1_i] : [X_i^n] \to [X_i^n]$. It is also clear that the

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map $\psi_\alpha : \omega([X_i]^n) \to \omega([X_i])^* \otimes \omega([X_i])$ can be identified with the ultraproduct of the maps $\psi^i_{\alpha_i} : \omega_i(X_i)^n \to \omega_i(X_i)^* \otimes \omega_i(X_i)$, if we allow a factorization through the isomorphism $\prod_{\mathcal{U}} \omega_i(X_i)^* \otimes \omega_i(X_i) \simeq (\prod_{\mathcal{U}} \omega_i(X_i))^* \otimes \prod_{\mathcal{U}} \omega_i(X_i)$.

Next we are asked to find $P^\alpha_{[X_i]}$, the smallest subobject of $[X_i]^n$ such that $\psi_\alpha(\omega(P^\alpha_{[X_i]}))$ contains id $: \omega([X_i]) \to \omega([X_i])$. $P^\alpha_{[X_i]}$ is an object of $\prod_R \mathcal{C}_i$, and can be written as $P^\alpha_{[X_i]} = [Y_i]$ for some collection Y_i of objects of $\mathcal{C}_i$. We claim $Y_i = P^{\alpha_i}_{X_i}$ for al- most every i. We can identify the element id $\in \omega([X_i])^* \otimes \omega([X_i])$ with the element $[\mathrm{id}_i] \in \prod_{\mathcal{U}} \omega_i(X_i)^* \otimes \omega_i(X_i)$, and the concepts of "smallest" and "subobject of" are both first-order. Thus the following statements are equivalent:

1. $[Y_i]$ is the smallest subobject of $[X_i]^n$ such that $\psi_\alpha(\omega([Y_i]))$ contains id $: \omega([X_i]) \to \omega([X_i])$

2. For almost every i, Y_i is the smallest subobject of X_i^n such that $\psi^i_{\alpha_i}(\omega_i(Y_i))$ contains $\mathrm{id}_i : \omega_i(X_i) \to \omega_i(X_i)$

Then we must have $P^\alpha_{[X_i]} = [P^{\alpha_i}_{X_i}]$, whence

$$L_{[X_i]} = \omega(P^\alpha_{[X_i]}) = \prod_{\mathcal{U}} \omega_i(P^{\alpha_i}_{X_i}) = \prod_{\mathcal{U}} L_{X_i}$$

and claim 1. is proved.

Claim 2. is immediate, as the taking of duals is known to distribute over ultra-products for boundedly finite dimensional collections of algebras (proposition 6.3.2).

For claim 3. we note that since $[X_i] \in \langle [Y_i] \rangle$, $X_i \in \langle Y_i \rangle$ for almost every i (lemma 9.2.2), and so T_{X_i,Y_i} is defined for almost every i. To prove the claim we look to the definition of the transition mapping. Let $[Y_i] \in \langle [X_i] \rangle$, and suppose for example that $[Y_i]$ is a subobject of $[X_i]$, under the map $[\iota_i]$. By 1. above, every member of $L_{[X_i]}$ is of the form $[\phi_i]$, where $\phi_i \in L_{X_i}$ for almost every i. Then the image of $[\phi_i]$ under the

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transition mapping $T_{[Y_i],[X_i]}$ is the unique map $[\sigma_i]$ that makes

$$\prod_{\mathcal{U}}\omega_i(Y_i) \xrightarrow{[\iota_i]} \prod_{\mathcal{U}}\omega_i(X_i)$$

$$[\sigma_i] \downarrow \qquad\qquad \downarrow [\phi_i]$$

$$\prod_{\mathcal{U}}\omega_i(Y_i) \xrightarrow[{[\iota_i]}]{} \prod_{\mathcal{U}}\omega_i(X_i)$$

commute, which is equivalent to the almost everywhere commutativity of

$$\omega_i(Y_i) \xrightarrow{\iota_i} \omega_i(X_i)$$

$$\sigma_i \downarrow \qquad\qquad \downarrow \phi_i$$

$$\omega_i(Y_i) \xrightarrow[{\iota_i}]{} \omega_i(X_i)$$

which is equivalent to $\sigma_i = T_{Y_i,X_i}(\phi_i)$ for almost every i. Thus $T_{[Y_i],[X_i]}([\phi_i]) = [T_{Y_i,X_i}(\phi_i)]$, and claim 3. is proved.

Claim 4. is merely the statement that, for $[\phi_i] \in L_{[X_i]}$ and $[x_i] \in \omega([X_i])$, $[\phi_i]([x_i]) = [\phi_i(x_i)]$, which is true by definition. Claim 5. is similarly proved.

$\square$

Part 2. of the above proposition tells us that, instead of the direct limit diagram 9.2.2, we can write instead

$$\varinjlim_{\mathcal{C}} \prod_{\mathcal{U}} L^{\circ}_{X_i}$$

$$\phi_{[Y_i]} \nearrow \qquad \nwarrow \phi_{[Z_i]}$$

$$\prod_{\mathcal{U}} L^{\circ}_{Y_i} \xrightarrow[{[\phi_{Y_i,Z_i}]}]{} \prod_{\mathcal{U}} L^{\circ}_{Z_i}$$

with the understanding that $\phi_{[Z_i]}$ is factoring through the isomorphism $L^{\circ}_{[Z_i]} \simeq \prod_{\mathcal{U}} L^{\circ}_{Z_i}$.

Lemma 9.2.2. *If $[X_i], [Y_i]$ are objects of $\mathcal{C}$, and if $[X_i] \in \langle [Y_i] \rangle$, then $X_i \in \langle Y_i \rangle$ for*

almost every i.

Proof. All of the concepts "is a subobject of", "is a quotient of", and (for *fixed n*) "is isomorphic to an n-fold direct sum of" are first-order statements in the language of abelian tensor categories. To say that $[X_i] \in \langle [Y_i] \rangle$ means that, for some fixed n, $[X_i]$ is a subobject of a quotient of $[Y_i]^n$. Apply theorem C.0.15 to see that the same must be true for almost every i. $\qquad\square$

Now, let us take diagram 9.2.1 and apply ultraproducts:

$$
\begin{array}{ccc}
 & \prod_{\mathcal{U}} \varinjlim_{\mathcal{C}_i} L^\circ_{X_i} & \\
{}^{[\phi_{Y_i}]}\nearrow & & \nwarrow{}^{[\phi_{Z_i}]} \\
\prod_{\mathcal{U}} L^\circ_{Y_i} & \xrightarrow{\ [\phi_{Y_i,Z_i}]\ } & \prod_{\mathcal{U}} L^\circ_{Z_i}
\end{array}
$$

As it stands this diagram is a bit nonsensical: $\prod_{\mathcal{U}} \varinjlim_{\mathcal{C}_i} L^\circ_{X_i}$ is little more than a set, being the ultraproduct of a collection of Hopf algebras, and lacking any kind of coalgebra structure. We claim however that each of the maps $[\phi_{Y_i}]$ have their image inside $A_R \subset \prod_{\mathcal{U}} A_i = \prod_{\mathcal{U}} \varinjlim_{\mathcal{C}_i} L^\circ_{X_i}$, the restricted ultraproduct of the Hopf algebras A_i.

Proposition 9.2.3. *The image of each $[\phi_{Y_i}]$, for $[Y_i] \in \mathcal{C}$, is contained inside A_R.*

Proof. Consider, for fixed i, the diagram

$$
\begin{array}{ccc}
 & (\varprojlim_{\mathcal{C}_i} L_{X_i})^\circ & \\
 & \uparrow{}^{\phi_i} & \\
{}^{T^\circ_{Y_i}}\nearrow & \varinjlim_{\mathcal{C}_i} L^\circ_{X_i} & \nwarrow{}^{T^\circ_{Z_i}} \\
 & {}^{\phi_{Y_i}}\nearrow \qquad \nwarrow{}^{\phi_{Z_i}} & \\
L^\circ_{Y_i} & \xrightarrow{\ \phi_{Y_i,Z_i}\ } & L^\circ_{Z_i}
\end{array}
$$

We claim that if Y_i has dimension n, then for any $\alpha \in L_{Y_i}^\circ$, $\phi_{Y_i}(\alpha)$ has rank no larger than n^2. Commutativity of the above gives

$$\phi_i(\phi_{Y_i}(\alpha)) = T_{Y_i}^\circ(\alpha)$$

which is equal to the composition

$$\varprojlim_{\mathcal{C}_i} L_{X_i} \xrightarrow{T_{Y_i}} L_{Y_i} \xrightarrow{\alpha} k$$

Now L_{Y_i}, being a subalgebra of $\mathrm{End}_{\mathrm{Vec}_{k_i}}(\omega_i(Y_i))$, certainly has dimension no larger than n^2; further, the kernel of $T_{Y_i} \circ \alpha$ contains the kernel of T_{Y_i}. But T_{Y_i} is an algebra map, and so its kernel *is* an ideal of $\varprojlim L_{X_i}$, and has codimension no larger than n^2. Thus $\phi_i(\phi_{Y_i}(\alpha))$ has rank no larger than n^2, hence by definition $\phi_{Y_i}(\alpha)$ has rank no larger than n^2.

Then if $[Y_i] \in \mathcal{C}$, say of constant dimension n, and if $[\alpha_i] \in \prod_\mathcal{U} L_{X_i}^\circ$, then $[\phi_{Y_i}]([\alpha_i]) \stackrel{\mathrm{def}}{=} [\phi_{Y_i}(\alpha_i)]$ has bounded rank, each being no larger than n^2; thus it is contained in A_R. $\qquad\square$

We now have two diagrams

where, in the second diagram we have replaced $\prod_\mathcal{U} \varinjlim_{\mathcal{C}_i} L_{X_i}^\circ$ with A_R, as we may by the previous proposition; some routine arguing shows that since $\phi_{Y_i} : L_{Y_i}^\circ \to A_i$ is a coalgebra map for every i, then $[\phi_{Y_i}] : \prod_\mathcal{U} L_{Y_i}^\circ \to A_R$ is also a coalgebra map. Some care must be taken here; on the right, the map $[\phi_{Y_i,Z_i}]$ is defined whenever $Y_i \in \langle Z_i \rangle$

for almost every i, while on the left, it is only defined when $[Y_i] \in \langle [Z_i] \rangle$. Nonetheless, by lemma 9.2.2, whenever it is defined on the left, it is defined on the right. We can now appeal to the universal property of direct limits to invoke the existence of a unique coalgebra map Ω making the following diagram commute:

This $\Omega : A_\infty \to A_R$ is our claimed isomorphism of coalgebras, later to be shown, of Hopf algebras.

9.3 Ω is an Isomorphism

This Ω, while difficult to define, is not that difficult to describe. A typical element of $A_\infty = \varinjlim \prod_{\mathcal{U}} L_{X_i}^\circ$ looks like

$$[[\alpha_i : L_{X_i} \to k_i]_{\mathcal{U}}]_{\mathcal{C}}$$

That is, it is an equivalence class of equivalence classes of linear functionals, with $\mathcal{U}$ denoting the equivalence defined by the ultraproduct with respect to the ultrafilter $\mathcal{U}$, and $\mathcal{C}$ denoting the equivalence defined by the direct limit over $\mathcal{C}$. Each α_i is an arbitrary linear functional, subject only to the restriction that the objects X_i have bounded dimension.

A typical element of $A_R \subset \prod_{\mathcal{U}} \varinjlim L_{X_i}^\circ$ on the other hand looks like

$$[[\alpha_i : L_{X_i} \to k_i]_{\mathcal{C}_i}]_{\mathcal{U}}$$

where $\mathcal{C}_i$ denotes the equivalence defined by the direct limit over each $\mathcal{C}_i$. The X_i here are not assumed to have bounded dimension; only that the functionals α_i have bounded rank.

The action of Ω is simple then:

$$\Omega : [[\alpha_i : L_{X_i} \to k_i]_{\mathcal{U}}]_{\mathcal{C}} \longmapsto [[\alpha_i : L_{X_i} \to k_i]_{\mathcal{C}_i}]_{\mathcal{U}} \tag{9.3.1}$$

To see it this way, it is not at all obvious that it is well-defined, or that it is a map of coalgebras, but we know it is, via the way we constructed it.

Lemma 9.3.1. *Let $\mathcal{C}$ be the category $Comod_A$, and let $[\alpha : L_Y \to k]$ be an element of $\varinjlim_{\mathcal{C}} L_X^\circ$ which has rank no greater than m. Then $[\alpha : L_Y \to k]$ can be written as*

$$[\beta : L_C \to k]$$

where C is some subcoalgebra of A having dimension no greater than m.

Proof. Y is in the principal subcategory generated by some subcoalgebra C of A, and so we have the map $L_Y^\circ \xrightarrow{\phi_{Y,C}} L_C^\circ$; this shows that we may as well take $[\alpha : L_Y \to k]$ to be $[\gamma : L_C \to k]$ for some $\gamma \in L_C^\circ$. To say that this element has rank no greater than m is to say that the composition

$$\varprojlim_{\mathcal{C}} L_X \xrightarrow{T_C} L_C \xrightarrow{\gamma} k$$

kills an ideal $I \lhd \varprojlim L_X$ of codimension no greater than m. Let J be the kernel of T_C. We can assume that I contains J; if not, enlarge I to $I + J$, which is still an

ideal contained in $\ker(T_C \circ \gamma)$ having codimension no larger than m. Then we have a commutative diagram

$$
\begin{array}{ccc}
\varprojlim\limits_{\mathcal{C}} L_X & \xrightarrow{\;T_C\;} & L_C \\
& \searrow{\scriptstyle \pi} & \downarrow{\scriptstyle \pi'} \\
& & (\varprojlim\limits_{\mathcal{C}} L_X)/I
\end{array}
$$

where π is the natural projection, and π' is the unique surjective map gotten because $J \subset I$. As $(\varprojlim L_X)/I$ is a quotient algebra of L_C, theorem 8.1.2 guarantees that it is isomorphic to L_D for some subcoalgebra D of C, and that under this identification we can take π' to be the transition map $T_{D,C}$. Thus we have the commutative diagram

$$
\begin{array}{ccc}
\varprojlim\limits_{\mathcal{C}} L_X & \xrightarrow{\;T_C\;} & L_C \\
& \searrow{\scriptstyle T_D} & \downarrow{\scriptstyle T_{D,C}} \\
& & L_D
\end{array}
$$

with L_D of dimension no greater than m. And since $\ker(\gamma) \supset T_C(I) = \ker(T_{D,C})$, there exists a linear functional β making

$$
\begin{array}{ccc}
L_C & \xrightarrow{\;\gamma\;} & k \\
\downarrow{\scriptstyle T_{D,C}} & \nearrow{\scriptstyle \beta} & \\
L_D & &
\end{array}
$$

commute. By definition then, γ and β are equal in the direct limit. We can thus write

$$[\gamma : L_C \to k] = [\beta : L_D :\to k]$$

Finally, since L_D has dimension no greater than m and D is a coalgebra, theorem

8.0.6 tells us that $D \simeq L_D^{\circ}$ has dimension no greater than m. This completes the proof. $\qquad \square$

Proposition 9.3.2. *The map Ω is surjective.*

Proof. Again, a typical element of A_R looks like

$$[[\alpha_i : L_{X_i} \to k_i]_{\mathcal{C}_i}]_{\mathcal{U}}$$

with the α_i having constant rank, say m. Then lemma 9.3.1 shows we can write this instead as

$$[[\beta_i : L_{D_i} \to k_i]_{\mathcal{C}_i}]_{\mathcal{U}}$$

where each D_i is a subcoalgebra of A_i having dimension no larger than m. Then the formula given for Ω at the beginning of this section (equation 9.3.1) shows that

$$[[\beta_i : L_{D_i} \to k_i]_{\mathcal{U}}]_{\mathcal{C}}$$

qualifies as a pre-image for our typical element under Ω. $\qquad \square$

Lemma 9.3.3. *Let G be an affine group scheme represented by the Hopf algebra A over a field k. Let (V, ρ) be an n-dimensional A-comodule, fix a basis $e_1, \ldots, e_n$ for V, and let (a_{ij}) be the matrix formula of the representation of G it defines in that basis. Then $C = span_k(a_{ij} : 1 \le i, j \le n)$ is a no more than n^2-dimensional subcoalgebra of A. Further, (V, ρ) can be embedded, as a comodule, into C^n.*

Proof. Apply the comodule identity $\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$ (equation 10.0.2) to see that $\Delta(C) \subset C \otimes C$, whence C is a subcoalgebra of A. For the embedding claim, we examine the embedding $V \to A^n$ (n-fold direct sum of the regular representation) defined in section 3.5 of [16]; we claim that the image of this embedding is in fact contained in $C^n \subset A^n$. Let $\Psi : V \otimes A \to A^n$ denote the vector space isomorphism

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$e_i \otimes a \mapsto (0, \ldots, a, \ldots, 0)$ (a in the ith slot, zeroes elsewhere). Consider

$$
\begin{array}{ccccc}
V & \xrightarrow{\rho} & V \otimes A & \xrightarrow{\Psi} & A^n \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle 1 \otimes \Delta} & & \downarrow{\scriptstyle \Delta^n} \\
 & & & & (A \otimes A)^n \\
 & & & & \downarrow{\scriptstyle \simeq} \\
V \otimes A & \xrightarrow[\rho \otimes 1]{} & V \otimes A \otimes A & \xrightarrow[\Psi \otimes 1]{} & A^n \otimes A
\end{array}
$$

Commutativity of the left rectangle is a comodule identity (see diagram 2.2.1), and commutativity of the right rectangle is obvious, whence this entire diagram commutes. Note that the composition that starts at the top right and goes directly down is by definition the comodule structure on A^n (see definition 2.3.3). Looking at a condensed version of the outermost rectangle

$$
\begin{array}{ccc}
V & \xrightarrow{\rho \circ \Psi} & A^n \\
\downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \Delta^n} \\
 & & (A \otimes A)^n \\
 & & \downarrow{\scriptstyle \simeq} \\
V \otimes A & \xrightarrow[(\rho \circ \Psi) \otimes 1]{} & A^n \otimes A
\end{array}
$$

we see that $\rho \circ \Psi$ is an embedding of V into A^n. And if we chase the basis element e_j from V to A^n we arrive at

$$
e_j \xmapsto{\rho} \sum_i e_i \otimes a_{ij} \xmapsto{\Psi} (a_{1j}, a_{2j}, \ldots, a_{nj})
$$

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which is an element of C^n. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 9.3.4. Ω *is injective.*

Proof. Let $[[\alpha_i : L_{X_i} \to k_i]_{\mathcal{U}}]_{\mathcal{C}}$ and $[[\beta_i : L_{Y_i} \to k_i]_{\mathcal{U}}]_{\mathcal{C}}$ be two typical elements of A_∞ such that Ω maps them to the same thing. This means that

$$[[\alpha_i : L_{X_i} \to k_i]_{\mathcal{C}_i}]_{\mathcal{U}} = [[\beta_i : L_{Y_i} \to k_i]_{\mathcal{C}_i}]_{\mathcal{U}}$$

which is to say that, for almost every i,

$$[\alpha_i : L_{X_i} \to k_i]_{\mathcal{C}_i} = [\beta_i : L_{Y_i} \to k_i]_{\mathcal{C}_i}$$

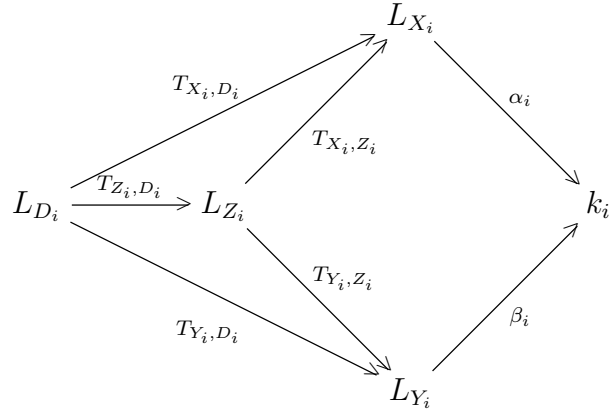which is to say that, for almost every i, there is a Z_i such that $X_i, Y_i \in \langle Z_i \rangle$ and
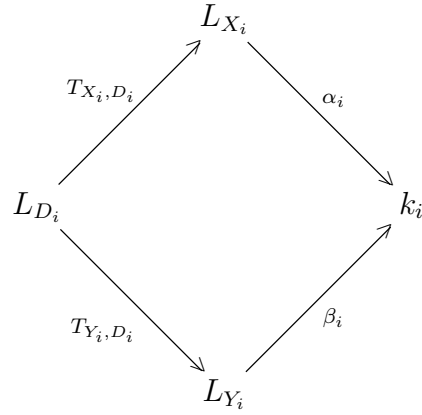

commutes. Now the Z_i are, as far as we know, not of bounded dimension, so we have some work to do. By lemma 9.3.3, for each i let C_i be a subcoalgebra of A_i such that C_i has dimension no larger than $\dim(X_i \oplus Y_i)^2$, and such that $X_i \oplus Y_i$ is embeddable in $C_i^{\dim(X_i \oplus Y_i)}$. Note in particular that this implies that both $[X_i]$ and $[Y_i]$ belong to the principal subcategory generated by $[C_i]$ (since they are both subobjects of a subobject of $[C_i^m] = [C_i]^m$ for a fixed m).

For each i let D_i be a subcoalgebra generating all of the X_i, Y_i, Z_i and containing C_i, which of course we cannot assume is of bounded dimension. Then for every i we
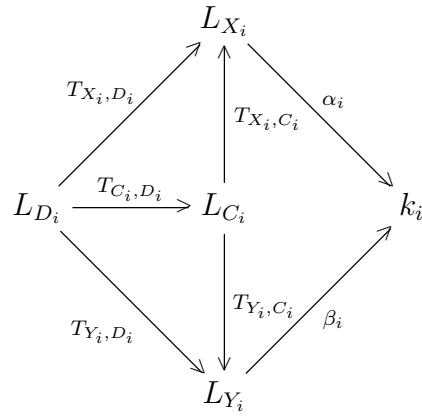
have a commutative diagram

$$L_{D_i} \xrightarrow{T_{Z_i,D_i}} L_{Z_i} \qquad L_{X_i} \qquad k_i \qquad L_{Y_i}$$

(diagram with vertices L_{X_i}, L_{D_i}, L_{Z_i}, k_i, L_{Y_i}; arrows labelled T_{X_i,D_i}, T_{X_i,Z_i}, α_i, T_{Z_i,D_i}, T_{Y_i,Z_i}, T_{Y_i,D_i}, β_i)

and in particular, the outermost diamond commutes:

(diamond with vertices L_{X_i}, L_{D_i}, k_i, L_{Y_i}; arrows labelled T_{X_i,D_i}, α_i, T_{Y_i,D_i}, β_i)

and hence so does

(diagram with vertices L_{X_i}, L_{D_i}, L_{C_i}, k_i, L_{Y_i}; arrows labelled T_{X_i,D_i}, T_{X_i,C_i}, α_i, T_{C_i,D_i}, T_{Y_i,C_i}, β_i, T_{Y_i,D_i})

As $C_i \subset D_i$ are subcoalgebras, proposition 8.1.1 tells us that T_{C_i,D_i} is surjective. Then commutativity of the above gives $T_{C_i,D_i} \circ T_{X_i,C_i} \circ \alpha_i = T_{C_i,D_i} \circ T_{Y_i,C_i} \circ \beta_i$, and since

T_{C_i,D_i} is surjective, this gives us commutativity of

$$
\begin{array}{ccc}
 & L_{X_i} & \\
T_{X_i,C_i} \nearrow & & \searrow \alpha_i \\
L_{C_i} & & k_i \\
T_{Y_i,C_i} \searrow & & \nearrow \beta_i \\
 & L_{Y_i} &
\end{array}
$$

Now apply ultraproducts to yield a commutative diagram

$$
\begin{array}{ccc}
 & \prod_{\mathcal{U}} L_{X_i} & \\
[T_{X_i,C_i}] \nearrow & & \searrow [\alpha_i] \\
\prod_{\mathcal{U}} L_{C_i} & & \prod_{\mathcal{U}} k_i \\
[T_{Y_i,C_i}] \searrow & & \nearrow [\beta_i] \\
 & \prod_{\mathcal{U}} L_{Y_i} &
\end{array}
$$

Note that $[C_i]$, being of bounded dimension, is an object of $\mathcal{C}$. Then if we identify $\prod_{\mathcal{U}} L_{C_i}$ with $L_{[C_i]}$, $[T_{X_i,C_i}]$ with $T_{[X_i],[C_i]}$, etc. (as we may by proposition 9.2.1), commutativity of the above implies the equality of $[\alpha_i]$ and $[\beta_i]$ in the direct limit over $\mathcal{C}$; that is

$$
[[\alpha_i : L_{X_i} \to k_i]_{\mathcal{U}}]_{\mathcal{C}} = [[\beta_i : L_{Y_i} \to k_i]_{\mathcal{U}}]_{\mathcal{C}}
$$

as desired. $\qquad \square$

Theorem 9.3.5. *The representing Hopf algebra of the restricted ultraproduct $\prod_R Comod_{A_i}$ is coalgebra-isomorphic to the restricted ultraproduct A_R of the Hopf algebras A_i.*

Proof. Apply propositions 9.3.2 and 9.3.4. $\qquad \square$

9.4 The Equivalence $\prod_R \mathcal{C}_i \simeq \mathbf{Comod}_{A_R}$

For a collection of Hopf algebras A_i, the previous section shows that A_R is coalgebra isomorphic to A_∞, the representing Hopf algebra of $\prod_R \mathrm{Comod}_{A_i}$. Then as A_∞ is a Hopf algebra, so is A_R, under whatever multiplication and antipode map are induced on it by Ω. We would like to of course prove that this induced multiplication and antipode are exactly those inherited by being a subset of $\prod_{\mathcal{U}} A_i$; i.e. that they are the ultraproduct of the individual multiplications and antipodes on the A_i restricted to A_R.

We will prove this for multiplication; we do not prove it for antipode, but believe a similar proof to the one we give for multiplication (using instead the dual construction instead of the tensor product) could be constructed.

Our first step is to build the equivalence from the category $\prod_R \mathcal{C}_i$ to Comod_{A_R} induced by the isomorphism Ω; here our work will finally start to pay off, as this equivalence is quite natural and easy to describe. Examination of this equivalence will further yield the required multiplication on A_R, as we examine the tensor product on Comod_{A_R} induced by this equivalence.

First, following the construction mentioned in theorem 3.3.4, we build the equivalence $G : \prod_R \mathrm{Comod}_{A_i} \to \mathrm{Comod}_{A_\infty}$. To keep notation simple we use the same symbol X for an object of Comod_A and its image under the fibre functor, and similarly for a morphism.

Let $[X_i, \rho_i : X_i \to X_i \otimes A_i]$ be an object of $\prod_R \mathcal{C}_i$. The remarks before theorem 3.3.4 tell us we should define the A_∞-comodule structure on $\prod_{\mathcal{U}} X_i$ to be the composition

$$\prod_{\mathcal{U}} X_i \xrightarrow{\rho'} \prod_{\mathcal{U}} X_i \otimes L^\circ_{[X_i]} \xrightarrow{1 \otimes \phi_{[X_i]}} \left(\prod_{\mathcal{U}} X_i\right) \otimes A_\infty$$

where ρ' is the natural $L^\circ_{[X_i]}$-comodule structure on $\prod_{\mathcal{U}} X_i$. But proposition 9.2.1 says that we can replace $L^\circ_{[X_i]}$ with $\prod_{\mathcal{U}} L^\circ_{X_i}$, and in so doing can define ρ' in terms of the

individual $L_{X_i}^{\circ}$-comodule structures on each X_i, whom we call ρ_i'; that is

$$\prod_{\mathcal{U}} X_i \xrightarrow{\qquad \rho' \qquad} \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} L_{X_i}^{\circ} \xrightarrow{1 \otimes \phi_{[X_i]}} (\prod_{\mathcal{U}} X_i) \otimes A_{\infty}$$

with $[\rho_i']$ down to $\prod_{\mathcal{U}} X_i \otimes L_{X_i}^{\circ}$ and Φ^{-1} up.

commutes. The A_{∞}-comodule structure on $\prod_{\mathcal{U}} X_i$, call it ρ, is thus the composition

$$\prod_{\mathcal{U}} X_i \xrightarrow{[\rho_i']} \prod_{\mathcal{U}} X_i \otimes L_{X_i}^{\circ} \xrightarrow{\Phi^{-1}} \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} L_{X_i}^{\circ} \xrightarrow{1 \otimes \phi_{[X_i]}} (\prod_{\mathcal{U}} X_i) \otimes A_{\infty}$$

$G([X_i])$ is thus $\prod_{\mathcal{U}} X_i$, with the above A_{∞} comodule structure. For a morphism $[\psi_i : X_i \to Y_i]$ in $\prod_R \mathcal{C}_i$, we define of course $G([\psi_i])$ to be $[\psi_i] : \prod_{\mathcal{U}} X_i \to \prod_{\mathcal{U}} Y_i$.

The next step is to pass to the isomorphism Ω to obtain an equivalence of categories $\prod_R \mathcal{C}_i \to \mathrm{Comod}_{A_R}$.

Theorem 9.4.1. *Define a functor $F : \prod_R \mathcal{C}_i \to \mathrm{Comod}_{A_R}$ as follows. F sends the object $[X_i, \rho_i : X_i \to X_i \otimes A_i]$ to the vector space $\prod_{\mathcal{U}} X_i$ with the A_R-comodule structure*

$$\prod_{\mathcal{U}} X_i \xrightarrow{[\rho_i]} \prod_{\mathcal{U}} X_i \otimes A_i \xrightarrow{\Phi^{-1}} \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} A_i \supset (\prod_{\mathcal{U}} X_i) \otimes A_R$$

and sends the morphism $[\psi : X_i \to Y_i]$ to $[\psi_i] : \prod_{\mathcal{U}} X_i \to \prod_{\mathcal{U}} Y_i$. Then F is the equivalence of categories induced on G by Ω.

Proof. Consider the diagram

$$\prod_{\mathcal{U}} X_i \xrightarrow{[\rho_i']} \prod_{\mathcal{U}} X_i \otimes L_{X_i}^{\circ} \xrightarrow{\Phi^{-1}} \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} L_{X_i}^{\circ} \xrightarrow{1 \otimes \phi_{[X_i]}} (\prod_{\mathcal{U}} X_i) \otimes A_{\infty}$$

with $[\rho_i]$ down to $\prod_{\mathcal{U}} X_i \otimes A_i$, maps $[1 \otimes \phi_{X_i}]$, $1 \otimes [\phi_{X_i}]$, $1 \otimes [\phi_{X_i}]$, $1 \otimes \Omega$, and bottom row

$$\prod_{\mathcal{U}} X_i \otimes A_i \xrightarrow{\Phi^{-1}} \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} A_i \xleftarrow{\quad} (\prod_{\mathcal{U}} X_i) \otimes A_R$$

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The composition that starts at the top left, goes all the away across, and then down, is the functor gotten from G by Ω; that is, the top line is the A_∞-comodule structure on $[X_i]$ under G, and then we tack on $1 \otimes \Omega$ to obtain an A_R-comodule structure. The composition that starts at the top left, goes diagonally down, and then all the way across, is the composition referenced in the statement of the theorem. We want to see then that this diagram commutes. Commutativity of the left-most triangle is equivalent to the almost everywhere commutativity of it, which in turn is simply the statement that the A_i-comodule structure on X_i can be factored through the $L_{X_i}^\circ$-comodule structure for it, and through the canonical injection ϕ_{X_i}. The next square follows automatically from the naturality of Φ (proposition 6.2.14). Commutativity of the next triangle is obvious, as $[\phi_{X_i}]$ is known to point to A_R, and the last triangle follows from the definition of Ω. The theorem is proved. $\qquad \square$

Thus we have an equivalence of categories $F : \prod_R \mathcal{C}_i \to \text{Comod}_{A_R}$ given by the previous theorem. As $\prod_R \mathcal{C}_i$ is a tensor category under $\otimes$, it induces a similar structure on Comod_{A_R} through F, which we call $\overline{\otimes}$, whose action is given as follows. Any two objects of Comod_{A_R} pull back under F to objects of $\prod_R \mathcal{C}_i$ which look like

$$[X_i, \rho_i : X_i \to X_i \otimes A_i]$$

$$[Y_i, \mu_i : Y_i \to Y_i \otimes A_i]$$

Their tensor product in $\prod_R \mathcal{C}_i$ is defined as

$$[X_i \otimes Y_i, X_i \otimes Y_i \xrightarrow{\rho_i \otimes \mu_i} X_i \otimes A_i \otimes Y_i \otimes A_i \xrightarrow{1 \otimes T_i \otimes 1} X_i \otimes Y_i \otimes A_i \otimes A_i \xrightarrow{1 \otimes 1 \otimes \text{mult}_i} X_i \otimes Y_i \otimes A_i]$$

and we push this new object back through F to yield a new object in Comod_{A_R}, having

underlying vector space $\prod_{\mathcal{U}} X_i \otimes Y_i$ and comodule map given by the composition

$$\prod_{\mathcal{U}} X_i \otimes Y_i \xrightarrow{[\rho_i \otimes \mu_i]} \prod_{\mathcal{U}} X_i \otimes A_i \otimes Y_i \otimes A_i \xrightarrow{[1 \otimes T_i \otimes 1]} \prod_{\mathcal{U}} X_i \otimes Y_i \otimes A_i \otimes A_i$$

$$\xrightarrow{[1 \otimes 1 \otimes \mathrm{mult}_i]} \prod_{\mathcal{U}} X_i \otimes Y_i \otimes A_i \xrightarrow{\Phi^{-1}} (\prod_{\mathcal{U}} X_i \otimes Y_i) \otimes \prod_{\mathcal{U}} A_i \supset (\prod_{\mathcal{U}} X_i \otimes Y_i) \otimes A_R$$

Playing the same game we see that two morphisms in Comod_{A_R} pull back to morphisms $[\psi_i : X_i \to V_i], [\xi_i : Y_i \to W_i]$ in $\prod_R \mathcal{C}_i$, and upon taking their tensor product in $\prod_R \mathcal{C}_i$ and pushing them back through F we obtain the image under $\overline{\otimes}$ of these morphisms:

$$\prod_{\mathcal{U}} X_i \otimes Y_i \xrightarrow{[\psi_i \otimes \xi_i]} \prod_{\mathcal{U}} V_i \otimes W_i$$

Now let us modify $\overline{\otimes}$ a bit; simply tack on Φ to both ends of the above to yield

$$\prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} Y_i \xrightarrow{\Phi}$$

$$\prod_{\mathcal{U}} X_i \otimes Y_i \xrightarrow{[\rho_i \otimes \mu_i]} \prod_{\mathcal{U}} X_i \otimes A_i \otimes Y_i \otimes A_i \xrightarrow{[1 \otimes T_i \otimes 1]} \prod_{\mathcal{U}} X_i \otimes Y_i \otimes A_i \otimes A_i$$

$$\xrightarrow{[1 \otimes 1 \otimes \mathrm{mult}_i]} \prod_{\mathcal{U}} X_i \otimes Y_i \otimes A_i \xrightarrow{\Phi^{-1}} (\prod_{\mathcal{U}} X_i \otimes Y_i) \otimes \prod_{\mathcal{U}} A_i \supset (\prod_{\mathcal{U}} X_i \otimes Y_i) \otimes A_R$$

$$\xrightarrow{\Phi^{-1} \otimes 1} \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} Y_i \otimes A_R$$

and instead of $[\psi_i \otimes \xi_i]$, write

$$\prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} Y_i \xrightarrow{[\psi_i] \otimes [\xi_i]} \prod_{\mathcal{U}} V_i \otimes \prod_{\mathcal{U}} W_i$$

The naturality of the isomorphism Φ guarantees that this new functor is naturally isomorphic to $\overline{\otimes}$; let us relabel this new functor as $\overline{\otimes}$.

The next proposition simplifies the description of $\overline{\otimes}$, one which doesn't require first pulling an object back to $\prod_R \mathcal{C}_i$.

Proposition 9.4.2. *If (X, ρ), (Y, μ) are objects of $Comod_{A_R}$, then $\overline{\otimes}$ sends this pair*

to the vector space $X \otimes Y$, with comodule map given by the composition

$$X \otimes Y \xrightarrow{\rho \otimes \mu} X \otimes A_R \otimes Y \otimes A_R \subset X \otimes \prod_{\mathcal{U}} A_i \otimes Y \otimes \prod_{\mathcal{U}} A_i$$

$$\xrightarrow{1 \otimes T \otimes 1} X \otimes Y \otimes \prod_{\mathcal{U}} A_i \otimes \prod_{\mathcal{U}} A_i \xrightarrow{1 \otimes 1 \otimes mult} X \otimes Y \otimes \prod_{\mathcal{U}} A_i \supset X \otimes Y \otimes A_R$$

where mult denotes the natural coordinate wise multiplication on $\prod_{\mathcal{U}} A_i$.

Proof. As $F : \prod_R \mathcal{C}_i \to \mathrm{Comod}_{A_R}$ is an equivalence, there is an object $[(X_i, \rho_i)]$ of $\prod_R \mathcal{C}_i$ such that $X = \prod_{\mathcal{U}} X_i$ and ρ is equal to the composition

$$\prod_{\mathcal{U}} X_i \xrightarrow{[\rho_i]} \prod_{\mathcal{U}} X_i \otimes A_i \xrightarrow{\Phi^{-1}} \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} A_i \supset \prod_{\mathcal{U}} X_i \otimes A_R$$

and similarly for (Y, μ). Consider

$$\begin{array}{ccccc}
X \otimes Y & =\!=\!=\!=\!=\!= & \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} Y_i & =\!=\!=\!=\!=\!= & \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} Y_i
\end{array}$$

with vertical maps $\rho\otimes\mu$ (left), $[\rho_i]\otimes[\mu_i]$ (middle), Φ (right):

$$\prod_{\mathcal{U}} X_i \otimes A_i \otimes \prod_{\mathcal{U}} Y_i \otimes A_i \qquad \prod_{\mathcal{U}} X_i \otimes Y_i$$

$\Phi^{-1}\otimes\Phi^{-1}$, $[\rho_i\otimes\mu_i]$

$$\prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} A_i \otimes \prod_{\mathcal{U}} Y_i \otimes \prod_{\mathcal{U}} A_i \overset{\Phi}{\longleftrightarrow} \prod_{\mathcal{U}} X_i \otimes A_i \otimes Y_i \otimes A_i$$

$\subset\otimes\subset$, $[1\otimes T_i\otimes 1]$

$$X \otimes A_R \otimes Y \otimes A_R \;=\!=\!=\; \prod_{\mathcal{U}} X_i \otimes A_R \otimes \prod_{\mathcal{U}} Y_i \otimes A_R \qquad \prod_{\mathcal{U}} X_i \otimes Y_i \otimes A_i \otimes A_i$$

$\subset$, $\subset$, $[1\otimes1\otimes\mathrm{mult}_i]$

$$X \otimes \prod_{\mathcal{U}} A_i \otimes Y \otimes \prod_{\mathcal{U}} A_i \;=\; \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} A_i \otimes \prod_{\mathcal{U}} Y_i \otimes \prod_{\mathcal{U}} A_i \;\;{}_\Phi \qquad \prod_{\mathcal{U}} X_i \otimes Y_i \otimes A_i$$

$1\otimes T\otimes1$, $1\otimes T\otimes1$, Φ

$$X \otimes Y \otimes \prod_{\mathcal{U}} A_i \otimes \prod_{\mathcal{U}} A_i \;=\; \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} Y_i \otimes \prod_{\mathcal{U}} A_i \otimes \prod_{\mathcal{U}} A_i \qquad \prod_{\mathcal{U}} X_i \otimes Y_i \otimes \prod_{\mathcal{U}} A_i$$

$1\otimes1\otimes\mathrm{mult}$, $1\otimes1\otimes\mathrm{mult}$, $\Phi\otimes1$, $\subset$

$$X \otimes Y \otimes \prod_{\mathcal{U}} A_i \;=\!=\!=\!=\; \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} Y_i \otimes \prod_{\mathcal{U}} A_i \qquad \left(\prod_{\mathcal{U}} X_i \otimes Y_i\right) \otimes A_R$$

$\subset$, $\subset$, $\Phi\otimes1$

$$X \otimes Y \otimes A_R \;=\!=\!=\!=\; \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} Y_i \otimes A_R \;=\!=\!=\!=\; \prod_{\mathcal{U}} X_i \otimes \prod_{\mathcal{U}} Y_i \otimes A_R$$

which the author would at this time like to nominate to the Academy as the ugliest diagram of all time. The top left rectangle commutes by the previous remarks, and all of the other simple sub-polygons, though numerous, are easy to check; thus this entire diagram commutes. The composition that starts at the top right and goes all

the way down is the image of (X, ρ) and (Y, μ) under $\overline{\otimes}$ as described previously; the composition that starts at the top left and goes all the way down is the composition given in the statement of the proposition. These are equal, and the proposition is proved. $\square$

So then, we have a bifunctor $\overline{\otimes}$ on Comod_{A_R} which sends two comodules to a new comodule whose underlying vector space is the tensor product of the underlying vector spaces of the comodules. Proposition 3.3.5 tells us that this functor is induced by a unique k-homomorphism $u : A_R \otimes A_R \to A_R$; that is, $\overline{\otimes}$ sends the objects (X, ρ) and (Y, μ) to $X \otimes Y$, with A_R-comodule structure given by the composition

$$X \otimes Y \xrightarrow{\rho \otimes \mu} X \otimes A_R \otimes Y \otimes A_R \xrightarrow{1 \otimes T \otimes 1} X \otimes Y \otimes A_R \otimes A_R$$

$$\xrightarrow{1 \otimes 1 \otimes u} X \otimes Y \otimes A_R$$

We claim that this u is nothing more than the natural multiplication on $\prod_u A_i$ restricted to A_R.

Lemma 9.4.3. *Let (C, Δ) be a coalgebra. Define the subset S of C to consist of those elements $a \in C$ with the following property: there exists a finite dimensional comodule (X, ρ) over C such that, for some element $x \in X$, $\rho(x) = \sum_i x_i \otimes a_i$, with the x_i linearly independent and $a = a_i$ for some i. Then S spans C.*

Proof. Let $c \in C$, and $C' \subset C$ the finite dimensional subcoalgebra it generates. Write $\Delta(c) = \sum_i a_i \otimes b_i$ with the a_i linearly independent. Then obviously all of the b_i are in S if we view C' as a finite dimensional comodule over C. Apply the coalgebra

identity

$$
\begin{array}{ccc}
C & \xrightarrow{\ \Delta\ } & C \otimes C \\
 & \searrow{\scriptstyle \simeq} & \downarrow{\scriptstyle \varepsilon \otimes 1} \\
 & & k \otimes C
\end{array}
$$

to yield $c = \sum_i \varepsilon(a_i) b_i$, showing c to be in the span of the b_i. $\qquad \square$

Theorem 9.4.4. *u is equal to the natural multiplication on $\prod_{\mathcal{U}} A_i$ restricted to A_R.*

Proof. Let (X, ρ),(Y, μ) be finite dimensional comodules over A_R, and consider

$$
\begin{array}{ccc}
X \otimes Y \xrightarrow{\ \rho \otimes \mu\ } X \otimes A_R \otimes Y \otimes A_R \xhookrightarrow{\ } X \otimes \prod_{\mathcal{U}} A_i \otimes Y \otimes \prod_{\mathcal{U}} A_i \\
\end{array}
$$

$$
\downarrow{\scriptstyle \rho \otimes \mu} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle 1 \otimes T \otimes 1}
$$

$$
X \otimes Y \otimes \prod_{\mathcal{U}} A_i \otimes \prod_{\mathcal{U}} A_i
$$

$$
X \otimes A_R \otimes Y \otimes A_R \qquad\qquad\qquad\qquad \downarrow{\scriptstyle 1 \otimes 1 \otimes \mathrm{mult}}
$$

$$
\downarrow{\scriptstyle 1 \otimes T \otimes 1} \qquad\qquad\qquad\qquad\qquad\qquad X \otimes Y \otimes \prod_{\mathcal{U}} A_i
$$

$$
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \uparrow{\scriptstyle \subset}
$$

$$
X \otimes Y \otimes A_R \otimes A_R \xrightarrow{\quad 1 \otimes 1 \otimes u \quad} X \otimes Y \otimes A_R
$$

The composition that starts at the top left, goes across, and then all the way down is the image of the pair (X, ρ), (Y, μ) under $\overline{\otimes}$ as proved in proposition 9.4.2. The one that starts at the top left, goes down, and then across is the bifunctor induced by u; this diagram commutes by assumption.

In the notation of the previous lemma, let $a, b \in S \subset A_R$. This means that there is an (X, ρ) such that, for some $x \in X$, $\rho(x) = \sum_i x_i \otimes a_i$ with the x_i linearly independent and $a = a_i$ for some i. Similarly, there is a (Y, μ) and $y \in Y$ such that

$\mu(y) = \sum_j y_j \otimes b_j$, with the y_j linearly independent and $b = b_j$ for some j. Now if we chase the element $x \otimes y$ in the above diagram around both ways, it gives

$$\sum_{i,j} x_i \otimes y_j \otimes u(a_i \otimes b_j) = \sum_{i,j} x_i \otimes y_j \otimes \mathrm{mult}(a_i \otimes b_j)$$

Since the $x_i \otimes y_j$ are linearly independent, by matching coefficients we must have $u(a_i \otimes b_j) = \mathrm{mult}(a_i \otimes b_j)$ for every i and j; in particular, $u(a \otimes b) = \mathrm{mult}(a \otimes b)$.

Thus we have shown that u and mult are equal on elements of the form $a \otimes b$, with $a, b \in S$. But the previous lemma shows that $S \otimes S$ spans $A_R \otimes A_R$. As both u and mult are k-linear, they must be equal everywhere. This completes the proof. $\square$

9.5 Examples

9.5.1 Finite Groups

Let G be a finite group defined over $\mathbb{Z}$, A its representing Hopf algebra, k_i a sequence of fields, $A_i = A \otimes k_i =$ the representing Hopf algebra of G over k_i, and $\mathcal{C}_i = \mathrm{Comod}_{A_i}$. We claim that the representing Hopf algebra of $\prod_R \mathcal{C}_i$ is nothing more than $A \otimes \prod_{\mathcal{U}} k_i$. The following observation shows this not to be surprising.

Proposition 9.5.1. *The category $\prod_R \mathcal{C}_i$ is equal to the principal subcategory generated by $[A \otimes k_i]$.*

Proof. By $[A \otimes k_i]$ we mean the object of $\prod_R \mathcal{C}_i$ consisting of the regular representation of G over k in every slot; it is clearly of constant finite dimension. Let $[X_i]$ be an object of $\prod_R \mathcal{C}_i$, say of dimension n. Then by theorem 2.2.2 each X_i is a subobject of a quotient of $(A \otimes k_i)^n$; this is a first-order statement, and so $[X_i]$ is a subobject of a quotient of $[A \otimes k_i]^n$, showing $[X_i]$ to be in the principal subcategory generated by $[A \otimes k_i]$. $\square$

Proposition 2.20 of [5] tells us that the property of being singularly generated in fact characterizes those neutral tannakian categories whose representing Hopf algebra is finite dimensional. In fact we can identify this Hopf algebra as $\text{End}(\omega|\langle[A \otimes k_i]\rangle)^\circ$, since $[A \otimes k_i]$ generates the entire category. Then by proposition 9.2.1 and theorem 8.0.6 we have that A_∞, as a coalgebra, can be identified as

$$A_\infty = \text{End}(\omega|\langle[A \otimes k_i]\rangle)^\circ \simeq \prod_\mathcal{U} \text{End}(\omega_i|\langle A \otimes k_i\rangle)^\circ = \prod_\mathcal{U} A \otimes k_i$$

Thus A_∞ is equal to the full ultraproduct of the $A \otimes k_i$. Note that in this case A_R, the restricted ultraproduct of the $A \otimes k_i$, is in fact equal to the *full* ultraproduct, since the $A \otimes k_i$ are of constant finite dimension.

Since the $A \otimes k_i$ are of constant finite dimension, we have an isomorphism $\prod_\mathcal{U} A \otimes k_i \simeq \prod_\mathcal{U} A \otimes \prod_\mathcal{U} k_i$; it is not hard to see that the latter can be further identified as $A \otimes \prod_\mathcal{U} k_i$ as an algebra, coalgebra, and indeed Hopf algebra.

Thus, for G finite, the category $\prod_R \text{Rep}_{k_i} G$ can be identified with $\text{Rep}_{\prod_\mathcal{U} k_i} G$.

9.5.2 The Multiplicative Group

Here we compute A_∞ for the multiplicative group G_m using the work done in this chapter. Let $\mathcal{C}_i = \text{Rep}_{k_i} G_m$ with k_i being some sequence of fields. Let A_i denote the representing Hopf algebra of G_m over the field k_i, which we identify as

$$A_i = k_i[x, x^{-1}]$$

$$\Delta_i : x \mapsto x \otimes x$$

$$\text{mult} : x^r \otimes x^s \mapsto x^{r+s}$$

We know then that A_∞ is isomorphic to A_R, the restricted ultraproduct of the Hopf algebras A_i, which we set about now identifying.

Fix a field k, and let $A = k[x, x^{-1}]$, with Δ, mult defined as above. Then A can be realized as the increasing union of the finite dimensional subcoalgebras $B_0 \subset B_1 \subset \ldots$, defined as

$$B_n = \operatorname{span}_k(x^{-n}, x^{-(n-1)}, \ldots, x^{-1}, 1, x, x^2, \ldots, x^n)$$

The dual algebra B_n^* to B_n we identify as

$$B_n^* = \operatorname{span}_k(\alpha_{-n}, \ldots, \alpha_0, \ldots, \alpha_n)$$

$$\operatorname{mult}_n : \alpha_r \otimes \alpha_s \mapsto \delta_{rs}\alpha_r$$

As the B_n form a direct system under the inclusion mappings, the B_n^* form an inverse system under the duals to these inclusion mappings. This map $B_n^* \leftarrow B_{n+1}^*$ is given by, for $\alpha_r : B_{n+1} \to k$, the image of α_r is α_r if $|r| \leq n$ and 0 otherwise. Then we leave it to the reader to verify

Proposition 9.5.2. *The inverse limit $\varprojlim B_n^*$ of the B_n^* can be identified as $\operatorname{span}_k(\alpha_i : i \in \mathbb{Z})$, with mult defined by*

$$mult : \alpha_r \otimes \alpha_s \mapsto \delta_{rs}\alpha_r$$

and the canonical mapping $T_m : \varprojlim B_n^* \to B_m^*$ given by

$$\alpha_r \mapsto \begin{cases} \alpha_r \text{ if } |r| \leq m \\ 0 \text{ otherwise} \end{cases}$$

Next we must identify the map $\phi : A \to \left(\varprojlim B_n^*\right)^\circ$ giving us the notion of 'rank' in A. For $x^r \in A$, we pull it back to $x^r \in B_m$ for some $m \geq |r|$, pass to the isomorphism $B_m \simeq B_m^{*\circ}$, and then up through T_m°. Thus we obtain

Proposition 9.5.3. *The linear functional $\phi(x^r)$ acts on $\varprojlim B_n^*$ by*

$$\phi(x^r) : \alpha_s \mapsto \delta_{rs}$$

Let $f = c_1 x^{m_1} + c_2 x^{m_2} + \ldots + c_n x^{m_n}$ be an arbitrary element of A, with $m_i \in \mathbb{Z}$ and $c_i \in k$. Then $\phi(f)$ kills the ideal

$$I = \operatorname{span}_k(\alpha_r : r \in \mathbb{Z} - \{m_1, m_2, \ldots, m_n\}) \lhd \varprojlim B_n^*$$

This ideal is the largest ideal that $\phi(f)$ kills and it has codimension n. Therefore

Proposition 9.5.4. *The rank of the element* $f = c_1 x^{m_1} + c_2 x^{m_2} + \ldots + c_n x^{m_n} \in A$ *is equal to n, the number of distinct monomials occurring as terms.*

Let $A_i = k_i[x, x^{-1}]$ be the representing Hopf algebra of G_m over the field k_i. According to the previous proposition, the rank of a polynomial in A_i is equal the number of monomial terms occurring in it. Thus, in order for an element $[f_i] \in \prod_\mathcal{U} A_i$ to have bounded rank, it is necessary and sufficient for it to have almost everywhere bounded monomial length. If this bound is n, then in almost every slot f_i has length among the finite set $\{0, 1, \ldots, n\}$, and so by lemma B.0.10 we may as well assume that the f_i have constant length. Then we have

Proposition 9.5.5. *The restricted ultraproduct A_R of the A_i can be identified as*

$$A_R = \{[f_i] \in \prod_\mathcal{U} A_i : f_i \text{ has constant length }\}$$

$$\Delta : [f_i] \mapsto [\Delta_i(f_i)]$$

$$mult : [f_i] \otimes [g_i] \mapsto [mult_i(f_i \otimes g_i)]$$

Let us find a tighter description of A_R. Let $\prod_\mathcal{U} \mathbb{Z}$ denote the ultrapower of the integers and let A' denote the $k = \prod_\mathcal{U} k_i$-span of the formal symbols $x^{[z_i]}$, $[z_i] \in \prod_\mathcal{U} \mathbb{Z}$.

Define the following Hopf algebra structure on this vector space as follows:

$$A' = \mathrm{span}_k(x^{[z_i]} : [z_i] \in \prod_{\mathcal{U}} \mathbb{Z})$$

$$\Delta : x^{[z_i]} \mapsto x^{[z_i]} \otimes x^{[z_i]}$$

$$\mathrm{mult} : x^{[z_i]} \otimes x^{[w_i]} \mapsto x^{[z_i + w_i]}$$

$$\varepsilon : x^{[z^i]} \mapsto 1$$

$$S : x^{[z_i]} \mapsto x^{[-z_i]}$$

Now every element of A_R looks like

$$[a_i^1 x^{z_i^1} + \ldots + a_i^m x^{z_i^m}]$$

with $a_i^j \in k_i$ for every i and j, and $z_i^j \in \mathbb{Z}$ with $z_i^1 < z_i^2 < \ldots < z_i^m$. Then define a map A_R to A' by

$$[a_i^1 x^{z_i^1} + \ldots + a_i^m x^{z_i^m}] \mapsto [a_i^1] x^{[z_i^1]} + \ldots + [a_i^m] x^{[z_i^m]}$$

We leave it to the reader to verify

Proposition 9.5.6. *The map just defined is an isomorphism of Hopf algebras.*

Finally, let us build the equivalence of categories $\prod_R \mathcal{C}_i \to \mathrm{Comod}_{A_R}$ using the description of A_R given above. Let $[X_i, \rho_i]$ be an object of $\prod_R \mathcal{C}_i$ of dimension m. It is well known that any module for G_m over a field is simply a diagonal sum of

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characters; that is, in some basis, it has matrix formula

$$\begin{pmatrix} x^{z_1} & & & \\ & x^{z_2} & & \\ & & \ddots & \\ & & & x^{z_m} \end{pmatrix}$$

for some collection of non-negative integers $z_1, \ldots, z_m$. For each i then, fix a basis $e_i^1, \ldots, e_i^m$ of X_i for which the action of G_m is a diagonal sum of characters. Then we can write

$$\rho_i : e_i^j \mapsto e_i^j \otimes x^{z_i^j}$$

The vectors $[e_i^1], \ldots, [e_i^m]$ form a basis for $\prod_{\mathcal{U}} X_i$, and in this basis the A_R-comodule structure of $\prod_{\mathcal{U}} X_i$ is given by

$$\rho : [e_i^j] \mapsto [e_i^j] \otimes x^{[z_i^j]}$$

That is, the action of $G_\infty =$ the group represented by A_R on $\prod_{\mathcal{U}} X_i$ is simply

$$\begin{pmatrix} x^{[z_i^1]} & & & \\ & x^{[z_i^2]} & & \\ & & \ddots & \\ & & & x^{[z_i^m]} \end{pmatrix}$$

Chapter 10

A Combinatorial Approach to the Representation Theory of Unipotent Algebraic Groups

As promised in the introduction, for the next several chapters we take a break entirely from working with ultraproducts, and instead focus on working out the concrete representation theories of certain unipotent algebraic groups. Here we outline the approach we will be taking for all of the proofs constructed throughout.

Let G be an algebraic group over the field k with Hopf algebra $A = k[x_1, \ldots, x_n]/I$, where I is the ideal of $k[x_1, \ldots, x_n]$ generated by the defining polynomial equations of G (e.g., if $G = \mathrm{SL}_2$, then $A = k[x_1, x_2, x_3, x_4]/(x_1 x_4 - x_2 x_3 - 1)$). If (V, ρ) is a comodule over A with basis $e_1, \ldots, e_m$, we can write

$$\rho : e_j \mapsto \sum_i e_i \otimes a_{ij}$$

$$a_{ij} = \sum_{\vec{r}=(r_1, \ldots, r_n)} c_{ij}^{\vec{r}} x_1^{r_1} \ldots x_n^{r_n}$$

where $c_{ij}^{\vec{r}}$ is a scalar for every i, j and $\vec{r}$, and the summation runs over some finite

collection of n-tuples of non-negative integers. If for each $\vec{r}$ we think of $(c_{ij})^{\vec{r}}$ as an $m \times m$ matrix over k, its significance is that it consists of the coefficients of the monomial $x_1^{r_1} \ldots x_n^{r_n}$ in the matrix formula for the representation in the basis $e_1, \ldots, e_m$. For example, consider the group $G_a \times G_a$, where G_a denotes the additive group (see chapter 12). $G_a \times G_a$ has as its representing Hopf algebra $k[x, y]$. Consider the representation defined by

$$\begin{pmatrix} 1 & 2y + x & 2y^2 + 2yx + \frac{1}{2}x^2 \\ 0 & 1 & 2y + x \\ 0 & 0 & 1 \end{pmatrix}$$

Then in this basis the (c_{ij}) matrices are given by

$$(c_{ij})^{(0,0)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (c_{ij})^{(1,0)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(c_{ij})^{(0,1)} = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \qquad (c_{ij})^{(2,0)} = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(c_{ij})^{(0,2)} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (c_{ij})^{(1,1)} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with $(c_{ij})^{\vec{r}} = 0$ for all other $\vec{r}$. Now the two diagrams asserting that a given vector

space V and linear map $\rho : V \to V \otimes A$ is a comodule over A are

$$
\begin{array}{ccc}
V \xrightarrow{\ \rho\ } V \otimes A & \qquad & V \xrightarrow{\ \rho\ } V \otimes A \\
\ \searrow \ \ \downarrow{\scriptstyle 1 \otimes \varepsilon} & & \downarrow{\scriptstyle \rho} \qquad \downarrow{\scriptstyle 1 \otimes \Delta} \\
{\scriptstyle \sim} \quad V \otimes k & & V \otimes A \xrightarrow[\rho \otimes 1]{} V \otimes A \otimes A
\end{array}
$$

In the first diagram, if we chase e_j along both paths we arrive at

$$
e_j \otimes 1 = \sum_i e_i \otimes \varepsilon(a_{ij})
$$

and we see by matching up coefficients that $\varepsilon(a_{ij}) = 1$ if $i = j$, zero otherwise. This simply says that

$$
(\varepsilon(a_{ij})) = \mathrm{Id} \tag{10.0.1}
$$

which will be some matrix expression among the (c_{ij}) matrices. For the second diagram we chase e_j along both paths and arrive at

$$
\sum_i e_i \otimes \left(\sum_k a_{ik} \otimes a_{kj} \right) = \sum_i e_i \otimes \Delta(a_{ij})
$$

and again, by matching coefficients, this reduces to

$$
\sum_k a_{ik} \otimes a_{kj} = \Delta(a_{ij}) \tag{10.0.2}
$$

Notice that the left hand side is precisely the $(i, j)^{\mathrm{th}}$ entry of the matrix product $(a_{ij} \otimes 1)(1 \otimes a_{ij})$. In practice these two equations will allow us to derive matrix equalities between the various $(c_{ij})^{\vec{r}}$, which will serve as necessary and sufficient conditions for them to define representations over a given group.

This approach is particularly amenable to the study of unipotent groups. It is well

known that all unipotent algebraic groups (and quite obviously the ones we will be studying) have Hopf algebras which are algebra-isomorphic to $A = k[x_1, x_2, \ldots, x_n]$; that is, isomorphic to a polynomial algebra with no relations. As such, the collection of all monomial tensors $x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n} \otimes x_1^{s_1} x_2^{s_2} \ldots x_n^{s_n}$ form a basis for $A \otimes A$, and a great deal of our work will involve looking at equalities between large summations in $A \otimes A$ and trying to match coefficients on a basis. Thus, for unipotent groups, a logical choice of basis with which to attempt this is always available.

We would like to briefly mention that, in a round-about way, what all this amounts to is working with the Lie algebra of the group in the characteristic 0 case, and the distribution algebra in the characteristic $p > 0$ case (see chapter 7 of [13] for a good account of the latter). In fact, for a given G-module, these $(c_{ij})^{\vec{r}}$ matrices correspond to the images of certain distributions under the associated $\mathrm{Dist}(G)$ module, and these matrix equalities we shall be deriving essentially amount to working out the multiplication law in $\mathrm{Dist}(G)$; compare for instance equation 12.1.2 with equation (1) of page 101 of [13]. We have chosen however to proceed without this machinery; it gives us no advantage to our purposes, and in any case puts less of a burden on the reader (and the author for that matter).

10.1 Morphisms

None of this work would be worth much if we couldn't say something about morphisms between comodules.

Again let G be an algebraic group over the field k, with Hopf algebra $A = k[x_1, \ldots, x_n]/I$, and this time fix a monomial basis $\{x_1^{r_1} \ldots x_n^{r_n} : \vec{r} \in R\}$ of A. For brevity, for $\vec{r} \in R$, denote by $x^{\vec{r}}$ the monomial $x_1^{r_1} \ldots x_n^{r_n}$. If (V, ρ) is a comodule over

A with basis $e_1, \ldots, e_n$, we can write

$$\rho : e_j \mapsto \sum_{i=1}^{n} e_i \otimes a_{ij}$$

$$a_{ij} = \sum_{\vec{r} \in R} c_{ij}^{\vec{r}} x^{\vec{r}}$$

with each $c_{ij}^{\vec{r}}$ a scalar. Similarly let (W, μ) be a comodule over A with basis $f_1, \ldots, f_m$, and write

$$\mu : f_j \mapsto \sum_{i=1}^{m} f_i \otimes b_{ij}$$

$$b_{ij} = \sum_{\vec{r} \in R} d_{ij}^{\vec{r}} x^{\vec{r}}$$

Theorem 10.1.1. *Let $\phi : V \to W$ be a linear map, and write it as the matrix (k_{ij}) in the given bases. Then ϕ is a morphism of V and W as A-comodules if and only if for every $\vec{r} \in R$*

$$(k_{ij})(c_{ij})^{\vec{r}} = (d_{ij})^{\vec{r}}(k_{ij})$$

Proof. Consider the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\ \phi\ } & W \\
{\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \mu} \\
V \otimes A & \xrightarrow[\ \phi \otimes 1\ ]{} & W \otimes A
\end{array}
$$

whose commutativity is equivalent to ϕ being a morphism between V and W. If we chase e_j along both paths we arrive at

$$\sum_{i=1}^{n} \left(\sum_{s=1}^{m} k_{si} f_s \right) \otimes a_{ij} = \sum_{i=1}^{m} k_{ij} \left(\sum_{s=1}^{m} f_s \otimes b_{si} \right)$$

Replacing the $a's$ and $b's$ with their definitions in terms of the $c's$ and $d's$ and re-

arranging a bit, we have

$$\sum_{s,\vec{r}} \left(\sum_{i=1}^{n} k_{si} c_{ij}^{\vec{r}} \right) f_s \otimes x^{\vec{r}} = \sum_{s,\vec{r}} \left(\sum_{i=1}^{m} d_{si}^{\vec{r}} k_{ij} \right) f_s \otimes x^{\vec{r}}$$

As $f_s \otimes x^{\vec{r}}$ for varying $s = 1 \ldots m$ and $\vec{r} \in R$ is a free basis for $W \otimes A$, we must have, for every j, s and $\vec{r} \in R$

$$\sum_{i=1}^{n} k_{si} c_{ij}^{\vec{r}} = \sum_{i=1}^{m} d_{si}^{\vec{r}} k_{ij}$$

But the left hand side is the $(s,j)^{\text{th}}$ entry of $(k_{ij})(c_{ij})^{\vec{r}}$, and the right the $(s,j)^{\text{th}}$ entry of $(d_{ij})^{\vec{r}}(k_{ij})$.

$\square$

A combinatorial lemma we will need later:

Lemma 10.1.2. *Let $(V, \rho),(V, \mu)$ be two representation of G on the finite dimensional vector space V, given by*

$$\rho : e_j \mapsto \sum_i e_i \otimes a_{ij} \qquad\qquad a_{ij} = \sum_{\vec{r}} c_{ij}^{\vec{r}} x^{\vec{r}}$$

$$\mu : e_j \mapsto \sum_i e_i \otimes b_{ij} \qquad\qquad b_{ij} = \sum_{\vec{r}} d_{ij}^{\vec{r}} x^{\vec{r}}$$

Then the matrices $(a_{ij} \otimes 1)$ and $(1 \otimes b_{ij})$ (taking their entries from $A \otimes A$) commute if and only if for every $\vec{r}, \vec{s} \in R$, the matrices $(c_{ij})^{\vec{r}}$ and $(d_{ij})^{\vec{s}}$ commute.

Proof. The $(i,j)^{\text{th}}$ entry of $(a_{ij} \otimes 1)(1 \otimes b_{ij})$ is

$$\sum_k a_{ik} \otimes b_{kj} = \sum_k \left(\sum_{\vec{r}} c_{ik}^{\vec{r}} x^{\vec{r}} \right) \otimes \left(\sum_{\vec{s}} d_{kj}^{\vec{s}} x^{\vec{s}} \right) = \sum_{\vec{r},\vec{s}} \left(\sum_k c_{ik}^{\vec{r}} d_{kj}^{\vec{s}} \right) x^{\vec{r}} \otimes x^{\vec{s}}$$

while the $(i,j)^{\text{th}}$ entry of $(1 \otimes b_{ij})(a_{ij} \otimes 1)$ is

$$\sum_k a_{kj} \otimes b_{ik} = \sum_k \left(\sum_{vecr} c_{kj}^{\vec{r}} x^{\vec{r}} \right) \otimes \left(\sum_{\vec{s}} d_{ik}^{\vec{s}} x^{\vec{s}} \right) = \sum_{\vec{r},\vec{s}} \left(\sum_k d_{ik}^{\vec{s}} c_{kj}^{\vec{r}} \right) x^{\vec{r}} \otimes x^{\vec{s}}$$

By matching coefficients on the free basis $x^{\vec{r}} \otimes x^{\vec{s}}$ for $A \otimes A$ we must have, for every $i, j \leq n$ and $\vec{r}, \vec{s} \in R$

$$\left(\sum_k c_{ik}^{\vec{r}} d_{kj}^{\vec{s}} \right) = \left(\sum_k d_{ik}^{\vec{s}} c_{kj}^{\vec{r}} \right)$$

i.e.

$$(c_{ij})^{\vec{r}} (d_{ij})^{\vec{s}} = (d_{ij})^{\vec{s}} (c_{ij})^{\vec{r}}$$

$\square$

Chapter 11

Representation Theory of Direct Products

Here we are interested in the following: given that one has a handle on the representation theory of the algebraic groups G and H, what can be said about the representation theory of $G \times H$? Any representation of $G \times H$ is evidently a representation of both G and H in a natural way. It is also evident that one cannot paste together any two representations of G and H to get one for $G \times H$; they must somehow be compatible. The goal of this section is to prove necessary and sufficient conditions for representations of G and H to together define one for $G \times H$, and to provide a formula for it.

I will spoil the suspense: two representations Φ and Ψ for G and H on the vector space V define one for $G \times H$ on V if and only if they commute; that is, the linear maps $\Phi(g)$ and $\Psi(h)$ commute for every pair $g \in G$, $h \in H$. The matrix formula for the $G \times H$-module they define is nothing more than the product of the matrix formulas of the constituent modules. Morphisms for the new module are exactly those that are morphisms for both of the constituent modules, and likewise direct sums and tensor products behave as we hope they will.

We leave it to the reader to verify the following: if $(A, \Delta_A, \varepsilon_A)$ and $(B, \Delta_B, \varepsilon_B)$ are the representing Hopf algebras of G and H, then the Hopf algebra for $G \times H$ is $(A \otimes B, \Delta, \varepsilon)$, defined by

$$\Delta : A \otimes B \xrightarrow{\Delta_A \otimes \Delta_B} A \otimes A \otimes B \otimes B \xrightarrow{1 \otimes \text{Twist} \otimes 1} A \otimes B \otimes A \otimes B$$

$$\varepsilon : A \otimes B \xrightarrow{\varepsilon_A \otimes \varepsilon_B} k \otimes k \simeq k$$

The natural embedding $G \to G \times H$ is induced by the Hopf algebra map $A \otimes B \xrightarrow{1 \otimes \varepsilon_B}$ $A \otimes k \simeq A$, similarly for H. Then for any vector space V and $A \otimes B$-comodule structure $\rho : V \to V \otimes A \otimes B$, we get an A-comodule structure on V given by the composition

$$V \xrightarrow{\rho} V \otimes A \otimes B \xrightarrow{1 \otimes 1 \otimes \varepsilon_B} V \otimes A \otimes k \simeq V \otimes A$$

Similarly we get a B-comodule structure by instead tacking on $1 \otimes \varepsilon_A \otimes 1$.

Let $(V, \rho), (V, \mu)$ be two comodule structures on the vector space V, the first for G, the second for H. Write

$$\rho : e_j \mapsto \sum_i e_i \otimes a_{ij}$$

$$\mu : e_j \mapsto \sum_i e_i \otimes b_{ij}$$

Theorem 11.0.3. *Let (V, ρ), (V, μ) be modules for G and H as above. Define*

$$z_{ij} = \sum_k a_{ik} \otimes b_{kj}$$

Then the map $\sigma : V \to V \otimes A \otimes B$ defined by

$$\sigma : e_j \mapsto \sum_i e_i \otimes z_{ij}$$

is a valid module structure for $G \times H$ if and only if the matrices (a_{ij}) and (b_{ij}) commute

with one another. σ restricts naturally to ρ and μ via the canonical embeddings, and this is the only possible comodule structure on V that does so.

Some explanation is in order. The matrices (a_{ij}) and (b_{ij}) take their entries from different algebras, so it doesn't make much sense to say they commute. What we really mean is that the matrix products $(a_{ij} \otimes 1)(1 \otimes b_{ij})$ and $(1 \otimes b_{ij})(a_{ij} \otimes 1)$, which take their entries from $A \otimes B$, are equal. The content of the theorem then is that, if they do commute, the matrix $(z_{ij}) = (a_{ij} \otimes 1)(1 \otimes b_{ij})$ provides a valid module structure for $G \times H$, and this is the only one that restricts to ρ and μ. In what follows we shall simply write a_{ij} and b_{ij}, even when we wish to consider them as elements of $A \otimes B$.

Proof. Given that this actually is a representation, the representation induced on G from it is given by

$$e_j \overset{\sigma}{\mapsto} \sum_i e_i \otimes z_{ij} \overset{1 \otimes 1 \otimes \varepsilon_B}{\mapsto} \sum_i e_i \otimes \sum_k a_{ik} \otimes \varepsilon_B(b_{kj})$$
$$= \sum_i e_i \otimes \sum_k a_{ik} \otimes \delta_{kj} = \sum_i e_i \otimes a_{ij}$$

and similarly for H. Thus σ does indeed restrict to ρ and μ.

To prove that this actually is a representation, as always, we must check that the equations $\varepsilon(z_{ij}) = \delta_{ij}$ and $\sum_k z_{ik} \otimes z_{kj} = \Delta(z_{ij})$ are satisfied. For the first, we have

$$\varepsilon(z_{ij}) = \varepsilon\left(\sum_k a_{ik} \otimes b_{kj}\right)$$
$$= \sum_k \varepsilon_A(a_{ik})\varepsilon_B(b_{kj})$$
$$= \sum_k \delta_{ik}\delta_{kj}$$
$$= \delta_{ij}$$

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as required. For the second equation, we have

$$\Delta(z_{ij}) = \Delta(\sum_k a_{ik} \otimes b_{kj})$$

$$= \sum_k \Delta_A(a_{ik}) \otimes \Delta_B(b_{kj})$$

$$= \sum_k (\sum_l a_{il} \otimes a_{lk}) \otimes (\sum_m b_{km} \otimes b_{mj})$$

$$\stackrel{\text{Twist}}{\simeq} \sum_{k,l,m} a_{il} \otimes b_{km} \otimes a_{lk} \otimes b_{mj}$$

where we have used the fact that ρ and μ are comodule structures for G and H, and applied the twist at the end. Note that the last expression is equal to the $(i, j)^{\text{th}}$ entry of the matrix $(a_{ij})(b_{ij})(a_{ij})(b_{ij})$. For the other side:

$$\sum_k z_{ik} \otimes z_{kj} = \sum_k (\sum_l a_{il} \otimes b_{lk}) \otimes (\sum_m a_{km} \otimes b_{mj})$$

$$= \sum_{k,l,m} a_{il} \otimes b_{lk} \otimes a_{km} \otimes b_{mj}$$

This is equal to the $(i, j)^{\text{th}}$ entry of the matrix $(a_{ij})(a_{ij})(b_{ij})(b_{ij})$. Thus, all this reduces to the matrix equality

$$(a_{ij})(b_{ij})(a_{ij})(b_{ij}) = (a_{ij})(a_{ij})(b_{ij})(b_{ij})$$

And since the matrices (a_{ij}) and (b_{ij}) are necessarily invertible, we can multiply both sides on the left by $(a_{ij})^{-1}$ and on the right by $(b_{ij})^{-1}$, and we are left with

$$(b_{ij})(a_{ij}) = (a_{ij})(b_{ij})$$

This proves the first part of the theorem. Lastly, we wish to see that σ is the only possible comodule structure on V restricting to ρ and μ. Let $\tau : V \to V \otimes A \otimes B$ be

any such comodule, and write

$$\tau : e_j \mapsto \sum_i e_i \otimes w_{ij}$$

τ restricting to ρ and μ means that $1 \otimes \varepsilon_B : w_{ij} \mapsto a_{ij}$ and that $\varepsilon_A \otimes 1 : w_{ij} \mapsto b_{ij}$. By virtue of τ being a comodule map we have

$$(\Delta_A \otimes \Delta_B)(w_{ij}) = \sum_k w_{ik} \otimes w_{kj}$$

and, using the Hopf algebra identity $(\varepsilon \otimes 1) \circ \Delta = (1 \otimes \varepsilon) \circ \Delta = 1$,

$$(\Delta_A \otimes \Delta_B)(\sum_k a_{ik} \otimes b_{kj}) = (\Delta_A \otimes \Delta_B)(\sum_k (1 \otimes \varepsilon_B)(w_{ik}) \otimes (\varepsilon_A \otimes 1)(w_{kj})$$

$$= \sum_k w_{ik} \otimes w_{kj}$$

But $\Delta_A \otimes \Delta_B = \Delta$ is a 1-1 map (as all co-multiplication maps are), and sends w_{ij} and $\sum_i a_{ik} \otimes b_{kj}$ to the same thing. We conclude then that $w_{ij} = \sum_k a_{ik} \otimes b_{kj}$. This completes the proof.

$\square$

11.1   Constructions and Morphisms

Here we record some facts about certain constructions and morphisms on representations of $G \times H$, relative to the induced representations on G and H. Throughout, let A and B be the representing Hopf algebras of G and H, V and W fixed finite dimensional vector spaces spanned by $\{e_i\}$ and $\{f_j\}$ respectively, and endowed with

$A \otimes B$-comodule structures σ and τ, given by

$$\sigma : e_j \mapsto \sum_i e_i \otimes z_{ij}$$

$$\tau : f_j \mapsto \sum_i f_i \otimes w_{ij}$$

Further, let $\sigma_A, \sigma_B, \tau_A, \tau_B$ be the induced comodule structures on A and B, given by

$$\sigma_A : e_j \mapsto \sum_i e_i \otimes a_{ij} \qquad \tau_A : f_j \mapsto \sum_i f_i \otimes l_{ij}$$

$$\sigma_B : e_j \mapsto \sum_i e_i \otimes b_{ij} \qquad \tau_B : f_j \mapsto \sum_i f_i \otimes m_{ij}$$

Then we know from the previous section that the matrix (z_{ij}) is equal to the matrix product $(a_{ij})(b_{ij})$, and similarly $(w_{ij}) = (l_{ij})(m_{ij})$, that the matrix (a_{ij}) commutes with (b_{ij}), and that (l_{ij}) commutes with (m_{ij}). Since σ and τ are the unique representations restricting to the given ones, we shall say that they are *induced* by σ_A, σ_B and τ_A, τ_B respectively.

Proposition 11.1.1. *The direct sum $\sigma \oplus \tau$ is induced by the direct sums $\sigma_A \oplus \tau_A$ and $\sigma_B \oplus \tau_B$.*

Proof. If we view (z_{ij}) and (w_{ij}) as the matrix formulas for the representations σ and τ, then the matrix formula for $\sigma \oplus \tau$ in the basis $\{e_i\} \cup \{f_i\}$ is

$$\begin{pmatrix} (z_{ij}) & \\ & (w_{ij}) \end{pmatrix}$$

which we can write as

$$\begin{pmatrix} (a_{ij})(b_{ij}) & \\ & (l_{ij})(m_{ij}) \end{pmatrix} = \begin{pmatrix} (a_{ij}) & \\ & (l_{ij}) \end{pmatrix} \begin{pmatrix} (b_{ij}) & \\ & (m_{ij}) \end{pmatrix}$$

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This last formula is exactly that of the representation on $G \times H$ induced by that of $\sigma_A \oplus \tau_A$ and $\sigma_B \oplus \tau_B$. $\qquad \square$

Proposition 11.1.2. *The tensor product of representations $\sigma \otimes \tau$ is induced by $\sigma_A \otimes \tau_A$ and $\sigma_B \otimes \tau_B$.*

Proof. This is merely the observation that tensor product of matrices commutes with matrix multiplication. The matrix formula for the representation $\sigma \otimes \tau$ is

$$(z_{ij}) \otimes (w_{ij}) = (a_{ij})(b_{ij}) \otimes (l_{ij})(m_{ij}) = [(a_{ij}) \otimes (l_{ij})][(b_{ij}) \otimes (m_{ij})]$$

which is the representation induced by $\sigma_A \otimes \tau_A$ and $\sigma_B \otimes \tau_B$. $\qquad \square$

Proposition 11.1.3. *A linear map $\phi : V \to W$ is a morphism between the representations (V, σ) and (W, τ) in the category $\mathrm{Rep}_k G \times H$ if and only if it is both a morphism between the representations (V, σ_A) and (W, τ_A) in the category $\mathrm{Rep}_k G$ and between (V, σ_B) and (W, τ_B) in the category $\mathrm{Rep}_k H$.*

Proof. Write ϕ as the matrix (k_{ij}) in the relevant bases. Recall that, ϕ being a morphism in $\mathrm{Rep}_k G \times H$ is equivalent to the matrix equality $(k_{ij})(z_{ij}) = (w_{ij})(k_{ij})$. Then the 'if' direction of the theorem is obvious, since by assumption $(z_{ij}) = (a_{ij})(b_{ij}) = (b_{ij})(a_{ij})$, similarly for (w_{ij}).

Conversely, consider the diagram

$$
\begin{array}{ccc}
V & \xrightarrow{\ \ \phi\ \ } & W \\[2pt]
\sigma \downarrow & & \downarrow \tau \\[2pt]
V \otimes A \otimes B & \xrightarrow{\ \phi \otimes 1 \otimes 1\ } & V \otimes A \otimes B \\[2pt]
\scriptstyle 1 \otimes 1 \otimes \varepsilon_B \downarrow & & \downarrow \scriptstyle 1 \otimes 1 \otimes \varepsilon_B \\[2pt]
V \otimes A & \xrightarrow{\ \phi \otimes 1\ } & V \otimes A
\end{array}
$$

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Commutativity of the top rectangle is the statement that ϕ is a morphism between (V, σ) and (W, τ), and commutativity of the bottom rectangle is true no matter what ϕ is. This gives us commutativity of the outermost rectangle, which is the assertion that ϕ is a morphism between (V, σ_A) and (W, τ_A) in $\mathrm{Rep}_k G$. An identical result holds in $\mathrm{Rep}_k H$ if we consider ε_A instead.

$\square$

Chapter 12

The Additive Group

In this chapter we give a complete characterization of the finite dimensional representations of the additive group G_a over any field, using no more than the combinatorial methods described in chapter 10. The goal, as with both of the unipotent groups we'll be investigating, is to show that, for characteristic $p > 0$ large with respect to dimension, modules for G_a in characteristic p look exactly like modules for G_a^n (direct product of copies of G_a) in characteristic zero. This is by far the easiest case we'll consider, as even when $p << \dim$, the analogy is still very strong (which is atypical; modules for the Heisenberg group, discussed later, in $\dim >> p$ do not share this property).

We start by considering the group G_a^∞, the countable direct product of G_a; all we need for the group G_a will emerge as a special case. For a fixed field k we identify the Hopf algebra A for the group G_a^∞ as $k[x_1, x_2, \ldots]$, the free algebra on countably many commuting variables, with Δ and ε defined by

$$A = k[x_1, x_2, \ldots]$$

$$\Delta : x_i \mapsto 1 \otimes x_i + x_i \otimes 1$$

$$\varepsilon : x_i \mapsto 0$$

Let (V, ρ) be a comodule over A, fix a basis $\{e_i\}$ of V, and write

$$\rho : e_j \mapsto \sum_i e_i \otimes a_{ij}$$

$$a_{ij} = \sum_{\vec{r}=(r_1,\ldots,r_n)} c_{ij}^{\vec{r}} x^{\vec{r}}$$

where we adopt the notation $x^{\vec{r}} = x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}$. Notice that only finitely many of the variables x_i can show up in any of the a_{ij}, so for all intents and purposes, this is really just a comodule over a finitely generated slice of A, namely $k[x_1, \ldots, x_n]$, the representing Hopf algebra of the group G_a^n.

12.1 Combinatorics

We look first at equation 10.0.1:

$$(\varepsilon(a_{ij})) = \mathrm{Id}$$

Note that ε acts on a_{ij} by simply picking off its constant term $c_{ij}^{\vec{0}}$. The above thus reduces to the matrix equality

$$(c_{ij})^{\vec{0}} = \mathrm{Id}$$

This equality is intuitively obvious, for when we evaluate the matrix formula of the representation at $x_1 = \ldots = x_n = 0$ we are left with $(c_{ij})^{(0,\ldots,0)}$, which should indeed be the identity matrix.

Next we look equation 10.0.2, namely $\sum_k a_{ik} \otimes a_{kj} = \Delta(a_{ij})$. Working with the

right hand side first we have, making repeated use of the binomial theorem

$$\Delta(a_{ij}) = \sum_{\vec{r}} c_{ij}^{\vec{r}} \Delta(x_1)^{r_1} \dots \Delta(x_n)^{r_n}$$

$$= \sum_{\vec{r}} c_{ij}^{\vec{r}} (1 \otimes x_1 + x_1 \otimes 1)^{r_1} \dots (1 \otimes x_n + x_n \otimes 1)^{r_n}$$

$$= \sum_{\vec{r}} c_{ij}^{\vec{r}} \left(\sum_{k_1+l_1=r_1} \binom{k_1+l_1}{l_1} x_1^{k_1} \otimes x_1^{l_1} \right) \dots \left(\sum_{k_n+l_n=r_n} \binom{k_n+l_n}{l_n} x_n^{k_n} \otimes x_n^{l_n} \right)$$

$$= \sum_{\vec{r}} c_{ij}^{\vec{r}} \sum_{\vec{k}+\vec{l}=\vec{r}} \binom{k_1+l_1}{l_1} \dots \binom{k_n+l_n}{l_n} x_1^{k_1} \dots x_n^{k_n} \otimes x_1^{l_1} \dots x_n^{l_n}$$

$$= \sum_{\vec{r}} c_{ij}^{\vec{r}} \sum_{\vec{k}+\vec{l}=\vec{r}} \binom{\vec{k}+\vec{l}}{\vec{l}} x^{\vec{k}} \otimes x^{\vec{l}}$$

Here we have adopted the notation, for two n-tuples of non-negative integers $\vec{m} = (m_1, \dots, m_n)$ and $\vec{r} = (r_1, \dots, r_n)$,

$$\vec{m} + \vec{r} = (m_1 + r_1, \dots, m_n + r_n)$$

and

$$\binom{\vec{m}}{\vec{r}} = \binom{m_1}{r_1} \binom{m_2}{r_2} \dots \binom{m_n}{r_n}$$

For the left hand side we have

$$\sum_k a_{ik} \otimes a_{kj} = \sum_k \left(\sum_{\vec{r}} c_{ik}^{\vec{r}} x^{\vec{r}} \right) \otimes \left(\sum_{\vec{s}} c_{kj}^{\vec{s}} x^{\vec{s}} \right)$$

$$= \sum_k \sum_{\vec{r},\vec{s}} c_{ik}^{\vec{r}} c_{kj}^{\vec{s}} x^{\vec{r}} \otimes x^{\vec{s}}$$

$$= \sum_{\vec{r},\vec{s}} \left(\sum_k c_{ik}^{\vec{r}} c_{kj}^{\vec{s}} \right) x^{\vec{r}} \otimes x^{\vec{s}}$$

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Thus, equation 10.0.2 reduces to

$$\sum_{\vec{r},\vec{s}} \left(\sum_k c_{ik}^{\vec{r}} c_{kj}^{\vec{s}} \right) x^{\vec{r}} \otimes x^{\vec{s}} = \sum_{\vec{r}} c_{ij}^{\vec{r}} \sum_{\vec{k}+\vec{l}=\vec{r}} \binom{\vec{k}+\vec{l}}{\vec{l}} x^{\vec{k}} \otimes x^{\vec{l}}$$

Now the polynomial ring $k[x_1, \ldots, x_n]$ has no relations, whence the collection of all monomial tensors $x^{\vec{r}} \otimes x^{\vec{s}}$ for varying $\vec{r}$ and $\vec{s}$ constitutes a free basis for $k[x_1, \ldots, x_n] \otimes k[x_1, \ldots, x_n]$; we can therefore simply match coefficients. In the left hand side of the above equation, clearly each monomial tensor occurs exactly once, and its coefficient is $\sum_k c_{ik}^{\vec{r}} c_{kj}^{\vec{s}}$. In the right hand side it is also true that each monomial tensor occurs exactly once, for if you choose $\vec{k}$ and $\vec{l}$, there is only one $\vec{r}$ in whose summation the term $x^{\vec{k}} \otimes x^{\vec{l}}$ will occur, and there it occurs exactly once. The coefficient of the monomial tensor $x^{\vec{r}} \otimes x^{\vec{s}}$ on the right hand side is thus $\binom{\vec{r}+\vec{s}}{\vec{s}} c_{ij}^{(\vec{r}+\vec{s})}$. Then we have

$$\binom{\vec{r}+\vec{s}}{\vec{s}} c_{ij}^{(\vec{r}+\vec{s})} = \sum_k c_{ik}^{\vec{r}} c_{kj}^{\vec{s}}$$

For every $i, j, \vec{r}$ and $\vec{s}$. But the right hand side is simply the $(i, j)^{\text{th}}$ entry of the matrix $(c_{ij})^{\vec{r}}(c_{ij})^{\vec{s}}$, and the left hand side is the $(i, j)^{\text{th}}$ entry of the matrix $\binom{\vec{r}+\vec{s}}{\vec{s}}(c_{ij})^{\vec{r}+\vec{s}}$. Equation 10.0.2 is therefore equivalent to the matrix equality

$$(c_{ij})^{\vec{r}}(c_{ij})^{\vec{s}} = \binom{\vec{r}+\vec{s}}{\vec{s}} (c_{ij})^{(\vec{r}+\vec{s})} \tag{12.1.1}$$

for every $\vec{r}$ and $\vec{s}$. This equation, along with $(c_{ij})^{\vec{0}} = \text{Id}$, and the requirement that $(c_{ij})^{\vec{r}}$ should vanish for all but finitely many $\vec{r}$, are necessary and sufficient for a collection of matrices $(c_{ij})^{\vec{r}}$ to define a representation of G_a^{∞} or G_a^n over any field k.

In the case of $G_a = G_a^1$ the above equations reduces to

$$(c_{ij})^0 = \text{Id}$$

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and

$$(c_{ij})^r (c_{ij})^s = \binom{r+s}{r} (c_{ij})^{r+s} \qquad (12.1.2)$$

Again, these equations, along with the requirement that $(c_{ij})^r$ vanish for large r, are necessary and sufficient to define a representation of G_a.

For the rest of this chapter we restrict to the case of G_a, and treat the case of zero and positive characteristic separately.

12.2   Characteristic Zero

Let k have characteristic zero.

Theorem 12.2.1. *Every n-dimensional representation of G_a over k is of the form e^{xN}, where N is an $n \times n$ nilpotent matrix with entries in k. Further, any $n \times n$ nilpotent matrix over k gives a representation according to this formula.*

Proof. By e^{xN} we mean the sum

$$1 + xN + \frac{x^2 N^2}{2} + \ldots + \frac{x^m N^m}{m!}$$

which of course terminates since N is nilpotent. Obviously such a formula gives a representation, in view of the matrix identity $e^{xN} e^{yN} = e^{(x+y)N}$ (see lemma 13.5.2). For the converse let $\rho : e_j \mapsto \sum_i e_i \otimes a_{ij}$ be any representation, and set $N = (c_{ij})^1$. Then examination of equation 12.1.2 gives, for any $r > 0$

$$(c_{ij})^r = \frac{N^r}{r!}$$

Since $(c_{ij})^r$ must vanish for large r, $N^r = 0$ for large r, whence N is nilpotent.

Recalling that the matrix formula for this representation is

$$(c_{ij})^0 + (c_{ij})^1 x + \ldots + (c_{ij})^n x^n$$

where n is the largest non-zero (c_{ij}), this representation is indeed of the form e^{xN}. $\quad\square$

In the preceding proof we used the fact that $\mathrm{char}(k) = 0$ in the form of assuming that $\frac{1}{r!}$ is defined for all non-negative integers r.

Example: take the 4×4 nilpotent matrix $N =$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then the representation it defines is $e^{xN} =$

$$\begin{pmatrix} 1 & x & \frac{x^2}{2} & \frac{x^3}{6} \\ & 1 & x & \frac{x^2}{2} \\ & & 1 & x \\ & & & 1 \end{pmatrix}$$

Proposition 12.2.2. *Two representation e^{xN}, e^{xM} are isomorphic if and only if the nilpotent matrices N and M are conjugate.*

Proof. Sufficiency is immediate. For the necessity, suppose e^{xN} and e^{xM} are isomorphic via a base change matrix P. Then

$$P e^{xN} P^{-1} = 1 + xPNP^{-1} + \ldots = 1 + xM + \ldots$$

which forces $PNP^{-1} = M$. $\quad\square$

Corollary 12.2.3. *Let k have characteristic zero.*

1. *For an n-dimensional representation of G_a over k, the polynomials occurring as matrix entries cannot have degree larger than $n - 1$*

2. *For a given dimension n, there are only finitely many non-isomorphic n-dimensional representations of G_a over k*

Proof. If N is nilpotent, it is nilpotent of order no greater than the dimension of the representation; this proves 1. Any nilpotent matrix is conjugate to a Jordan matrix with $0's$ on the main diagonal, of which there are only finitely many for a given dimension; this proves 2. $\square$

12.3 Characteristic p

Let k have characteristic $p > 0$. In this section we prove

Theorem 12.3.1. *Any representation of G_a over k is of the form*

$$e^{xN_0}e^{x^p N_1}e^{x^{p^2} N_2}\ldots e^{x^{p^m} N_m}$$

with each of the factors commuting (and so necessarily and sufficiently all of the N_i commuting), and each N_i being nilpotent of order $\leq p$. Further, any finite collection of commuting, p-nilpotent matrices defines a representation according to the above formula.

Note that, for a sequence of assignments to the $(c_{ij})^n$ as in equation 12.1.2, satisfying the relations $(c_{ij})^0 = \mathrm{Id}$ along with

$$(c_{ij})^r(c_{ij})^s = \binom{r+s}{r}(c_{ij})^{r+s}$$

are still necessary and sufficient to determine a representation, no matter the characteristic. The gist of the theorem is that, while in characteristic 0 the entire representation is determined by the nilpotent matrix $(c_{ij})^1$, in characteristic p the binomial coefficient $\binom{r+s}{r}$ can and often does vanish, which relaxes the relations that the (c_{ij}) matrices must satisfy. This is to be expected, since for example

$$\begin{pmatrix} 1 & x^{p^m} \\ 0 & 1 \end{pmatrix}$$

defines a perfectly respectable representation of G_a in characteristic p, for any m as large as we like. This illustrates that, for example, $(c_{ij})^{p^m}$ is not determined by the matrix $(c_{ij})^1$.

It turns out that one can choose the matrices $(c_{ij})^{p^m}$ freely, subject only to the condition that they commute and are nilpotent of order $\leq p$, and that these completely determine the rest of the representation. These $(c_{ij})^{p^m}$ matrices for $m \geq 1$ should thus be thought of as accounting for the "Frobenius" parts of the representation.

To start, we need some number theory concerning the behavior of binomial and multinomial coefficients modulo a prime. The notation $\binom{n}{a,b,\ldots,z}$ denotes the usual multinomial expression $\frac{n!}{a!b!\ldots z!}$.

Theorem 12.3.2. *(Lucas' theorem) Let n and $a, b, \ldots, z$ be non-negative integers with $a + b + \ldots + z = n$, p a prime. Write $n = n_m p^m + n_{m-1} p^{m-1} + \ldots + n_0$ in p-ary notation, similarly for $a, b, \ldots, z$. Then, modulo p,*

$$\binom{n}{a, b, \ldots, z} = \begin{cases} 0 & \text{if for some } i, a_i + b_i + \ldots + z_i \geq p \\ \binom{n_0}{a_0, b_0, \ldots, z_0} \binom{n_1}{a_1, b_1, \ldots, z_1} \ldots \binom{n_m}{a_m, b_m, \ldots, z_m} & \text{otherwise} \end{cases}$$

Some corollaries we will need later:

Corollary 12.3.3. *Let p be a prime, n, r and s non-negative integers.*

1. The binomial coefficient $\binom{n}{r}$ is non-zero if and only if every p-digit of n is greater than or equal to the corresponding digit of r

2. If n is not a power of p, then for some $0 < r < n$, $\binom{n}{r}$ is non-zero

3. If n is a power of p, for every $0 < r < n$, $\binom{n}{r}$ is zero

4. $\binom{r+s}{r}$ is non-zero if and only if there is no p-digit rollover (i.e. carrying) for the sum $r + s$.

See [6] for a proof of these facts.

Theorem 12.3.4. *A representation of G_a over k given by the matrices $(c_{ij})^n$ is completely determined by the assignments*

$$(c_{ij})^{p^0} = X_0, \quad (c_{ij})^{p^1} = X_1, \quad \ldots, (c_{ij})^{p^m} = X_m$$

(with the understanding that $(c_{ij})^{p^k} = 0$ for $k > m$). The X_i must necessarily commute with each other and satisfy $X_i^p = 0$.

Proof. All is proved by examining equation 12.1.2. It is easy to see by induction that the values of $(c_{ij})^n$ are determined by the X_i. If n is a power of p then its value is given, and if not, by 2. of corollary 12.3.3 there is $0 < r < n$ with $\binom{n}{r} \neq 0$ forcing

$$(c_{ij})^n = \binom{n}{r}^{-1} (c_{ij})^r (c_{ij})^{n-r}$$

For the commutativity condition, if $n \neq m$, then by theorem 12.3.2 $\binom{p^m + p^n}{p^m}$ is non-zero, and we must have

$$(c_{ij})^{p^m + p^n} = \binom{p^m + p^n}{p^m}^{-1} X_m X_n = \binom{p^m + p^n}{p^n}^{-1} X_n X_m$$

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To prove the nilpotency claim, consider

$$(c_{ij})^{p^m} = X_m$$

$$(c_{ij})^{2p^m} = \binom{2p^m}{p^m}^{-1} X_m^2$$

$$\vdots$$

$$(c_{ij})^{(p-1)p^m} = \left[\prod_{k=1}^{p-1} \binom{kp^m}{p^m}\right]^{-1} X_m^{p-1}$$

Noting that there is carrying in computing the sum $(p^{m+1} - p^m) + p^m$, corollary 12.3.3 tells us that $\binom{p^{m+1}}{p^m} = 0$, and we have

$$0 = \binom{p^{m+1}}{p^m} X_{m+1} = (c_{ij})^{(p-1)p^m}(c_{ij})^{p^m} = \left[\prod_{k=2}^{p-1} \binom{kp^m}{p^m}\right]^{-1} X_m^p$$

forcing $X_m^p = 0$.

$\square$

We have shown thus far that commutativity and p-nilpotency of the X_i are necessary to define a representation; we must now show sufficiency. This will become clear once we have a closed expression for $(c_{ij})^n$ in terms of the X_i. For $n = n_m p^m + n_{m-1} p^{m-1} + \ldots + n_0$ in p-ary notation, let $\Gamma(n) = n_0! n_1! \ldots n_m!$. Obviously $\Gamma(n)$ is always non-zero mod p.

Proposition 12.3.5. *Let $X_0, \ldots, X_m$ be pair-wise commuting p-nilpotent matrices, and let $n = n_m p^m + \ldots + n_0$ be the p-ary expansion of n. Then the assignment*

$$(c_{ij})^n = \Gamma(n)^{-1} X_0^{n_0} X_1^{n_1} \ldots X_m^{n_m}$$

defines a representation of G_a over k.

Proof. Obviously these assignments satisfy $(c_{ij})^0 = \mathrm{Id}$, with $(c_{ij})^n$ vanishing for large

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n (we define $X_i = 0$ for $i > m$). Then it remains to check the equation

$$\binom{r+s}{r}(c_{ij})^{r+s} = (c_{ij})^r(c_{ij})^s$$

Let $r = r_m p^m + \ldots + r_0$, $s = s_m p^m + \ldots + s_0$, and suppose first that $\binom{r+s}{r} = 0$. This means, by corollary 12.3.3, that there is some digit rollover in the computation of $r + s$, i.e. $r_i + s_i \geq p$ for some i. Looking at the right hand side in view of the given assignments we see that $X_i^{r_i + s_i}$ will occur as a factor. But X_i is nilpotent of order less than or equal to p, so the right hand side will be zero as well.

On the other hand, if $\binom{r+s}{r} \neq 0$ let $r + s = z_m p^m + \ldots + z_0$, so that necessarily $r_i + s_i = z_i$ for all i. Then the given assignments give the same power of each X_i on either side, so it only remains to check the coefficients. This reduces to

$$\binom{r+s}{r}\Gamma(r)\Gamma(s) = \Gamma(r+s)$$

After applying theorem 12.3.2 for the term $\binom{r+s}{r}$, the equality is clear.

$\square$

We can now prove theorem 12.3.1. Let $X_0, \ldots, X_m$ be commuting, p-nilpotent matrices over k. Then according to the previous theorem, the representation they define is

$$\sum_{r=0}^{p^{m+1}-1} (c_{ij})^r x^r = \sum_{r=0}^{p^{m+1}-1} \Gamma(r)^{-1} X_0^{r_0} X_1^{r_1} \ldots X_m^{r_m} x^{r_0 + r_1 p + \ldots + r_m p^m}$$

$$= \sum_{r_0=0}^{p-1}\sum_{r_1=0}^{p-1}\ldots\sum_{r_m=0}^{p-1} \frac{1}{r_0!} \ldots \frac{1}{r_m!} X_0^{r_0} \ldots X_m^{r_m} x^{r_0} x^{r_1 p} \ldots x^{r_m p^m}$$

$$= \left(\sum_{r_0=0}^{p-1} \frac{1}{r_0!} X_0^{r_0} x^{r_0}\right)\left(\sum_{r_1=0}^{p-1} \frac{1}{r_1!} X_1^{r_1} x^{r_1 p}\right)\ldots\left(\sum_{r_m=0}^{p-1} \frac{1}{r_m!} X_m^{r_m} x^{r_m p^m}\right)$$

$$= e^{x X_0} e^{x^p X_1} \ldots e^{x^{p^m} X_m}$$

as claimed. All of the factors of course commute, since the X_i do.

Example: define the following matrices:

$$X_0 = X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

These all commute and are nilpotent of order less than or equal to 3. Then the representation they define in characteristic 3 is $e^{xX_0}e^{x^3X_1}e^{x^9X_2} =$

$$\begin{pmatrix} 1 & x + x^3 & 2x^2 + x^4 + 2x^6 + 2x^9 \\ & 1 & x + x^3 \\ & & 1 \end{pmatrix}$$

With a view toward defining the height-restricted ultraproduct later, we make the following simple but important observation.

Theorem 12.3.6.

1. *Let k have characteristic zero. Then the n-dimensional representations of G_a^∞ over k are in $1-1$ correspondence with the finite ordered sequences N_i of $n \times n$ commuting nilpotent matrices over k, according to the formula*

$$e^{x_0 N_0}e^{x_1 N_1}\ldots e^{x_m N_m}$$

2. *Let k have positive characteristic p. Then if $p \gg n$, the n-dimensional representations of G_a over k are in $1-1$ correspondence with the finite ordered sequences N_i of $n \times n$ commuting nilpotent matrices over k, according to the formula*

$$e^{xN_0}e^{x^p N_1}\ldots e^{x^{p^m} N_m}$$

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Proof. By the work done in chapter 11, all representations of G_a^∞ over k are given by commuting finite products of individual representations of G_a over k. It is easy to see that the representations e^{xN} and e^{xM} commute if and only if N and M do; this proves 1. 2. follows immediately from 12.3.1, with the additional realization that if p is greater than or equal to dimension, being nilpotent and p-nilpotent are identical concepts.

$\square$

Chapter 13

The Heisenberg Group

In this chapter we investigate the group H_1 of all 3×3 unipotent upper triangular matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

for arbitrary x, y and z. Our intent is to prove a theorem analogous to theorem 12.3.6 for the group G_a; that is, if one is content to keep p large with respect to dimension (larger than twice the dimension in fact), modules for H_1 in characteristic p 'look like' modules for H_1^∞ in characteristic zero. This result will be more precisely stated in a later chapter, as we consider the 'height-restricted ultraproduct' of the categories $\mathrm{Rep}_{k_i} H_1$ for a collection of fields k_i of increasing positive characteristic. The results of this chapter are perhaps more surprising than the previous in that, unlike modules for G_a, modules for H_1 in characteristic $p << \dim$ look hardly at all like representation for H_1^∞ in characteristic zero; it is only when p becomes large enough with respect to dimension that the resemblance is apparent.

It was the author's original intent to prove these results for all of the generalized Heisenberg groups H_n, but that is not attempted here. Nonetheless, we shall at least

work out the fundamental relations for the groups H_n, as all we need for H_1 shall arise as a special case.

13.1 Combinatorics for H_n

Let H_n, the nth generalized Heisenberg group, be the group of all $(n+2) \times (n+2)$ matrices of the form

$$\begin{pmatrix} 1 & x_1 & \cdots & x_n & z \\ & 1 & & 0 & y_1 \\ & & \ddots & & \vdots \\ & & & 1 & y_n \\ & & & & 1 \end{pmatrix}$$

That is, upper triangular matrices with free variables on the top row, right-most column, $1's$ on the diagonal, and $0's$ elsewhere. The Hopf algebra for H_n is

$$A = k[x_1, \ldots, x_n, y_1, \ldots, y_n, z]$$

$$\Delta : x_i \mapsto 1 \otimes x_i + x_i \otimes 1, \quad y_i \mapsto 1 \otimes y_i + y_i \otimes 1, \quad z \mapsto z \otimes 1 + 1 \otimes z + \sum_{i=1}^{n} x_i \otimes y_i$$

$$\varepsilon : x_i, y_i, z \mapsto 0$$

Let us adopt the notation, for an n-tuple of non-negative integers $\vec{r}$, $x^{\vec{r}} = x_1^{r_1} \ldots x_n^{r_n}$, and similarly for $y^{\vec{r}}$. Let (V, ρ) be a comodule for G with basis $\{e_i\}$, and ρ given by

$$\rho : e_j \mapsto \sum_i e_i \otimes a_{ij}$$

and write

$$a_{ij} = \sum_{(\vec{r}, \vec{s}, t)} c_{ij}^{(\vec{r}, \vec{s}, t)} x^{\vec{r}} y^{\vec{s}} z^t$$

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Lemma 13.1.1. *With a_{ij} as above, $\Delta(a_{ij})$ is equal to*

$$\sum_{\substack{(\vec{r}_1,\vec{s}_1,a)\\(\vec{r}_2,\vec{s}_2,b)}} \left(\sum_{0\leq\vec{t}\leq\vec{r}_1,\vec{s}_2} \binom{\vec{r}_1+\vec{r}_2-\vec{t}}{\vec{r}_2}\binom{\vec{s}_1+\vec{s}_2-\vec{t}}{\vec{s}_1}\binom{a+b+|\vec{t}|}{a,b,t_1,\ldots,t_n}c_{ij}^{(\vec{r}_1+\vec{r}_2-\vec{t},\vec{s}_1+\vec{s}_2-\vec{t},a+b+|\vec{t}|)} \right)$$

$$x^{\vec{r}_1}y^{\vec{s}_1}z^a \otimes x^{\vec{r}_2}y^{\vec{s}_2}z^b$$

Remark: the summation condition $0 \leq \vec{t} \leq \vec{r}_1, \vec{s}_2$ is understood to mean all $\vec{t}$ such that every entry in $\vec{t}$ is no larger than either of the corresponding entries of in $\vec{r}_1$ or $\vec{s}_2$. $|\vec{t}|$ means $t_1 + \ldots + t_n$.

Proof. We start by computing

$$\Delta(a_{ij}) = \sum_{\vec{r},\vec{s},t} c_{ij}^{(\vec{r},\vec{s},\vec{t})}\Delta(x^{\vec{r}})\Delta(y^{\vec{s}})\Delta(z)^t$$

We have

$$\Delta(x^{\vec{r}}) = \Delta(x_1)^{r_1} \ldots \Delta(x_n)^{r_n}$$

$$= (x_1 \otimes 1 + 1 \otimes x_1)^{r_1} \ldots (x_n \otimes 1 + 1 \otimes x_n)^{r_n}$$

$$= \left(\sum_{l_1+m_1=r_1} \binom{l_1+m_1}{m_1}x_1^{l_1} \otimes x_1^{m_1} \right) \ldots \left(\sum_{l_n+m_n=r_n} \binom{l_n+m_n}{m_n}x_n^{l_n} \otimes x_n^{m_n} \right)$$

$$= \sum_{\vec{l}+\vec{m}=\vec{r}} \binom{\vec{l}+\vec{m}}{\vec{m}}x^{\vec{l}} \otimes x^{\vec{m}}$$

where $\binom{\vec{l}+\vec{m}}{\vec{m}}$ is shorthand for the product $\binom{l_1+m_1}{m_1} \ldots \binom{l_n+m_n}{m_n}$. Similarly we have

$$\Delta(y^{\vec{s}}) = \sum_{\vec{f}+\vec{g}=\vec{s}} \binom{\vec{f}+\vec{g}}{\vec{g}}y^{\vec{f}} \otimes y^{\vec{g}}$$

and

$$\Delta(z^t) = (z \otimes 1 + 1 \otimes z + \sum_i x_i \otimes y_i)^t$$

$$= \sum_{a+b+c=t} \binom{a+b+c}{a,b,c} (z \otimes 1)^a (1 \otimes z)^b (\sum_i x_i \otimes y_i)^c$$

$$= \sum_{a+b+c=t} \binom{a+b+c}{a,b,c} z^a \otimes z^b \sum_{|\vec{t}|=c} \binom{|\vec{t}|}{t_1,\dots,t_n} x^{\vec{t}} \otimes y^{\vec{t}}$$

$$= \sum_{a+b+|\vec{t}|=t} \binom{a+b+|\vec{t}|}{a,b,t_1,\dots,t_n} x^{\vec{t}} z^a \otimes y^{\vec{t}} z^b$$

where $|\vec{t}| \overset{\text{def}}{=} t_1 + \dots + t_n$. Thus $\Delta(a_{ij})$ is equal to

$$\sum_{\vec{r},\vec{s},t} c_{ij}^{\vec{r},\vec{s},t} \sum_{\substack{\vec{l}+\vec{m}=\vec{r} \\ \vec{f}+\vec{g}=\vec{s} \\ a+b+|t|=t}} \binom{\vec{l}+\vec{m}}{\vec{m}} \binom{\vec{f}+\vec{g}}{\vec{g}} \binom{a+b+|\vec{t}|}{a,b,t_1,\dots,t_n} x^{\vec{l}+\vec{t}} y^{\vec{f}} z^a \otimes x^{\vec{m}} y^{\vec{g}+\vec{t}} z^b$$

We seek to write this as a sum over distinct monomial tensors, i.e. in the form

$$\sum_{\substack{\vec{r}_1,\vec{s}_1,a \\ \vec{r}_2,\vec{s}_2,b}} \chi \begin{pmatrix} \vec{r}_1, \vec{s}_1, a \\ \vec{r}_2, \vec{s}_2, b \end{pmatrix} x^{\vec{r}_1} y^{\vec{s}_1} z^a \otimes x^{\vec{r}_2} y^{\vec{s}_2} z^b$$

for some collection of scalars χ, which is merely a question of how many times a given monomial tensor shows up as a term in our summation expression of $\Delta(a_{ij})$. That is, how many solutions are there to

$$x^{\vec{l}+\vec{t}} y^{\vec{f}} z^a \otimes x^{\vec{m}} y^{\vec{g}+\vec{t}} z^b = x^{\vec{r}_1} y^{\vec{s}_1} z^a \otimes x^{\vec{r}_2} y^{\vec{s}_2} z^b$$

Clearly the values of $\vec{f}, \vec{m}, a$ and b are determined. Further, once one chooses $\vec{t}$, the values of both $\vec{l}$ and $\vec{g}$ follow; thus, we can parameterize by $\vec{t}$. For the n-tuple $\vec{t}$ to induce a solution, it is necessary and sufficient that none of its entries be larger than the corresponding entries in $\vec{r}_1$ or $\vec{s}_2$; we shall express this condition by $\vec{0} \leq \vec{t} \leq \vec{r}_1, \vec{s}_2$.

Then

$$\chi\begin{pmatrix}\vec{r}_1,\vec{s}_1,a\\\vec{r}_2,\vec{s}_2,b\end{pmatrix}=\sum_{\vec{0}\le\vec{t}\le\vec{r}_1,\vec{s}_2}c_{ij}^{(\vec{r},\vec{s},t)}\begin{pmatrix}\vec{l}+\vec{m}\\\vec{m}\end{pmatrix}\begin{pmatrix}\vec{f}+\vec{g}\\\vec{g}\end{pmatrix}\begin{pmatrix}a+b+|\vec{t}|\\a,b,t_1,\ldots,t_n\end{pmatrix}$$

and upon substituting

$$\vec{m}=\vec{r}_2\quad\vec{f}=\vec{s}_1\quad\vec{s}=\vec{s}_1+\vec{s}_2-\vec{t}\quad\vec{r}=\vec{r}_1+\vec{r}_2-\vec{t}\quad t=a+b+|\vec{t}|$$

we get

$$\chi\begin{pmatrix}\vec{r}_1,\vec{s}_1,a\\\vec{r}_2,\vec{s}_2,b\end{pmatrix}$$

$$=\sum_{0\le\vec{t}\le\vec{r}_1,\vec{s}_2}\begin{pmatrix}\vec{r}_1+\vec{r}_2-\vec{t}\\\vec{r}_2\end{pmatrix}\begin{pmatrix}\vec{s}_1+\vec{s}_2-\vec{t}\\\vec{s}_1\end{pmatrix}\begin{pmatrix}a+b+|\vec{t}|\\a,b,t_1,\ldots,t_n\end{pmatrix}c_{ij}^{(\vec{r}_1+\vec{r}_2-\vec{t},\vec{s}_1+\vec{s}_2-\vec{t},a+b+|\vec{t}|)}$$

which proves the lemma.

$\square$

Theorem 13.1.2. *A finite collection of (c_{ij}) matrices defines a module for H_n if and only if $(c_{ij})^{(\vec{0},\vec{0},0)}=\mathrm{Id}$, and for all $\vec{r}_1,\vec{r}_2,a,\vec{s}_1,\vec{s}_2,b$, the following matrix equation holds:*

$$(c_{ij})^{(\vec{r}_1,\vec{s}_1,a)}(c_{ij})^{(\vec{r}_2,\vec{s}_2,b)}\tag{13.1.1}$$

$$=\sum_{0\le\vec{t}\le\vec{r}_1,\vec{s}_2}\begin{pmatrix}\vec{r}_1+\vec{r}_2-\vec{t}\\\vec{r}_2\end{pmatrix}\begin{pmatrix}\vec{s}_1+\vec{s}_2-\vec{t}\\\vec{s}_1\end{pmatrix}\begin{pmatrix}a+b+|\vec{t}|\\a,b,t_1,\ldots,t_n\end{pmatrix}(c_{ij})^{(\vec{r}_1+\vec{r}_2-\vec{t},\vec{s}_1+\vec{s}_2-\vec{t},a+b+|\vec{t}|)}$$

Proof. This follows by matching coefficients for the equation $\Delta(a_{ij})=\sum_k a_{ik}\otimes a_{kj}$. The coefficient of the monomial tensor $x^{\vec{r}_1}y^{\vec{s}_1}z^a\otimes x^{\vec{r}_2}y^{\vec{s}_2}z^b$ for $\Delta(a_{ij})$ is the right hand side of the above equation, as proved in the previous lemma, while for $\sum_k a_{ik}\otimes a_{kj}$ it is the left hand side of the above equation, as is easy to verify. $\square$

13.2   Combinatorics for H_1

The Hopf algebra (A, Δ, ε) for the group H_1 over the field k is

$$A = k[x, y, z]$$

$$\Delta : x \mapsto 1 \otimes x + x \otimes 1, \quad y \mapsto 1 \otimes y + y \otimes 1, \quad z \mapsto 1 \otimes z + x \otimes y + z \otimes 1$$

$$\varepsilon : x, y, z \mapsto 0$$

Let V be a finite dimensional vector space over k, $\rho : V \to V \otimes A$ a k-linear map. Fix a basis $\{e_i\}$ of V, and write

$$\rho : e_j \mapsto \sum_i e_i \otimes a_{ij}$$

Each $a_{ij} \in A$ is a polynomial in the variables x, y and z, so write

$$a_{ij} = \sum_{\vec{r}} c_{ij}^{\vec{r}} x^{r_1} y^{r_2} z^{r_3}$$

where the summation is over all 3-tuples of non-negative integers, remembering of course that $c_{ij}^{\vec{r}} = 0$ for all but finitely many $\vec{r}$.

Theorem 13.2.1. *A finite collection of $(c_{ij})^{\vec{r}}$ matrices defines a module for H_1 if and only if they satisfy $(c_{ij})^{(0,0,0)} = Id$, and for all 3-tuples $\vec{r}$ and $\vec{s}$*

$$(c_{ij})^{\vec{r}}(c_{ij})^{\vec{s}} = \sum_{l=0}^{min(r_1,s_2)} \binom{r_1 + s_1 - l}{s_1} \binom{r_2 + s_2 - l}{r_2} \binom{r_3 + s_3 + l}{r_3, s_3, l} (c_{ij})^{\vec{r}+\vec{s}+(-l,-l,l)}$$

(13.2.1)

Proof. Apply theorem 13.1.2 to the case of $n = 1$. $\qquad\qquad\square$

We will also make good use of the fact that G contains three copies of the additive group G_a, one for every coordinate. This says that, for example, the collection of all

matrices of the form $(c_{ij})^{(r,0,0)}$ (the matrices representing the x-coordinate in the representation) must in isolation satisfy equation 12.1.2:

$$(c_{ij})^{(r,0,0)}(c_{ij})^{(s,0,0)} = \binom{r+s}{r}(c_{ij})^{(r+s,0,0)}$$

An identical statement holds for matrices of the form $(c_{ij})^{(0,r,0)}$ and those of the form $(c_{ij})^{(0,0,r)}$, the y and z parts respectively. These relations could of course just as well have been read off of equation 13.2.1.

From here on we treat the cases of zero and prime characteristic separately.

13.3 Characteristic Zero

Let k be a field of characteristic zero, and (V, ρ) a representation of H_1 as in the previous section. Set $X = (c_{ij})^{(1,0,0)}$, $Y = (c_{ij})^{(0,1,0)}$, $Z = (c_{ij})^{(0,0,1)}$.

Theorem 13.3.1. *A representation of H_1 over k is completely determined by the assignments X and Y. Necessarily $Z = [X, Y]$, each of X and Y must commute with Z, and X, Y and Z must all be nilpotent. Further, any X and Y satisfying these relations defines a representation of G over k.*

Proof. We know from our previous work with the additive group G_a that the following identities must hold:

$$(c_{ij})^{(r,0,0)} = \frac{1}{r!}X^r \qquad (c_{ij})^{(0,r,0)} = \frac{1}{r!}Y^r \qquad (c_{ij})^{(0,0,r)} = \frac{1}{r!}Z^r$$

We work with the fundamental relation for H_1, equation 13.2.1:

$$(c_{ij})^{\vec{r}}(c_{ij})^{\vec{s}} = \sum_{l=0}^{\min(r_1,s_2)} \binom{r_1+s_1-l}{s_1}\binom{r_2+s_2-l}{r_2}\binom{r_3+s_3+l}{r_3,s_3,l}(c_{ij})^{(r_1+s1-l,r_2+s_2-1,l)}$$

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We have

$$(c_{ij})^{(0,m,0)}(c_{ij})^{(n,0,0)} = \sum_{l=0}^{0} \binom{n-l}{n}\binom{m-l}{m}\binom{l}{l}(c_{ij})^{(n-l,m-l,l)}$$

which says that

$$(c_{ij})^{(n,m,0)} = \frac{1}{m!n!}Y^m X^n$$

Using again the fundamental relation, we also have

$$(c_{ij})^{(n,m,0)}(c_{ij})^{(0,0,k)} = (c_{ij})^{(n,m,k)}$$

which together with the last equation gives

$$(c_{ij})^{(n,m,k)} = \frac{1}{n!m!k!}Y^m X^n Z^k$$

Thus, all of the (c_{ij}) are determined by X, Y and Z, according to the above formula. Further,

$$XY = \sum_{l=0}^{1}\binom{1-l}{0}\binom{1-l}{0}\binom{l}{l}(c_{ij})^{1-l,1-1,l)} = YX + Z$$

and so $Z = [X, Y]$ as claimed. Each of Y and X must commute with Z, for if we apply the fundamental relation to each of XZ and ZX, in each case we obtain $(c_{ij})^{(1,0,1)}$, showing $XZ = ZX$, and an identical computation shows $YZ = ZY$. And by our work on G_a we know that each of X, Y and Z must be nilpotent.

We must now show sufficiency of the given relations. Let X, Y and Z be any three nilpotent matrices satisfying $Z = XY - YX$, with each of X and Y commuting with Z. We need to show that the fundamental relation, equation 13.2.1, is always satisfied. We assign

$$(c_{ij})^{(n,m,k)} = \frac{1}{n!m!k!}Z^k Y^m X^n$$

Since each of X, Y and Z are nilpotent, $(c_{ij})^{\vec{r}}$ will vanish for all by finitely many

$\vec{r}$, as required. The fundamental relation, with these assignments, reduces to (after shuffling all coefficients to the right-hand side and some cancellation)

$$Z^{r_3+s_3}Y^{r_2}X^{r_1}Y^{s_2}X^{s_1} = \sum_{l=0}^{\min(r_1,s_2)} l! \binom{r_1}{l}\binom{s_2}{l} Z^{r_3+s_3+l}Y^{r_2+s_2-l}X^{r_1+s_1-l}$$

Each term in the summation has the term $Z^{r_3+s_3}$ in the front and X^{s_1} in the rear, and so does the left-hand side. So it suffices to show

$$Y^{r_2}X^{r_1}Y^{s_2} = \sum_{l=0}^{\min(r_1,s_2)} l! \binom{r_1}{l}\binom{s_2}{l} Z^l Y^{r_2+s_2-l}X^{r_1-l}$$

and since Y commutes with Z, the summation term (minus coefficients) can be written as $Y^{r_2}Z^l Y^{s_2-l}X^{r_1-l}$. We can now take off the Y^{r_2} term from the front of either side, so it suffices to show

$$X^n Y^m = \sum_{l=0}^{\min(n,m)} l! \binom{n}{l}\binom{m}{l} Z^l Y^{m-l}X^{n-l}$$

where we have renamed r_1 and s_2 with the less cumbersome n and m.

We proceed by a double induction on n and m. The case of n or m being zero is trivial, and if $n = m = 1$, the above equation is $XY = Z + YX$, which is true by assumption. Consider then $X^n Y$, and by induction suppose that the equation holds for $X^{n-1}Y$, so that $X^{n-1}Y = YX^{n-1} + (n-1)ZX^{n-2}$. Then using the relation $XY = Z + YX$ and X commuting with Z, we have

$$X^n Y = X^{n-1}XY$$
$$= X^{n-1}(Z + YX)$$
$$= ZX^{n-1} + (X^{n-1}Y)X$$
$$= ZX^{n-1} + (YX^{n-1} + (n-1)ZX^{n-2})X$$
$$= nZX^{n-1} + YX^n$$

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and so the equation is true when $m = 1$. Now suppose that $m \leq n$, so that $\min(n, m) = m$. Then we have

$$X^n Y^m = (X^n Y) Y^{m-1}$$

$$= (YX^n + nZX^{n-1})Y^{m-1}$$

$$= Y(X^n Y^{m-1}) + nZ(X^{n-1}Y^{m-1})$$

which by induction is equal to

$$Y\left(\sum_{l=0}^{m-1} l! \binom{n}{l}\binom{m-1}{l} Z^l Y^{m-1-l} X^{n-l}\right) + nZ\left(\sum_{l=0}^{m-1} l! \binom{n-1}{l}\binom{m-1}{l} Z^l Y^{m-1-l} X^{n-1-l}\right)$$

$$= \sum_{l=0}^{m-1} l! \binom{n}{l}\binom{m-1}{l} Z^l Y^{m-l} X^{m-l} + \sum_{l=0}^{m-1} nl! \binom{n-1}{l}\binom{m-1}{l} Z^{l+1} Y^{m-1-l} X^{n-1-l}$$

$$= Y^m X^n + \sum_{l=1}^{m-1} l! \binom{n}{l}\binom{m-1}{l} Z^l Y^{m-l} X^{n-l} + \sum_{l=1}^{m} n(l-1)! \binom{n-1}{l-1}\binom{m-1}{l-1} Z^l Y^{m-l} X^{n-l}$$

where, in the last step, we have chopped off the first term of the first summation and shifted the index l of the second summation. If we chop off the last term of the second summation we obtain

$$= Y^m X^n + \sum_{l=1}^{m-1} l! \binom{n}{l}\binom{m-1}{l} Z^l Y^{m-l} X^{n-l}$$

$$+ \sum_{l=1}^{m-1} n(l-1)! \binom{n-1}{l-1}\binom{m-1}{l-1} Z^l Y^{m-l} X^{n-l} + n(m-1)! \binom{n-1}{m-1}\binom{m-1}{m-1} Z^m X^{n-m}$$

and upon merging the summations, we have

$$= Y^m X^n + \sum_{l=1}^{m-1} \left[l! \binom{n}{l} \binom{m-1}{l} + n(l-1)! \binom{n-1}{l-1} \binom{m-1}{l-1} \right] Z^l Y^{m-1} X^{n-l}$$

$$+ n(m-1)! \binom{n-1}{m-1} \binom{m-1}{m-1} Z^m X^{n-m}$$

$$= Y^m X^n + \sum_{l=1}^{m-1} \left[l! \binom{n}{l} \binom{m-1}{l} + n(l-1)! \binom{n-1}{l-1} \binom{m-1}{l-1} \right] Z^l Y^{m-1} X^{n-l}$$

$$+ n! \binom{n}{m} \binom{m}{m} Z^m X^{n-m}$$

The two terms outlying the summation are exactly the first and last terms of what the fundamental relation predicts them to be. To finish then, it suffices to show that the term in brackets is equal to $l! \binom{n}{l} \binom{m}{l}$, which is a straightforward computation left to the reader. This completes the case of $m \leq n$, and the case $n \geq m$ is hardly any different, and left to the reader.

$\square$

13.4   Characteristic p

Here we are not interested in giving a complete combinatorial classification of characteristic p representations of H_1. Rather, we shall only be interested in the case where p is sufficiently large when compared to the dimension of the module. Doing so, we obtain a result analogous to theorem 12.3.6 for the group G_a, namely that such representations 'look like' representations of H_1^n in characteristic zero.

Let (V, ρ) be a comodule for G over the field k of characteristic $p > 0$, given by the matrices $(c_{ij})^{\vec{r}}$ over k. Again, matrices of the form $(c_{ij})^{(r,0,0)}$, $(c_{ij})^{(0,r,0)}$ and $(c_{ij})^{(0,0,r)}$, the x, y and z parts of the representation respectively, must in isolation define representations of G_a over k. We know then that (proposition 12.3.5), for example, all

matrices of the form $(c_{ij})^{(r,0,0)}$ are completely determined by the assignments

$$X_0 = (c_{ij})^{(p^0,0,0)}, \quad X_1 = (c_{ij})^{(p^1,0,0)}, \ldots, X_m = (c_{ij})^{(p^m,0,0)}$$

and abide by the formula, for $r = r_m p^m + r_{m-1} p^{m-1} + \ldots r_0$ in p-ary notation

$$(c_{ij})^{(r,0,0)} = \Gamma(r)^{-1} X_0^{r_0} X_1^{r_1} \ldots X_m^{r_m}$$

and that the X_i must commute and be p-nilpotent. An identical statement holds for the matrices $Y_m = (c_{ij})^{(0,p^m,0)}$ and $Z_m = (c_{ij})^{(0,0,p^m)}$.

From here on we adopt the notation $X_{(i)} = (c_{ij})^{(i,0,0)}$, similarly for $Y_{(i)}$ and $Z_{(i)}$. Note that $X_{(i)}$ and X_i are not the same thing.

Theorem 13.4.1. *Let k have characteristic $p > 0$. A representation of H_1 over k is completely determined by the X_i and Y_i. The X_i must commute with one another, same for the Y_i and Z_i, and each X_i must commute with every Z_j, same for Y_i and Z_j.*

Proof. We work again with the fundamental relation for H_1, equation 13.2.1:

$$(c_{ij})^{\vec{r}}(c_{ij})^{\vec{s}} = \sum_{l=0}^{\min(r_1,s_2)} \binom{r_1 + s_1 - l}{s_1} \binom{r_2 + s_2 - l}{r_2} \binom{r_3 + s_3 + l}{r_3, s_3, l} (c_{ij})^{(r_1+s1-l,r_2+s_2-1,l)}$$

taking care of course to realize when a given binomial coefficient is or is not zero mod p. We begin with

$$Y_{(m)} X_{(n)} = (c_{ij})^{(0,m,0)} (c_{ij})^{(n,0,0)}$$
$$= \sum_{l=0}^{0} \binom{n}{0} \binom{m}{0} \binom{l}{l} (c_{ij})^{(n-l,m-l,l)}$$
$$= (c_{ij})^{(n,m,0)}$$

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and

$$Y_{(m)}X_{(n)}Z_{(k)} = (c_{ij})^{(n,m,0)}(c_{ij})^{(0,0,k)}$$

$$= \sum_{l=0}^{0} \binom{r_1}{l}\binom{r_2}{l}\binom{k+l}{k,l}(c_{ij})^{(n-l,m-l,k+l)}$$

$$= (c_{ij})^{(n,m,k)}$$

Thus we have a formula for an arbitrary (c_{ij}) matrix:

$$(c_{ij})^{(n,m,k)} = Y_{(m)}X_{(n)}Z_{(k)}$$

We now show that each of Z_i are determined by the X_i and Y_i. The fundamental relation gives, just as in characteristic zero

$$X_0 Y_0 = (c_{ij})^{(1,0,0)}(c_{ij})^{(0,1,0)} = Y_0 X_0 + Z_0$$

showing $Z_0 = [X_0, Y_0]$. Now assume by induction that Z_i is determined by the X_i and Y_i for $i < m$, and we have

$$X_m Y_m = (c_{ij})^{(p^m,0,0)}(c_{ij})^{(0,p^m,0)}$$

$$= \sum_{l=0}^{p^m} \binom{p^m-l}{0}\binom{p^m-l}{0}\binom{l}{l}Y_{(p^m-l)}X_{(p^m-l)}Z_{(l)}$$

$$= \left(\sum_{l=0}^{p^m-1} \binom{p^m-l}{0}\binom{p^m-l}{0}\binom{l}{l}Y_{(p^m-l)}X_{(p^m-l)}Z_{(l)}\right) + Z_m$$

For $l < p^m$, $Z_{(l)}$ is determined by the Z_i for $i < m$, who in turn, by induction, are determined by the X_i and Y_i. Every term in the summation is thus determined by the X_i and Y_i, hence so is Z_m, the outlying term. This shows that the entire representation is determined by the X_i and Y_i.

To see that each X_i commutes with every Z_j, simply apply the fundamental

relation to both $X_i Z_j$ and $Z_j X_i$, for which you get the same answer. Do the same for Y_i and Z_j, and this completes the proof.

□

We ask the reader to note that it is *not* generally the case that $Z_m = [X_m, Y_m]$ for $m > 0$, nor is it the case that X_i and Y_j commute for $i \neq j$ (the author verified this with several counter-examples which he will not burden you with). However, we will see now that these relations do in fact hold so long as p is sufficiently large when compared to the dimension of a module.

Lemma 13.4.2. *Suppose that p is greater than twice the dimension of a module, and that the sum $r + s$ carries. Then at least one of $P_{(r)}$ or $Q_{(s)}$ must be zero, where P and Q can be any of X, Y or Z.*

Proof. The key fact is that since the X_i, Y_i, and Z_i are all nilpotent, they are nilpotent of order less than or equal to the the dimension of the module, which we assume is no greater than $p/2$. Since the sum $r + s$ carries, we have $r_i + s_i \geq p$ for some i, whence, say, $r_i \geq p/2$. Then

$$P_{(r)} = \Gamma(r)^{-1} P_0^{r_0} \dots P_i^{r_i} \dots P_i^{r_m}$$

is zero, since $P_i^{r_i}$ is.

□

Proposition 13.4.3. *Suppose p is greater than or equal to twice the dimension of a module. Then the following relations must hold: $Z_m = [X_m, Y_m]$ for every m, and $X_m Y_n = Y_n X_m$ for every $m \neq n$.*

Proof. Consider the fundamental relation, equation 13.2.1, applied to $X_m Y_m$:

$$X_m Y_m = Y_m X_m + \left(\sum_{l=1}^{p^m - 1} \binom{p^m - l}{0} \binom{p^m - l}{0} \binom{l}{l} Z_{(l)} Y_{(p^m - l)} X_{(p^m - l)} \right) + Z_m$$

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For every $0 < l < p^m$ there is clearly some carrying in computing the sum $(p^m - l) + l$, so lemma 13.4.2 says that the summation term $Z_{(l)} Y_{(p^m - l)} X_{(p^m - l)}$ is always zero, since at least one of $Z_{(l)}$ or $Y_{(p^m - l)}$ is zero. This gives $Z_m = [X_m, Y_m]$ as claimed.

Now let $n \neq m$, and consider the fundamental relation applied to $X_m Y_n$:

$$X_m Y_n = Y_n X_m + \left(\sum_{l=1}^{\min(p^n, p^m)} \binom{p^m - l}{0} \binom{p^n - l}{0} \binom{l}{l} Z_{(l)} Y_{(p^n - l)} X_{(p^m - l)} \right)$$

In case $m < n$, for every value of l in the above summation, $(p^n - l) + l$ has digit rollover, again forcing at least one of $Z_{(l)}$ or $Y_{(p^n - l)}$ to be zero, forcing every term in the summation to be zero. A similar statement holds in case $n < m$. This proves $X_m Y_n = Y_n X_m$, as claimed.

$\square$

Thus far we have shown that, for $p \geq 2d$, every d-dimensional module must satisfy at least those relations that representations of $G \times G \times \ldots$ over a field of characteristic zero must satisfy. We now show sufficiency.

Lemma 13.4.4. *If p is greater than or equal to twice the dimension of a module, then for any r and s*

$$Z_{(r)} Z_{(s)} = \binom{r + s}{r} Z_{(r+s)}$$

The same holds if we replace Z with X or Y.

Proof. In case the sum $r + s$ does not carry, we know from our previous work with G_a (or direct verification) that the equation is true, just by checking the assignments of the $Z_{(i)}$ in terms of the Z_i (This is true even without the hypothesis that p be large). If on the other hand the sum does carry, then the binomial coefficient on the right is zero by corollary 12.3.3. But so is the product on the left, by lemma 13.4.2. $\square$

We can now prove

Theorem 13.4.5. *Suppose $p \geq 2d$. Let X_i, Y_i and Z_i be a finite sequence of $d \times d$ matrices satisfying*

1. *The X_i, Y_i, and Z_i are all nilpotent*

2. *$Z_i = [X_i, Y_i]$ for every i*

3. *$[X_i, Z_i] = [Y_i, Z_i] = 0$ for every i*

4. *For every $i \neq j$, X_i, Y_i, Z_i all commute with X_j, Y_j, Z_j*

Let $n = n_m p^m + n_{m-1} p^{m-1} + \ldots + n_1 p + n_0$, and assign

$$X_{(n)} = \Gamma(n)^{-1} X_m^{n_m} \ldots X_0^{n_0}$$

and similarly for $Y_{(n)}$ and $Z_{(n)}$. Set

$$(c_{ij})^{(n,m,k)} = Z_{(k)} Y_{(m)} X_{(n)}$$

Then these assignments define a valid d-dimensional representation of G over k.

Proof. For arbitrary n, m, k, r, s and t, the equation we must verify is

$$(c_{ij})^{(n,m,k)} (c_{ij})^{(r,s,t)}$$

$$= \sum_{l=0}^{\min(n,s)} \binom{n+r-l}{r} \binom{m+s-l}{m} \binom{k+t+l}{k,t,l} (c_{ij})^{(n+r-l,m+s-l,k+t+l)}$$

which, with the given assignments and assumptions, can be written

$$Z_{(k)} Z_{(t)} Y_{(m)} X_{(n)} Y_{(s)} X_{(r)}$$

$$= \sum_{l=0}^{\min(n,s)} \binom{n+r-l}{r} \binom{m+s-l}{m} \binom{k+t+l}{k,t,l} Z_{(k+t+l)} Y_{(m+s-l)} X_{(n+r-l)}$$

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Lemma 13.4.4 gives the identities

$$Z_{(k)}Z_{(t)} = \binom{k+t}{t} Z_{(k+t)}$$

$$Y_{(m)}Y_{(s-l)} = \binom{m+s-l}{m} Y_{(m+s-l)}$$

$$X_{(n-l)}X_{(r)} = \binom{n+r-l}{r} X_{(n+r-l)}$$

so we can rewrite our equation as

$$\binom{k+t}{t} Z_{(k+t)}Y_{(m)}X_{(n)}Y_{(s)}X_{(r)} = \sum_{l=0}^{\min(n,s)} \binom{k+t+l}{k,t,l} Z_{(k+t+l)}Y_{(m)}Y_{(s-l)}X_{(n-l)}X_{(r)}$$

First suppose that the sum $k + t$ carries. In this case the equation is true, since the left hand side binomial coefficient vanishes, and the right hand side multinomial coefficient vanishes for every l, causing both sides to be zero. We assume then that $k + t$ does not carry, so we can divide both sides by $\binom{k+t}{t}$ to yield

$$Z_{(k+t)}Y_{(m)}X_{(n)}Y_{(s)}X_{(r)} = \sum_{l=0}^{\min(n,s)} \binom{k+t+l}{l} Z_{(k+t+l)}Y_{(m)}Y_{(s-l)}X_{(n-l)}X_{(r)}$$

Now apply $\binom{k+t+l}{l} Z_{(k+t+l)} = Z_{(k+t)}Z_{(l)}$:

$$Z_{(k+t)}Y_{(m)}X_{(n)}Y_{(s)}X_{(r)} = \sum_{l=0}^{\min(n,s)} Z_{(k+t)}Z_{(l)}Y_{(m)}Y_{(s-l)}X_{(n-l)}X_{(r)}$$

We have $Z_{(k+t)}$ in the front and $X_{(r)}$ in the rear of both sides, so it suffices to show

$$Y_{(m)}X_{(n)}Y_{(s)} = \sum_{l=0}^{\min(n,s)} Z_{(l)}Y_{(m)}Y_{(s-l)}X_{(n-l)}$$

and since $Y_{(m)}$ commutes with $Z_{(l)}$, we can move it to the front of the right hand side,

and then take it off both sides, so it suffices to show

$$X_{(n)}Y_{(m)} = \sum_{l=0}^{\min(n,m)} Z_{(l)}Y_{(m-l)}X_{(n-l)} \tag{13.4.1}$$

where we have replaced s with the more traditional m.

Now we begin to replace the $X'_{(i)}s$ with their definitions in terms of the $X'_i s$, similarly for Y and Z, so that the left hand side of equation 13.4.1 is

$$[\Gamma(n)\Gamma(m)]^{-1} X_0^{n_0} \dots X_k^{n_k} Y_0^{m_0} \dots Y_k^{m_k}$$

and since everything commutes except X_i and Y_j when $i = j$, we can write

$$[\Gamma(n)\Gamma(m)]^{-1} (X_0^{n_0} Y_0^{m_0}) \dots (X_k^{n_k} Y_k^{m_k})$$

Moving all coefficients to the right, we must show

$$(X_0^{n_0} Y_0^{m_0}) \dots (X_k^{n_k} Y_k^{m_k}) = \Gamma(n)\Gamma(m) \sum_{l=0}^{\min(n,m)} Z_{(l)}Y_{(m-l)}X_{(n-l)}$$

We proceed by induction on k, maximum number of p-digits of either m or n. If $k = 0$ the equation is

$$X_0^{n_0} Y_0^{m_0} = n_0! m_0! \sum_{l=0}^{\min(n_0,m_0)} Z_{(l)}Y_{(m_0-l)}X_{(n_0-l)}$$

$$= \sum_{l=0}^{\min(n_0,m_0)} \frac{n_0! m_0!}{(m_0 - l)!(n_0 - l)! l!} Z_0^l Y_0^{m_0-l} X_0^{n_0-l}$$

$$= \sum_{l=0}^{\min(n_0,m_0)} l! \binom{n_0}{l} \binom{m_0}{l} Z_0^l Y_0^{m_0-l} X_0^{n_0-l}$$

The reader may recall that this was exactly the equation to be verified halfway through the proof of theorem 13.3.1 in the characteristic zero case for X, Y and Z. Nowhere in

that section of the proof did we use the characteristic of the field; the same hypotheses hold here for X_0, Y_0 and Z_0, and the proof goes through just the same, so we do not repeat it. Now suppose the equation is true when n and m have no more than $k-1$ digits. Let $n = n_{k-1}p^{k-1} + \ldots + n_0$ and let $n' = n_k p^k + n_{k-1}p^{k-1} + \ldots + n_0$, and similarly for m. Then by induction we have

$$\Gamma(n')\Gamma(m')X_{(n')}Y_{(m')} = \left[(X_0^{n_0}Y_0^{m_0})\ldots(X_{k-1}^{n_{k-1}}Y_{k-1}^{m_{k-1}})\right](X_k^{n_k}Y_k^{m_k})$$

$$= \left(\Gamma(n)\Gamma(m)\sum_{l=0}^{\min(n,m)}Z_{(l)}Y_{(m-l)}X_{(n-l)}\right)\left(\sum_{l'=0}^{\min(n_k,m_k)}l'!\binom{n_k}{l'}\binom{m_k}{l'}Z_k^{l'}Y_k^{m_k-l'}X_k^{n_k-l'}\right)$$

$$= n_k!\Gamma(n)m_k!\Gamma(m)\sum_{l,l'}\left(\frac{Z_{(l)}Z_k^{l'}}{l'!}\right)\left(\frac{Y_{(m-l)}Y_k^{m_k-l'}}{(m_k-l')!}\right)\left(\frac{X_{(n-l)}X_k^{n_k-l'}}{(n_k-l')!}\right)$$

Note that these divisions are valid, since for every value of l' in the summation, $l' \le m_k, n_k < p$. Note also that, since $l \le p^{k-1}$ and $l' < p$ for all values of l, l' in the summation, Lucas' theorem gives that $\binom{l+l'p^k}{l} = 1$ for all such l and l'. For similar reasons we have $\binom{(m-l)+(m_k-l')p^k}{m-l} = \binom{(n-l)+(n_k-l')p^k}{n-l} = 1$. Then we have the identities

$$n_k!\Gamma(n) = \Gamma(n') \qquad\qquad m_k!\Gamma(m) = \Gamma(m')$$

$$\frac{Z_{(l)}Z_k^{l'}}{l'!} = Z_{(l)}Z_{(l'p^k)} = \binom{l+l'p^k}{l}Z_{(l+l'p^k)} = Z_{(l+l'p^k)}$$

$$\frac{Y_{(m-l)}Y_k^{m_k-l'}}{(m_k-l')!} = Y_{(m-l)}Y_{((m_k-l')p^k)} = \binom{(m-l)+(m_k-l')p^k}{m-l}Y_{((m+m_kp^k)-(l+l')p^k)} = Y_{(m'-(l+l'p^k))}$$

and similarly

$$\frac{X_{(n-l)}X_k^{n_k-l'}}{(n_k-l')!} = X_{(n'-(l+l'p^k))}$$

These substitutions transform the right hand side of our equation into

$$= \Gamma(n')\Gamma(m')\sum_{l,l'}Z_{(l+l'p^k)}Y_{(m'-(l+l'p^k))}X_{(n'-(l+l'p^k))}$$

But, if we look at the summation limits of $l = 0\ldots\min(n,m)$ and $l' = 0\ldots\min(n_k,m_k)$,

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we see that it is really a single summation running from 0 to $\min(n', m')$, with $l + l'p^k$ as the summation variable. That is

$$= \Gamma(n')\Gamma(m') \sum_{l=0}^{\min(n',m')} Z_{(l)}Y_{(m'-l)}X_{(n'-l)}$$

which finally gives

$$X_{(n')}Y_{(m')} = \sum_{l=0}^{\min(n',m')} Z_{(l)}Y_{(m'-l)}X_{(n'-l)}$$

as required. This completes the proof.

□

13.5 Baker-Campbell-Hausdorff Formula For H_1 in Positive Characteristic

There is a much more compact way to state all of this. What we have really recovered is, in characteristic zero, the familiar Baker-Campbell-Hausdorff formula for the group H_1, and for characteristic $p >>$ dimension, something very close to it.

In theorem 3.1 of [10] it is proven that, if X, Y and Z are matrices over $\mathbb{R}$ such that $Z = [X, Y]$ and $[Z, X] = [Z, Y] = 0$, then

$$e^X e^Y = e^{X+Y+\frac{1}{2}Z}$$

Our first step is to extend this result to the case of a field of sufficiently large characteristic when compared to the dimensions of the matrices X, Y and Z, and under the additional hypothesis that they be nilpotent. The proof we give is an almost exact replica of that given in [10]; the only difference is that we replace the notion of derivative with 'formal derivative' of polynomials.

For the remainder, by a polynomial $f(t)$, we shall mean a polynomial in the com-

muting variable t with coefficients which are matrix expressions among the matrices X,Y and Z over a given field; for example, $f(t) = XY + 2(Z - YX)t + \frac{Y}{2}t^2 - t^3$. We define the **formal derivative** of $f(t)$ in the usual manner; for example, $f'(t) = 2(Z - YX) + Yt - 3t^2$. Then the following facts hold just as well for formal differentiation as they do for standard differentiation.

Lemma 13.5.1. *Let $f(t)$, $g(t)$ be polynomials, and suppose that the field is either of characteristic zero, or of positive characteristic greater than the degrees of both $f(t)$ and $g(t)$.*

1. *(product rule) $(fg)' = f'g + fg'$*

2. *(uniqueness of antiderivatives) If $f'(t) = g'(t)$, and if $f(0) = g(0)$, then $f(t) = g(t)$*

3. *(uniqueness of solutions to differential equations) Let M be some matrix expression among X,Y and Z. Then if $f'(t) = Mf(t)$, and if $g'(t) = Mg(t)$, and if $f(0) = g(0)$, then $f(t) = g(t)$*

Remark: the assumption that $\text{char}(k) > $ degree is essential. For example, in characteristic 2, the derivatives of the polynomials t^2 and 0 are both zero, and they are both zero when evaluated at $t = 0$, but they are obviously not themselves equal.

Proof. 1. is true even without any hypothesis on the characteristic. Let $f = \sum_{k=0}^{m} a_k t^k$,

$g = \sum_{k=0}^{m} b_k t^k$, where the a_i, b_i are matrix expressions in X, Y and Z. Then

$$(fg)' = \left[\left(\sum_{k=0}^{m} a_k t^k \right) \left(\sum_{l=0}^{m} b_l t^l \right) \right]'$$

$$= \left[\sum_{r=0}^{2m} \left(\sum_{k+l=r} a_k b_l \right) t^r \right]'$$

$$= \sum_{r=0}^{2m} r \left(\sum_{k+l=r} a_k b_l \right) t^{r-1}$$

$$= \sum_{r=1}^{2m} r \left(\sum_{k+l=r} a_k b_l \right) t^{r-1}$$

$$= \sum_{r=0}^{2m-1} (r+1) \left(\sum_{k+l=r+1} a_k b_l \right) t^r$$

and

$$f'g + fg' = \left(\sum_{k=0}^{m} k a_k t^{k-1} \right) \left(\sum_{l=0}^{m} b_l t^l \right) + \left(\sum_{k=0}^{m} a_k t^k \right) \left(\sum_{l=0}^{m} l b_l t^{l-1} \right)$$

$$= \sum_{k,l=0}^{m} (k a_k b_l) t^{k-1+l} + \sum_{k,l=0}^{m} (l a_k b_l) t^{k-1+l}$$

$$= \sum_{k,l=0}^{m} (k+l)(a_k b_l) t^{k+l-1}$$

$$= \sum_{r=0}^{2m-1} \left(\sum_{k+l-1=r} (k+l) a_k b_l \right) t^r$$

$$= \sum_{r=0}^{2m-1} (r+1) \left(\sum_{k+l=r+1} a_k b_l \right) t^r$$

which proves 1.

For 2., let f and g be as before. To say that $f' = g'$ is to say that $n a_n = n b_n, (n-1) a_{n-1} = (n-1) b_{n-1}, \ldots, a_1 = b_1$, and to say that $f(0) = g(0)$ is to say that $a_0 = b_0$. Under the given hypotheses all of $n, n-1, \ldots, 1$ are invertible, which forces $a_n = b_n, a_{n-1} = b_{n-1}, \ldots a_1 = b_1$ and $a_0 = b_0$, whence $f = g$. This proves 2.

For 3., suppose $f' = Mf$ and $g' = Mg$. Then by matching coefficients for the

various powers of t this forces the equalities

$$Ma_n = 0 \qquad\qquad Mb_n = 0$$

$$na_n = Ma_{n-1} \qquad\qquad nb_n = Mb_{n-1}$$

$$(n-1)a_{n-1} = Ma_{n-2} \qquad\qquad (n-1)b_{n-1} = Mb_{n-2}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$2a_2 = Ma_1 \qquad\qquad 2b_2 = Mb_1$$

$$a_1 = Ma_0 \qquad\qquad b_1 = Mb_0$$

$f(0) = g(0)$ again forces $a_0 = b_0$. Noting again that all of $n, n-1, \ldots, 1$ are invertible, we can work backwards to see that $a_1 = Ma_0 = Mb_0 = b_1$, that $a_2 = \frac{1}{2}Ma_1 = \frac{1}{2}Mb_1 = b_2, \ldots, a_n = \frac{1}{n}Ma_{n-1} = \frac{1}{n}Mb_{n-1} = b_n$, whence $f = g$. This proves 3.

$\square$

Lemma 13.5.2. *Let X and Y be commuting nilpotent matrices over a field k such that k is of characteristic zero, or of positive characteristic greater than or equal to the dimension of X and Y. Then*

1. $\left(e^{tX}\right)' = e^{tX}X$

2. $\left(e^{t^2X}\right)' = e^{t^2X}(2tX)$

3. $e^X e^Y = e^{X+Y}$

Remark: the first two are obvious corollaries to the usual chain rule for differentiation, but the chain rule is in general *not* valid for polynomials in non-commuting coefficients. It is convenient for our purposes just to treat these cases separately.

Proof. We note firstly that, if $\mathrm{char}(k) = p \geq \dim$, then all of the above expressions make sense, since their series expansions will vanish before we get to see denominators

divisible by p. For 1., compute:

$$(e^{tX})' = \left(1 + tX + \frac{t^2 X^2}{2!} + \ldots + \frac{t^n X^n}{n!}\right)'$$

$$= X + \frac{2tX^2}{2!} + \ldots + \frac{nt^{n-1}X^n}{n!}$$

$$= X\left(1 + tX + \ldots + \frac{t^{n-1}X^{n-1}}{(n-1)!} + \frac{t^n X^n}{n!}\right)$$

$$= Xe^{tX}$$

Note that, in the second to last expression, we are justified in tacking on the term $\frac{t^n X^n}{n!}$ since multiplication by X will annihilate it anyway. This proves 1.

For 2., compute again:

$$(e^{t^2 X})' = \left(1 + t^2 X + \frac{t^4 X^2}{2!} + \ldots + \frac{t^{2n} X^n}{n!}\right)'$$

$$= 2tX + \frac{4t^3 X^2}{2!} + \ldots + \frac{2nt^{2n-1}X^n}{n!}$$

$$= 2tX\left(1 + \frac{2t^2 X}{2!} + \ldots + \frac{nt^{2(n-1)}X^n}{n!}\right)$$

$$= 2tX\left(1 + t^2 X + \frac{t^4 X^2}{2!} + \ldots + \frac{t^{2(n-1)}X^{n-1}}{(n-1)!} + \frac{t^{2n} X^n}{n!}\right)$$

$$= 2tXe^{t^2 X}$$

where, again, in the second to last expression, we are justified in tacking on the term $\frac{t^{2n} X^n}{n!}$ since X will annihilate it anyhow. This proves 2.

For 3., we shall prove that $e^{tX}e^{tY} = e^{t(X+Y)}$ as polynomials; evaluating at $t = 1$ gives the desired result. Note that the right hand side is defined; if X and Y commute, they can be put in simultaneous upper triangular form, and so $X + Y$ is nilpotent. By 3. of lemma 13.5.1, since they are equal when evaluated at $t = 0$, it is enough to

show that they satisfy the same differential equation:

$$\left(e^{tX}e^{tY}\right)' = \left(e^{tX}\right)'e^{tY} + e^{tX}\left(e^{tY}\right)'$$

$$= Xe^{tX}e^{tY} + e^{tX}Ye^{tY}$$

$$= e^{tX}e^{tY}(X+Y)$$

and

$$\left(e^{t(X+Y)}\right)' = e^{t(X+Y)}(X+Y)$$

This completes the proof.

$\square$

Lemma 13.5.3. *Let X and Y be nilpotent matrices over a field, commuting with their nilpotent commutator Z. If the field is either of characteristic zero, or of positive characteristic larger than twice the dimension of the matrices, then*

$$e^X e^Y = e^{X+Y+\frac{1}{2}Z}$$

Proof. We shall prove something stronger, namely that

$$e^{tX}e^{tY} = e^{tX+tY+\frac{t^2}{2}Z}$$

as polynomials; evaluating at $t = 1$ will give the desired result.

We note first that if char $= p \geq 2$dim, all of the above expressions make sense, since e.g. the series expansion for e^{tX} will vanish before we get to see denominators divisible by p. Note also that the results of the previous two lemmas apply, since the maximum degree of any of the above polynomials is 2dim $- 2$.

Note also that $tX + tY + \frac{t^2}{2}Z$ must be also be nilpotent. If X,Y and Z are matrices satisfying the given hypotheses, they define a representation of H_1 according to either theorem 13.3.1 or theorem 13.4.5. As any representation of a unipotent algebraic

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group can be put in upper triangular form, it follows that X, Y and Z can be put in simultaneous upper triangular form. It is obvious then that any linear combination of X, Y and Z is nilpotent, and so the right hand side makes sense as well (except when $p = 2$; but this forces dimension to be ≤ 1, and in this case the result is trivial).

The proof proceeds exactly as in theorem 3.1 of [10] for the Lie group case. Since Z commutes with both X and Y, we can rewrite the above equation as

$$e^{tX} e^{tY} e^{\frac{-t^2}{2} Z} = e^{t(X+Y)}$$

Denote by $A(t)$ the left hand side of this equation, $B(t)$ the right hand side. These are both equal to 1 when evaluated at $t = 0$, so by 3. of lemma 13.5.1 it suffices to show that they both satisfy the same linear differential equation. Working first with $A(t)$, using the iterated product rule we have

$$A'(t) = e^{tX} X e^{tY} e^{\frac{-t^2}{2} Z} + e^{tX} e^{tY} Y e^{\frac{-t^2}{2} Z} + e^{tX} e^{tY} e^{\frac{-t^2}{2} Z}(-tZ)$$

$$= e^{tX} e^{tY} \left(e^{-tY} X e^{tY} \right) e^{\frac{-t^2}{2} Z} + e^{tX} e^{tY} Y e^{\frac{-t^2}{2} Z} + e^{tX} e^{tY} e^{\frac{-t^2}{2} Z}(-tZ)$$

We claim that $e^{-tY} X e^{tY}$ is equal to $X + tZ$. They are both equal to X when evaluated at $t = 0$, and $(X + tZ)' = Z$, so it suffices to show by part 2. of lemma 13.5.1 that the derivative of $e^{-tY} X e^{tY}$ is equal to Z:

$$\left(e^{-tY} X e^{tY} \right)' = e^{-tY}(-YX)e^{tY} + e^{-tY} XY e^{tY}$$

$$= e^{-tY}(XY - YX)e^{tY}$$

$$= e^{-tY} Z e^{tY}$$

$$= Z$$

as required. Thus

$$A'(t) = e^{tX}e^{tY}\left(X + tZ\right)e^{\frac{-t^2}{2}Z} + e^{tX}e^{tY}Ye^{\frac{-t^2}{2}Z} + e^{tX}e^{tY}e^{\frac{-t^2}{2}Z}(-tZ)$$

$$= e^{tX}e^{tY}e^{\frac{-t^2}{2}Z}(X + tZ + Y - tZ)$$

$$= e^{tX}e^{tY}e^{\frac{-t^2}{2}Z}(X + Y)$$

$$= A(t)(X + Y)$$

To finish then, it suffices to show that $B'(t) = B(t)(X + Y)$; but this is obvious by part 1. of lemma 13.5.2. This completes the proof.

□

Theorem 13.5.4. *Let $M(x, y, z)$ be (the matrix formula for) a finite dimensional module for H_1 in characteristic zero given by the nilpotent matrices X, Y, and Z, in the notation of theorem 13.3.1. Then*

$$M(x, y, z) = e^{xX + yY + (z - xy/2)Z}$$

Proof. We will first prove that the formula given is actually a representation of H_1, which amounts to verifying the matrix equality

$$e^{xX + yY + (z - xy/2)Z}e^{rX + sY + (t - rs/2)Z} = e^{(x+r)X + (y+s)Y + (z + xs + t - (x+r)(y+s)/2)Z}$$

If X and Y are nilpotent and commute with their nilpotent commutator Z, then $xX + yY$ and $rX + sY$ are also nilpotent, and also commute with their nilpotent commutator $(xs - yr)Z$, and so lemma 13.5.3 applies:

$$\exp(xX + yY)\exp(rX + sY) = \exp((x + r)X + (y + s)Y + \frac{(xs - yr)}{2}Z)$$

Recalling that $e^R e^S = e^{R+S}$ whenever R and S commute, the left hand side of our

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first equation can be written

$$\exp(xX + yY)\exp(rX + sY)\exp((z + t - (xy + rs)/2)Z)$$

$$= \exp((x + r)X + (y + s)Y + \frac{xs - yr}{2}Z)\exp((z + t - (xy + rs)/2)Z)$$

$$= \exp((x + r)X + (y + s)Y + (z + xs + t - (x + r)(y + s)/2)Z)$$

The expression given in the statement of the theorem is therefore indeed a representation of H_1. To see that they are equal, simply verify that, in the notation of theorem 13.3.1, the matrix $(c_{ij})^{(1,0,0)}$ is actually X, and the matrix $(c_{ij})^{(0,1,0)}$ is actually Y; as these completely determine the rest of the representation, we conclude that $M(x, y, z)$ and the given expression are in fact equal. $\qquad\square$

In the characteristic $p > 0$ case, if we assume $p \geq 2\dim$, we obtain a result analogous to theorem 12.3.1 for the group G_a.

Theorem 13.5.5. *Let k have characteristic $p > 0$, and suppose $p \geq 2d$. Then every d-dimensional representation of H_1 over k is of the form*

$$e^{xX_0 + yY_0 + (z - xy/2)Z_0}e^{x^p X_1 + y^p Y_1 + (z^p - x^p y^p/2)Z_1}\dots e^{x^{p^m} X_m + y^{p^m} Y_m + (z^{p^m} - x^{p^m} y^{p^m}/2)Z_m}$$

with all of the factors commuting. Further, any collection X_i, Y_i, Z_i of d-dimensional matrices satisfying the hypotheses of theorem 13.4.5 gives a representation according to the above formula.

Proof. In the notation of theorem 13.4.5, let $X_0, \dots, X_s$, $Y_0, \dots, Y_s$, $Z_0, \dots, Z_s$ be

given. Then the matrix formula for the representation they define is

$$M(x,y,z) = \sum_{n,m,k} (c_{ij})^{(n,m,k)} x^n y^m z^k = \sum_{n,m,k} Z_{(k)} Y_{(m)} X_{(n)} x^n y^m z^k$$

$$= \sum_{n,m,k} \Gamma(n)^{-1} \Gamma(m)^{-1} \Gamma(k)^{-1} Z_0^{k_0} \dots Z_s Y_0^{m_0} \dots Y_s^{m_s} X_0^{n_0} \dots X_s^{n_s}$$

$$x^{n_0+n_1 p+\dots+n_s p^s} y^{m_0+m_1 p+\dots+m_s p^s} z^{k_0+k_1 p+\dots k_s p^s}$$

$$= \left(\sum_{n_0,m_0,k_0=0}^{p-1} \frac{1}{n_0! m_0! k_0!} Z_0^{k_0} Y_0^{m_0} X_0^{n_0} x^{n_0} y^{m_0} z^{m_0} \right)$$

$$\dots \left(\sum_{n_s,m_s,k_s=0}^{p-1} \frac{1}{n_s! m_s! k_s!} Z_s^{k_s} Y_s^{m_s} X_s^{n_s} x^{n_s p^s} y^{m_s p^s} z^{k_s p^s} \right)$$

We note that, for fixed r, the matrices X_r, Y_r, and Z_r, by theorem 13.4.5, satisfy the hypotheses of lemma 13.5.3. Working through the proof of theorem 13.5.4, we see that nowhere was the characteristic of the field used; only that lemma 13.5.3 was satisfied. In other words, theorem 13.5.4 establishes a purely combinatorial fact that, if X_0, Y_0, Z_0 satisfy lemma 13.5.3, then

$$\sum_{n_0,m_0,k_0=0}^{m} \frac{1}{n_0! m_0! k_0!} Z_0^{k_0} Y_0^{m_0} X_0^{n_0} x^{n_0} y^{m_0} z^{m_0} = \exp(xX_0 + yY_0 + (z - xy/2)Z_0)$$

whenever m is greater than or equal to the nilpotent orders of X_0, Y_0, and Z_0. We conclude that

$$\sum_{n_0,m_0,k_0=0}^{p-1} \frac{1}{n_0! m_0! k_0!} Z_0^{k_0} Y_0^{m_0} X_0^{n_0} x^{n_0} y^{m_0} z^{m_0} = \exp(xX_0 + yY_0 + (z - xy/2)Z_0)$$

We can of course replace x, y and z with x^{p^r}, y^{p^r} and z^{p^r} to likewise obtain

$$\sum_{n_r,m_r,k_r=0}^{p-1} \frac{1}{n_r! m_r! k_r!} Z_r^{k_r} Y_r^{m_r} X_r^{n_r} x^{n_r p^r} y^{m_r p^r} z^{k_r p^r} = \exp(x^{p^r} X_r + y^{p^r} Y_r + (z^{p^r} - x^{p^r} y^{p^r}/2)Z_r)$$

for any r, and hence

$$\left(\sum_{n_0,m_0,k_0=0}^{p-1} \frac{1}{n_0!m_0!k_0!} Z_0^{k_0} Y_0^{m_0} X_0^{n_0} x^{n_0} y^{m_0} z^{m_0} \right)$$

$$\dots \left(\sum_{n_s,m_s,k_s=0}^{p-1} \frac{1}{n_s!m_s!k_s!} Z_s^{k_s} Y_s^{m_s} X_s^{n_s} x^{n_s p^s} y^{m_s p^s} z^{k_s p^s} \right)$$

$$= \exp(xX_0 + yY_0 + (z - xy/2)Z_0)\exp(x^p X_1 + y^p Y_1 + (z^p - x^p y^p/2)Z_1)$$

$$\dots \exp(x^{p^s} X_s + y^{p^s} Y_s + (z^{p^s} - x^{p^s} y^{p^s}/2)Z_s)$$

which proves the theorem. Note that all of the factors commute, since so do X_i, Y_i, Z_i and X_j, Y_j, Z_j when $i \neq j$.

$\square$

Chapter 14

The Height-Restricted

Ultraproduct

In the previous chapters, we saw that for the unipotent groups G that we studied, the representation theories of G^n in characteristic zero and for G in characteristic $p >>$ dimension are in perfect analogy. The appropriate context, we believe, in which to interpret these results is in consideration of the so-called height-restricted ultraproduct, which we formally define now.

Let G be any of our so far studied unipotent groups, and let k be a field of characteristic $p > 0$. They all have Hopf algebras isomorphic to $k[x_1, \ldots, x_n]$ for some n, so the following definition makes sense:

Definition 14.0.1. The **height** of a representation M of G over k is the largest m such that, for some i, $x_i^{p^{m-1}}$ occurs as a coefficient in the matrix formula of M. In case no such occurs (i.e. M is a trivial representation), we say M has height zero.

Since all of the Hopf algebras at issue are isomorphic to $k[x_1, \ldots, x_n]$, height is an isomorphism invariant. Lemma 8.0.10 shows that applying any base change to the matrix formula of a representation yields two matrices, each of whose entries will be linear combinations of the entries of the other.

Example: the representation

$$
\begin{pmatrix}
1 & x + x^{p^2} \\
0 & 1
\end{pmatrix}
$$

for G_a has height 3.

If p is large with respect to dimension, we know that every representation of G_a or H_1 can be factored into a commuting product of representations, each accounting for one of its Frobenius layers. In this case the height of M is equal to the number of these layers (even in the case of a trivial representation, which has no layers), hence the motivation for the definition.

Let k_i be a collection of fields of strictly increasing positive characteristic, $\mathcal{C}_i = \mathrm{Rep}_{k_i} G$.

Definition 14.0.2. The **height** of an object $[X_i]$ of $\prod_R \mathcal{C}_i$ is defined to be the essential supremum of $\{\mathrm{height}(X_i) : i \in I\}$. The **height-restricted ultraproduct** of the $\mathcal{C}_i$, denoted $\prod_H \mathcal{C}_i$, is the full subcategory of $\prod_R \mathcal{C}_i$ consisting of those objects having finite height. For $n \in \mathbb{N}$, we denote by $\prod_{H \leq n} \mathcal{C}_i$ the full subcategory of $\prod_H \mathcal{C}_i$ consisting of those objects of height no greater than n.

Note that, as a subcategory of $\prod_R \mathcal{C}_i$, we demand the objects of $\prod_H \mathcal{C}_i$ to be of bounded dimension as well as height.

The remainder of this chapter is devoted to proving

Theorem 14.0.6. *Let G be any of the so far studied unipotent groups, k_i a sequence of fields of strictly increasing positive characteristic. Then $\prod_H \mathrm{Rep}_{k_i} G$ is a neutral tannakian subcategory of $\prod_R \mathrm{Rep}_{k_i} G$, and is tensorially equivalent to $\mathrm{Rep}_{\prod_\mathcal{U} k_i} G^\infty$. Likewise, for any $n \in \mathbb{N}$, $\prod_{H \leq n} \mathrm{Rep}_{k_i} G$ is tensorially equivalent to $\mathrm{Rep}_{\prod_\mathcal{U} k_i} G^n$.*

Note that the groups G^∞ and G^n in this theorem are *independent* of the choice of non-principal ultrafilter, and while the field $\prod_\mathcal{U} k_i$ does indeed vary, it will in all

cases have characteristic zero.

We shall prove the theorem for the group H_1, leaving it to the reader to convince himself that the same proof applies to the group G_a. The proof is quite straightforward; we shall construct an explicit equivalence between the two categories, show that it is tensor preserving, and it will be immediate that the usual forgetful functor on $\text{Rep}_{\prod_\mathcal{U} k_i} G^\infty$ can be identified as the restriction of the fibre functor $\omega : \prod_R \mathcal{C}_i \to \text{Vec}_{\prod_\mathcal{U} k_i}$ defined in chapter 7 to $\prod_H \text{Rep}_{k_i} G$.

For the remainder, let G denote the group H_1, k_i a sequence of fields of strictly increasing positive characteristic, $\mathcal{C}_i = \text{Rep}_{k_i} G$, and $k = \prod_\mathcal{U} k_i$.

14.1 Labelling the Objects of $\prod_H \mathcal{C}_i$

Now, what does an object $[V_i]$ of $\prod_H \mathcal{C}_i$ actually look like? Firstly, the vector spaces V_i are of bounded dimension. And since the k_i are of strictly increasing characteristic, this tells us that, for all but finitely many i, V_i is of the form given by theorem 13.4.5; that is, it is determined by a finite sequence of nilpotent transformations on V_i

$$X_0^i, \ldots, X_m^i, Y_0^i, \ldots, Y_m^i, Z_0^i, \ldots, Z_m^i$$

according to the formula given in theorem 13.5.5, and under the conditions given in theorem 13.4.5. Secondly, it is of finite height, whence we can take m to be constant for almost every i.

Theorem 14.1.1. *Each object $[V_i]$ of $\prod_H \mathcal{C}_i$ is completely determined by a sequence $X_0, \ldots, X_m, Y_0, \ldots, Y_m, Z_0, \ldots, Z_m$ of linear transformations on $\prod_\mathcal{U} V_i$ satisfying*

1. *The X_j, Y_j, and Z_j are all nilpotent*

2. *$Z_j = [X_j, Y_j]$ for every j*

3. *$[X_j, Z_j] = [Y_j, Z_j] = 0$ for every j*

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4. For every $i \neq j$, the matrices X_i, Y_i, Z_i commute with X_j, Y_j, Z_j

Further, any such sequence of linear transformations on a finite dimensional vector space over k gives an object of $\prod_H \mathcal{C}_i$.

Proof. We define these transformations as the ultraproduct of the transformations given above:

$$X_0 = [X_0^i], \ldots, X_m = [X_m^i]$$

$$Y_0 = [Y_0^i], \ldots, Y_m = [X_m^i]$$

$$Z_0 = [Z_0^i], \ldots, Z_m = [Z_m^i]$$

By theorem 6.2.9, all four of the above conditions are valid among the X_j, Y_j and Z_j if and only if, for almost every i, all four are valid among the X_j^i, Y_j^i and Z_j^i. Thus every object of $\prod_H \mathcal{C}_i$ determines such a collection of transformations on a $\prod_{\mathcal{U}} k_i$-vector space.

Conversely, suppose that we are given a sequence $X_0, \ldots, X_m, Y_0, \ldots, Y_m, Z_0, \ldots, Z_m$ of linear transformations on an n-dimensional $\prod_{\mathcal{U}} k_i$-vector space V satisfying all the above; we claim there is an object $[V_i]$ of $\prod_H \mathcal{C}_i$, unique up to isomorphism, to which these transformations correspond. By proposition 6.2.4 let V_i be a collection of n-dimensional k_i-vector spaces such that $\prod_{\mathcal{U}} V_i \simeq V$. By proposition 6.2.6, X_0 is uniquely of the form $[X_0^i]$, where each X_0^i is a linear transformation on V_i; the same goes for all of the X_k, Y_k and Z_k. Finally, note that given relations among the X_k, Y_k and Z_k amount to a finite number of equations involving composition of maps, and so by theorem 6.2.9 these relations are almost everywhere valid among the X_k^i, Y_k^i and Z_k^i. As such, almost everywhere, they define a valid H_1-module structure on V_i according to theorem 13.4.5. The object we seek then is $[V_i]$. That $[V_i]$ is unique up to isomorphism is clear from the description of morphisms in $\prod_H \mathcal{C}_i$ given in the following paragraphs. $\qquad\square$

And what about morphisms? By definition, a morphism $[\phi_i] : [V_i] \to [W_i]$ in the category $\prod_H \mathcal{C}_i$ is such that, for almost every i, $\phi_i : V_i \to W_i$ is a morphism in the category $\mathcal{C}_i$. And by theorem 10.1.1, for large enough i, such ϕ_i are exactly those which commute with the X_j^i, Y_j^i, and Z_j^i for every j. Again by theorem 6.2.9 this is equivalent to saying

Theorem 14.1.2. *Let V, W be objects of $\prod_H \mathcal{C}_i$, given by (according to the previous theorem) the transformations X_j, Y_j and Z_j for V and R_j, S_j and T_j for W. Then $\phi = [\phi_i]$ is a morphism between V and W if and only if, for every j, ϕ satisfies*

$$X_j \circ \phi = \phi \circ R_j \qquad Y_j \circ \phi = \phi \circ S_j \qquad Z_j \circ \phi = \phi \circ T_j$$

We can therefore identify the category $\prod_H \mathcal{C}_i$ as the collection of all finite dimensional vector spaces V over $k = \prod_{\mathcal{U}} k_i$, each endowed with a collection of linear transformations

$$X_0, \ldots, X_m, Y_0, \ldots, Y_m, Z_0, \ldots, Z_m$$

satisfying the relations given in theorem 14.1.1, with morphisms being those linear maps commuting with the $X's$, $Y's$, and $Z's$.

14.2 Labelling the Objects of $\mathrm{Rep}_k G^\infty$

Let $k = \prod_{\mathcal{U}} k_i$. What does the category $\mathrm{Rep}_k G^\infty$ look like? By theorem 11.0.3 representations of G^∞ on the k-vector space V are exactly the finite commuting products of representations of G on V. And as k has characteristic zero, according to theorem 13.3.1, an individual representation of G on V is determined by a triple X, Y, Z of nilpotent linear transformations on V satisfying $Z = [X, Y]$, $XZ = ZX$, $YZ = ZY$, and any such triple gives a representation. Thus, every object of $\mathrm{Rep}_k G^\infty$

is a finite dimensional vector space V with an attached collection

$$X_0, \ldots, X_m, Y_0, \ldots, Y_m, Z_0, \ldots, Z_m$$

of nilpotent linear transformations on V, such that the representation determined by X_i, Y_i, Z_i commutes with the representation determined by X_j, Y_j, Z_j for $i \neq j$. By theorem 10.1.2, commutativity of these representations is equivalent to requiring that X_i, Y_i, Z_i all commute with X_j, Y_j, Z_j for $i \neq j$. Thus

Theorem 14.2.1. *Each object of $Rep_k G^\infty$ is a finite dimensional k-vector space V with an attached sequence $X_0, \ldots, X_m, Y_0, \ldots, Y_m, Z_0, \ldots, Z_m$ of linear transformations on V satisfying*

1. *The X_j, Y_j, and Z_j are all nilpotent*

2. *$Z_j = [X_j, Y_j]$ for every j*

3. *$[X_j, Z_j] = [Y_j, Z_j] = 0$ for every j*

4. *For every $i \neq j$, the matrices X_i, Y_i, Z_i commute with the matrices X_j, Y_j, Z_j*

Further, any such sequence of linear transformations on a finite dimensional vector space over k gives an object of $Rep_k G^\infty$.

Let V, W be objects of $Rep_k G^\infty$, given by the nilpotent transformations $X_0, \ldots, X_m$, $Y_0, \ldots, Y_m$, $Z_0, \ldots, Z_m$ and $R_0, \ldots, R_m$, $S_0, \ldots, S_m$, $T_0, \ldots, T_m$ respectively. What is a morphism between these two objects? By proposition 11.1.3 it is a linear map $\phi : V \to W$ such that, for every j, ϕ is a morphism between the representations of G on V determined by X_j, Y_j, Z_j and R_j, S_j, T_j. And by theorem 10.1.1 this is equivalent to

Theorem 14.2.2. *Let V and W be objects of $Rep_k G^\infty$, given by the transformations X_j, Y_j and Z_j for V and R_j, S_j and T_j for W. Then a linear map $\phi : V \to W$ is a*

morphism between V and W if and only if, for every j, ϕ satisfies

$$X_j \circ \phi = \phi \circ R_j \qquad Y_j \circ \phi = \phi \circ S_j \qquad Z_j \circ \phi = \phi \circ T_j$$

We can therefore identify the category $\text{Rep}_k G^\infty$ as the collection of all finite dimensional vector spaces V over k, each endowed with a collection of linear transformations

$$X_0, \ldots, X_m, Y_0, \ldots, Y_m, Z_0, \ldots, Z_m$$

satisfying the relations given in theorem 14.2.1, with morphisms being those linear maps commuting with the $X's$, $Y's$, and $Z's$.

14.3   The Equivalence $\prod_H \mathcal{C}_i \to \text{Rep}_{\prod_{\mathcal{U}} k_i} G^\infty$

With the characterization for objects and morphisms in the categories $\prod_H \text{Rep}_{k_i} G$ and $\text{Rep}_{\prod_{\mathcal{U}} k_i} G^\infty$ given in the previous two sections, the equivalence (as k-linear abelian categories) is obvious.

What is left to verify is the not quite obvious fact that this equivalence is tensor preserving. First, let us examine the tensor product on $\text{Rep}_k G^1$. Fix two objects V and W, given by the transformations X, Y, Z and R, S, T on V and W respectively. Then their tensor product has the matrix formula

$$\left(\sum_{n,k,k} (c_{ij})^{(n,m,k)} x^n y^m z^k \right) \otimes \left(\sum_{r,s,t} (d_{ij})^{(r,s,t)} x^r y^s z^t \right)$$

$$= \sum_{n,m,k)} \sum_{r,s,t} (c_{ij})^{(n,m,k)} \otimes (d_{ij})^{(r,s,t)} x^{n+r} y^{m+s} z^{k+t}$$

Recalling that $(c_{ij})^{(n,m,k)} = \frac{1}{n!m!k!} Z^k Y^m Z^n$ and similarly for (d_{ij}), we see that the coefficient matrix for x in the representation $V \otimes W$ is actually $X \otimes R$, that for y is

$Y \otimes S$, and for z is $Z \otimes T$.

Now consider two representations V and W for G^∞ over k, given by the transformations X_j, Y_j, Z_j and R_j, S_j, T_j respectively. Proposition 11.1.2 tells that the 'layers' for the tensor product of V and W is just the tensor product of the individual layers. That is

Proposition 14.3.1. *The tensor product of the representations V and W is given by the sequence of transformations $X_j \otimes R_j$, $Y_j \otimes S_j$, $Z_j \otimes T_j$.*

In positive characteristic the situation is slightly more delicate. Consider for example the natural representation of H_1

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}$$

where we consider it as a representation of G in characteristic 2, that is, as a height-1 representation given by the matrices

$$X_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, Z_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The tensor product of this representation with itself is a 9×9 representation in which x^2 will occur as an entry, causing it to have height 2. We see then that, in general, the taking of tensor products in positive characteristic often causes Frobenius layers to 'spill over' into one another, and we do not always have a situation analogous to proposition 14.3.1.

However, because objects of $\prod_H \mathrm{Rep}_{k_i} G$ are demanded to be of bounded dimension, for large enough i this difficulty will always vanish.

Proposition 14.3.2. *Let k have characteristic $p > 0$, and let V and W be G-modules over k, say of height m, given by the transformations X_j, Y_j, Z_j and R_j, S_j, T_j. Then if p is large compared to the dimensions of both V and W, the representation $V \otimes W$ will also be of height m, given by the transformations*

$$X_j \otimes R_j, Y_j \otimes S_j, Z_j \otimes T_j$$

for all $j = 0, \dots, m$.

Proof. We will first prove the theorem in the case where V and W contain a single, mutual non-zero Frobenius layer, say the rth layer, given by X, Y, Z and R, S, T; the case of an arbitrary number of layers is an easy corollary. We show that, if p is large compared to the dimensions of both, there is no possibility of that layer 'spilling over' into the next one. We can assume that p is large enough so that all of V, W, and $V \otimes W$ are of the form given by theorem 13.4.5, and in particular, that all the relevant matrices have nilpotent order $\leq p/2$. This means that we can write the representation V as

$$\sum_{n,m,k=0}^{p/2-1} \frac{1}{n!m!k!} Z^k Y^m X^n x^{np^r} y^{mp^r} z^{kp^r}$$

and W as

$$\sum_{a,b,c=0}^{p/2-1} \frac{1}{a!b!c!} T^c S^b R^a x^{ap^r} y^{bp^r} z^{cp^r}$$

and their tensor product as

$$= \sum_{n,m,k,a,b,c=0}^{p/2-1} \frac{1}{n!m!k!a!b!c!} (Z^k Y^m X^n \otimes T^c S^b R^a) x^{(n+a)p^r} y^{(m+b)p^r} z^{(k+c)p^r}$$

Note that this new representation still only has a single non-zero Frobenius layer, the rth one, since, e.g., $(n+a)p^r < p^{r+1}$ for every n and a in the summation. Notice also the coefficient matrix for the monomial x^{p^r} is exactly $X \otimes R$, that for y^{p^r} is exactly

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$Y \otimes S$, and that for z^{p^r} is $Z \otimes T$. Thus the theorem is true in the single layer case, and the case of an arbitrary number of layers easily follows. $\square$

So, given two objects $[V_i], [W_i]$ of $\prod_H \mathrm{Rep}_{k_i} G$, for large enough i the previous proposition applies, whence the tensor products on $\prod_H \mathrm{Rep}_{k_i} G$ and $\mathrm{Rep}_{\prod_{\mathcal{U}} k_i} G^\infty$, via our equivalence, are compatible, and this equivalence is indeed tensor preserving.

We leave it to the reader to convince himself that all of the arguments of this chapter can be slightly modified to prove

Theorem 14.3.3. *If k_i is a sequence of fields of strictly increasing positive characteristic, then for any $n \in \mathbb{N}$, the category $\prod_{H \leq n} Rep_{k_i} G$ is tensorially equivalent to $Rep_{\prod_{\mathcal{U}} k_i} G^n$.*

Chapter 15

Height-Restricted Generic Cohomology

One application of theorem 14.0.6 of the previous chapter is to give quick and intuitive large characteristic, 'height-restricted' generic cohomology results for the two unipotent groups we have studied, at least in the case of Ext^1.

15.1 First-Order Definability of Ext^1

Let M and N be objects in some tannakian category over the field k, fix n, and let $\xi_1, \ldots, \xi_m$ be a sequence of diagrams of the form

$$\xi_j : 0 \to N \to X_j \to M \to 0$$

To prevent ourselves from having to repeat the same long-winded sentence over and over again, we define the formula $LISE(\xi_1, \ldots, \xi_m, M, N)$ to mean " $\xi_1, \ldots, \xi_m$ is a linearly independent sequence of 1-fold extensions of M by N."

Theorem 15.1.1. *For fixed m, the formula $LISE(\xi_1, \ldots, \xi_m, M, N)$, modulo the theory of tannakian categories, is expressible as a first-order formula in the language of*

abelian tensor categories.

Proof. A full proof that LISE is a first-order formula would be an unnecessarily mind-numbing exercise; we shall instead be content to give an outline of such a proof, leaving it to the reader to fill in the necessary details.

Define the formula $\mathrm{Exten}(N, \iota, X, \pi, M)$ to mean that these objects and morphisms comprise a 1-fold extension of M by N. This amounts to demanding that N, X, and M are objects, that ι and π are morphisms, that the morphisms point between the objects we want them to, that ι is injective, that π is surjective, and that the sequence is exact at X. This can be translated into a first-order sentence.

Now the formula $\mathrm{LISE}(\xi_1, \ldots, \xi_n, M, N)$ doesn't make sense on its face, since we are treating the extensions ξ_i as if they were elements of our category, which they are not. If we were being strictly formal, we should instead use the objects and morphisms comprising the extensions as the variables, and make the additional assertions that they are all extensions of M by N. But since Exten is first-order, this can certainly be done, so we are justified in using this abbreviation.

The formula $\mathrm{LISE}(\xi_1, \ldots, \xi_m, M, N)$ should go something like, "$\xi_1, \ldots, \xi_m$ are extensions of M by N, and for any scalars $k_1, \ldots, k_n$, if $k_1\xi_1 \oplus \ldots \oplus k_n\xi_n$ is equivalent to the trivial extension, then $k_1 = \ldots = k_n = 0$." It is then simply a matter of showing that the concepts of being a scalar, of scalar multiplication of extensions, of Baer sum of extensions, of being a trivial extension, and of being the zero scalar are all first-order.

The first is obvious; to be a scalar simply means that it is an endomorphism of the identity object, which is clearly first-order. If $\phi : X \to Y$ is a morphism and k a scalar, then we define the scalar multiplication of k on ϕ to be the composition

$$X \xrightarrow{\mathrm{unit}_X} \underline{1} \otimes X \xrightarrow{k \otimes \phi} \underline{1} \otimes Y \xrightarrow{\mathrm{unit}_Y^{-1}} Y$$

which again is first-order. This allows us, under the definition given in section 2.4, to define the scalar multiplication of an extension in a first-order way. (We would of course have to make separate definitions for the case when $k = 0$ or $k \neq 0$; but this is no problem, since "k is the zero scalar" and "ξ is a trivial extension" are both first-order, as is shown below.)

As for the Baer sum, we ask the reader to see the definition of it given in section 2.4. It involves such concepts as "being a pullback", "being the unique map pushing through a pullback", "being a cokernel", "being the unique map pushing through a cokernel", etc. All of these concepts are expressed in terms of universal properties, which are quite amenable to being expressed in a first-order fashion. They simply state that, given a collection of morphisms making such and such a diagram commute, there is a unique morphism making such and such a diagram commute. These types of statements are plainly first-order.

The statement that two extensions are equivalent is first-order; it is merely the assertion that there exists a morphism making an equivalence diagram between the two extensions commute (this is the only point in the proof at which it is necessary to restrict to Ext^1 as opposed to higher Ext; see section 15.3 for more on this). Then to say that χ is equivalent to the trivial extension would go something like "if ξ is any extension, then $\xi \oplus \chi$ is equivalent to ξ".

Finally, to say that the scalar k is the zero scalar is simply to say that it is the additive identity of the field $\mathrm{End}(\underline{1})$, which is clearly first-order.

$\square$

Corollary 15.1.2. *For fixed n, the formula "$\dim \mathrm{Ext}^1(M, N) = n$" is first-order.*

Proof. It is equivalent to "there exist n linear independent 1-fold extensions of M by N, and there do not exist $n + 1$ of them". $\square$

15.2   Generic Cohomology for Ext[1]

In this section G denotes any unipotent group, defined over $\mathbb{Z}$, for which the conclusion of theorem 14.0.6 is true (and in particular, has Hopf algebra isomorphic to $A = \mathbb{Z}[x_1, \ldots, x_n]$ for some n), k_i is a sequence of fields of strictly increasing positive characteristic, $\mathcal{C}_i = \operatorname{Rep}_{k_i} G$, and k is the ultraproduct of the fields k_i.

Definition 15.2.1. Let k be a field of characteristic $p > 0$, M and N modules for G over k, and let $n, h \in \mathbb{N}$. Then we define

$$\operatorname{Ext}_{G(k)}^{n,h}(M, N)$$

to be the subset of $\operatorname{Ext}_{G(k)}^{n}(M, N)$ consisting of those (equivalence classes of) n-fold extensions of M by N such that, up to equivalence, each of the extension modules can be taken to have height less than or equal to h.

Example: the (equivalence class of) the extension

$$0 \to k \to \begin{pmatrix} 1 & x^{p^2} \\ 0 & 1 \end{pmatrix} \to k \to 0$$

is a member of $\operatorname{Ext}_{G_a(k)}^{1,3}(k, k)$, but not of $\operatorname{Ext}_{G_a(k)}^{1,2}(k, k)$.

Lemma 15.2.1. *If M is a G-module of height no greater than h, then any submodule or quotient of M also has height no greater than h. If M and N have height no greater than h, so does $M \oplus N$.*

Proof. The case of subobjects and quotients follows immediately from lemma 8.0.10: any subobject or quotient of M will have matrix formula with entries who are linear combinations of the entries of M. The case of $M \oplus N$ is even easier to see, examining the usual matrix representation for a direct sum. $\square$

Theorem 15.2.2. *Let M and N be modules for G over a field k of characteristic $p > 0$, of height no greater than h. Then $\mathrm{Ext}^{1,h}_{G(k)}(M, N)$ is a subspace of $\mathrm{Ext}^1_{G(k)}(M, N)$.*

Proof. Let ξ, χ be extensions in $\mathrm{Ext}^{1,h}_{G(k)}(M, N)$, with the extension modules of both ξ and χ having height no greater than h. We examine the definitions given for the Baer sum and scalar multiplications in section 2.4. Clearly a non-zero scalar multiple of either of them is still in $\mathrm{Ext}^{1,h}_{G(k)}(M, N)$. As for the Baer sum $\xi \oplus \chi$, we recall the concrete constructions of a pullback or pushout of G-modules. The former is defined as a certain subobject of the direct sum of two modules, and the other a certain quotient of their direct sum. By the previous lemma both of these constructions yield modules of height no greater than those of the originals. Thus $\mathrm{Ext}^{1,h}_{G(k)}(M, N)$ is closed under the Baer sum. The trivial extension has extension module isomorphic to the direct sum of M and N, again by the previous lemma, of height no greater than that of M or N. $\qquad\square$

Let M and N be modules for G over $\mathbb{Z}$. Then it of course makes sense to consider them as modules for G over any field. Further, for any $n \in \mathbb{N}$, and indeed for $n = \infty$, we can consider them as modules for G^n over any field, that is, as representations with a single layer, according to theorem 11.0.3.

Our goal for the rest of this section is to prove

Theorem 15.2.3. *Let $h \in \mathbb{N}$, M, N modules for G over $\mathbb{Z}$. Suppose that the computation $\dim \mathrm{Ext}^1_{G^h(k)}(M, N) = m$ is the same for any characteristic zero field k. Let k_i be a sequence of fields of strictly increasing characteristic.*

  1. If m is finite, then for sufficiently large i

$$\dim \mathrm{Ext}^{1,h}_{G(k_i)}(M, N) = m$$

  2. If $m = \infty$, then for any sequence of fields k_i of strictly increasing characteristic,

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$\dim \operatorname{Ext}_{G(k_i)}^{1,h}(M, N)$ *diverges to infinity with increasing* i.

For each i, suppose we have a diagram in $\mathcal{C}_i$

$$\xi^i : 0 \to N_i \xrightarrow{\iota^i} X_i \xrightarrow{\pi^i} M_i \to 0$$

Denote by $[\xi^i]$ corresponding diagram in $\prod_{\mathcal{U}} \mathcal{C}_i$:

$$[\xi^i] : 0 \to [N_i] \xrightarrow{[\iota^i]} [X_i] \xrightarrow{[\pi^i]} [M_i] \to 0$$

Proposition 15.2.4. *For each* i, *let* $\xi_i^1, \ldots, \xi_i^m$ *be the sequence of diagrams in* $\mathcal{C}_i$

$$\xi_i^1 : 0 \to N_i \to X_i^1 \to M_i \to 0$$

$$\vdots$$

$$\xi_i^m : 0 \to N_i \to X_i^m \to M_i \to 0$$

Then the formula $LISE(\xi_i^1, \ldots, \xi_i^m, M_i, N_i)$ *holds in almost every* $\mathcal{C}_i$ *if and only if the formula* $LISE([\xi_i^1], \ldots, [\xi_i^m], [M_i], [N_i])$ *holds in* $\prod_{\mathcal{U}} \mathcal{C}_i$.

Proof. Apply theorems 15.1.1 and C.0.15. $\qquad\square$

Now fix two modules M and N for G over $\mathbb{Z}$, and for each i, let $\xi_i^1, \ldots, \xi_i^m$ be the sequence of diagrams in $\mathcal{C}_i$

$$\xi_i^1 : 0 \to N \to X_i^1 \to M \to 0$$

$$\vdots$$

$$\xi_i^m : 0 \to N \to X_i^m \to M \to 0$$

Further, assume that each ξ_i^j is a member of $\operatorname{Ext}_{G(k_i)}^{1,h}(M, N)$; this is merely the assertion that every X_i^j has height no greater than h. As M and N are constant over i and

have height 1, each of $[M], [N]$, and $[X_i^j]$ have bounded height, whence $[\xi_i]_j \overset{\text{def}}{=} [\xi_i^j]$ is an extension in $\prod_{H \leq h} \mathcal{C}_i$.

Proposition 15.2.5. *For fixed $n, h \in \mathbb{N}$ and modules M and N for G over $\mathbb{Z}$, the statement "dim $\text{Ext}_{G(k_i)}^{1,h}(M, N) \geq n$" holds for almost every i if and only if dim $\text{Ext}_{G^h(\prod_{\mathcal{U}} k_i)}^1(M, N) \geq n$.*

Proof. Suppose dim $\text{Ext}_{G(k_i)}^{1,h}(M, N) \geq n$ holds for almost every i. This means that, for almost every i, we have a linearly independent sequence of 1-fold extensions of M by N

$$\xi_i^1 : 0 \to N \to X_i^1 \to M \to 0$$

$$\vdots$$

$$\xi_i^m : 0 \to N \to X_i^m \to M \to 0$$

with each X_i^j being of height $\leq h$. Note that the objects $[X_i]^1, \ldots, [X_i]^m$ are of bounded height and dimension; then these project to the sequence of diagrams $[\xi_i]^1, \ldots, [\xi_i]^m$ in $\prod_{H \leq h} \mathcal{C}_i$. As the formula LISE is first-order, these extensions, considered as diagrams in the full ultracategory $\prod_{\mathcal{U}} \mathcal{C}_i$, are also linearly independent, and by proposition 5.0.4, so also are they in the undercategory $\prod_{H \leq h} \mathcal{C}_i$. Note also that, under the equivalence $\prod_{H \leq h} \mathcal{C}_i \simeq \text{Rep}_{\prod_{\mathcal{U}} k_i} G^h$ given in section 14.3 the objects $[M]$ and $[N]$ in $\prod_{H \leq h} \mathcal{C}_i$ actually correspond to the objects M and N in $\text{Rep}_{\prod_{\mathcal{U}} k_i} G^h$. This gives a collection of m linearly independent extensions of N by M in the category $\text{Rep}_{\prod_{\mathcal{U}} k_i} G^h$; thus, dim $\text{Ext}_{G^h(\prod_{\mathcal{U}} k_i)}^1(M, N) \geq n$.

The converse is proved similarly; if dim $\text{Ext}_{G^h(\prod_{\mathcal{U}} k_i)}^1(M, N) \geq n$, take a linearly independent sequence of extensions $[\xi_i]^1, \ldots, [\xi_i]^n$ of N by M in $\prod_{H \leq h} \mathcal{C}_i$, which project back to, for almost i, a linearly independent sequence of extensions $\xi_i^1, \ldots, \xi_i^n$ of M by N in $\mathcal{C}_i$, showing dim $\text{Ext}_{G(k_i)}^{1,h}(M, N) \geq n$ for almost every i. $\square$

Corollary 15.2.6. *The statement* $\dim \text{Ext}^1_{G^h(\prod_{\mathcal{U}} k_i)}(M, N) = n$ *holds if and only if* $\dim \text{Ext}^{1,h}_{G(k_i)}(M, N) = n$ *holds for almost every* i.

Proof. The above statement is equivalent to the conjunction "$\dim \text{Ext}^1_{G^h(\prod_{\mathcal{U}} k_i)}(M, N) \geq n$" and NOT "$\dim \text{Ext}^1_{G^h(\prod_{\mathcal{U}} k_i)}(M, N) \geq n + 1$". Apply the previous proposition. $\square$

We can now prove theorem 15.2.3. Suppose that the computation $\dim \text{Ext}^1_{G^h(k)}(M, N) = n$ is both finite and the same for *any* characteristic zero field k. Then in particular, it is the same for the field $\prod_{\mathcal{U}} k_i$ for *any* choice of non-principal ultrafilter. Let $J \subset I$ be the set on which $\dim \text{Ext}^{1,h}_{G(k_i)}(M, N) = n$ is true. By the previous corollary, J is large for every choice of non-principal ultrafilter, and by corollary B.0.8, J is cofinite. This proves the first part of the theorem.

If instead $\dim \text{Ext}^1_{G^h(k)}(M, N)$ is infinite, then for any $n \in \mathbb{N}$, the statement $\dim \text{Ext}^{1,h}_{G(k_i)} = n$ is false for almost every i, for every choice of non-principal ultrafilter, whence the statement $\dim \text{Ext}^{1,h}_{G(k_i)} = n$ is false on a cofinite set. This goes for every $n \in \mathbb{N}$, whence $\dim \text{Ext}^{1,h}_{G(k_i)}$ is divergent.

15.2.1 An Example

We shall illustrate an application of theorem 15.2.3 with a simple, easily verifiable example.

Let $G = G_a$ and consider $\text{Ext}^1_{G_a(k)}(k, k)$, where k has characteristic $p > 0$. As the extension module of any extension of k by k has dimension 2, and as $p \geq 2$ for all primes, theorem 12.3.1 applies, whence any 2-dimensional representation of G_a is given by a finite sequence $X_0, \ldots, X_m$ of commuting nilpotent matrices over k. We can take X_0 to be in Jordan form times some scalar, namely

$$\begin{pmatrix} 0 & c_0 \\ 0 & 0 \end{pmatrix}$$

for some scalar c_0. The centralizers of this matrix are exactly those of the form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

and if we demand them to be nilpotent, we must have $a = 0$. Thus the X_i can be taken to be

$$X_0 = \begin{pmatrix} 0 & c_0 \\ 0 & 0 \end{pmatrix}, \ldots, X_m = \begin{pmatrix} 0 & c_m \\ 0 & 0 \end{pmatrix}$$

for some scalars $c_0, \ldots, c_m$. The representation they generate according to theorem 12.3.1 is

$$\begin{pmatrix} 1 & c_0 x + c_1 x^p + \ldots + c_m x^{p^m} \\ 0 & 1 \end{pmatrix}$$

These are all extensions of k by k with the obvious injection $1 \mapsto (1, 0)$ and projection $(1, 0) \mapsto 0, (0, 1) \mapsto 1$, and any extension of k by k must be of this form. Therefore extensions of the form

$$0 \to k \to \begin{pmatrix} 1 & c_0 x + c_1 x^p + \ldots + c_m x^{p^m} \\ 0 & 1 \end{pmatrix} \to k \to 0$$

constitute all extensions of k by k. Denote by ξ_m the extension

$$\xi_m : 0 \to k \to \begin{pmatrix} 1 & x^{p^m} \\ 0 & 1 \end{pmatrix} \to k \to 0$$

Then direct computation shows that the Baer sum of ξ_m and ξ_n is the extension

$$\xi_m \oplus \xi_n : 0 \to k \to \begin{pmatrix} 1 & x^{p^m} + x^{p^n} \\ 0 & 1 \end{pmatrix} \to k \to 0$$

and that, for $c \neq 0$, the scalar multiplication $c\xi_m$ is (equivalent to)

$$c\xi_m : 0 \to k \to \begin{pmatrix} 1 & cx^{p^m} \\ 0 & 1 \end{pmatrix} \to k \to 0$$

A basis for $\mathrm{Ext}^1_{G_a(k)}(k,k)$ is therefore given by $\xi_1, \xi_1, \ldots$. If instead we restrict to $\mathrm{Ext}^{1,h}_{G_a(k)}(k,k)$, then this is a finite dimensional subspace spanned by $\xi_0, \ldots, \xi_{h-1}$.

Now consider $\mathrm{Ext}^1_{G_a^h(k)}(k,k)$, where k now has characteristic zero. Using theorems 12.2.1 and 11.0.3 virtually identical computations to the above show it to be spanned by the linearly independent extensions $\chi_0, \ldots, \chi_{h-1}$, given by

$$\chi_m : 0 \to k \to \begin{pmatrix} 1 & x_m \\ 0 & 1 \end{pmatrix} \to k \to 0$$

where x_m denotes the mth free variable of the Hopf algebra $k[x_0, \ldots, x_{h-1}]$, and that the Baer sum and scalar multiplication of extensions give analogous results to that of the above. We see then that

$$\dim \mathrm{Ext}^{1,h}_{G_a(k)}(k,k) = \dim \mathrm{Ext}^1_{G_a^h(k')}(k',k')$$

when k has characteristic p and k' has characteristic zero. In particular we conclude that, if k_i is a sequence of fields of increasing positive characteristic, then

$$\dim \mathrm{Ext}^{1,h}_{G_a(k_i)}(k_i,k_i) \ \longrightarrow \ \dim \mathrm{Ext}^1_{G_a^h(\Pi_{\mathcal{U}} k_i)}(\prod_{\mathcal{U}} k_i, \prod_{\mathcal{U}} k_i)$$

which is predicted by theorem 15.2.3.

The reader should note that this example is misleading, in that the generic value of $\mathrm{Ext}^{1,h}_{G_a(k)}(k,k)$ is attained for *any* positive characteristic $p \geq 2$. This was simply due to the fact that theorem 12.3.6 applies to all characteristics in dimension 2,

i.e. because any 2×2 nilpotent matrix is nilpotent of order ≤ 2. If instead we were to consider $\mathrm{Ext}^{1,h}_{G_a(k_i)}(k_i \oplus k_i, k_i \oplus k_i)$, then (assuming the computation on the right does not depend on the particular characteristic zero field) we would still have

$$\dim \mathrm{Ext}^{1,h}_{G_a(k_i)}(k_i \oplus k_i, k_i \oplus k_i) \longrightarrow \dim \mathrm{Ext}^1_{G_a^h(\Pi_{\mathcal{U}} k_i)}(\prod_{\mathcal{U}} k_i \oplus \prod_{\mathcal{U}} k_i, \prod_{\mathcal{U}} k_i \oplus \prod_{\mathcal{U}} k_i)$$

only this time we would have to wait for $\mathrm{char}(k_i) = 5$ for the generic value to be obtained.

15.3   The Difficulty with Higher Ext

To finish, we mention a few of the reasons why our attempts to apply this machinery to Ext^n for $n > 1$ have so far proved unfruitful.

In the previous section we saw that there is a $1 - 1$ correspondence between extensions in $\mathrm{Ext}^1_{G^h(\Pi_{\mathcal{U}} k_i)}(M, N)$ and almost everywhere extensions in $\mathrm{Ext}^{1,h}_{G(k_i)}(M, N)$. But for higher Ext, this does not always work. Here is what can go wrong. For concreteness' sake consider $\mathrm{Ext}^{2,h}_{G(k_i)}(M, N)$, and suppose that, for each i, we have an element $\xi_i \in \mathrm{Ext}^{2,h}_{G(k_i)}(M, N)$. This means that each ξ_i is of the form

$$\xi_i : 0 \to M \to X_i \to Y_i \to N \to 0$$

with every X_i and Y_i being, up to equivalence of extensions, of height $\leq h$. But this says nothing about the *dimensions* of X_i and Y_i, and indeed there is every reason to suspect that $\dim(X_i)$ and $\dim(Y_i)$ diverge as i becomes large. As the objects of $\prod_{H \leq h} \mathrm{Rep}_{k_i} G$ are demanded to have bounded dimension as well as height, the objects $[X_i]$ and $[Y_i]$ will not belong to $\prod_{H \leq h} \mathrm{Rep}_{k_i} G$, and hence the extension $[\xi_i] \in \prod_{\mathcal{U}} \mathrm{Rep}_{k_i} G$ will not belong to $\prod_{H \leq h} \mathrm{Rep}_{k_i} G$.

Another problem we face in the case of higher Ext is in trying to define equivalence

of extensions in a first-order way. For Ext^1 this was no problem, since if two 1-fold extensions are equivalent, there is necessarily an actual equivalence map between them, which is easily asserted in a first-order way. But this is not so for n-fold extensions in general; equivalent extensions need not have an actual equivalence mapping between them (see section 2.4).

To illustrate the problem, suppose we have, for each i, two equivalent extensions ξ_i and χ_i in the category $\mathcal{C}_i$. What this says is that, for each i, there exists a finite sequence of m_i extensions $\rho_i^1, \rho_i^2, \ldots, \rho_i^{m_i}$ forming a chain of concrete equivalencies leading from ξ_i to χ_i. But there is every reason to suspect that m_i diverges to infinity as i becomes large. As such, these equivalencies between ξ_i and χ_i in the categories $\mathcal{C}_i$ do not necessarily project to an equivalence between the extensions $[\xi_i]$ and $[\chi_i]$ in the category $\prod_u \mathcal{C}_i$. If such a collection were found (and we have none in mind), this would in fact prove that the property of being equivalent is not first-order.

With these difficulties in mind, we tried instead to prove the following inequality:

Theorem 15.3.1. *Let $n, h \in \mathbb{N}$, and let M and N be modules for G over $\mathbb{Z}$. Suppose that the computation $\dim \text{Ext}^n_{G^h(k)}(M, N) = m$ (where m could possibly be infinite) is the same for every characteristic zero field k. Then for any sequence of fields k_i of increasing positive characteristic*

$$\dim \text{Ext}^{n,h}_{G(k_i)}(M, N) \geq m$$

for all sufficiently large i.

But the obvious attempt at a proof of this falls apart as well. Suppose we had a sequence $[\xi_i]^1, \ldots, [\xi_i]^n$ of linearly independent extensions in the category $\prod_{H \leq h} \text{Rep}_{k_i} G$; then we would like to see that these project back to an almost everywhere sequence of linearly independent extensions $\xi_i^1, \ldots, \xi_i^n$ in the categories $\mathcal{C}_i$. But even this, as far as we can tell, is not guaranteed. Linear independence means that, whenever

$a_i^1 \xi_i^1 \oplus \ldots \oplus a_i^n \xi_i^n$ is a trivial extension, then $a_i^1 = \ldots = a_i^n = 0$. Being a trivial extension in turn means that, whenever χ_i is any extension, $\chi_i \oplus (a_i^1 \xi_i^1 \oplus \ldots \oplus a_i^n \xi_i^n)$ is equivalent to χ_i. But again, equivalence of extensions is not necessarily first-order, and so neither is the property of being a trivial extension. We see then that a linear dependence among the $\xi_i^1, \ldots, \xi_i^n$ does not necessarily project to a linear dependence among the $[\xi_i]^1, \ldots, [\xi_i]^n$, and we cannot automatically conclude that the $a_i^1, \ldots, a_i^n$ are equal to zero for almost every i.

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Appendix A

Model Theory and First-Order Languages

Here is a very basic and sometimes imprecise introduction to the notions of models and first-order languages; it will be just enough to get by. The reader is encouraged to consult [11] for an excellent introduction to the subject.

We abandon the term 'model' for the moment and instead focus on the notion of **relational structure**. This is by definition a set X (called the domain of the structure) endowed with the following: a collection $\{f_i\}$ of n_i-ary functions on X (functions from X^{n_i} to X, where $n_i \in \mathbb{N}$), a collection $\{r_j\}$ of n_j-ary relations on X (subsets of X^{n_j}) , and a collection of 'constants' $\{c_k\}$, certain distinguished elements of the domain. The various labels given to these functions, relations and constants is called the **signature** of the structure. Many (but not all) of the usual mathematical structures one comes across can be realized as relational structures. We do not at all demand that a signature be finite, but all of the examples given in this dissertation will have finite signatures.

Example: a field k can be realized as a relational structure. A natural choice for signature might be the two binary functions $+$ and $*$, the unary function $-$, and the

two constants 0 and 1, representing the obvious. We abusively call this the 'signature of fields', realizing that a random structure in this signature is not at all guaranteed to be a field.

The whole point of bothering with which symbols you choose to attach to a relational structure is three-fold. Firstly, it determines the definition of a 'homomorphism' of relational structures (always assumed to be between structures in the same signature); namely, a homomorphism is demanded to preserve relations, functions, and send constants to constants. Secondly, it determines the notion of a 'substructure' A of a structure B, which by definition must contain all constants, be closed under all functions, and such that the relations on A are compatible with those on B. Note for instance that we included the symbol $-$ in the language of fields, whence any substructure of a field must be closed under negation. If we were to omit this symbol, this would no longer be the case; e.g. $\mathbb{N}$ would now qualify as a substructure of $\mathbb{Q}$.

Thirdly, and most importantly for us, the signature of a structure determines the structure's **first-order language**. Roughly speaking, the first-order language of a structure is the collection of all meaningful 'formulae' one can form, in certain prescribed ways, using the symbols of the signature as the primitive elements of the language.

Any language, at the least, needs certain primitive verbs and nouns. In the context of first-order languages verbs are called **predicates** and nouns are called **terms**. For a given signature we define the terms of our language as follows:

1. Any variable is a term (a variable is any convenient symbol you might choose not being used by the language already, e.g. x, y, a, b, etc.)

2. Any constant symbol is a term

3. If f is an n-ary function symbol in the language and $t_1, \ldots, t_n$ are terms, then so is $f(t_1, \ldots, t_n)$.

In the case of fields, 1 is a term, so is x, so is $0 * x$, and so is $(x + y) * (1 + (-z))$. These represent the 'nouns' of our language.

Next we need predicates, ways to say stuff about our nouns. This is the role fulfilled by the *relational* symbols of our language, as well as the binary relational symbol '$=$', representing equality, which we always reserve for ourselves. We define the **atomic formulae**, which one can think of as the most basic sentences belonging to our language, as follows:

1. If s and t are terms, then $s = t$ is an atomic formula.

2. If r is an n-ary relational symbol, and $t_1, \ldots, t_n$ are terms, then $r(t_1, \ldots, t_n)$ is an atomic formula.

In the case of fields, $1 = 0$ is an atomic formula, and so is $x + y = 1 * z$. The signature we chose for fields did not include any relational symbols other than '$=$', so all atomic formulae in this signature must be built from this.

We are of course not content to restrict ourselves to these primitive formulae; we want to able to put them together using the usual logical symbols. Our primitive logical symbols are $\wedge$, $\vee$, and $\neg$, representing 'and', 'or', and 'not'. Thus, the following are all formulae in the first-order language of fields: $\neg(1 = 0) \wedge x = y$, $\neg(1 + x = x)$, and $\neg(1 = 0) \wedge \neg(1 + 1 = 0) \wedge \neg(1 + 1 + 1 = 0) \wedge \neg(1 + 1 + 1 + 1 = 0)$. In higher order languages, there is indeed a notion of conjunction or disjunction of an infinite collection of formulae, but the definition of a first-order language explicitly disallows this. All logical combinations of formulae take place over finite collections of formulae.

We finally have two more symbols, namely $\forall$ and $\exists$, representing universal and existential quantification. For any formulae Φ in our language, and any variable x, we also have the formula $\forall x \Phi$ and $\exists x \Phi$. So, for example, in the language of fields, the following are formulae: $\forall x (0 * x = 0)$, $(\forall x)(\forall y)(x * y = y * x)$, and $\neg(\exists x)(x * 0 = 1)$.

It is important to remember that quantification is always understood to be over the *elements* of a structure; in particular, we have no concept in first-order logic of quantification over *subsets* of a structure.

To make things manageable, we shall not hesitate to use abbreviations. For two formulae Φ and Ψ, $\Phi \implies \Psi$ is shorthand for $\Psi \vee \neg\Phi$, and $\Phi \iff \Psi$ is shorthand for $(\Phi \implies \Psi) \wedge (\Psi \implies \Phi)$. If x is a variable and $\Phi(x)$ is a formula in which the free variable (unbound by quantification) x occurs, then $(\exists!x)\Phi(x)$ is shorthand for $(\exists x)(\Phi(x) \wedge (\forall y)(\Phi(y) \implies x = y))$. We shall be making several such abbreviations as we go along, and usually leave it the reader to convince himself that the intended meaning can be achieved using only the primitive symbols of our language.

We say that a first-order formula is a **sentence** if it has no free variables. A (perhaps infinite) collection of sentences in a given first-order language is called a **theory**. If M is a relational structure and $\Phi = \{\phi_i : i \in I\}$ is a theory, we say that M is a **model** of Φ if every sentence of Φ is true in the structure M. Obviously not all collections of sentences have models; $\{1 = 0, \neg(1 = 0)\}$ obviously has no model, whatever you interpret 0 and 1 to be.

Theorem A.0.2. *(Compactness theorem for first-order logic) Let Φ be a collection of first-order sentences such that every finite subset of Φ has a model. Then Φ has a model.*

Proof. See theorem 5.1.1 of [11]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Some oft used corollaries:

Proposition A.0.3. *The following are all corollaries of the compactness theorem.*

1. *If the first-order sentence ϕ is equivalent to the infinite conjunction of the first-order sentences $\{\psi_i : i \in I\}$, then ϕ is equivalent to some finite conjunction of them.*

2. *If the first-order sentence ϕ is implied by the infinite conjunction of the first-order sentences $\{\psi_i : i \in I\}$, then ϕ is implied by some finite conjunction of them.*

As an easy example of an application of compactness, let $+, *, -, 0, 1$ be the language of fields. See proposition 6.1.1 for the fairly obvious observation that "is a field" is expressible by a first-order sentence in this language.

Proposition A.0.4. *Let L be the language of fields.*

1. *The statement "has characteristic zero", modulo the theory of fields, is not expressible by a first-order sentence of L.*

2. *If ϕ is a first-order L-sentence which is true of every characteristic zero field, then ϕ is true for all fields of sufficiently large positive characteristic.*

Proof. For a fixed prime number p, define char$_p$ to be the first-order sentence $1 + 1 + \ldots + 1 = 0$ (p-occurrences of 1). Modulo the theory of fields, this is obviously equivalent to the assertion that the field is of characteristic p. Now the statement "is of characteristic zero" is by definition equivalent to the infinite conjunction of the sentences $\neg$char$_p$ for $p = 2, 3, 5, \ldots$. By 1. of proposition A.0.3, if this were expressible as a first-order sentence, it would be equivalent to some finite subset of this collection. But we know this is absurd; no finite collection of the sentences $\neg$char$_p$ can guarantee a field to be of characteristic zero. We conclude that "is of characteristic zero" is not first-order.

Now suppose that the first-order sentence ϕ were true in every characteristic zero field. This means that the infinite conjunction of the sentences $\neg$char$_p$ implies ϕ. By 2. of proposition A.0.3, ϕ is implied by some finite subset of them. Any field of large enough positive characteristic satisfies this finite collection of sentences, and hence satisfies ϕ as well. $\square$

Appendix B

Ultrafilters

The notion of a filter is sometimes given as a slightly more general definition then we give, but it suffices for our purposes.

Definition B.0.1. Let I be a set. A **filter** on I is a non-empty collection $\mathcal{F}$ of subsets of I satisfying

1. $\mathcal{F}$ is closed under the taking of pairwise intersections

2. If Y is a superset of some element of $\mathcal{F}$, then Y is in $\mathcal{F}$

3. The empty set is not in $\mathcal{F}$

A filter is called an **ultrafilter** if it is maximal with respect to inclusion among all filters. An ultrafilter is called **principal** if it is of the form $\{X \subset I : x \in X\}$ for some element x of I.

We sometimes call the elements of a filter **large** sets. If $\phi(i)$ is some statement about elements of I we say that ϕ holds **almost everywhere** or **for almost every** i if the set on which $\phi(i)$ is true is large.

Proposition B.0.5. *A filter $\mathcal{F}$ on I is an ultrafilter if and only if for any subset X of I, either X or its complement is in $\mathcal{F}$.*

Proof. Suppose $\mathcal{F}$ does not contain X or its complement; we claim that $\mathcal{F}$ can be enlarged to a new filter containing one or the other. First suppose that $R, S \in \mathcal{F}$ are such that $X \cap R = \emptyset$ and $X^C \cap S = \emptyset$. Then $R \subset X^C$ and $S \subset X$, whence $R \cap S = \emptyset$; but this cannot happen since $\mathcal{F}$ is closed under intersections and does not contain $\emptyset$. Thus at least one of X or its complement is not disjoint from anything in $\mathcal{F}$, let's say X. Then define $\mathcal{F}' = \mathcal{F} \cup \{S \subset I : S \supset X\} \cup \{S \cap R : R \in \mathcal{F}, S \supset X\}$, which is easily seen to be a new filter properly containing $\mathcal{F}$.

Conversely, if $\mathcal{F}$ contains every set or its complement, then it is necessarily maximal, since there are no new sets we can throw in; any such X would intersect with its complement to arrive at $\emptyset \in \mathcal{F}$. $\qquad\square$

Examples: The collection of all subsets having Lebesgue measure 1 is a filter on the interval $[0, 1]$, and the collection of all cofinite subsets is a filter on $\mathbb{N}$. These are both obviously non-principal and non-ultra.

Principal ultrafilters are boring and useless; we need non-principal ultrafilters.

Proposition B.0.6. *An ultrafilter $\mathcal{U}$ is non-principal if and only if it contains no finite sets if and only if it contains no singleton sets.*

Proof. If $\mathcal{U}$ is principal, say generated by $x \in I$, then obviously $\mathcal{U}$ contains the singleton set $\{x\}$. Conversely, suppose $\mathcal{U}$ contains the finite set $X = \{x_1, \ldots, x_n\}$, $n > 1$. Then at least one of the sets $\{x_1, \ldots, x_{n-1}\}$ or its complement is in $\mathcal{U}$. In the latter case we intersect with X to obtain $\{x_n\} \in \mathcal{U}$, and in either case we have a new subset with less than n elements. Applying this process finitely many times will eventually yield some singleton $\{x\}$ in $\mathcal{U}$. Then any subset containing x is in $\mathcal{U}$, no subset not containing x can be in $\mathcal{U}$, and thus $\mathcal{U}$ is principal. $\qquad\square$

Proposition B.0.7. *Let I be an infinite set, and $X \subset I$ any infinite subset of I. Then there exists a non-principal ultrafilter on I containing X.*

Proof. Let $\mathcal{C}$ be the filter on I consisting of all cofinite sets, and enlarge it, as in the proof of proposition B.0.5, to contain X. Partially order the collection of all filters containing X by inclusion, which we just showed is non-empty. The union over any chain of filters qualifies as an upper bound for that chain; take a maximal element by Zorn's Lemma. It is guaranteed to be non-principal since it contains no finite sets, by proposition B.0.6. $\qquad\square$

Corollary B.0.8. *The subsets of I that are contained in every non-principal ultrafilter are exactly the cofinite subsets.*

Proof. If X is not cofinite, the previous proposition shows that its complement is contained in some non-principal ultrafilter, necessarily not containing X. If X is cofinite, then proposition B.0.6 shows that every non-principal ultrafilter does not contain X^C, and so contains X. $\qquad\square$

Lemma B.0.9. *Let $\mathcal{U}$ be an ultrafilter on I, J a member of $\mathcal{U}$, and $X_1, \ldots, X_n$ a finite collection of subsets of I which cover J. Then at least one of the X_i is in $\mathcal{U}$.*

Proof. Suppose none of them are in $\mathcal{U}$. Then all of their complements are in $\mathcal{U}$, as well as the intersection of their complements, which is contained in J^C; but this cannot be, since $J^C \notin \mathcal{U}$. $\qquad\square$

Lemma B.0.10. *If $\mathcal{U}$ is an ultrafilter on I, J a member of $\mathcal{U}$, and $X_1, \ldots, X_n$ a finite disjoint partition of J, then exactly one of the X_i is contained in $\mathcal{U}$.*

Proof. At least one of them is in $\mathcal{U}$ by the previous lemma, and no two of them can be, lest we take their intersection and arrive at $\emptyset \in \mathcal{U}$. $\qquad\square$

With a view towards defining ultraproducts in the next section, we close with

Proposition B.0.11. *Let X_i be a collection of sets indexed by I, $\mathcal{U}$ an ultrafilter on I. Define a relation on $\prod_{i \in I} X_i$ (cartesian product of the X_i) as follows. For tuples*

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$(x_i), (y_i) \in \prod_{i \in I} X_i$, $(x_i) \sim (y_i)$ *if and only if the set* $\{i \in I : x_i = y_i\}$ *is in* $\mathcal{U}$. *Then* $\sim$ *is an equivalence relation.*

Proof. Reflexivity is clear since necessarily $I \in \mathcal{F}$, and symmetry is obvious. For transitivity, suppose $(x_i) \sim (y_i)$ and $(y_i) \sim (z_i)$. Then the set

$$\{i \in I : x_i = z_i\}$$

contains at least the set

$$\{i \in I : x_i = y_i\} \cap \{i \in I : y_i = z_i\}$$

which is in $\mathcal{U}$ by intersection closure. Then so is $\{i \in I : x_i = z_i\}$, by superset closure. $\square$

We say that two such tuples are equal **almost everywhere** or **on a large set** if they are related through this relation, and we denote by $[x_i]$ the equivalence class of the tuple (x_i).

Appendix C

Ultraproducts

Let M_i be a collection of relational structures in a common signature L, indexed by the set I, and fix a non-principal ultrafilter $\mathcal{U}$ on I. Then we define the **ultraproduct** of these structures relative to $\mathcal{U}$, denoted $M = \prod_{\mathcal{U}} M_i$, to be a new L-structure defined as follows.

The domain of M is the collection of all equivalence classes $[x_i]$ of tuples $(x_i) \in \prod_{i \in I} M_i$ (cartesian product of the M_i) as defined in proposition B.0.11. For an n-ary relation symbol r, we define $r([x_i]_1, \ldots, [x_i]_n)$ to hold if and only if, for almost every i, $r(x_{i,1}, \ldots, x_{i,n})$ holds in the structure M_i. For an n-ary function symbol f, $f([x_i]_1, \ldots, [x_i]_n)$ is the element $[f(x_{i,1}, \ldots, x_{i,n})]$ of M, and the constant c corresponds to the element $[x_i]$, where x_i is the element of X_i corresponding to the constant c.

Proposition C.0.12. *For any ultrafilter $\mathcal{U}$, the definition just given for $M = \prod_{\mathcal{U}} M_i$ is well-defined.*

Proof. We must show that the definitions given are independent of the choice of tuple (x_i) one uses to represent the equivalence class $[x_i]$. Suppose then that $(x_i)_1 \sim (y_i)_1, \ldots, (x_i)_n \sim (y_i)_n$, with $x_{i,1} = y_{i,1}$ holding on the large set J_1, similarly for $J_2, \ldots, J_n$. Let r be an n-ary relational symbol, and suppose that the relation $r(x_{i,1}, \ldots, x_{i,n})$ holds for almost every i, say on the large set $J \subset I$. Then the relation

$r(y_{i,1}, \ldots, y_{i,n})$ holds at least on the set $J_1 \cap \ldots \cap J_n \cap J$, which is large by intersection closure. Thus deciding if $r([x_i]_1, \ldots, [x_i]_n)$ holds in M is independent of the choice of representatives. Identical arguments hold for function and constant symbols. $\square$

Since we are assumed to be working over an ultrafilter, we can say something stronger:

Proposition C.0.13. *The relation* $r([x_i]_1, \ldots, [x_i]_n)$ *does not hold in* M *if and only if, for almost every* i, $r(x_{i,1}, \ldots, x_{i,n})$ *does not hold in* M_i.

Proof. The 'if' direction is true even in a non-ultra filter. For the converse, If $r([x_i]_1, \ldots, [x_i]_n)$ does not hold, it is because the set on which $r(x_{i,1}, \ldots, x_{i,n})$ holds is not large. Then as $\mathcal{U}$ is an ultrafilter, its complement is large, namely the set on which $r(x_{i,1}, \ldots, x_{i,n})$ does not hold. $\square$

This is the reason we demand our filters to be ultra; otherwise M preserves the primitive relations r_i, but not necessarily their negations. The reason we demand our ultrafilters to be non-principal is because

Proposition C.0.14. *If* $\mathcal{U}$ *is a principal ultrafilter, say generated by* $j \in I$, *then* M *is isomorphic to* M_j.

Proof. Two tuples $(x_i), (y_i)$ are then equivalent if and only if the set on which they are equal contains j, if and only if $x_j = y_j$. The map $[x_i] \mapsto x_j$ is thus easily seen to be an isomorphism of L-structures, preserving all relations and whatnot. $\square$

The 'fundamental theorem of ultraproducts', what makes them worth studying at all, would have to be

Theorem C.0.15. *(Łoś' Theorem) Let* $\mathcal{U}$ *be an ultrafilter on* I, $M = \prod_{\mathcal{U}} M_i$ *the ultraproduct of the structures* M_i *with respect to* $\mathcal{U}$. *Let* $\Phi(x_1, \ldots, x_n)$ *be a first-order formula in the language* L *in the variables* $x_1, \ldots, x_n$, *and let* $[a_i]_1, \ldots, [a_i]_n$ *be*

a collection of elements of M. Then $\Phi([a_i]_1, \ldots, [a_i]_n)$ is true of M if and only if $\Phi(a_{i,1}, \ldots, a_{i,n})$ is true of M_i for almost every i.

Proof. See theorem 8.5.3 of [11]. □

We've proved this theorem already in the case of atomic formulae or their negations. The rest of the proof proceeds by induction on the complexity (i.e. length) of the formula. For example, if the theorem is true for the formulae $\Phi(x_1, \ldots, x_n)$ and $\Psi(y_1, \ldots, y_m)$ then it is also true for their conjunction, by considering the intersection of two large sets, which is also large.

Corollary C.0.16. *If Φ is a first-order statement in the language L, then Φ is true of $\prod_{\mathcal{U}} M_i$ if and only if it is true of almost every i.*

Proof. Sentences are just a particular type of formulae; apply Łos' theorem. □

Ultimately, we are not particularly interested in what sorts of statements might hold in $\prod_{\mathcal{U}} M_i$ for a particular choice of non-principal ultrafilter, but rather those first-order statements that hold for *every* non-principal ultrafilter.

Proposition C.0.17. *Let Φ be a first-order statement that holds in $\prod_{\mathcal{U}} M_i$ for every choice of non-principal ultrafilter on I. Then Φ holds in M_i for all but finitely many $i \in I$.*

Proof. Apply corollary C.0.16 and corollary B.0.8. □

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