The Elements of Special Relativity

Amanda Maxwell

Einstein's Special Theory of Relativity is founded upon the curious fact that the speed of light as measured by an observer is independent of that observer's relative motion to the source of the light. In this talk we will discuss some of the surprising and counterintuitive predictions of this theory, in particular the prediction that different observers can, and often do, measure different time intervals between the exact same two events.

1 Newtonian Relativity and Galilean transformations of coordinates

Newtonian relativity describes the physics of our "every-day world". It predicts, for example, that observers in relative motion to each other will measure different velocities for the same object.

Example:

Suppose a man (observer S) is standing next to a railroad track, and another man (observer S') is traveling on a train heading east, and that observer S measures the speed of the train as being 40 mph. Observer S measures a truck, also heading east, as traveling at 60 mph. What will observer S' measure as the speed of the truck?

Answer: $60 \text{ mph} - 40 \text{ mph} = 20 \text{ mph}$

We formalize this reasoning using **Galilean transformation of coordinates**.

Suppose two observers, S and S', start at the same position at time $t = 0$, and begin to move at a constant velocity v relative to one another along the x-axis. Suppose that an event is observed by S as occurring at a point (x, y, z) in space and at a point t in time. Then S' measures this event as occurring at the point (x', y', z') at time t', where:

$$x' = x - vt$$

$$y' = y$$

$$z' = z$$

$$t' = t$$

Further, if S measures the speed of an object in the x-direction as u_x, y-direction as u_y, and z-direction as u_z, then we have

$$u'_x = u_x - v$$

$$u'_y = u_y$$

$$u'_z = u_z$$

Example:

$$u_x = 60 \text{ mph}$$

$$u_y = 0$$

$$u_z = 0$$

$$v = 40 \text{ mph}$$

Then

$$u'_x = u_x - v = 60 \text{ mph} - 40 \text{ mph} = 20 \text{ mph}$$

$$u'_y = 0$$

$$u'_z = 0$$

*** In short: in our everyday experience, observers moving relative to one another will *always* measure different velocities for the same object.

2 Constancy of the Speed of Light

All experimental evidence to date indicates that, *no matter the relative motion of two observers*, they will always measure the *same* speed for the same light pulse, namely

$$\begin{aligned} c &= 299,792,458\ \frac{\mathrm{m}}{\mathrm{s}} \\ &= 186,282\ \frac{\text{miles}}{\text{sec}} \end{aligned}$$

This is a blatant contradiction of Newtonian relativity.

Example:

Suppose a man (observer S) is standing next to a railroad track, and another man (observer S') is traveling on a train heading east, and that observer S measures the speed of the train as being $v = .7c$. Observer S measures a pulse of light, also heading east, as traveling at $u_x = c$. What will observer S' measure as the speed of the light pulse?

The answer is *not* $u'_x = c - .7c = .3c$, as Newtonian relativity would predict; it is $u'_x = c$!

*** Conclusion: we need an entirely new physics to deal with this puzzling phenomenon, courtesy of Albert Einstein.

3 Special Relativity and the Relativity of Time

Einstein's Special Theory of Relativity, published in 1905, is founded upon two postulates:

- The laws of physics are the same in all inertial reference frames.
- The speed of light in a vacuum is equal to the value c, independent of the observer's motion to the source.

These postulates, especially the second, have astounding consequences for the universe we live in, not the least of which is the *relativity of time*.

Example:

Suppose a man (observer S) is standing next to a railroad track, and another man (observer S') is traveling on a train heading east, and that observer S measures the speed of the train as being v.

Suppose that observer S' shines a light pulse on a mirror traveling on the train with him, at a distance D from him, and allows it to return. Then S and S' will clearly measure different distances for the light pulse to travel:

Let Δt be the time interval measured for the light pulse's trip in the S frame, similarly for S'. Then from the figure we have

$$\left(\frac{c\Delta t}{2}\right)^2 = D^2 + \left(\frac{v\Delta t}{2}\right)^2$$

and

$$2D = c\Delta t'$$

Solving for D in both equations and equating, this yields

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}$$

In other words, S measures a *longer* time interval for the light pulse's trip than does S'!

For example, if $v = .9c$ and $\Delta t' = 1$ sec, then

$$\Delta t' = 2.29 \text{ sec}$$

*** The moral: different observers, in relative motion to one another, can and do measure different time intervals between the exact same two events.

Example:

An observer on earth measures the speed of a spacecraft as $v = .99c$, bound for the nearest star Proxima Centauri, which is (about) 4 light years away. The observer on earth therefore measures the trip as taking a little over 4 years, or

$$\Delta t = 4/.99 = 4.04 \text{ years}$$

How long will an observer on the spacecraft measure the trip as taking?

Answer: We have

$$\Delta t = \gamma \Delta t'$$

or

$$\begin{aligned}\Delta t' &= \frac{\Delta t}{\gamma} \\ &= \sqrt{1 - v^2/c^2}\Delta t \\ &= .57 \text{ years}\end{aligned}$$

or just a little over 6 months.

4 The Lorentz Transformations

We have seen that Galilean transformation of coordinates is not correct within a relativistic framework; the correct transformations are the **Lorentz transformations**.

Suppose two observers, S and S', start at the same position at time $t = 0$, and begin to move at a constant velocity v relative to one another along the x-axis. Suppose that an event is observed by S as occurring at a point (x, y, z) in space and at a point t in time. Then S' measures this event as occurring at the point (x', y', z') at time t', where:

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma(t - \frac{vx}{c^2})$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Notice that, when $v << c$, $\gamma \approx 1$, and the above transformations reduce to the Galilean transformations.

Further, if S measures the speed of an object in the x-direction as u_x, y-direction as u_y, and z-direction as u_z, then we have

$$
\begin{aligned}
u'_x &= \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \\
u'_y &= \frac{u_y}{\gamma(1 - \frac{vu_x}{c^2})} \\
u'_z &= \frac{u_z}{\gamma(1 - \frac{vu_x}{c^2})}
\end{aligned}
$$

Notice that, if $u_x = c$, then $u'_x = c$ as well. That is, if any observer measures an object's speed as c, then *all* observers will measure the speed of that object as c.

Example:

The starship Enterprise is chasing a Klingon vessel. An earthbound observer S measures the Enterprise as having a speed $v = .8c$, and the Klingon vessel as $u_x = .6c$. What will the Enterprise measure as the speed of the Klingon vessel?

Answer: We have

$$v = .8c$$

$$u_x = .6c$$

and thus

$$u_x' = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

$$= -.38c$$

That is, the Enterprise will measure itself as gaining on the Klingon vessel at a speed of $.38c$.

5 Further remarks

- Special relativity is the setting of Einstein's famous equation $E = mc^2$, showing an equivalence between *mass* and *energy*.
- It can be shown that, just as the time interval between two events can be different for different observers, so too can the *distance*; this is the phenomenon of **length contraction**.
- It can be shown that c is the *universal speed limit* of the universe. That is, no observer will measure any object as traveling at a speed greater than c, and that the only particles that can travel at c are massless.
- Special relativity is called "special" because it only makes predictions for *inertial* (i.e., non-accelerated) frames, and only the absence of any gravitational field. For this, one needs Einstein's *general* theory of relativity.

Thank you!