The Black-Scholes formula for stock option pricing

Jingyuan Chen

The University of Findlay

1 Stock Options

Definition. A (European style) **call option** is the right of one party to purchase from another party a specific stock (called the **underlying**) at a specific price (called the **strike price**) on a specific date (the **exercise** date). A **put option** is the right to sell such a stock.

Example:

You purchase a call option for stock XYZ with:

$S_0 =$ current price $= 50$

$K =$ strike price $= 50$

$C =$ cost of option $= 2.50$

$T =$ time to exercise $= 60$ days

If, at time $T = 60$, the price of XYZ is $S_T = 55$, the writer will owe the purchaser $S_T - K = 5$. If $S_T < 50$, the writer will owe the purchaser nothing.

*** A call option is essentially a bet that the stock price will rise to a certain level in the future.

You purchase a put option for stock XYZ with:

$S_0 =$ current price $= 50$

$K =$ strike price $= 50$

$P =$ cost of option $= 2.42$

$T =$ time to exercise $= 60$ days

If, at time $T = 60$, the price of XYZ is $S_T = 40$, the writer will owe the purchaser $K - S_T = 10$. If $S_T > 50$, the writer will owe the purchaser nothing.

*** A put option is essentially a bet that the stock price will *fall* to a certain level in the future.

The price of call options and put options are related by

Theorem. ***(Put-Call parity)*** *If* $C =$ *cost of a call option* , $P =$ *cost of a put option* , *both on the same underlying XYZ currently trading at* S_0, *same strike price* K, *and same time to exercise* T, *then*

$$C - P = S_0 - Ke^{-rT}$$

where r *is the risk-less rate of return.*

Proof. The purchase of a put and sale of a call (i.e. the left hand side of this equation) can be seen as a forward contract to sell stock XYZ at price K at time T. The fair price of a such a contract is known to be given by the right hand side of this equation.

$\square$

Example:

The previous two examples satisfy put-call parity for a risk-less rate of return of $r = .01$:

$$\begin{aligned} C - P &= 2.5 - 2.42 \\ &= .08 \\ &= 50 - 50e^{-.01\frac{60}{360}} \\ &= S_0 - Ke^{-rT} \end{aligned}$$

2 The Black-Scholes model of stock movements

The price of a call option can be seen as compensation to the writer for the possible losses incurred due to movements in the underlying stock price. Thus, one needs a probabilistic model of stock prices to adequately price an option. That is, we would like to realize the price of a stock at a given time as an appropriate continuous random variable.

The market assumptions of the Black-Scholes model

1. There is an underlying *riskless rate of return*, r, at which anyone can lend or borrow freely
2. There are no *arbitrage opportunities* (a riskless profit at zero net investment)
3. Stocks pay no dividends during the life of an option
4. The market is *frictionless* (no fees, taxes, commissions, etc.)
5. The stock may be bought or sold at any fractional amount
6. The instantaneous logarithmic return of a stock's price exhibits *Brownian motion*

Definition. The **volatility** of a stock, σ, is the standard deviation of the yearly logarithmic returns of a stock. It is a measure of the tendency of the stock's price to exhibit unpredictable movements over a given period of time. It can often be inferred from historical data.

Proposition. *(The Black-Scholes model)*

Over a small period of time Δt*, the change in the price of a stock can be approximated as*

$$\Delta S = \mu S \Delta t + \sigma S \sqrt{\Delta t}\varepsilon$$

where

$\Delta S =$ *change in stock price*

S $=$ *current stock price*

μ $=$ *drift rate*

$\sigma =$ *volatility*

ε $=$ *a fair coin flip with* 1 *on one side,* -1 *on the other*

From this we immediately infer

Proposition. *Divide the time interval* $[0, T]$ *into a large number* N *of small time intervals of size* Δt. *Then the stock price* S_T *at time* T *is approximated by the random variable*

$$S_T = S_0 \left(1 + \mu\Delta t + \sigma\sqrt{\Delta t}\right)^X \left(1 + \mu\Delta t - \sigma\sqrt{\Delta t}\right)^{N-X}$$

where X *is the binomial random variable counting the number of heads on* N *fair coin flips.*

Proof. We have

$$
\begin{aligned}
S_{t+1} &= S_t + \Delta S \\
&= S_t + \mu S_t \Delta t + \sigma S_t \sqrt{\Delta t}\varepsilon \\
&= S_t(1 + \mu \Delta t + \sigma \sqrt{\Delta t}\varepsilon)
\end{aligned}
$$

Thus

$$
S_T = S_0 \left(1 + \mu \Delta t + \sigma \sqrt{\Delta t}\right)^X \left(1 + \mu \Delta t - \sigma \sqrt{\Delta t}\right)^{N-X}
$$

where X is the number of $1's$ which occurred, $N - X$ the number of $-1's$ which occurred. $\square$

If we pass to the limit as $\Delta t \to 0$, using the Central Limit Theorem we can show

Proposition. *Let S_T be the continuous random variable giving the price of a stock at time T. Then*

$$S_T = S_0 e^{(\mu - \frac{\sigma^2}{2})T} e^{\sigma \sqrt{T} Z}$$

where Z is a standard normal random variable.

*** $\sqrt{T}Z$ can actually be replaced by standard *Brownian Motion* W_T.

3 Deriving the Black-Scholes formula for option prices

Theorem. *The fair value of a European call option is given by*

$$C = S_0 N(d_1) - K e^{-rT} N(d_2)$$

where

$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S_0}{K}\right) + (r + \sigma^2/2)T\right]$

$d_2 = d_1 - \sigma\sqrt{T}$

$N(x) =$ *standard normal cummulative distribution*

$S_0 =$ *current stock price*

$K =$ *strike price*

$T =$ *time to expiration*

$r =$ *risk-free rate of return*

$\sigma =$ *volatility of the underlying*

Proof. (Outline) The payout of the writer to the purchaser at time of expiry is the random variable

$$\text{payout} = \max(S_T - K, 0)$$

The expected value of this payout, discounted to the present value, is then the fair value of the option:

$$C = e^{-rT} E\left[\text{payout}\right]$$

From our model of stock prices, payout has cummulative distribution

$$\begin{aligned}
L(x) &= P(S_T - K \leq x) \\
&= P(S_0 e^{(r-\sigma^2/2)T} e^{\sigma\sqrt{T}Z} - K leq x) \\
&= P\left(Z \leq \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{x}{S_0}\right) - (r - \sigma^2/2)T\right]\right) \\
&= N(\lambda)
\end{aligned}$$

and thus

$$e^{-rT} E\left[\text{payout}\right] = e^{-rT} \int_K^\infty (x - K) N(\lambda)' dx$$

Standard integration techniques yield the desired result.

$\square$

4 Properties of the Black-Scholes fomrula

Basic differentiation shows that

$\frac{dC}{d\sigma} > 0$

$\frac{dC}{dT} > 0$

$\frac{dC}{dK} < 0$

That is, the cost of a European call option is increasing with σ and T, and decreasing with K.

5 Final remarks

- The Black-Scholes model of stock prices, and so also the Black-Scholes formula, is valid only insofar as its assumptions are valid (which they are not in a good many cases)

- Deviation from the Black-Scholes formula for option prices has been observed even in nearly ideal situations; the *option smile* is a well studied and common anomaly

- A more general equation, valid for *any* option (European or not), is the *Black-Scholes* equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Thank you!