The stochastic neural field model

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1 Second order neural field model

The neural field model is popular due to being parsimonious yet having a strong link with the underlying physiology. The model relates the average number of action potentials $g(\mathbf{r},t)$ arriving at time $t \in \mathbb{R}^+$ at position $\mathbf{r} \in \mathcal{O} \subset \mathbb{R}^d$ to the local post synaptic membrane voltage $v(\mathbf{r},t)$. It is well known that the post-synaptic potentials generated at a neuronal population at location $\mathbf{r}$ by action potentials arriving from all other connected populations in $\mathcal{O}$ can be described by []

$$v(\mathbf{r},t) = e^{-\alpha t} v_0 (t\alpha + 1) + t e^{-\alpha t} v_0' + \int_0^t h(t-t') g(\mathbf{r},t') dt' \tag{1}$$

where v_0 is the initial membrane voltage and v_0' is the time derivative of the initial voltage. The post-synaptic response kernel $h(t)$ can be described by an exponentially decaying function of the form []

$$h(t) = t \exp(-\alpha t) \tag{2}$$

Let $W(\mathbf{r},t)$ be a space-time Wiener process. Then, for $\sigma \in \mathbb{R}^+$, the non-local interactions $g(\mathbf{r},t)$ between cortical populations are described by

$$[g(\mathbf{r},t) - \tilde{g}(\mathbf{r},t)]dt = \sigma dW(\mathbf{r},t) \tag{3}$$

In (3) we have divided the action potentials into i) a deterministic part $\tilde{g}(\mathbf{r},t)$ described by an induced input $u(\mathbf{r},t)$ and nonlocal interactions and ii) the random component or disturbance $W(\mathbf{r},t)$. As a result of the latter component, the membrane voltage $v(\mathbf{r},t)$ is itself a stochastic process and in fact v_0' may not be well defined. We will tackle this issue later. The deterministic part $\tilde{g}(\mathbf{r},t)$ is usually modelled as an integral equation with exogenous deterministic inputs of the form []

$$\tilde{g}(\mathbf{r},t) = \int_{\mathcal{O}} w(\mathbf{r},\mathbf{r}') f(v(\mathbf{r}',t)) d\mathbf{r}' + u(\mathbf{r},t) \tag{4}$$

We substitute this quantity into (1) to get the following expression for $v(\mathbf{r},t)$.

$$v(\mathbf{r},t) = e^{-\alpha t} v_0 (t\alpha + 1) + t e^{-\alpha t} \text{``}v_0'\text{''} + \int_0^t h(t-t') \tilde{g}(\mathbf{r},t') dt' + \int_0^t h(t-t') \sigma dW(\mathbf{r},t') \tag{5}$$

where we have used double quotes to denote the informal setting. We stress that σ does not depend on the field $v(\mathbf{r},t)$ and hence the noise in (5) is strictly additive. One may postulate that $v(\mathbf{r},t)$ is a solution to the initial value problem

$$\begin{cases} dv(\mathbf{r},t) = \tilde{v}(\mathbf{r},t)dt \\ d\tilde{v}(\mathbf{r},t) + 2\zeta\tilde{v}(\mathbf{r},t)dt + \alpha^2 v(\mathbf{r},t)dt = \quad \tilde{g}(\mathbf{r},t)dt + dW(\mathbf{r},t) \qquad t \geq 0, v(\mathbf{r},0) = v_0, \text{``}v'(0)\text{''} = \text{``}v_0'\text{''} \end{cases} \tag{6}$$

By considering the homogeneous problem one can assume that the solution to the coupled SDE system is given as

$$v(\mathbf{r},t) = \kappa_1(\mathbf{r},t) v_1(\mathbf{r},t) + \kappa_2(\mathbf{r},t) v_2(\mathbf{r},t) \tag{7}$$

where v_1 and v_2 would solve the homogeneous function and are hence deterministic. By applying Ito's product rule we hence have that

$$dv(\mathbf{r},t) = d\kappa_1(\mathbf{r},t) v_1(\mathbf{r},t) + dv_1(\mathbf{r},t)\kappa_1(\mathbf{r},t) + dv_2(\mathbf{r},t)\kappa_2(\mathbf{r},t) + d\kappa_2(\mathbf{r},t) v_2(\mathbf{r},t)$$

To restrict our set of solutions, as is typical of variation of parameter methods, at this point we can impose the constraint

$$d\kappa_1(\mathbf{r},t)v_1(\mathbf{r},t) + d\kappa_2(\mathbf{r},t)v_2(\mathbf{r},t) = 0$$

so that

$$dv(\mathbf{r},t) = dv_1(\mathbf{r},t)\kappa_1(\mathbf{r},t) + dv_2(\mathbf{r},t)\kappa_2(\mathbf{r},t)$$

Now $d\tilde{v}(\mathbf{r},t)dt = dv(\mathbf{r},t)$ so that $d\tilde{v}_1(\mathbf{r},t)dt = dv_1(\mathbf{r},t)$ and $d\tilde{v}_2(\mathbf{r},t)dt = dv_2(\mathbf{r},t)$. Therefore we have that $\tilde{v}(\mathbf{r},t) = \tilde{v}_1(\mathbf{r},t)\kappa_1(\mathbf{r},t) + \tilde{v}_2(\mathbf{r},t)\kappa_2(\mathbf{r},t)$ and re-applying the product rule we get

$$d\tilde{v}(\mathbf{r},t) = d\tilde{v}_1(\mathbf{r},t)\kappa_1(\mathbf{r},t) + \tilde{v}_1(\mathbf{r},t)d\kappa_1(\mathbf{r},t) + d\tilde{v}_2(\mathbf{r},t)\kappa_2(\mathbf{r},t) + \tilde{v}_2(\mathbf{r},t)d\kappa_2(\mathbf{r},t) \tag{8}$$

Substitute (8) in (6) to get

$$\begin{aligned} \kappa_1(\mathbf{r},t)[d\tilde{v}_1(\mathbf{r},t) + 2\alpha\tilde{v}_1(\mathbf{r},t) + \alpha^2 v_1(\mathbf{r},t)] & \\ +\kappa_2(\mathbf{r},t)[d\tilde{v}_2(\mathbf{r},t) + 2\alpha\tilde{v}_2(\mathbf{r},t) + \alpha^2 v_2(\mathbf{r},t)] & \\ +\tilde{v}_1(\mathbf{r},t)d\kappa_1(\mathbf{r},t) + \tilde{v}_2 d\kappa_2(\mathbf{r},t) &= \tilde{g}(\mathbf{r},t) + dW(\mathbf{r},t) \end{aligned}$$

The terms in the brackets, by presupposition, solve the homogeneous problem exactly and are hence equal to zero and we are left with the coupled problem

$$\begin{cases} \tilde{v}_1(\mathbf{r},t)d\kappa_1(\mathbf{r},t) + \tilde{v}_2(\mathbf{r},t)d\kappa_2(\mathbf{r},t) = & \tilde{g}(\mathbf{r},t)dt + dW(\mathbf{r},t) \\ v_1(\mathbf{r},t)d\kappa_1(\mathbf{r},t) + v_2(\mathbf{r},t)d\kappa_2(\mathbf{r},t) = & 0 \quad t \geq 0, v(\mathbf{r},0) = v_0, \text{“}v'(\mathbf{r},0)\text{”} = \text{“}v_0'\text{”} \end{cases}$$

Rearranging, we get the following expressions

$$d\kappa_1(\mathbf{r},t) = \frac{g(\mathbf{r},t)v_2(\mathbf{r},t)}{\tilde{v}_1(\mathbf{r},t)v_2(\mathbf{r},t) - \tilde{v}_2(\mathbf{r},t)v_1(\mathbf{r},t)} \quad d\kappa_2(\mathbf{r},t) = -\frac{g(\mathbf{r},t)v_1(\mathbf{r},t)}{\tilde{v}_1(\mathbf{r},t)v_2(\mathbf{r},t) - \tilde{v}_2(\mathbf{r},t)v_1(\mathbf{r},t)}$$

and upon integrating from 0 to t we get

$$\begin{aligned} \kappa_1(\mathbf{r},t) &= \kappa_1(\mathbf{r},0) + \int_0^t \frac{\tilde{g}(\mathbf{r},t')v_2(\mathbf{r},t')}{\tilde{v}_1(\mathbf{r},t')v_2(\mathbf{r},t') - \tilde{v}_2(\mathbf{r},t')v_1(\mathbf{r},t')}dt' + \int_0^t \frac{v_2(\mathbf{r},t')}{\tilde{v}_1(\mathbf{r},t')v_2(\mathbf{r},t') - \tilde{v}_2(\mathbf{r},t')v_1(\mathbf{r},t')}dW(\mathbf{r},t') \\ \kappa_2(\mathbf{r},t) &= \kappa_2(\mathbf{r},0) - \int_0^t \frac{\tilde{g}(\mathbf{r},t')v_1(\mathbf{r},t')}{\tilde{v}_1(\mathbf{r},t')v_2(\mathbf{r},t') - \tilde{v}_2(\mathbf{r},t')v_1(\mathbf{r},t')}dt' - \int_0^t \frac{v_1(\mathbf{r},t')}{\tilde{v}_1(\mathbf{r},t')v_2(\mathbf{r},t') - \tilde{v}_2(\mathbf{r},t')v_1(\mathbf{r},t')}dW(\mathbf{r},t') \end{aligned} \tag{9}$$

In our case, terms for v_1 and v_2 which solve the homogeneous problem are given by $v_1(\mathbf{r},t) = te^{-\alpha t}, v_2(\mathbf{r},t) = e^{-\alpha t}$ (note these are independent of the spatial variable). so that

$$\begin{aligned} dv_1(\mathbf{r},t) &= (-\alpha te^{-\alpha t} + e^{-\alpha t})dt, \quad & \tilde{v}_1(\mathbf{r},t) = -\alpha te^{-\alpha t} + e^{-\alpha t} \\ dv_2(\mathbf{r},t) &= (-\alpha e^{-\alpha t})dt, & \tilde{v}_2(\mathbf{r},t) = -\alpha e^{-\alpha t} \end{aligned}$$

so that $\tilde{v}_1(\mathbf{r},t)v_2(\mathbf{r},t) - \tilde{v}_2(\mathbf{r},t)v_1(\mathbf{r},t) = e^{-2\alpha t}$. Using this to evaluate $\kappa_1(\mathbf{r},t)$ and $\kappa_2(\mathbf{r},t)$ we can now substitute into (7) to get

$$\begin{aligned} v(\mathbf{r},t) &= \int_0^t te^{-\alpha(t-t')}\tilde{g}(\mathbf{r},t')dt' + \int_0^t te^{-\alpha(t-t')}dW(\mathbf{r},t') \\ &\quad -\int_0^t t'e^{-\alpha(t-t')}\tilde{g}(\mathbf{r},t')dt' - \int_0^t t'e^{-\alpha(t-t')}dW(\mathbf{r},t') + te^{-\alpha t}\kappa_1(\mathbf{r},0) + e^{-\alpha t}\kappa_2(\mathbf{r},0) \end{aligned}$$

$\kappa_1(\mathbf{r},0)$ and $\kappa_2(\mathbf{r},0)$ are found by considering initial field and initial field increments and are given as

$$\begin{aligned}\kappa_2(\mathbf{r},0) &= v_0 \\ \kappa_1(\mathbf{r},0)dt &= dv_0 + \alpha v_0 dt\end{aligned} \tag{10}$$

The constant $\kappa_1(\mathbf{r},0)$ can be informally expressed in terms of the field derivative to give

$$\kappa_1(\mathbf{r},0) = \text{``}v_0'\text{''} + \alpha v_0 \tag{11}$$

to give the IDE model of (5)

$$v(\mathbf{r},t) = te^{-\alpha t}(\text{``}v_0'\text{''}+\alpha v_0)+e^{-\alpha t}v_0+\int_0^t (t-t')e^{-\alpha(t-t')}\tilde{g}(\mathbf{r},t')dt'+\int_0^t (t-t')e^{-\alpha(t-t')}dW(\mathbf{r},t') \tag{12}$$

1.1 Working with differences

Since the first derivative of $v(\mathbf{r},t)$ cannot be assumed to exist at time $t=0$, the strict way to handle this problem would be to work with the quantity

$$dv(\mathbf{r},t) = dv_1(\mathbf{r},t)\kappa_1(\mathbf{r},t) + dv_2(\mathbf{r},t)\kappa_2(\mathbf{r},t) \tag{13}$$

Effecting the relevant substitutions, it is not difficult to show that this problem is solved as

$$dv(\mathbf{r},t) = dv_0 e^{-\alpha t}[1-\alpha t]-\alpha^2 v_0 t e^{-\alpha t}+\int_0^t (1-\alpha[t-t'])e^{-\alpha(t-t')}\tilde{g}(\mathbf{r},t')dt'+\int_0^t (1-\alpha[t-t'])e^{-\alpha(t-t')}dW(\mathbf{r},t') \tag{14}$$

By working with the frst difference, the solution remains valid and well-posed. Note that this is just a different form of representation for the same problem and hence, the kernel $h(t)$ remains unchanged.

References