Non-linear Spatio-temporal modelling using the Integro-Difference Equation

Parham Aram*, Dean R. Freestone, *Graduate Student Member, IEEE*, Kenneth Scerri, Michael Dewar and Visakan Kadirkamanathan, *Member, IEEE*

Abstract

Index Terms

Dynamic spatio-temporal modelling, Integro-Difference Equation (IDE), ...

June 29, 2010 DRAFT

1

Non-linear Spatio-temporal modelling using the Integro-Difference Equation

I. INTRODUCTION

Non-linear spatio-temporal behaviour...

II. NEURAL FIELD MODEL

III. STATE SPACE REPRESENTATION OF THE SPATIO-TEMPORAL NON-LINEAR IDE

We define the pth order non-linear, stochastic, spatio-temporal IDE as

$$\rho_t(\mathbf{r}) = \int_{\Omega} w(\mathbf{r}, \mathbf{r}') f(v_{t-p}(\mathbf{r}')) d\mathbf{r}' + \mathbf{e}_{t-p}(\mathbf{r})$$
(1)

where $\rho_t(\mathbf{r})$ is a lag polynomial defined as

$$\rho_t(\mathbf{r}) = \left(1 + \sum_{i=1}^p \varphi_i L_i\right) v_t(\mathbf{r}) \tag{2}$$

where $v_t(\mathbf{r})$ denotes the spatial field at time t and at spatial location \mathbf{r} , f(.) is some nonlinear map, and w(.) is the spatial kernel. The disturbance $\mathbf{e}_{t-p}(\mathbf{r})$ is a zero-mean normally distributed noise process, spatially coloured but temporally independent, with covariance

$$cov\left(e_{t}\left(\mathbf{r}\right), e_{t+\tau}\left(\mathbf{r}'\right)\right) = \gamma\left(\mathbf{r} - \mathbf{r}'\right)\delta(t - \tau)$$
(3)

where $\delta(.)$ is the Dirac-delta function. The spatial field is observed by

$$\mathbf{y}_{t} = \int_{\Omega} m \left(\mathbf{r}_{n_{y}} - \mathbf{r}' \right) v_{t} \left(\mathbf{r}' \right) d\mathbf{r}' + \boldsymbol{\varepsilon}_{t}, \tag{4}$$

where the observation vector \mathbf{y}_t , compiled at n_y spatial locations and at time t, is corrupted by an i.i.d normally distributed zero-mean white noise $\boldsymbol{\varepsilon}_t \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Sigma}_{\varepsilon}\right)$ with $\boldsymbol{\Sigma}_{\varepsilon} = \sigma_{\varepsilon}^2 \mathbf{I}_{n_y}$. It is assumed here that $\boldsymbol{\Sigma}_{\varepsilon}$ and $\gamma\left(\mathbf{r} - \mathbf{r}'\right)$ are known. The output kernel $m(\mathbf{r} - \mathbf{r}')$ governs the sensor pick-up geometry and is defined by

$$m(\mathbf{r} - \mathbf{r}') = \exp\left(-\frac{(\mathbf{r} - \mathbf{r}')^{\top}(\mathbf{r} - \mathbf{r}')}{\sigma_m^2}\right),$$
 (5)

where σ_m sets the sensor width. The superscript \top denotes the transpose operator. We proceed by writing the spatio-temporal IDE model in a parameterized state space form which allows the application

June 29, 2010 DRAFT

of standard estimation techniques. We use a decomposition of the filed and the spatial kernel using a set of Gaussian basis functions

$$v_t(\mathbf{r}) \approx \boldsymbol{\phi}^{\top}(\mathbf{r}) \, \mathbf{x}_t$$
 (6)

$$w(\mathbf{r}, \mathbf{r}') \approx \boldsymbol{\psi}^{\top}(\mathbf{r}, \mathbf{r}') \boldsymbol{\theta}$$
 (7)

where $\phi(\mathbf{r})$ and $\psi(\mathbf{r})$ are vectors of Gaussian basis functions that are scaled by the state vector, \mathbf{x}_t and the scaling parameter vector $\boldsymbol{\theta}$ respectively. Substituting (2) in (1) and using the above decomposition we have

$$\boldsymbol{\phi}^{\top}(\mathbf{r})\left(1 + \sum_{i=1}^{p} \varphi_{i} L_{i}\right) \mathbf{x}_{t} = \int_{\Omega} \boldsymbol{\psi}^{\top}(\mathbf{r}, \mathbf{r}') f(\boldsymbol{\phi}^{\top}(\mathbf{r}') \mathbf{x}_{t-p}) d\mathbf{r}' \boldsymbol{\theta} + \mathbf{e}_{t-p}(\mathbf{r})$$
(8)

multiplying (8) by $\phi(\mathbf{r})$ and integrate over the spatial domain Ω we get

$$\Gamma(1 + \sum_{i=1}^{p} \varphi_{i} L_{i}) \mathbf{x}_{t} = \int_{\Omega} \boldsymbol{\phi}(\mathbf{r}) \int_{\Omega} \boldsymbol{\psi}^{\top} (\mathbf{r}, \mathbf{r}') f(\boldsymbol{\phi}^{\top} (\mathbf{r}') \mathbf{x}_{t-p}) d\mathbf{r}' \boldsymbol{\theta} + \int_{\Omega} \boldsymbol{\phi}(\mathbf{r}) \mathbf{e}_{t-p}(\mathbf{r}) d\mathbf{r}$$
(9)

where

$$\mathbf{\Gamma} \triangleq \int_{\Omega} \boldsymbol{\phi}(\mathbf{r}) \, \boldsymbol{\phi}^{\top}(\mathbf{r}) \, d\mathbf{r} \tag{10}$$

cross-multiplying (9) by Γ^{-1} gives

$$(1 + \sum_{i=1}^{p} \varphi_{i} L_{i}) \mathbf{x}_{t} = \mathbf{\Gamma}^{-1} \int_{\Omega} \boldsymbol{\phi}(\mathbf{r}) \int_{\Omega} \boldsymbol{\psi}^{\top} (\mathbf{r}, \mathbf{r}') f(\boldsymbol{\phi}^{\top} (\mathbf{r}') \mathbf{x}_{t-p}) d\mathbf{r}' \boldsymbol{\theta} + \mathbf{\Gamma}^{-1} \int_{\Omega} \boldsymbol{\phi}(\mathbf{r}) \mathbf{e}_{t-p}(\mathbf{r}) d\mathbf{r}$$
(11)

Here we consider the case where p = 1, therefore we have

$$x_{t} + \varphi_{1}x_{t-1} = \mathbf{\Gamma}^{-1} \int_{\Omega} \boldsymbol{\phi}(\mathbf{r}) \int_{\Omega} \boldsymbol{\psi}^{\top} (\mathbf{r}, \mathbf{r}') f(\boldsymbol{\phi}^{\top} (\mathbf{r}') \mathbf{x}_{t-1}) d\mathbf{r}' \boldsymbol{\theta} + \mathbf{\Gamma}^{-1} \int_{\Omega} \boldsymbol{\phi}(\mathbf{r}) \mathbf{e}_{t-1}(\mathbf{r}) d\mathbf{r}$$
(12)

IV. ESTIMATION OF THE NON-LINEAR SPATIO-TEMPORAL IDE MODEL

- A. Unscented Kalman Filter
- B. Unscented RTS Smoother
- C. Parameter Estimation
- D. Computational Complexity

V. EXAMPLE

VI. CONCLUSION

APPENDIX

Lemma 1:

Proof:

June 29, 2010 DRAFT