Non-linear Spatio-temporal modelling using the Integro-Difference Equation

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Abstract

Index Terms

Dynamic spatio-temporal modelling, Integro-Difference Equation (IDE), ...

Non-linear Spatio-temporal modelling using the Integro-Difference Equation

I. Introduction

Non-linear spatio-temporal behaviour...

II. Neural Field Model

III. State space Representation of the Spatio-temporal Non-linear IDE

We define the pth order non-linear, stochastic, spatio-temporal IDE as

$$\rho_t(\mathbf{r}) = \int_\Omega w(\mathbf{r}, \mathbf{r}') f(v_{t-p}(\mathbf{r}')) d\mathbf{r}' + \mathbf{e}_{t-p}(\mathbf{r}) \tag{1}$$

where $\rho_t(\mathbf{r})$ is a lag polynomial defined as

$$\rho_t(\mathbf{r}) = (1 + \sum_{i=1}^{p} \varphi_i L_i) v_t(\mathbf{r}) \tag{2}$$

where $v_t(\mathbf{r})$ denotes the spatial field at time t and at spatial location $\mathbf{r}$, $f(.)$ is some nonlinear map, and $w(.)$ is the spatial kernel. The disturbance $\mathbf{e}_{t-p}(\mathbf{r})$ is a zero-mean normally distributed noise process, spatially coloured but temporally independent, with covariance

$$cov\left(e_t(\mathbf{r}), e_{t+\tau}(\mathbf{r}')\right) = \gamma\left(\mathbf{r} - \mathbf{r}'\right) \delta(t - \tau) \tag{3}$$

where $\delta(.)$ is the Dirac-delta function. The spatial field is observed by

$$\mathbf{y}_t = \int_\Omega m\left(\mathbf{r}_{n_y} - \mathbf{r}'\right) v_t\left(\mathbf{r}'\right) d\mathbf{r}' + \boldsymbol{\varepsilon}_t, \tag{4}$$

where the observation vector $\mathbf{y}_t$, compiled at n_y spatial locations and at time t, is corrupted by an i.i.d normally distributed zero-mean white noise $\boldsymbol{\varepsilon}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_\varepsilon)$ with $\boldsymbol{\Sigma}_{\boldsymbol{\varepsilon}} = \sigma_\varepsilon^2 \mathbf{I}_{n_y}$. It is assumed here that $\boldsymbol{\Sigma}_\varepsilon$ and $\gamma(\mathbf{r} - \mathbf{r}')$ are known. The output kernel $m(\mathbf{r} - \mathbf{r}')$ governs the sensor pick-up geometry and is defined by

$$m\left(\mathbf{r} - \mathbf{r}'\right) = \exp\left(-\frac{(\mathbf{r} - \mathbf{r}')^\top (\mathbf{r} - \mathbf{r}')}{\sigma_m^2}\right), \tag{5}$$

where σ_m sets the sensor width. The superscript $\top$ denotes the transpose operator. We proceed by writing the spatio-temporal IDE model in a parameterized state space form which allows the application

of standard estimation techniques. We use a decomposition of the filed and the spatial kernel using a set of Gaussian basis functions

$$v_t\left(\mathbf{r}\right) \approx \boldsymbol{\phi}^{\top}\left(\mathbf{r}\right)\mathbf{x}_t \tag{6}$$

$$w\left(\mathbf{r},\mathbf{r}'\right) \approx \boldsymbol{\psi}^{\top}\left(\mathbf{r},\mathbf{r}'\right)\boldsymbol{\theta} \tag{7}$$

where $\boldsymbol{\phi}(\mathbf{r})$ and $\boldsymbol{\psi}(\mathbf{r})$ are vectors of Gaussian basis functions that are scaled by the state vector, $\mathbf{x}_t$ and the scaling parameter vector $\boldsymbol{\theta}$ respectively. Substituting (2) in (1) and using the above decomposition we have

$$\boldsymbol{\phi}^{\top}\left(\mathbf{r}\right)\left(1+\sum_{i=1}^{p}\varphi_i L_i\right)\mathbf{x}_t = \int_{\Omega}\boldsymbol{\psi}^{\top}\left(\mathbf{r},\mathbf{r}'\right)f(\boldsymbol{\phi}^{\top}\left(\mathbf{r}'\right)\mathbf{x}_{t-p})d\mathbf{r}'\boldsymbol{\theta} + \mathbf{e}_{t-p}(\mathbf{r}) \tag{8}$$

multiplying (8) by $\boldsymbol{\phi}(\mathbf{r})$ and integrate over the spatial domain Ω we get

$$\boldsymbol{\Gamma}(1+\sum_{i=1}^{p}\varphi_i L_i)\mathbf{x}_t = \int_{\Omega}\boldsymbol{\phi}(\mathbf{r})\int_{\Omega}\boldsymbol{\psi}^{\top}\left(\mathbf{r},\mathbf{r}'\right)f(\boldsymbol{\phi}^{\top}\left(\mathbf{r}'\right)\mathbf{x}_{t-p})d\mathbf{r}'\boldsymbol{\theta} + \int_{\Omega}\boldsymbol{\phi}(\mathbf{r})\mathbf{e}_{t-p}(\mathbf{r})d\mathbf{r} \tag{9}$$

where

$$\boldsymbol{\Gamma} \triangleq \int_{\Omega}\boldsymbol{\phi}\left(\mathbf{r}\right)\boldsymbol{\phi}^{\top}\left(\mathbf{r}\right)d\mathbf{r} \tag{10}$$

cross-multiplying (9) by $\boldsymbol{\Gamma}^{-1}$ gives

$$(1+\sum_{i=1}^{p}\varphi_i L_i)\mathbf{x}_t = \boldsymbol{\Gamma}^{-1}\int_{\Omega}\boldsymbol{\phi}(\mathbf{r})\int_{\Omega}\boldsymbol{\psi}^{\top}\left(\mathbf{r},\mathbf{r}'\right)f(\boldsymbol{\phi}^{\top}\left(\mathbf{r}'\right)\mathbf{x}_{t-p})d\mathbf{r}'\boldsymbol{\theta} + \boldsymbol{\Gamma}^{-1}\int_{\Omega}\boldsymbol{\phi}(\mathbf{r})\mathbf{e}_{t-p}(\mathbf{r})d\mathbf{r} \tag{11}$$

Here we consider the case where $p=1$, therefore we have

$$x_t + \varphi_1 x_{t-1} = \boldsymbol{\Gamma}^{-1}\int_{\Omega}\boldsymbol{\phi}(\mathbf{r})\int_{\Omega}\boldsymbol{\psi}^{\top}\left(\mathbf{r},\mathbf{r}'\right)f(\boldsymbol{\phi}^{\top}\left(\mathbf{r}'\right)\mathbf{x}_{t-1})d\mathbf{r}'\boldsymbol{\theta} + \boldsymbol{\Gamma}^{-1}\int_{\Omega}\boldsymbol{\phi}(\mathbf{r})\mathbf{e}_{t-1}(\mathbf{r})d\mathbf{r} \tag{12}$$

IV. Estimation of the Non-linear Spatio-temporal IDE Model

A. Unscented Kalman Filter

B. Unscented RTS Smoother

C. Parameter Estimation

D. Computational Complexity

V. Example

VI. Conclusion

Appendix

Lemma 1:

Proof:

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