Non-linear Spatio-temporal modelling using the Integro-Difference Equation

Parham Aram*, Dean R. Freestone, *Graduate Student Member, IEEE,* Kenneth Scerri, Michael Dewar and Visakan Kadirkamanathan, *Member, IEEE*

Abstract

Index Terms

Dynamic spatio-temporal modelling, Integro-Difference Equation (IDE), ...

Non-linear Spatio-temporal modelling using the Integro-Difference Equation

I. Introduction

Non-linear spatio-temporal behaviour...

II. Neural Field Model

III. State space Representation of the Spatio-temporal Non-linear IDE

We define the pth order non-linear, stochastic, spatio-temporal IDE as

$$\rho_t(\mathbf{r}) = \int_\Omega w(\mathbf{r},\mathbf{r}')f(v_{t-p}(\mathbf{r}'))d\mathbf{r}' + \mathbf{e}_{t-p}(\mathbf{r}) \tag{1}$$

where $\rho_t(\mathbf{r})$ is a lag polynomial defined as

$$\rho_t(\mathbf{r}) = (1+\sum_{i=1}^{p}\varphi_i L_i)v_t\left(\mathbf{r}\right) \tag{2}$$

where $v_t(\mathbf{r})$ denotes the spatial field at time t and at spatial location $\mathbf{r}$, $f(.)$ is some nonlinear map, and $w(.)$ is the spatial kernel. The disturbance $\mathbf{e}_{t-p}(\mathbf{r})$ is a zero-mean normally distributed noise process, spatially coloured but temporally independent, with covariance

$$cov\left(e_t\left(\mathbf{r}\right), e_{t+\tau}\left(\mathbf{r}'\right)\right) = \gamma\left(\mathbf{r}-\mathbf{r}'\right)\delta(t-\tau) \tag{3}$$

where $\delta(.)$ is the Dirac-delta function. The spatial field is observed by

$$\mathbf{y}_t = \int_\Omega m\left(\mathbf{r}_{n_y}-\mathbf{r}'\right)v_t\left(\mathbf{r}'\right)d\mathbf{r}' + \boldsymbol{\varepsilon}_t, \tag{4}$$

where the observation vector $\mathbf{y}_t$, compiled at n_y spatial locations and at time t, is corrupted by an i.i.d normally distributed zero-mean white noise $\boldsymbol{\varepsilon}_t \sim \mathcal{N}\left(\mathbf{0},\mathbf{\Sigma}_\varepsilon\right)$ with $\mathbf{\Sigma}_{\boldsymbol{\varepsilon}} = \sigma_\varepsilon^2\mathbf{I}_{n_y}$. It is assumed here that $\mathbf{\Sigma}_\varepsilon$ and $\gamma\left(\mathbf{r}-\mathbf{r}'\right)$ are known. The output kernel $m(\mathbf{r}-\mathbf{r}')$ governs the sensor pick-up geometry and is defined by

$$m\left(\mathbf{r}-\mathbf{r}'\right) = \exp\left(-\frac{(\mathbf{r}-\mathbf{r}')^\top(\mathbf{r}-\mathbf{r}')}{\sigma_m^2}\right), \tag{5}$$

where σ_m sets the sensor width. The superscript $\top$ denotes the transpose operator. We proceed by writing the spatio-temporal IDE model in a parameterized state space form which allows the application of standard estimation techniques. We use a decomposition of the filed and the spatial kernel using a set of Gaussian basis functions

$$v_t\left(\mathbf{r}\right) \approx \boldsymbol{\phi}^\top\left(\mathbf{r}\right)\mathbf{x}_t \tag{6}$$

$$w\left(\mathbf{r},\mathbf{r}'\right) \approx \boldsymbol{\psi}^\top\left(\mathbf{r},\mathbf{r}'\right)\boldsymbol{\theta} \tag{7}$$

where $\boldsymbol{\phi}(\mathbf{r})$ and $\boldsymbol{\psi}(\mathbf{r})$ are vectors of Gaussian basis functions that are scaled by the state vector, $\mathbf{x}_t$ and the scaling parameter vector $\boldsymbol{\theta}$ respectively. Substituting (2) in (1) and using the above decomposition we have

$$\boldsymbol{\phi}^\top\left(\mathbf{r}\right)(1+\sum_{i=1}^{p}\varphi_i L_i)\mathbf{x}_t = \int_\Omega \boldsymbol{\psi}^\top\left(\mathbf{r},\mathbf{r}'\right)f(\boldsymbol{\phi}^\top\left(\mathbf{r}'\right)\mathbf{x}_{t-p})d\mathbf{r}'\boldsymbol{\theta} + \mathbf{e}_{t-p}(\mathbf{r}) \tag{8}$$

multiplying (8) by $\boldsymbol{\phi}(\mathbf{r})$ and integrate over the spatial domain Ω we get

$$\boldsymbol{\Gamma}(1+\sum_{i=1}^{p}\varphi_i L_i)\mathbf{x}_t = \int_\Omega \boldsymbol{\phi}(\mathbf{r})\int_\Omega \boldsymbol{\psi}^\top\left(\mathbf{r},\mathbf{r}'\right)f(\boldsymbol{\phi}^\top\left(\mathbf{r}'\right)\mathbf{x}_{t-p})d\mathbf{r}'\boldsymbol{\theta} + \int_\Omega \boldsymbol{\phi}(\mathbf{r})\mathbf{e}_{t-p}(\mathbf{r})d\mathbf{r} \tag{9}$$

where

$$\boldsymbol{\Gamma} \triangleq \int_\Omega \boldsymbol{\phi}\left(\mathbf{r}\right)\boldsymbol{\phi}^\top\left(\mathbf{r}\right)d\mathbf{r} \tag{10}$$

cross-multiplying (11) by $\boldsymbol{\Gamma}^{-1}$ gives

$$(1+\sum_{i=1}^{p}\varphi_i L_i)\mathbf{x}_t = \boldsymbol{\Gamma}^{-1}\int_\Omega \boldsymbol{\phi}(\mathbf{r})\int_\Omega \boldsymbol{\psi}^\top\left(\mathbf{r},\mathbf{r}'\right)f(\boldsymbol{\phi}^\top\left(\mathbf{r}'\right)\mathbf{x}_{t-p})d\mathbf{r}'\boldsymbol{\theta} + \boldsymbol{\Gamma}^{-1}\int_\Omega \boldsymbol{\phi}(\mathbf{r})\mathbf{e}_{t-p}(\mathbf{r})d\mathbf{r} \tag{11}$$

Here we consider the case where $p=1$, therefore we have

$$v_t(\mathbf{r}) + \varphi_1 v_{t-1}(\mathbf{r}) = \int_\Omega w(\mathbf{r},\mathbf{r}')f(v_{t-1}(\mathbf{r}'))d\mathbf{r}' + \mathbf{e}_{t-1}(\mathbf{r}) \tag{12}$$

IV. Estimation of the Non-linear Spatio-temporal IDE Model

A. *Unscented Kalman Filter*

B. *Unscented RTS Smoother*

C. *Parameter Estimation*

D. *Computational Complexity*

V. Example

VI. Conclusion

Appendix

Lemma 1:

Proof:

■