Model-Based Estimation of Intracortical Spatial Properties

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***Abstract*—**

I. Introduction

Discuss model-based estimation

II. Method

A. Stochastic Neural Field Model

The model relates the average number of action potentials $g(\mathbf{r},t)$ arriving at time t and position $\mathbf{r}$ to the local post-synaptic membrane voltage $v(\mathbf{r},t)$. The post-synaptic potentials generated at a neuronal population at location $\mathbf{r}$ by action potentials arriving from all other connected populations is described by

$$v\left(\mathbf{r},t\right) = \int_{-\infty}^{t} h\left(t-t'\right) g\left(\mathbf{r},t'\right)\, dt'. \tag{1}$$

The post-synaptic response kernel $h(t)$ is described by

$$h(t) = \eta(t)\exp\left(-\zeta t\right), \tag{2}$$

where $\zeta = \tau^{-1}$, τ is the synaptic time constant and $\eta(t)$ is the Heaviside step function. Non-local interactions between cortical populations at positions $\mathbf{r}$ and $\mathbf{r}'$ are described by

$$g\left(\mathbf{r},t\right) = \int_{\Omega} w\left(\mathbf{r},\mathbf{r}'\right) f\left(v\left(\mathbf{r}',t\right)\right)\, d\mathbf{r}', \tag{3}$$

where $f(\cdot)$ is the firing rate function, $w(\cdot)$ is the spatial connectivity kernel and Ω is the spatial domain representing a cortical sheet or surface. The firing rate of the presynaptic neurons is related to the post-synaptic membrane potential by the sigmoidal activation function

$$f\left(v\left(\mathbf{r}',t\right)\right) = \frac{1}{1+\exp\left(\varsigma\left(v_0 - v\left(\mathbf{r}',t\right)\right)\right)}. \tag{4}$$

The parameter v_0 describes the firing threshold of the neural populations and ς governs the slope of the sigmoid. By substituting equation 3 into equation 1 we get the spatiotemporal model

$$v\left(\mathbf{r},t\right) = \int_{-\infty}^{t} h\left(t-t'\right) \int_{\Omega} w\left(\mathbf{r},\mathbf{r}'\right) f\left(v\left(\mathbf{r}',t'\right)\right)\, d\mathbf{r}' dt'. \tag{5}$$

To obtain the standard integro-differential equation form of the model, we use the fact that the synaptic response kernel is a Green's function of a linear differential equation defined by the differential operator $\mathrm{D} = d/dt + \zeta$. A Green's function satisfies

$$\mathrm{D}h\left(t\right) = \delta\left(t\right), \tag{6}$$

where $\delta(t)$ is the Dirac-delta function. Applying the differential operator D to equation 1 gives

$$\begin{aligned} \mathrm{D}v\left(\mathbf{r},t\right) &= \mathrm{D}\left(h * g\right)\left(\mathbf{r},t\right) &(7)\\ &= \left(\mathrm{D}h * g\right)\left(\mathbf{r},t\right) &(8)\\ &= \left(\delta\left(t\right) * g\right)\left(\mathbf{r},t\right) &(9)\\ &= g\left(\mathbf{r},t\right) &(10) \end{aligned}$$

where $*$ denotes the convolution operator. This gives the standard form of the model

$$\frac{dv\left(\mathbf{r},t\right)}{dt} + \zeta v\left(\mathbf{r},t\right) = \int_{\Omega} w\left(\mathbf{r},\mathbf{r}'\right) f\left(v\left(\mathbf{r}',t\right)\right)\, d\mathbf{r}'. \tag{11}$$

To arrive at the integro-difference equation (IDE) form of the model, we discretize time using a first-order Euler method giving

$$v_{t+T_s}\left(\mathbf{r}\right) = \xi v_t\left(\mathbf{r}\right) + T_s \int_{\Omega} w\left(\mathbf{r},\mathbf{r}'\right) f\left(v_t\left(\mathbf{r}'\right)\right)\, d\mathbf{r}' + e_t\left(\mathbf{r}\right), \tag{12}$$

where T_s is the time step, $\xi = 1 - T_s\zeta$ and $e_t(\mathbf{r})$ is an $i.i.d.$ disturbance such that $e_t(\mathbf{r}) \sim \mathcal{GP}(\mathbf{0}, \gamma(\mathbf{r}-\mathbf{r}'))$. Here $\mathcal{GP}(\mathbf{0}, \gamma(\mathbf{r}-\mathbf{r}'))$ denotes a zero mean Gaussian process with spatial covariance function $\gamma(\mathbf{r}-\mathbf{r}')$ ([**?**]). The disturbance is added to account for model uncertainty and unmodeled inputs. To simplify the notation, the index of the future time sample, $t+T_s$, shall be referred to as $t+1$ throughout the rest of the paper.

The mapping between the membrane voltage and the electrophysiological data, denoted by $\mathbf{y}_t$, is modeled using the observation function that incorporates sensors with a spatial extent by

$$y_t(\mathbf{r}_n) = \int_{\Omega} m\left(\mathbf{r}_n - \mathbf{r}'\right) v_t\left(\mathbf{r}'\right)\, d\mathbf{r}' + \varepsilon_t(\mathbf{r}_n), \tag{13}$$

where $m(\mathbf{r}_n - \mathbf{r}')$ is the observation kernel, $\mathbf{r}_n$ defines the location of the electrodes in the field, $n = 0, ..., n_y - 1$ indexes the sensors and $\varepsilon_t(\mathbf{r}_n) \sim \mathcal{N}(0, \mathbf{\Sigma}_\varepsilon)$ denotes a multivariate normal distribution with mean zero and the covariance matrix $\mathbf{\Sigma}_\varepsilon = \sigma_\varepsilon^2 \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. Since we are considering intracranial measurements recorded directly from the surface of the cortex or within the brain, the lead field is not modeled by the observation equation.

B. Maybe: Basis Function Decomp

C. Estimation of Connectivity Kernel Support

The spatial relationship between consecutive observations is governed by the connectivity kernel. Therefore, the cross-correlation between consecutive observations is used to estimate the support of the connectivity kernel. The cross-correlation is defined as

$$R_{y_{t+1}y_t}(\boldsymbol{\tau}) = \int_\Omega y_{t+1}(\mathbf{r}) y_t(\mathbf{r} + \boldsymbol{\tau}) d\mathbf{r}, \tag{14}$$

where τ is the spatial shift. Now substituting equation 13 for $y_{t+1}(\mathbf{r})$ and assuming noise-free observations gives

$$R_{y_{t+1}y_t}(\boldsymbol{\tau}) = \int_\Omega \int_\Omega m(\mathbf{r} - \mathbf{r}') v_{t+1}(\mathbf{r}') d\mathbf{r}' y_t(\mathbf{r} + \boldsymbol{\tau}) d\mathbf{r}. \tag{15}$$

Next the equation 12 is substituted in for $v_{t+1}(\mathbf{r}')$ giving

$$\begin{aligned} R_{y_{t+1}y_t}(\boldsymbol{\tau}) = \int_\Omega \int_\Omega m(\mathbf{r} - \mathbf{r}') \left(\xi v_t(\mathbf{r}') + g_t(\mathbf{r}') \right. \\ \left. + \, e_t(\mathbf{r}')\right) d\mathbf{r}' y_t(\mathbf{r} + \boldsymbol{\tau}) d\mathbf{r}. \end{aligned} \tag{16}$$

The cross-correlation is simplified by recognizing that

$$\int_\Omega \int_\Omega m(\mathbf{r} - \mathbf{r}') \xi v_t(\mathbf{r}') d\mathbf{r}' y_t(\mathbf{r} + \boldsymbol{\tau}) d\mathbf{r} = \xi R_{y_t y_t}(\boldsymbol{\tau}) \tag{17}$$

$$\mathbf{E}\left[\int_\Omega \int_\Omega m(\mathbf{r} - \mathbf{r}') e_t(\mathbf{r}') d\mathbf{r}' y_t(\mathbf{r} + \boldsymbol{\tau}) d\mathbf{r}\right] = 0, \tag{18}$$

giving

$$\begin{aligned} R_{y_{t+1}y_t}(\boldsymbol{\tau}) &= \xi R_{y_t y_t}(\boldsymbol{\tau}) \\ &+ \int_\Omega \int_\Omega m(\mathbf{r} - \mathbf{r}') g_t(\mathbf{r}') d\mathbf{r}' y_t(\mathbf{r} + \boldsymbol{\tau}) d\mathbf{r}. \end{aligned} \tag{19}$$

Rearranging and substituting in equation 3 for $g_t(\mathbf{r}')$ gives

$$\begin{aligned} R_{y_{t+1}y_t}(\boldsymbol{\tau}) - \xi R_{y_t y_t}(\boldsymbol{\tau}) = T_s \int_\Omega \int_\Omega m(\mathbf{r} - \mathbf{r}') \\ \times \int_\Omega w(\mathbf{r}' - \mathbf{r}'') f\left(v_t(\mathbf{r}'')\right) d\mathbf{r}'' d\mathbf{r}' y_t(\mathbf{r} + \boldsymbol{\tau}) d\mathbf{r}. \end{aligned} \tag{20}$$

Using the commutativity property of convolution, the order can be rearranged to

$$\begin{aligned} R_{y_{t+1}y_t}(\boldsymbol{\tau}) - \xi R_{y_t y_t}(\boldsymbol{\tau}) = T_s \int_\Omega \int_\Omega w(\mathbf{r} - \mathbf{r}') \\ \times \int_\Omega m(\mathbf{r}' - \mathbf{r}'') f\left(v_t(\mathbf{r}'')\right) d\mathbf{r}'' d\mathbf{r}' y_t(\mathbf{r} + \boldsymbol{\tau}) d\mathbf{r}. \end{aligned} \tag{21}$$

Next, the nonlinear function $f(\cdot)$ is approximated by the simple linear relationship

$$f\left(v_t(\mathbf{r})\right) \approx \varsigma v_t(\mathbf{r}) \tag{22}$$

giving

$$\begin{aligned} R_{y_{t+1}y_t}(\boldsymbol{\tau}) - \xi R_{y_t y_t}(\boldsymbol{\tau}) = \varsigma T_s \int_\Omega \int_\Omega w(\mathbf{r} - \mathbf{r}') \\ \times \int_\Omega m(\mathbf{r}' - \mathbf{r}'') v_t(\mathbf{r}'') d\mathbf{r}'' d\mathbf{r}' y_t(\mathbf{r} + \boldsymbol{\tau}) d\mathbf{r}, \end{aligned} \tag{23}$$

which simplifies to

$$\begin{aligned} R_{y_{t+1}y_t}(\boldsymbol{\tau}) - \xi R_{y_t y_t}(\boldsymbol{\tau}) = \varsigma T_s \int_\Omega \int_\Omega w(\mathbf{r} - \mathbf{r}') y_t(\mathbf{r}') d\mathbf{r}' \\ \times y_t(\mathbf{r} + \boldsymbol{\tau}) d\mathbf{r} \end{aligned} \tag{24}$$

A property of cross-correlation and convolution is $(a * b) \star c = a(-) * (b \star c)$, so we can write

$$\begin{aligned} R_{y_{t+1}y_t}(\boldsymbol{\tau}) - \xi R_{y_t y_t}(\boldsymbol{\tau}) &= \varsigma T_s \int_\Omega w(\boldsymbol{\tau} - \boldsymbol{\tau}') \\ &\times \int_\Omega y_t(\mathbf{r}) y_t(\mathbf{r} + \boldsymbol{\tau}') d\mathbf{r} d\boldsymbol{\tau}' \qquad (25) \\ &= \varsigma T_s \int_\Omega w(\boldsymbol{\tau} - \boldsymbol{\tau}') R_{y_t y_t}(\boldsymbol{\tau}') d\boldsymbol{\tau}'. \qquad (26) \end{aligned}$$

The solution of the above equation for the connectivity kernel is a deconvolution. This can be approached from a number of different standpoints. The simplest solution is to use the convolution theorem and find the solution by

$$w(\boldsymbol{\tau}) = \frac{1}{\varsigma T_s} \mathcal{F}^{-1} \left\{ \frac{\mathcal{F}\left(R_{y_{t+1}y_t}(\boldsymbol{\tau}) - \xi R_{y_t y_t}(\boldsymbol{\tau})\right)}{\mathcal{F}\left(R_{y_t y_t}(\boldsymbol{\tau})\right)} \right\}. \tag{27}$$

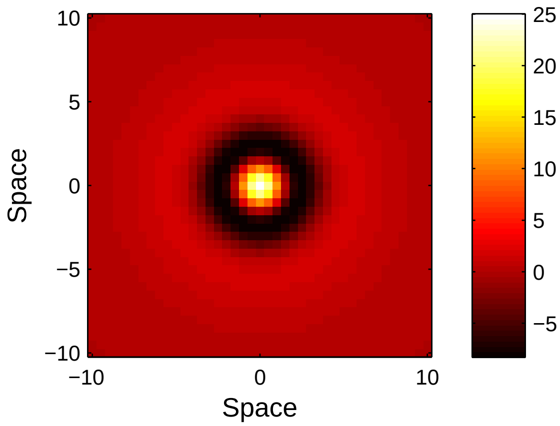
Alternatively, the convolution can be written as a system of linear equations by forming the convolution (Toeplitz) matrix. The solution of the convolution equation for $w(\cdot)$ can then be found by either inverting the convolution matrix or directly solving the system of equations. Directly solving the system is the most numerically stable and less computationally demanding then inverting the convolution matrix.

III. Results

A. Linear Neural Field Model

Show the exact estimation of the kernel.

B. Nonlinear Neural Field Model



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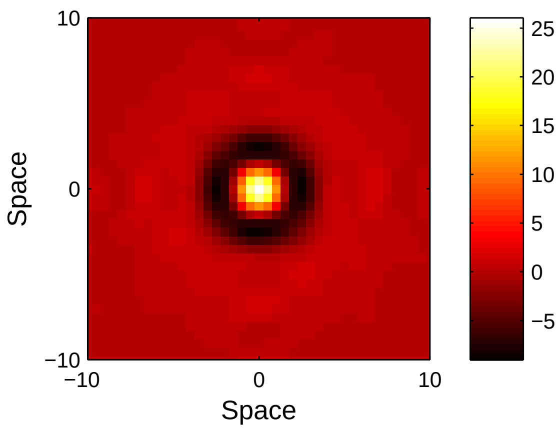

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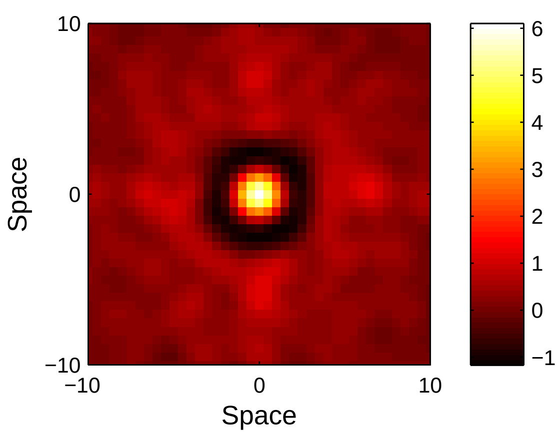
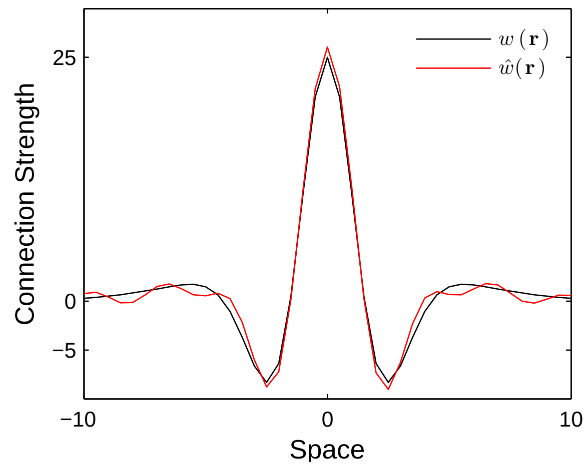


(c)

Fig. 2. a) The actual kernel. b) Estimate of kernel when generating data using the linear model. b) Estimate of the kernel when using the nonlinear model.



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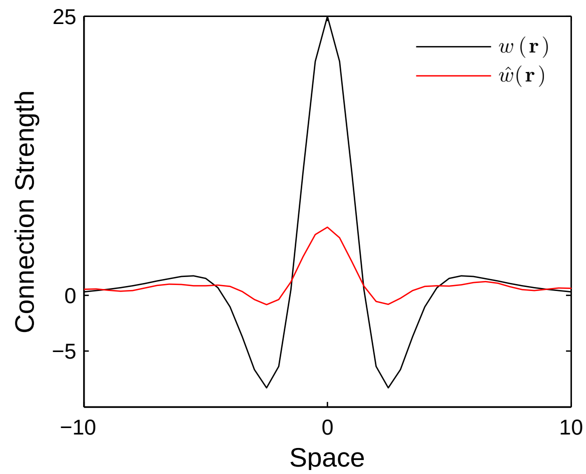


(b)

Fig. 3. a) From the linear model. b) from the nonlinear model.

IV. Discussion

V. Conclusion

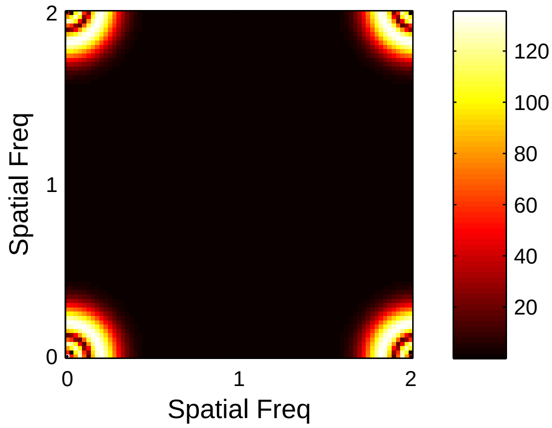
The conclusion goes here.

Acknowledgment

The authors would like to thank...

References

[1] H. Kopka and P. W. Daly, *A Guide to LaTeX*, 3rd ed. Harlow, England: Addison-Wesley, 1999.



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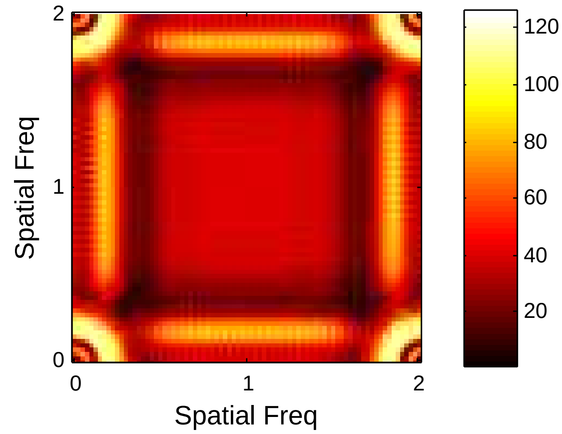

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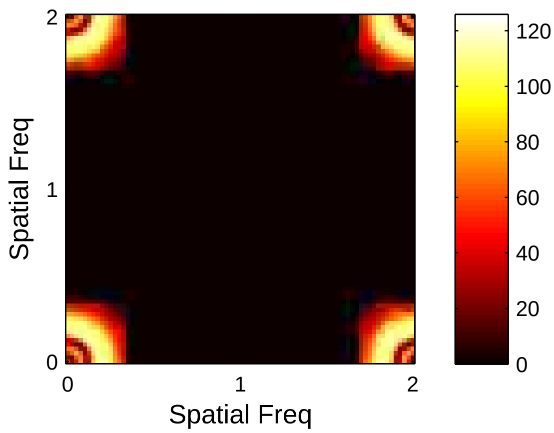

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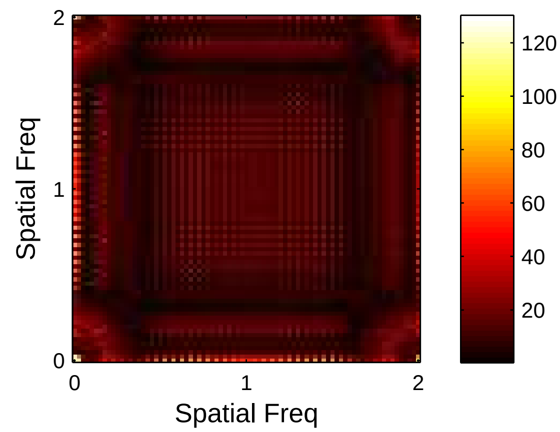

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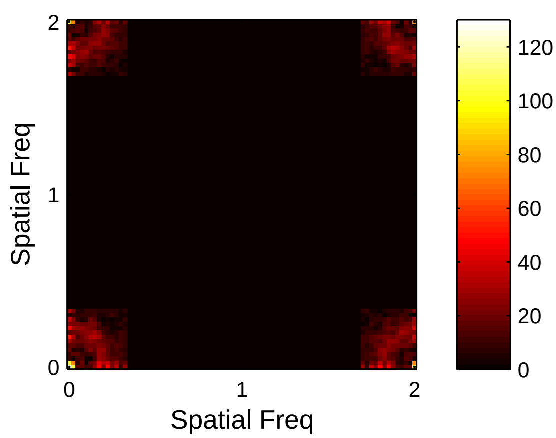
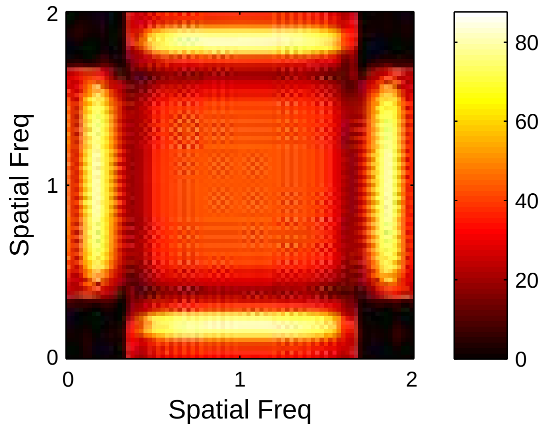


(e)

Fig. 1. a) FFT of kernel. b) FFT of kernel estimate when generating data with linear model showing numerical issues. c) Thresholded FFT of kernel estimate when generating data with linear model. d) FFT of kernel estimate when generating data with nonlinear model. e) Thresholded FFT of kernel estimate when generating data with nonlinear linear model.



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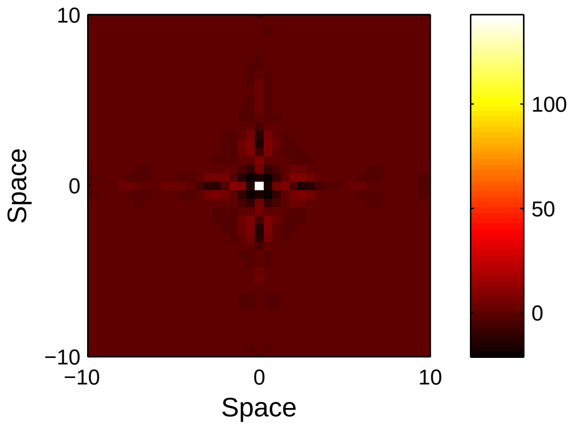

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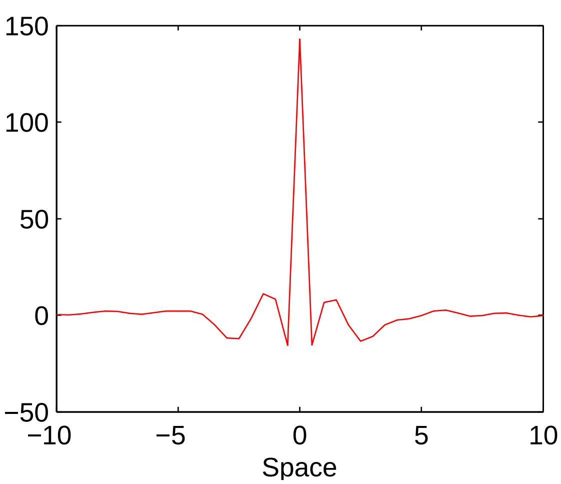

(c)

Fig. 4. This part is for debugging. a) Removing the kernel from the spectrum of the estimate. b) Kernel left overs. c) Cross section of b.