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Open boundary conditions for wave propagation problems on unbounded domains

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Graz, 12. September 2007

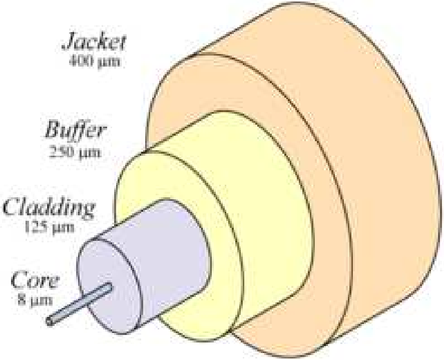
Wave propagation in “unbounded domains” – applications:

- water waves:

$\longrightarrow$ shallow water equation, t-dependent wave equation

- open / non-reflecting boundary conditions needed

- fiber-optic cables:

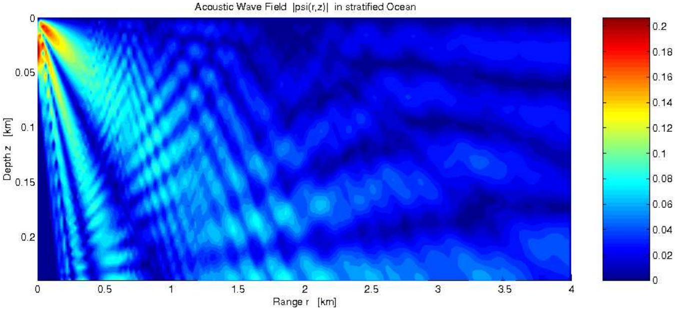


- for core diameter $< 10\mu$m :
  Maxwell equations, 1-way wave equation (Schrödinger/parabolic type)
- wave propagation mainly in core $\Rightarrow$ limit the computational domain with open BC

- underwater acoustics:

ocean surface:

sound source→



sea bottom:

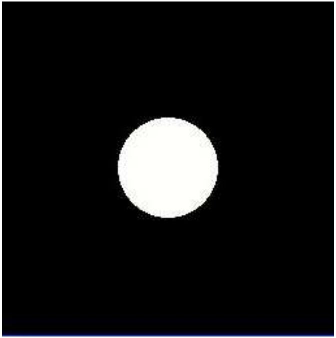
→ reflection / transmission

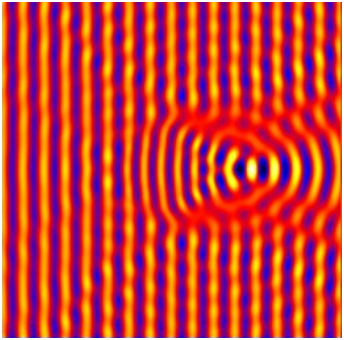
→ open BC to artificially limit computational domain

- original model: 2D–Helmholtz equation (time-harmonic solution)
- simplified model: 1–way wave equation

Scattering problems:

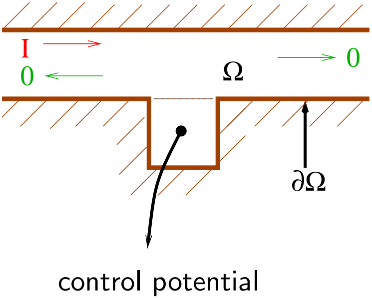
- electromagnetic wave scattered by “soft” ball:

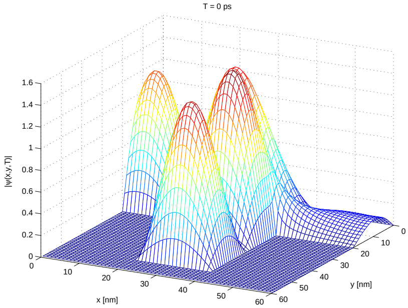




$\longrightarrow$ Helmholtz equation with incoming plane wave (from left)

- T–shaped quantum waveguide for electron flow:





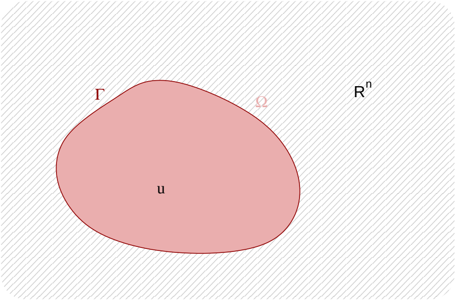
$\rightarrow$ Schrödinger equation with incoming plane wave (from left)

- open / non-reflecting boundary condition needed

Outline:

1. non-reflecting boundary conditions
    - approaches, derivation
2. transparent BCs for Schrödinger equation
    - rectangular geometry
    - fast algorithms
    - circular geometry
3. perfectly matched layer
    - Klein Gordon equation
4. perspectives

(1) Non-reflecting boundary conditions



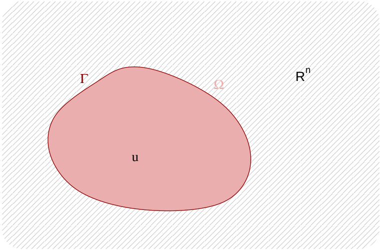
- original PDE for $u(x, t)$ (stationary or transient) on unbounded domain (e.g. $\mathbb{R}^n$)
- for numerics: introduce artificial boundary Γ
  $\longrightarrow$ encloses finite domain Ω
- derive non-reflecting boundary condition on/around Γ

Necessary features of non-reflecting BCs:

1. yield a well-posed BVP or IBVP on Ω
2. no (or little) spurious wave reflections due to BC — compared to whole space problem
3. allow for efficient numerical implementation

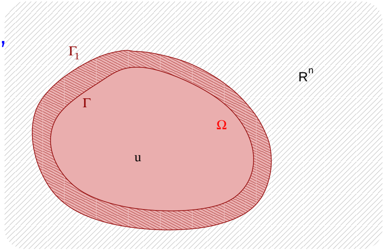
3 basic approaches for non-reflecting BCs:

1. PDE-based: exact transparent BC
   (DtN map: non-local operator in x, t)
   $\longrightarrow$ tricky implementation, high accuracy



2. PDE-based: local absorbing BC
   (low order differential operator)
   $\longrightarrow$ simple implementation, low accuracy
   [Engquist-Majda, 1977] for wave equ.

3. material-based: 'perfectly matched layer' (PML)
   $\longrightarrow$ add dissipative media around Ω
   $\Rightarrow$ damp outgoing waves
   [Bérenger, 1994] for Maxwell



Derivation of exact / transparent BCs:

- simple idea:
  - factorize wave-like PDE into incoming/outgoing modes
  - add BC $\Rightarrow$ outgoing waves cross Γ freely
- difficult realization:
  - exact factorization only for linear PDEs on Ω with 'regular shape' — else: approximate factorization
  - BC is pseudo-differential (non-local in x, t)
  - tricky & expensive discretization

Transparent boundary condition for 1D wave equation:

- computational domain $\Omega = (0, L)$
- assume: constant coefficients on left/right exterior domains Ω^c
- assume: initial condition supported in Ω
- factorization:

$$0 = u_{tt} - u_{xx} = \underbrace{(\partial_t - \partial_x)}_{\leftarrow} \underbrace{(\partial_t + \partial_x)}_{\rightarrow} u$$

$$\left. \begin{array}{rcl} u_t = u_x & , & x = 0 \\ u_t = -u_x & , & x = L \end{array} \right\} \text{“perfect”} = \text{transparent BCs}$$

$\Rightarrow$ Waves leave the domain Ω without being reflected back.

Transparent boundary conditions for 2D wave equation:

- computational domain $\Omega = \{(x, y),\ x > 0\}$
- factorization:

$$u_{xx} - (u_{tt} - u_{yy}) = \underbrace{\left(\partial_x - \sqrt{\partial_t^2 - \partial_y^2}\right)}_{\leftarrow} \underbrace{\left(\partial_x + \sqrt{\partial_t^2 - \partial_y^2}\right)}_{\rightarrow} u = 0$$

<u>TBC at $x = 0$:</u>

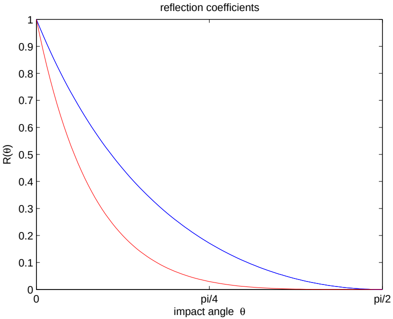
$$u_x - \underbrace{\sqrt{\partial_t^2 - \partial_y^2}}_{\Psi\text{DO; nonlocal in } y,t} u = 0 \quad \text{or } \hat{u}_x - i\sqrt{\omega^2 - \eta^2}\ \underbrace{\hat{u}}_{y-t-\text{Fouriertransform}} = 0$$

Absorbing boundary conditions for 2D wave equation:

- rational approximation of ΨDO–symbol $\sqrt{\omega^2 - \eta^2}$ (Taylor, Padé) :
- high frequency approximation, i.e. $\left|\frac{\eta}{\omega}\right| \ll 1$
  $\Rightarrow$ hierarchy of local ABCs at $x = 0$ [Engquist-Majda 1977] :

$$u_x - u_t = 0, \qquad u_{xt} - u_{tt} + \frac{1}{2} u_{yy} = 0 \qquad \longrightarrow \text{ well-posed IBVP}$$

- reflection coefficient depends on impact angle at $\partial\Omega$
- $|R(\theta)| < 1 \Rightarrow$ "absorbing BC"



2^{nd} order Taylor approximation $\longrightarrow$ strongly ill-posed IBVP

(2) Transparent boundary cond. for Schrödinger equation:

- Example: 1D-Schrödinger equation:

wavefunction: $\psi(x,t) \in \mathbb{C}$

$$\begin{cases} i\psi_t = -\psi_{xx} + V(x,t)\psi; \quad x \in \mathbb{R}, t > 0 \\ \mathrm{supp}\psi^0 \subset (0,L) \quad \ldots \quad \text{computational domain} \\ V(x,t) = V_l(t), x \leq 0; \quad V(x,t) = V_r(t), x \geq L \end{cases}$$

Goal: reproduce $\psi_{[0,L]}$ with artificial BCs at $x = 0,\ L$

1. exterior potential $V_I = const = 0 \rightarrow$ factorization:

$$0 = \psi_{xx} - (-i)\psi_t = \underbrace{\left(\partial_x - \sqrt{-i}\sqrt{\partial_t}\right)}_{\leftarrow}\underbrace{\left(\partial_x + \sqrt{-i}\sqrt{\partial_t}\right)}_{\rightarrow}\psi$$

TBC at $x = 0$: $\quad \psi_x(0,t) = \sqrt{-i\partial_t}\psi = \dfrac{e^{-\frac{\pi}{4}i}}{\sqrt{\pi}}\dfrac{d}{dt}\displaystyle\int_0^t \dfrac{\psi(0,\tau)}{\sqrt{t-\tau}}d\tau$

- BC non-local in t (memory-type $\rightarrow$ store $\psi(t)\big|_\Gamma$)
  [Papadakis '82, Baskakov-Popov '91, Hellums-Frensley '94]

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2. exterior potential $V_l = V_l(t) \neq 0 \Rightarrow$ <u>TBC at $x = 0$:</u>

$$\psi_x(0,t)\, e^{i\int_0^t V_l(\tau)d\tau} = \frac{e^{-\frac{\pi}{4}i}}{\sqrt{\pi}}\frac{d}{dt}\int_0^t \frac{\psi(0,\tau)}{\sqrt{t-\tau}}\, e^{i\int_0^\tau V_l(s)ds}\, d\tau$$

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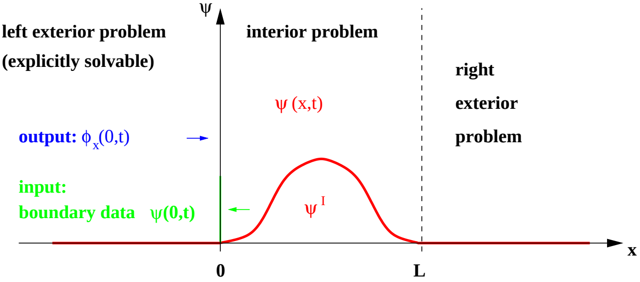
3. scattering problem with incoming wave $\psi^{in}(0,t)$:

$$\underline{\text{inhomogeneous TBC :}} \quad \left(\partial_x - \sqrt{-i\partial_t}\right)\left(\psi(0,t) - \psi^{in}(0,t)\right) = 0$$

rigorous derivation of transparent boundary conditions



- elimination of left exterior problem —
  analytically (via Fourier/Laplace transforms) or numerically
  $\Rightarrow$ left TBC $\psi_x(0,t) = (T\,\psi)(0,t)$ from :

$$\begin{cases} i\phi_t &= -\phi_{xx}, \quad x<0 \\ \phi^0(x) &= 0 \\ \phi(0,t) &= \psi(0,t), \quad \phi(-\infty,t)=0 \end{cases}$$

$\Rightarrow\ (T\,\psi)(t) = \phi_x(0,t), \qquad T$ ... Dirichlet–to–Neumann (DtN) operator

Schrödinger boundary value problem

$$\begin{cases} i\psi_t = -\psi_{xx} + V(x,t)\psi; \quad x \in (0,L),\ t > 0 \\ \text{supp}\psi^0 \subset (0,L) \\ \Big(V(x,t) = V_l(t), x \le 0; \quad V(x,t) = V_r(t), x \ge L\Big) \\ \text{TBC at } x = 0,\ L \end{cases} \tag{1}$$

Theorem (DiMenza, 1995)

Let $\psi^0 \in H^1(0,L) \Rightarrow$ unique solution of (1) is whole-space solution.

- [Ben Abdallah-Méhats-Pinaud '04] : extension to 2D, 3D + scattering problem for $\psi^0 \in H^2(\Omega)$

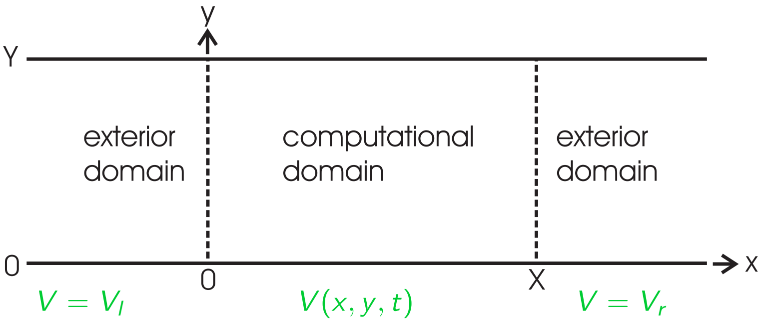
2D Schrödinger equation – waveguide geometry

- factorization of $\qquad \psi_{xx} = -\psi_{yy} - i\psi_t + V\psi$

  $\Rightarrow$ TBC at $x = 0$ is non-local (pseudo-differential) in t and y :

$$\psi_x(0, y, t) = \sqrt{-\partial_{yy} - i\partial_t + V}\,\psi$$

- BC for waveguides: $\psi(x, 0, t) = \psi(x, Y, t) = 0$:



2D Schrödinger equation – waveguide geometry

- Fourier series in y: $\psi(x,y,t) = \sum_{m\in\mathbb{N}} \hat{\psi}^m(x,t) \sin \frac{m\pi y}{Y}$
- $V = const = V_l$ in left exterior domain $\Rightarrow$ y-modes are decoupled
- TBC is local in y for each sine-mode [AA-Ehrhardt-Sofronov '03] :

$$
\begin{aligned}
\hat{\psi}^m_x(0,t) &= \sqrt{-i\partial_t + V^m}\,\hat{\psi}^m \\
&= \sqrt{\frac{-i}{\pi}}\, e^{-iV^m t} \frac{d}{dt} \int_0^t \frac{\hat{\psi}^m(0,\tau) e^{iV^m \tau}}{\sqrt{t-\tau}} d\tau \\
V^m &= V_l + \left(\frac{m\pi}{Y}\right)^2, \qquad m \in \mathbb{N}
\end{aligned}
$$

Discretization of analytic TBC

- Dangers :
  - may destroy the (unconditional) stability of the whole-space scheme
  - may introduce numerical reflections at the boundary

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- one option: "discrete transparent boundary conditions"

  STRATEGY :

  - discretize left exterior problem ($j \leq 0$)
  - derivation of the discrete TBC (for discrete scheme)
    instead of: discretization of the analytic TBC
  - for many linear equ, many discretization schemes

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- alternatives :
  - semidiscrete TBCs [Alonso-Mallo-Reguera '03], [Lubich-Schädle '02]
  - TBCs from 'pole condition' [Ruprecht-Schädle-Schmidt-Zschiedrich '07]
  - ...

discrete transparent boundary conditions (DTBCs)

- Ex: Crank-Nicolson finite difference-scheme for free Schrödinger equ: $\psi_j^n \approx \psi(j\Delta x, n\Delta t)$, unconditionally stable: $\|\psi^n\|_2 = \|\psi^0\|_2$

$$i\frac{\psi_j^{n+1} - \psi_j^n}{\Delta t} = -\frac{1}{2}\frac{\psi_{j-1}^{n+1} - 2\psi_j^{n+1} + \psi_{j+1}^{n+1}}{\Delta x^2} - \frac{1}{2}\frac{\psi_{j-1}^n - 2\psi_j^n + \psi_{j+1}^n}{\Delta x^2}$$

- Z-transformed exterior problem ($\psi_j^0 = 0,\ j \le 0$) with $\mathcal{Z}\{\psi_j^n\} = \hat{\psi}_j(z) = \sum_{n=0}^{\infty} \psi_j^n\, z^{-n},\ z \in \mathbb{C}$:

$$\hat{\psi}_{j-1} - 2\Big(1 - i\frac{\Delta x^2}{\Delta t}\,\frac{z-1}{z+1}\Big)\,\hat{\psi}_j + \hat{\psi}_{j+1} = 0, \quad j \le 1$$

- choose decaying solution as $j \to -\infty$: $\hat{\psi}_j(z) = \alpha(z)^j$
$\Rightarrow$ transformed DTBC :

$$\hat{\psi}_1(z) = \alpha(z)\hat{\psi}_0(z), \quad |\alpha(z)| > 1$$

discrete transparent boundary conditions

transformed DTBC : $\hat{\psi}_1(z) = \alpha(z)\hat{\psi}_0(z)$

- inverse Z-transform (explicit or numerical): $(s_n) := \mathcal{Z}^{-1}\{\frac{z+1}{z}\alpha(z)\}$
- discrete TBC: $\psi_1^n = \sum_{k=1}^n \psi_0^k s_{n-k} - \psi_1^{n-1}$ ... discrete convolution
- 3-point recursion for (s_n)
- $s_n = O(n^{-3/2})$

Theorem (AA '98)

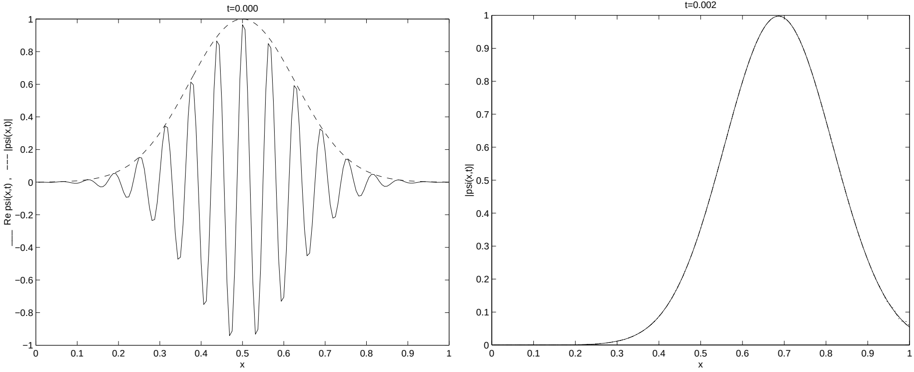
CN - FD scheme for Schrödinger equation with discrete TBC is unconditionally stable:

$$\|\psi^n\|_2^2 := \Delta x \sum_{j=1}^{J} |\psi_j^n|^2 \leq \|\psi^0\|_2^2, \quad n \geq 1$$

$\longrightarrow$ no numerical reflections

$\longrightarrow$ same numerical effort as 'conventional' discretizations of $\psi_x(0,t) = C\frac{d}{dt}\int_0^t \frac{\psi(0,\tau)}{\sqrt{t-\tau}}d\tau$

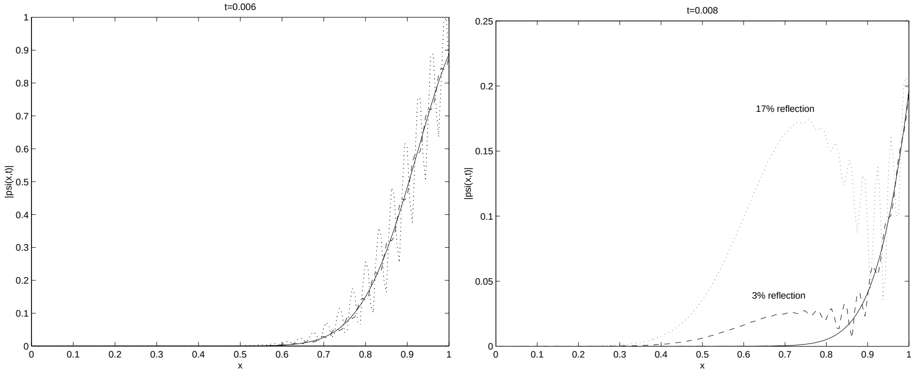
free Schrödinger equation ($V = 0$) with TBC



Gaussian beam $|\psi(x,t)|$, $x \in \mathbb{R}$; right-traveling

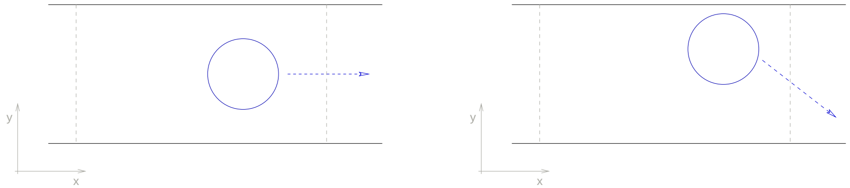
$\Delta x = \frac{1}{160}$, $\Delta t = 2 \cdot 10^{-5}$ (rather coarse discretization)

free Schrödinger equation ($V = 0$) with TBC



$\cdots$	[B. Mayfield 1989]	}	discretization of
- - -	[Baskakov & Popov 1991]		the analytic TBC
——	[AA 1998]		discrete TBC

Gaussian beam in 2D waveguide:



right traveling Gaussian beam on $\Omega = (0,1)^2$

aligned with waveguide   at 45° with waveguide

- $\Delta x = \Delta y = 1/120$, $\Delta t = 5 \cdot 10^{-5}$
- TBC local in y for each sine-mode
  $\Rightarrow$ implementation of TBC in y-Fourier space faster [Schulte-AA '07]

Gaussian wave aligned with waveguide:

Gaussian wave at 45° with waveguide:

- advantage of discrete TBCs:
  - absolutely reflection–free
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- disadvantage (for long–time calculations):
  - compute boundary–convolutions: $O(N^2)$–effort;
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- goal:
  approximate convolution kernel $(s_n) \longrightarrow O(N)$–effort

approximate TBCs – fast evaluation of convolutions

- if $s_k = q^{-k}$: trivial update of convolutions:

$$\sum_{k=0}^{n} u_k\, s_{n-k} = \frac{1}{q}\left(\sum_{k=0}^{n-1} u_k\, s_{n-1-k}\right) + u_n\, s_0$$

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- Idea: approximation of s_n by sum of exponentials; $s_n = O(n^{-3/2})$ — discrete analogue of [Grote-Keller '95]:

$$s_n \approx \tilde{s}_n = \sum_{l=1}^{L} b_l q_l^{-n}, \quad n \in \mathbb{N},\ |q_l| > 1,\ L \sim 10 - 20$$

$$\mathcal{Z}\{\tilde{s}_n\} = s_0 + \sum_{l=1}^{L} \frac{b_l}{q_l z - 1}, \quad |z| \geq 1.$$

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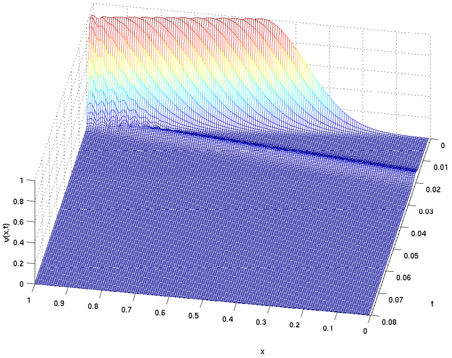
- b_l, q_l from Padé approximation of

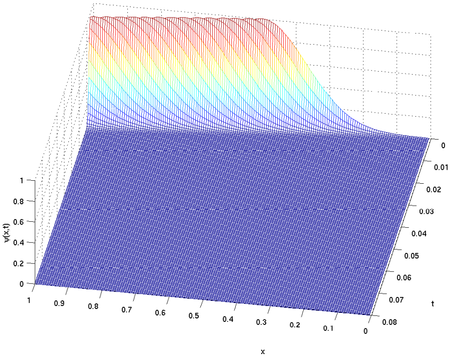
$$f(x) = \sum_{n=0}^{2L-1} s_n x^n, \quad x = \frac{1}{z}$$

- convolution update: linear effort, constant memory requirement

approximate TBCs – 1D Schrödinger equation ($V = 0$)



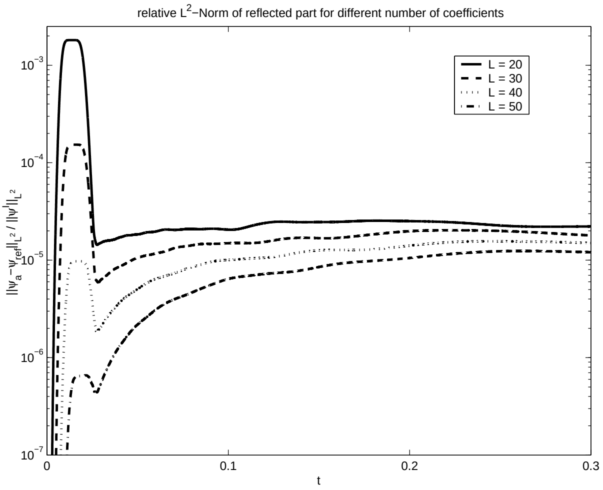
solution with $L = 10$



solution with $L = 20$

[AA-Ehrhardt-Sofronov, 2003]

approximate TBCs: error decreases with L



relative $L^2(0,1)$-error with approximate TBCs, up to 15.000 time steps
[AA-Ehrhardt-Sofronov, 2003]

approximate TBCs: error estimate

(ψ^n) ... solution with exact TBC
(ψ_L^n) ... solution with approximate TBC

convolution coefficients:

$$(s_n) = \mathcal{Z}^{-1}\left(\frac{z+1}{z}\alpha(z)\right), \qquad (s_n^L) = \mathcal{Z}^{-1}\left(\frac{z+1}{z}\alpha_L(z)\right) \quad \text{... } L \text{ exp. terms}$$

Theorem (AA-Ehrhardt-Sofronov, 2003)

$$\|\psi^n - \psi_L^n\|_{L^2(0,1)} \leq C(n)\,\|\psi^0\|_{H^1_{discr}} \left\| \frac{1}{\alpha(e^{i\varphi})} - \frac{1}{\alpha_L(e^{i\varphi})} \right\|_{L^\infty(0,2\pi)}, \qquad n \in \mathbb{N}_0$$

Example for $\Delta x = 1/160$, $\Delta t = 2 \cdot 10^{-5}$:

$L =$	5	10	15	20
error $\|\frac{1}{\alpha} - \frac{1}{\alpha_L}\|_\infty$	1.8247e-04	1.2808e-07	6.4439e-11	2.962e-14

approximate TBCs: stability

TBC with approximate convolution coefficients:

$$(s_n^L) = \mathcal{Z}^{-1}\left(\frac{z+1}{z}\alpha_L(z)\right) \quad \ldots\ L \text{ exp. terms}$$

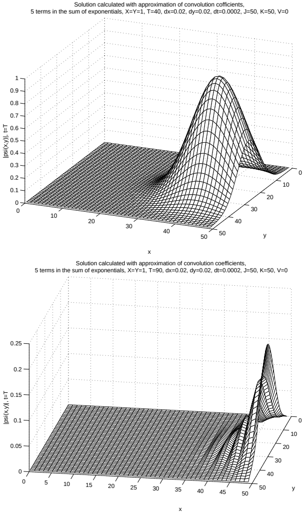
$\longrightarrow$ stability not trivial !

Lemma (AA-Ehrhardt-Sofronov, 2003)

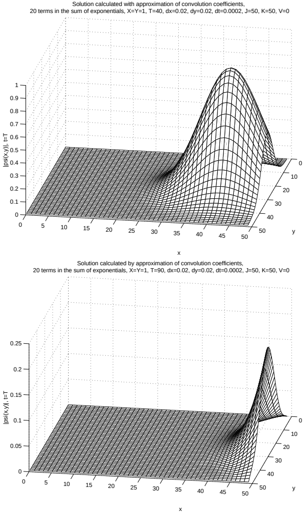
If $\Im\,\alpha_L(\beta e^{i\varphi}) \leq 0 \quad \forall\, 0 \leq \varphi \leq 2\pi$ *and* α_L *analytic for* $|z| > \beta$

$$\Rightarrow \|\psi^n\|_2 \leq \|\psi^0\|_2 \beta^n, \quad n \in \mathbb{N}$$

approximate TBCs – 2D Schrödinger equation ($V = 0$)



solution with $L = 5$



solution with $L = 20$

TBC at $x = 50$; right-traveling Gaussian beam [Schulte-AA, 2007]

2D Schrödinger equation: discrete TBC for circular domain

in polar coordinates, $V = 0$:

$$i\psi_t = -\frac{1}{2}\left(\frac{1}{r}(r\psi_r)_r + \frac{1}{r^2}\psi_{\theta\theta}\right)$$

- uniform radial off-set grid $r_j = (j + \frac{1}{2})\Delta r$, $j \in \mathbb{N}_0$; uniform angular grid

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  $\Rightarrow$ finite difference equation (variable coefficients!) for each mode $\hat{\psi}^m$:

$$a_j\hat{\psi}_{j-1}(z) + b_j^m(z)\hat{\psi}_j(z) + c_j\hat{\psi}_{j+1}(z) = 0, \; j \geq J-1 \qquad (2)$$

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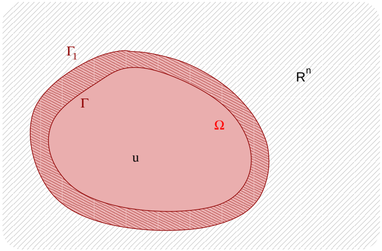
- for stability: solve (2) numerically from $j =$ "∞" back to $j = J$;
  initial condition at $j =$ "∞": 1D–TBC
- numerical / discrete inverse Z-transformation of $\frac{z+1}{z}\alpha_{J+1}(z)$, $z \in \mathbb{C}$
  $\rightarrow$ convolution coefficients (s_n) for each mode m

Gaussian wave in circle, traveling south-east

symmetric polar grid: $\Delta r = \frac{1}{128}$, $\Delta\theta = \frac{2\pi}{128}$; $\Delta t = \frac{1}{128}$
[AA-Ehrhardt-Schulte-Sofronov, 2007]

(3) perfectly matched layer (PML)

- surround computational domain Ω with layer of artificial damping medium
- interface Γ : zero reflection ($\forall$ angles)
  $\rightarrow$ "perfectly matched layer"
- attenuate all outgoing waves
- waves reflected from Γ_1 :
  amplitude very small on Γ
- Bérenger-PML for Maxwell :
  numerically great;
  weakly well-posed [Bécache-Joly, 2001]



PML for 1D stationary equation: derivation

- solution u ; computational domain $\Omega = \mathbb{R}^+$
- define "modified" (complex) solution on $\mathbb{R}$, strong decay as $x \to -\infty$:

$$u^m(x) := \begin{cases} u(x) e^{f(x)}, & x < 0 \\ u(x), & x \geq 0 \end{cases}$$

damping factor: $\Re f(x) \nearrow$, $f(0) = 0$

- find equation for $u^m(x)$
- truncate layer at $x = -a < 0$
- construction not unique

PML for linear 1D Klein-Gordon equation: derivation

$$\text{Example:} \qquad u_{tt} = u_{xx} - u_t\,, \quad x \in \mathbb{R},\ t > 0$$

- reformulate as hyperbolic system,
  modified solution for each t–Laplace mode
  $\Rightarrow$ modified hyperbolic PML–system :

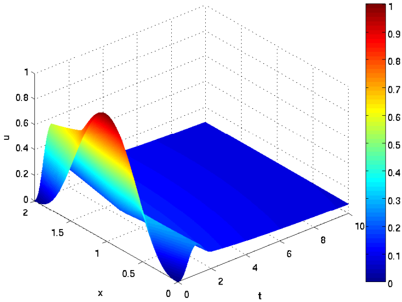
$$\begin{cases} u_t^m = w^m\,, & x \in \mathbb{R} \\ v_t^m = w_x^m - \sigma(x) p^m & \\ w_t^m = v_x^m - u_t^m - \sigma(x) q^m & \\ p_t^m = w_x^m - (\alpha(x) + \sigma(x)) p^m & \\ q_t^m = v_x^m - (\alpha(x) + \sigma(x)) q^m & \end{cases}$$

- α, σ ... damping parameters on Ω^c (related to f)
  p, q ... auxiliary PML variables
- well-posed; $u(x) = u^m(x)$ on Ω by construction (for ∞ layer)

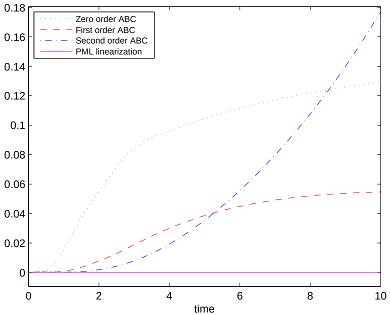
PML for linear 1D Klein-Gordon equation: better than ABC

$$\begin{cases} u_{tt} = u_{xx} - u_t \,, & x \in \mathbb{R},\ t > 0 \\ \text{supp } u^0,\ u_t^0 \subset \Omega = (0,2) \end{cases}$$

2 PML-layers; thickness $= 0.1$



exact solution



relative $L^2(0,2)$–errors

- local ABC of [Engquist-Majda, 1979]
- max. PML–error on $t \in [0,10]$: $4 \cdot 10^{-8}$ [AA-Amro-Zheng, 2007]

PML for nonlinear 1D Klein-Gordon equation: derivation

$$\text{Example:} \qquad u_{tt} = u_{xx} - \varphi(u, u_t, u_x)$$

- first – linear equation:

$$u_{tt} = u_{xx} - (a\,u + b\,u_t + c\,u_x)$$

- PML–system for linear equation :

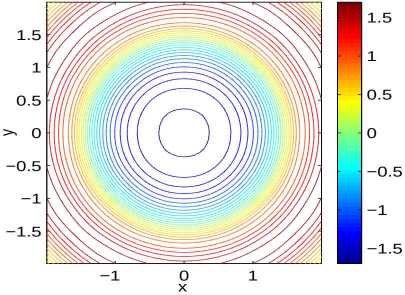
$$\begin{cases} u_t = w \\ v_t = w_x - \sigma(x)p \\ w_t = v_x - (a\,u + b\,u_t + c\,u_x) - \sigma(x)q \\ p_t = w_x - (\alpha(x) + \sigma(x))p \\ q_t = v_x - (\alpha(x) + \sigma(x))q \end{cases} \tag{3}$$

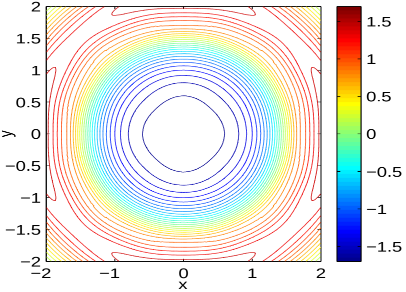
- "PML-linearization":
  replace $(a\,u + b\,u_t + c\,u_x)$ by $\varphi(u, u_t, u_x)$ in (3)

nonlinear Klein-Gordon equation with PML: circular wave

$$u_{tt} = \Delta u - u^3, \quad x, y \in \mathbb{R} \tag{4}$$

PML with layer thickness 0.8:





$u(x, y, t = 3.5)$: "PML linearization", (i.e. nonlinear PML-system)

direct linearization of (4) about $u \equiv 0$ in PML-layer

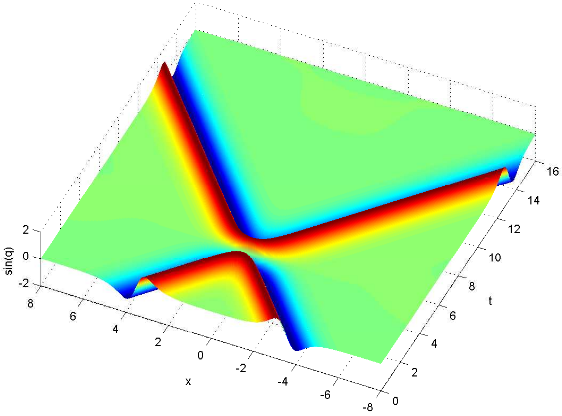
[AA-Amro-Zheng, 2007], well-posedness ?

(4) Perspectives / first solution steps:

- inhomogeneous / anisotropic exterior problems :
  - PML sometimes unstable [Bécache-Fauqueux-Joly, 2003]
  - [Sofronov-Zaitsev, 2006] : TBC for anisotropic elastic waves, numerical construction of DtN-map in t-Laplace domain
- nonlinear models :
  - [Szeftel, 2006] : ABCs by pseudo-/paradifferential calculus (special linearization) for many nonlinear wave equations
  - [Zheng-Amro, 2007] : PML for 2D Euler (compressible) — well-posedness ?

Perspectives / transparent BCs for fully integrable systems

sine-Gordon equation: $u_{tt} - u_{xx} + \sin u = 0, \quad x \in \mathbb{R},\ t > 0$
exact TBC exists (as solution to nonlinear ODE), based on inverse scattering theory [Fokas, 2002]



interaction of 2 solitons, no boundary reflections [Zheng, 2007]
(also for KdV, cubic Schrödinger)

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- Tareq Amro (Münster)
- Matthias Ehrhardt (Berlin)
- Maike Schulte (Münster)