Data Visualization

Lecture 5
Mathamatics Visualization

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Content

Plot a mathmatics graph

- Introduction to Linear Algebra 4

Vector Spaces and Vector Subspaces

Definition 27 *A* **scalar multiplication** *is an operation such that*

$$c \cdot v$$

where c is a scalar and v is an element of a set V.

A vector space is a foundation in many areas of mathematics and it is a well-defined set with some nice properties. A vector space seems to be very intuitive, however, we have to be very careful with the definition.

Vector Spaces and Vector Subspaces

Definition 28 *A* **vector space** V *is a non-empty set with two operations:* **addition** *and* **scalar multiplication** *such that the following are true:*

- $v + u$ *is in* V *if* v *and* u *are in* V.
- $v + u = u + v$ *for all* v *and* u *in* V.
- $(v + u) + w = v + (u + w)$ *for all* v, u, *and* w *in* V.
- *There exists a* **zero vector** $\mathbf{0}$ *in* V *such that* $u + \mathbf{0} = u$ *for any* u *in* V.
- *For any* v *in* V *there exists a vector* $-v$ *in* V *such that* $v + (-v) = (-v) + v = \mathbf{0}$.
- *If* v *is in* V *and if* c *is a scalar, then* $c \cdot v$ *is in* V.
- $c \cdot (v + u) = c \cdot v + c \cdot u$ *for any* v, u *in* V *and for any scalar* c.
- $(c + d) \cdot v = c \cdot v + d \cdot v$ *for any scalar* c, d *and any* v *in* V.
- $c \cdot (d \cdot v) = (c \cdot d) \cdot v$ *for any scalar* c, d *and any* v *in* V.
- $1 \cdot v = v$ *for any* v *in* V.

Vector Spaces and Vector Subspaces

Proposition 4.1 *For any vector v in a vector space V,*

- $0 \cdot v = \mathbf{0}$,
- $c \cdot \mathbf{0} = \mathbf{0}$ *for any scalar* c,
- $(-1) \cdot v = -v$.

Example 90 *Suppose* $V = \mathbb{R}^3 := \{(x, y, z) | x, y, z,$ *are real numbers*$\}$. *Now, we check all properties of a vector space so we can verify that* V *is a vector space.*

- *Want to verify:* $v + u$ *is in* V *if* v *and* u *are in* V. *Suppose we have* (x_1, y_1, z_1), (x_2, y_2, z_2) *in* V *for* $x_1, x_2, y_1, y_2, z_1, z_2$, *which are real numbers. Then* $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1, +y_2, z_1 + z_2)$ *is in* V *since* $x_1 + x_2, y_1 + y_2, z_1 + z_2$ *are real numbers.*
- *Want to verify:* $v + u = u + v$ *for all* v *and* u *in* V. *Suppose we have* (x_1, y_1, z_1), (x_2, y_2, z_2) *in* V *for* $x_1, x_2, y_1, y_2, z_1, z_2$, *which are real numbers. Then* $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1, +y_2, z_1 + z_2) = (x_2 + x_1, y_2, +y_1, z_2 + z_1) = (x_2, y_2, z_2) + (x_1, y_1, z_1)$.

Vector Spaces and Vector Subspaces

- *Want to verify:* $(v + u) + w = v + (u + w)$ *for all* v, u, *and* w *in* V. *Suppose we have* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) *in* V *for* $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$, *which are real numbers. Then*

$$\begin{aligned}
&((x_1, y_1, z_1) + (x_2, y_2, z_2)) + (x_3, y_3, z_3) \\
&= ((x_1 + x_2) + x_3, (y_1, +y_2) + y_3, (z_1 + z_2) + z_3) \\
&= (x_1 + (x_2 + x_3), y_1, +(y_2 + y_3), z_1 + (z_2 + z_3)) \\
&= (x_1, y_1, z_1) + ((x_2, y_2, z_2) + (x_3, y_3, z_3))
\end{aligned}$$

.

- *Want to verify: There exists a* **zero vector 0** *in* V *such that* $u + \mathbf{0} = u$ *for any* u *in* V. *In this example, we have* $\mathbf{0} = (0, 0, 0)$.

- *Want to verify: For any* v *in* V *there exists a vector* $-v$ *in* V *such that* $v + (-v) = (-v) + v = \mathbf{0}$. *Suppose we have* $v = (x_1, y_1, z_1)$ *in* V *for* x_1, y_1, z_1, *which are real numbers. Then* $-v = (-x_1, -y_1, -z_1)$.

- *Want to verify: If* v *is in* V *and if* c *is a scalar, then* $c \cdot v$ *is in* V. *Suppose we have* $v = (x_1, y_1, z_1)$ *in* V *for* x_1, y_1, z_1, *which are real numbers and a scalar* c *which is a real number. Then,* $c \cdot v = (c \cdot x_1, c \cdot x_2, c \cdot x_3)$ *is in* V *since* $c \cdot x_1, c \cdot y_1, c \cdot z_1$ *are real numbers.*

- *Want to verify:* $c \cdot (v + u) = c \cdot v + c \cdot u$ *for any* v, u *in* V *and for any scalar* c. *Suppose we have* $v = (x_1, y_1, z_1)$, $u = (x_2, y_2, z_2)$ *in* V *for* $x_1, x_2, y_1, y_2, z_1, z_2$, *which are real numbers and* c *is real number scalar. Then*

$$\begin{aligned}
c \cdot (v + u) &= c \cdot (x_1 + x_2, y_1 + y_2, z_1 + z_2) \\
&= (c \cdot x_1 + c \cdot x_2, c \cdot y_1 + c \cdot y_2, c \cdot z_1 + c \cdot z_2) \\
&= c \cdot v + c \cdot u.
\end{aligned}$$

- *Want to verify:* $(c + d) \cdot v = c \cdot v + d \cdot v$ *for any scalar* c, d *and any* v *in* V. *Suppose we have* $v = (x_1, y_1, z_1)$ *in* V *for* x_1, y_1, z_1, *which are real numbers and* c, d *are real number scalars. Then*

$$\begin{aligned}
(c \cdot d) \cdot v &= ((c \cdot d) \cdot x_1, (c \cdot d) \cdot y_1, (c \cdot d) \cdot z_1) \\
&= (c \cdot x_1 + d \cdot x_1, c \cdot y_1 + d \cdot y_1, c \cdot z_1 + d \cdot z_1) \\
&= c \cdot v + d \cdot v.
\end{aligned}$$

- *Want to verify:* $c \cdot (d \cdot v) = (c \cdot d) \cdot v$ *for any scalar* c, d *and any* v *in* V. *Suppose we have* $v = (x_1, y_1, z_1)$ *in* V *for* x_1, y_1, z_1 *are real numbers and* c, d *are real number scalars. Then*

$$\begin{aligned}
c \cdot (d \cdot v) &= (c \cdot (d \cdot x_1), c \cdot (d \cdot y_1), c \cdot (d \cdot z_1)) \\
&= ((c \cdot d) \cdot x_1, (c \cdot d) \cdot y_1, (c \cdot d) \cdot z_1) \\
&= (c \cdot d) \cdot v.
\end{aligned}$$

Vector Spaces and Vector Subspaces

- *Want to verify:* $1 \cdot v = v$ *for any* v *in* V*. Suppose we have* $v = (x_1, y_1, z_1)$ *in* V *for* x_1, y_1, z_1*, which are real numbers. Then we have* $1 \cdot v = 1 \cdot (x_1, y_1, z_1) = (1 \cdot x_1, 1 \cdot y_1, 1 \cdot z_1) = (x_1, y_1, z_1) = v$.

Therefore, V *is a vector space.*

Example 91 *Let* V *be a set of all* 2×2 *matrices with real numbers. Then* V *is a vector space.*

Example 92 *Let* V *be a set of all symmetric* 3×3 *matrices with real numbers. Then* V *is a vector space.*

Example 93 *Let* V *be a set of all polynomials* $a \cdot x^2 + b \cdot x \cdot y + c \cdot y^2 + d \cdot x + e \cdot y + f = 0$ *for* a*,* b*,* c*,* d*,* e*,* f*, which are real numbers. Then* V *is a vector space.*

Vector Subspaces

Definition 29 *A* **vector subspace** W *is a subset of a vector space* V *such that*

- *the zero vector* $\mathbf{0}$ *in* V *is also in* W,
- W *is closed under addition, namely, for any vectors* v, u *in* W, $v + u$ *is also in* W, *and*
- W *is closed under the scalar multiplication, namely, for any vector* v *in* W *and for any scalar* c, $c \cdot v$ *is also in* W.

Example 94 *This is from Example 90. Suppose* $V = \{(x, y, z) | x, y, z$ *are real numbers*$\}$, *i.e., the* 3*-dimensional real numbers. Then let* $W = \{(x, y, 0) | x, y$ *are real numbers*$\}$. *Now we want to show that* W *is a vector subspace.*

- *Want to verify: The zero vector* $\mathbf{0}$ *in* V *is also in* W. *When* $x = y = 0$, *then* $\mathbf{0}$ *is in* W.
- *Want to verify:* W *is closed under addition, namely, for any vectors* v, u *in* W, $v + u$ *is also in* W. *Suppose* $v = (x_1, y_1, 0)$, $u = (x_2, y_2)$ *for* x_1, x_2, y_1, y_2, *which are real numbers. Then* $v + u = (x_1 + x_2, y_1 + y_2, 0)$, *which is in* W.

Vector Subspaces

- *Want to verify: W is closed under scalar multiplication, namely, for any vector v in W and for any scalar c, $c{\cdot}v$ is also in W. Suppose $v = (x_1, y_1, 0)$ for x_1, y_1, which are real numbers, and let c be a scalar of a real number. Then, $c \cdot v = (c \cdot x_1, c \cdot y_1, 0)$ is in W.*

All conditions are satisfied. Therefore, W is a vector subspace of V.

Example 95 *This is from Example 91. Let V be a set of all 2×2 matrices with real numbers and W be a set of all 2×2 symmetric matrices with real numbers. Then V is a vector space and W is a vector subspace of V.*

Example 96 *This is from Example 92. Let V be a set of all symmetric 3×3 matrices with real numbers and W be a set of all 3×3 diagonal matrices with real numbers. Then V is a vector space and W is a vector subspace of V.*

Example 97 *This is from Example 93. Let V be a set of all polynomials $a \cdot x^2 + b \cdot x \cdot y + c \cdot y^2 + d \cdot x + e \cdot y + f = 0$ for a, b, c, d, e, f and let W be a set of all polynomials $d \cdot x + e \cdot y + f = 0$ for e, f, which are real numbers. Then V is a vector space and W is a vector subspace of V.*

Example 98 *The vector space V is defined in Example 90. Suppose $V = \{(x, y, z) | x, y, z \text{ are real numbers}\}$, i.e., 3-dimensional real numbers. Then let $W = \{(x, y, z) | z = 2 \cdot x - 3 \cdot y\}$. Then W is a vector subspace.*

Vector Subspaces

To visualize W from Example 98 in 3-dimensional space, we can use the **rgl** package in R [2].

```
library(rgl)
```

Then we initialize the plot:

```
# Create some dummy data
dat <- replicate(2, 1:3)
# Initialize the scene, no data plotted
plot3d(dat, type = 'n', xlim = c(-1, 8), ylim = c(-1, 8),
zlim = c(-10, 20), xlab = '', ylab = '', zlab = '')
```

Then we define the plane by a linear equation and define the origin:

```
# Define the linear plane
planes3d(2, 3, -1, 0, col = 'red', alpha = 0.6)
# Define the origin
points3d(x=0, y=0, z=0)
```

Vector Subspaces

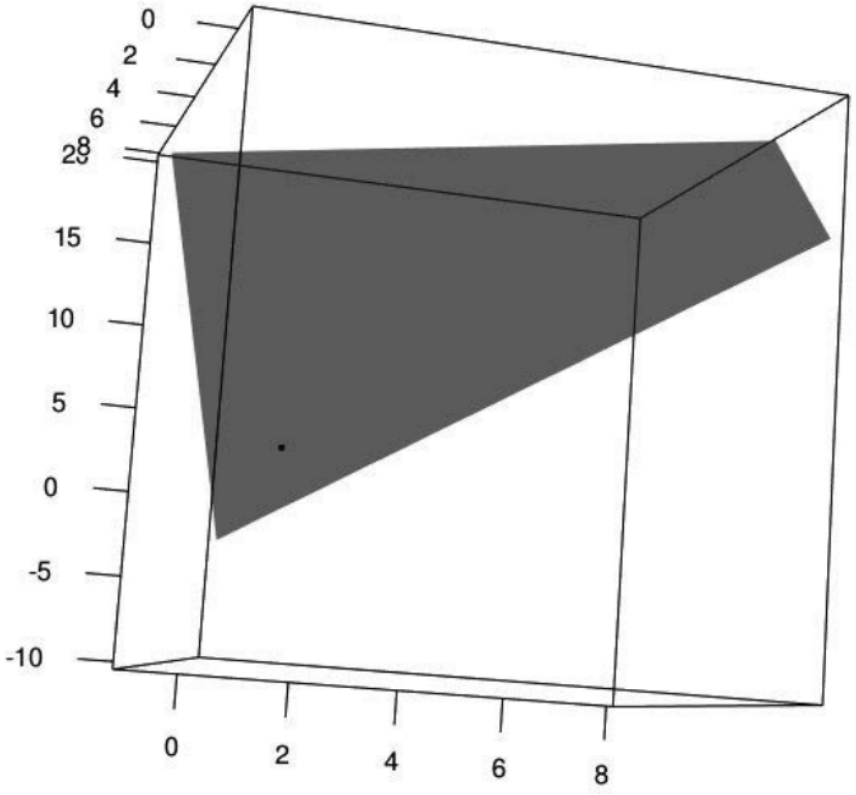

FIGURE 4.2
A vector subspace in the 3-dimensional space defined by a linear equation $z = 2 \cdot x - 3 \cdot y$ plotted by the `rgl` package in `R`. The black point in the plot is the origin $(0, 0, 0)$.

Vector Subspaces

Remark 4.2 *If we have a linear plane through the origin* $(0, 0, 0)$ *in the 3-dimensional space, then the set of all points on the linear plane becomes a vector subspace.*

Remark 4.3 *A vector subspace is also a vector space itself.*

From Example 98 we can represent the vector subspace as $W = \{(x, y, z)|z = 2 \cdot x - 3 \cdot y\}$. Also we can write this vector subspace as

$$W = \{(x, y, z) = \alpha_1(1, 1, -1) + \alpha_2(3, 1, 3)|\alpha_1, \alpha_2 \text{ are real numbers}\}.$$

Writing this form

$$W = \{\alpha_1 v_1 + \alpha_2 v_2 | \alpha_1, \alpha_2 \text{ are real numbers}\}$$

is called **spanned** by vectors v_1 and v_2.

Vector Subspaces

Example 99 *The 3-dimensional space $\mathbb{R}^3$ is spanned by vectors*

$$\begin{aligned} v_1 &= (1,0,0) \\ v_2 &= (0,1,0) \\ v_3 &= (0,0,1). \end{aligned}$$

Example 100 *This is from Example 95. Let V be a set of all 2×2 matrices with real numbers and W be a set of all 2×2 symmetric matrices with real numbers. Then V is a vector space and W is a vector subspace of V. V is spanned by the matrices:*

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

W is spanned by the matrices:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example

The fundamental problem of linear algebra is to solve n linear equations in n unknowns; for example:

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3. \end{aligned}$$

In this first lecture on linear algebra we view this problem in three ways.

The system above is two dimensional ($n = 2$). By adding a third variable z we could expand it to three dimensions.

Row Picture

Plot the points that satisfy each equation. The intersection of the plots (if they do intersect) represents the solution to the system of equations. Looking at Figure 1 we see that the solution to this system of equations is $x = 1, y = 2$.

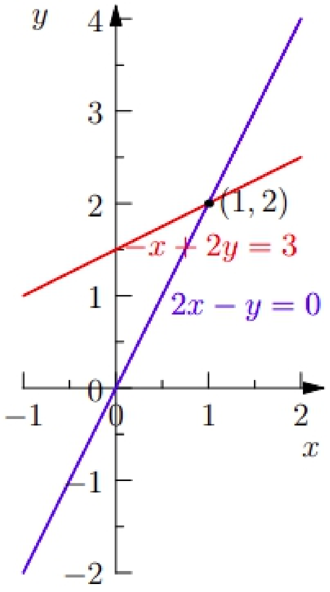

Figure 1: The lines $2x - y = 0$ and $-x + 2y = 3$ intersect at the point $(1, 2)$.

We plug this solution in to the original system of equations to check our work:

$$\begin{aligned} 2 \cdot 1 - 2 &= 0 \\ -1 + 2 \cdot 2 &= 3. \end{aligned}$$

The solution to a three dimensional system of equations is the common point of intersection of three planes (if there is one).

Example

Column Picture

In the column picture we rewrite the system of linear equations as a single equation by turning the coefficients in the columns of the system into vectors:

$$x\begin{bmatrix}2\\-1\end{bmatrix}+y\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}.$$

Given two vectors **c** and **d** and scalars x and y, the sum $x\mathbf{c}+y\mathbf{d}$ is called a *linear combination* of **c** and **d**. Linear combinations are important throughout this course.

Geometrically, we want to find numbers x and y so that x copies of vector $\begin{bmatrix}2\\-1\end{bmatrix}$ added to y copies of vector $\begin{bmatrix}-1\\2\end{bmatrix}$ equals the vector $\begin{bmatrix}0\\3\end{bmatrix}$. As we see from Figure 2, $x=1$ and $y=2$, agreeing with the row picture in Figure 2.

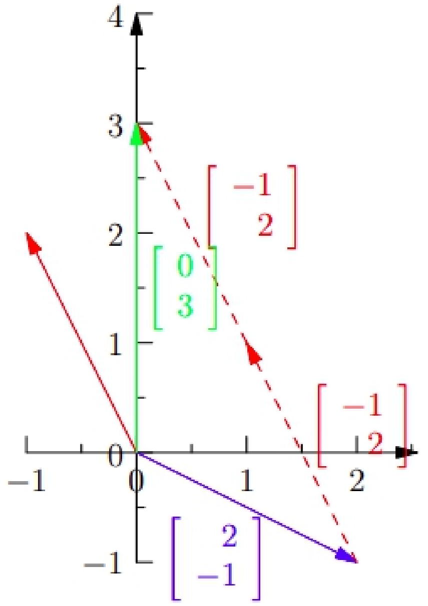

Figure 2: A linear combination of the column vectors equals the vector **b**.

In three dimensions, the column picture requires us to find a linear combination of three 3-dimensional vectors that equals the vector **b**.

Matrix Picture

We write the system of equations

$$\begin{aligned}2x-y &= 0\\-x+2y &= 3\end{aligned}$$

as a single equation by using matrices and vectors:

$$\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\3\end{bmatrix}.$$

The matrix $A=\begin{bmatrix}2&-1\\-1&2\end{bmatrix}$ is called the *coefficient matrix*. The vector $\mathbf{x}=\begin{bmatrix}x\\y\end{bmatrix}$ is the vector of unknowns. The values on the right hand side of the equations form the vector **b**:

$$A\mathbf{x}=\mathbf{b}.$$

The three dimensional matrix picture is very like the two dimensional one, except that the vectors and matrices increase in size.

Example

Matrix Multiplication

How do we multiply a matrix A by a vector $\mathbf{x}$?

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \quad ?$$

One method is to think of the entries of $\mathbf{x}$ as the coefficients of a linear combination of the column vectors of the matrix:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

This technique shows that $A\mathbf{x}$ is a linear combination of the columns of A.

You may also calculate the product $A\mathbf{x}$ by taking the dot product of each row of A with the vector $\mathbf{x}$:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 5 \cdot 2 \\ 1 \cdot 1 + 3 \cdot 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}.$$

Linear Independence

In the column and matrix pictures, the right hand side of the equation is a vector $\mathbf{b}$. Given a matrix A, can we solve:

$$A\mathbf{x} = \mathbf{b}$$

for every possible vector $\mathbf{b}$? In other words, do the linear combinations of the column vectors fill the xy-plane (or space, in the three dimensional case)?

If the answer is "no", we say that A is a *singular matrix*. In this singular case its column vectors are *linearly dependent*; all linear combinations of those vectors lie on a point or line (in two dimensions) or on a point, line or plane (in three dimensions). The combinations don't fill the whole space.

Practice: solve linear equations

https://cran.r-project.org/web/packages/matlib/vignettes/linear-equations.html

Null Space, Column Space, and Row Space

Suppose we have the following system of 3 linear equations with the variables x_1, x_2, x_3:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & -1 \\ -x_1 & - & x_2 & + & 5x_3 & = & 2. \end{array}$$

Then we want to solve the system of linear equations. In addition to solving the system of linear equations, we would like to find **all** solutions if they exist.

We discussed how to solve a system of linear equations using the reduced echelon form of the augmented matrix in Chapter 1 and using Cramer's rule in Section 3.5. Here, we use the **null space** of the coefficient matrix to find all solutions of the system of linear equations.

Definition 30 *Suppose we have an $m \times n$ matrix A. The* **null space** *of the matrix A is the set of all solutions for the system of linear equations such that*

$$A \cdot x = \mathbf{0}.$$

Going back to the system of linear equations above, here we have

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 0 & -1 \\ -1 & -1 & 5 \end{bmatrix}, b = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Null Space, Column Space, and Row Space

In order to find the null space of the matrix A, we have to solve the system of linear equations:

$$A \cdot X = b$$

that is

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & 4x_3 & = & 0 \\ 2x_1 & & & - & x_3 & = & 0 \\ -x_1 & - & x_2 & + & 5x_3 & = & 0. \end{array}$$

Null Space, Column Space, and Row Space

With the help of `R`, we solve this system of linear equations using the reduced echelon form of the matrix A with the rref() function from the `pracma` package:

```
library(pracma)
A <- matrix(c(1, -1, 4, 0, 2, 0, -1, 0, -1, -1, 5, 0),
nrow=3, ncol=4, byrow=TRUE)
rref(A)
```

`R` outputs as follows:

```
> rref(A)
     [,1] [,2] [,3] [,4]
[1,]    1    0 -0.5    0
[2,]    0    1 -4.5    0
[3,]    0    0  0.0    0
```

This means the reduced echelon form of the system is

$$\begin{array}{rcrcl} x_1 & & - & 0.5x_3 & = & 0 \\ & x_2 & - & 4.5x_3 & = & 0. \end{array}$$

The solutions of the system of linear equations are

$$\begin{array}{rcl} x_1 & & = & 0.5t \\ & x_2 & = & 4.5t \end{array}$$

Null Space, Column Space, and Row Space

where t is any real number. Therefore, the null space of the matrix A is spanned by the vector

$$v = \begin{bmatrix} 0.5 \\ 4.5 \\ 1 \end{bmatrix}$$

and the null space of the matrix A is the set

$$\left\{ t \cdot \begin{bmatrix} 0.5 \\ 4.5 \\ 1 \end{bmatrix} \middle| -\infty < t < \infty \right\}.$$

Now we have the following theorem about the null space of a matrix.

Theorem 4.4 *The null space of a matrix A is a vector space.*

Then we have the following lemma:

Lemma 4.5 *Suppose we have an $m \times n$ matrix A. Then the null space of the matrix A is a vector subspace of $\mathbb{R}^n$.*

Null Space, Column Space, and Row Space

Let $\mathrm{Null}(A)$ be the null space of the matrix A. Here, we go through how the null space of the matrix A relates to the system of linear equations

$$A \cdot x = b.$$

Suppose y is a vector in the null space of the matrix A. Then y satisfies the system of linear equations

$$A \cdot y = \mathbf{0}.$$

Suppose x_0 is a solution which satisfies the system of linear equations above. Then, x_0 satisfies the system of linear equations

$$A \cdot x_0 = b.$$

Therefore, $x_0 + y$ satisfies the system of linear equations

$$A \cdot (x_0 + y) = A \cdot x_0 + A \cdot y = b + \mathbf{0} = b.$$

Thus, $x_0 + y$ is also another solution of the system of linear equations $A \cdot x = b$. Therefore, if the system of linear equations $A \cdot x = b$ has a unique solution, then the null space of the matrix A consists of the zero vector, i.e., $\mathrm{Null}(A) = \{\mathbf{0}\}$.

Null Space, Column Space, and Row Space

Theorem 4.6 *Suppose we have a system of linear equations such that*

$$A \cdot x = b.$$

Then the system of linear equations has a unique solution if and only if $Null(A) = \{\mathbf{0}\}$.

Going back to the example in the beginning, we have the null space of the coefficient matrix A of the system is

$$\text{Null}\,(A) = \left\{ t \cdot \begin{bmatrix} 0.5 \\ 4.5 \\ 1 \end{bmatrix} \middle| -\infty < t < \infty \right\}.$$

Therefore, the system of linear equations has infinitely many solutions since $\text{Null}\,(A) \neq \{\mathbf{0}\}$. Also by solving a system of linear equations, note that the vector

$$x_0 = \begin{bmatrix} -1/2 \\ -3/2 \\ 0 \end{bmatrix}$$

is a solution to the system of linear equations. Therefore the solution set of the system is

$$\left\{ \begin{bmatrix} -1/2 \\ -3/2 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0.5 \\ 4.5 \\ 1 \end{bmatrix} \middle| -\infty < t < \infty \right\}.$$

Null Space, Column Space, and Row Space

In R we can use the Null() function from the MASS package to compute the vectors spanning the null space of a matrix A. We will use the example in the beginning to demonstrate how to use the function in R. First we load the package and define the matrix:

```
library(MASS)
A <- matrix(c(1, -1, 4, 2, 0, -1, -1, -1, 5),
nrow=3, ncol=3, byrow=TRUE)
```

Then we use the NULL() function to the **transpose** of the matrix A:

```
Null(t(A))
```

Example 101 *Then* R *returns as follows:*

```
> Null(t(A))
          [,1]
[1,] 0.1078328
[2,] 0.9704950
[3,] 0.2156655
```

Suppose we have the following system of 3 *linear equations with the variables* x_1, x_2, x_3, x_4*:*

$$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & 4x_3 & + & x_4 & = & 1 \\ 2x_1 & & & - & x_3 & - & 2x_4 & = & -1 \\ -x_1 & - & x_2 & + & 5x_3 & + & 3x_4 & = & 2. \end{array}$$

Null Space, Column Space, and Row Space

Then we can re-write this system as

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & -1 & 4 & 1 \\ 2 & 0 & -1 & -2 \\ -1 & -1 & 5 & 3 \end{bmatrix}, b = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

First we will find the null space of the matrix A and then find the set of all solutions to the system.

Using the rref() function from the `pracma` *package:*

```
library(pracma)
A <- matrix(c(1, -1, 4, 1, 0, 2, 0, -1, -2, 0, -1, -1, 5, 3, 0),
nrow=3, ncol=5, byrow=TRUE)
rref(A)
```

`R` *outputs as follows:*

```
> rref(A)
     [,1] [,2] [,3] [,4] [,5]
[1,]    1    0 -0.5   -1    0
[2,]    0    1 -4.5   -2    0
[3,]    0    0  0.0    0    0
```

This means the reduced echelon form of the system is

$$\begin{array}{rcrcrcl} x_1 & & - & 0.5x_3 & - & x_4 & = & 0 \\ & x_2 & - & 4.5x_3 & - & 2x_2 & = & 0. \end{array}$$

This means the solutions of the system of linear equations are

$$\begin{array}{llll} x_1 & & & & = & 0.5t + s \\ & x_2 & & & = & 4.5t + 2s \\ & & x_3 & & = & t \\ & & & x_4 & = & s \end{array}$$

where t and s are any real numbers. The null space of the coefficient matrix A is

$$Null(A) = \left\{ t \cdot \begin{bmatrix} 0.5 \\ 4.5 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} | -\infty < t, s < \infty \right\}.$$

Therefore $Null(A)$ is a vector subspace of $\mathbb{R}^4$ spanned by two vectors

$$\begin{bmatrix} 0.5 \\ 4.5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Null Space, Column Space, and Row Space

Now we notice that the vector

$$x_0 = \begin{bmatrix} -1/2 \\ -3/2 \\ 0 \\ 0 \end{bmatrix}$$

is a solution to the system of linear equations. Thus, the set of all solutions for the system of linear equations is:

$$\left\{ \begin{bmatrix} -1/2 \\ -3/2 \\ 0 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0.5 \\ 4.5 \\ 1 \\ 0 \end{bmatrix} + s \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \middle| -\infty < t, s < \infty \right\}.$$

Now we shift our focus to the **column space** of the matrix A. The column space of the matrix A is also a vector space and if the right hand side vector b is in the column space, then the system of linear equations $A \cdot x = b$ is feasible, i.e., there exists a solution.

Null Space, Column Space, and Row Space

Definition 31 *Suppose we have an $m \times n$ matrix A and let $\mathbf{a_i}$ be the ith column of the matrix A. Then the* **column space** *of the matrix A, denoted by $Col(A)$, is a vector space spanned by $\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n}$, i.e.,*

$$Col(A) = \{\alpha_1 \cdot \mathbf{a_1} + \alpha_2 \cdot \mathbf{a_2} + \ldots + \alpha_n \cdot \mathbf{a_n} | -\infty < \alpha_1, \alpha_2, \ldots \alpha_n < \infty\}.$$

Theorem 4.7 *The system of linear equations $A \cdot x = b$ has a solution if and only if the vector b is in the column space of the matrix A, $Col(A)$.*

Going back to the example in the beginning, recall that we have the following system of linear equations:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & -1 \\ -x_1 & - & x_2 & + & 5x_3 & = & 2. \end{array}$$

For this example, the column space of the coefficient matrix A is

$$\mathrm{Col}(A) = \left\{ \alpha_1 \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \alpha_2 \cdot \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} + \alpha_3 \cdot \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} | -\infty < \alpha_1, \alpha_2, \alpha_3 < \infty \right\}.$$

If $\alpha_1 = -1/2$, $\alpha_2 = -3/2$, $\alpha_3 = 0$, we have

$$(-1/2) \cdot \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + (-3/2) \cdot \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}.$$

Therefore, this system has a solution.

Finally, we introduce the **row space** of the matrix A.

Null Space, Column Space, and Row Space

Definition 32 *Suppose we have an $m \times n$ matrix A and let $\mathbf{a^i}$ be the ith row of the matrix A. Then the* **row space** *of the matrix A, denoted by $Row(A)$, is a vector space spanned by $\mathbf{a^1}, \mathbf{a^2}, \ldots, \mathbf{a^m}$, i.e.,*

$$Col(A) = \left\{\beta_1 \cdot \mathbf{a^1} + \beta_2 \cdot \mathbf{a^2} + \ldots + \beta_m \cdot \mathbf{a^m} | -\infty < \beta_1, \beta_2, \ldots \beta_m < \infty\right\}.$$

Theorem 4.8 *Suppose we have a system of linear equations $A \cdot x = b$. Let B be the augmented matrix of the system of linear equations, i.e.,*

$$B = [A|b].$$

Suppose v is in the row space of B, $Row(B)$, and let B' be the matrix created by adding v to B as its row. Then the system of linear equations whose augmented matrix is B' has the same solutions as the system of linear equations $A \cdot x = b$.

Null Space, Column Space, and Row Space

Going back to the example in the beginning, recall that we have the following system of linear equations:

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & -1 \\ -x_1 & - & x_2 & + & 5x_3 & = & 2. \end{array}$$

Recall that the solution set of the system is

$$\left\{ \begin{bmatrix} -1/2 \\ -3/2 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0.5 \\ 4.5 \\ 1 \end{bmatrix} \middle| -\infty < t < \infty \right\}.$$

For this example, the row space of the augmented matrix B is

$\mathrm{Row}\,(B) =$
$\{\beta_1 \cdot [1,-1,4,1] + \beta_2 \cdot [2,0,-1,-1] + \beta_3 \cdot [-1,-1,5,2] \,|\, -\infty < \beta_1, \beta_2, \beta_3 < \infty\}.$

Let $\beta_1 = \beta_2 = \beta_3 = 1$. Then we have

$$[1,-1,4,1] + [2,0,-1,-1] + [-1,-1,5,2] = [2,-2,8,2]\,.$$

The new matrix B' is

$$B' = \begin{bmatrix} 1 & -1 & 4 & 1 \\ 2 & 0 & -1 & -1 \\ -1 & -1 & 5 & 2 \\ 2 & -2 & 8 & 2 \end{bmatrix}.$$

The new system of linear equations is

$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & 4x_3 & = & 1 \\ 2x_1 & & & - & x_3 & = & -1 \\ -x_1 & - & x_2 & + & 5x_3 & = & 2 \\ 2x_1 & - & 2x_2 & + & 8x_3 & = & 2. \end{array}$$

Let

$$v = \begin{bmatrix} -1/2 \\ -3/2 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0.5 \\ 4.5 \\ 1 \end{bmatrix}$$

for t is a real number. Then we plug v into the new system of linear equations:

$$\begin{array}{rcrcrcr} (-1/2+0.5t) & - & (-3/2+4.5t) & + & 4t & = & 1 \\ 2(-1/2+0.5t) & & & - & t & = & -1 \\ -(-1/2+0.5t) & - & (-3/2+4.5t) & + & 5t & = & 2 \\ 2(-1/2+0.5t) & - & 2(-3/2+4.5t) & + & 8t & = & 2. \end{array}$$

Therefore the solution set for this new system of linear equations is also

$$\left\{ \begin{bmatrix} -1/2 \\ -3/2 \\ 0 \end{bmatrix} + t \cdot \begin{bmatrix} 0.5 \\ 4.5 \\ 1 \end{bmatrix} \middle| -\infty < t < \infty \right\}.$$

Practice

Now we go back to the introductory example, the British woman's tea problem. Let 1 = Milk and 2 = Tea for notation. As we discussed in Section 1.1, in order to test the hypotheses, we fix the sum of each row and the sum of each column. Then we assign a variable x_{ij} for each (i, j)th entry in the 2×2 contingency table.

Guess/True	Milk	Tea	Total
Milk	x_{11}	x_{12}	4
Tea	x_{21}	x_{22}	4
Total	4	4	8

This can be written as a system of linear equations such that:

$$\begin{array}{ccccccccc} x_{11} & + & x_{12} & & & & & = & 4 \\ & & & & x_{21} & + & x_{22} & = & 4 \\ x_{11} & & & + & x_{21} & & & = & 4 \\ & & x_{12} & & & + & x_{22} & = & 4. \end{array}$$

Therefore we have the system of linear equations

$$A \cdot x = b$$

where

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix}.$$

Here we compute the null space of the matrix A with the Null function from the `MASS` package in `R`. First, we upload the `MASS` library and then define the coefficient matrix A and the right hand side vector b:

```
library(MASS)
A <- matrix(c(1,1,0,0,0,0,1,1,1,0,1,0,0,1,0,1), 4, 4, byrow=TRUE)
b <- c(4, 4, 4, 4)
```

Then we use the Null() function to compute the null space of the matrix A:

```
v <- Null(t(A))
```

`R` outputs the vector spanning the null space of A:

```
> v
     [,1]
[1,]  0.5
[2,] -0.5
[3,] -0.5
[4,]  0.5
```

Practice

Therefore, the vector

$$v = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

spans the null space of A. Note that the observed table is a feasible solution. Thus, we set

$$x_0 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

and we know that

$$x_0 + t \cdot v \tag{4.1}$$

For any real numbers, t is a feasible solution for the system of linear equations. However, note that the number in each cell in the table must be a non-negative integer. Thus, we have

$$x_0 + t \cdot v \geq 0$$

for some integer t.

Fisher's exact test is to enumerate all possible contingency tables satisfying the given row sums and column sums. Thus we enumerate all tables satisfying the given row and column sums using the conditions in (4.1). For $t = 1$ we have

$$x_1 = x_0 + v = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix}$$

which represents the table

Guess/True	Milk	Tea	Total
Milk	4	0	4
Tea	0	4	4
Total	4	4	8

.

For $t = 10$ we have the observed table

$$x_2 = x_0 + 0 \cdot v = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

which represents the table

Guess/True	Milk	Tea	Total
Milk	3	1	4
Tea	1	3	4
Total	4	4	8

.

Practice

For $t = -1$ we have

$$x_3 = x_0 - v = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

which represents the table

Guess/True	Milk	Tea	Total
Milk	2	2	4
Tea	2	2	4
Total	4	4	8

.

For $t = -2$ we have

$$x_4 = x_0 - 2 \cdot v = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

which represents the table

Guess/True	Milk	Tea	Total
Milk	1	3	4
Tea	3	1	4
Total	4	4	8

.

For $t = -3$ we have

$$x_5 = x_0 + v = \begin{bmatrix} 0 \\ 4 \\ 4 \\ 0 \end{bmatrix}$$

which represents the table

Guess/True	Milk	Tea	Total
Milk	0	4	4
Tea	4	0	4
Total	4	4	8

.

Now we compute the probability to observing each table with given row sums and column sums under the **hypergeometric distribution**. We can use the dhyper() function in R. For $t = 1$, we have the probability to observing the table x_1 is

```
> dhyper(4, 4, 4, 4)
[1] 0.01428571
```

For $t = 0$, we have the probability to observing the table x_2 is

Practice

```
> dhyper(3, 4, 4, 4)
[1] 0.2285714
```

The p-value for the hypotheses is $P(x_1) + P(x_2) = 0.01428571 + 0.2285714 = 0.2428571$.

If we use the fisher.test() function, we can conduct Fisher's exact test. For this we create the table as a matrix and call the fisher.test() function:

```
Tea <- matrix(c(3, 1, 1, 3), 2, 2, byrow = TRUE)
fisher.test(Tea, alternative = "greater")
```

Then `R` outputs as follows:

```
> fisher.test(Tea, alternative = "greater")

Fisher's Exact Test for Count Data

data:  Tea
p-value = 0.2429
alternative hypothesis: true odds ratio is greater than 1
95 percent confidence interval:
 0.3135693       Inf
sample estimates:
odds ratio
  6.408309
```

Thus we have the same p-value. If we set the significance level of 0.05, then the p-value is bigger than 0.05. Thus, we fail to reject the null hypothesis and this means that her guess cannot be differentiated from a random guess.

Practice

Exercise 4.12 *Suppose we have a matrix*

$$A = \begin{bmatrix} 0 & 0 & 3 \\ -3 & -1 & -2 \\ 2 & 1 & 3 \end{bmatrix}.$$

1. *Compute the null space of A.*
2. *Compute the column space of A.*
3. *Compute the row space of A.*

Exercise 4.13 *Suppose we have a matrix*

$$A = \begin{bmatrix} 3 & -1 & -3 \\ 0 & 3 & 1 \\ 3 & 2 & -2 \end{bmatrix}.$$

1. *Compute the null space of A.*
2. *Compute the column space of A.*
3. *Compute the row space of A.*