Topologically irreducible representations and radicals in Banach algebras

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1 Introduction.

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The Jacobson radical of an associative algebra is the intersection of the kernels of the strictly irreducible representations. It is natural, when studying normed algebras, to consider continuous 'topologically irreducible' representations on Banach spaces; i.e. continuous homomorphisms of the algebra onto algebras of bounded operators on Banach spaces for which no non-trivial *closed* subspace is invariant, (where 'non-trivial' means having non-zero dimension and codimension). Again, one looks at the intersection of the kernels of all these representations of a given algebra. We shall show (Theorem 8.1) that this is, in a reasonable sense, a 'topological radical'.

For Banach algebras, topological irreducibility is more general than strict irreducibility, so our long-term aspiration is to use topologically irreducible representations to study Jacobson radical Banach algebras. However, whilst it is easy to find continuous topologically irreducible representations which are not strictly irreducible, it is not immediately clear that the intersection of the kernels of these can be strictly smaller than the Jacobson radical.

One way to construct a topologically irreducible representation of a normed algebra A is to find a continuous homomorphism $\phi : A \to B$ into a Banach algebra B such that $\phi(A)$

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is dense in B and B has strictly irreducible representations. Then every strictly irreducible representation of B induces a continuous topologically irreducible representation of A. This construction is due to Meyer [12], who calls such representations *standard*. We shall use it in Section 9 to produce a non-commutative Banach algebra in which the radical described above is strictly smaller than the Jacobson radical.

During the preparation of this paper, the author asked Charles Read whether his work on the Invariant Subspace Problem could be extended to produce a *quasi-nilpotent* operator on a Banach space with no closed invariant subspace. Read was able to do this [17] and his example gives a second Banach algebra in which the radical associated with topologically irreducible representations is strictly smaller than the Jacobson radical. Both examples are important in our theory. Read's example has the merit of being commutative; ours, which is substantially easier, distinguishes the Jacobson radical from the radical associated with the stronger condition of 'topological transitivity'.

A normed representation π of an algebra A on a normed space X is said to be *topologically transitive* if, whenever $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ are finite subsets of X with $\{x_1, \ldots, x_n\}$ linearly independent and $\varepsilon > 0$, there is an element $a \in A$ with $\|\pi(a)x_i - y_i\| < \varepsilon$ $(1 \leqslant i \leqslant n)$. It follows from Jacobson's Density Theorem that all standard representations of normed algebras have this property. It is natural to ask ([14] p.460, [2] p.132) whether every topologically irreducible representation is topologically transitive. We shall observe (Corollary 5.5) that Enflo's solution of the Invariant Subspace Problem for Banach spaces gives a counterexample, because topologically transitive representations of commutative algebras must be one-dimensional. Generalizing this, we show (Corollary 5.10) that all topologically transitive representations of PI-algebras are finite-dimensional. Read's example shows that the radicals associated with topologically irreducible and with topologically transitive representations are distinct.

Our discussion of the radicals associated with these various types of representation requires an abstract theory of 'topological radicals' in topological algebras. We devote Section 6 to setting up such a theory. The main problem is to choose the correct definitions: the theory seems to have unusually 'sensitive dependence on initial conditions', to borrow a phrase from Chaos Theory. Many reasonable variants on our chosen axioms seem not to provide the desired results, (though we have not searched for counterexamples to establish this, since our principal concern is with specific radicals rather than the axiomatics). With this theory in place, we can produce topological radicals from maps which satisfy most but not all of the axioms (UTRs and OTRs). This enables us to relate the new radicals to each other and to a topological radical derived from the Baer radical.

Our theory of topological radicals has a variant which applies to all normed algebras, not just to Banach algebras. This is useful in order to have an axiom about the radical of a continuous homomorphic image. Unfortunately, the Jacobson radical is not a 'topological radical' in this version: it is not necessarily closed! However, as we show in Section 10, the intersection of the kernels of the *continuous* strictly irreducible representations on Banach spaces provides a good alternative which coincides with the Jacobson radical in Banach algebras.

In Section 11 we note the consequences of not requiring the representation space to be complete and the paper concludes with a list of open questions.

I should like to thank Dr. John Rennison for pointing out errors in an earlier draft of this paper.

2 Definitions and abbreviations

All algebras considered will be linear associative algebras over the complex field. They will not necessarily be commutative or unital.

A *representation* of an algebra A is a homomorphism π of A into the algebra of all operators on a vector space X. We shall call π a *normed representation* of the algebra A if X is a normed space and π is a homomorphism of A into the algebra $\mathcal{L}(X)$ of all bounded operators on X. We can look at various refinements of this concept: we may make A a normed or Banach algebra, we may then require the representation to be *continuous* (with respect to the given norm on A and the operator norm on $\mathcal{L}(X)$). We shall generally do this; otherwise, we should be ignoring the topology on A. Also, we may require X to be a Banach space. This too is a sensible option, though we shall consider, in Section 11, the consequences of using incomplete spaces.

A representation π of an algebra A on a vector space X is said to be *strictly irreducible* if there is no subspace $Y \subseteq X$ with $\{0\} \neq Y \neq X$ and $\pi(a)(Y) \subseteq Y$ for all $a \in A$. A normed representation $\pi : A \to \mathcal{L}(X)$ of an algebra A on a normed space X is said to be *topologically irreducible (TI)* is there is no closed subspace $Y \subseteq X$ with $\{0\} \neq Y \neq X$ and $\pi(a)(Y) \subseteq Y$ for all $a \in A$.

For any algebra A, we denote by $J(A)$ the *Jacobson radical* of A, which is the intersection of the kernels of all the strictly irreducible representations of A. For a normed algebra A, we define the *TI radical* $T(A)$ to be the intersection of the kernels of all the continuous TI representations of A on Banach spaces. Equivalently, we can define a left Banach A-module to be *topologically simple* if it has no closed submodule. Then $T(A)$ is the intersection of the annihilators of the topologically simple left Banach A-modules.

A representation π of an algebra A on a vector space X is said to be *transitive* (or *strictly dense*) if, whenever $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ are finite subsets of X with $\{x_1, \ldots, x_n\}$ linearly independent, there is an element $a \in A$ with $\pi(a)x_i = y_i$ $(1 \leqslant i \leqslant n)$. In fact, if this holds for $n = 2$, it holds for all n and the topological version of Jacobson's Density Theorem ([14] 4.2.13, [18] (2.4.7)) says that, for Banach algebras, every strictly irreducible representation is transitive. There is an obvious topological analogue: for each positive integer n, a normed representation π of an algebra A on a normed space X is said to be *topologically n-transitive* (n-TT) if, whenever $\{x_1, \ldots, x_n\}$, $\{y_1, \ldots, y_n\}$ are subsets of X with $\{x_1, \ldots, x_n\}$ linearly independent and $\varepsilon > 0$, there is an element $a \in A$ with $\|\pi(a)x_i - y_i\| < \varepsilon$ $(1 \leqslant i \leqslant n)$. A representation is said to be *topologically transitive* (TT) if it is topologically n-transitive for all positive integers n. This is 'topologically completely irreducible' in Palmer's terminology and is equivalent to saying that $\pi(A)$ is dense in $\mathcal{L}(X)$ in the strong operator topology (the topology given by the seminorms $T \mapsto \|Tx\|$ $(x \in X)$). It is not known whether or not n-TT for some $n \geqslant 2$ implies TT. We write $T_n(A)$, $T_\infty(A)$ for the intersection of the kernels of all the continuous n-TT, TT (respectively) representations of A on Banach spaces.

A particular type of continuous TT representation arises as follows. Let ρ be a strictly irreducible representation of a Banach algebra B on a linear space X. Then there is a unique Banach space norm on X making the representation normed and continuous ([14], 4.2.16(a), 4.2.15). Let A be a normed algebra and $\phi : A \to B$ a continuous homomorphism such that $\phi(A)$ is dense in B. Then $\pi = \rho\phi$ is a TT representation of A on X. Following Meyer [12], we call such TT representations *standard*. The intersection of the kernels of all the standard TT representations of a given normed algebra A will be denoted $S(A)$.

For a normed algebra A, the inclusions

$$T(A) \subseteq T_m(A) \subseteq T_n(A) \subseteq T_\infty(A) \subseteq J(A) \qquad (m \leqslant n)$$

are clear. What is not immediately clear is whether any of these inclusions can be strict.

3 Elementary properties of TI representations

Some of our later examples will produce, *inter alia*, TI representations which are not strictly irreducible, but it is worth noting now that satisfying these requirements alone is quite easy.

Example 3.1 Let $A = \ell^1(S_2)$ be the semigroup algebra of the free semigroup on two generators X, Y. Let T be the unilateral shift on $H = \ell^2$:

$$\begin{aligned} T(\xi_1, \xi_2, \xi_3, \ldots) &= (0, \xi_1, \xi_2, \ldots) \\ T^*(\xi_1, \xi_2, \xi_3, \ldots) &= (\xi_2, \xi_3, \xi_4, \ldots) \end{aligned}$$

Let π be the continuous representation of A on ℓ^2 defined by $\pi(\delta_X) = T$, $\pi(\delta_Y) = T^*$. It is easy to see that $\pi(A)$ is a *-subalgebra of $\mathcal{L}(H)$ with scalar commutant, so, by von Neumann's Double Commutant Theorem, its strong closure is $\mathcal{L}(H)$, i.e. π is TT; but

$$\pi(A)((1, 0, 0, \ldots)) = \ell^1,$$

so π is not strictly irreducible.

Remark 3.2 For *-representations of C*-algebras, Kadison's Transitivity Theorem says that TI implies strictly irreducible ([11], see also [13] 5.2.2, [20] 1.21.17). In the example above, $\pi(A)$ is not closed in $\mathcal{L}(H)$.

We shall be seeking to relate the TI radical to radicals definable without reference to representations. In one direction this is easy, provided the algebra is complete: every strictly irreducible representation of a Banach algebra A has the same kernel as some continuous strictly irreducible representation of A on a Banach space ([14] 4.2.9, [18] (2.4.7)). Hence the TI radical of a Banach algebra is contained in the Jacobson radical, which has many characterizations not directly involving representations (largest quasi-regular ideal, largest ideal of topologically nilpotent elements, intersection of the maximal modular left ideals). It is not immediately clear that this inclusion can be strict—in Example 3.1 above, the algebra A is semisimple so there are many other representations which are strictly irreducible—but we shall give an example later where this is so.

In the other direction, the only results we know stem from the following proposition.

Proposition 3.3 ([14] 4.2.5(a), 4.4.9(a)) *The kernel of a TI representation of an algebra A on a normed space is a prime ideal of A. Hence, the intersection of the kernels of the TI representations of an algebra A contains the Baer radical of A.*

The *Baer radical* or *prime radical* $\beta(A)$ of an algebra A is the intersection of all the prime ideals of A; equivalently, it is the smallest ideal I of A such that A/I has no non-zero nilpotent ideals ([14] 4.4.6). It is the smallest of three radicals, the others being the Levitzki radical and the nil radical, that coincide for Banach algebras [4]. However, Corollary 9.4 below shows that, for incomplete normed algebras, the TI radical does not necessarily contain the other two radicals.

When we consider *continuous* TI representations of a normed algebra, we have the further information that the kernels of the representations are closed. Consequently, the TI radical of a normed algebra A contains the closure of the Baer radical $\overline{\beta(A)}$. However, $A/\overline{\beta(A)}$ might fail to be semiprime, in which case the preimage in A of its Baer radical is also included in the TI radical, as is its closure, and so on. This leads us to construct (in Corollary 6.8) a new radical, the *closed-Baer radical* $\overline{\beta}^*$, to give a good lower bound for the TI radical.

4 Classical problems

The difficulty of working with TI representations is well illustrated by their relation to some famous problems of functional analysis.

Proposition 4.1 *The following are equivalent (and true):*

(1) *there is a singly-generated (as a Banach algebra) Banach algebra with a continuous faithful TI representation on an infinite-dimensional Banach space;*

(2) *there is a singly-generated Banach algebra with a continuous non-zero TI representation on an infinite-dimensional Banach space;*

(3) *there is an operator on an infinite-dimensional Banach space with no non-trivial closed invariant subspace.*

Proof. The truth of (3) is Enflo's solution of the Invariant Subspace Problem for Banach Spaces [7] (see also [15], [1] Chapter XIV). The proof of the equivalence of (1), (2) and (3) is straightforward.

Note the sharp contrast with the situation for strictly irreducible representations: it follows from Schur's Lemma that strictly irreducible representations of commutative Banach algebras must be one-dimensional ([14] 4.2.19).

Proposition 4.2 *The following are equivalent (and true):*

(1) *there is a singly-generated (as a Banach algebra) radical Banach algebra with a continuous faithful TI representation on an infinite-dimensional Banach space;*

(2) *there is a singly-generated (as a Banach algebra) radical Banach algebra with a continuous non-zero TI representation on an infinite-dimensional Banach space;*

(3) *there is a quasi-nilpotent operator on an infinite-dimensional Banach space with no non-trivial closed invariant subspace.*

Proof. The truth of (3) is a recent result of Read [17]. We prove the equivalence of (1), (2) and (3).

In (2)$\Rightarrow$(3), if $\pi : A \to \mathcal{L}(X)$ is a continuous, TI representation with $\pi(a) \neq 0$ for some $a \in A$, then the desired operator $\pi(a)$ is quasi-nilpotent. In (3)$\Rightarrow$(1), if T is the given quasi-nilpotent operator on X, then the closed subalgebra of $\mathcal{L}(X)$ that it generates is radical.

Remark 4.3 Since finite-dimensional subspaces are automatically closed, all finite-dimensional TI representations of algebras are strictly irreducible. Hence, if a radical algebra has TI representations, they must be infinite-dimensional.

Finally, we note that the famous problem of the existence of a topologically simple commutative radical Banach algebra is equivalent to asking for a commutative radical Banach algebra for which the left regular representation is TI.

5 Topologically transitive representations

The obvious first question about TT representations is whether there are TI representations which are not TT. One way in which such representations might occur is as the left regular representations of radical Banach algebras with no non-trivial closed left ideals, if such exist.

Theorem 5.1 *If A is a Banach algebra of dimension greater than 1, then the left regular representation of A on A is not 2-TT.*

Proof. We begin by proving this under the assumption
(*) there are elements $x, y \in A$ such that x and xy are linearly independent.

Suppose the representation is 2-TT. Then, for every $\varepsilon > 0$ there is an element $a \in A$ such that

$$\|axy - x\| < \varepsilon \qquad \text{and} \qquad \|ax\| < \varepsilon.$$

Then, for every $\varepsilon > 0$,

$$\|x\| \leqslant \|axy - x\| + \|ax\| \, \|y\| < \varepsilon(1 + \|y\|).$$

Thus $x = 0$, contradicting (*).

Now assume that (*) is false. Then, for every $a, x_1, x_2 \in A$ with $\{x_1, x_2\}$ linearly independent, the elements ax_1 and ax_2 lie in the same 1-dimensional subspace (spanned by a). Thus we can not make choices of a which bring ax_1, ax_2 indefinitely close to two given linearly independent vectors y_1, y_2; so the left regular representation is not 2-TT.

Remark 5.2 The problem of whether there exists a radical Banach algebra with no non-trivial closed left ideals lies between two unsolved problems: the existence of a topologically simple radical Banach algebra and the existence of a topologically simple commutative radical Banach algebra.

Another approach to constructing TI, non-TT representations leads to the Invariant Subspace Problem, and therefore succeeds. The following theorem is probably the best topological analogue of Schur's Lemma on strictly irreducible representations. (Remember that $\mathcal{L}(X)$ here denotes the algebra of all *bounded* operators on X.)

Theorem 5.3 *If (π, X) is a 2-TT representation of a (not necessarily normed) algebra A on a normed space X, then*

$$\{T \in \mathcal{L}(X) : T\pi(a) = \pi(a)T \quad (a \in A)\} = \mathbb{C}I.$$

In particular, if A is commutative then $\dim X = 1$.

Proof. Suppose $T\pi(a) = \pi(a)T$ and T is not a multiple of the identity. Let $\xi \in X$ be such that ξ and $T\xi$ are linearly independent. Let $\eta, \zeta \in X$ be arbitrary. If π were 2-TT, we could find a sequence (b_n) in A with $\pi(b_n)\xi \to \eta$ and $\pi(b_n)(T\xi) \to \zeta$. However,

$$\pi(b_n)(T\xi) = T(\pi(b_n)\xi) \to T\eta.$$

Therefore $\zeta = T\eta$, contradicting the arbitrariness of ζ.

Corollary 5.4 *For a commutative Banach algebra, $T_2(A) = J(A)$.*

Applying Theorem 5.3 to the infinite-dimensional TI representation derived from the solution to the Invariant Subspace Problem (Proposition 4.1(1)), we obtain the following corollary.

Corollary 5.5 *There is a commutative Banach algebra with a continuous TI representation which is not 2-TT.*

Read's new example (Proposition 4.2) yields a significantly stronger statement.

Corollary 5.6 *There is a commutative Banach algebra with $T(A) \neq T_2(A)$.*

Remark 5.7 Beauzamy ([1] Chapter XIV) and Read's papers [16], [17] on the invariant subspace problem give examples where the Banach space is ℓ_1, so, in these corollaries, the pathology may be confined to the algebra (whose structure is unclear) and the representation, rather than the Banach space. We conjecture that there are examples with straightforward algebras and Banach spaces, the pathology being confined just to the representations.

It is interesting to explore generalizations of Theorem 5.3 to algebras satisfying polynomial identities. Bearing in mind the Amitsur–Levitzki Theorem, that $M_n(\mathbb{C})$ satisfies the standard polynomial identities S_k for $k \geqslant 2n$, we make the following conjecture.

Conjecture 5.8 Let $n \geqslant 1$. If A is an algebra satisfying the standard polynomial identity S_{2n-1}, then every n-TT representation of A on a normed space X has $\dim X < \infty$, (and is therefore a strictly irreducible representation with $\dim X < n$).

The obvious approach goes as follows. Suppose $\pi : A \to \mathcal{L}(X)$ is a n-TT representation with $\dim X \geqslant n$. Let $\{e_1, \ldots e_n\}$ be a linearly independent set in X. Now the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices does not satisfy S_{2n-1}. Let $T_1, \ldots, T_{2n-1}$ be linear mappings on the span of $\{e_1, \ldots e_n\}$ such that $S_{2n-1}(T_1, \ldots, T_{2n-1}) \neq 0$. Since π is n-TT, we can find $a_1, \ldots, a_{2n-1} \in A$ with $\pi(a_i)e_j$ approximating $T_i e_j$ ($1 \leqslant i \leqslant 2n-1,\ 1 \leqslant j \leqslant n$). Unfortunately, we have no control over the norms $\|a_i\|$ and so the elements $\pi(a_{i_1}) \ldots \pi(a_{i_{2n-1}})e_j$ might not approximate to $T_{i_1} \ldots T_{i_{2n-1}} e_j$.

Theorem 5.9 *Let $n \geqslant 1$. If A is an algebra satisfying the standard polynomial identity S_{n+1}, then every 2^n-TT representation of A on a normed space X has $\dim X < \infty$, (and is therefore a strictly irreducible representation with $\dim X \leqslant [(n+1)/2]$).*

Proof. The following notation will be useful: if $B = \{b_1, \ldots, b_k\} \subseteq A$ and π is our representation of A, then

$$S(B) = \pm S_k(\pi(b_1), \ldots, \pi(b_k)),$$

where we shall ignore the sign. (This is a convenient shorthand; a more detailed proof merely requires a straightforward, but obfuscating, replacement of this notation by one dependent on particular orderings of the subsets B for which $S(B)$ is used.) We write $S(\emptyset) = I$, the identity operator.

We shall prove, by induction on n, that if $\pi : A \to \mathcal{L}(X)$ is a 2^n-TT representation of an algebra A on an infinite-dimensional normed space X and $x \in X \setminus \{0\}$, then there exist $a_1, \ldots, a_{n+1} \in A$ such that $\{S(B)x : B \subseteq \{a_1, \ldots, a_{n+1}\}\}$ is linearly independent. In particular, this implies that

$$\pi(S_{n+1}(a_1, \ldots, a_{n+1})) = \pm S(\{a_1, \ldots, a_{n+1}\}) \neq 0,$$

so A does not satisfy the identity S_{n+1}.

The induction starts trivially at $n = 0$. Suppose the result has been proved for $n-1$, where $n \geqslant 1$. Given a 2^n-TT representation $\pi : A \to \mathcal{L}(X)$ and $x \in X \setminus \{0\}$, we use the induction hypothesis to find $a_1, a_2, \ldots, a_n \in A$ such that the set $E_0 = \{S(B)x : B \subseteq \{a_1, \ldots, a_n\}\}$ is linearly independent.

As a first approximation to a_{n+1}, we find an element $b_1 \in A$ such that $\pi(b_1)x \notin \mathrm{span} E_0$, i.e. the set

$$E_1 = \{S(B)x : B \subseteq \{a_1, \ldots, a_n\}, B = \{b_1\}\}$$

is linearly independent.

Let $W_1 = \emptyset, W_2, W_3, \ldots, W_{2^n}$ be an enumeration of all the subsets of $\{a_1, \ldots, a_n\}$, ordered so that $|W_i| \leqslant |W_j|$ $(i \leqslant j)$, (where $|W|$ denotes the cardinality of W). We construct successive approximations b_k to the desired a_{n+1} such that

$$E_k = \{S(W_i \cup \{b_k\})x : 1 \leqslant i \leqslant k\} \cup E_0$$

is linearly independent. We will then set $a_{n+1} = b_{2^n}$.

We have already described the construction of b_1. Suppose b_{k-1} has been constructed as above. Each $S(W_i \cup \{b_k\})x$ $(i < k)$ may be written

$$S(W_i \cup \{b_k\})x = \sum \pm S(U)\pi(b_k)S(V)x,$$

where the summation is over all partitions $U \cup V = W_i$ and is therefore a continuous function of the vectors $\pi(b_k)S(W_j)x \in X$ $(j \leqslant i)$. The induction hypothesis on b_{k-1} means that the set

$$F_k = \{S(W_i \cup \{b_k\})x : 1 \leqslant i \leqslant k-1\} \cup \{S(B)x : B \subseteq \{a_1, \ldots, a_n\}\}$$

is linearly independent when $b_k = b_{k-1}$ (making $F_k = E_{k-1}$). If $S(W_k \cup \{b_{k-1}\})x$ is not in the linear span of E_{k-1}, then we may set $b_k = b_{k-1}$. Suppose otherwise. We shall set $b_k = b_{k-1} + d_k$ for some perturbation d_k which is 'small' *in the sense that* the vectors $\pi(d_k)S(W_j)x$ are small for all $j < k$. Our basic idea is that by the stability of linear independence under small perturbations, (see e.g. [10] Corollary 20.7), the fact that F_k is linearly independent remains true as b_k is perturbed away from b_{k-1}, provided that d_k is sufficiently small in the above sense.

Choose a vector y outside the span of E_{k-1}. Then the set $E_{k-1} \cup \{S(W_k \cup \{b_{k-1}\})x + y\}$ is linearly independent. By the induction hypothesis on a_n, the points $S(W_j)x$ $(1 \leqslant j \leqslant 2^n)$ are linearly independent. These 2^n points may be separated by π; we can therefore find 'small' d_k so that $\pi(d_k)S(W_k)x$ approximates y. Now

$$S(W_k \cup \{b_k\})x = S(W_k \cup \{b_{k-1}\})x + \pi(d_k)S(W_k)x + \sum\nolimits' \pm S(U)\pi(d_k)S(V)x$$

where the sum $\sum'$ is taken over all partitions $U \cup V = W_k$ with $U \neq \emptyset$. The last term is a continuous function of the vectors $\pi(d_k)S(W_j)x$ $(j < k)$. Therefore, if we choose sufficiently 'small' d_k with $\pi(d_k)S(W_k)x$ sufficiently close to y, then we can ensure that the perturbation from

$$E_{k-1} \cup \{S(W_k \cup \{b_{k-1}\})x + y\}$$

to

$$E_k = F_k \cup \{S(W_k \cup \{b_k\})x\}$$

is small enough to preserve linear independence. This completes the induction step in the construction of the b_k. Putting $a_{n+1} = b_{2^n}$ then completes the induction step for the whole proof.

Corollary 5.10 *Every TT representation of a PI-algebra is finite-dimensional and hence strictly irreducible.*

Proof. If A is a PI-algebra, then $A/\beta(A)$ is PI and so, by a corollary of Kaplansky's Theorem ([19] Theorem 6.1.28), satisfies a standard identity.

Every TT representation $\pi : A \to \mathcal{L}(X)$ has $\ker \pi \supseteq \beta(A)$ and so induces a representation $\pi' : A/\beta(A) \to \mathcal{L}(X)$ with $\pi'(A/\beta(A)) = \pi(A)$. In particular, π' is TT. By Theorem 5.9, π' is strictly irreducible, so π is strictly irreducible.

6 A general theory of radicals

In this section, we develop a little of a general theory of radicals in normed algebras. The calculations are generally straightforward once the correct definitions are in place; but the theory is quite sensitive to these. Our attempts based on only slightly different definitions, such as the obvious analogue of Divinsky's algebraic definition or a definition including non-closed ideals in (4) below have foundered on seemingly insignificant technicalities.

Nevertheless, several variations do work. In the following definition we present simultaneously our algebraic and topological notions of a 'radical', and some of our theorems will exist in both contexts. In Section 10 we shall discuss the variant of the topological version in which radicals are defined for incomplete algebras.

Definition 6.1 By a *radical* (respectively, a *topological radical*), we mean a map R associating with each algebra (Banach algebra) A a (closed) ideal $R(A) \lhd A$ such that the following hold.

(1) $R(R(A)) = R(A)$.

(2) $R(A/R(A)) = \{0\}$, where $\{0\}$ denotes the zero coset in $A/R(A)$.

(3) If A, B are (Banach) algebras and $\phi : A \twoheadrightarrow B$ is a (continuous) epimorphism, then $\phi(R(A)) \subseteq R(B)$. ('Epimorphism' here means just 'surjective homomorphism'.)

(4) If I is a (closed) ideal of A, then

  (a) $R(I)$ is a (closed) ideal of A and
  (b) $R(I) \subseteq R(A) \cap I$.

We say that R is a *hereditary (topological) radical* if it satisfies (2), (3), (4) and

(5) If I is a (closed) ideal of A, then $R(I) \supseteq R(A) \cap I$.

(Note that (5) $\Rightarrow$ (1).)

We say that an algebra A is *R-semisimple* if $R(A) = \{0\}$ and *R-radical* if $R(A) = A$.

We have preferred to cast our theory in terms of maps, rather than 'radical property' used by other authors (e.g. Divinsky [3] p.3), to facilitate generalizations (Definition 6.2). Translation between the two forms is easy: the 'property' corresponding to a radical R is A being equal to $R(A)$; the map corresponding to a given property associates with a (Banach) algebra A the largest (closed) ideal of A with the given property. Divinsky's definition and its obvious topological analogue, stated in terms of a map R, are our definitions with (3) and (4) replaced by

(3)′ If A, B are (Banach) algebras, $A = R(A)$ and $\phi : A \twoheadrightarrow B$ is a (continuous) epimorphism, then $B = R(B)$.

(4)′ if I is a (closed) ideal of A with $R(I) = I$, then $I \subseteq R(A)$.

In the algebraic case, this is easily equivalent to our definition, but it is not clear whether this is so in the topological case. (Clearly (3) ⇒ (3)′ and (4) ⇒ (4)′, in both cases. Also the implication ((2) & (3)′ & (4)′) ⇒ (4) can be proved in the topological case by the method of Divinsky's Theorem 47 if the radical satisfies a weak non-triviality condition: that all Banach algebras with zero multiplication be radical.) We claim that our definition is at least as aesthetically satisfying as Divinsky's and is easier to work with in the topological case and we leave the detailed investigation of the relation between the two for others to study.

Condition (5) has the following equivalent (in the presence of (4)) formulation (see [3] p.123, Lemma 68).

(5)′ Every (closed) ideal I of A with $I \subseteq R(A)$ has $I = R(I)$.

We shall order radical and other such maps by inclusion: we write $R \leqslant S$ to mean $R(A) \subseteq S(A)$ for all algebras A (all Banach algebras A, in the topological case).

Definition 6.2 We shall say that a map $A \mapsto R(A)$ which associates with each (Banach) algebra a (closed) ideal is an *under radical (UR)*, (respectively, *under topological radical (UTR)*) if it satisfies (1), (3) and (4). We shall say that it is an *over radical (OR)*, (respectively, *over topological radical (OTR)*) if it satisfies (2), (3) and (4).

The reason for the terminology is that we shall show how a radical can be constructed above a given UR (Theorem 6.6) and below a given OR (Theorem 6.10). (We avoid the Latin prefixes 'sub' and 'super' lest common usage of the latter should suggest something stronger than radical.)

One way in which UTRs arise is in trying to convert algebraic radicals into topological radicals by taking closures.

Theorem 6.3 *Let R be a UR. For Banach algebras A, define*

$$\overline{R}(A) = \overline{R(A)}.$$

Then $\overline{R}$ is a UTR.

The proof is straightforward. Unfortunately, the map $\overline{R}$ need not satisfy (2), even if R does. The natural example of this is the following.

Example 6.4 Let A be the Banach algebra $(C[0,1], *)$ of all bounded, continuous, complex-valued functions on the unit interval, with convolution multiplication and let β be the Baer radical map. Then, using the Titchmarsh Convolution Theorem, $\beta(A)$ is the ideal of all functions vanishing on a neighbourhood of zero. Hence $\overline{\beta}(A)$ is the ideal of functions vanishing at zero. The quotient $A/\overline{\beta}(A)$ is the one-dimensional algebra with zero multiplication, so condition (2) fails.

If $\overline{R}(A/\overline{R}(A)) \neq \{0\}$, then we have to look at the inverse image in A of $\overline{R}(A/\overline{R}(A))$ under the quotient map $A \twoheadrightarrow A/\overline{R}(A)$. There is no reason why the quotient of A by this ideal should be $\overline{R}$-semisimple, so we again look at the inverse image of the radical. We may expect to have to continue this process transfinitely to get a topological radical.

Definition 6.5 Let R be a map associating with each (Banach) algebra A a (closed) ideal $R(A)$. We define a transfinite sequence of such maps (R_α) by:

(i) $R_0(A) = \{0\}$;

(ii) $R_{\alpha+1}(A) = q^{-1}\left(R(A/R_\alpha(A))\right)$, where $q : A \to A/R_\alpha(A)$ is the quotient map, (so, for example, $R_1 = R$);

(iii) for limit ordinals λ,

$$R_\lambda(A) = \bigcup_{\alpha<\lambda} R_\alpha(A)$$

in the algebraic case and

$$R_\lambda(A) = \overline{\bigcup_{\alpha<\lambda} R_\alpha(A)}$$

in the topological case.

The transfinite sequence of sets $(R_\alpha(A))$ is monotonic non-decreasing, so it must stabilise at the αth stage, where α is at most the cardinality of A. We then write $R^*(A) = R_\alpha(A) = R_{\alpha+1}(A)$.

For example, if $R = \overline{\beta}$, then $R^*(A) = R_2(A)$ for the algebra A of Example 6.4 above.

Theorem 6.6 *If R is a UR (respectively, a UTR), then so is R_α, for every ordinal α, and R^* is a radical (topological radical).*

Proof. We prove the topological case, which is the one of most interest to us now. The algebraic case is similar and easier.

The fact that R^* satisfies condition (2) is easy: because the sequence has stabilised,

$$R^*(A) = q^{-1}(R(A/R^*(A))),$$

i.e. $R(A/R^*(A)) = \{0\}$. It follows that $R_\alpha(A/R^*(A)) = \{0\}$ for all α, and so $R^*(A/R^*(A)) = \{0\}$.

The rest of the proof consists of showing by transfinite induction on α, that R_α satisfies conditions (1), (3) and (4). The result is trivially true for $\alpha = 0$, which starts the induction. Most of the work lies in the induction step to successor ordinals. Suppose R_α is a topological radical. We write $(n)_\alpha$ to mean condition (n) on R_α.

(1)

$$\begin{aligned} R_{\alpha+1}(R_{\alpha+1}(A)) &= q^{-1}(R(R_{\alpha+1}(A)/R_\alpha(A))) \\ &= q^{-1}(R(R(A/R_\alpha(A)))) \\ &= q^{-1}(R(A/R_\alpha(A))), \qquad \text{by } (1)_1, \\ &= R_{\alpha+1}(A). \end{aligned}$$

(3) Suppose $\phi : A \twoheadrightarrow B$ is a continuous epimorphism between Banach algebras. By $(3)_\alpha$, we have $\phi(R_\alpha(A)) \subseteq R_\alpha(B)$. Hence ϕ induces a continuous homomorphism $\psi : A/R_\alpha(A) \twoheadrightarrow B/R_\alpha(B)$. By $(3)_1$,

$$\psi(R(A/R_\alpha(A))) \subseteq R(B/R_\alpha(B)).$$

Writing $q_A : A \twoheadrightarrow A/R_\alpha(A)$ and $q_B : B \twoheadrightarrow B/R_\alpha(B)$, we have

$$\begin{aligned} R_{\alpha+1}(B) &= q_B^{-1}(R(B/R_\alpha(B))) \\ &\supseteq q_B^{-1}(\psi(R(A/R_\alpha(A)))) \\ &= q_B^{-1}(\{\psi(x + R_\alpha(A)) : x \in R_{\alpha+1}(A)\}) \\ &= \{\phi(x) + y : x \in R_{\alpha+1}(A), y \in R_\alpha(B)\}. \end{aligned}$$

Therefore $\phi(R_{\alpha+1}(A)) \subseteq R_{\alpha+1}(B)$.

(4) Suppose I is a closed ideal of the Banach algebra A. Then $(4)_\alpha$ implies that $R_\alpha(I)$ is a closed ideal of A and $R_\alpha(I) \subseteq R_\alpha(A) \cap I$. Now $I/R_\alpha(I)$ is a closed ideal of $A/R_\alpha(I)$, so $R(I/R_\alpha(I))$ is a closed ideal of $A/R_\alpha(I)$ by $(4)_1$(a). By definition,

$$R_{\alpha+1}(I) = q_I^{-1}\left(R(I/R_\alpha(I))\right),$$

where $q_I : A \twoheadrightarrow A/R_\alpha(I)$ is the quotient map. Therefore $R_{\alpha+1}(I)$ is a closed ideal of $q_I^{-1}\left(A/R_\alpha(I)\right) = A$.

By $(4)_1$(b),

$$R(I/R_\alpha(I)) \subseteq R(A/R_\alpha(I)) \cap I/R_\alpha(I).$$

Let us write q_A for the quotient map $A \twoheadrightarrow A/R_\alpha(A)$. Then the fact that $R_\alpha(I) \subseteq R_\alpha(A)$ produces a natural map $p : A/R_\alpha(I) \twoheadrightarrow A/R_\alpha(A)$. Applying $(3)_1$ to p gives

$$R(A/R_\alpha(A)) \supseteq p(R(A/R_\alpha(I))) = q_A q_I^{-1}(R(A/R_\alpha(I))),$$

whence

$$R_{\alpha+1}(A) = q_A^{-1}(R(A/R_\alpha(A))) \supseteq q_I^{-1}(R(A/R_\alpha(I))) \supseteq R_{\alpha+1}(I).$$

(It is in proving $(4)_{\alpha+1}$ that a theory which allows non-closed ideals in (4) has problems. If I is not closed, then we cannot guarantee that $R_\alpha(I)$ is closed in A, without which $A/R_\alpha(I)$ is not a normed algebra in a quotient norm.)

The induction step at limit ordinals is easier and is therefore omitted.

The proof of Theorem 6.6 is now completed by observing that each of properties (1), (3) and (4) for R^* follows from the corresponding property for R_α by choosing α sufficiently large so that $R^*(\Gamma) = R_\alpha(\Gamma)$ for the two algebras Γ involved; i.e. for $\Gamma = A$ and $R^*(A)$ in (1), $\Gamma = A$ and B in (3) and $\Gamma = I$ and A in (4).

Corollary 6.7 *The topological radical* $\overline{R}^*(A)$ *of a Banach algebra* A *is the smallest* closed *ideal* I *of* A *such that* A/I *is* R*-semisimple.*

Corollary 6.8 *The Baer radical* β *gives rise to the topological radical* $\overline{\beta}^*$ *and* $\overline{\beta}^*(A)$*, for a Banach algebra* A*, is the smallest closed ideal* I *of* A *such that* A/I *contains no non-zero nilpotent ideals.*

(We recall that the Baer radical $\beta(A)$ can be characterized as the smallest ideal I of A such that A/I contains no non-zero nilpotent ideals.)

Definition 6.9 Let R be a map associating with each (Banach) algebra A, a (closed) ideal $R(A)$. We define a transfinite sequence (R^α) by:

(i) $R^0(A) = A$;

(ii) $R^{\alpha+1}(A) = R(R^\alpha(A))$, (so, for example, $R^1 = R$);

(iii) for limit ordinals λ,
$$R^\lambda(A) = \bigcap_{\alpha<\lambda} R^\alpha(A).$$

The transfinite sequence of sets $(R^\alpha(A))$ is monotonic non-increasing, so it must stabilise at the αth stage, where α is at most the cardinality of A. We then write $R_*(A) = R^\alpha(A) = R^{\alpha+1}(A)$.

Theorem 6.10 *If* R *is an OR (respectively, an OTR), then so is* R^α*, for every ordinal* α*, and* R_* *is a radical (topological radical).*

Proof. Again, we prove the topological case, the algebraic case being similar and easier. First we observe that (4) implies that all of the $R_\alpha(A)$ are closed ideals of A and hence so is $R_*(A)$.

The fact that R_* satisfies condition (1) is easy: because the sequence has stabilised,
$$R_*(A) = R(R_*(A)),$$
It follows that $R^\alpha(R_*(A)) = R_*(A)$ for all α, and so $R_*(R_*(A)) = R_*(A)$.

The rest of the proof consists of showing by transfinite induction on α, that R^α satisfies conditions (2), (3) and (4). The result is trivially true for $\alpha = 0$, which starts the induction. Suppose R^α is an OTR. Again, we write $(n)_\alpha$ to mean condition (n) on R^α.

(2) Since $R^{\alpha+1}(A) \subseteq R^\alpha(A)$, there is a natural continuous homomorphism
$$A/R^{\alpha+1}(A) \twoheadrightarrow A/R^\alpha(A).$$
Applying $(3)_\alpha$ to this map shows that $R^\alpha(A/R^{\alpha+1}(A))$ maps into $R^\alpha(A/R^\alpha(A))$, which is the zero coset, by $(2)_\alpha$. Therefore $R^\alpha(A/R^{\alpha+1}(A)) \subseteq R^\alpha(A)/R^{\alpha+1}(A)$, in fact, $R^\alpha(A/R^{\alpha+1}(A))$ is a closed ideal of $R^\alpha(A)/R^{\alpha+1}(A)$. Applying $(4)_1$ to this ideal,
$$R^{\alpha+1}(A/R^{\alpha+1}(A)) = R(R^\alpha(A/R^{\alpha+1}(A))) \subseteq R(R^\alpha(A)/R^{\alpha+1}(A)) = \{0\},$$
where the last step uses $(2)_1$.

(3) Suppose $\phi : A \twoheadrightarrow B$ is a continuous epimorphism between Banach algebras. By $(3)_\alpha$, we have $\phi(R^\alpha(A)) \subseteq R^\alpha(B)$. The argument in the algebraic case, continues with

$$\phi(R^{\alpha+1}(A)) = \phi(R(R^\alpha(A))) \subseteq R(\phi(R^\alpha(A))) \subseteq R(R^\alpha(B)),$$

but this fails in the topological case because, for the last step, we need $\phi(R^\alpha(A))$ to be a *closed* ideal of $R^\alpha(B)$ to apply $(4)_1$. Instead, we argue that $\overline{\phi(R^\alpha(A))}$ is a closed ideal of $R^\alpha(B)$, so, by $(4)_1$,

$$R\left(\overline{\phi(R^\alpha(A))}\right) \subseteq R(R^\alpha(B)) = R^{\alpha+1}(B).$$

Now let

$$I = \phi^{-1}\left(\overline{\phi(R^\alpha(A))}\right) \supseteq R^\alpha(A).$$

Then $R^\alpha(A)$ is a closed ideal of I, so $R^{\alpha+1}(A) = R(R^\alpha(A)) \subseteq R(I)$, by $(4)_1$. Thus, by applying $(3)_1$ to the mapping $\phi : I \twoheadrightarrow \overline{\phi(R^\alpha(A))}$, we obtain

$$\phi(R^{\alpha+1}(A)) \subseteq \phi(R(I)) \subseteq R\left(\overline{\phi(R^\alpha(A))}\right) \subseteq R^{\alpha+1}(B).$$

(4) Suppose I is a closed ideal of the Banach algebra A. Then, first, $(4)_\alpha$(a) implies that $R^\alpha(I)$ is a closed ideal of A and so, applying $(4)_1$(a) to this ideal, we see that $R^{\alpha+1}(I) = R(R^\alpha(I))$ is a closed ideal of A. Secondly, $(4)_\alpha$(b) implies that $R^\alpha(I) \subseteq R^\alpha(A)$, so $R^\alpha(I)$ is a closed ideal of $R^\alpha(A)$, and $(4)_1$(b) applied to this ideal gives

$$R^{\alpha+1}(I) = R(R^\alpha(I)) \subseteq R(R^\alpha(A)) = R^{\alpha+1}(A).$$

Again, we omit the induction step at limit ordinals, which is straightforward. The proof is completed by observing that each of properties (1), (3) and (4) for R_* follows from the corresponding property for R^α by choosing α sufficiently large so that $R_*(\Gamma) = R^\alpha(\Gamma)$ for the two algebras Γ involved.

Theorem 6.11 *Let R be a UR (UTR) and S an OR (OTR), with $R \leqslant S$. Then $R^* \leqslant S_*$. In particular, if S is a radical (topological radical), then $R^* \leqslant S$; if R is a radical (topological radical), then $R \leqslant S_*$. If R is a radical and S is a topological radical, with $R \leqslant S$, then $\overline{R}^* \leqslant S$.*

Proof. The proof consists of three steps.

(a) We first show, by transfinite induction on α, that $R_\alpha \leqslant S$. This is trivial for $\alpha = 0$ and the step to limit ordinals is easy. For the successor step, suppose $R_\alpha \leqslant S$. Then

$$R_{\alpha+1}(A)/R_\alpha(A) = R\left(A/R_\alpha(A)\right) \subseteq S\left(A/R_\alpha(A)\right).$$

Since $R_\alpha(A) \subseteq S(A)$, we have a natural (continuous) epimorphism of $A/R_\alpha(A)$ onto $A/S(A)$. This maps $S\left(A/R_\alpha(A)\right)$ into $S\left(A/S(A)\right) = \{0\}$. Therefore

$$S\left(A/R_\alpha(A)\right) \subseteq S(A)/R_\alpha(A),$$

so

$$R_{\alpha+1}(A)/R_\alpha(A) \subseteq S(A)/R_\alpha(A),$$

so $R_{\alpha+1}(A) \subseteq S(A)$.

(b) Next, we show that $R \leqslant S^\beta$ for all β. This is easier, the successor step being that if $R \leqslant S^\beta$ then $R(A)$ is a (closed) ideal in $S^\beta(A)$, so

$$R(A) = R(R(A)) \subseteq S(R(A)) \subseteq S(S^\beta(A)) = S^{\beta+1}(A).$$

(c) Since every R_α is UR (UTR), we can apply step (b) to R_α in place of R to get $R_\alpha(A) \leqslant S^\beta(A)$ for all ordinals α, β and all (Banach) algebras A. Hence $R^*(A) \subseteq S_*(A)$.

Corollary 6.12 *If R is a UR (UTR), then R^* is the smallest (topological) radical greater than or equal to R: i.e. $R^* \geqslant R$ and if S is a (topological) radical with $S \geqslant R$, then $R^* \leqslant S$. Likewise, if S is an OR (OTR), then S_* is the greatest (topological) radical less than or equal to S.*

7 Hereditary radicals

It is natural to ask whether, in Theorem 6.3, the map R satisfying axiom (5) would imply $\overline{R}$ satisfying (5). The answer is negative, as the following example shows.

Example 7.1 Let A be the commutative Banach algebra generated by $\{X_n : n = 1, 2, 3, \ldots\}$ subject to the relations $X_n^{n+1} = 0$ $(n = 1, 2, 3, \ldots)$ and $X_i X_j = 0$ $(i \neq j)$. That is, if A_0 is the algebra defined, algebraically, by these generators and relations, with the norm given by

$$\left\| \sum_{1 \leqslant j \leqslant i} \lambda_{ij} X_i^j \right\| = \sum_{1 \leqslant j \leqslant i} |\lambda_{ij}|,$$

then A is the completion of $(A_0, \|.\|)$. A typical element of A is just an infinite sum $x = \sum_{1 \leqslant j \leqslant i} \lambda_{ij} X_i^j$ with $\|x\| = \sum_{1 \leqslant j \leqslant i} |\lambda_{ij}| < \infty$.

Let $y = \sum_{i=1}^\infty 2^{-i} X_i$ and let B be the set of elements of A of the form $\sum_{2 \leqslant j \leqslant i} \lambda_{ij} X_i^j$. Then $I = \mathbb{C}y + B$ is a closed ideal of A. Now an element $\sum_{1 \leqslant j \leqslant i} \lambda_{ij} X_i^j$ of A is nilpotent if and only if $\sup\{i/j : \lambda_{ij} \neq 0\} < \infty$ and, since A is commutative, $\beta(A)$ and $\beta(I)$ are just the sets of nilpotent elements of A and I, respectively. Thus $\overline{\beta}(A) = A$ and $\overline{\beta}(I) = B$; so $\overline{\beta}(A) \cap I \not\subseteq \overline{\beta}(I)$.

Remark 7.2 In this example (5) seems to fail in a rather trivial way. Indeed, for every hereditary radical R, if I is a closed ideal in a Banach algebra A, we have $(\overline{R}(A) \cap I)^2 \subseteq \overline{R}(I)$. One consequence of this is that if $R \geqslant \beta$ then $\overline{R}(A) \cap I \subseteq \overline{R}_2(I)$, where $\overline{R}_2$ is constructed from $\overline{R}$ as in Definition 6.5. It is tempting to conjecture that this will form the start of a transfinite induction leading to $\overline{R}^*$ being hereditary, but our attempts to carry out this plan have been thwarted by that perennial problem of Banach algebra theory: the fact that the sum of two closed ideals is not necessarily closed. We do not even know whether or not $\overline{\beta}^*$ is hereditary.

8 The TT radicals

Theorem 8.1 *For $n = 1, 2, \ldots, \infty$, the map $A \to T_n(A)$ which associates with every Banach algebra A its topologically n-transitive radical is a hereditary topological radical.*

Proof.

(1) follows from (5) below and (2) and (3) are straightforward.

(4) (a) We must show that if I is a closed ideal of A, then $T_n(I)$ is an ideal of A. Suppose $a \in A$ and $b \in T_n(I)$; then $\pi(b) = 0$ for every continuous n-TT representation π of I and we need to show that $\pi(ab) = 0$ and $\pi(ba) = 0$ for all such representations.

Consider $\pi(ab)$ where $\pi : I \to \mathcal{L}(X)$ is n-TT. If $x \in X$ and $c \in I$ then

$$\pi(c)\pi(ab)x = \pi(cab)x = \pi(ca)\pi(b)x = 0.$$

This, for all $c \in I$, implies $\pi(ab)x = 0$, because π is TI. Hence $\pi(ab) = 0$. Likewise, $\pi(ba)\pi(c)x = \pi(b)\pi(ac)x = 0$ and $\{\pi(c)x : c \in I,\ x \in X\}$ is dense in X, so $\pi(ba) = 0$.

(4) (b) Every continuous representation π of A restricts to a continuous representation of a closed ideal I. It suffices to show that, for $n < \infty$, if π is n-TT then $\pi|I$ is n-TT or zero. It will then follow that $T_n(I) \subseteq T_n(A)$ for all n.

We begin by showing that if $\pi|I$ is non-zero, then it is TI. To see this, suppose $x, y \in X$ with $x \neq 0$ and $\varepsilon > 0$; let b be any element of I such that $\pi(b)x \neq 0$. (If no such b exists, then $\pi(I)(\pi(A)x) \subseteq \pi(IA)x \subseteq \pi(I)x = \{0\}$, so $\pi(I)X = 0$, contrary to assumption.) We may then find $a \in A$ such that $\|\pi(a)(\pi(b)x) - y\| < \varepsilon$, so we have $\|\pi(ab)x - y\| < \varepsilon$ and $ab \in I$.

We now show that $\pi|I$ is n-TT. Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ with $x_1, \ldots, x_n$ linearly independent and $\varepsilon > 0$. Choose $\eta > 0$ such that every set $\{z_1, \ldots, z_n\}$ with $\|z_i - x_i\| < \eta$ $(1 \leqslant i \leqslant n)$ is linearly independent. Since π restricts to a TI representation on I, by the above, we can find, for each i an element $b_i \in I$ such that $\|\pi(b_i)x_i - x_i\| < \eta/2n$. We then find $c_i \in A$ $(1 \leqslant i \leqslant n)$ such that

$$\|\pi(c_i)x_j - \delta_{ij}x_i\| < \eta/2n\|\pi(b_i)\| \qquad (1 \leqslant i, j \leqslant n).$$

Then

$$\|\pi(b_ic_i)x_j - \delta_{ij}\pi(b_i)x_i\| < \eta/2n \qquad (1 \leqslant i, j \leqslant n).$$

So

$$\|\pi(b_ic_i)x_j - \delta_{ij}x_i\| < \eta/n \qquad (1 \leqslant i, j \leqslant n).$$

Let

$$b = \sum_{i=1}^{n} b_ic_i \in I.$$

Then

$$\|\pi(b)x_j - x_j\| < \eta \qquad (1 \leqslant j \leqslant n).$$

Therefore the set $\{\pi(b)x_1, \ldots, \pi(b)x_n\}$ is linearly independent. Let $a \in A$ be such that

$$\|\pi(a)\pi(b)x_j - y_j\| < \varepsilon$$

and $ab \in I$ is the desired element such that $\pi(ab)$ sends x_j near to y_j for all j.

(5) The fact that, for every closed ideal I of a Banach algebra, $T_n(I) \supseteq T_n(A) \cap I$ follows from Lemma 8.2 below.

Lemma 8.2 *If I is a closed ideal of a Banach algebra A and $1 \leqslant n \leqslant \infty$, then for every continuous n-TT representation π of I on a Banach space $(X, \|.\|)$ there is a continuous n-TT representation ρ of A on a Banach space $(W, |.|)$ such that* $\ker \pi = \ker \rho \cap I$.

Proof of Lemma. Let $\pi : I \to \mathcal{L}(X)$ be a continuous TI representation of the ideal I on a Banach space X. We shall describe the construction of a continuous representation $\rho : A \to \mathcal{L}(W)$ and then show that, for every $k < \infty$, if π is k-TT, then so is ρ.

Let

$$Y = \pi(I)X = \left\{ \sum_{i=1}^{n} \pi(b_i)x_i : b_i \in I,\ x_i \in X\ (1 \leqslant i \leqslant n),\ n = 1, 2, 3, \ldots \right\}$$

with norm

$$|y| = \inf \left\{ \sum_{i=1}^{n} \|b_i\|\,\|x_i\| : y = \sum_{i=1}^{n} \pi(b_i)x_i \text{ as above} \right\}.$$

Let Z be the completion of $(Y, |.|)$. Then the inclusion map $Y \to X$ is continuous of norm at most $\|\pi\|$ and so extends to a continuous map $\theta : Z \to X$. Let $(W, |.|) = (Z, |.|)/\ker\theta$ and denote by $\tilde{\theta} : W \to X$ the injective map induced by θ.

We wish to define $\pi_1 : A \to \mathcal{L}(Y)$ by

$$\pi_1(a)\left(\sum_{i=1}^{n} \pi(b_i)x_i\right) = \sum_{i=1}^{n} \pi(ab_i)x_i \qquad (b_i \in I,\ x_i \in X\ (1 \leqslant i \leqslant n),\ n = 1, 2, 3, \ldots),$$

but we first need to show that this is well defined. If

$$\sum_{i=1}^{n} \pi(b_i)x_i = \sum_{j=1}^{m} \pi(b'_j)x'_j,$$

then, for all $c \in I$,

$$\begin{aligned}
\pi(c)\left(\sum_{i=1}^{n} \pi(ab_i)x_i\right) &= \sum_{i=1}^{n} \pi(cab_i)x_i \\
&= \pi(ca)\sum_{i=1}^{n} \pi(b_i)x_i \\
&= \pi(ca)\sum_{j=1}^{m} \pi(b'_j)x'_j \\
&= \sum_{j=1}^{m} \pi(cab'_j)x'_j \\
&= \pi(c)\left(\sum_{j=1}^{m} \pi(ab'_j)x'_j\right).
\end{aligned}$$

Because π is TI, it follows that

$$\sum_{i=1}^{n} \pi(ab_i)x_i = \sum_{j=1}^{m} \pi(ab'_j)x'_j.$$

To prove that $\pi_1(a) \in \mathcal{L}(Y)$ for all $a \in A$ and that π_1 is continuous, we note that, for every presentation

$$y = \sum_{i=1}^{n} \pi(b_i)x_i$$

of a given element $y \in Y$ we have

$$\begin{aligned} |\pi_1(a)y| &= \left|\sum_{i=1}^n \pi(ab_i)x_i\right| \\ &\leqslant \sum_{i=1}^n \|ab_i\|\,\|x_i\| \\ &\leqslant \|a\| \sum_{i=1}^n \|b_i\|\,\|x_i\|. \end{aligned}$$

It follows that

$$|\pi_1(a)y| \leqslant \|a\|\,|y|.$$

We can now extend each $\pi_1(a) \in \mathcal{L}(Y)$ to $\pi_2(a) \in \mathcal{L}(Z)$ by continuity, thus defining a continuous representation $\pi_2 : A \to \mathcal{L}(Z)$. For $b \in I$, and $y = \sum_{i=1}^n \pi(b_i)x_i \in Y$ we have

$$\pi_1(b)(y) = \sum_{i=1}^n \pi(bb_i)x_i = \sum_{i=1}^n \pi(b)\pi(b_i)x_i = \pi(b)y;$$

i.e.

$$\pi_2(b)(y) = \pi(b)(\theta(y)).$$

By continuity (i.e. $\pi_2(b) \in \mathcal{L}(Z)$, $\pi(b) \in \mathcal{L}(X)$ and $\theta : Z \to X$ being continuous) and the fact that Y is dense in Z, it follows that

$$\pi_2(b)(z) = \pi(b)(\theta(z)) \qquad (z \in Z,\ b \in I). \tag{1}$$

Let $a \in A$, $z \in Z$; then for all $c \in I$ we have

$$\pi(c)\theta(\pi_2(a)z) = \pi_2(c)\pi_2(a)z = \pi_2(ca)z = \pi(ca)\theta(z).$$

Therefore, if $\theta(z) = 0$ then $\pi(c)\theta(\pi_2(a)z) = 0$ for all $c \in I$ and so, because π is TI on X, we have $\theta(\pi_2(a)z) = 0$. Thus, for each $a \in A$, there is a well-defined mapping $\rho(a) : W \to W$ such that

$$\rho(a)(z + \ker\theta) = \pi_2(a)z + \ker\theta.$$

Clearly ρ is a continuous representation of A on the Banach space W. Equation (1) implies that

$$\rho(b)(w) = \pi(b)(\tilde{\theta}(w)) \qquad (w \in W,\ b \in I). \tag{2}$$

Now suppose that π is k-TT with $k < \infty$. We show that ρ is k-TT. Since $Y/\ker\theta$ is dense in W, it suffices to show that for every linearly independent $w^{(1)}, \ldots, w^{(k)} \in W$, every $y^{(1)}, \ldots, y^{(k)} \in Y/\ker\theta$ and $\varepsilon > 0$ there is an $a \in A$ with $|\rho(a)w^{(j)} - y^{(j)}| < \varepsilon$ $(1 \leqslant j \leqslant k)$. We shall, in fact, show that this can be done with an $a \in I$: thus (by (2)) we shall show

$$|\pi(a)x^{(j)} - y^{(j)}| < \varepsilon \qquad (1 \leqslant j \leqslant k),$$

where $x^{(j)} = \tilde{\theta}(w^{(j)})$. Since $\tilde{\theta}$ is injective, the set $\{x^{(1)}, \ldots, x^{(k)}\}$ is linearly independent.

We can write $y^{(j)} = \sum_{i=1}^n \pi(b_i)x_i^{(j)}$ $(1 \leqslant j \leqslant k)$, with the $b_i \in I$, $x_i^{(j)} \in X$, by the simple expedient of making the sets $N_j = \{i : x_i^{(j)} \neq 0\}$ disjoint. Because the original representation π is k-TT, we can find $c_i \in I$ $(1 \leqslant i \leqslant n)$ so that

$$\|\pi(c_i)x^{(j)} - x_i^{(j)}\| < \frac{\varepsilon}{n\|b_i\|} \qquad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant k).$$

Let $a = \sum_{i=1}^n b_i c_i$. Then

$$\begin{aligned}
|\pi(a)x^{(j)} - y^{(j)}| &= \left| \sum_{i=1}^n \pi(b_i c_i) x^{(j)} - \sum_{i=1}^n \pi(b_i) x_i^{(j)} \right| \\
&= \left| \sum_{i=1}^n \pi(b_i) \left(\pi(c_i) x^{(j)} - x_i^{(j)} \right) \right| \\
&\leqslant \sum_{i=1}^n \|b_i\| \, \|\pi(c_i)x^{(j)} - x_i^{(j)}\|, \qquad \text{by the definition of } |.|, \\
&< \varepsilon.
\end{aligned}$$

Finally, equation (2), together with the fact that $\tilde{\theta}(W) = \theta(Z)$ is dense in X shows that if $b \in I$, then $\rho(b) = 0$ if and only if $\pi(b) = 0$. Thus $\ker \pi = \ker \rho \cap I$.

9 Standard TI representations

Let us recall that a standard TI representation of a normed algebra A is a representation π of A on a Banach space X which is of the form $\pi = \phi\rho$ where ϕ is a continuous homomorphism of A onto a dense subalgebra of a Banach algebra B and ρ is a strictly irreducible representation of B on X.

Notice that there is no point in broadening this definition by dropping the completeness requirements on either X or B, separately. We have already remarked that if X were incomplete, or just a general linear space then it would automatically have a suitable Banach space structure. If B were incomplete, we should require that ρ be continuous, in order to make π continuous. This being so, we need only complete B and extend ρ to the completion to regain our original scenario.

Since ρ may be factored through the primitive algebra $B/\ker \rho$, we can characterize $S(A)$ as the intersection of the kernels of the continuous homomorphisms of A onto dense subalgebras of primitive (or semisimple) Banach algebras.

Theorem 9.1 *The map $A \to S(A)$, defined for Banach algebras A, is an OTR. Hence S_* is a topological radical.*

Proof. Notation: throughout this proof, B will be an arbitrary semisimple Banach algebra and ϕ mapping into B will be a continuous homomorphism with dense range. We shall use the characterisation of $S(A)$ as the intersection of the kernels of such mappings $\phi : A \to B$.

(2) Every $\phi : A \to B$ induces a continuous homomorphism $\tilde{\phi} : A/S(A) \to B$ with the same range. The intersection of the kernels of these induced homomorphisms is zero, so $S(A/S(A)) = \{0\}$.

(3) If A_1, A_2 are Banach algebras and $\psi : A_1 \to A_2$ is a continuous epimorphism, then every $\phi : A_2 \to B$ gives rise to a continuous homomorphism $\phi\psi : A_1 \to B$ with dense range. The fact that $\psi(S(A_1)) \subseteq S(A_2)$ follows immediately.

(4) (a) Let I be a closed ideal of A and let $a \in A$, $b \in S(I)$; we show that $ab \in S(I)$, the case of ba being similar. Given $\phi : I \to B$ as above, we have $\phi(b) = 0$. For arbitrary $c \in I$ we have

$$\phi(c)\phi(ab) = \phi(cab) = \phi(ca)\phi(b) = 0.$$

Since $\phi(I)$ is dense in B, it follows that $B\phi(ab) = \{0\}$ and hence, since B is semisimple, that $\phi(ab) = 0$.

(4) (b) Let I be a closed ideal of A. Let $\phi : A \to B$ as usual. Now $I \lhd A$ implies $\phi(I) \lhd B$, (because $\phi(A)$ is dense in B). Therefore $\overline{\phi(I)} \lhd B$. Therefore $\overline{\phi(I)}$ is semisimple; (the Jacobson radical is hereditary). Thus $\phi : I \to \overline{\phi(I)}$ is a continuous homomorphism into a semisimple Banach algebra with dense range. The fact that $S(I) \subseteq S(A) \cap I$ follows immediately.

We can summarize the relationships between our various topological radicals as follows.

Theorem 9.2 *The topological radicals S_* and T_n $(1 \leqslant n \leqslant \infty)$ are related to the closed-Baer radical $\overline{\beta}^*$ and the Jacobson radical J by the inequalities*

$$\overline{\beta}^* \leqslant T_1 \leqslant T_m \leqslant T_n \leqslant T_\infty \leqslant S_* \leqslant J \qquad (1 \leqslant m \leqslant n \leqslant \infty).$$

Proof. The inequality $\overline{\beta} \leqslant T_1$ is Proposition 3.3 and the other inequalities are trivial. The result follows by Theorem 6.11.

The main result of this section is that the last of these inequalities is proper.

Example 9.3 We construct an example of a radical Banach algebra A which has an injective, dense embedding ϕ into a semisimple Banach algebra B. Hence, $S(A) = \{0\}$.

The algebra B is the algebra regrettably called A in [5]. Let A_0 be the algebra on symbols $X_1, X_2, \ldots$ subject to the following relations: every monomial $X_{i_1} \ldots X_{i_r}$ containing more than n occurrences of X_n, where $n = \max\{i_1, \ldots, i_r\}$, must vanish. It follows that $X_{i_1} \ldots X_{i_r} = 0$ if $r \geqslant (n+1)!$, where $n = \max\{i_1, \ldots, i_r\}$. This algebra is given a norm $\|.\|$ by

$$\left\| \sum_{i=0}^{n} \lambda_i M_i \right\| = \sum_{i=0}^{n} |\lambda_i|,$$

where the λ_i are scalars and the M_i monomials. The Banach algebra B is the completion of $(A_0, \|.\|)$.

We shall construct the radical Banach algebra A as the completion of A_0 in a larger norm, so that there is a natural continuous embedding of A into B, whose range contains A_0 and is therefore dense.

For each monomial $M = X_{i_1} \ldots X_{i_r}$, we define $n(M) = \max\{i_1, \ldots, i_r\}$ and $|M| = ((n+1)!)^{(n+1)!}$. Consider a product of $2k$ monomials $M_1 \ldots M_{2k}$, where $k \geqslant 1$. We distinguish three cases.

(a) If $(n(M_i)+1)! \leqslant k$ $(1 \leqslant i \leqslant k)$, then $M_1 \ldots M_k = 0$.

(b) If $(n(M_i)+1)! \leqslant k$ $(k+1 \leqslant i \leqslant 2k)$, then $M_{k+1} \ldots M_{2k} = 0$.

(c) If neither (a) nor (b) hold, then there are at least two values of i for which $(n(M_i)+1)! > k$ and it follows from the definition of $|M_i|$ that

$$|M_1 \ldots M_{2k}| \leqslant k^{-k} |M_1| \ldots |M_{2k}|. \tag{3}$$

In all three cases, (3) holds.

We extend the norm to general elements of A_0 by defining

$$\left|\sum_{i=0}^{n} \lambda_i M_i\right| = \sum_{i=0}^{n} |\lambda_i||M_i|,$$

where the λ_i are scalars and the M_i monomials. We then define A to be the completion of A_0 in this norm, which is clearly identified with the set of infinite sums

$$\sum_{i=0}^{\infty} \lambda_i M_i$$

such that

$$\sum_{i=0}^{n} |\lambda_i||M_i| < \infty.$$

Hence the natural embedding of A_0 into B extends to a natural embedding of A into B.

It follows from (3) that

$$|x_1 \dots x_{2k}| \leqslant k^{-k}|x_1| \dots |x_{2k}|$$

for all $x_1, \dots, x_{2k} \in A$. Since

$$\left(k^{-k}\right)^{1/2k} \to 0$$

as $k \to \infty$, we see that A is topologically nilpotent, and so *a fortiori* radical.

Notice that the TI representations of A restrict to TI representations of the (incomplete) normed algebra A_0. This is interesting, since A_0 is locally nilpotent and hence nil, but not semiprime. This example prevents us from extending Proposition 3.3 to the Levitzki and nil radicals.

Corollary 9.4 *There is a locally nilpotent normed algebra with a separating family of continuous TI representations.*

It would be interesting to know how general this construction can be made, so as to produce a wide variety of S_*-semisimple, Jacobson-radical algebras. The following theorem is a first step in that direction.

Theorem 9.5 *Let $(B, \|.\|)$ be a Banach algebra with an increasing family of nilpotent subalgebras M_n $(n = 1, 2, 3, \dots)$ such that $M_n^n = \{0\}$. Then there is a radical Banach algebra $(A, |.|)$ and a continuous injective homomorphism $\phi : A \to B$ such that $\bigcup_{n=1}^{\infty} M_n \subseteq \phi(A) \subseteq B$.*

Proof. Let $M = \bigcup_{n=1}^{\infty} M_n$. For $x \in M$ define

$$|x| := \inf \left\{ \sum_{i=1}^{k} \nu(i)\|m_i\| : x = \sum_{i=1}^{k} m_i, \quad m_i \in M_i \right\},$$

where the increasing sequence of positive real numbers $\nu(i)$ will be defined later.

The function $|.|$ is clearly a norm on M. If $\nu(i) \geqslant 1$ for all i then $|.|$ is submultiplicative and $\|x\| \leqslant |x|$ for all $x \in M$. Let A be the completion of $(M, |.|)$.

Now consider $x^{(1)}, \ldots, x^{(2N)} \in M$ with

$$x^{(n)} = \sum_{i_n} m_{i_n}^{(n)} \qquad (1 \leqslant n \leqslant 2N),$$

for some $m_{i_n}^{(n)} \in M_{i_n}$. Then $m_{i_1}^{(1)} \ldots m_{i_{2N}}^{(2N)} \in M_r$ where $r = \max\{i_1, \ldots, i_{2N}\}$, so

$$\begin{aligned}
\left| x^{(1)} \ldots x^{(2N)} \right| &\leqslant \sum_{i_1} \cdots \sum_{i_{2N}} \max_n \nu(i_n) \| m_{i_1}^{(1)} \ldots m_{i_{2N}}^{(2N)} \| \\
&\leqslant \sum_{i_1} \cdots \sum_{i_{2N}} \frac{\nu(i_1) \ldots \nu(i_{2N})}{\nu(N)} \| m_{i_1}^{(1)} \ldots m_{i_{2N}}^{(2N)} \| \qquad (4) \\
&\leqslant \sum_{i_1} \cdots \sum_{i_{2N}} \frac{\nu(i_1) \ldots \nu(i_{2N})}{\nu(N)} \| m_{i_1}^{(1)} \| \ldots \| m_{i_{2N}}^{(2N)} \|,
\end{aligned}$$

where (4) holds because at least two of the i_j exceed N; for otherwise, the sequence $m_{i_1}^{(1)}, \ldots, m_{i_{2N}}^{(2N)}$ would contain at least N consecutive terms belonging to M_N and therefore $m_{i_1}^{(1)} \ldots m_{i_{2N}}^{(2N)} = 0$, since $M_N^N = \{0\}$. Thus

$$|x^{(1)} \ldots x^{(2N)}| \leqslant \frac{1}{\nu(N)} |x^{(1)}| \ldots |x^{(2N)}|$$

for all $x^{(1)}, \ldots, x^{(2N)}$ in M and hence, by continuity, in A. Taking $\nu(n) = n^n$ makes A topologically nilpotent, and hence radical.

The inequality $\|x\| \leqslant |x|$ $(x \in M)$ shows that the natural embedding $\phi : (M, |.|) \to (B, \|.\|)$ is continuous and therefore extends to a continuous homomorphism $\phi : A \to B$. However, it is not clear that $\phi : A \to B$ is necessarily injective. If not, we obtain an injective map by simply replacing A by $A/\ker\phi$.

10 Radicals in incomplete algebras

Our theory of topological radicals can be developed equally well in the context of incomplete normed algebras: simply replace 'Banach algebra' by 'normed algebra' throughout Section 6. Section 8 may be treated likewise: the maps T_n are topological radicals for normed algebras. In Section 9, the same recipe applies, except that the algebra B must remain Banach and the inequality $S_* \leqslant J$ of Corollary 9.2 no longer applies (see Example 10.6 below).

This alternative theory has the advantage that its axiom (3) gives information about the behaviour of the radical under all continuous homomorphisms, not just those with complete range. However, it has a major disadvantage: it excludes the Jacobson radical because the Jacobson radical of an incomplete normed algebra is not necessarily closed.

Example 10.1 Let B be the subalgebra of $(C[0,1], *)$ consisting of the polynomials. Then B is algebraically isomorphic to the subalgebra of $\mathbb{C}[X]$ consisting of polynomials without constant term. Therefore $J(B) = \{0\}$, indeed, B has a separating family of (discontinuous) transitive 1-dimensional representations.

Now let A be the subalgebra of $(C[0,1], *)$ consisting of all functions which are polynomial on a neighbourhood of 0. There is an obvious homomorphism of A onto B and so the Jacobson radical $J(A)$ is contained in the inverse image of $\{0\}$; that is, the set of functions vanishing in a neighbourhood of 0. The reverse inclusion is obvious: in fact, if $f \in A$ vanishes on a neighbourhood of 0, then f is nilpotent in A. Thus

$$J(A) = \{f \in A : f \equiv 0 \text{ on a neighbourhood of } 0\},$$

which is not closed: $\overline{J(A)}$ is the ideal of functions vanishing at zero.

This example is very similar to Example 6.4. As there, the quotient $A/\overline{J}(A)$ is the one-dimensional algebra with zero multiplication, so $\overline{J}$ does not satisfy axiom (2), and we have to go to $\overline{J}^*$ to obtain a topological radical. However, this is only one of the possible ways to get a topological radical for normed algebras which reduces to the Jacobson radical for Banach algebras. In order to study these, we need some basic information about representations of incomplete algebras. This is a little-studied topic. It is well-known that strictly irreducible representations of Banach algebras are transitive, (which means that we do not have to consider radicals based on different degrees of transitivity), but the following variation in which the completeness hypothesis is on the space rather than the algebra seems not to be available in the literature.

Theorem 10.2 *Every strictly irreducible normed representation of an algebra on a Banach space is transitive.*

Proof. Let $\pi : A \to \mathcal{L}(X)$ be a strictly irreducible representation of A on the Banach space X. Let us write $\mathcal{U}(X)$ for the algebra of all linear endomorphisms of X. Then Schur's Lemma tells us that

$$\{S \in \mathcal{U}(X) : S\pi(a) = \pi(a)S \quad (a \in A)\}$$

is a division algebra. We are interested in the set

$$\pi(A)' = \{S \in \mathcal{L}(X) : S\pi(a) = \pi(a)S \quad (a \in A)\}.$$

Then every $S \in \pi(A)'$ is bijective, and therefore, by Banach's Isomorphism Theorem, has an inverse in $\pi(A)'$. Thus $\pi(A)'$ is a normed division algebra. By the Gelfand-Mazur Theorem, $\pi(A)'$ consists of the scalar multiples of the identity. The remainder of the proof is standard (see [14] Theorem 4.2.13).

Remark 10.3 If $\pi : A \to \mathcal{L}(X)$ is a continuous strictly irreducible representation of a normed algebra A on a Banach space X, then we may regard $\pi : A \to \overline{\pi(A)}$ as a continuous homomorphism of A into the Banach algebra $\overline{\pi(A)}$ and the identity map $\overline{\pi(A)} \to \mathcal{L}(X)$ as a strictly irreducible representation of $\overline{\pi(A)}$. Thus π is standard.

In looking for topological radicals of normed algebras corresponding to the Jacobson radical, let us restrict our attention to those based on continuous representations on Banach spaces. The natural concept to define is the following.

Definition 10.4 For a normed algebra A, let $I(A)$ denote the intersection of the kernels of the continuous strictly irreducible representations of A on Banach spaces.

If A is Banach, $J(A) = I(A)$. We shall show that I is a (hereditary) topological radical, different from $\overline{J}^*$.

Theorem 10.5 *The mapping I is a hereditary topological radical.*

Proof. The proof proceeds as in Theorem 8.1, the analogue of Lemma 8.2 going as follows.

If I is a closed ideal of a normed algebra A and π is a continuous strictly irreducible representation of I on a Banach space X, we show that there is a continuous strictly irreducible representation $\hat{\pi}$ of A on X such that $\ker \pi = \ker \hat{\pi} \cap I$.

Let $\tilde{A}$ be the completion of A and let $\tilde{I}$ be the closure of I in $\tilde{A}$. Then $\tilde{I}$ is a closed ideal in $\tilde{A}$. Since π is continuous, it extends to a continuous strictly irreducible representation $\tilde{\pi} : \tilde{I} \to \mathcal{L}(X)$. We use the usual algebraic method to extend this to a representation $\hat{\pi}$ mapping $\tilde{A}$ into the algebra of endomorphisms of the vector space X: we choose any nonzero $x_0 \in X$; every $y \in X$ may be written in the form $y = \tilde{\pi}(b)x_0$ for some $b \in \tilde{I}$; for $a \in \tilde{A}$ we define $\hat{\pi}(a)$ by $\hat{\pi}(a)y = \tilde{\pi}(ab)x_0$. It is easy to check that $\hat{\pi}$ is a well-defined homomorphism. The restriction $\hat{\pi}|A$ to A is strictly irreducible, since it is an extension of the strictly irreducible representation π. It remains to show that $\hat{\pi}(a) \in \mathcal{L}(X)$ and that $\hat{\pi}$ is continuous.

The mapping $b \mapsto \tilde{\pi}(b)x_0 : \tilde{I} \to X$ is continuous, surjective, and therefore open. Thus, there is a constant $K > 0$ such that for every $y \in X$ there is a $b \in \tilde{I}$ with $\|b\| \leqslant K\|y\|$ such that $y = \tilde{\pi}(b)x_0$. Then

$$\|\hat{\pi}(a)y\| \leqslant \|\tilde{\pi}\| \, \|ab\| \, \|x_0\| \leqslant K\|\tilde{\pi}\| \, \|a\| \, \|y\| \, \|x_0\| \qquad (a \in A, \ y \in X),$$

$$\|\hat{\pi}(a)\| \leqslant K\|\tilde{\pi}\| \, \|a\| \, \|x_0\| \qquad (a \in A).$$

Thus $\hat{\pi}(a) \in \mathcal{L}(X)$ and $\hat{\pi}$ is continuous of norm at most $K\|\tilde{\pi}\| \, \|x_0\|$.

We have $J(A) \subseteq I(A)$ for all normed algebras A. Therefore, by Theorem 6.11, $\overline{J}^* \leqslant I$. The following example shows that we can have $\overline{J}^* \neq I$.

Example 10.6 Let B be the subalgebra of $(C[0,1], *)$ consisting of the polynomials, discussed in Example 10.1. Then $J(B) = \{0\}$, and so $\overline{J}^*(A) = \{0\}$. On the other hand, any continuous TI representation of B would extend by continuity to a TI representation of $(C[0,1], *)$, which is impossible, since $\overline{\beta}^*(C[0,1]) = C[0,1]$. Therefore $T_1(B) = B$, so $I(B) = B$.

This example shows that to make Theorem 9.2 work for incomplete normed algebras we have to replace J by I:

Theorem 10.7 *The topological radicals of general normed algebras are related by*

$$\overline{\beta}^* \leqslant T_1 \leqslant T_m \leqslant T_n \leqslant T_\infty \leqslant S_* \leqslant I \qquad (1 \leqslant m \leqslant n \leqslant \infty).$$

$$\overline{\beta}^* \leqslant \overline{J}^* \leqslant I$$

Proof. Every continuous, strictly irreducible representation $\pi : A \to \mathcal{L}(X)$ extends to an irreducible representation $\tilde{\pi} : \tilde{A} \to \mathcal{L}(X)$, and is therefore a standard representation of A. Thus $S \leqslant I$ and it follows from Theorem 6.11 that $S_* \leqslant I$. The other inequalities are obvious.

Remark 10.8 It is tempting to try to define a topological radical for normed algebras by $J'(A) = A \cap J(\tilde{A})$, where $\tilde{A}$ is the completion of A. However, while this J' does satisfy axioms (1), (2), (4) and (5), it fails to satisfy (3).

Let $(A_0, |.|)$, $(A_0, \|.\|)$ be as in Example 9.3, with completions A, B respectively, and let $\phi : (A_0, |.|) \to (A_0, \|.\|)$ be the identity map. Then $J'(A_0, |.|) = A_0 \cap J(A) = A_0$ and $J'(A_0, \|.\|) = A_0 \cap J(B) = \{0\}$, so $\phi(J'(A_0, |.|))$ is not contained in $J'(\phi(A_0, |.|))$.

11 Representations on incomplete spaces

Let us now consider representations of Banach algebras on incomplete spaces. We define $U_n(A)$ to be the intersection of the kernels of the continuous n-TT representations $\pi : A \to \mathcal{L}(X)$ of A on normed spaces X. As before, U_n is a topological radical, the proof being the same except for the construction of the space Z in Lemma 8.2, where we simply put $Z = Y$.

Generally, we expect representations on incomplete spaces to be less interesting, but the relaxed condition does make it easier to construct examples. The Banach space analogue of the following theorem is the example (Proposition 4.2) based on Read's recent work [17].

Theorem 11.1 *There is a commutative, singly-generated, Jacobson-radical Banach algebra A with a faithful continuous TI representation on a normed space; hence $\{0\} = U_1(A) \subset U_2(A) = A$.*

Proof. Our algebra A will be the disc algebra with convolution multiplication

$$(f * g)(z) = \int_0^z f(w)g(z-w)\,dw, \tag{5}$$

the integral being taken along any path in the unit disc from 0 to z. The norm is the usual supremum norm

$$\|f\|_A = \sup_{|z|\leqslant 1} |f(z)|.$$

This is a well-known example of a commutative Jacobson-radical Banach algebra which is an integral domain (see [8] p.478, [14] 4.8.3, [18] A.2.11). The restriction map $\phi : A \to L^1[0,1]$ is injective. Let $X = \phi(A)$. Let $\pi : A \to \mathcal{L}(X)$ be defined by

$$\pi(a)(x) = \phi(a * \phi^{-1}(x)) = \phi(a) * x \qquad (a \in A, x \in X).$$

(The integral (5) may be taken along the real axis when $z \in [0,1]$.) It is easy to see that $\|\pi(a)(x)\|_1 \leqslant \|a\|_A \|x\|_1$ $(a \in A, x \in X)$, so π is a continuous normed representation. Since π is, algebraically, the left regular representation of an integral domain, π is faithful.

We must show that π has no nontrivial closed invariant subspaces. This is equivalent to saying that the algebra $(X, *)$ has no nontrivial closed ideals. Since X is dense in $(L^1[0,1], *)$, the closed ideals of X are of the form $X \cap I$ where I is a closed ideal of $(L^1[0,1], *)$. Now the only proper closed ideals I of $(L^1[0,1], *)$ consist of functions vanishing in a neighbourhood of zero [6] and so have $X \cap I = \{0\}$. The result follows.

12 Open questions

The open questions in this subject outnumber the theorems and we can pick out only a few here. Let us begin with a wild conjecture.

Conjecture 12.1 For all Banach algebras A, $T_1(A) = \overline{\beta}^*(A)$.

This would mean that TI representations play the same rôle for semiprime Banach algebras that strictly irreducible representations do for Jacobson semisimple algebras. Note that this conjecture implies that the topological radical $\overline{\beta}^*$ is hereditary.

Topologically transitive representations are little understood. The following are two of the basic questions.

Question 12.2 Is every continuous 2-TT representation of a Banach algebra on a Banach space topologically transitive?

Question 12.3 Are the radicals T_n $(2 \leqslant n \leqslant \infty)$ distinct?

Moving further up the list of radicals we ask:

Question 12.4 Is every continuous TT representation of a Banach algebra on a Banach space standard?

Question 12.5 Are the radicals T_∞ and S_* distinct?

Finally, there are also many questions on the general theory of radicals in normed algebras that we have left open. We have not been concerned to make an exhaustive study of all the possible choices of axioms, we only wanted to find one that worked. Nevertheless, it would be interesting to know if any of the alternatives we rejected can be made to work and if others can be proved inequivalent to ours.

$$\begin{aligned} |\pi(a)x^{(j)} - y^{(j)}| &= \left| \sum_{i=1}^{n} \pi(b_i c_i)x^{(j)} - \sum_{i=1}^{n} \pi(b_i)x_i^{(j)} \right| \\ &= \left| \sum_{i=1}^{n} \pi(b_i) \left(\pi(c_i)x^{(j)} - x_i^{(j)} \right) \right| \\ &\leqslant \sum_{i=1}^{n} \|b_i\| \, \|\pi(c_i)x^{(j)} - x_i^{(j)}\|, \qquad \text{by the definition of } |.|, \\ &< \varepsilon. \end{aligned}$$

$$B_r = -\frac{\partial f}{\partial z} B_{0z} G - r \frac{\partial B_{bz}}{\partial z}, \quad B_\phi = 0, \quad B_z = \frac{\partial f}{\partial r} B_{0z} G + 2B_{bz}, \tag{6}$$

$$\rho_{\mathrm{v}} = \rho_{\mathrm{ref}}(z) + \rho_0 \exp\left(-\frac{z}{z_\alpha}\right), \tag{7}$$

$$\frac{\mathrm{d}p_{\mathrm{v}}}{\mathrm{d}z} = \rho_{\mathrm{v}} g \Rightarrow \quad p_{\mathrm{v}}(z) = p_{\mathrm{ref}}(z_{\min}) + \int_{z_{\min}}^{z} \rho_{\mathrm{v}}(z^*) g \, \mathrm{d}z^*, \tag{8}$$

$$\nabla P = \nabla p_{_{\mathrm{v}}} + \nabla p_{_{\mathrm{h}}} + \nabla \frac{|B|^2}{2} + (B \cdot \nabla) B = (\rho_h + \rho_v) g, \tag{9}$$

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