Algebraic Expressions

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1 Introduction.

Practice manipiulation of algebraic expressions

will use computer algebra progam but should practice this craft by hand!

Examples will be highlighted using green text boxes:

After simlification the expression

$$5x^2y + 6xy^2 + 12xy \tag{1}$$

becomes

$$xy(5x + 6y + 12) \tag{2}$$

Examples using maxima will be shown in a red textbox these will feature examples of how to use maxima

A simplification example using the ratsimp maxima command. Simplify the expression:

$$5x^2y + 6xy^2 + 12xy \tag{3}$$

```
(%i1)f:6*x*y^2+5*x^2*y+12*x*y;
```

(%o1) $$6\,x\,y^2 + 5\,x^2\,y + 12\,x\,y$$

```
(%i2)ratsimp(f);
```

(%o2) $$6\,x\,y^2 + \left(5\,x^2 + 12\,x\right)\,y$$

- Multiply and divide integer powers
- Expand a single term over brackets and collect like terms
- Expand the product of multiple expressions
- Factorise linear, quadratic and cubic expressions

1991 *Tuition notes for AS and A level Maths* 46H15, 46H25, 16Nxx.

- Knowledge and use of the law of indices
- Simplify and use the rules of surds
- Rationalise denominatorsd

Extra information from [1]

2 Revision

An important skill for mathematicians is the use of algebra to solve problems and develop models. A basic skill is to be able to manipulate simple expressions. Simplify expressions such as

$$5x^2y + 6x^2y^2 + 12xy + 4x^2y^2 - 2xy \tag{4}$$

or

$$4ab + 2a^2b^2 + 4b^2 + 4a^2 \tag{5}$$

Understand indices and be able to write the following expressions as a single power

$$3^3 \times 3^6 \tag{6}$$

or

$$(2^2)^3 \tag{7}$$

$$\frac{(4^5)}{(4^2)} \tag{8}$$

Expand expressions such as

$$2(a + 2b) \tag{9}$$

or

$$5(2 - 2x) \tag{10}$$

$$4(3x + 4y) \tag{11}$$

Identify the highest common factors for expressions such as

36 and 108
or

$$6x$$

and

$$36x^3$$

or

$$12x^2y$$

and

$$48xy^2$$

Simplify the following

$$\frac{15x}{3}$$

or

$$\frac{40a}{8}$$

3 Index Laws

Simplification of operations involving powers requires the laws of indices. The rules for simplifying powers of the same base is shown below.

$$a^m \times a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n}$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

Some example expansions are shown below

```
(%i1)2*x^3*3*x^7;
```

(%o1) $$6\,x^{10}$$

An expansion example

$$3x(5x+4) - 4(3x+4)$$

```
(%i4)f:3*x*(5*x+4)-4*(3*x+4);

(%i5)ratsimp(f);
```

(%o5) $$15\,x^2 - 16$$

An expansion requiring index subtraction

$$\frac{6x^4 - 12x^7}{3x^4}$$

```
(%i6)g:(6*x^4-12*x^7)/(3*x^4);

(%i7)ratsimp(%o6);
```

(%o7) $$2 - 4\,x^3$$

4 Expanding Brackets

When we manipulate and expand expressions we need to understand the rule for finding the product of expressions enclosed within brackets. An example of such an expression is shown below:

$$(2x + 4y)(4x - 2y + 1)$$

To determine the product we multiply each term in one bracket with every term in the other bracket.

$$(2x + 4y)(4x - 2y + 1)$$

$$2x(4x - 2y + 1) + 4y(4x - 2y + 1)$$

Multiply the first bracket

$$8x^2 - 4xy + 2x + 4y(4x - 2y + 1)$$

Multiply the second bracket

$$8x^2 - 4xy + 2x + 16xy - 8y^2 + 4y$$

Group similar terms together

$$8x^2 - 8y^2 + 2x + 4y + 16xy - 4xy$$

Simplify last two terms

$$8x^2 - 8y^2 + 2x + 4y + 12xy$$

Or use the expand command in maxima

```
(%i11) f:(-2*y+4*x+1)*(4*y+2*x);
```

$$(\%o11) \quad (-2\,y+4\,x+1)\,(4\,y+2\,x)$$

```
(%i12) expand(f);
```

$$(\%o12) \quad -8\,y^2+12\,x\,y+4\,y+8\,x^2+2\,x$$

Later expansions such as the Binomial and Taylor expansions will be used to expand expressions.

5 Factorising

As well as expanding brackets it is necessary to perform the reverse operation. The reverse operation of expansion is factorisation.

The expression:

$$8x^2 - 8y^2 + 2x + 4y + 12xy$$

when factorised becomes:

$$(2x + 4y)(4x - 2y + 1)$$

We will focus later on quadratic expressions such as

$$ax^2 + bx + c$$

For example consider the quadratic equation

$$2x^2 + 7x - 4$$

Here,

$$a = 2$$

,

$$b = 7$$

and

$$c = -3$$

. Find two factors

$$f_1$$

and

$$f_2$$

such that

$$f_1 + f_2 = b$$

Using this rule, the quadratic can be written as

$$2x^2 + 8x - x - 4$$

The terms can be grouped as

$$(2x - 1)x + (2x - 1)4$$

Finally, take out the comman factor to give

$$(2x - 1)(x + 4)$$

$$4x + 8 = 4(x + 2)$$

$$2x^2 - x = x(2x - 1)$$

$$3xy + 9xy^2 = 3xy(1 + 3y)$$

6 Negative Indices, Fractional Indices and Surds

Rational numbers are numbers which can be written as a ratio of the form

$$\frac{a}{b}$$

where a and b are integers. Rules for the laws of indices with any rational power are as follows

$$a^{\frac{1}{m}} = \sqrt[m]{a}$$

$$a^{\frac{n}{m}} = \sqrt[m]{(a^n)}$$

$$a^{-m} = \frac{1}{a^m}$$

$$a^0 = 1$$

Consider the examples

$$x^{\frac{2}{3}} x^{\frac{4}{3}} = x^2$$

$$(x^3)^{\frac{2}{3}} = x^2$$

$$\sqrt[3]{125x^6} = (125x^6)^{\frac{1}{3}}$$

$$\frac{2x^2 - x}{x^5} = \frac{2}{x^3} - \frac{1}{x^4}$$

7 Surds and Rationalising Denominators

Surds are examples of irrational numbers i.e. numbers which cannot be written in the form

$$\frac{a}{b}$$

Rules for manipulating surds

$$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

A fraction with a surd can be rearranged so that the denominator is a rational number. Rules for rationalising denominators are as follows.

$$\frac{1}{\sqrt{a}}$$

, multiply numerator and denominator by

$$\sqrt{a}$$

$$\frac{1}{a+\sqrt{b}}$$

, mu;tiply numerator and denominator by

$$a-\sqrt{b}$$

$$\frac{1}{a-\sqrt{b}}$$

, mu;tiply numerator and denominator by

$$a+\sqrt{b}$$

An example of simplifying surds

$$(2-\sqrt{3})(5+\sqrt{3})$$

Expanding this becomes

$$= 2(5+\sqrt{3}) - \sqrt{3}(5+\sqrt{3})$$

Collect like terms together

$$= 10 + 2\sqrt{3} - 5\sqrt{3} - \sqrt{9}$$

$$7 - 3\sqrt{3}$$

Factorising the denominator

$$\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$

$$= \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}+\sqrt{2}}$$

Use

$$\sqrt{2}\sqrt{5} = \sqrt{10}$$

$$= \frac{5+\sqrt{5}\sqrt{2}+\sqrt{2}\sqrt{5}+2}{5-2}$$

$$= \frac{7+2\sqrt{10}}{3}$$

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