

# Algebraic Expressions

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## 1 Introduction.

Practice manipulation of algebraic expressions

will use computer algebra program but should practice this craft by hand!

- Multiply and divide integer powers
- Expand a single term over brackets and collect like terms
- Expand the product of multiple expressions
- Factorise linear, quadratic and cubic expressions
- Knowledge and use of the law of indices
- Simplify and use the rules of surds
- Rationalise denominators

Extra information from [1]

## 2 Revision

An important skill for mathematicians is to Some simple expressions to start with

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 $E = mc^2$  Formula of the universe  
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(%i1) f:6\*x\*y^2+5\*x^2\*y+12\*x\*y;

(%o1)  $6xy^2 + 5x^2y + 12xy$

My box.  
 $E = mc^2$  Formula of the universe

### 3 Index Laws

Some of our later examples will produce, *inter alia*, TI representations which are not strictly irreducible, but it is worth noting now that satisfying these requirements alone is quite easy.

**Example 3.1** Let  $A = \ell^1(S_2)$  be the semigroup algebra of the free semigroup on two generators  $X, Y$ . Let  $T$  be the unilateral shift on  $H = \ell^2$ :

$$\begin{aligned} T(\xi_1, \xi_2, \xi_3, \dots) &= (0, \xi_1, \xi_2, \dots) \\ T^*(\xi_1, \xi_2, \xi_3, \dots) &= (\xi_2, \xi_3, \xi_4, \dots) \end{aligned}$$

Let  $\pi$  be the continuous representation of  $A$  on  $\ell^2$  defined by  $\pi(\delta_X) = T$ ,  $\pi(\delta_Y) = T^*$ . It is easy to see that  $\pi(A)$  is a \*-subalgebra of  $\mathcal{L}(H)$  with scalar commutant, so, by von Neumann's Double Commutant Theorem, its strong closure is  $\mathcal{L}(H)$ , i.e.  $\pi$  is TT; but

$$\pi(A)((1, 0, 0, \dots)) = \ell^1,$$

so  $\pi$  is not strictly irreducible.

**Remark 3.2** For \*-representations of C\*-algebras, Kadison's Transitivity Theorem says that TI implies strictly irreducible ([12], see also [14] 5.2.2, [21] 1.21.17). In the example above,  $\pi(A)$  is not closed in  $\mathcal{L}(H)$ .

We shall be seeking to relate the TI radical to radicals definable without reference to representations. In one direction this is easy, provided the algebra is complete: every strictly irreducible representation of a Banach algebra  $A$  has the same kernel as some continuous strictly irreducible representation of  $A$  on a Banach space ([15] 4.2.9, [19] (2.4.7)). Hence the TI radical of a Banach algebra is contained in the Jacobson radical, which has many characterizations not directly involving representations (largest quasi-regular ideal, largest ideal of topologically nilpotent elements, intersection of the maximal modular left ideals). It is not immediately clear that this inclusion can be strict—in Example 3.1 above, the algebra  $A$  is semisimple so there are many other representations which are strictly irreducible—but we shall give an example later where this is so.

### 4 Expanding Brackets

### 5 Factorising

The obvious first question about TT representations is whether there are TI representations which are not TT. One way in which such representations might occur is as the left regular representations of radical Banach algebras with no non-trivial closed left ideals, if such exist.

**Remark 5.1** The problem of whether there exists a radical Banach algebra with no non-trivial closed left ideals lies between two unsolved problems: the existence of a topologically simple radical Banach algebra and the existence of a topologically simple commutative radical Banach algebra.

### 6 Negative and Fractional Indices

In this section, we develop a little of a general theory of radicals in normed algebras. The calculations are generally straightforward once the correct definitions are in place;

## 7 Rationalising Denominators

It is natural to ask whether, in Theorem ??, the map  $R$  satisfying axiom (5) would imply  $\overline{R}$  satisfying (5). The answer is negative, as the following example shows.

## 8 Summary of Key Points

## References

- [1] Khan Academy, *Algebra1*, (<https://www.khanacademy.org/math/algebra> **42**, 2018 Khan Academy).
- [2] B. Beauzamy, *Introduction to operator theory and invariant subspaces*, (North-Holland, North-Holland Mathematical Library **42**, Amsterdam 1988).
- [3] F. F. Bonsall and J. Duncan, *Complete normed algebras*. (Springer, Berlin–Heidelberg–New York, 1973).
- [4] N. J. Divinsky, *Rings and radicals*, (George Allen & Unwin, London, 1965).
- [5] P. G. Dixon, “Semiprime Banach algebras”, *J. London Math. Soc.* (2), **6** (1973), 676–678.
- [6] P. G. Dixon, “A Jacobson-semisimple Banach algebra with a dense nil subalgebra”, *Colloq. Math.*, **37** (1977), 81–82.
- [7] W. F. Donoghue Jr., “The lattice of invariant subspaces of a completely continuous quasi-nilpotent transformation” *Pacific J. Math.*, **7** (1957) 1031–1035.
- [8] P. Enflo, “On the invariant subspace problem in Banach spaces”, *Acta Math.*, **158** (1987), 213–313.
- [9] E. Hille, *Functional analysis and semigroups*, (American Math. Soc., Colloquium Publications **31**, Providence, R.I., 1948)
- [10] N. Jacobson, *Structure of Rings*, third edition (Amer. Math. Soc. Coll. Publ. **37**, Providence, R.I., 1968).
- [11] G. J. O. Jameson, *Topology and normed spaces*, (Chapman & Hall, 1974). *Bull. Amer. Math. Soc.*, **73** (1967), 537–539.
- [12] R. V. Kadison, “Irreducible operator algebras”, *Proc. Nat. Acad. Sci. U.S.A.*, **43** (1957), 273–276.
- [13] M. J. Meyer, “Continuous dense embeddings of strong Moore algebras”, *Proc. Amer. Math. Soc.*, **116** (1992), 727–735.
- [14] G. J. Murphy, *C\*-algebras and operator theory* (Academic Press, London, 1990).
- [15] Th. W. Palmer, *Banach algebras and the general theory of \*-algebras, volume I: algebras and Banach algebras* (C.U.P., Cambridge, 1994)
- [16] C. J. Read, “A solution to the invariant subspace problem”, *Bull. London Math. Soc.*, **16** (1984), 337–401.

- [17] C. J. Read, “A solution to the invariant subspace problem on the space  $\ell_1$ ”, *Bull. London Math. Soc.*, **17** (1985), 305–317.
- [18] C. J. Read, “Quasinilpotent operators and the invariant subspace problem”, (preprint, Trinity College, Cambridge, 1995).
- [19] C. E. Rickart, *General theory of Banach algebras* (van Nostrand, Princeton, 1960).
- [20] L. H. Rowen, *Ring theory: student edition*. (Academic Press, San Diego, Ca., 1991).
- [21] S. Sakai,  *$C^*$ -algebras and  $W^*$ -algebras*, (Springer, Ergebnisse **60**, Berlin–Heidelberg–New York 1971).

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