# Algebraic Expressions

M. K. Griffiths

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## 1 Introduction.

Practice manipiulation of algebraic expressions
will use computer algebra progam but should practice this craft by hand!

- Multiply and divide integer powers
- Expand a single term over brackets and collect like terms
- Expand the product of multiple expressions
- Factorise linear, quadratic and cubic expressions
- Knowledge and use of the law of indices
- Simplify and use the rules of surds
- Rationalise denominatorsd

## 2 Revision

Some simple expressions to start with

## 3 Index Laws

Some of our later examples will produce, *inter alia*, TI representations which are not strictly irreducible, but it is worth noting now that satisfying these requirements alone is quite easy.

**Example 3.1** Let $A = \ell^1(S_2)$ be the semigroup algebra of the free semigroup on two generators $X, Y$. Let $T$ be the unilateral shift on $H = \ell^2$:

$$\begin{aligned} T(\xi_1, \xi_2, \xi_3, \ldots) &= (0, \xi_1, \xi_2, \ldots) \\ T^*(\xi_1, \xi_2, \xi_3, \ldots) &= (\xi_2, \xi_3, \xi_4, \ldots) \end{aligned}$$

Let $\pi$ be the continuous representation of $A$ on $\ell^2$ defined by $\pi(\delta_X) = T$, $\pi(\delta_Y) = T^*$. It is easy to see that $\pi(A)$ is a *-subalgebra of $\mathcal{L}(H)$ with scalar commutant, so, by von Neumann's Double Commutant Theorem, its strong closure is $\mathcal{L}(H)$, i.e. $\pi$ is TT; but

$$\pi(A)((1, 0, 0, \ldots)) = \ell^1,$$

1991 *Tuition notes for AS and A level Maths* 46H15, 46H25, 16Nxx.

so $\pi$ is not strictly irreducible.

**Remark 3.2** For *-representations of C*-algebras, Kadison's Transitivity Theorem says that TI implies strictly irreducible ([11], see also [13] 5.2.2, [20] 1.21.17). In the example above, $\pi(A)$ is not closed in $\mathcal{L}(H)$.

We shall be seeking to relate the TI radical to radicals definable without reference to representations. In one direction this is easy, provided the algebra is complete: every strictly irreducible representation of a Banach algebra $A$ has the same kernel as some continuous strictly irreducible representation of $A$ on a Banach space ([14] 4.2.9, [18] (2.4.7)). Hence the TI radical of a Banach algebra is contained in the Jacobson radical, which has many characterizations not directly involving representations (largest quasi-regular ideal, largest ideal of topologically nilpotent elements, intersection of the maximal modular left ideals). It is not immediately clear that this inclusion can be strict—in Example 3.1 above, the algebra $A$ is semisimple so there are many other representations which are strictly irreducible—but we shall give an example later where this is so.

# 4 Expanding Brackets

# 5 Factorising

The obvious first question about TT representations is whether there are TI representations which are not TT. One way in which such representations might occur is as the left regular representations of radical Banach algebras with no non-trivial closed left ideals, if such exist.

**Remark 5.1** The problem of whether there exists a radical Banach algebra with no non-trivial closed left ideals lies between two unsolved problems: the existence of a topologically simple radical Banach algebra and the existence of a topologically simple commutative radical Banach algebra.

# 6 Negative and Fractional Indices

In this section, we develop a little of a general theory of radicals in normed algebras. The calculations are generally straightforward once the correct definitions are in place;

# 7 Rationalising Denominators

It is natural to ask whether, in Theorem **??**, the map $R$ satisfying axiom (5) would imply $\overline{R}$ satisfying (5). The answer is negative, as the following example shows.

# 8 Summary of Key Points

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Pure Mathematics Section,
School of Mathematics and Statistics,
University of Sheffield,
SHEFFIELD, S3 7RH,
England.

e-mail: P.Dixon@sheffield.ac.uk