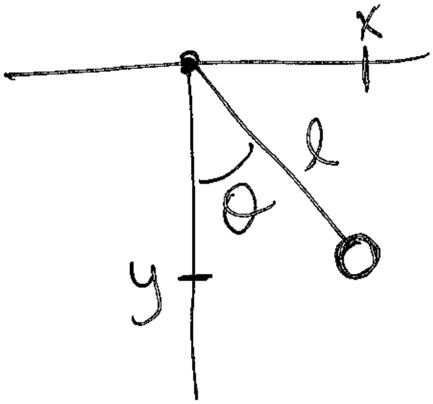
MAP4371_020618 Double Pendulum.

Derive the pendulum equation using the Euler-Lagrange approach.



$$x = l\sin\theta \rightarrow \dot{x} = l\cos\theta \cdot \dot{\theta}$$

$$y = -l\cos\theta \rightarrow \dot{y} = l\sin\theta \cdot \dot{\theta}$$

potential energy $V = mgy = -mgl\cos\theta$

kinetic energy $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$

$$= \frac{1}{2}m(l^2\cos^2\theta + l^2\sin^2\theta)\dot{\theta}^2$$

$$= \frac{1}{2}ml^2\dot{\theta}^2$$

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

Euler-Lagrange:

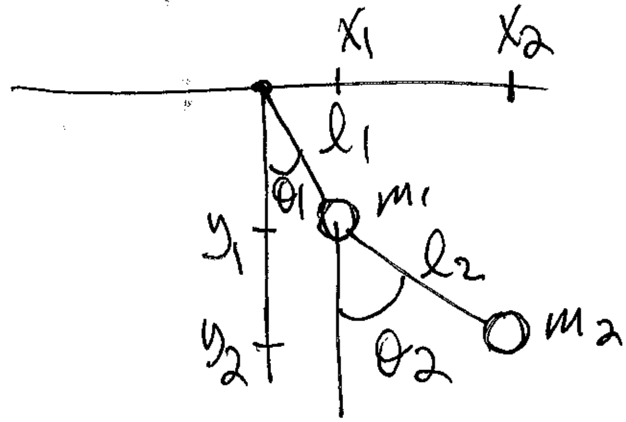
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Double pendulum



$$x_1 = l_1 \sin\theta_1 \quad ; \quad y_1 = -l_1 \cos\theta_1$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

$$V = m_1 g y_1 + m_2 g y_2$$

$$= -m_1 l_1 g \cos\theta_1 + m_2 g(-l_1 \cos\theta_1 - l_2 \cos\theta_2)$$

$$T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \Big[(l_1 \cos\theta_1 \cdot \dot{\theta}_1 + l_2 \cos\theta_2 \cdot \dot{\theta}_2)^2$$

$$+ (l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \cdot \dot{\theta}_2)^2 \Big]$$

$$= \frac{1}{2} \underbrace{(m_1 + m_2)}_{M} l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2$$

$$+ \frac{1}{2} m_2 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \overbrace{(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2)}^{\cos\Delta\theta}$$

$$L = T - V = \tfrac{1}{2} M l_1^2 \dot\theta_1^2 + \tfrac{1}{2} m_2 l_2^2 \dot\theta_2^2 + m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos\Delta\theta$$
$$+ M l_1 g \cos\theta_1 + m_2 l_2 g \cos\theta_2 .$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} - \frac{\partial L}{\partial \theta_1} = 0 \quad ; \quad \frac{d}{dt}\frac{\partial L}{\partial \dot\theta_2} - \frac{\partial L}{\partial \theta_2} = 0$$

$$M l_1 \ddot\theta_1 + m_2 l_2 \ddot\theta_2 \cos\Delta\theta - m_2 l_2 \dot\theta_2^2 \sin\Delta\theta + M g \sin\theta_1 = 0$$
$$l_2 \ddot\theta_2 + l_1 \ddot\theta_1 \cos\Delta\theta + l_1 \dot\theta_1^2 \sin\Delta\theta + g \sin\theta_2 = 0$$

$$x_1 = \theta_1 \; ; \; x_2 = \theta_2 \; ; \; x_3 = \dot\theta_1 \; ; \; x_4 = \dot\theta_2$$

$$M l_1 \dot x_3 + m_2 l_2 \dot x_4 \cos\Delta\theta - m_2 l_2 x_4^2 \sin\Delta\theta + M g \sin x_1 = 0$$
$$l_1 \dot x_3 \cos\Delta\theta + l_2 \dot x_4 + l_1 x_3^2 \sin\Delta\theta + g \sin x_2 = 0$$

$$\begin{pmatrix} M l_1 & m_2 l_2 C \\ l_1 C & l_2 \end{pmatrix} \begin{pmatrix} \dot x_3 \\ \dot x_4 \end{pmatrix} = \begin{pmatrix} m_2 l_2 x_4^2 S - M g \sin x_1 \\ -l_1 x_3^2 S - g \sin x_2 \end{pmatrix}$$

$C = \cos\Delta\theta$
$S = \sin\Delta\theta$
$\Delta\theta = \theta_2 - \theta_1$

Multiply by inverse of

$$\begin{pmatrix} M l_1 & m_2 l_2 C \\ l_1 C & l_2 \end{pmatrix} \rightarrow \begin{pmatrix} l_2 & -m_2 l_2 C \\ -l_1 C & M l_1 \end{pmatrix} \Big/ D$$

$$D = M l_1 l_2 - m_2 l_1 l_2 C^2$$
$$= l_1 l_2 M \left(1 - \frac{m_2}{M} C^2\right) > 0$$

$$\begin{pmatrix} \dot{X}_3 \\ \dot{X}_4 \end{pmatrix} = \frac{1}{D} \begin{pmatrix} l_2 & -m_2 l_2 C \\ -l_1 C & M l_1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} F_3 \\ F_4 \end{pmatrix}$$

Our total system

$$\begin{pmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \\ \dot{X}_4 \end{pmatrix} = \begin{pmatrix} X_3 \\ X_4 \\ F_3 \\ F_4 \end{pmatrix} \quad ; \quad \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}(0) = \underline{X}_0$$

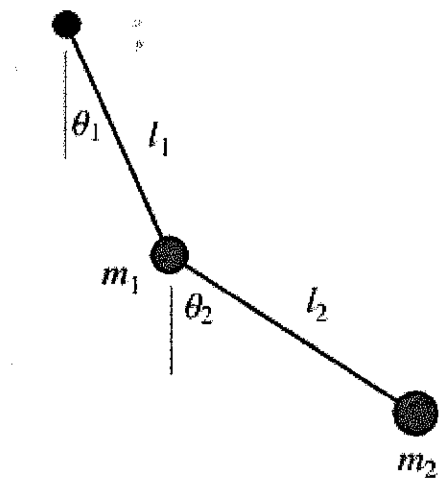
or

$$\underline{X}' = F(t, \underline{X}) \qquad \underline{X}(0) = \underline{X}_0$$

Use RK4 to solve it.

Try different initial conditions $\underline{X}_0$

Apendix



A double pendulum consists of one pendulum attached to another. Double pendula are an example of a simple physical system which can exhibit chaotic behavior. Consider a double bob pendulum with masses m_1 and m_2 attached by rigid massless wires of lengths l_1 and l_2. Further, let the angles the two wires make with the vertical be denoted θ_1 and θ_2, as illustrated above. Finally, let gravity be given by g. Then the positions of the bobs are given by

$$x_1 = l_1 \sin\theta_1 \quad (1)$$

$$y_1 = -l_1 \cos\theta_1 \quad (2)$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \quad (3)$$

$$y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2. \quad (4)$$

The potential energy of the system is then given by

$$V = m_1 g y_1 + m_2 g y_2 \quad (5)$$

$$= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2, \quad (6)$$

and the kinetic energy by

$$T = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 \quad (7)$$

$$= \tfrac{1}{2} m_1 l_1^2 \dot\theta_1^2 + \tfrac{1}{2} m_2 [l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2)]. \quad (8)$$

The Lagrangian is then

$$L \equiv T - V$$

$$= \tfrac{1}{2}(m_1 + m_2) l_1^2 \dot\theta_1^2 + \tfrac{1}{2} m_2 l_2^2 \dot\theta_2^2 + m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2)$$

$$+ (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2. \quad (9)$$

Therefore, for θ_1,

$$\frac{\partial L}{\partial \dot\theta_1} = m_1 l_1^2 \dot\theta_1 + m_2 l_1^2 \dot\theta_1 + m_2 l_1 l_2 \dot\theta_2 \cos(\theta_1 - \theta_2) \tag{10}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) = (m_1 + m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot\theta_2 \sin(\theta_1 - \theta_2)(\dot\theta_1 - \dot\theta_2) \tag{11}$$

$$\frac{\partial L}{\partial \theta_1} = -l_1 g (m_1 + m_2) \sin\theta_1 - m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2), \tag{12}$$

so the Euler-Lagrange differential equation for θ_1 becomes

$$(m_1 + m_2) l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot\theta_2^2 \sin(\theta_1 - \theta_2) + l_1 g (m_1 + m_2) \sin\theta_1 = 0. \tag{13}$$

Dividing through by l_1, this simplifies to

$$(m_1 + m_2) l_1 \ddot\theta_1 + m_2 l_2 \ddot\theta_2 \cos(\theta_1 - \theta_2) + m_2 l_2 \dot\theta_2^2 \sin(\theta_1 - \theta_2) + g (m_1 + m_2) \sin\theta_1 = 0. \tag{14}$$

Similarly, for θ_2,

$$\frac{\partial L}{\partial \dot\theta_2} = m_2 l_2^2 \dot\theta_2 + m_2 l_1 l_2 \dot\theta_1 \cos(\theta_1 - \theta_2) \tag{15}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) = m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot\theta_1 \sin(\theta_1 - \theta_2)(\dot\theta_1 - \dot\theta_2) \tag{16}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot\theta_1 \dot\theta_2 \sin(\theta_1 - \theta_2) - l_2 m_2 g \sin\theta_2, \tag{17}$$

so the Euler-Lagrange differential equation for θ_2 becomes

$$m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot\theta_1^2 \sin(\theta_1 - \theta_2) + l_2 m_2 g \sin\theta_2 = 0. \tag{18}$$

Dividing through by l_2, this simplifies to

$$m_2 l_2 \ddot\theta_2 + m_2 l_1 \ddot\theta_1 \cos(\theta_1 - \theta_2) - m_2 l_1 \dot\theta_1^2 \sin(\theta_1 - \theta_2) + m_2 g \sin\theta_2 = 0. \tag{19}$$