# Numerical Methods HW 10.2

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**Problem 10.2.10**

*Proof.*
We have that the second order Runge Kutta is:

$$x(t+h) = x(t) + w_1 hf(t,x) + w_2 hf(t+\alpha h, x+\beta hf(t,x)$$

We can further expand this using the second order Taylor series on functions of two variables:

$$f(x+h,y+k) = \sum_{i=0}^{\infty}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^i f(x,y)$$

Letting $t, \alpha h, x$ and $\beta hf$ be $x, h, y, k$ respectively we are obtain:

$$f(t+\alpha h, x+\beta hf(t,x)) = f + \alpha hf_t + \beta hff_x + \frac{1}{2}\left(\alpha h\frac{\partial}{\partial x} + \beta hf\frac{\partial}{\partial y}\right)^2 f$$

Substituting this result into our second order Runge-Kutta and letting $\beta = \alpha, w_1 = 1 - \frac{1}{2\alpha}, w_2 = \frac{1}{2\alpha}$ we obtain:

$$x(t+h) = x(t) + hf + \frac{1}{2}h^2 f + \frac{1}{2}h^2 ff_x + \frac{\alpha}{4}h^3\left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}\right)^2 f$$

Next we take the third order Taylor approximation:

$$x(t+h) = x(t) + hx'(t) + \frac{h^2}{2}x''(t) + \frac{h^3}{6}x'''(t)$$

Using the fact that $x'(t) = f(t,x)$ we then have that

$$x'' = \frac{dx'}{dt} = \frac{df(t,x)}{dt} = \left(\frac{\partial}{\partial x}\right)\left(\frac{dt}{dt}\right) + \left(\frac{\partial f}{\partial x}\right)\left(\frac{dx}{dt}\right) = f_t + f_x f = \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}\right)f$$

$$x''' = \frac{dx''}{dt} = \frac{d}{dt}\left(\left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}\right)f\right) = \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}\right)^2 + f_x\left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}\right)f$$

This changes our Third Order Taylor to:

$$x(t+h) = x(t) + hf + \frac{1}{2}h^2 f + \frac{1}{2}h^2 f f_x + \frac{1}{6}h^3 \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}\right)^2 + \frac{1}{6}h^3 f_x \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}\right) f$$

Utilizing the fact that the terms for each of these approximation are the same until the $h^3$ terms we calculate the error as the difference in these terms:

$$\frac{1}{6}h^3 \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}\right)^2 + \frac{1}{6}h^3 f_x \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}\right) f - \frac{\alpha}{4}h^3 \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}\right)^2 f$$

Which is equivalent to:

$$\frac{1}{4}h^3 \left(\frac{2}{3} - \alpha\right) \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}\right)^2 f + \frac{1}{6}h^3 f_x \left(\frac{\partial}{\partial t} + f\frac{\partial}{\partial x}\right) f$$

$\square$