Numerical Methods HW

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Double Pendulum Problem

Derive a closed derivation for the motion of a double pendulum in vertor form.

Proof.
We start with some definitions: let l_1, l_2 be the length of the rigid wires, m_1, m_2 the mass of the bobs, and θ_1, θ_2 the angl each of the wires makes with the defined vertical.
The positions of the bobs in a Cartesian plane can then be defined as:

$$x_1 = l_1 sin(\theta_1) \tag{1}$$
$$y_1 = -l_1 cos(\theta_1) \tag{2}$$
$$x_2 = l_1 sin(\theta_1) + l_2 sin(\theta_2) \tag{3}$$
$$y_2 = -l_1 cos(\theta_1) - l_2 cos(\theta_2) \tag{4}$$

We then have the potential energy of the system given by:

$$\begin{aligned} V &= m_1 g y_1 + m_2 g y_2 \\ &= -(m_1 + m_2) g l_1 g cos(\theta_1) - m_2 g l_2 cos(\theta_2) \end{aligned} \tag{5}$$

And the Kinetic Energy:

$$\begin{aligned} T &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \\ &= \frac{1}{2} m_1 (x_1^2 + y_1^2) + \frac{1}{2} m_2 (x_2^2 + y_2^2) \\ &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} (m_2 l_1^2 \dot{\theta}_1^2 + m_2 l_2^2 \dot{\theta}_2^2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 cos(\theta_1 - \theta_2) \end{aligned} \tag{6}$$

The Langragian of this system is then:

$$\begin{aligned} L &\equiv T - V \\ &= \frac{1}{2} (m_1 + m_2) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta}_2^2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 cos(\theta_1 - \theta_2) \\ &+ (m_1 + m_2) g l_1 g cos(\theta_1) + m_2 g l_2 cos(\theta_2) \end{aligned} \tag{7}$$

Now for θ_1:

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \theta_1 + m_2 l_1^2 \theta_1 + m_2 l_1 l_2 \theta_2 cos(\theta_1 - \theta_2) \tag{8}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}_1}) = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \tag{9}$$

$$\frac{\partial L}{\partial \theta_1} = -l_1 g(m_1 + m_2) sin(\theta_1) - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2) \tag{10}$$

Then the Euler Lagrange equation for θ_1 is $\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}_1}) - \frac{\partial L}{\partial \theta_1} = 0$ which is equivalent to:

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 sin(\theta_1 - \theta_2) + l_1 g(m_1 + m_2) sin(\theta_1) = 0$$

We can divide through by l_1 yielding:

$$(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 cos(\theta_1 - \theta_2) + m_2 l_2 \dot{\theta}_2^2 sin(\theta_1 - \theta_2) + g(m_1 + m_2) sin(\theta_1) = 0$$

Repeating this process for θ_2:

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \theta_1 + m_2 l_1 l_2 \theta_2 cos(\theta_1 - \theta_2) \tag{11}$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}_2}) = (m_1 + m_2) l_2^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 sin(\theta_1 - \theta_2)(\dot{\theta}_1 - \dot{\theta}_2) \tag{12}$$

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 sin(\theta_1 - \theta_2) - l_2 g m_2 sin(\theta_2) \tag{13}$$

Then the Euler Lagrange equation for θ_2 is:

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_2 cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2^2 sin(\theta_1 - \theta_2) + l_2 m_2 g sin(\theta_2) = 0$$

Dividing through by l_2 and m_2 this time yields:

$$l_2 \ddot{\theta}_2 + l_1 \ddot{\theta}_2 cos(\theta_1 - \theta_2) - l_1 l_2 \dot{\theta}_2^2 sin(\theta_1 - \theta_2) + g sin(\theta_2) = 0$$

Now we can substitute variables x_1, x_2, x_3, x_4 in for $\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2$ respectively and letting $\Delta\theta = \theta_1 - \theta_2$ and $M = m_1 + m_2$ to obtain system of equations:

$$\begin{aligned} M l_1^2 \dot{x}_3 + m_2 l_2 \dot{x}_4 cos(\Delta\theta) - m_2 l_2 x_4^2 sin(\theta_1 - \theta_2) + M g sin(x_1) = 0 \\ l_2 \dot{x}_4 + l_1 \dot{x}_2 cos(\Delta\theta) - l_1 x_3^2 sin(\Delta\theta) + g sin(x_2) = 0 \end{aligned}$$

Then we can use the substitutions $C = cos(\Delta\theta), S = sin(\Delta\theta)$ and turn this into a matrix of equations:

$$\begin{pmatrix} M l_1 & m_2 l_2 C \\ l_1 C & l_2 \end{pmatrix} \begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} m_2 l_2 x_4^2 S - M g sin(x_1) \\ -l_1 x_3^2 S - g sin(x_2) \end{pmatrix}$$

Multiplying both sides by the inverse of the first matrix with determinant:

$$D = M l_1 l_2 - m_2 l_1 l_2 C^2 = l_1 l_2 M (1 - \frac{m_2}{M} C^2) > 0$$

We obtain the system:

$$\begin{pmatrix} \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \frac{1}{D} \begin{pmatrix} l_2 & -m_2 l_2 C \\ -l_1 C & M l_1 \end{pmatrix} \begin{pmatrix} m_2 l_2 x_4^2 S - Mgsin(x_1) \\ -l_1 x_3^2 S - gsin(x_2) \end{pmatrix} \equiv \begin{pmatrix} F_3 \\ F_4 \end{pmatrix}$$

Thus our total system is:

$$\mathbb{X} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ F_3 \\ F_4 \end{pmatrix};\ \mathbb{X}_0 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}(0)$$

Where

$$\mathbb{X}' = F(t, \mathbb{X});\ \mathbb{X}_0 = \mathbb{X}(0)$$

$\square$