Numerical Methods Hw 3

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Problem 3.2: 40. Newtons method can be defined for the equation $f(z) = g(x,y) + ih(x,y)$, where $f(z)$ is an analytic function of the complex variable $z = x + iy$ (x and y real) and $g(x,y)$ and $h(x,y)$ are real functions for all x and y. The derivative $f'(z)$ is given by $f'(z) = g_x + ih_x = h_y - ig_y$ because the Cauchy-Riemann equations $g_x = h_y$ and $h_x = g_y$ hold. Here the partial derivatives are defined as $g_x = \frac{\partial g}{\partial x}$, $g_y = \frac{\partial g}{\partial y}$, and so on. Show that Newtons method:

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}$$

can be written in the form:

$$x_{n+1} = x_n - \frac{gh_y - hg_y}{g_x h_y - g_y h_x}, \quad y_{n+1} = y_n - \frac{hg_x - gh_x}{g_x h_y - g_y h_x}$$

Here all functions are evaluated at $z_n = x_n + iy_n$.

Proof.

Starting from Newtons Method and substituting $z_n = x_n + iy_n$ and $f(z) = g(x,y) + ih(x,y)$

$$z_{n+1} = z_n \frac{f(z_n)}{f'(z_n)} \Rightarrow \ x_{n+1} + iy_{n+1} = x_n + iy_n - \frac{g + ih}{g_x + ih_x}$$

Utilizing the fact that $g_x = h_y$ and $h_x = g_y$ and multiplying the numerator and the denominator by $h_y + ig_y$

$$\frac{g + ih}{g_x + ih_x} = \frac{(g + ih)(h_y + ig_y)}{(g_x + ih_x)(h_y + ig_y)} = \frac{gh_y - hg_y + ihh_y + igg_y}{g_x h_y - g_y h_x} = \frac{gh_y - hg}{g_x h_y - g_y h_x} + i\frac{hg_x - gh_x}{g_x h_y - g_y h_x}$$

Substituting yields the equation

$$x_{n+1} + iy_{n+1} = x_n + iy_n - \frac{gh_y - hg}{g_x h_y - g_y h_x} - i\frac{hg_x - gh_x}{g_x h_y - g_y h_x}$$

And then by separating the real parts from the imaginary

$$x_{n+1} = x_n - \frac{gh_y - hg_y}{g_x h_y - g_y h_x}, \quad y_{n+1} = y_n - \frac{hg_x - gh_x}{g_x h_y - g_y h_x}$$

$\square$