IMT Atlantique
Master of Sciences
Année 2021-2022

March 2022

Maths Tools: Matlab Project

Read each question carefully, and answer each question completely. Show all of your work. No credit will be given for unsupported answers. Write your solutions clearly and legibly. No credit will be given for illegible solutions.

I - Matlab Project

Part I.1: Eigenvalues and eigenvectors

(hand): Calculate the eigenvalues for each of the following matrices:

$$\boldsymbol{A}_1 = \begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}$$

$$\boldsymbol{A}_2 = \begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}$$

$$\boldsymbol{A}_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\boldsymbol{A}_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\boldsymbol{A}_5 = \begin{bmatrix} 25 & 0 & 20 \\ 0 & 1 & 0 \\ 20 & 0 & 41 \end{bmatrix}$$

(Matlab): For each matrix $\boldsymbol{A}_i$, use **eig** to get a matrix $\boldsymbol{D}$ whose diagonal entries are the eigenvalues of $\boldsymbol{A}_i$, and a matrix $\boldsymbol{P}$ whose columns are associated eigenvectors. Record $\boldsymbol{P}$ and $\boldsymbol{D}$, and inspect $\boldsymbol{D}$ to be sure the eigenvalues produced by **eig** agree with what you calculated by hand in the previous question.

Calculate the product $\boldsymbol{AP}$ and $\boldsymbol{PD}$ for each $\boldsymbol{A}_i$, to verify that $\boldsymbol{AP} = \boldsymbol{PD}$.

Part I.2: Matrix inverses and infinite series

Purpose: To see examples for which the matrix series $\boldsymbol{I}+\boldsymbol{C}+\boldsymbol{C}^2+\ldots+\boldsymbol{C}^k$ does converge to $(\boldsymbol{I}-\boldsymbol{C})^{-1}$ and examples for which it does not.
Let the matrix $\boldsymbol{C}$ given by:

$$\boldsymbol{C} = \frac{1}{10}\begin{bmatrix} 1 & -1 & 5 \\ 0 & 1 & -2 \\ 1 & 3 & -3 \end{bmatrix}$$

Type the following lines to calculate $\boldsymbol{I}+\boldsymbol{C}+\boldsymbol{C}^2+\ldots+\boldsymbol{C}^k$ for several values of k.

$$\boldsymbol{I} = \text{eye}(3); \boldsymbol{S} = \boldsymbol{I};$$

$$\boldsymbol{S} = \boldsymbol{I} + \boldsymbol{C} * \boldsymbol{S}$$

1) Use the up arrow key to execute the sum line repeatedly and watch to see that this series does seem to converge. (The first time you execute this line, you get $\boldsymbol{S}=\boldsymbol{I}+\boldsymbol{C}$; the second time you get $\boldsymbol{I}+\boldsymbol{C}+\boldsymbol{C}^2$; etc.) How many times must you repeat it until the matrix $\boldsymbol{S}$ seems to stop changing, at least as far as what you see on the screen?

2) One way to check whether $\boldsymbol{S}$ is close to the inverse of $(\boldsymbol{I}-\boldsymbol{C})$ is to see whether the norm of $(\boldsymbol{I}-\boldsymbol{C})\boldsymbol{S}-\boldsymbol{I}$ is small. Experiment with different values of k to find how many terms of the series you must use in $\boldsymbol{S}=\boldsymbol{I}+\boldsymbol{C}+\boldsymbol{C}^2+\ldots+\boldsymbol{C}^k$ in order to get the norm of $(\boldsymbol{I}-\boldsymbol{C})\boldsymbol{S}-\boldsymbol{I}$ to be 10^{-10} or smaller.

(hand): Find under which conditions the series $\boldsymbol{I}+\boldsymbol{C}+\boldsymbol{C}^2+\ldots+\boldsymbol{C}^k$ converge to $(\boldsymbol{I}-\boldsymbol{C})^{-1}$ and provide a simple example where the matrix does not converge.

Part I.3: Fishing in George's Bank

This model is concerned with the New England fishing industry, especially George's Bank, which has fallen on desperate times of late. The model covers most of the 20th century as there is a multitude of data available. It is based upon the age specific Leslie model.

Here, we assume we are looking at the haddock population of the north Atlantic, particularly George's Bank. This is a critical area, for in the early and mid 20th century, New England fishing produced consistently over 80% of the United States fish. Also,in recent times, the diet of Americans has slowly moved away from red meat, for health reasons, and has included more fish.

Matrix algebra and the basic Leslie population model. In this model, we assume that haddock live to 10 years of age, and may have offspring as early as 3 years old. The Leslie model for this situation, with a time period of 1 year, is of the form

$$\boldsymbol{x}(k+1) = \boldsymbol{A}\,\boldsymbol{x}(k) \qquad \boldsymbol{x}(1) \text{ given} \qquad k = 1,2,3,\ldots \tag{1}$$

where $\boldsymbol{A}$ is a 10×10 matrix which is initially assumed to have values:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 1.5 & 4.2 & 3.8 & 2.5 & 1.5 & 0 & 0 & 0 \\ 0.65 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.68 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.75 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.55 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.35 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 \end{bmatrix}$$

Equation (1) may be solved iteratively to give

$$\boldsymbol{x}(k+1) = \boldsymbol{A}^k \, \boldsymbol{x}(1) \qquad k = 1, 2, 3, \ldots \tag{2}$$

1. If the haddock population in 1900 is $\boldsymbol{x}(1) = [100, 0, 0, 0, 0, 0, 0, 0, 0, 0]^T$ (units are millions of pounds), what will the population be like in 1910? 1920? 1930? 1950? 1990? 2000? Based upon what you have seen, do you believe the population to be stable or unstable?

2. Suppose pollution has the effect of lowering each birth rate by 10% of the value given above and each survival coefficient by 15% beginning in 1950. What effect does this have on the population in 1990 as compared to having no pollution effects?

3. Beginning in 1925, assume that fish 3 years old and older are caught at a rate such that each year, 25% of those groups are taken. Fish under 3 years old may not be taken (the distinction is based upon size).

   (a) What is the matrix form of the model now? (modification of equation (1), above). Did you assume the fish were harvested before or after the annual birth process occured? Please generalize to show a model reflecting any harvesting rate which you may denote by h. Thus above, $h = 0.20$.

   (b) What does the population now look like for 1930, 1950, 1995, 2000? Is this a good strategy?
