Part 1.2: Matrix Inverses and Infinite Series

(hand): Find under which conditions the series $\boldsymbol{I} + \boldsymbol{C} + \boldsymbol{C}^2 + \ldots + \boldsymbol{C}^k$ converge to $(\boldsymbol{I} - \boldsymbol{C})^{-1}$ and provide a simple example where the matrix does not converge.

Soln: Let define spectral radius of matrix $C \in C_n$ defined as

$$\rho(C) := \max\{|\lambda|, \lambda \text{ eigenvalue of } C\}$$

* Let C be a square matrix and $C \in C_n\,(C_{n \times n})$ then

$$\lim_{k \to \infty} C^k = 0 \iff \rho(C) < 1$$

and moreover, the geometric series $\sum_{k=0}^{\infty} C^k$ is convergent iff $\rho(C) < 1$

$$\text{Therefore; } \sum_{k=0}^{\infty} C^k = (I - C)^{-1} \star$$

Proof: Let $\rho(C) < 1$, then $\exists\, \varepsilon > 0$ such that $\rho(C) < 1 - \varepsilon$ and thus, $\exists$ an induced matrix normal $\|\cdot\|$ such that $\|C\| \leq \rho(C) + \varepsilon < 1$. From the fact that; $\|C^k\| \leq \|C\|^k < 1$ and as $k \to \infty$ the sequence $\{C^k\} \to 0$. Conversely, assume that, $\lim_{k \to \infty} C^k \to 0$ and let λ eigenvalue of C.

Then $C^k . x = \lambda^k . x$ $(x \neq 0)$ an eigenvector associated with λ so that $\lim_{k \to \infty} \lambda^k = 0$. As a consequence, $|\lambda| < 1$ and because this is true for generic eigenvalue one gets $\rho(C) < 1$

$\rightarrow$ The eigenvalues of $I - C$ are given by $I - \lambda(C)$, $\lambda(C)$ being generic eigenvalue of C. On the other hand, since $\rho(C) < 1$, we deduced that $I - C$ is nonsingular

Then; $(I-C)(I+C+\dots+C^k)=(I-C^{k+1})$ and taking limit for $k\to\infty$ then the theorem follows since

$$(I-C)\sum_{k=0}^{\infty} C^k = I \Rightarrow \sum_{k=0}^{\infty} C^k = (I-C)^{-1}$$

Ex: Take $C=\begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}$ eigenvalues $\lambda_1=7$, $\lambda_2=1$

since $\rho(C)=\max\{|\lambda_1|,|\lambda_2|\} \Rightarrow \rho(C)=7>1$

eigenvectors; $\begin{bmatrix} -2 & 2 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1=x_2$ say $v_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\lambda_1=7$

$\begin{bmatrix} 4 & 2 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad 4x_1=-2x_2$, $2x_1=-x_2$ say $v_2=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ for $\lambda_2=1$

$\Lambda=\begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$, $C=\begin{bmatrix} 5 & 2 \\ 4 & 3 \end{bmatrix}$

$\Rightarrow C^2=P\begin{bmatrix} 7^2 & 0 \\ 0 & 1 \end{bmatrix}P^{-1}$, $C^3=P\begin{bmatrix} 7^3 & 0 \\ 0 & 1 \end{bmatrix}P^{-1}\dots$

$C^k=P\begin{bmatrix} 7^k & 0 \\ 0 & 1 \end{bmatrix}P^{-1}$ as $k\to\infty$

$\lim_{k\to\infty} C^k \cong \infty$ because of that $\sum_{k=0}^{\infty} C^k \cong \infty$ not convergent since $\rho(C)=7>1$.

$\blacksquare$